

SUBSTRATE INTEGRATED WAVEGUIDE RESONANT CAVITY SENSOR

by

Richard Aaron Revia

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Master of Science

in

Electrical Engineering

MONTANA STATE UNIVERSITY
Bozeman, Montana

November, 2013

© COPYRIGHT

by

Richard Aaron Revia

2013

All Rights Reserved

APPROVAL

of a thesis submitted by

Richard Aaron Revia

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to The Graduate School.

Dr. James P. Becker

Approved for the Department of Electrical and Computer Engineering

Dr. Robert C. Maher

Approved for The Graduate School

Dr. Ronald W. Larsen

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a master's degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library.

If I have indicated my intention to copyright this thesis by including a copyright notice page, copying is allowable only for scholarly purposes, consistent with "fair use" as prescribed in the U.S. Copyright Law. Requests for permission for extended quotation from or reproduction of this thesis in whole or in parts may be granted only by the copyright holder.

Richard Aaron Revia

November, 2013

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to all those who made this project possible. Without the unwavering resolve and keen insight of Dr. Jim Becker, my graduate chair and advisor, I would not have been able to complete this work, or even start it for that matter. I would also like to thank the members of my graduate committee, Dr. David Dickensheets and Andy Olson. Their input and suggestions were much appreciated. I would also like to thank Professor Mark Burr for discussions about the applicability of this project with regards to biofilm interrogation.

I must acknowledge my parents, for without their love and support, I would certainly not have been able to achieve all that I have. Lastly, I want to thank all of my colleagues and classmates at Montana State University, especially Stacie Smith for proofing early drafts of this thesis; their encouragement and friendship made my time at MSU invaluable.

TABLE OF CONTENTS

1. INTRODUCTION	1
1.1 Permittivity Measurements and Biosensing Techniques at Radio and Microwave Frequencies	8
1.1.1 Microwave Resonators for Biosensing	11
1.1.2 Cavity Resonators as Permittivity Sensors	14
1.2 A Novel Resonator Structure: Substrate Integrated Waveguide Resonant Cavity	14
1.3 Thesis Overview	15
2. SENSOR DESIGN	17
2.1 Design of an SIW Resonator	18
2.1.1 Microstrip and Microstrip-to-Coaxial Transitions	21
2.1.2 Substrate Integrated Waveguide Design	31
2.1.3 Microstrip-to-SIW Transition	35
2.1.4 High- Q Resonant Cavity	38
2.2 Fabricated Device and Model Comparison	43
2.3 Shortcomings of the Designed SIW Cavity Sensor	50
2.4 Design Summary	53
3. ESTIMATING ANALYTE PERMITTIVITY	55
3.1 Perturbation Theory	56
3.2 Applying the Perturbation Technique to the SIW Cavity	63
3.3 Sensitivity Assessments	75
3.4 Summary of Permittivity Estimation with the SIW Resonator	78
4. CONCLUSION	80
4.1 Suggested Design Outline	81
4.2 Application Space for SIW Biosensors	83
4.3 Recommendations for Future Work	85
REFERENCES CITED	86
APPENDICES	92
APPENDIX A: Scattering Parameters and the SOLT Calibration Method	93
APPENDIX B: Propagation and Resonance in Rectangular Waveguides and Cavities	98

LIST OF TABLES

Table	Page
2.1 The Effect of $B_{\perp}$ on f_0 and Q	41
2.2 The Effect of A on f_0 and Q	42
2.3 The Effect of D_R on f_0 and Q	44
2.4 Finite Element Model Correction.....	48
2.5 Teflon Plot Data.....	49
2.6 Reported Literature Complex Permittivity Values for Reference Liquids at $T = 20^{\circ}\text{C}$ and $f = 2.8\text{ GHz}$	50
2.7 Simulated SIW Response Values to Lossy Reference Liquids.....	51
2.8 Summary of Design Values	54
2.9 Comparison of f_0 and Q Values	54
3.1 Select Estimates of ϵ'_{rs}	69
3.2 Estimated ϵ''_{rs} Values.....	70
3.3 Q Values Corresponding to the Estimates of ϵ''_{rs}	71
3.4 Permittivity of Reference Liquids at $T = 20^{\circ}\text{C}$ and $f = 2.8\text{ GHz}$	73
3.5 Simulated Estimates Calibrated with Teflon.....	75
3.6 Simulated Estimates Calibrated with Butanol	75
3.7 Salient Data from the $ S_{21} $ Responses of the HFSS Simulations for the $D_R = 39\text{ mils}$ Case	77
3.8 Determining Factors for the Range of Measurable ϵ'_{rs} and ϵ''_{rs}	78
B.1 Summary of EM Wave Propagation in Lossless Rectangular Waveguides.....	118

LIST OF FIGURES

Figure	Page
1.1 The polarization of an atom.....	4
1.2 Response of a dielectric material to electrical disturbances	5
2.1 Schematic of the SIW resonant cavity sensor.....	19
2.2 Microstrip transmission line	22
2.3 HFSS microstrip model.....	27
2.4 Microstrip Z_0 versus width W as a function of frequency.....	27
2.5 Coaxial and microstrip field lines.....	28
2.6 Coaxial to microstrip transition.....	30
2.7 Metallic fixtures and N type coaxial connector	30
2.8 Rectangular waveguide to SIW	31
2.9 SIW topology	33
2.10 Finite element analysis of an SIW.....	35
2.11 Microstrip-to-SIW transition	36
2.12 Finite element analysis of the microstrip-to-SIW transition	36
2.13 HFSS comparison of the transmission structures.....	37
2.14 Rectangular waveguide to SIW	38
2.15 HFSS model of an SIW cavity resonator	40
2.16 The effect of $B_{\perp}$ on f_0 and Q	41
2.17 The effect of A on f_0 and Q	42
2.18 HFSS models of the resonant cavity with different receptacle sizes	44
2.19 The effect of D_R	44
2.20 HFSS model of the full SIW cavity sensor	44
2.21 Finite element analysis of the full SIW cavity resonator	45
2.22 Photographs of the fabricated sensor	46

LIST OF FIGURES – CONTINUED

Figure	Page
2.23 Testing setup for the SIW cavity sensor.....	46
2.24 Measured $ S_{21} $ response of the fabricated SIW cavity resonator.....	47
2.25 HFSS model correction.....	48
2.26 Measured and simulated responses for the unloaded SIW cavity and a Teflon test sample.....	49
2.27 HFSS simulations of the SIW sensor loaded with lossy reference liquids	51
2.28 Measured $ S_{21} $ response to lossy reference liquids.....	52
3.1 Perturbation of a resonant cavity.....	57
3.2 HFSS SIW cavity model	64
3.3 Perturbation theory estimates of ϵ'_{rs} on HFSS simulations of the SIW sensor with $D_R = 39$ mils	65
3.4 The effect of choosing an appropriate calibration standard for perturbation theory	66
3.5 The effect of D_R on the accuracy of measurements of ϵ'_{rs}	68
3.6 The effect of D_R on the accuracy of measurements of ϵ''_{rs}	70
3.7 Permittivity estimates for lossy liquids.....	74
3.8 Simulated $ S_{21} $ responses for the lossy liquids with $D_R = 39$ mils	76
A.1 2-Port network scattering parameters	94
B.1 Electromagnetic wave classifications	100
B.2 Geometry of a rectangular waveguide	101
B.3 A rectangular cavity resonator	119
B.4 Frequency response of a rectangular waveguide.....	124
B.5 Frequency response of a rectangular cavity.....	125

ABSTRACT

A current area of active research is the development of biosensors. Biosensors have been constructed to examine a large range of target analytes such as enzymes, antibodies, DNA, and cells. The majority of currently developed biosensors require the use of labels which attach to an analyte to enhance the sensitivity of the sensor to a significant level. Labeling targets introduces many drawbacks: added complexity to the detection process, increased preparation time, and most importantly, the possible modification of analyte properties due to the attachment of the label. For these reasons, there has been much attention on electronic means of label-free biological agent detection.

One such electronic biosensing method is the use of a resonant circuit operating at microwave frequencies for impedance and dielectric spectroscopy. In these spectroscopic measurements, changes in the resonant frequency of the circuit are detected and correlated to the presence of a specific analyte. Various resonator circuits have been utilized in dielectric spectroscopy and biomolecule detection; however, these biosensors, although label-free, possess their own idiosyncratic complications such as imprecise and convoluted test sample deposition schemes.

To address some of the challenges associated with existing biosensors, a device is presented demonstrating the potential to be used for label-free biosensing and promises a convenient sample deposition procedure. The instrument is based on the construction of a substrate integrated waveguide analog of an enclosed section of rectangular waveguide. Classical microwave engineering principles were used to give an outline of key electrical characteristics and dimensions, and full-wave finite element analysis software was utilized to further refine and optimize the device. A fabricated prototype was tested through measurement of scattering parameters using a network analyzer.

The archetypal resonant circuit discussed herein can be used to extract the complex permittivity from test materials. Discussions of the cardinal design parameters, sensitivity analysis, and permittivity extraction techniques are provided.

Suggestions for continued development are presented based on experience gained from the design of the prototype sensor. Proposed future work includes a scaled-down version of the substrate integrated waveguide resonator and testing with biological agents such as biofilms and single cells.

CHAPTER 1

INTRODUCTION

Examples abound in which precise knowledge of the permittivity and permeability of a material is required, as the equations governing how an electromagnetic (EM) field behaves in free-space versus inside some arbitrary material at a macroscopic level, Maxwell's equations, can be completely specified by these two parameters. With all other factors held constant (i.e., frequency, electric field intensity, and magnetic field intensity), the difference in the EM fields between two substances is described by the important material properties known as the permittivity ϵ and permeability μ . Maxwell's equations can be written in phasor form as¹

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}} - \tilde{\mathbf{M}}, \quad (1.1a)$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}} + \tilde{\mathbf{J}}, \quad (1.1b)$$

$$\nabla \cdot \tilde{\mathbf{D}} = \rho, \quad (1.1c)$$

$$\nabla \cdot \tilde{\mathbf{B}} = 0, \quad (1.1d)$$

with the constitutive relations:

$$\tilde{\mathbf{D}} = \epsilon\tilde{\mathbf{E}}, \quad (1.2a)$$

$$\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}, \quad (1.2b)$$

where

$\tilde{\mathbf{E}}$ = the electric field,

$\tilde{\mathbf{M}}$ = the magnetic current density,

$\tilde{\mathbf{H}}$ = the magnetic field,

$\tilde{\mathbf{J}}$ = the electric current density,

$\tilde{\mathbf{D}}$ = the electric flux density,

ρ = the electric charge density.

$\tilde{\mathbf{B}}$ = the magnetic flux density,

¹The notation conventions used herein are as follows: medium-weight italic font for symbols denoting scalar quantities (e.g., R); boldface roman font for symbols denoting vectors (e.g., $\mathbf{F}_e$); unit vectors printed with a circumflex above the symbol (e.g., $\hat{\mathbf{R}}$); phasor quantities printed with a tilde above the symbol (e.g., $\tilde{\mathbf{E}}$).

To give some practical motivation and to further illustrate the need for accurate measurements of permittivity and permeability, consider the following examples:

- The characteristic impedances of electrical transmission lines operating at radio and microwave frequencies are directly dependent upon ϵ and μ . (See equation 2.3 for example.)
- The radiative transfer of microwave energy carrying communications signals is dependent upon the permittivities of the media through which it travels, which includes the Earth's atmosphere and water [1].
- The capacitance of a parallel plate capacitor is directly proportional to the permittivity of the material between the two conductive plates of the device. Thus, precise knowledge of a material's electrical properties will allow for the efficacious construction of crucial components.

Given this need to accurately characterize the electrical properties of materials, much effort has been applied to obtaining precise measurements of the permittivity and permeability of arbitrary materials. For free-space, $\epsilon = \epsilon_0 \approx 8.854 \times 10^{-12}$ [F/m] and $\mu = \mu_0 = 4\pi \times 10^{-7}$ [H/m]. Most naturally occurring materials are nonmagnetic, and as such their permeability is the same as that of free-space, $\mu = \mu_0$ [2]; common exceptions are ferromagnetic materials such as yttrium iron garnet and other materials containing aluminum, cobalt, manganese, and nickel [3].

Equation 1.2a gives the constitutive relation between the electric field intensity $\tilde{\mathbf{E}}$ and the electric flux density $\tilde{\mathbf{D}}$. To appreciate the significance of the permittivity of a material, it is beneficial to delve deeper into the meaning of equation 1.2a. All electrically charged entities (e.g., individual electrons and protons, ions, charged molecules) are sources of an electric force $\mathbf{F}_e$. The electric force acting on some charged body due to another charged

body is given by Coulomb's law:

$$\mathbf{F}_e = \hat{\mathbf{R}}_{12} \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2} \quad (\text{in free-space}), \quad (1.3)$$

where q_1 is the charge associated with the source entity, q_2 is the charge associated with the second body, R_{12} is the distance between the two charged bodies, $\hat{\mathbf{R}}_{12}$ is a unit vector pointing from charge q_1 to charge q_2 , and ϵ_0 is a universal constant known as the permittivity of free-space. In equation 1.3 it is assumed that the two charged bodies are separated by free-space (i.e., there is no material between the two charges other than free-space). The electric field $\mathbf{E}$ can then be defined as the force per unit charge acting at a given point due to the source body:

$$\mathbf{E} = \frac{\mathbf{F}_e}{q_2} = \hat{\mathbf{R}} \frac{q_1}{4\pi\epsilon_0 R^2} \quad (\text{in free-space}), \quad (1.4)$$

where R is the distance between the source charge and the observation point, and $\hat{\mathbf{R}}$ is the radial unit vector pointing away from the charge.²

If the source charge is not in free-space, but has been placed inside of some medium, the net electric field inside the medium will be that given by equation 1.4 but reduced by a factor of ϵ_r , where ϵ_r is the relative permittivity of the medium, a property inherent to the material in which the source of charge has been placed [4]. To understand this reduction in the strength of the electric field, consider that, in the absence of a disturbance, the atoms of a nonpolar dielectric material will have a cloud of negative charge situated around a region of positive charge such that the net charge of the material is zero (the positive charge will be the protons in the nuclei of the atoms and the negative charge will be the electrons). Upon exposing such atoms to an external electric field, individual regions of charge will be displaced from their equilibrium positions and orient themselves in such a way as to counteract the effects of the introduced electrical disturbance. This rearrangement of the

²As in equation 1.3, equation 1.4 assumes all bodies of interest are in vacuum.

structure of the atoms is called polarization. Figure 1.1 gives a graphical depiction of the polarization of an atom in a dielectric material.³

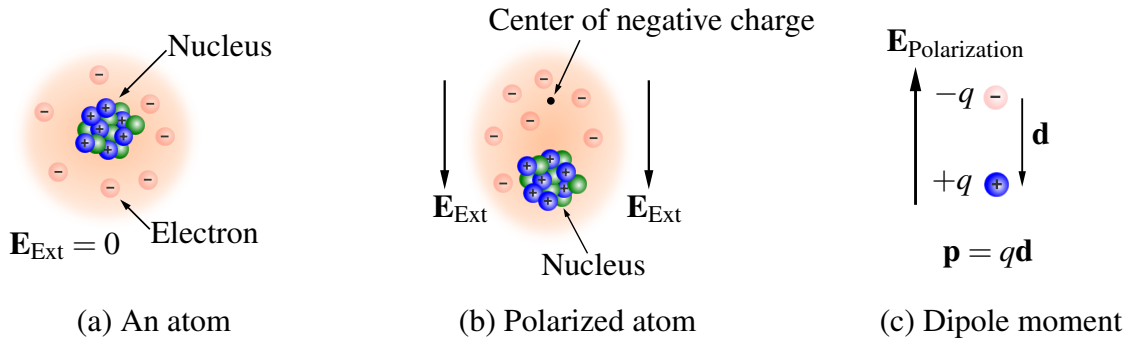


Figure 1.1: The polarization of an atom. (a) In an atom with no external electric field, the centers of positive and negative charge are co-located and there is no dipole moment. (b) The polarization of an atom due to an applied electric field. The centers of positive and negative charge are displaced from their coincident location in the presence of an external electric field. (c) The resulting dipole moment $\mathbf{p}$ due to the polarization of the atom. An electric field associated with the dipole moment now exists and is in direct opposition to the applied external field.

The net macroscopic electric field located in a material with a source charge placed inside it is less than the strength of the electric field due to the source charge in free-space because the polarization of regions of charge in the material produces electric fields via dipole moments which are in direct opposition to the source charge's electric field. Figure 1.2 presents a simplified view of the polarization of a nonpolar dielectric material induced by both a point charge and a uniform external electric field.⁴ In both cases, polarization of the dielectric material reduces the net electric field inside the medium.

³In general, a *dielectric* material refers to a nonconductive substance in which the constituent atoms or molecules *polarize* when exposed to an external electric field (charges are bound to their atoms) as opposed to a conductor in which charge *flows* in the presence of an applied electric field (charges are loosely held to atoms).

⁴While Figure 1.2 shows the polarization of a nonpolar material, a similar situation occurs in materials with constituent molecules that have permanent dipole moments such as water.

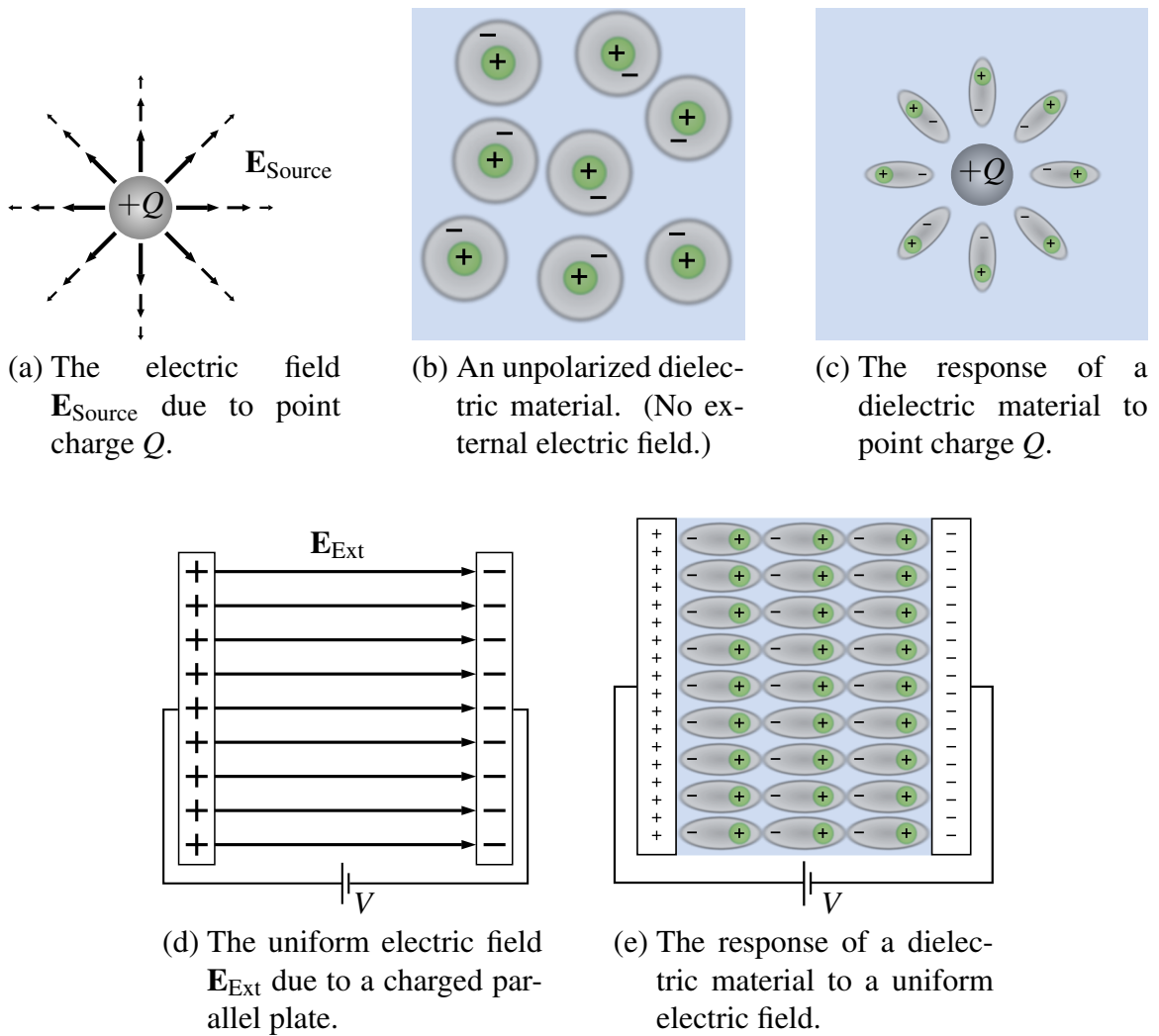


Figure 1.2: The response of a dielectric material to electrical disturbances: (a) For a positive point charge, the electric field is defined to radiate away from the source in the radial direction $\hat{\mathbf{R}}$. (b) On average, the centers of positive charge (atomic nuclei) coincide with the centers of negative charge (orbital electrons within each atom), resulting in zero dipole moments (on average) and thus no net electric field within the material. (c) Polarization of the atoms in the dielectric displace centers of positive and negative charge from being co-located, resulting in a dipole moment associated with each atom. These dipole moments counteract the electric field of the source charge, thus reducing the overall net electric field inside the dielectric material. (d) The electric field between two conductive parallel plates connected to a DC voltage source is uniform. (e) Again, the polarization of the dielectric medium between the parallel plates causes the net electric field inside the dielectric to be less than what the electric field would be in free-space. This reduction in the strength of the electric field is characterized by the material's permittivity ϵ .

It is of paramount importance to note that the term ϵ in equation 1.2a is complex (and likewise for μ in equation 1.2b):

$$\text{complex permittivity: } \epsilon = \epsilon' - j\epsilon'', \quad (1.5)$$

$$\text{complex permeability: } \mu = \mu' - j\mu''. \quad (1.6)$$

The parameters ϵ and μ are properties specific to a given material. Furthermore, they are not constant, but rather are material properties that depend upon temperature, humidity, material orientation, and the magnitude and frequency of the applied EM field. The frequency dependency of ϵ arises from the fact that polarization phenomena in materials cannot form and change instantaneously. It takes some time for the constituents of a material to respond to the applied electric field. Representing the permittivity as a complex variable allows for convenient inclusion of the frequency dependency of ϵ .

The electric fields that arise in a medium due to the displacement of charge caused by the introduction of a source charge (or an applied external field) are accounted for by the electric polarization field $\tilde{\mathbf{P}}$. In a linear and isotropic material, which most ordinary materials can be approximated as where macroscopic effects are concerned, the polarization field is directly proportional to (linear) and in the same direction as (isotropic) the applied electric field such that

$$\tilde{\mathbf{P}} = [\epsilon' - j\epsilon'' - \epsilon_0] \tilde{\mathbf{E}}. \quad (1.7)$$

Furthermore, the polarization field defines the electric flux density $\tilde{\mathbf{D}}$ by

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}. \quad (1.8)$$

Substituting equation 1.7 into equation 1.8 and by use of equation 1.5

$$\begin{aligned}
 \tilde{\mathbf{D}} &= \epsilon_0 \tilde{\mathbf{E}} + \epsilon' \tilde{\mathbf{E}} - j\epsilon'' \tilde{\mathbf{E}} - \epsilon_0 \tilde{\mathbf{E}}, \\
 &= \epsilon' \tilde{\mathbf{E}} - j\epsilon'' \tilde{\mathbf{E}}, \\
 &= (\epsilon' - j\epsilon'') \tilde{\mathbf{E}}, \\
 &= \epsilon \tilde{\mathbf{E}},
 \end{aligned}$$

which is equation 1.2a. From equations 1.8 and 1.7 it is seen that $\tilde{\mathbf{D}} \propto \epsilon' \tilde{\mathbf{E}}$ and $\tilde{\mathbf{P}} \propto \epsilon' \tilde{\mathbf{E}}$. Thus, ϵ' may be viewed as a measure of the ease with which a material can be polarized by an applied electric field or as a measure of how much opposition a material puts up to an applied electric field. The real part of ϵ , which is ϵ' , is proportional to the polarization field that arises in a material; thus, ϵ' is also a measure of the stored energy in the material. Energy is required in order to displace charges during polarization; once the applied external electric field vanishes, the charges return to their original locations, releasing the energy that displaced the charges. By convention, $\epsilon' = \epsilon_0 \epsilon_r$ where ϵ_r is known as the relative permittivity.^{5,6} The imaginary part of ϵ , which is ϵ'' , accounts for dissipative loss (i.e., energy loss in the form of heat due to damping of the vibrating dipole moments) in the medium [3]. In many cases, ϵ'' is not of direct interest, but rather another quantity known as the loss tangent, which is defined as

$$\tan \delta = \frac{\omega \epsilon'' + \sigma}{\omega \epsilon'}, \quad (1.9)$$

where σ quantifies the conductive loss in the medium (heating due to free charge conduction). The term $\omega \epsilon'' + \sigma$ can be viewed as the total effective conductivity of a material.⁷

⁵ ϵ_r is also referred to as the dielectric constant, but this term has been deprecated by some standards organizations due to possible ambiguity with its use [5].

⁶A similar convention exists for μ' with $\mu' = \mu_0 \mu_r$.

⁷ ϵ'' accounts for the loss due to *bound* charge whereas σ is attributed to the loss caused by *free* charge.

For macroscopic considerations the loss mechanisms are not distinctly discernible.

At the macroscopic level, all materials will permit EM fields to pass through them to some extent, even good conductors [6]. In the design of electrical and optical systems, it is often necessary to know how EM fields will propagate through a given material; that is, how do the EM fields reflect, transmit, and scatter when they reach an interface of two materials? Understanding these characteristics requires knowledge of the complex permittivity and permeability of a given material.

1.1 Permittivity Measurements and Biosensing Techniques at Radio and Microwave Frequencies

The previous discussion provided the motivation for measurement of the complex permittivity of arbitrary materials. As mentioned, the complex permittivity is a function of frequency. As such, various measurement schemes are needed in order to characterize ϵ over a broad range of frequencies. Frequencies of particular concern are the radio and microwave frequencies. Radio frequencies (RF) typically refer to signals in the 30 MHz–3 GHz range, whereas microwave frequencies typically correspond to the 3 GHz–300 GHz range. The electrical wavelength of microwave signals range from $\lambda = c/f = 10$ cm to $\lambda = 1$ mm.⁸ Once the electrical wavelength of the signals propagating through a device are on the order of the dimensions of the electrical components that make up the device, phase differences exist over individual components for both voltage and current. Classical circuit theory cannot be used in these situations as the phase difference for voltage and current must be accounted for as well as the possibility for reflected signals occurring in the device. Circuits operating at frequencies below radio frequencies can generally be analyzed using standard circuit theory relationships such as Ohm’s law and lumped-element circuit components such as resistors, capacitors, and inductors. At higher frequencies, in which the electri-

⁸The speed of light is denoted as $c \approx 3.0 \times 10^8$ [m/s], electrical wavelength by λ , and frequency by f .

cal wavelength is much shorter than the device dimensions, optical engineering principles may be used to analyze the systems. At these two extremes of the frequency spectrum, approximations allow Maxwell's equations to be simplified into scalar theories describing the physical operation of a system as opposed to vector theories. For the engineering of microwave systems, the vector differential form of Maxwell's equations must often be used directly, and as such, precise knowledge of the permittivity and permeability of a material is required.

Even for common materials, the range of values for both ϵ_r and $\tan \delta$ encountered is broad. Values for ϵ_r can vary from 1.0 for air, 2–4 for plastics, 5–10 for glass and ceramics, 80 for water, 5 to greater than 1000 for biological tissues, and up 2000 for ferromagnetics [6]. Likewise, the loss tangent can range from 1×10^{-4} for quartz to 0.157 for distilled water [3]. In addition to a wide range of values for the complex permittivity, the physical makeup of the materials under study can be diverse: the materials can be solid, liquid, or gas, they may be hard or soft, etc. Thus, to analyze the electrical properties of all possible materials, many different measurement techniques have been developed to accommodate certain categories of materials:

- Admittance method (also called the quasi static method): the admittance between two conductive plates is measured with and without a test material between the two plates. From these two admittance measurements, the permittivity of the test material is determined. Essentially, the capacitance of a parallel-plate capacitor is measured with a known dielectric material and then with an unknown material. The change in capacitance is attributed to the change in permittivity introduced by the test material. This method is limited to RF frequencies and below due to the vanishingly small size of the parallel plate capacitor for frequencies above about 1 GHz.

- Cavity resonator techniques: a resonant cavity is loaded with a test material, and the resulting shift in resonant peak frequency and quality factor determined from the scattering parameters are related to the complex permittivity of the test material.
- Transmission line methods:
 - A test material can be placed between the conductors of a transmission line (e.g., coaxial line, rectangular waveguide, microstrip) and the scattering parameters can be used to determine both the permittivity and permeability of the material.
 - The flat-faced end of a coaxial probe can be placed in direct contact with a test material. The EM wave propagating through the coaxial line will launch fringe fields into the test material. The measured reflection coefficient can be related to the permittivity of the test material through a modal analysis of the fields in the test material. Similar techniques may be carried out with alternative waveguide geometries such as rectangular waveguides.
- Free-field methods: as with transmission line methods, either scattering parameters or reflection coefficients are measured and related to the complex permittivity, but instead of employing guided waves, antennas in air are used.

The methods outlined above serve to provide an overview of the classifications usually given to the plethora of techniques that have been developed for measuring the complex permittivity of materials. The majority of the methods in use are variations of those described above. Reviews of the techniques used to measure the complex permittivity of materials at RF and microwave frequencies can be found in [6–9].

1.1.1 Microwave Resonators for Biosensing

In addition to the methods and tools developed for ascertaining complex permittivity information of arbitrary materials to be applied to more conventional aspects of RF and microwave engineering such as communications and remote sensing, techniques for measuring the permittivity of biological substances have been investigated for many decades. Initially, values for the complex permittivity of various human tissues were interrogated to gain an understanding of the interaction between such tissues and electromagnetic radiation oscillating at microwave frequencies [10–13]. These initial measurements on the permittivities of human tissues were made by filling a small section of a coaxial line or waveguide with a sample; then, by extracting the resulting signal propagation properties (specifically the attenuation constant α and the phase constant β) from measurements of the resulting standing wave pattern, the complex permittivity of the sample was determined.

More recently, researchers have been exploiting the dielectric properties of biological materials at RF and microwave frequencies for label-free medical diagnostics. Devices designed to detect the presence of biological agents are classified as biosensors. Electronic techniques for biodetection hold great potential for use in various settings in which low-cost, miniaturized, and robust biomaterial sensing is needed such as in point-of-care diagnostics and biowarfare agent detection [14]. One of many possible classification schemes for biosensors is to separate the sensors into two categories: those that require the use of labels and those that are label-free. The majority of presently developed biosensors employ labels in some aspect of the detection strategy. Some sensors use immobilized label molecules which selectively attach to target analytes; when a target agent attaches to a label, a change in one of the sensor's parameters (e.g., current or voltage) is measured and assumed to correlate with the presence of the target. Alternatively, a sensor may employ a label to increase the sensitivity of the device, for without a label, the detection limits

would be insufficient for practical use. For example, fluorophores are often used to target specific proteins in solution and fluorescence microscopy is used to detect the presence of the proteins of interests. However, if the fluorophores attach to other extraneous proteins, then an intermediate label that attaches more selectively to the proteins of interest can be first bonded to the fluorophores to increase the sensitivity of the device [15]. However, the use of labels presents a number of drawbacks such as increased sample preparation complexity, longer detection times, and most detrimentally, the possible alteration of target properties due to the binding of a label [16].

Given these disadvantages, label-free biosensing has garnered much interest. One promising label-free biosensing modality is to interrogate the dielectric properties (i.e., measure the complex permittivity) of a target. Information about the dielectric properties of biological structures can be used in medical diagnostics since the electrical permittivity of any material is directly related to its chemical makeup and underlying microstructure. Changes in tissue microstructure arising from pathological sources (e.g., cancer) cause changes in the dielectric behavior as compared to healthy tissue. For example, due to the higher water content of tumorous tissue, healthy tissue of the same type will exhibit lower permittivity and conductivity values [17].

Foregoing the use of labels for biosensing results in a situation where target analytes interface directly with a sensor's architecture, as opposed to the situation that arises with the use of a label in which target entities indirectly interact with the sensor structure. This condition introduces a design criterion on any label-free biosensor: target analyte dimensions must be commensurate with the size of the sensor's detection region. Furthermore, when making a case for the development of a mass-producible, low cost biosensor, the ease and economics of manufacturability must be considered. These guidelines match the label-free interrogation of bulk bio-tissues and cells to ideal use with devices that utilize millimeter and centimeter wavelength non-ionizing electromagnetic radiation. The choice

of microwave devices for such biosensors allows for the convenient use of standard printed circuit board manufacturing and microfabrication techniques for device construction. For designs operating at the lower end of the microwave spectrum, centimeter and millimeter dimensions will define the device architecture and provide a device with the ability to analyze samples of similar dimensions (e.g., bulk tissue samples and cell cultures). Devices of this size are well suited for printed circuit board manufacturing processes. Operating at the upper end of the microwave region calls for dimensions on the micron scale which will promote the examination of single cells and thin-film biological tissues. Bulk micromachining procedures facilitate the construction of such devices. In addition to the benefits of having dimensions amenable to certain biological structures and being cheaply and relatively simple to fabricate, microwave biosensors also have the potential to be noninvasive and nondestructive, have quick measurement times, facilitate real-time monitoring, utilize biocompatible materials, and use low sample volumes.

Label-free biosensing schemes based on dielectric measurements made at RF and microwave frequencies have been used to assess skin edema [18], monitor cell growth and differentiation for tissue engineering [19], and distinguish between healthy and cancerous tissue [20, 21]. At the cell-scale, RF and microwave dielectric measurements have been used to separate and sort cell subpopulations from human blood [22], estimate microbial biomass of yeast cell suspensions [23], and evaluate the aggressiveness level of cancerous cells [24].

In an effort to determine the dielectric properties of single cells, many methods have been developed such as electrorotation and dielectrophoretic techniques [25, 26], impedance cytometry [27], scanning dielectric microscopy [28], coplanar waveguide devices [29, 30], and resonant circuits [31, 32].

1.1.2 Cavity Resonators as Permittivity Sensors

A widely used and accurate method of determining the complex permittivity of a sample at microwave frequencies is the cavity perturbation technique. In this method a resonant cavity, typically cylindrical or rectangular in geometry, is created in which a small volume of the resonator is designated to hold a sample, referred to as a cell. First, the resonant frequency f_0 and quality factor Q of the unloaded (empty) resonator is measured. Then, a sample material is loaded into the cell and the resonant frequency and quality factor of the loaded resonator are measured. The change in resonant frequency and quality factor can then be related to the complex permittivity of the sample loaded into the cell of the resonant cavity.

1.2 A Novel Resonator Structure: Substrate Integrated Waveguide Resonant Cavity

In Section 1.1.1 a case was made for the use of a biosensor operating at microwave frequencies for determination of the complex permittivity of a biological sample, while Section 1.1.2 described a method of measuring the dielectric parameters of an analyte at microwave frequencies. A few examples of microwave biosensors were referenced in Section 1.1.1. Whereas these methods capitalize on the many advantages of using high frequency non-ionizing radiation for biosensing (e.g., noninvasive, nondestructive, quick measurement times, real-time monitoring, biocompatible materials, low sample volumes, etc.), they suffer from convoluted sample deposition procedures and cumbersome data manipulation and calibration steps in order to obtain the complex permittivity of an analyte.

To address some of the challenges associated with existing sensors, a novel architecture for a microwave biosensor is proposed that employs a planar high- Q resonant cavity to determine the complex permittivity of biological agents using the perturbation technique.

The geometry of this prototypical sensor facilitates simple analyte deposition procedures, and analytic expressions for the relationship between the real and imaginary parts of the complex permittivity and directly measurable sensor response parameters (i.e., resonant frequency and quality factor) are obtainable. Device dimensions scale with frequency, and thus miniaturization of this device can be envisioned allowing for the employment of microfabrication procedures for inexpensive, highly repeatable device production, and the incorporation of microfluidic channels for precise, contamination-free analyte handling.

The proposed resonant cavity structure utilizes a substrate integrated waveguide (SIW) analog of a closed section of rectangular waveguide. The SIW allows for the construction of a resonant circuit that is readily compatible with other planar devices, unlike the standard metal cavity. Furthermore, the SIW design is easily scalable over a broad frequency range (1–100 GHz) giving such devices a large application space.

1.3 Thesis Overview

In order to investigate the use of a planar-compatible resonant structure for permittivity analysis in the microwave and millimeter-wave range as a means of label-free biosensing, a variation on a rectangular cavity resonator was designed based on a substrate integrated waveguide architecture. The specific sensor geometry considered here has dimensions that preclude it from the ability to study microscopic entities such as single cells or biomolecules; however, the intent of the present sensor design is to demonstrate the potential such an architecture has for biosensing capabilities. Large component dimensions were intentionally selected for this prototype to facilitate inexpensive manufacturing through printed circuit board techniques and facile testing with readily available dielectric reference standards. Based on the success of this initial design, scaled-down versions are envisioned that employ the same overall device structure and design approach, but require photolitho-

graphic and other microfabrication techniques involving clean room environments to construct a device with the dimensions appropriate for cell-scale and biomolecule detection.

Chapter 2 discusses the design and testing of a prototype sensor. Classical microwave engineering principles were used to provide a starting framework for the design of the sensor. A full-wave finite element analysis tool, ANSYS's High Frequency Structural Simulator (HFSS), was used to further refine and optimize device operation. The fabricated prototype sensor was then tested using a vector network analyzer to measure the scattering parameters of the device.⁹ Experimental results are compared to the finite element models. Chapter 3 discusses a modified sensor and describes the permittivity extraction method. Finite element models are used to show the efficacy of this sensor design in determining the complex permittivity of samples. A sensitivity analysis of the sensor is given to provide guidelines for the application limitations of this device. Finally, Chapter 4 provides a summary of the results of the design and testing of the SIW biosensor. Suggestions for future directions for development of this device are given based on experience with prototype designs.

⁹See Appendix A for a review of scattering parameters.

CHAPTER 2

SENSOR DESIGN

As discussed in section 1.1.1, previously developed resonant structures operating at microwave frequencies have been successfully utilized to measure the permittivity of various analytes as well as perform in biosensing applications. In section 1.2 the benefits of using a substrate integrated waveguide resonant circuit were outlined. In the present chapter, the design of a high- Q SIW resonator operating over a frequency range of approximately 1.5 to 3.0 GHz is described. The design components include a high- Q resonant cavity within a substrate integrated waveguide, a microstrip-to-SIW transition, and a coaxial-to-microstrip transition.

While the design approach that follows is valid and can be used as a guide for building similar versions of this device, it will be shown that the volume in which samples to be tested are placed (referred to as the analyte receptacle) used in this particular design violates the conditions of the perturbation technique. As such, the complex permittivity of samples analyzed with this sensor geometry cannot be extracted using the simple relationships that follow from perturbation theory. Of course, a full-wave analysis or use of a finite element simulator could be used to determine the dielectric properties of samples from experimental data measured with this sensor layout, but such procedures go against the spirit of convenience this sensor structure is touted to inspire. However, as will be discussed in Chapter 3, by keeping all other dimensions constant, reducing the diameter of the analyte receptacle alone allows for the use of the analytic equations of perturbation theory for the easy computation of the complex permittivity of a sample.

It is the goal of this chapter to demonstrate a possible design approach for describing the geometry of all necessary sensor components, with accompanying equations, *except for*

the analyte receptacle. The design constraints of the analyte receptacle will be discussed in Chapter 3. Agreement between design equations, finite element simulations, and measurements made on a fabricated device will be used to verify the design approach. It should be noted that the specific sensor architectures discussed in Chapters 2 and 3 are much too large to accommodate any type of biosensing other than perhaps large samples of bulk tissues. The present objective is to provide a proof of concept by showing a large-scale version of a device that may ultimately be miniaturized to allow for single cell biosensing.

2.1 Design of an SIW Resonator

The SIW resonant cavity sensor includes a number of discrete structures that are linked together to form a complete device. A cursory sketch of the sensor's operation provides some insight into a possible design approach. The sensor's transducer is a resonant cavity which is excited by a time-varying electromagnetic field. Electromagnetic energy is introduced into the resonant cavity via a chain of transmission lines. First, a coaxial cable carries a signal to a coaxial connector or launcher. A coaxial-to-microstrip transition is then used to transfer power from the coaxial transmission line to a microstrip. The microstrip then feeds into a substrate integrated waveguide structure which in turn feeds power into the resonant cavity.

Transmission line transitions are needed at each structural interface to minimize reflective losses to achieve robust power transfer throughout the device. A successful transition compensates for both the mechanical mismatch between the spatial distributions of the electric and magnetic fields and the electrical disparity between the impedances of the two media. Electromagnetic field compensation is generally achieved through a mechanical design which configures the dimensions and geometry of the transition to facilitate a smooth

conversion of the field distribution. Electrical impedance matching is obtained through quarter-wave transformers and tapered transmission lines [33].

The resonant cavity of this sensor is based on the design of an enclosed section of rectangular waveguide (which is itself a cavity resonator [3]) adapted to an equivalent substrate integrated circuit of such a resonator.¹ In the center of the resonant cavity, a small volume of the underlying substrate is removed to form an analyte receptacle inside which test materials are placed to be probed by the energy introduced into the resonant cavity. Inserting a test material into the analyte receptacle results in a change in the permittivity content of the overall sensor causing a detectable shift in the resonant peak frequency and a change in the quality factor of the device which can then be correlated to the permittivity of the analyte. The receptacle position is chosen to be in the center of the cavity as the electric field is at a maximum at this location and the magnetic field a minimum when considering the dominant resonant mode (TE_{101}); as such, changes in the resonant frequency may be primarily attributed to changes in permittivity as opposed to permeability. Shown below in Figure 2.1 is a schematic of the circuit to be designed.

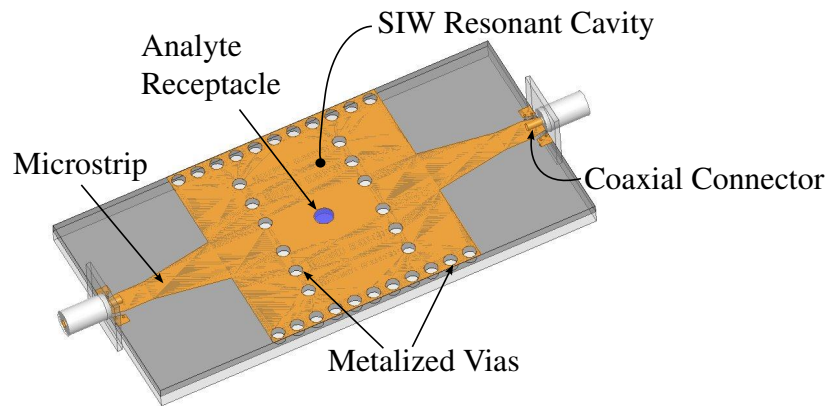


Figure 2.1: Schematic of the SIW resonant cavity sensor.

¹For a brief review of electromagnetic wave propagation in rectangular waveguides and rectangular resonant cavities, see Appendix B on page 99.

Each portion of the sensor is designed using classical microwave engineering guidelines, then further refined and optimized via a full-wave simulation tool, ANSYS Corporation's High Frequency Structure Simulator (HFSS), a finite element method solver. Before delving into the design of each component, a few constraints need to be set in order to provide a starting point.

Two design criteria are held with the utmost regard throughout the design process:

1. The sensor should be user friendly.
2. The sensor should be simple and inexpensive to fabricate and test.

These requirements exemplify the improvements on existing microwave biosensors this design involves. Keeping in mind that what is to be designed at present is a prototype that will not be used specifically for biosensing, but serve as a conceptual example, the structure sizes selected here are intentionally chosen to be quite large compared to what is ultimately needed for biosensing. Specifically, two design parameters are chosen to both provide a starting point and to satisfy design criteria #1 and #2:

- Use a relatively thick substrate; let the substrate thickness $h = 125$ mils.
 - Since this circuit will operate with microwave frequencies, a substrate is needed to serve as the supporting material for the overall structure. The thicker the substrate, the larger the analyte receptacle can be, and the larger the analyte receptacle is, the easier it will be for a user to introduce test materials into the device. This addresses design criterion #1.
- Choose a resonant frequency f_0 at the lower end of the microwave spectrum; design for $2.5 \text{ GHz} \leq f_0 \leq 3.0 \text{ GHz}$.
 - The lower the operating frequencies are, the larger the dimensions of the circuit will be. This allows for the circuit size to facilitate a user friendly device, thus meeting design criterion #1. That is, the device will be handheld, lightweight,

portable, and have an analyte receptacle of convenient dimensions for the deposition of testing materials.

- Also, this allows for a circuit that can be fabricated with commonplace printed circuit board techniques, thus satisfying design criterion #2.

To be sure, the exact values selected for the substrate thickness and unloaded resonant frequency are somewhat arbitrary, but since this is a preliminary design, there is a lack of information at this point to inform on possible optimal values for these two design parameters.

With these constraints in place, the sensor design continues in the following order

1. microstrip line,
2. microstrip-to-coax transition,
3. substrate integrated waveguide,
4. substrate integrated waveguide-to-microstrip transition,
5. resonant cavity,
6. combine the individual components, and
7. design the analyte receptacle.²

2.1.1 Microstrip and Microstrip-to-Coaxial Transitions

Ultimately, this sensor will be used to quantify the complex permittivity of a test analyte by measuring the change in resonant frequency and quality factor that results due to the presence of the test material in the resonant cavity. The values of the resonant frequency

²While the approach taken for designing the analyte receptacle geometry in this chapter results in a sensor that can be used to measure the complex permittivity of an analyte, an alternative procedure is discussed in Chapter 3 which adheres to the parameters of perturbation theory, allowing for the application of a much simpler method for determining sample permittivities.

and quality factor will be obtained through scattering parameter measurements using a vector network analyzer (VNA). Coaxial cables are used to connect the VNA to devices under test. Such coaxial cables and their connectors typically have a designed characteristic impedance of $50\ \Omega$.

Microstrip transmission lines are commonly found in microwave circuits. They consist of a conductor of width W printed on top of a grounded substrate of thickness h and relative permittivity ϵ as shown in Figure 2.2a. The reason for their widespread use is that their miniaturization and fabrication can be simply and inexpensively achieved using photolithographic techniques, and their planar architecture allows them to interface with other planar passive and active microwave components for the construction of monolithic microwave integrated circuits [33].

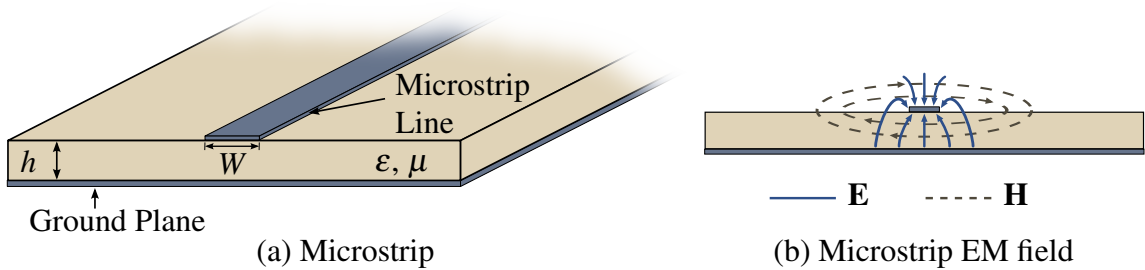


Figure 2.2: (a) Microstrip layout. (b) Electric and magnetic fields for a microstrip exist in two disparate media at the same time.

As seen in Figure 2.1, a microstrip transmission line is used as an intermediate waveguide structure to facilitate the propagation of the EM field from the coaxial line connected to the VNA to the resonant cavity. Mismatches between the characteristic impedances of transmission lines produce reflections, and reflections reduce the accuracy of scattering parameter measurements [6]. Thus, the microstrip line must be designed to have a characteristic impedance of $50\ \Omega$ to match the characteristic impedance of the coaxial cable.

The electric and magnetic field lines associated with a microstrip are shown in Figure 2.2b. Microstrip transmission lines support hybrid TM-TE modes which, in addition to consisting of electric and magnetic field components transverse to the direction of propagation, contain $\mathbf{E}$ and $\mathbf{H}$ field components along the direction of propagation. Perfect TEM fields are not supported by microstrip lines because their geometry requires the EM field to simultaneously exist inside the substrate between the microstrip and ground plane and in the medium above the substrate (usually air). Because the permittivity of the substrate is not the same as that of the medium above the microstrip, the EM fields will propagate at different velocities in each medium; this situation inhibits the propagation of purely transverse EM waves.

It may be shown through rigorous full-wave analysis that if the substrate height h is much less than the electrical wavelength of the propagating EM wave (i.e., $h \ll \lambda$), then the majority of both the electric and magnetic fields are contained in the substrate between the metal strip and ground plane. In this case, the longitudinal EM field components are small compared to the transverse components, and the field configuration is not too far from being purely TEM. The EM waves in this situation are called quasi-TEM [33]. For the quasi-TEM case, approximations to the exact solutions describing the EM field in a microstrip line provide useful design formulas. Many methods can be employed for the study of microstrip lines as discussed at length in [33]. A common practice in the analysis of microstrip lines is to define an effective permittivity ϵ_e that combines the influence of the air and substrate permittivities into a single parameter; from [34]

$$\epsilon_e = \begin{cases} \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \left[\frac{1}{\sqrt{1 + \frac{12h}{W}}} + 0.04 \sqrt{1 - \frac{W}{h}} \right], & \text{for } w/h \leq 1, \\ \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \left[\frac{1}{\sqrt{1 + \frac{12h}{W}}} \right], & \text{for } w/h \geq 1. \end{cases} \quad (2.1)$$

At present, a design formula is presented that allows for the synthesis of a microstrip line by providing the ratio of the microstrip width W to the substrate height h (i.e., W/h) needed to achieve a microstrip line of desired characteristic impedance Z_0 on a substrate characterized by a relative permittivity of ϵ_r . The expressions shown in equations 2.1–2.3 were first presented in [34] and are often used in textbooks devoted to microwave engineering principles such as [3] and [33].

$$\frac{W}{h} = \begin{cases} \frac{8}{e^A - 2e^{-A}}, & \text{for } W/h \leq 2, \\ \frac{2}{\pi} \left[B - 1 - \ln(2B - 1) + \frac{\epsilon_r - 1}{2\epsilon_r} \left\{ \ln(B - 1) + 0.39 - \frac{0.61}{\epsilon_r} \right\} \right], & \text{for } W/h \geq 2, \end{cases} \quad (2.2)$$

where

$$A = \frac{\pi Z_0}{\eta_0} \sqrt{2(\epsilon_r + 1)} + \frac{\epsilon_r - 1}{\epsilon_r + 1} \left(0.23 + \frac{0.11}{\epsilon_r} \right),$$

$$B = \frac{\pi \eta_0}{2Z_0 \sqrt{\epsilon_r}},$$

and

$\eta_0 = 377 \, \Omega$ is the intrinsic impedance of free-space.

Conversely, for a known microstrip width W and substrate height h , the characteristic impedance of the microstrip line is given as

$$Z_0 = \begin{cases} \frac{60}{\sqrt{\epsilon_e}} \ln \left(\frac{8h}{W} + \frac{W}{4h} \right), & \text{for } W/h \leq 1, \\ \frac{120\pi}{\sqrt{\epsilon_e} [W/h + 1.393 + 0.667 \ln(W/h + 1.444)]}, & \text{for } W/h \geq 1. \end{cases} \quad (2.3)$$

The relations in equations 2.1–2.3 are approximations giving errors in $\epsilon_e \leq 1\%$ and $Z_0 \leq 0.4\%$ for the constraints $0.05 \leq \frac{W}{h} \leq 20$ and $\epsilon_r \leq 16$ (see [34]).

Before equation 2.2 may be applied to find the microstrip width W that will result in the required microstrip characteristic impedance of $Z_0 = 50 \, \Omega$ to match the characteristic

impedance of the coaxial line, a substrate must be selected. As discussed previously, a few more or less arbitrary choices need to be made at the beginning of the design process, as is the case here again with the substrate. In preliminary literature reviews on high- Q microwave resonators, two particular papers ([35] and [36]) were selected to provide guidance for the design of this resonator. Specifically, in [35], a microwave resonator was constructed with a resonant frequency $f_0 = 10.77$ GHz and quality factor $Q = 750$ in which the substrate was Rogers Duroid 5880 ($\epsilon_r = 2.2$) with height $h = 3.05$ mm (~ 120 mils).³ This device serves as an example of a high- Q resonator achieved using a relatively thick substrate material. Furthermore, it was known at the outset of the design process that the fabrication of the resonator was to be completed by Prototron Circuits, Inc. due to a familiarity with this company's quality of work. According to the advertised capabilities of Prototron and Rogers Corporation, Rogers Duroid 5880 with thickness $h = 125$ mils is a readily available product. Thus, the substrate chosen for this design is Rogers Duroid 5880 ($\epsilon_r = 2.20 \pm 0.02$, $\tan \delta = 0.0009$ [37]) with thickness $h = 125$ mils.

Using equation 2.2 with $\epsilon_r = 2.2$ and $h = 125$ mils, the required value for the microstrip width W to achieve a characteristic impedance $Z_0 = 50 \Omega$ is $W = 385$ mils. As mentioned, equation 2.2 is approximate and is used here only as a baseline design from which to start. In truth, equations 2.1–2.3 are only valid at DC and very low frequencies. Due to the non-TEM nature of the EM field associated with a microstrip line, the effective permittivity of the air-substrate composite is a function of frequency as are other parameters like the characteristic impedance and propagation constant of a microstrip line.

An *extremely* powerful tool for obtaining accurate analyses of complex three-dimensional physical systems is the use of finite element analysis (FEA) [38]. FEA is used extensively

³Note that the resonators constructed in [35] and [36] were not SIW resonators based on a rectangular cavity, but electromagnetic bandgap devices.

in many areas of science and engineering such as structural analysis, fluid flow dynamics, heat transfer problems, biomechanics, and others [39]. In FEA, a mathematical model of a system is built from a computer aided design (CAD) tool, usually in the form of a two- or three-dimensional representation of the system. This CAD model is then split up into many discrete components (i.e., “finite elements”) called a “mesh” in FEA parlance. This meshing process allows for the computation of approximate solutions to simpler equations for each element as opposed to the complex overall structure [40].

The high frequency structural simulator (HFSS) produced by ANSYS Corporation is a finite element software tool specifically designed for the analysis of RF and microwave electromagnetic devices. By specifying the three-dimensional geometry and electrical parameters of a design, HFSS can provide a range of analytical information concerning device operation such as scattering parameters, simulations of steady-state EM fields, and impedance values.

Among the many advantages of HFSS is the program’s optimization tools (called “Optimetrics” within HFSS) that allow the user to perform parametric sweeps of design variables (e.g., a component dimension or electrical parameter) while simultaneously defining a target goal. HFSS then attempts to autonomously find the optimal solution by varying the parameter of interest to best satisfy the user-defined design criteria.

Figure 2.3 is a screenshot of a microstrip line modeled in HFSS. In this model, the substrate is Rogers Duroid 5880 with a specified relative permittivity of $\epsilon_r = 2.2$ (as per the manufacturer’s datasheet [37]) and substrate height $h = 125$ mils. Using this model, a finite element analysis is performed to acquire a final design value for the microstrip width W . To begin, W is set to the value obtained by use of equation 2.2, $W = 385$ mils. Then, using HFSS’s optimization capabilities, the microstrip width is set as a variable, and HFSS completes numerous simulations of the microstrip model for various values of W to find

the optimal value for W that will produce $Z_0 = 50 \Omega$. The FEA is performed over the range of frequencies this device is required to operate (approximately 1.5–3.0 GHz).

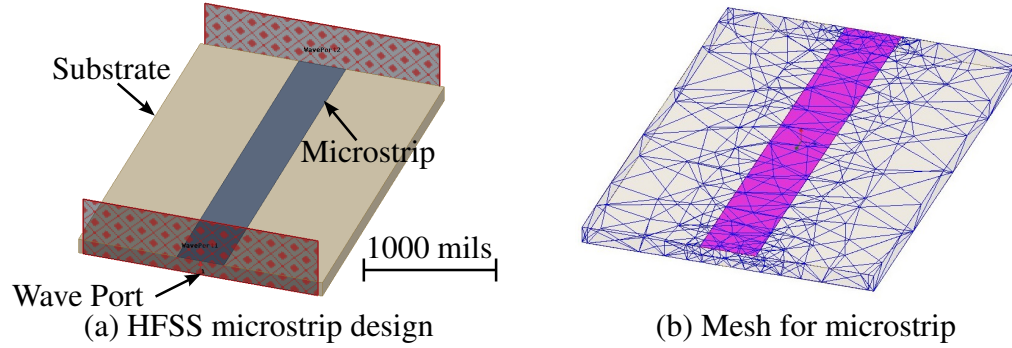


Figure 2.3: (a) HFSS model of a microstrip line. (b) Mesh created for finite element analysis in HFSS; HFSS divides the model into many tetrahedral elements.

Figure 2.4 is a plot of the microstrip's characteristic impedance as a function of frequency computed by HFSS simulations for three values of microstrip width W : $W = 385$ mils (as determined from equation 2.2 above), $W = 365$ mils, and $W = 345$ mils. The differences in the widths is 20 mils, which is a little over 0.5 mm. From this figure, it is seen that a more appropriate value for the microstrip width is $W = 365$ mils as opposed to $W = 385$ mils since the smaller width produces characteristic impedance values closer to the desired 50Ω over the frequency range of interest. Thus, a microstrip width of $W = 365$ mils is selected for this design.

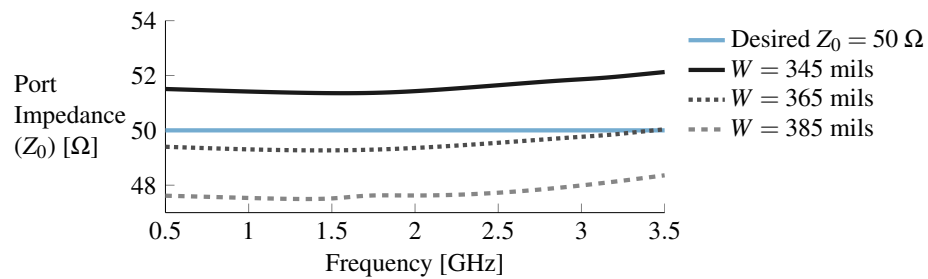


Figure 2.4: Microstrip Z_0 versus width W as a function of frequency.

Now that the microstrip width and characteristic impedance have been set, the mechanical aspect of the microstrip-to-coax transition must be considered. There are two goals to be met by the mechanical portion of a transmission line transition:

1. To facilitate the physical rearrangement of the spatial distribution of the EM fields between the two transmission lines, and
2. to minimize any additional impedance between the transmission lines due to the insertion of the transition.

First, consider goal #1 by observing the distribution of the electric and magnetic fields in a coaxial line and a microstrip; Figure 2.5 depicts the fields associated with a coaxial line and a microstrip.

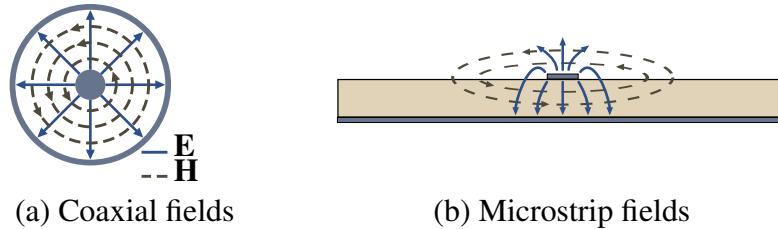


Figure 2.5: (a) Electric and magnetic fields for a coaxial line are TEM. (b) Electric and magnetic fields for a microstrip are quasi-TEM.

While the two sets of field distributions appear quite different, it is actually not too difficult to match them. The coaxial line fields are TEM and are confined between two conductors; the microstrip fields are quasi-TEM and are also mostly confined between two conductors.

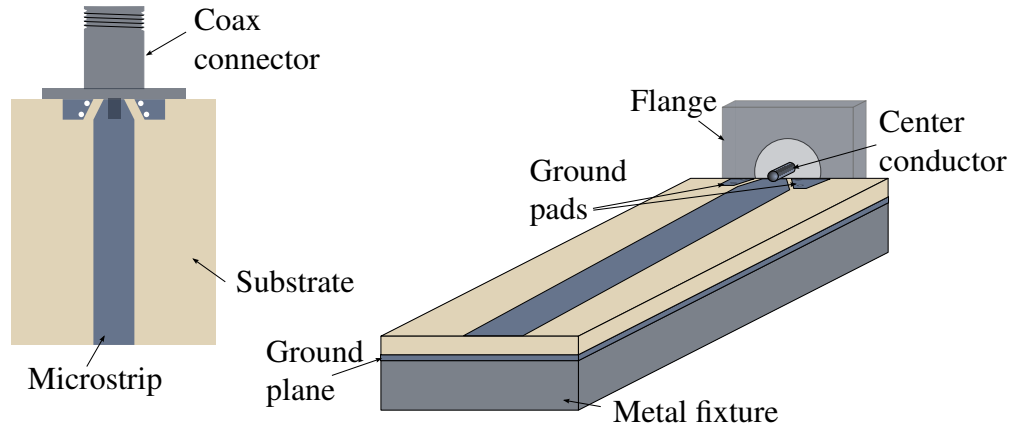
In consideration of goal #2 it is important to note that all conductors have an associated resistance R and inductance L , which can be represented as an impedance $Z = R + j\omega L$. When ω is small, R dominates the impedance, and thus, the impedance associated with the conductors of a DC or low frequency (e.g., audio frequencies) circuit can usually be approximated as being composed of just a resistance. However, as ω approaches the radio frequencies and above, the inductance associated with all conductors becomes non-

negligible. Since inductance increases with the length of a conductor, in RF and microwave circuits it is desirable to minimize the size of a transition so as to avoid the addition of any unwanted impedance [41]. In an effort to meet goal #2, extreme care should be taken in the design of RF and microwave devices to avoid any gaps between conductors when combining two disparate transmission lines. Any such gaps introduce discontinuity reactances by lengthening the path of current flow [33].

One of the most popular configurations for a coaxial line to microstrip transition is an edge-mounted coaxial connector [41]. In this transition, the center pin of a coaxial connector is placed directly on top of a microstrip line and soldered in place for higher reliability. The outer flange of the coaxial connector is fastened to a metallic fixture that is in direct contact with the microstrip ground plane. Extra care must be taken to ensure flush contact between the flange of the coax connector and the fixture as well as between the microstrip ground plane and the fixture's top surface. The edge-mount transition used for this sensor is shown schematically in Figure 2.6a, and photos of the coax-to-microstrip portion of a fabricated prototype are shown in Figures 2.6b–2.6d.

The coaxial connector used in this design is an N type flange mount jack from Emerson Network Power. The metallic fixture is composed of two identical aluminum blocks that lie flush with the ground plane of the sensor and have screw holes for the attachment of the coaxial connector. Photographs of the coaxial connector and metallic fixtures are shown in Figure 2.7. The ground pads located on either side of the center pin of the coaxial connector help to compensate for the difference in the geometry of the EM fields between the coaxial line and the microstrip. Note that the microstrip line begins to taper as the line approaches the coaxial connector. The reason for this is to help provide a smooth redistribution of the electric and magnetic fields, and also because without a taper, the microstrip would touch the flange of the coaxial connector on either side of the center pin, effectively shorting out

the microstrip. HFSS was used to optimize the angle of the taper as well as the locations of the ground pads.



(a) Schematic of the coaxial-to-microstrip transition.

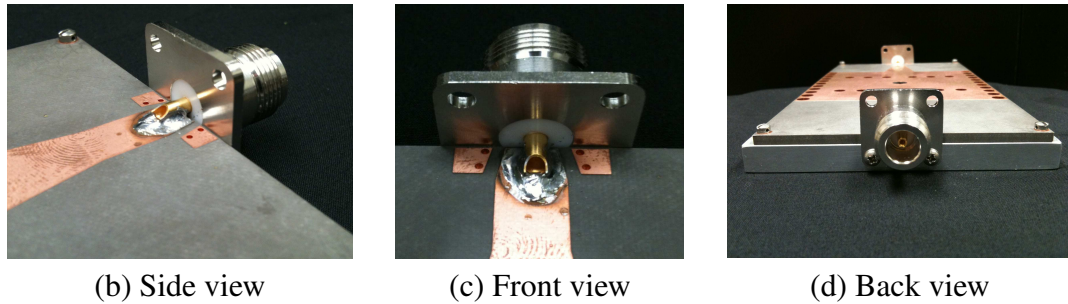


Figure 2.6: The edge-mount coax-to-microstrip transition used in this sensor design. (a) Schematic and (b)–(d) photographs of the actualized transition.

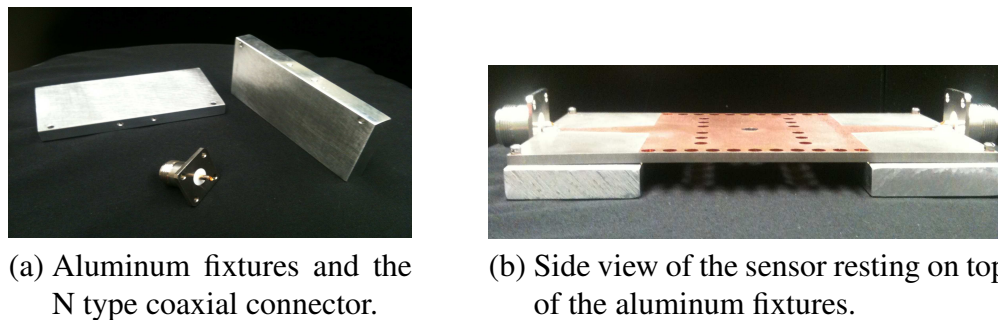


Figure 2.7: Photographs of the aluminum fixtures and N type coaxial connector.

2.1.2 Substrate Integrated Waveguide Design

The concept of a substrate integrated waveguide was first introduced by Deslandes and Wu in [42] as a solution to the problem of integrating planar microwave circuit components such as microstrip lines with nonplanar devices like rectangular waveguides via a technique that facilitates mass production. In an SIW circuit, rectangular waveguides are synthesized by compressing the waveguide height to equal that of the substrate supporting the other planar components of the circuit and replacing the waveguide sidewalls with rows of metalized vias or grooves as shown in Figure 2.8.

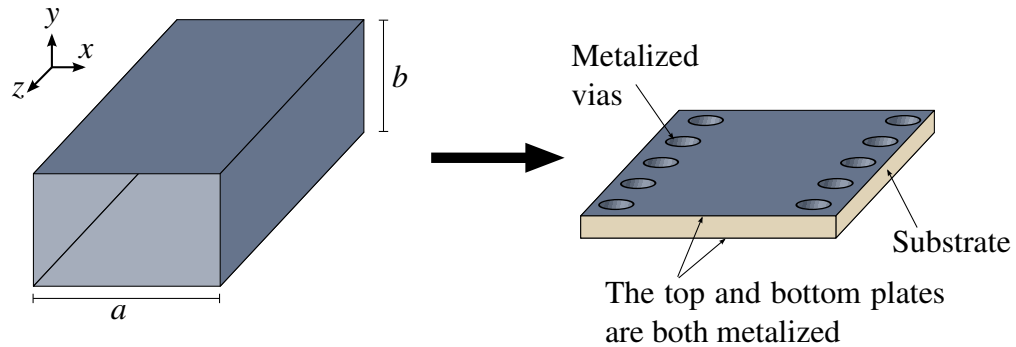


Figure 2.8: A nonplanar rectangular waveguide becomes a planar substrate integrated waveguide.

The result is a synthesized planar waveguide that has superior loss characteristics compared to traditional planar counterparts (e.g., microstrip, coplanar waveguide, etc.) while being less expensive and easier to fabricate than bulky nonplanar waveguide structures (e.g., rectangular waveguides) [43].

Just as with a rectangular waveguide, an SIW cannot support transverse electromagnetic (TEM) waves.⁴ Consider the constraints on EM wave propagation in both a rectangular waveguide and an SIW using Figure 2.8 as a reference. Since both structures are open-

⁴An in depth discussion of electromagnetic wave propagation in rectangular waveguides, including a derivation of the equations describing the EM waves they support, is given in Appendix B.

ended (i.e., neither waveguide has any conductive components parallel to the x -direction), the only boundary conditions are due to the metallic walls on the top and bottom of the waveguides and the sidewalls. In going from the rectangular waveguide to the SIW, the top and bottom walls do not change, just the distance between the two walls decreases. However, what does change are the sidewalls. What were once continuous conductive sheets become rows of periodically spaced hollow cylinders with metallic walls. In the SIW, there are gaps in the sidewalls which allow for EM power to leak through. If the diameter of the vias D and the spacing between the vias B adhere to the conditions given by equations 2.4a and 2.4b, then the radiation loss due to the nonhomogeneous structure of the rows of vias is low enough to allow the SIW to be modeled by a rectangular waveguide supporting TE modes.

SIW Design rules:

$$D < \frac{\lambda_g}{5}, \quad (2.4a)$$

$$B \leq 2D, \quad (2.4b)$$

where λ_g is the guided wavelength (the wavelength of the EM wave inside the substrate); $\lambda_g = \lambda_0 / \sqrt{\epsilon_r}$, where λ_0 is the free-space wavelength. The SIW design constraints of equations 2.4a and 2.4b were derived from empirical results of finite element analyses [44].

For a rectangular waveguide of width a and height b as shown in Figure 2.8, the cutoff frequency corresponding to each TE mode of propagation is given by

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \quad (2.5)$$

where μ and ϵ are the permeability and permittivity associated with the substrate and m and n take on integer values greater than or equal to zero (excluding $m = n = 0$).⁵ Equations 2.4a, 2.4b, and 2.5 may be used to design the dimensions of an SIW, as shown in

⁵Recall the definitions of μ and ϵ from Chapter 1; $\mu = \mu' - j\mu'' = \mu_0\mu_r - j\mu''$ and $\epsilon = \epsilon' - j\epsilon'' = \epsilon_0\epsilon_r - j\epsilon''$, where $\mu_0 = 4\pi \times 10^{-7}$ [H/m] and $\epsilon_0 \approx 8.854 \times 10^{-12}$ [F/m].

Figure 2.9, for a desired cutoff frequency. For an SIW, the height b in equation 2.5 becomes the substrate height h , but the width a is not so easily defined. One plausible option is the distance a_{SIW} as shown in Figure 2.9, another is the distance between the center of two vias (i.e., $a_{\text{SIW}} + D$). Deslandes and Wu performed finite element analyses of various SIW designs to find that the effective width a_{eff} is somewhere between a_{SIW} and $a_{\text{SIW}} + D$ as shown in Figure 2.9 [44].

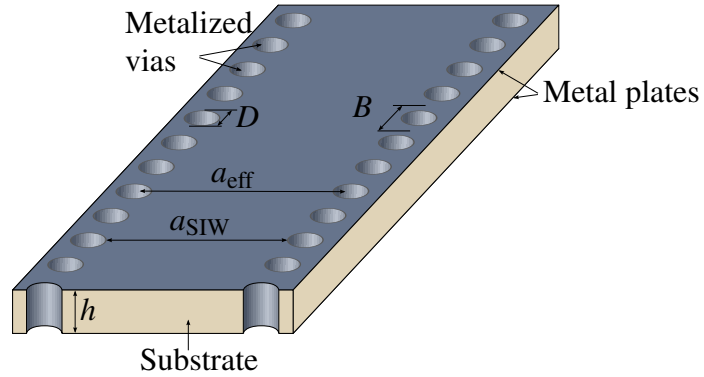


Figure 2.9: The topology of an SIW on a substrate characterized by ϵ and μ . The constraints for the values of the via diameter D and via spacing B are given in equations 2.4a and 2.4b.

Thus, equation 2.5 should only be used as a guideline from which to start from (just as equation 2.2 was used in the design of the microstrip characteristic impedance) when trying to find the cutoff frequency for an SIW. For the design of the prototype sensor, a is chosen to be a_{SIW} as shown in Figure 2.9, the via diameter and spacing are based off of equations 2.4a and 2.4b, and then HFSS is used to fine tune all three parameters.

The dominant TE mode of propagation is chosen for this design.⁶ For the TE_{10} mode $m = 1$, $n = 0$, and equation 2.5 becomes

$$f_{c_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}}. \quad (2.6)$$

⁶A discussion of using multiple TE modes simultaneously is taken up in Chapter 4.

Equation 2.6 allows for the design of the substrate width and spacing between the two rows of vias. Since the desired range of operation of the sensor was chosen at the outset to be 1.5–3.0 GHz, the cutoff frequency for the SIW must be lower than 1.5 GHz. For this design, a cutoff frequency of $f_{c10} = 1.25$ GHz is selected. Rearranging equation 2.6 to solve for a in terms of f_{c10} yields $a = 3183$ mils. Rounding up to an even number, for this design the separation between the rows of vias is selected to be $a_{SIW} = 3200$ mils (equation 2.6 gives $f_{c10} = 1.24$ GHz for $a_{SIW} = 3200$ mils).

To find appropriate values for the via diameter D and spacing B , the guided wavelength λ_g must be known. It will be shown in Chapter 3 that the resonant frequency f_0 will always decrease when a sample is inserted into the sensor. Thus, the highest f_0 will be is when the sensor is empty. The precise value for the unloaded resonant frequency has yet to be determined, but a range was decided on earlier such that $2.5 \text{ GHz} \leq f_0 \leq 3.0 \text{ GHz}$. The via diameter D is limited in size by the guided wavelength; the smaller λ_g is, the smaller D must be to minimize radiation loss. To be safe, assume $f_0 = 3.0$ GHz for now since the larger f_0 is, the smaller D must be. For $f_0 = 3.0$ GHz, $\lambda_g = c/f_0\sqrt{\epsilon_r} = 3 \times 10^8 / (3 \times 10^9) \sqrt{2.2} = 67.42 \text{ mm} = 2654$ mils. With this value of λ_g , equation 2.4a indicates that the via diameter must be 530 mils or less. This is truly not much of a design constraint as $1/2$ inch vias would be very large indeed. In fact, the largest possible via holes Prototron is capable of normally producing is $1/4$ inch (or 250 mils). As such, 220 mil via diameters are selected for this design. With $D = 220$ mils, equation 2.4b requires that the via spacing is $B \leq 440$ mils. For this design B is selected to be $B = 330$ mils.

Figure 2.10 shows the results of an HFSS simulation of an SIW with the selected design parameters, $a_{SIW} = 3200$ mils, $D = 220$ mils, and $B = 330$ mils. Figure 2.10a is a model of the SIW and Figure 2.10b is a plot of the insertion loss IL ($IL = -20 \log_{10}(|S_{21}|)$, or simply $|S_{21}|$ [dB]) versus frequency as computed by HFSS. The

cutoff frequency computed by HFSS is $f_{c10} = 1.277$ GHz, a 3% difference from the value given by equation 2.6.⁷

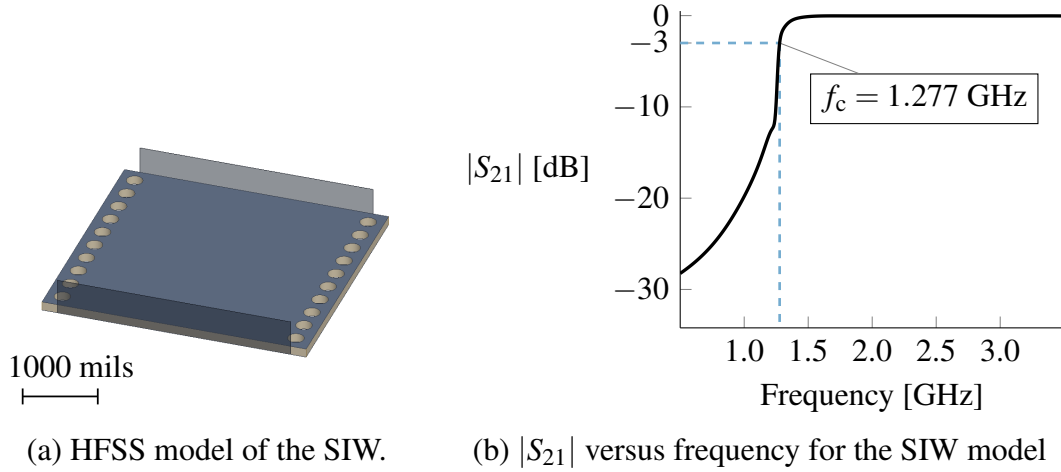


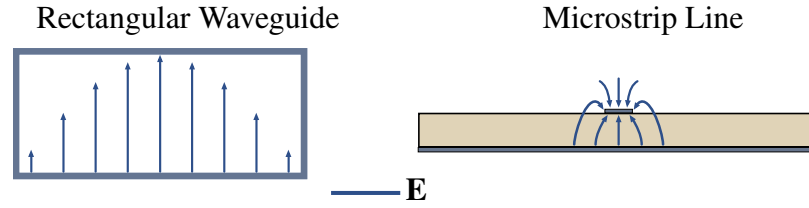
Figure 2.10: HFSS SIW CAD model and the computed insertion loss. The SIW acts as a high-pass filter where the cutoff frequency is set by the width of the SIW.

2.1.3 Microstrip-to-SIW Transition

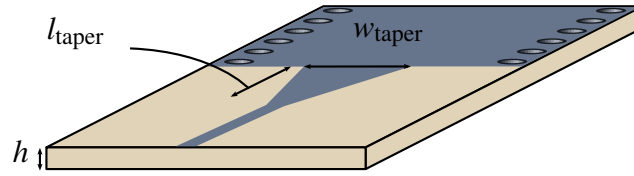
A simple transition from a microstrip to an SIW is realized by linearly widening the width of the microstrip as it approaches the SIW. This transition was proposed by Deslandes and Wu in their original paper introducing the SIW [42].

Figure 2.11a depicts the electric field lines associated with the dominant modes of operation in a rectangular waveguide (which should be *very* similar to the electric field distribution in an SIW given their comparable geometries) and a microstrip. Since the fields are similar in distribution, a smooth transition between a microstrip line and an SIW can be achieved without a complicated apparatus. As such, the transition structure of Figure 2.11b in which a tapered microstrip feeds into an SIW provides adequate field matching over a reasonably broad bandwidth [42].

⁷See Appendix B for a discussion of the convention used to define the cutoff frequency as the frequency at which the magnitude of $|S_{21}|$ falls 3 dB down from the maximum value of $|S_{21}|$ in the passband.



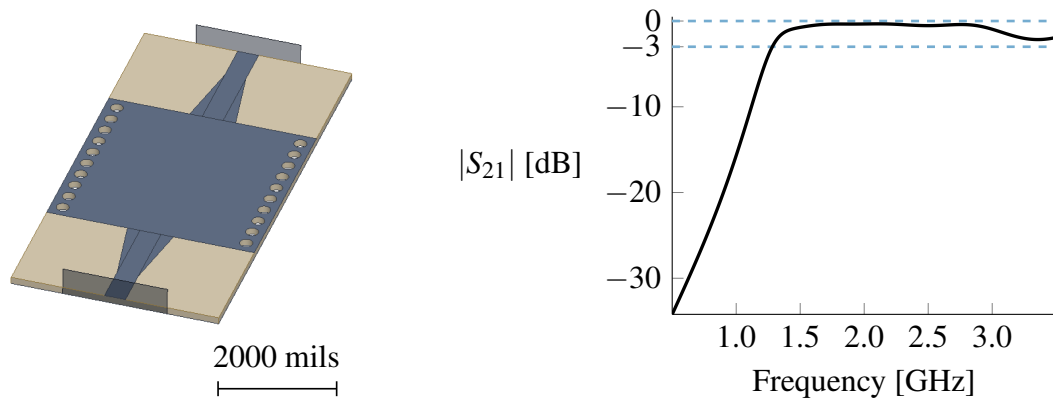
(a) Electric fields associated with a rectangular waveguide and a microstrip line.



(b) Schematic of the microstrip-to-SIW transition.

Figure 2.11: Microstrip-to-SIW transition. (a) The electric fields of the dominant mode of a rectangular waveguide and a microstrip are similar in distribution. (b) The microstrip-to-SIW transition is realized by a tapered microstrip geometry.

The taper length l_{taper} and width w_{taper} are optimized over the frequency range of interest through finite element simulations in HFSS. Figure 2.12 is a plot of the resulting transmission characteristic as computed by HFSS for the final transition design.



(a) HFSS model of the SIW-to-microstrip transition.

(b) $|S_{21}|$ versus frequency for the microstrip-to-SIW transition.

Figure 2.12: HFSS $|S_{21}|$ calculation of the microstrip-to-SIW transition.

This completes the “transmission line” portion of the design work. So far, the goal has been to guide an EM wave from one coaxial launcher, through a microstrip, onto an SIW, back through a microstrip, and finally out another coaxial connector with as little loss as possible. Mitigating loss is achieved by developing a transition structure at each transmission line interface. As a check, Figure 2.13b is a plot of HFSS simulations of just the transmission characteristics of the SIW alone, the microstrip-to-SIW transition, and the full transmission structure shown in Figure 2.13a. Up to now, the HFSS simulations have been performed assuming perfect conductors, and the only loss was associated with the loss tangent of the substrate. Figure 2.13b also contains a plot of the transmission characteristic of the full transmission structure with both dielectric loss and a finite conductivity of $\sigma = 58 \times 10^6$ [S/m] modeled for all conductors.

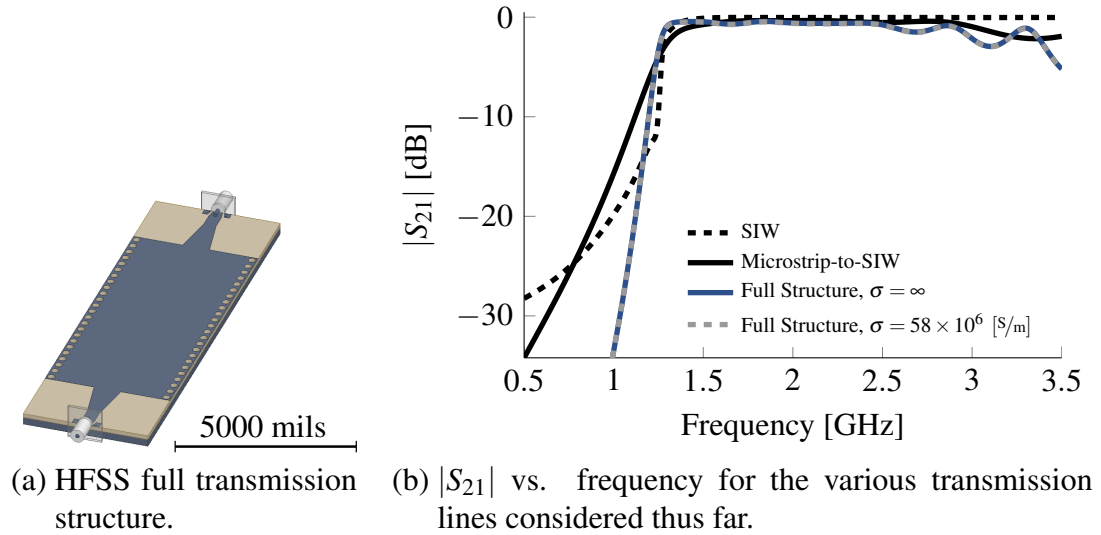


Figure 2.13: HFSS $|S_{21}|$ calculation of the full transmission line from input to output.

Figure 2.13b indicates that the coax-to-microstrip and microstrip-to-SIW transitions designed are acceptable since $|S_{21}|$ versus frequency is relatively flat over the necessary frequency range, and the minimum value for $|S_{21}|$ in the passband is better than -3 dB.

2.1.4 High- Q Resonant Cavity

As discussed in Section 2.1.2, a rectangular waveguide may be synthesized in a planar substrate by replacing the metallic sidewalls of a conventional rectangular waveguide with rows of metalized vias. If two more rows of vias are added perpendicularly to the first two rows, then a rectangular *cavity* may be synthesized as shown in Figure 2.14.

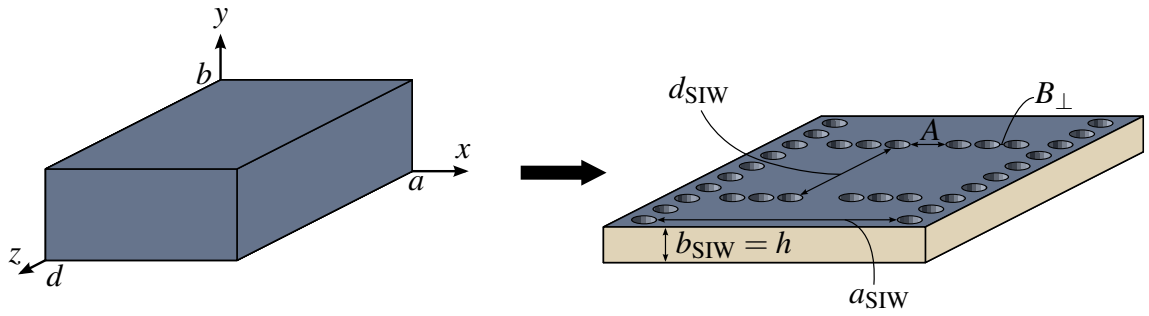


Figure 2.14: A nonplanar rectangular cavity becomes a planar substrate integrated waveguide resonant cavity.

Just as the equation for the cutoff frequency of a rectangular waveguide operating in a TE mode can be applied to an SIW, so can the equations for the resonant frequency and quality factor of a rectangular cavity be applied to an SIW cavity. For a closed section of rectangular waveguide, the resonant frequency is given by

$$f_{0_{mnp}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}, \quad (2.7)$$

(see [3], also derived in Appendix B). The quality factor Q for such a resonant cavity is given by

$$Q = \left(\frac{1}{Q_c} + \frac{1}{Q_d} \right)^{-1}, \quad (2.8)$$

where Q_c accounts for power dissipated by the metallic walls of the cavity, and Q_d accounts for the loss associated with the dielectric filling the rectangular cavity (or the dielectric

substrate where the SIW cavity is concerned) [3].

$$Q_c = \frac{(kad)^3 b \eta}{2\pi^2 R_s} \frac{1}{(2p^2 a^3 b + 2bd^3 + p^2 a^3 d + ad^3)}, \quad (2.9)$$

where

$$\begin{aligned} k &= \frac{2\pi}{\lambda}, & \text{the wavenumber,} \\ R_s &= \sqrt{\frac{\omega\mu_0}{2\sigma}}, & \text{the surface resistivity of the metallic walls,}^8 \\ \eta &= \frac{j\omega\mu}{j\omega\sqrt{\mu\epsilon}\sqrt{1-j\frac{\sigma}{\omega\epsilon}}}, & \text{the intrinsic impedance of the metallic walls,} \\ \eta &= \sqrt{\frac{\mu}{\epsilon}}, & \text{in the lossless case.} \end{aligned}$$

$$Q_d = \frac{1}{\tan \delta}. \quad (2.10)$$

Designing for the TE_{101} resonant mode, $m = 1$, $n = 0$, and $p = 1$ in equation 2.7; a (or a_{SIW}) represents the width of the SIW, and b (or b_{SIW}) is the same as the substrate height h . Thus, the resonant frequency, f_{101} , can be set by adjusting the length of the resonator d (or d_{SIW}):

$$f_{101} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2}. \quad (2.11)$$

The value for a was found in Section 2.1.2 as $a = 3200$ mils (81.28 mm). At the beginning of the design, it was decided that the resonant frequency should lie somewhere between 2.5 and 3.0 GHz. Substituting $a = 3200$ mils and $d = 1500$ mils (38.1 mm) into equation 2.11 gives $f_{101} = 2.93$ GHz; thus for this sensor $d = 1500$ mils.

Consulting Figure 2.14, the value for d sets the distance between the two rows of vias running transverse to the z -direction (i.e., $d = d_{SIW}$). Just as with the via rows running in the z -direction (call them the “SIW vias”), the purpose of these two rows of vias (call them

⁸ σ is the conductivity of the metallic walls

the “transverse vias”) is to confine any EM waves within the cavity, and as before, to ensure that radiative loss is minimized, the diameter and spacing of these vias should abide by the SIW design rules of equations 2.4a and 2.4b.

That being said, there is no constraint that dictates that the diameter and spacings of the SIW vias must be the *same* as those of the transverse vias. In fact, the diameter of the transverse vias $D_{\perp}$ and the spacing between them $B_{\perp}$ allows for the adjustment of the quality factor of the resonator (this will be shown momentarily). Furthermore, again consulting Figure 2.14, a relatively large gap in the center of the transverse vias is needed to allow for a significant amount of EM energy to enter and exit the resonant cavity. This gap, referred to as an aperture of width A , couples the resonant cavity to external circuitry.

For simplicity let $D = D_{\perp}$. To gain a feel for the effects of altering $B_{\perp}$ and A , observe the HFSS model of the SIW cavity resonator in Figure 2.15. In this model, the substrate is Rogers Duroid 5880, with height $h = 125$ mils; the SIW vias have diameter $D = 220$ mils and spacing $B = 330$ mils; the distance between the SIW vias is $a = 3200$ mils, and the distance between the transverse vias is $d = 1500$ mils (i.e., this is the designed resonator structure under current consideration). The transverse vias also have diameter $D_{\perp} = 220$ mils. Figure 2.16 is a plot of the transmission characteristics of the SIW cavity resonator for various values of $B_{\perp}$ while A is a constant 500 mils.

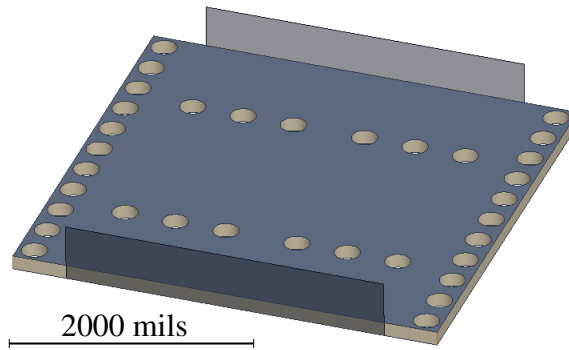
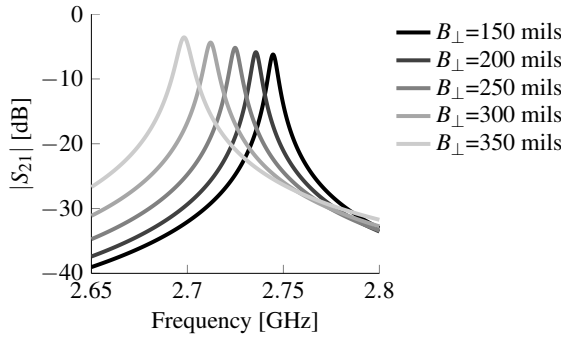


Figure 2.15: HFSS model of an SIW cavity resonator.

Table 2.1: The Effect of $B_{\perp}$ on f_0 and Q

$B_{\perp}$ [mils]	f_0 [GHz]	Q
150	2.7446	574
200	2.7357	551
250	2.7249	505
300	2.7121	444
350	2.6983	379

Figure 2.16: Effect of transverse via spacing $B_{\perp}$.

First and foremost, it should be noted that this structure is in fact a resonator with a clearly discernible resonant peak frequency somewhat close to the predicted value of $f_0 = 2.93$ GHz. Table 2.1 contains f_0 and Q values for the simulations corresponding to the curves of Figure 2.16. For $B_{\perp} = 150$ mils, $f_0 = 2.7446$ GHz which is only a 6% deviation from the predicted value determined through use of equation 2.11; an equation that was derived assuming homogenous conductive sheets of metal as opposed to rows of periodically spaced vias for the cavity walls. From Figure 2.16 and Table 2.1 it is evident that $B_{\perp}$ plays a role in the placement of f_0 , in the value of Q , and to a lesser extent, the maximum value of $|S_{21}|$. As $B_{\perp}$ gets larger, the distance between vias increases. Such an increase in the spacing between vias leads to a decrease in f_0 , a decrease in Q , and an increase in the maximum amount of transmitted energy from the input (port 1) to the output (port 2). The inverse relationship between Q and increasing $B_{\perp}$ is not surprising. The quality factor Q is a measure of the loss of a resonator in which higher values of Q represent systems with lower loss. The closer together the vias that form the “walls” of this SIW resonator are, the more EM energy they can confine. Wider spacings between the vias allows for EM power to radiate out of the resonant cavity; such a phenomenon was the reason for the imposition of the SIW desing rules of equations 2.4a and 2.4b. Along this same line of reasoning, the direct proportionality between the maximum value of $|S_{21}|$ and

increasing $B_{\perp}$ is expected since more EM energy is able to pass through the transverse rows of vias on the way from port 1 to port 2 and less energy is lost (i.e., absorbed or radiated) while traversing the cavity.

Figure 2.17 shows the transmission characteristics of the SIW cavity resonator of Figure 2.15 in which $B_{\perp} = 200$ mils while the coupling aperture A is altered. The apparent trends are essentially the same for increasing A as increasing $B_{\perp}$: a decrease in f_0 and Q , and an increase in the maximum transmission value.

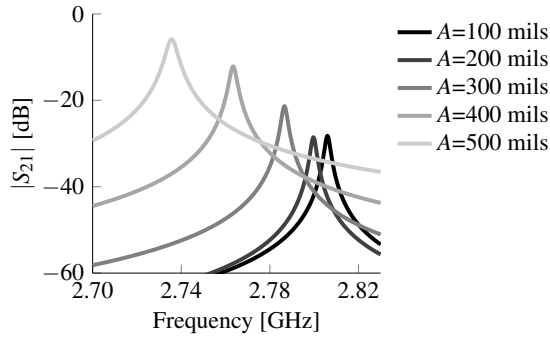


Table 2.2: The Effect of A on f_0 and Q

A [mils]	f_0 [GHz]	Q
100	2.8060	1080
200	2.7997	1077
300	2.7866	1032
400	2.7634	864
500	2.7357	547

Figure 2.17: Effect of coupling aperture A .

The message to take away from the results of Figures 2.16 and 2.17 is that $B_{\perp}$ and A are circuit parameters that give an SIW cavity designer the ability to adjust the specific values of f_0 , Q , and $|S_{21}|_{\max}$.

The values to choose for $B_{\perp}$ and A depend on the application. Larger values for $B_{\perp}$ and A will provide better coupling between the input and output at the expense of lower Q (i.e., higher loss). For resonators in applications where high Q is needed, such as a frequency meter, loose coupling (i.e., small A and $B_{\perp}$) is likely the best route, but if maximum power transfer is desired, then tighter coupling should be considered by selecting larger values of A and $B_{\perp}$.

At present, it is not known what exactly are appropriate values for Q and $|S_{21}|_{\max}$ for this sensor as this is the first prototype. It is hypothesized that both high Q and tight coupling

are desirable properties. Larger Q values should allow for smaller detections in the change in f_0 of the SIW cavity due to an analyte. Better coupling should facilitate simple and accurate measurements of the frequency response of the sensor with a VNA. Given this situation, intermediate values of Q and $|S_{21}|_{\max}$ are designed for and achieved by selecting moderate values of A and $B_{\perp}$. For this design $A = 400$ mils and $B_{\perp} = 220$ mils.

2.2 Fabricated Device and Model Comparison

The final design component for this sensor is the analyte receptacle. Recall the principle of operation for this sensor: the unloaded cavity's resonant frequency and quality factor are measured; then, the analyte receptacle is loaded with a material of interest and the cavity's resonant frequency and quality factor are measured again; the change in f_0 and Q is then used to determine the permittivity of the material loaded into the resonant cavity. Figure 2.18 shows HFSS models of the SIW resonant cavity with a cylindrical analyte receptacle with multiple diameters D_R . Figure 2.19 and Table 2.3 summarize the effect that increasing D_R has on f_0 and Q . The effect is small, much less pronounced than the effects of via spacing $B_{\perp}$ and resonator aperture A . The effect on Q is inconsistent, but there is a definite relationship between receptacle size and f_0 . As the diameter of the receptacle size increases, f_0 also increases. This result can be explained by considering equation 2.11. Increasing the analyte receptacle decreases the effective value of ϵ , thus increasing f_{101} . Without a receptacle $\Re\{\epsilon\} = \epsilon_0\epsilon_r$ where ϵ_r is the relative permittivity of the substrate, but with the receptacle, the permittivity content of the overall structure effectively decreases since a portion of the substrate with $\epsilon_r = 2.20$ is replaced with air where $\epsilon_{\text{air}} \approx 1.0$.

The only guidelines used for the choice of the receptacle was that it needed to facilitate ease of use. With this goal in mind, it was decided to use a cylindrical analyte receptacle with a diameter of 312.5 mils. This diameter was chosen because dielectric tubing with

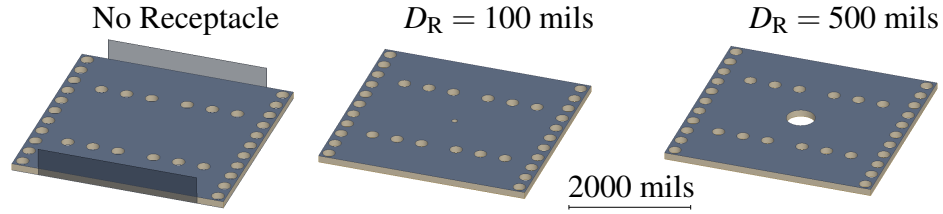


Figure 2.18: HFSS models of the resonant cavity with different receptacle sizes.

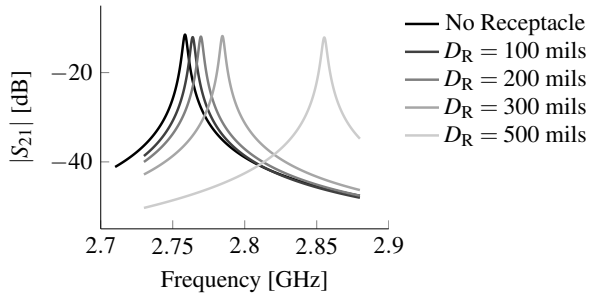


Table 2.3: The Effect of D_R on f_0 and Q

D_R [mils]	f_0 [GHz]	Q
No Receptacle	2.7585	823
100	2.7639	845
200	2.7695	852
300	2.7845	856
500	2.8553	807

Figure 2.19: Effect of receptacle diameter D_R .

an outside diameter of 312.5 mils ($5/16$ of an inch) is readily available. Plastic tubing (e.g., vinyl or polyethylene) can be inserted into the analyte receptacle and then filled with liquid for easy testing. Also, dielectric rods such as Teflon are inexpensive test materials that are easy to procure.

With the analyte receptacle geometry chosen, the sensor design is complete. As a check, the full sensor structure is modeled in HFSS as shown in Figure 2.20 and the $|S_{21}|$ response computed by HFSS is shown in Figures 2.21a and 2.21b.

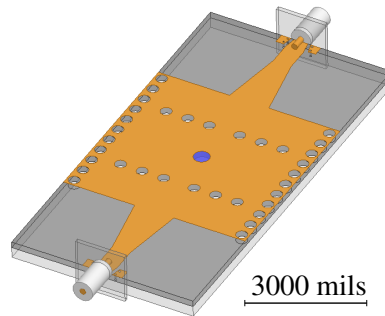
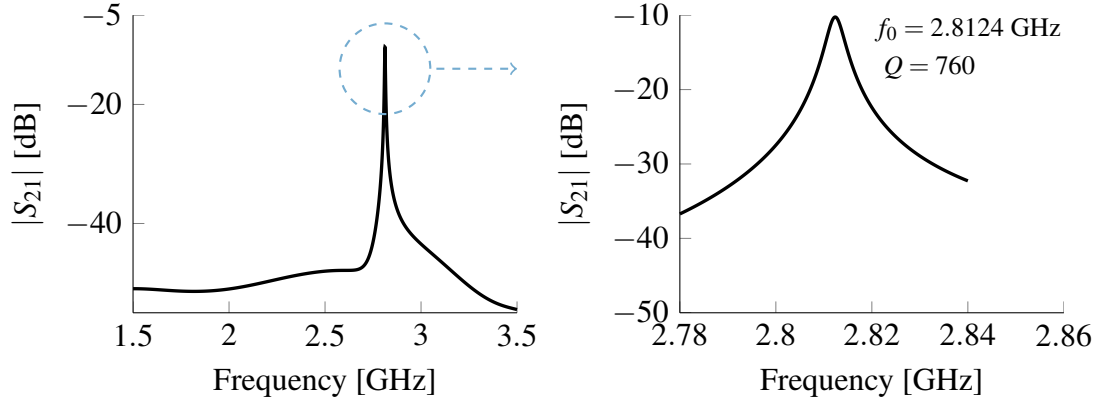


Figure 2.20: HFSS model of the full SIW cavity sensor.



(a) Broad view of $|S_{21}|$ vs. frequency for the simulated SIW cavity. (b) Zooming in on the resonant peak.

Figure 2.21: Computed $|S_{21}|$ response of the SIW resonant cavity model.

The predicted resonant frequency based on the use of equation 2.11, which is only an approximation when applied to the SIW cavity, yields a value very close to that given by the simulation: $f_{0_{\text{calc}}} = 2.93$ GHz and $f_{0_{\text{sim}}} = 2.8124$ GHz, a 4% difference. Through use of equations 2.8, 2.9, and 2.10, the quality factor for this circuit is predicted to be $Q_{\text{calc}} = 987$. The simulated value $Q_{\text{sim}} = 760$ is about only 23% lower than Q_{calc} . Furthermore, for the simulation producing the responses of Figure 2.21, perfect conductors were being modeled, and thus the only sources of loss were due to the dielectric loss associated with the substrate and radiative leakage between the vias composing the SIW cavity. The calculated value of Q does not account for the aperture nor via spacings of the SIW cavity (and therefore does not include radiative loss), as it is based on a rectangular cavity. As discussed previously, smaller values of coupling aperture A and tighter via spacing will achieve higher Q values. Thus, the large discrepancy between Q_{calc} and Q_{sim} may be explained by the presence of the gaps in the “walls” of the SIW cavity symbolized by $B_{\perp}$ and A .

The discussion thus far has only utilized theoretical considerations and finite element analyses for the design of a substrate integrated waveguide resonant cavity. To verify the

validity of the design equations and simulations used, a device was fabricated with the dimensions chosen herein. Figure 2.22 shows photographs of the fabricated prototype. The manufacture of this sensor was completed by Prototron Inc. at their Tuscon, AZ facilities.

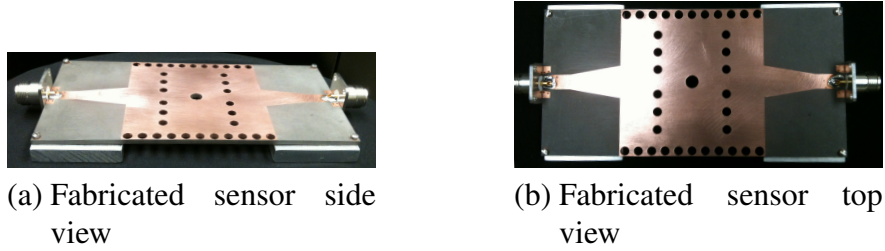


Figure 2.22: Photographs of the fabricated sensor.

Verification was performed by measuring the scattering parameters of the fabricated device with a vector network analyzer. The VNA used was an N5230A PNA-L Network Analyzer from Agilent Technologies. Prior to all measurements of the S -parameters, a short, open, line, thru (SOLT) calibration was performed to develop an error model of the test system (i.e., the VNA plus the cables connecting the device to the VNA). The SOLT calibration kit model number 8850Q made by Maury Microwave Corporation was used; see Appendix A for a discussion of the SOLT calibration technique.

Figure 2.23 is a photograph of the prototype sensor during measurement with the VNA. In the situation shown, the sensor is unloaded; that is, nothing is present in the analyte receptacle.

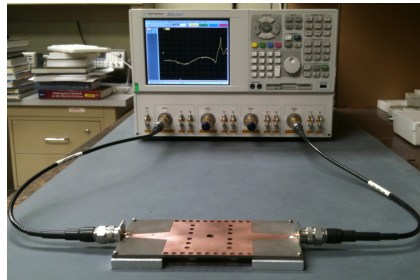
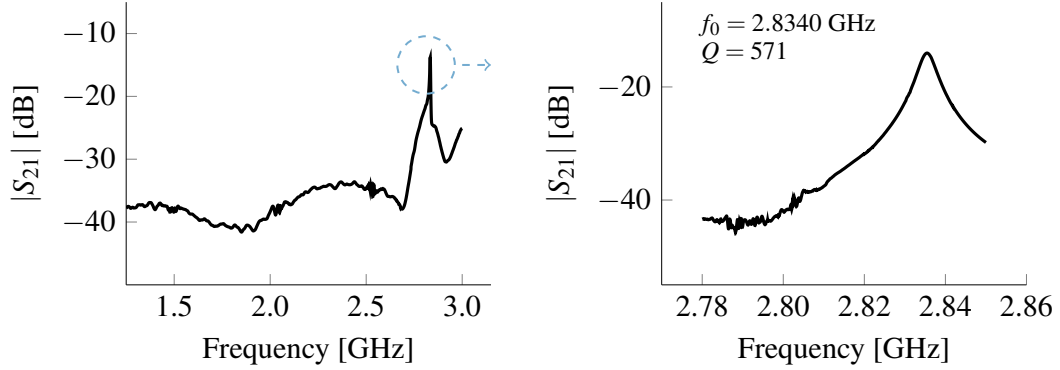


Figure 2.23: Testing setup for the SIW sensor.

Figure 2.24 contains plots of the measured $|S_{21}|$ response of the fabricated SIW cavity resonator. As indicated in Figure 2.24b, the measured resonant frequency and quality factor are $f_{0\text{meas}} = 2.8340$ GHz and $Q_{\text{meas}} = 571$. Comparing these values to those of the lossless HFSS simulations gives percent differences of 0.76% in f_0 and 33% in Q .



(a) Broad view of $|S_{21}|$ vs. frequency for the measured SIW cavity. (b) Zooming in on the resonant peak.

Figure 2.24: Measured $|S_{21}|$ response of the fabricated SIW cavity.

A more accurate model of the SIW cavity resonator may be achieved by defining a finite conductivity value for metal surfaces in the finite element model. Figure 2.25 contains plots of $|S_{21}|$ versus frequency for the measured unloaded SIW cavity as well as the response of the sensor as determined by various HFSS simulations. Specifically, three variations of the HFSS model are shown:

- The first case only considers loss associated with the dielectric substrate as the conductors are assumed to be ideal.
- The second case includes a finite conductivity value for all conductors with $\sigma = 58 \times 10^6$ [S/m].
- Finally, in addition to finite conductivity, the substrate relative permittivity is modeled to have a value of $\epsilon_r = 2.18$ instead of $\epsilon_r = 2.20$. The value of ϵ_r falls within the limits specified by the manufacturer of Rogers Duroid 5880 ($\epsilon_r = 2.20 \pm 0.02$).

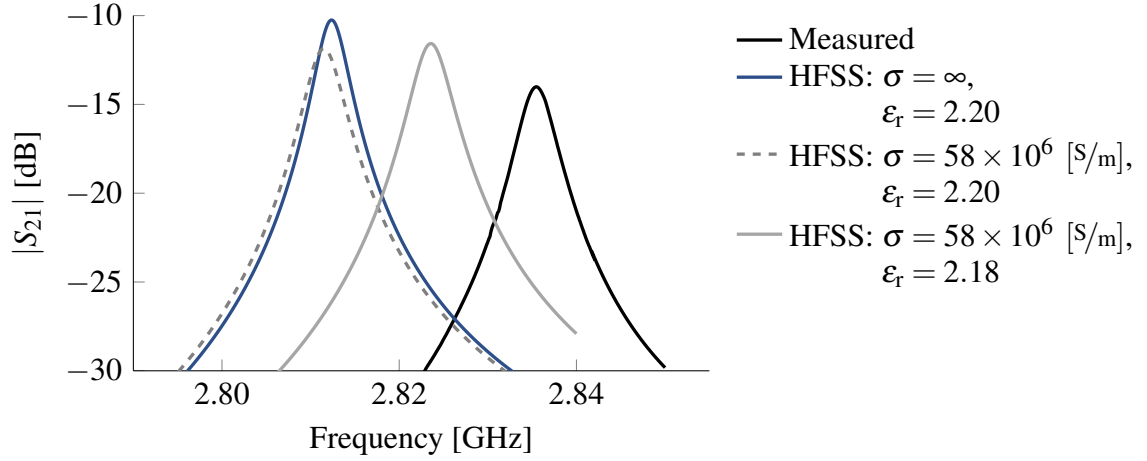


Figure 2.25: Adjusting the HFSS model of the SIW cavity by including finite conductivity loss and adjusting for a substrate with $\epsilon_r = 2.18$.

Table 2.4 lists the resonant frequencies and quality factors for the plots of Figure 2.25. Including finite conductivity provides a better model in terms of predicting f_0 , Q , and $|S_{21}|_{\max}$. Modeling the substrate relative permittivity as $\epsilon_r = 2.18$ instead of $\epsilon_r = 2.20$ could explain why the measured value for f_0 is higher than predicted.

Table 2.4: Finite Element Model Correction

	f_0 [GHz]	Q	$ S_{21} _{\max}$ [dB]
Measured	2.8340	571	-14.01
HFSS Lossless	2.8124	760	-10.25
HFSS Lossy	2.8155	626	-11.86
HFSS Lossy + Substrate Adjustment	2.8236	617	-11.58

Finally, to test the sensor's response to the presence of an analyte, the sensor was loaded with a Teflon rod. Figure 2.26 shows the measured responses of the unloaded sensor and the sensor loaded with Teflon. HFSS simulations of the unloaded and Teflon loaded sensor are provided for comparison. In the HFSS simulation, Teflon was modeled to have a relative permittivity of $\epsilon_r = 2.1$ and a loss tangent of $\tan \delta = 0.001$. Table 2.5 includes the resonant frequencies and Q values for the experiments shown in Figure 2.26.

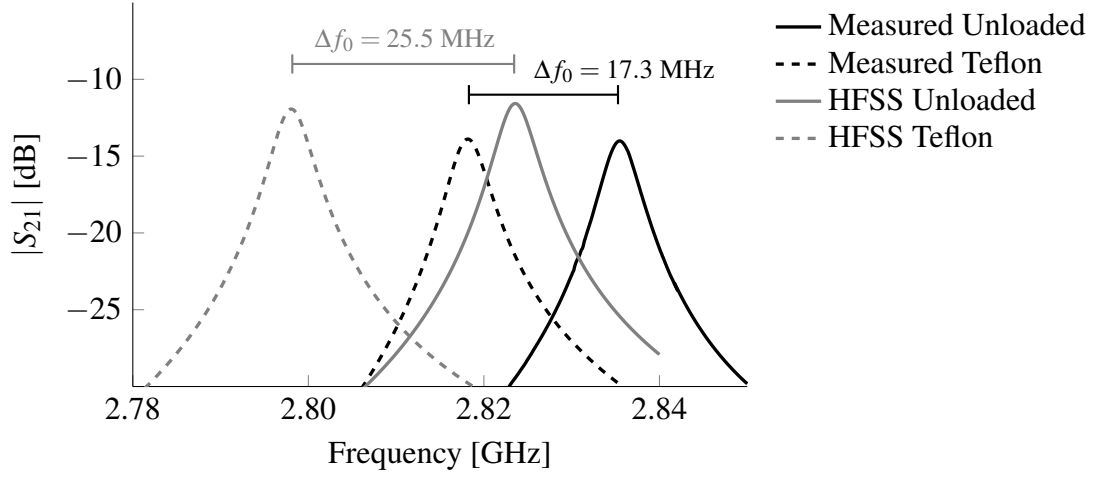


Figure 2.26: A comparison of the measured and simulated responses of the unloaded SIW cavity and for the analyte receptacle loaded with a Teflon rod.

Table 2.5: Teflon Plot Data

	f_0 [GHz]	Q
Measured Unloaded	2.8340	571
Measured Teflon	2.8167	530
HFSS Unloaded	2.8236	618
HFSS Teflon	2.7981	611

While the HFSS model does not correlate perfectly with the measured data, what is important is that a measurable shift in f_0 and Q due to the presence of an analyte can be detected with this sensor. The difference between the measured and simulated values in f_0 are 0.37% and 0.66% for the unloaded and Teflon cases respectively; similarly, the differences in Q are 8.2% and 15% for the unloaded and Teflon cases respectively. These percent errors are of similar magnitude to other such calculations between measured and simulated responses for microwave resonant cavities (see [35, 36] for example).

2.3 Shortcomings of the Designed SIW Cavity Sensor

The use of resonant cavities for the measurement of the permittivity of an analyte has been around for decades. Typical implementations have been made with cylindrical and rectangular cavities. What is different here is the employment of an SIW cavity that can be made much smaller than a rectangular cavity, especially where height is considered. The advantage of smaller dimensions is the type of analytes that may be interrogated. A specific application of such an SIW cavity resonator is for biosensing, in which the dimensions of the device are small enough to facilitate the measurement of biological substances on the order of microns in size.

Figure 2.26 shows that the sensor designed thus far could at least plausibly be used for the measurement of the permittivity of low loss dielectrics such as Teflon, but what about substances more comparable to biological entities? Easily attainable polar liquids such as methanol, propanol, and acetone are excellent reference materials as such liquids have permittivity values similar to those of human tissues [9, 45].

Based on the extensive work of Gregory and Clarke (see [9, 45]) on the characterization of reference liquids to be used for the calibration of dielectric measurement systems, the reference liquids and reported permittivity values shown in Table 2.6 were selected for testing with the SIW cavity resonator.

Table 2.6: Reported Literature Complex Permittivity Values for Reference Liquids at $T = 20^\circ\text{C}$ and $f = 2.8\text{ GHz}$ [45]

	Propan2ol	Ethanol	Methanol	Acetone
ϵ'	3.74	6.285	20.035	21.13
ϵ''	2.29	5.895	13.935	1.15
$\tan \delta$	0.6123	0.9379	0.6955	0.0544

Figure 2.27 shows the predicted sensor response when loaded with the lossy dielectric reference liquids of Table 2.6 as determined from HFSS simulations; Table 2.7 reports the f_0 and Q values of the plots in Figure 2.27. There is a consistent decrease in f_0 with increasing ϵ' . Similarly, there is a decreasing trend in Q for increasing $\tan \delta$. Such dependable trends suggests that this device could be used for permittivity measurement. However, from these simulations, it seems that the only reference liquid that should result in a measurable $|S_{21}|$ response is acetone; the other three liquids are characterized by high enough loss that HFSS predicts the $|S_{21}|$ values for these liquids to be about -35 dB or less for frequencies near the resonant peak. This is below what is measurable with the VNA on this device with the SOLT calibration performed.

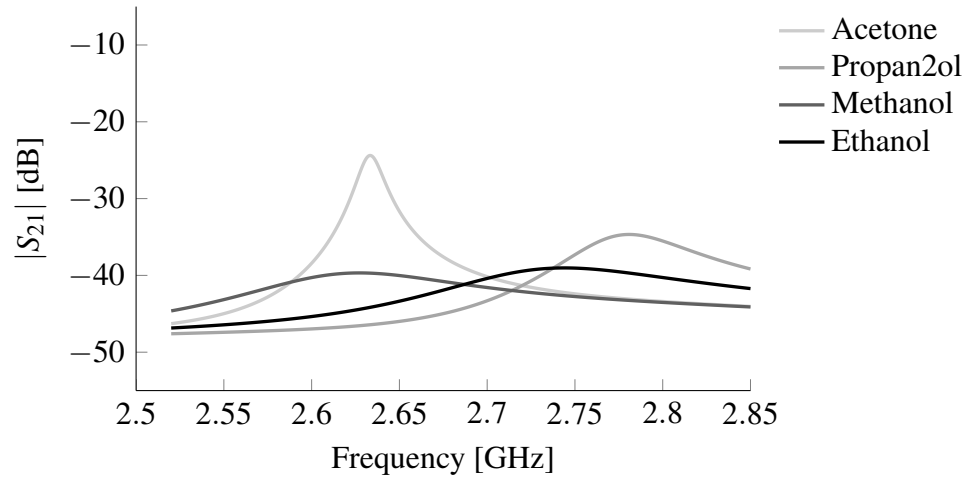


Figure 2.27: HFSS simulations of the SIW sensor when loaded with lossy reference liquids.

Table 2.7: Simulated SIW Response Values to Lossy Reference Liquids

	Propan2ol	Ethanol	Methanol	Acetone
f_0 [GHz]	2.7809	2.7440	2.6271	2.6334
Q	35	13	14	179

The measured results of Figure 2.28 confirm this prediction as the only resonant peak visible is that of the response of the SIW resonator when loaded with acetone. For ethanol, methanol, and propan2ol there is no discernible resonant frequency. The additional loss added to the resonant cavity when analytes with high loss tangent values are loaded into the analyte receptacle destroys the ability of the cavity to store energy. In such situations, Q diminishes to the point where the $|S_{21}|$ response appears flat (note the trending of the curves in Figure 2.27 to broaden with increasing $\tan \delta$). In practice, the responses obtained with the VNA with analytes of propan2ol, ethanol, and methanol do not contain useful information as seen from Figure 2.28.

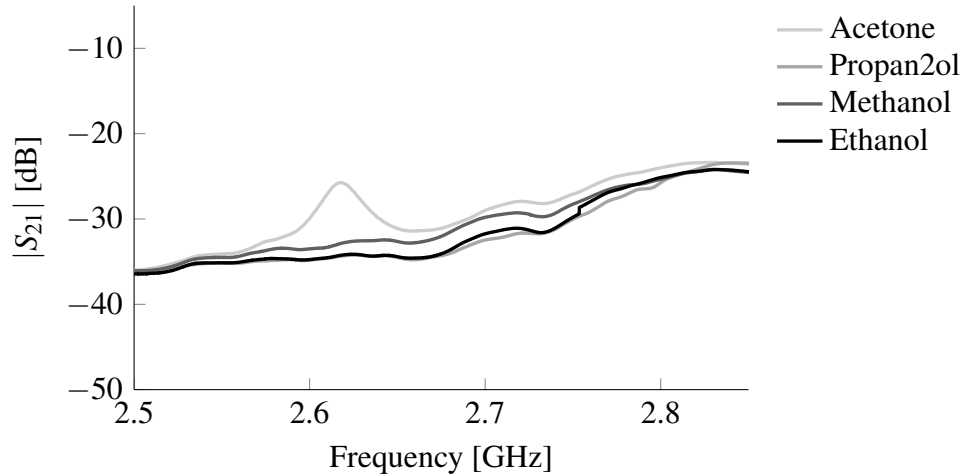


Figure 2.28: Measured $|S_{21}|$ response of the SIW sensor loaded with lossy liquids. The only visible response is that of acetone. The response to the other liquids is below the noise floor of the VNA.

Given the results of Figures 2.27 and 2.28, two plausible solutions to address the problem of the inability of this sensor to measure liquids with relatively high loss are⁹

⁹There is another option. The noise floor achieved with the SOLT calibration shown here is very poor. Superior noise floors are obtainable via thru, reflect, line (TRL) calibrations. This route was not pursued since, even in simulations, Q values were *very* low for the liquids considered (see Table 2.7).

1. to increase the transverse via spacing $B_{\perp}$ and the coupling aperture A in order to increase the amount of transmission through the cavity; this should bring the response to high loss analytes above the noise floor of the VNA, and
2. to decrease the volume of the analyte receptacle so that the effects of loading the resonant cavity are not as drastic.

If the analyte receptacle were to remain at the same volume as currently designed, but $B_{\perp}$ and A were increased, then the overall values of $|S_{21}|$ would increase as more EM energy would reach port 2 of the sensor. However, while increasing $B_{\perp}$ and A would certainly raise $|S_{21}|$ and possibly bring the responses of lossy materials above the noise floor, larger values of $B_{\perp}$ and A will also decrease the quality factor of the unloaded and loaded sensor. Consulting Table 2.7, it is seen that with the current values of $B_{\perp}$ and A , the measured quality factors for propan2ol, ethanol, and methanol are already very low (almost in the single digits). From Figure 2.27 it is seen that Q values this small make it difficult to clearly define f_0 .

The more promising solution is to reduce the diameter of the analyte receptacle. Scaling down D_R should diminish the effect of introducing an analyte into the SIW. Reducing the analyte receptacle volume will be explored in more detail in Chapter 3.

2.4 Design Summary

In this chapter, the design of an SIW cavity resonator for use as a permittivity measurement device was described. Starting from classical microwave engineering analysis of a rectangular waveguide and rectangular cavity, a substrate integrated waveguide analog of a rectangular cavity was designed. Optimized values for the length, width, via diameters, and via spacings of the SIW resonator were obtained through full-wave finite element simulations with HFSS.

Table 2.8 provides a summary of the chosen design values discussed in this chapter, and Table 2.9 reports the calculated, simulated, and measured resonant frequency and quality factor for the SIW cavity resonator device designed herein. Fair agreement between these values demonstrates that the rectangular cavity equations provide a reasonable approximation for the simulated and measured resonant peak and quality factor.

Finally, it was shown that creating a hole in the center of the SIW resonator allows for the introduction of test materials into the resonator. Such analytes perturb the EM field inside the cavity resulting in a measurable shift in f_0 and Q . In Chapter 3, guidelines for the design of the analyte receptacle volume will be provided that facilitates the use of simple algebraic relationships between the measured shift in f_0 and Q for the unloaded and loaded SIW cavity and the complex permittivity of an analyte.

Table 2.8: Summary of Design Values

	Parameter	Value	Note
Substrate: Rogers Duroid 5880	ϵ_r	2.2	± 0.02
	$\tan \delta$	0.0009	
Substrate height	h (or b_{SIW})	125 mils	
SIW width	a_{SIW}	3200 mils	Sets cutoff frequency and f_0
Resonator length	d_{SIW}	1500 mils	Sets f_0
Via diameter	D	220 mils	Controls radiative leakage
SIW via spacing	B	330 mils	Controls radiative leakage
Resonator via spacing	$B_{\perp}$	220 mils	Controls throughput and Q
Resonator aperture	A	400 mils	Controls throughput and Q
Analyte Receptacle Diameter	D_R	312.5 mils	

Table 2.9: Comparison of f_0 and Q Values

	Calculated	Simulated (HFSS)	Measured (VNA)
f_0	2.93 GHz	2.8235 GHz	2.8340
Q	987	617	571

CHAPTER 3

ESTIMATING ANALYTE PERMITTIVITY

In Section 2.3 it was demonstrated that the SIW cavity resonator design of Chapter 2 requires modification in order to facilitate testing of high-loss analytes. The present chapter is devoted to a discussion of the optimal analyte receptacle size.

By keeping all of the design parameters as were decided on in Chapter 2, it will now be shown that reducing the diameter of the analyte receptacle D_R will allow for the SIW cavity sensor to measure the complex permittivity of a wide range of analytes. Reducing D_R not only decreases the effect of high-loss analytes so that measurable $|S_{21}|$ responses may be achieved, but also promotes the use of the well established cavity perturbation equations to be employed for the determination of the complex permittivity of a sample loaded into the SIW resonator. The validity of the cavity perturbation equations depend on the fulfillment of one key condition; namely, that the change in the resonant frequency between the unloaded and loaded cavity be small [46]. This prerequisite is usually met by keeping the volume of the analyte receptacle much smaller than the volume of the resonant cavity. Specific guidelines for these volumes will be discussed, along with a derivation of the cavity perturbation equations, in Section 3.1.

The advantage of the perturbation technique is the simplicity of its application while maintaining a relatively high degree of accuracy and sensitivity when compared to nonresonant methods of measuring the complex permittivity of materials [47]. By keeping the fractional change in the resonant frequency small, a linear relationship between the real part of the relative permittivity of a material and the change in resonant frequency of the cavity may be obtained (a similar linear relationship exists between the imaginary part of the permittivity and the change in quality factor).

HFSS simulations of a modified SIW cavity resonator will be used to validate the cavity perturbation method. Simulations with multiple analyte receptacle diameters will provide insight into the type of sensitivities and range of complex permittivity values such an SIW permittivity sensor may be expected to achieve.

3.1 Perturbation Theory

Before a treatment of the perturbation of a resonant cavity is given, recall the following relationships from basic electromagnetic theory:

- When taking the conjugate of a vector equation involving phasors, replace j with $-j$ wherever j appears; denote the conjugate of phasor $\tilde{\mathbf{A}}$ by $\tilde{\mathbf{A}}^*$.
- The divergence theorem

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{A} dV = \oint_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s}. \quad (3.1)$$

- Vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}. \quad (3.2)$$

- The tangential component of the electric field at the surface of a perfect conductor is zero. That is, if $\hat{\mathbf{n}}$ denotes the direction perpendicular to the surface of an infinitely conductive material boundary, then $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = 0$.
- The component of the magnetic field perpendicular to the surface of a perfect conductor is zero. Thus, $\hat{\mathbf{n}} \cdot \tilde{\mathbf{H}} = 0$ at the surface of an infinitely conductive material boundary.

Figure 3.1 depicts two situations for a resonant cavity. The original cavity, shown in Figure 3.1a, supports the electric and magnetic fields $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_1$ oscillating at the resonant frequency ω_1 . The volume of the cavity is $\mathcal{V}_c$ and is composed of some material characterized by ϵ and μ ; the surface of the cavity resonator, denoted by $\mathcal{S}_c$, is assumed to

be a perfect conductor. Figure 3.1b depicts the situation in which a small portion of the volume of the original cavity is replaced with some small material. This material causes the overall permittivity and permeability content of the cavity to alter so that the cavity is now described by $\epsilon + \Delta\epsilon$ and $\mu + \Delta\mu$. Such a change in the material properties of the cavity will result in a change in the resonant frequency and quality factor. The perturbed cavity thus supports the electric and magnetic fields $\tilde{\mathbf{E}}_2$ and $\tilde{\mathbf{H}}_2$ at resonant frequency ω_2 .

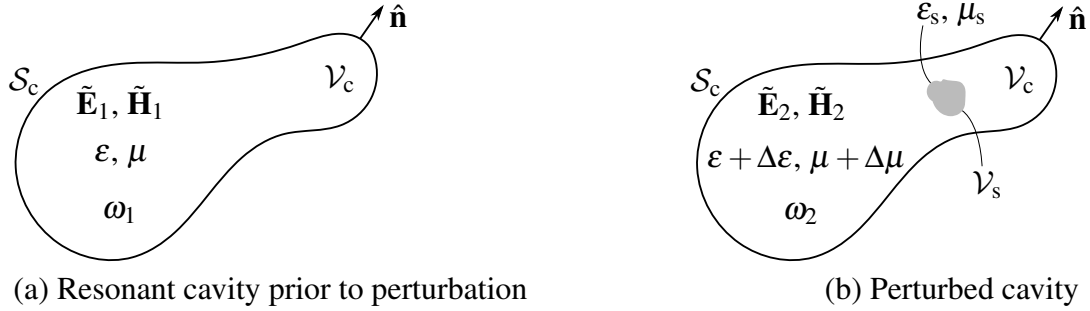


Figure 3.1: A metallic cavity of volume $\mathcal{V}_c$ and surface $\mathcal{S}_c$ is filled with a dielectric material characterized by ϵ and μ . (a) The original cavity supports the electric and magnetic fields $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_1$ at the resonant frequency ω_1 . (b) A small material of volume $\mathcal{V}_s$ is introduced into the cavity perturbing the permittivity and permeability of the original cavity leading to a change in the electric and magnetic fields as well as the resonant frequency.

In an effort to obtain equations describing the relationships between the change in resonant frequency and the change in ϵ and μ , consider the original and perturbed resonant cavities of Figure 3.1. Recall that the fields and resonant frequency of the unperturbed cavity are $\tilde{\mathbf{E}}_1$, $\tilde{\mathbf{H}}_1$, and ω_1 , and the perturbed cavity is defined by the electric field $\tilde{\mathbf{E}}_2$, magnetic field $\tilde{\mathbf{H}}_2$, and resonant frequency ω_2 . In both situations, Maxwell's equations must be satisfied. From equations 1.1a and 1.1b, prior to perturbation, the EM field must satisfy

$$\nabla \times \tilde{\mathbf{E}}_1 = -j\omega_1\mu\tilde{\mathbf{H}}_1, \quad (3.3a)$$

$$\nabla \times \tilde{\mathbf{H}}_1 = j\omega_1\epsilon\tilde{\mathbf{E}}_1, \quad (3.3b)$$

and after introduction of a perturbing material, the EM field must abide by

$$\nabla \times \tilde{\mathbf{E}}_2 = -j\omega_2 (\mu + \Delta\mu) \tilde{\mathbf{H}}_2, \quad (3.4a)$$

$$\nabla \times \tilde{\mathbf{H}}_2 = j\omega_2 (\varepsilon + \Delta\varepsilon) \tilde{\mathbf{E}}_2. \quad (3.4b)$$

Take the conjugate of equations 3.3a and 3.3b:

$$\nabla \times \tilde{\mathbf{E}}_1^* = j\omega_1 \mu \tilde{\mathbf{H}}_1^*, \quad (3.5a)$$

$$\nabla \times \tilde{\mathbf{H}}_1^* = -j\omega_1 \varepsilon \tilde{\mathbf{E}}_1^*. \quad (3.5b)$$

Postmultiply both sides of equation 3.5a by $\tilde{\mathbf{H}}_2$ and both sides of equation 3.5b by $\tilde{\mathbf{E}}_2$:

$$(\nabla \times \tilde{\mathbf{E}}_1^*) \cdot \tilde{\mathbf{H}}_2 = j\omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2, \quad (3.6a)$$

$$(\nabla \times \tilde{\mathbf{H}}_1^*) \cdot \tilde{\mathbf{E}}_2 = -j\omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2. \quad (3.6b)$$

Postmultiply both sides of equation 3.4b by the conjugate of $\tilde{\mathbf{E}}_1$ (i.e., $\tilde{\mathbf{E}}_1^*$) and both sides of equation 3.4a by $\tilde{\mathbf{H}}_1^*$:

$$(\nabla \times \tilde{\mathbf{H}}_2) \cdot \tilde{\mathbf{E}}_1^* = j\omega_2 (\varepsilon + \Delta\varepsilon) \tilde{\mathbf{E}}_2 \cdot \tilde{\mathbf{E}}_1^*, \quad (3.7a)$$

$$(\nabla \times \tilde{\mathbf{E}}_2) \cdot \tilde{\mathbf{H}}_1^* = -j\omega_2 (\mu + \Delta\mu) \tilde{\mathbf{H}}_2 \cdot \tilde{\mathbf{H}}_1^*. \quad (3.7b)$$

Subtract equation 3.7a from equation 3.6a and subtract equation 3.6b from equation 3.7b:

$$(\nabla \times \tilde{\mathbf{E}}_1^*) \cdot \tilde{\mathbf{H}}_2 - (\nabla \times \tilde{\mathbf{H}}_2) \cdot \tilde{\mathbf{E}}_1^* = j\omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - j\omega_2 (\varepsilon + \Delta\varepsilon) \tilde{\mathbf{E}}_2 \cdot \tilde{\mathbf{E}}_1^*, \quad (3.8a)$$

$$(\nabla \times \tilde{\mathbf{E}}_2) \cdot \tilde{\mathbf{H}}_1^* - (\nabla \times \tilde{\mathbf{H}}_1^*) \cdot \tilde{\mathbf{E}}_2^* = -j\omega_2 (\mu + \Delta\mu) \tilde{\mathbf{H}}_2 \cdot \tilde{\mathbf{H}}_1^* + j\omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2. \quad (3.8b)$$

Apply the vector identity of equation 3.2 to equations 3.8a and 3.8b:

$$\nabla \cdot (\tilde{\mathbf{E}}_1^* \times \tilde{\mathbf{H}}_2) = j\omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - j\omega_2 (\varepsilon + \Delta\varepsilon) \tilde{\mathbf{E}}_2 \cdot \tilde{\mathbf{E}}_1^*, \quad (3.9a)$$

$$\nabla \cdot (\tilde{\mathbf{E}}_2 \times \tilde{\mathbf{H}}_1^*) = -j\omega_2 (\mu + \Delta\mu) \tilde{\mathbf{H}}_2 \cdot \tilde{\mathbf{H}}_1^* + j\omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2. \quad (3.9b)$$

Add equations 3.9a and 3.9b:

$$\begin{aligned} & \nabla \cdot (\tilde{\mathbf{E}}_1^* \times \tilde{\mathbf{H}}_2 + \tilde{\mathbf{E}}_2 \times \tilde{\mathbf{H}}_1^*) \\ &= j\omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - j\omega_2 (\varepsilon + \Delta\varepsilon) \tilde{\mathbf{E}}_2 \cdot \tilde{\mathbf{E}}_1^* - j\omega_2 (\mu + \Delta\mu) \tilde{\mathbf{H}}_2 \cdot \tilde{\mathbf{H}}_1^* + j\omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2. \end{aligned} \quad (3.10)$$

Integrate equation 3.10 over the volume of the resonant cavity $\mathcal{V}_c$:

$$\begin{aligned} & \int_{\mathcal{V}_c} \nabla \cdot (\tilde{\mathbf{E}}_1^* \times \tilde{\mathbf{H}}_2 + \tilde{\mathbf{E}}_2 \times \tilde{\mathbf{H}}_1^*) dV \\ &= j \int_{\mathcal{V}_c} \{ [\omega_1 \mu - \omega_2 (\mu + \Delta\mu)] \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + [\omega_1 \varepsilon - \omega_2 (\varepsilon + \Delta\varepsilon)] \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV. \end{aligned} \quad (3.11)$$

The divergence theorem may be applied to the right-hand side of equation 3.11:

$$\int_{\mathcal{V}_c} \nabla \cdot (\tilde{\mathbf{E}}_1^* \times \tilde{\mathbf{H}}_2 + \tilde{\mathbf{E}}_2 \times \tilde{\mathbf{H}}_1^*) dV = \oint_{\mathcal{S}_c} (\tilde{\mathbf{E}}_1^* \times \tilde{\mathbf{H}}_2 + \tilde{\mathbf{E}}_2 \times \tilde{\mathbf{H}}_1^*) \cdot \hat{\mathbf{n}} ds = 0. \quad (3.12)$$

To see why the surface integral of equation 3.12 is equal to zero, consider the electric and magnetic fields just inside the metallic walls of the cavity. At the surface, the electric field is perpendicular and the magnetic field is parallel to the cavity walls. Thus, $\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}$ (and similarly for $\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}$ and $\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*$) is in a direction parallel to the surface and thus $(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}) \cdot \hat{\mathbf{n}} = 0$. Therefore,

$$j \int_{\mathcal{V}_c} \{ [\omega_1 \mu - \omega_2 (\mu + \Delta\mu)] \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + [\omega_1 \varepsilon - \omega_2 (\varepsilon + \Delta\varepsilon)] \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV = 0. \quad (3.13)$$

Expanding equation 3.13 gives

$$\begin{aligned} & \int_{\mathcal{V}_c} \{ \omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - \omega_2 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - \omega_2 \Delta\mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 \\ & \quad + \omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 - \omega_2 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 - \omega_2 \Delta\varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV = 0. \end{aligned}$$

Rearranging

$$\begin{aligned} & \int_{\mathcal{V}_c} \{ -\omega_2 \Delta\mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - \omega_2 \Delta\varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV \\ & \quad + \int_{\mathcal{V}_c} \{ \omega_1 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 - \omega_2 \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + \omega_1 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 - \omega_2 \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV = 0, \end{aligned}$$

$$\begin{aligned}
\int_{\mathcal{V}_c} \{ [\omega_1 - \omega_2] \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + [\omega_1 - \omega_2] \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV = \\
\int_{\mathcal{V}_c} \{ \omega_2 \Delta \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + \omega_2 \Delta \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV, \\
\frac{\omega_1 - \omega_2}{\omega_2} = \frac{\int_{\mathcal{V}_c} \{ \Delta \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + \Delta \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV}{\int_{\mathcal{V}_c} \{ \mu \tilde{\mathbf{H}}_1^* \cdot \tilde{\mathbf{H}}_2 + \varepsilon \tilde{\mathbf{E}}_1^* \cdot \tilde{\mathbf{E}}_2 \} dV}. \quad (3.14)
\end{aligned}$$

Equation 3.14 is exact but not very useful because, in general, the exact fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are not known.

If the changes in the overall permittivity and permeability content of the resonant cavity, $\Delta \varepsilon$ and $\Delta \mu$ respectively, are small, then the perturbed fields $\tilde{\mathbf{E}}_2$ and $\tilde{\mathbf{H}}_2$ may be approximated as equivalent to the unperturbed fields, $\tilde{\mathbf{E}}_2 \approx \tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_2 \approx \tilde{\mathbf{H}}_1$ [3, 47, 48]. With this assumption, equation 3.14 becomes

$$\frac{\omega_1 - \omega_2}{\omega_2} \approx \frac{\int_{\mathcal{V}_c} \{ \Delta \mu |\tilde{\mathbf{H}}_1|^2 + \Delta \varepsilon |\tilde{\mathbf{E}}_1|^2 \} dV}{\int_{\mathcal{V}_c} \{ \mu |\tilde{\mathbf{H}}_1|^2 + \varepsilon |\tilde{\mathbf{E}}_1|^2 \} dV}. \quad (3.15)$$

Note that equation 3.15 indicates that for any increase in ε and μ (i.e., positive $\Delta \varepsilon$ and $\Delta \mu$), the resonant frequency decreases since the right-hand side of equation 3.15 will be positive in such cases. The assumption made in simplifying the exact solution of equation 3.14 to the more useful form of equation 3.15 (i.e., that the geometry of the perturbed fields change only slightly compared to the fields of the unperturbed cavity) consequently requires that the difference in resonant frequency between the original and perturbed cavity be small, namely $\omega_1 - \omega_2 \ll \omega_1$ [46, 49, 50]. Practically speaking, these conditions are achieved by keeping the volume of the material introduced into the cavity very small relative to the volume of the cavity (i.e., $\mathcal{V}_s \ll \mathcal{V}_c$).

Consider the special case of a rectangular cavity operating in the TE_{101} mode. Through the application of a similar modification to the equations describing the TE mode of operation for a rectangular waveguide given in Table B.1 of Appendix B as they apply to a

rectangular cavity, as was performed for the y -component of $\mathbf{E}$ for the TM mode in Section B.2, the total fields for the TE_{101} resonant mode can be shown to be

$$\tilde{E}_x = 0, \quad (3.16a)$$

$$\tilde{E}_y = -\frac{2\omega\mu a}{\pi}A \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{d}z\right), \quad (3.16b)$$

$$\tilde{E}_z = 0, \quad (3.16c)$$

$$\tilde{H}_x = j\frac{2a}{d}A \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{d}z\right), \quad (3.16d)$$

$$\tilde{H}_y = 0, \quad (3.16e)$$

$$\tilde{H}_z = -j2A \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{d}z\right). \quad (3.16f)$$

If a small sample material of volume $\mathcal{V}_s$ is placed in the center of the rectangular cavity, where $\mathcal{V}_s \ll \mathcal{V}_c$, then equation 3.15 may be simplified to

$$\frac{\omega_1 - \omega_2}{\omega_2} \approx \frac{\int_{\mathcal{V}_s} \{\Delta\epsilon |\tilde{\mathbf{E}}_{1\max}|^2\}}{\int_{\mathcal{V}_c} \{\mu |\tilde{\mathbf{H}}_1|^2 + \epsilon |\tilde{\mathbf{E}}_1|^2\}}. \quad (3.17)$$

There are three differences between equations 3.15 and 3.17:

1. The $\Delta\mu |\tilde{\mathbf{H}}_1|^2$ term has been dropped from the numerator because the magnetic field is zero in the center of the cavity (plug $x = \frac{a}{2}$ into equations 3.16d and 3.16f).
2. The numerator of equation 3.17 is only integrated over the volume of the sample $\mathcal{V}_s$ because the change in permittivity $\Delta\epsilon$ is zero everywhere outside of the sample volume.
3. In the numerator, $|\tilde{\mathbf{E}}_1|$ is changed to $|\tilde{\mathbf{E}}_{1\max}|$ because the sample is introduced to the cavity where the electric field is maximum, and thus the effective interaction is greater than it would be if it were placed somewhere else [51].

Imagine a rectangular cavity that is to be used for measuring the complex permittivity of arbitrary sample materials. The rectangular cavity is filled with some dielectric medium characterized by ϵ and μ , and in the center of the cavity a very small cylindrical volume

$\mathcal{V}_s$ is removed to make room for the insertion of analytes of interest. In this situation, the change in the overall permittivity $\Delta\epsilon$ that occurs when an analyte is introduced into the cavity is $\Delta\epsilon = \epsilon_s - \epsilon_0$ (i.e., the change in permittivity is the difference between the permittivity of vacuum, describing the unloaded situation, and permittivity of the analyte, describing the loaded situation), where $\epsilon_s = \epsilon'_s - j\epsilon''_s$. Also note that $\epsilon'_s = \epsilon_0\epsilon'_{rs}$ and $\epsilon''_s = \epsilon_0\epsilon''_{rs}$.

Substituting equations 3.16a–3.16f into equation 3.17, taking frequency to be complex, and noting that $\Re\{\Delta\epsilon\} = \epsilon_0(\epsilon'_{rs} - 1)$ yields

$$\epsilon'_{rs} - 1 = \frac{\mathcal{V}_c}{2\mathcal{V}_s} \frac{\omega_1 - \omega_2}{\omega_2}, \quad (3.18)$$

where ϵ'_{rs} is the relative permittivity of the sample. Similarly, it may be shown that the imaginary part of ϵ_s is related to the change in the quality factor of cavity by [48]

$$\epsilon''_{rs} = \frac{\mathcal{V}_c}{4\mathcal{V}_s} \frac{Q_c - Q_s}{Q_c Q_s}, \quad (3.19)$$

where Q_c and Q_s are the quality factors of the empty and loaded resonator respectively.

The constants of $1/2$ in equation 3.18 and $1/4$ in equation 3.19 are specific to the geometry of a rectangular cavity. If a resonant cavity of a different geometry were used, for example a cylindrical cavity or the SIW cavity discussed in Chapter 2, then different constants would appear in equations 3.18 and 3.19. For a general resonant cavity subjected to a small material perturbation, the real and imaginary parts of the permittivity of the introduced material may be determined from the following equations, assuming that $\mathcal{V}_s \ll \mathcal{V}_c$,

$$\epsilon'_{rs} = M \frac{\mathcal{V}_c}{\mathcal{V}_s} \left[\frac{f_c - f_s}{f_s} \right] + 1, \quad (3.20)$$

$$\epsilon''_{rs} = N \frac{\mathcal{V}_c}{\mathcal{V}_s} \left[\frac{Q_c - Q_s}{Q_c Q_s} \right]. \quad (3.21)$$

For well defined cavity geometries such as a rectangular or cylindrical cavity, the constants M and N may be solved for analytically. For the SIW cavity however, the rows of

vias defining the parallel and transverse walls make solving for such constants analytically difficult at best. However, M and N may be obtained by testing the cavity resonator with calibration standards (i.e., analytes with known permittivity values).

Equations 3.20 and 3.21 provide an algebraic, linear set of equations relating the permittivity of an analyte to the shift in resonant frequency and quality factor of a resonant cavity. For equations 3.20 and 3.21 to remain valid, the fields in the perturbed cavity may only be slightly distorted from the unperturbed case, which is equivalent to requiring that the shift in the resonant frequency be small upon loading the cavity with some material [46]. To comport with this imposition, either large samples with very small values of permittivity may be interrogated, or very small sample volumes with a wider range of permittivity values may be investigated. Early researchers employing the cavity perturbation technique suggested that fractional changes in resonant frequency be $\frac{f_c - f_s}{f_s} < 0.001$ and $\frac{V_s}{V_c} < 2 \times 10^{-4}$ to achieve errors in permittivity and permeability estimates below 0.1% [49]. More recently, researchers exploring the use of the cavity perturbation technique for interrogation of high-loss analytes found that, in testing distilled water ($\epsilon'_{rs} \approx 82$, $\epsilon''_{rs} \approx 39$), the perturbation equations were valid (with estimate errors below 2.5%) for $\frac{f_c - f_s}{f_s} < 0.0056$ and $\frac{V_s}{V_c} < 8 \times 10^{-5}$ [52]. For the SIW cavity resonator design of Chapter 2, $\frac{V_s}{V_c} = 0.016$, and for the low permittivity valued analyte of Teflon, the measured fractional change in resonant frequency was $\frac{f_c - f_s}{f_s} = 0.006$. Thus, as designed, the analyte receptacle is too large for the perturbation equations to be valid. In the next section, analyte receptacle diameters that conform to the conditions of perturbation theory will be explored.

3.2 Applying the Perturbation Technique to the SIW Cavity

To examine the use of the small material perturbation method of permittivity analysis with the SIW cavity resonator, the sensor design of Chapter 2 is modeled in HFSS as shown

in Figure 3.2. The only dimension not consistent with those summarized in Table 2.8 is that of the analyte receptacle diameter. To adhere to the conditions of perturbation theory, the receptacle diameter is now set to $D_R = 39.0625$ mils in Figure 3.2a, a reduction by a factor of 8 compared to the design of Chapter 2.

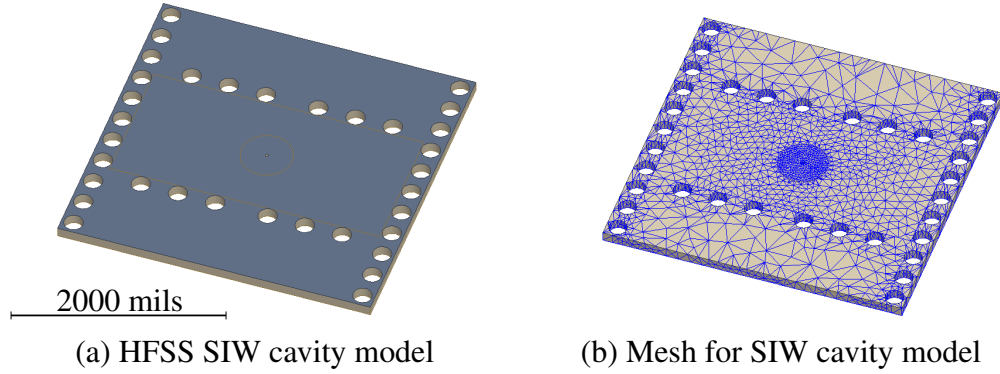


Figure 3.2: (a) HFSS model of the SIW cavity designed in Chapter 2 with a significantly reduced analyte receptacle diameter ($D_R \approx 39$ mils). (b) The mesh created for analysis of the SIW cavity. Because of the small dimensions of the analyte, special care must be taken to define a tight mesh around the sample volume.

Figure 3.2b shows the mesh used in the HFSS simulations of this design. To achieve meaningful results, it is necessary to force the simulator to densely mesh the volume near the analyte receptacle.

With $D_R \approx 39$ mils, $\frac{\nu_s}{\nu_c} = 2.5 \times 10^{-4}$, which is slightly larger than the ratio of suggested use from [49] and [52], but smaller values of D_R will be investigated shortly. First, the SIW sensor of Figure 3.2 was loaded with lossless (i.e., $\tan \delta = 0$) analytes with relative permittivities from $\epsilon'_{rs} = 1$ to $\epsilon'_{rs} = 100$ in steps of 1. HFSS calculated the S -parameters for each analyte, and from the $|S_{21}|$ responses, the resonant frequencies were obtained. To estimate the real part of the permittivity of the “analytes” inserted into the SIW cavity, equation 3.20 is used. Before application of equation 3.20, the constant M must be determined from a

calibration step via

$$M = \frac{\mathcal{V}_s}{\mathcal{V}_c} (\epsilon'_{\text{calibration}} - 1) \left[\frac{f_{\text{calibration}}}{f_{\text{unloaded}} - f_{\text{calibration}}} \right], \quad (3.22)$$

in which a well characterized sample material is placed into the analyte receptacle, and from the measured resonant frequencies of the unloaded cavity f_{unloaded} and the cavity loaded with the calibration sample $f_{\text{calibration}}$, as well as the known value of the sample permittivity $\epsilon'_{\text{calibration}}$, M is determined.

For this first experiment, the calibration step is performed, arbitrarily, for $\epsilon'_{\text{calibration}} = 2$; other values of $\epsilon'_{\text{calibration}}$ will be considered momentarily. Figure 3.3 displays the results from estimating the values of ϵ'_{rs} using perturbation theory on the simulated data.

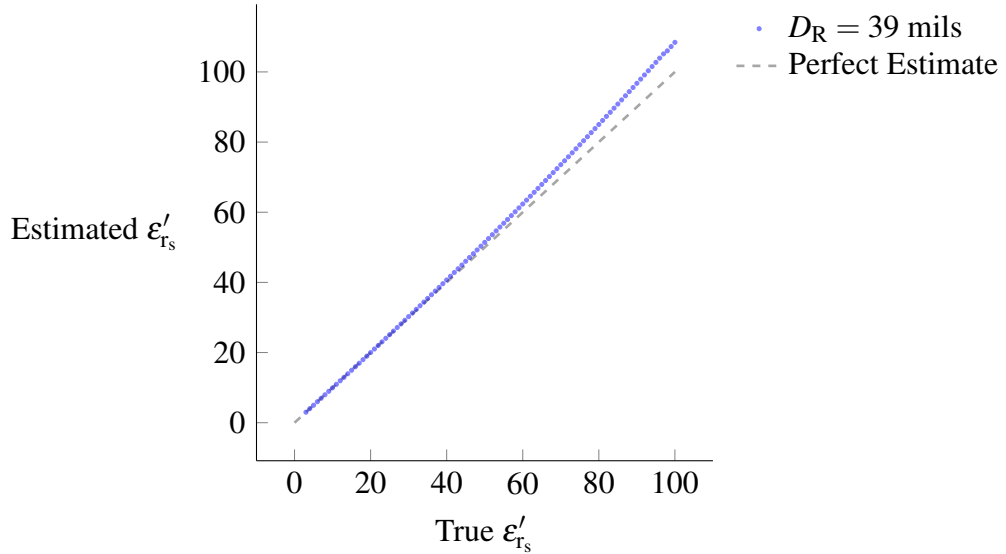


Figure 3.3: Perturbation theory estimates of ϵ'_{rs} on HFSS simulations of the SIW sensor with $D_R = 39$ mils. As the values of ϵ'_{rs} get larger, the estimation begins to lose accuracy.

Quantitative analyses of the estimation accuracy will be considered, but for now the key is to note the qualitative trend of Figure 3.3; excellent estimates of the real part of the permittivity of analytes are achieved for small values of ϵ'_{rs} , but the estimation accuracy begins to deteriorate for increasing values of ϵ'_{rs} . This trend can be explained by considering

the assumptions made in the derivation of equation 3.20. The only requirement needed for equation 3.20 to hold is that the change in the overall geometry of the EM fields in the resonant cavity must be small due to the introduction of some material into the cavity. Large values of ϵ'_{rs} disturb the EM fields relative to the unloaded situation more significantly than small values of ϵ'_{rs} and thus invalidate the perturbation equations.

What if a different data point is used to calibrate (i.e., determine M) for the same data set as shown in Figure 3.3? Figure 3.4 displays the results of simulations performed by HFSS with $D_R \approx 39$ mils, $\epsilon'_{rs} = 1, 2, 3, \dots, 100$, and $\tan \delta = 0$ for three different calibrations:

$$\epsilon'_{\text{calibration}_1} = 2, \epsilon'_{\text{calibration}_2} = 50, \text{ and } \epsilon'_{\text{calibration}_3} = 100.$$

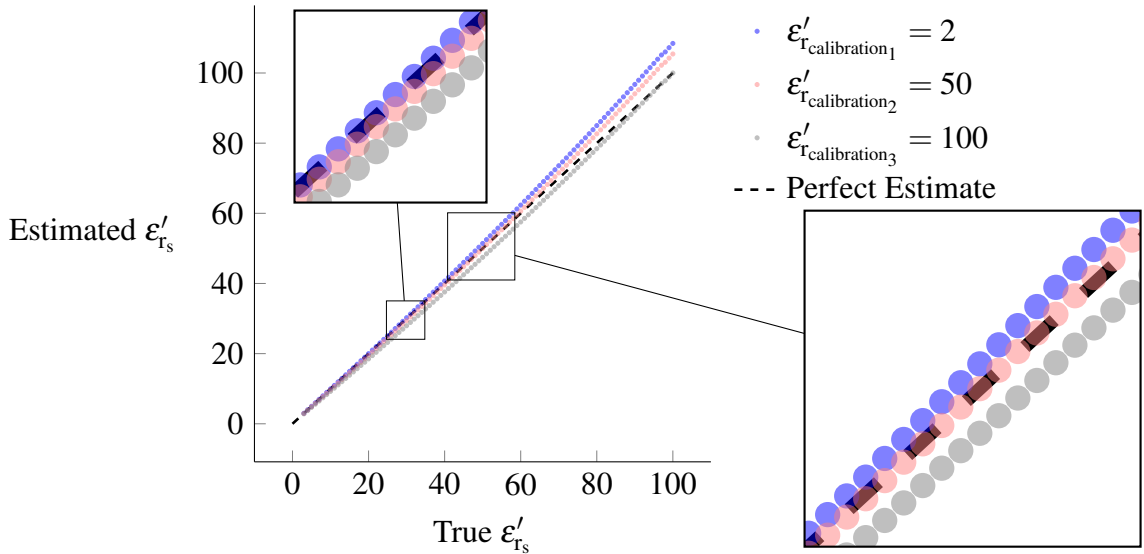


Figure 3.4: The effect of choosing a calibration standard on perturbation theory estimates of ϵ'_{rs} . Each data set is from the same simulation but was post-processed using a different calibration standard. Each set is more accurate over a certain region. The inset in the upper left shows that the $\epsilon'_{\text{calibration}_1} = 2$ calibration standard produces superior estimates for low permittivity values. The inset on the lower right shows that the $\epsilon'_{\text{calibration}_2} = 50$ calibration standard produces accurate estimates for middle-range permittivity values. At the upper right of the plot, the data set calibrated with $\epsilon'_{\text{calibration}_3} = 100$ tracks the “perfect estimate” line reasonably well, whereas the other two data sets clearly diverge from accurate estimates.

The $|S_{21}|$ data is the same for all three sets of estimates in Figure 3.4; the only change is the calibration step used for finding M . The values for M obtained from the three calibrations are $M_1 = 1.276$, $M_2 = 1.230$, and $M_3 = 1.181$. The conclusion to draw from Figure 3.4 is that the calibration standard used impacts the accuracy of the permittivity estimates; if the calibration standard has a permittivity value of similar magnitude to the analyte in question, then the permittivity estimate obtained from the perturbation technique will be more accurate than if a standard with a significantly different permittivity value was used in the calibration step to determine M . Consider the upper-right portion of the data composing Figure 3.4; clearly, the most accurate estimates of large values of ϵ'_{rs} (i.e., $\epsilon'_{rs} > 80$) are obtained using $\epsilon'_{\text{calibration}_3} = 100$. Likewise, the two insets of Figure 3.4 show that for low permittivity values ($\epsilon'_{rs} < 30$), the estimates from the $\epsilon'_{\text{calibration}_1} = 2$ data are superior, and the estimates from the $\epsilon'_{\text{calibration}_2} = 50$ data are superior for values of ϵ'_{rs} near 50. This suggests that if, prior to measurement, a range of hypothesized permittivity values of the analytes to be tested is known, then a calibration standard with a similar permittivity value should be used to determine M .

What else can be done to improve the accuracy of measurements using the perturbation technique? Recall that in simplifying the exact relationship of equation 3.14 to the more useful form of equation 3.15 that the assumption was $\tilde{\mathbf{E}}_2 \approx \tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_2 \approx \tilde{\mathbf{H}}_1$. This assumption is valid for small perturbations of the EM fields of the resonant cavity. Thus, as $\frac{\nu_s}{\nu_c}$ decreases, the perturbation equations should provide more accurate permittivity estimates.

Figure 3.5 shows the effect of decreasing the analyte receptacle diameter on the accuracy of the permittivity estimates. Again, the SIW cavity resonator design of Chapter 2 was simulated in HFSS and loaded with lossless analytes with relative permittivities of $\epsilon'_{rs} = 1, 2, 3, \dots, 100$. This simulation was performed for three different values of D_R :

$D_R = 39.0625$ mils, $D_R = 19.5$ mils, and $D_R = 13$ mils (these diameters correspond to a reduction in receptacle diameter of 8, 16, and 24 compared to the design of Chapter 2 respectively).¹ For clarity, only select data points are shown from the simulation to reduce clutter. As predicted, decreasing D_R , and thus decreasing $\mathcal{V}_s$, improves the estimates of ϵ'_{rs} ; this is most clearly evident by observing the upper right portion of Figure 3.5 where the data points for the $D_R = 13$ mils simulation are much closer to the “perfect estimate” line than the data sets corresponding to the larger diameter simulations.

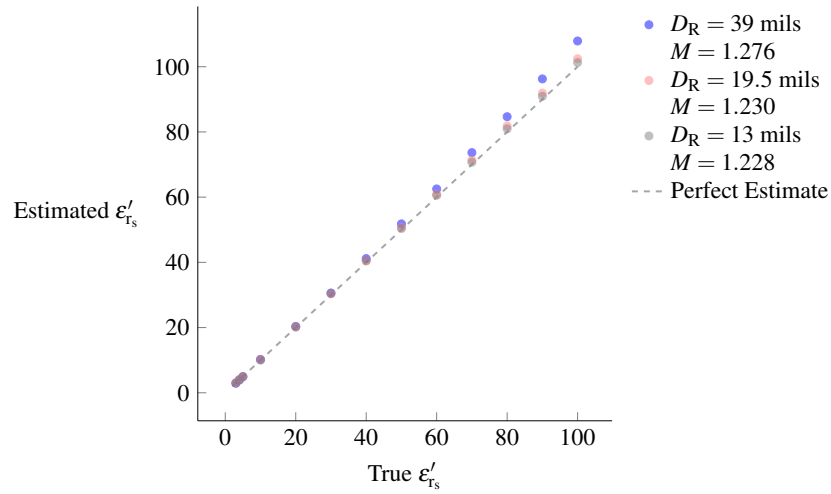


Figure 3.5: The effect of D_R on the accuracy of measurements of ϵ'_{rs} . As D_R decreases, the estimates of ϵ'_{rs} become more accurate.

It is helpful to observe an alternative representation of the data in Figure 3.5. Table 3.1 contains the values for the estimations plotted in Figure 3.5. For all estimates, the $D_R = 13$ mils data provide the most accurate estimates of ϵ'_{rs} . Similarly, the $D_R = 19.5$ mils data set yields better estimates of the real part of the relative permittivity compared to the estimates obtained when $D_R = 39$ mils. Also note that, for the data reported in Table 3.1, the average percent error is reduced by more than half in reducing the receptacle diameter

¹For all three values of D_R simulated, the calibration step was performed with $\epsilon'_{\text{calibration}} = 2$.

from $D_R = 39$ mils to $D_R = 19.5$ mils and similarly for the reduction from $D_R = 19.5$ mils to $D_R = 13$ mils.

Table 3.1: Select Estimates of ϵ'_{rs}

ϵ'_{rs}	Estimate ($D_R = 39$ mils)	Estimate ($D_R = 19.5$ mils)	Estimate ($D_R = 13$ mils)
3	2.9263	3.0715	3.0113
4	3.9085	4.0718	4.0114
5	4.9096	5.0006	5.0003
10	10.2371	10.2181	10.0129
20	20.3656	20.2321	20.1427
30	30.6093	30.3278	30.2985
40	41.1572	40.4339	40.3652
50	51.8052	50.6939	50.4023
60	62.5163	60.8927	60.5505
70	73.6527	71.3898	70.7148
80	84.6652	81.8261	80.9400
90	96.2624	91.9129	90.9105
100	107.8922	102.4430	101.2579
Average % Error	3.80%	1.65%	0.74%

Now, consider a similar experiment in which the SIW cavity resonator is simulated for three different values of D_R , but the imaginary part of the permittivity is varied while ϵ'_{rs} is held constant at a value of 100. As before, it is expected that as D_R decreases, the estimates of ϵ''_{rs} shall improve. Figure 3.6 displays the results of such an experiment. Again, consulting equation 3.21 it is observed that a calibration step is required in order to determine N ,

$$N = \frac{\mathcal{V}_s}{\mathcal{V}_c} \epsilon''_{r_{\text{calibration}}} \left[\frac{Q_{\text{unloaded}} Q_{\text{calibration}}}{Q_{\text{unloaded}} - Q_{\text{calibration}}} \right]. \quad (3.23)$$

Figure 3.6 displays the estimates of ϵ''_{rs} when N is determined from a calibration with $\epsilon''_{r_{\text{calibration}}} = 50$.

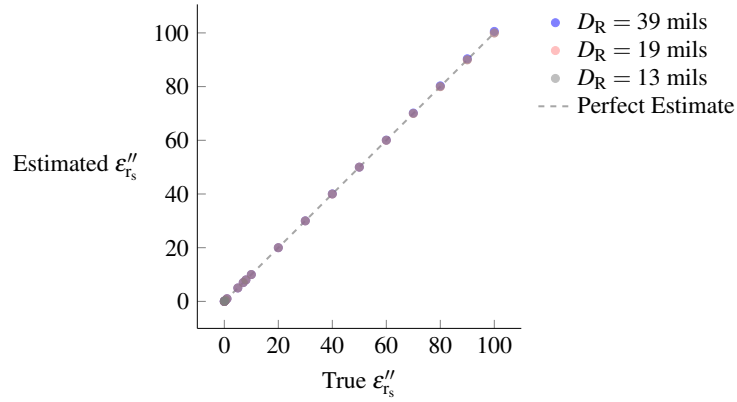


Figure 3.6: The effect of D_R on the accuracy of measurements of ε''_{rs} .

It is difficult to see any differentiation between the estimates from Figure 3.6 as all of the estimates are fairly accurate. However, upon a close inspection of the numerical values composing Figure 3.6, it is seen that there is more to the story. Table 3.2 contains the actual estimates constituting Figure 3.6.

Table 3.2: Estimated ε''_{rs} Values

ε''_{rs}	Estimate ($D_R = 39$ mils)	Estimate ($D_R = 19.5$ mils)	Estimate ($D_R = 13$ mils)
.005	.0058	.0065	.0312
.01	.0124	.0137	.0312
.05	.0517	.0544	.0624
.1	.0970	.1088	.1248
.5	.4994	.4979	.4897
1	1.0066	1.0022	.9988
5	5.0095	5.0059	4.9978
10	9.9952	10.0033	9.9978
20	20.0066	19.9944	19.9986
30	30.0084	29.9941	29.9979
40	40.0424	39.9631	39.9971
50	50.0000	50.0000	50.0000
60	60.0713	60.0027	60.0023
70	70.1617	69.9895	69.9899
80	80.2885	79.9694	79.9918
90	90.3639	89.9741	89.9781
100	100.5626	99.9709	99.9800

There are two trends to note from the data of Table 3.2:

1. For low values of ϵ''_{rs} ($\epsilon''_{rs} < 1$), larger receptacle diameters provide better estimates.
2. For large values of ϵ''_{rs} ($\epsilon''_{rs} > 1$), smaller receptacle diameters provide better estimates.

Trend #2 is to be expected since the equations from the perturbation technique depend on small sample volumes. However, trend #1 is not expected. As a clue toward explaining this trend, consider the Q values obtained from the experiment under consideration provided in Table 3.3.

Table 3.3: Q Values Corresponding to the Estimates of ϵ''_{rs}

ϵ''_{rs}	Q ($D_R = 39$ mils)	Q ($D_R = 19.5$ mils)	Q ($D_R = 13$ mils)
.005	859	853	837
.01	844	849	837
.05	731	820	825
.1	626	785	809
.5	292	587	703
1	230	522	660
5	208	494	447
10	175	447	604
20	97	302	471
30	67	228	386
40	51	183	327
50	42	153	284
60	35	131	251
70	30	115	224
80	26	103	203
90	24	92	186
100	21	84	171

In consideration of trend #1, observe the values of Q for the $\epsilon''_{rs} = .005$ and $\epsilon''_{rs} = .01$ cases for all three values D_R . Specifically, consider the difference between the values of Q . In the $D_R = 39$ mils case, $\Delta Q = 15$, for $D_R = 19.5$ mils, $\Delta Q = 4$, and for $D_R = 13$ mils, $\Delta Q = 0$. This indicates that the estimation error for low values of ϵ''_{rs} , in the case of very low

values of D_R (and hence small $\mathcal{V}_s$), is due to a lack of Q resolution. The minimum detectable change in Q is dependent upon the frequency resolution of the measurement apparatus, in this case HFSS, a finite element simulator. To overcome such a lack of resolution, more data points could be taken; however, this highlights a practical dilemma. Just as with the simulator, measurements made on a physical SIW cavity resonator with a VNA will also have finite frequency resolution, and thus will have a minimum resolvable change in Q . Since the frequency resolution is finite, there is a minimum detectable change in ϵ''_{rs} due to the imperfections of the measurement devices used.

Also note the values of Q for the trials corresponding to large values of ϵ''_{rs} . From Table 3.3 it is seen that, for the same value of ϵ''_{rs} , the sensor with a larger analyte volume will have a lower Q when considering moderate- to high-loss materials. Recall from Chapter 2 that low values of Q lead to unmeasurable $|S_{21}|$ responses. In fact, for HFSS predicted Q less than about 40, it was shown experimentally that no response was measured with the VNA on a fabricated SIW cavity (see Figures 2.27 and 2.28). Thus, for the $D_R = 39$ mils case, ϵ''_{rs} larger than about 40 lead to unmeasurable $|S_{21}|$ responses.

In summary, there is a minimum detectable change in ϵ''_{rs} due to the minimum frequency resolution of the device measuring the S -parameters. To improve the smallest detectable change in ϵ''_{rs} , either increase the frequency resolution of the measurement apparatus or increase the volume of the analyte receptacle. There is a maximum value of ϵ''_{rs} (truly it is $\tan \delta$) that may be detected by a cavity resonator. The limit is set by the minimum value of Q for which a measurable $|S_{21}|$ response may be obtained experimentally. To maximize the largest measurable value of ϵ''_{rs} , decrease the analyte receptacle volume. Here are two competing design choices: increase $\mathcal{V}_s$ to gain better resolution in ϵ''_{rs} (this also sets the low end of the measurable range of ϵ''_{rs}); decrease $\mathcal{V}_s$ to maximize the largest measurable value of ϵ''_{rs} (this sets the upper end of the measurable range of ϵ''_{rs}).

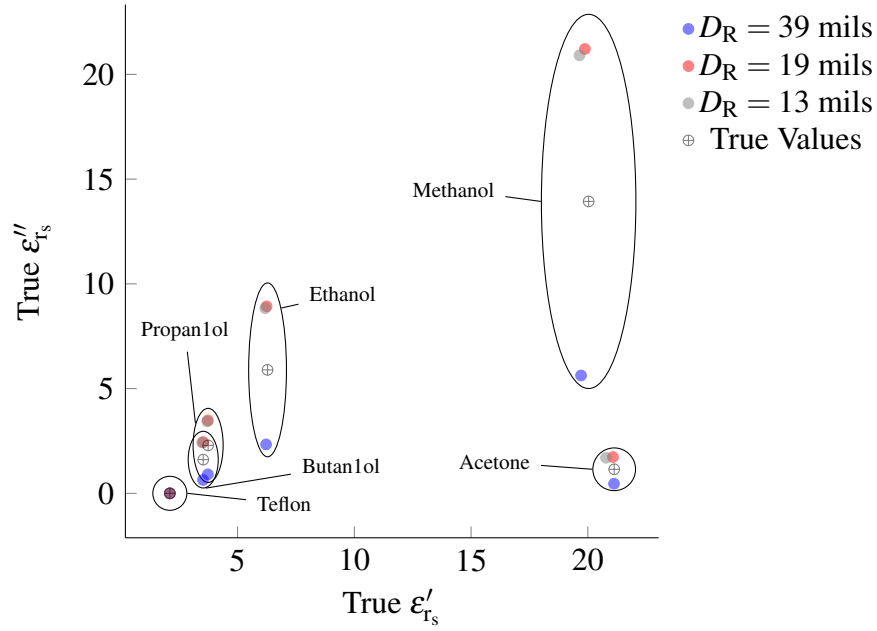
In applying the small material perturbation technique to idealistic data with the SIW cavity resonator, all seems well, but can this device be used to interrogate the lossy liquids considered at the end of Chapter 2? Table 3.4 again reports the permittivity values of some common liquids as determined by Gregory and Clarke [45], along with Teflon.

Table 3.4: Permittivity Values of Reference Liquids at $T = 20^\circ\text{C}$ and $f = 2.8\text{ GHz}$ [45]

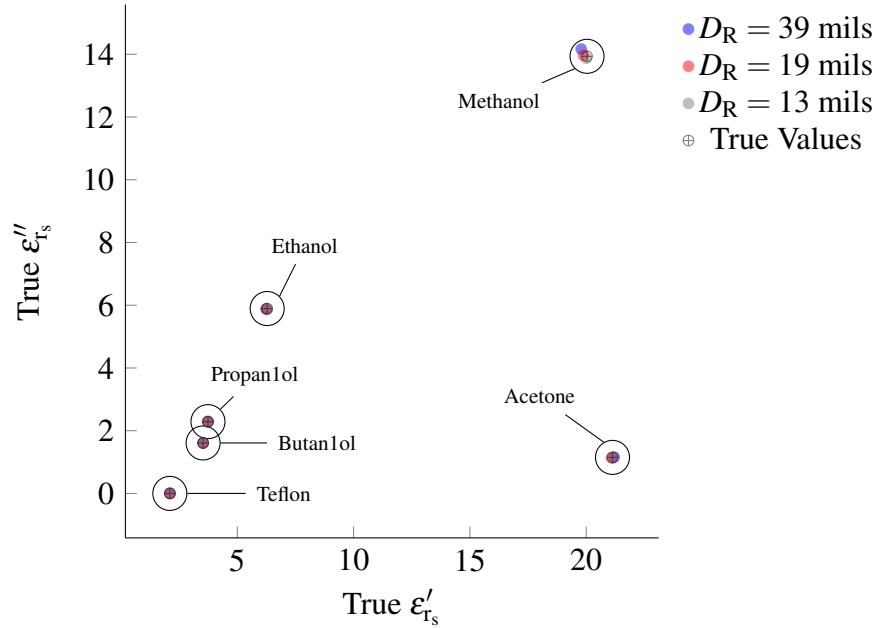
	Teflon	Butan1ol	Propan2ol	Ethanol	Methanol	Acetone
ϵ'	2.1	3.53	3.74	6.285	20.035	21.13
ϵ''	0.0021	1.61	2.29	5.895	13.935	1.15
$\tan \delta$	0.001	0.4561	0.6123	0.9379	0.6955	0.0544

Figure 3.7 shows the results of using the perturbation technique to estimate the permittivity of the lossy liquids of Table 3.4 via HFSS simulations of the SIW cavity resonator for $D_R = 39\text{ mils}$, $D_R = 19.5\text{ mils}$, and $D_R = 13\text{ mils}$. The results of Figure 3.7a were obtained using a Teflon calibration standard, and the results of Figure 3.7b were obtained using a butan1ol calibration standard.

These plots indicate the importance of choosing an appropriate calibration standard. For the Teflon calibrated case, the estimates of ϵ'_{rs} are not too bad; considering the $D_R = 13\text{ mils}$ data, the average percent error in estimating ϵ'_{rs} is 1.5%. However, the estimates of ϵ''_{rs} are unacceptably inaccurate. Butan1ol has real and imaginary permittivity values much more akin to those of the other lossy liquids in comparison to Teflon. For the butan1ol calibrated case, the estimates of both ϵ'_{rs} and ϵ''_{rs} are excellent for all three diameters of the analyte receptacle. For convenience, the tabulated values of the estimates contained in Figures 3.7a and 3.7b can be found in Tables 3.5 and 3.6. For the $D_R = 13\text{ mils}$ case calibrated with butan1ol, the average percent error in estimating ϵ'_{rs} is 0.14% and for ϵ''_{rs} the average percent error is 0.86% for the lossy liquids (i.e., excluding the estimate of Teflon). Thus, with an appropriate calibration standard, this sensor can be expected to give permittivity estimates with less than 1% error over a certain range of permittivity values.



(a) Permittivity estimates of lossy liquids with a Teflon calibration.



(b) Permittivity estimates of lossy liquids with a butan1ol calibration.

Figure 3.7: Permittivity estimates for lossy liquids. (a) When calibrated with Teflon, the estimates for ϵ''_{rs} are very poor. (b) Much improved estimates for both ϵ'_{rs} and ϵ''_{rs} are obtained by calibrating with a material more comparable to the analytes under investigation.

Table 3.5: Simulated Estimates Calibrated with Teflon

	True	$D_R = 39$ mils	$D_R = 19.5$ mils	$D_R = 13$ mils
ϵ'_{rs}	Teflon	2.1	2.1000	2.1000
	Butanol	3.53	3.5197	3.5288
	Propan2ol	3.74	3.7289	3.7288
	Ethanol	6.285	6.2219	6.2440
	Methanol	20.035	19.7099	19.8742
	Acetone	21.13	21.1210	21.0765
ϵ''_{rs}	Teflon	0.0021	0.0010	0.0010
	Butanol	1.61	0.6396	2.4450
	Propan2ol	2.29	0.9101	3.4812
	Ethanol	5.895	2.3358	8.9263
	Methanol	13.935	5.6273	21.2124
	Acetone	1.15	0.4604	1.7319

Table 3.6: Simulated Estimates Calibrated with Butanol

	True	$D_R = 39$ mils	$D_R = 19.5$ mils	$D_R = 13$ mils
ϵ'_{rs}	Teflon	2.1	2.1045	2.1005
	Butanol	3.53	3.5300	3.5300
	Propan2ol	3.74	3.7400	3.7301
	Ethanol	6.285	6.2432	6.2465
	Methanol	20.035	19.7863	19.8834
	Acetone	21.13	21.2032	21.0863
ϵ''_{rs}	Teflon	0.0021	0.0025	0.0007
	Butanol	1.61	1.6100	1.6100
	Propan2ol	2.29	2.2911	2.2923
	Ethanol	5.895	5.8799	5.8778
	Methanol	13.935	14.1654	13.9679
	Acetone	1.15	1.1589	1.1404

3.3 Sensitivity Assessments

It has been shown that reducing the analyte receptacle diameter allows the SIW cavity resonator to successfully measure the complex permittivity of lossy analytes. For the values of D_R considered in the present chapter, are *measurable* $|S_{21}|$ responses achievable?

Figure 3.8 shows the $|S_{21}|$ responses of the lossy liquids as computed by HFSS for the $D_R = 39$ mils case along with the measured, unloaded $|S_{21}|$ response of the fabricated sensor ($D_R = 312.5$ mils).

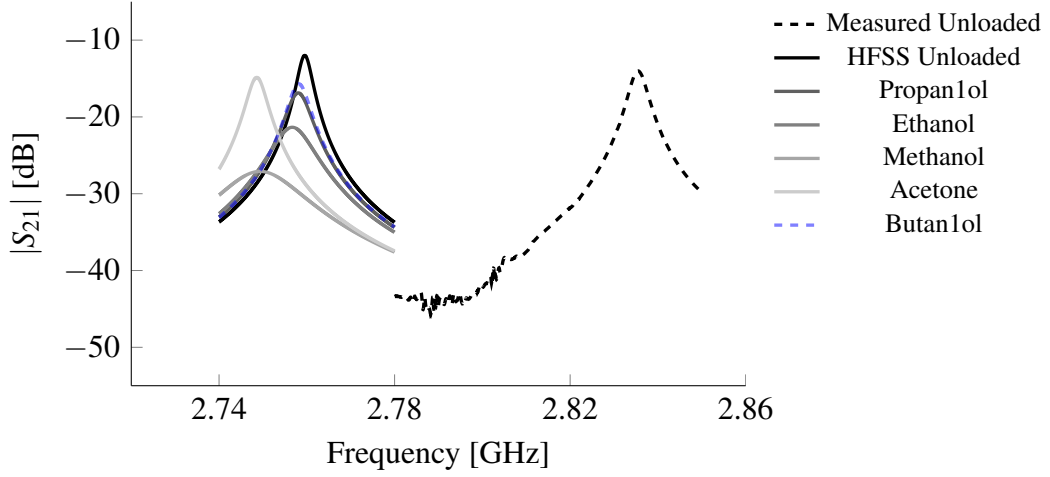


Figure 3.8: Simulated $|S_{21}|$ responses for the lossy liquids with $D_R = 39$ mils alongside the measured $|S_{21}|$ response of the unloaded, fabricated SIW cavity resonator.

Here, note that the $|S_{21}|$ responses for the lossy liquids are all well above the -40 dB noise floor of the VNA. This suggests that, with this reduction of analyte volume, indeed the lossy liquids considered here should have measurable responses. For the cases in which $D_R = 19.5$ mils and 13 mils, the $|S_{21}|$ responses will be even larger in overall magnitude.

However, Figure 3.8 brings forth another practical limit concerning measurement. The original dilemma of bringing the $|S_{21}|$ responses of the SIW resonator to high-loss materials above the noise floor was solved by reducing the volume of the analyte receptacle. This volume reduction works by reducing the effect that an introduced analyte has on the overall response of the sensor. Now the problem is whether the VNA can discern between two responses. Table 3.7 list the computed data of interest concerning the curves of Figure 3.8. Consider the two closest values of resonant frequency, that of butan1ol and propan2ol. The difference in f_0 for these two liquids is 114 kHz. Thus, the question now is, what is the

minimal frequency resolution of the measurement device? Network analyzers are available with frequency resolutions below 10 Hz (the exact frequency resolution depends on user adjustable settings such as the frequency sweep range, intermediate frequency bandwidth, amount of data averaging, and the number of data points collected), so the resonant frequencies of these liquids can certainly be discriminated.

Table 3.7: Salient Data from the $|S_{21}|$ Responses of the HFSS Simulations for the $D_R = 39$ mils Case

	Unloaded	Butanol	Propan2ol	Ethanol	Methanol	Acetone
f_0 [GHz]	2.759528	2.758154	2.75884	2.8756682	2.749358	2.748594
Δf [MHz]		1.374	1.488	2.846	1.017	1.0937
Q	840	549	479	287	149	608

What is the minimum resolvable change in ϵ'_{rs} achievable with this device. Consider equation 3.20 and take the derivate with respect to δf_0 , where $\delta f_0 = f_c - f_s$:

$$\epsilon'_{rs} = M \frac{\mathcal{V}_c}{\mathcal{V}_s} \left[\frac{\delta f_0}{f_s} \right] + 1,$$

$$\frac{\partial \epsilon'_{rs}}{\partial (\delta f_0)} = \frac{M \mathcal{V}_c}{f_c \mathcal{V}_s}. \quad (3.24)$$

Note that f_c was substituted for f_s in the denominator of equation 3.24 since perturbation theory requires that $f_s \approx f_c$. Consider the specific case of $D_R = 39$ mils:

$$M \approx 1.276,$$

$$\frac{\mathcal{V}_c}{\mathcal{V}_s} = \frac{1}{2.5 \times 10^{-4}},$$

$$f_c \approx 2.76 \times 10^9 \text{ Hz}.$$

Substituting these values into equation 3.24 yields

$$\Delta \epsilon'_{rs} = (1.85 \times 10^{-6}) \Delta (\delta f_0). \quad (3.25)$$

Thus, the minimum resolvable change in ϵ'_{rs} is dependent on the frequency resolution of the measurement apparatus. For butanol and propanol, $\Delta\epsilon'_{rs} = 3.74 - 3.53 = 0.21$. Equation 3.25 indicates that such a change in ϵ'_{rs} should result in a $\Delta(\delta f_0)$ value of 113.6 kHz for this device. The simulated shift is $\Delta(\delta f_0) = 114$ kHz, only a 0.4% difference.

Table 3.8 summarizes the factors determining the measurable limits of ϵ'_{rs} and ϵ''_{rs} when using the small material perturbation technique with a resonant cavity.

Table 3.8: Determining Factors for the Range of Measurable ϵ'_{rs} and ϵ''_{rs}

	Minimum Limit	How to Improve	Maximum Limit	How to Improve
ϵ'_{rs}	Limited by the frequency resolution of the device measuring the S -parameters	Increase frequency resolution of the measurement device; increase f_c ; increase $\mathcal{V}_s$	Limited by the constraints of perturbation theory which requires that $\delta f_0/f_0$ be small	Decrease $\mathcal{V}_s$
ϵ''_{rs}	Limited by the frequency resolution of the device measuring the S -parameters	Increase frequency resolution of the measurement device; increase Q_c ; increase $\mathcal{V}_s$	Limited by the maximum loss tangent value for which a measurable $ S_{21} $ response may be achieved	Decrease $\mathcal{V}_s$; Increase Q_c

3.4 Summary of Permittivity Estimation with the SIW Resonator

In Chapter 2 it was shown that the initial analyte receptacle design was too large, leading to unmeasurable $|S_{21}|$ responses when the sensor was loaded with significantly lossy analytes. The proposed solution of reducing the analyte diameter D_R was explored in this chapter. By choosing D_R such that the ratio of the volume of the analyte receptacle to the volume of the resonant cavity adheres to the conditions of perturbation theory, it was shown that not only is the SIW cavity sensor able to interrogate high-loss analytes, but

simple algebraic relations may then be used to determine the complex permittivity of an analyte from the change in f_0 and Q between the unloaded and loaded sensor.

Furthermore, finite element simulations have been used to explore the range of measurable permittivity values and the resolution limitations that pertain to the use of an SIW resonant cavity as a permittivity measurement device. It was shown that improving the lower limits of detection can be achieved by increasing the analyte volume, whereas decreasing $\mathcal{V}_s$ improves the upper limits of permittivity detection and increases the accuracy of estimation. It was also shown that, for maximum accuracy in estimates of both ϵ'_{rs} and ϵ''_{rs} , the required calibration steps in determining M and N should be performed with a reference standard characterized by ϵ' and ϵ'' values that are close to those of the analytes being interrogated. Such design considerations should be given high priority in the design of cavity resonators to be employed as permittivity sensors.

CHAPTER 4

CONCLUSION

An investigation of a novel resonant cavity structure employing a substrate integrated waveguide architecture for use as a permittivity sensor was explored. Design guidelines based on classical microwave engineering principles were used as a starting point to develop a prototype sensor. The sensor layout was optimized through finite element simulations. Verification of the response of the designed sensor was achieved by measuring the scattering parameters of a fabricated device. It was shown that the measured resonant frequency and quality factor agree well with the predicted values obtained through finite element analyses.

The shortcomings of the initial sensor design were illuminated through measurement of lossy dielectric reference liquids. A proposed solution of reducing the volume of the analyte receptacle was shown to overcome these deficiencies and provide a simple permittivity extraction method. An analysis of the limitations, sensitivity, and accuracy of the permittivity measurement capabilities of the SIW cavity resonator was also given.

While the use of a resonant cavity operating at microwave frequencies to determine the complex permittivity of analytes is not new, the use of a substrate integrated architecture in such an application is. SIW technology provides simple and inexpensive manufacturing capabilities, the potential for combination with other planar components on a monolithic substrate, and smaller geometries that facilitate the interrogation of biological entities.

4.1 Suggested Design Outline

The following is a summary of the design process used to develop the prototype sensor discussed herein:

1. Select the substrate thickness h and permittivity ϵ_r .
2. Select the desired unloaded resonant frequency f_0 .
3. Determine the microstrip width W to give a desired characteristic impedance (in this case $Z_0 = 50 \Omega$) based on the given substrate height and ϵ_r (Equation 2.2).
4. Design the microstrip taper and ground compensation for the microstrip-to-coax transition and optimize the transition through FEA simulations.
5. Based on the operational frequency range, determine the substrate integrated waveguide width using the cutoff frequency criterion (see equation 2.5).
6. From the desired unloaded resonant frequency, compute $\lambda_g = c/\sqrt{\epsilon_r}f_0$. Determine the SIW via diameters D and via spacings b by:
 - $D < \lambda_g/5$,
 - $b \leq 2D$.
7. Design a microstrip-to-SIW transition by use of a tapered microstrip line (essentially a quarter-wave transformer) and optimize via FEA simulations.
8. Based on the desired unloaded f_0 value, determine the distance between the transverse rows of vias that define the SIW resonant cavity from the equation for the resonant frequency of a rectangular cavity (equation 2.7).
9. Adjust the the precise value of f_0 and Q by altering the number of transverse SIW vias and their spacings, as well as adjusting the coupling aperture dimension, and verify with FEA simulations.
10. Choose an analyte receptacle size of convenient dimensions for sample deposition.

From the results of this initial design it was found that one of the most important features (if not *the* most important design component) is the volume of the analyte receptacle $\mathcal{V}_s$. Chapter 3 gave an in-depth analysis of the design trade-offs corresponding to the selection of the analyte receptacle volume. First and foremost, $\mathcal{V}_s$ should be selected such that the conditions of the small material perturbation technique are met so that a simple permittivity measurement approach may be used. In general, as $\mathcal{V}_s$ decreases the estimates of the complex permittivity improve and the smallest resolvable change in ϵ improves. However, along with this increased accuracy and improved resolution comes a higher demand on the frequency resolution of the device used to measure the scattering parameters of the SIW resonator.

Based on the experience gained from this research, a new design process for the development of an SIW cavity sensor is suggested:

1. Design $\mathcal{V}_s$ based on the analytes to be investigated.
 - Base $\mathcal{V}_s$ off of the size of the analytes of interest.
 - If there is any additional leeway in regards to defining this volume, the following guidelines provide assistance:
 - For low-loss analytes, maximize $\mathcal{V}_s$.
 - For high-loss analytes, minimize $\mathcal{V}_s$.
 - If high resolution in ϵ is of high value and the S -parameter measurement device to be used has excellent frequency resolution, maximize $\mathcal{V}_s$.
2. If the range of permittivity values to be investigated is unclear, consider fabricating multiple SIW resonators on a single substrate. Different size analyte volumes can be designed to preferably accommodate low-loss analytes or high-loss analytes all on the same substrate.

3. Choose the volume of the SIW resonant cavity $\mathcal{V}_c$ to ensure the validity of the perturbation equations (equations 3.20 and 3.21). As a rule of thumb, let $\frac{\mathcal{V}_s}{\mathcal{V}_c} < 2 \times 10^{-4}$, and note that as this ratio decreases, estimation accuracy increases and higher frequency resolution is requisite of the network analyzer performing the measurements.
4. From the known analyte receptacle volume and resonant cavity volume, the dimensions of the resonator width and length may be set. These dimensions also set the resonant frequency of the unloaded resonator. Use of the equations describing the resonant frequency and quality factor of a rectangular cavity may be used to give a rough estimate of f_0 and Q for the SIW cavity (see equations 2.7, 2.8, 2.9, and 2.10)
5. From the designed resonant frequency, the SIW via diameters and spacing may be selected through use of equations 2.4a and 2.4b.
6. At this point, if it has not already been implemented, finite element simulations should be used to verify and refine the design of the SIW cavity. Adjustments of the resonator width, length, via diameter and spacing, as well as the coupling aperture should be made to meet a design's needs.
7. Finally, design the SIW-to-microstrip and microstrip-to-coax transitions as discussed in Chapter 2.

4.2 Application Space for SIW Biosensors

An emphasis has been placed on the applicability of the SIW permittivity sensor toward small volume analytes with an ultimate goal of biomedical applications. However, there is nothing that prohibits the SIW cavity resonator from interrogating more commonplace materials that have historically been analyzed with conventional rectangular cavity sensors. In fact, such typical dielectric reference liquids were investigated with the SIW resonator

in Chapter 3. To highlight an advantage of using an SIW cavity resonator versus a conventional rectangular cavity for permittivity sensing, imagine a single substrate composing an array of SIW resonators. In such an array of sensors, each individual device could have a slightly different geometry, f_0 and Q value, or $\frac{\mathcal{V}_s}{\mathcal{V}_c}$ ratio, thereby allowing for a wide range of analytes to be interrogated by various sensor designs all on a single substrate. Such an array of sensors could also be used to obtain permittivity values over a range of frequencies.

At this point, to say that the SIW cavity resonator is *readily* applicable to biological applications is a great overstatement. Much more work into scaling down the dimensions of the proposed sensor layout is needed to validate such a claim. However, as a quick sanity check, consider whether it is even in the realm of possibility for the device to interrogate single cells. Cell diameters in plants and animals range from about 1 to 100 μm [53]. The volume of an approximately spherical cell with diameter $D_{\text{cell}} = 100 \mu\text{m}$ is about $V_{\text{cell}} = 5.24 \times 10^{-13} \text{ m}^3$. Consider an SIW cavity adhering to the constraints of perturbation theory via $\frac{\mathcal{V}_{\text{cell}}}{\mathcal{V}_c} = 1 \times 10^{-5}$.¹ This gives $\mathcal{V}_c = 5.24 \times 10^{-8} \text{ m}^3$. Assume the substrate height is $h = 127 \mu\text{m}$ (this is the thinnest Rogers Corp. manufactures Rogers Duroid 5880). For a square SIW cavity, the width and length will be about 2.03 cm, and for a substrate of Rogers Duroid 5880, equation 2.7 predicts the resonant frequency to be about 7.04 GHz and equations 2.8, 2.9, and 2.10 predict $Q = 195$.

These quick calculations are not definitive, but they do show plausibility. In Chapter 3 it was shown that even for Q values around 100, decent estimates of the complex permittivity can be achieved even for large values of loss tangent (see Figure 3.6 and Tables 3.2 and 3.3).

¹Note that for the $D_R = 13$ mils case considered in Chapter 3, $\frac{\mathcal{V}_s}{\mathcal{V}_c} = 2.8 \times 10^{-5}$, showing that $\frac{\mathcal{V}_s}{\mathcal{V}_c} = 1 \times 10^{-5}$ is a reasonable ratio to consider.

Furthermore, *single* cell analysis represents an extreme limit of biosensing. Other avenues of biosensing for bulk samples of tissue, cell cultures, and biofilms are worth exploring with the SIW cavity sensor.

4.3 Recommendations for Future Work

The next logical step for the SIW cavity resonator is to scale down the device to dimensions of a few centimeters or millimeters. Use of microelectronic fabrication techniques will be needed to achieve a scaled-down version of the sensor. The incorporation of a microfluidic channel composed of a biocompatible substance (e.g., polydimethylsiloxane) could facilitate the analysis of biological fluids and tissues.

Considering higher order resonant modes in the resonant cavity could provide multiple data points in a single device. Specifically, the TE_{10n} modes for odd n are promising since, for each of these modes, the electric field has a maximum located at the center of the cavity. Each mode occurs at a distinct frequency, thereby allowing this sensor to provide permittivity estimates at multiple frequencies.

REFERENCES CITED

- [1] T. Meissner and F. Wentz, “The complex dielectric constant of pure and sea water from microwave satellite observations,” *Geoscience and Remote Sensing, IEEE Transactions on*, vol. 42, no. 9, pp. 1836–1849, Sept. 2004.
- [2] F. T. Ulaby, *Fundamentals of Applied Electromagnetics*, 5th ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2007.
- [3] D. Pozar, *Microwave Engineering*, 4th ed. Hoboken, NJ: John Wiley & Sons, Inc., 2012.
- [4] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. Hoboken, NJ: John Wiley & Sons, Inc., 1999.
- [5] “IEEE standard definitions of terms for radio wave propagation,” *IEEE Std 211-1997*, pp. 1–39, 1998.
- [6] B. Clarke, “Measurement of the dielectric properties of materials at RF and microwave frequencies,” in *Microwave Measurements*, 3rd ed., R. Collier and D. Skinner, Eds. London, United Kingdom: The Institution of Engineering and Technology, 2007.
- [7] A. R. von Hippel, Ed., *Dielectric Materials and Applications*. New York: The Technology Press of MIT and John Wiley & Sons, 1954.
- [8] U. Kaatz and Y. Feldman, “Broadband dielectric spectrometry of liquids and biosystems,” *Measurement Science and Technology*, vol. 17, pp. R17–R35, 2006.
- [9] A. P. Gregory and R. N. Clarke, “A review of RF and microwave techniques for dielectric measurements on polar liquids,” *IEEE Transactions on Dielectrics and Electrical Insulation*, vol. 13, no. 4, pp. 727–743, Aug. 2006.
- [10] T. S. England and N. A. Sharples, “Dielectric properties of the human body in the microwave region of the spectrum,” *Nature*, vol. 163, pp. 487–488, Mar. 1949.
- [11] T. S. England, “Dielectric properties of the human body for wave-lengths in the 1 – 10 cm range,” *Nature*, vol. 166, pp. 480–481, Sep. 1950.
- [12] H. F. Cook, “Dielectric behaviour of human blood at microwave frequencies,” *Nature*, vol. 168, pp. 247–248, Aug. 1951.
- [13] —, “The dielectric behaviour of some types of human tissues at microwave frequencies,” *British Journal of Applied Physics*, vol. 2, pp. 295–300, Oct. 1951.
- [14] J. S. Daniels and N. Pourmand, “Label-free impedance biosensors: Opportunities and challenges,” *Electroanalysis*, vol. 19, no. 12, pp. 1239–1257, May 2007.

- [15] B. B. Haab, "Antibody arrays in cancer research," *Molecular & Cellular Proteomics*, vol. 4, no. 4, pp. 377–383, 2005.
- [16] K. D. Vos, I. Bartolozzi, E. Schacht, P. Bienstman, and R. Baets, "Silicon-on-insulator microring resonator for sensitive and label-free biosensing," *Optics Express*, vol. 15, no. 12, pp. 7610–7615, 2007.
- [17] K. R. Foster and H. P. Schwan, "Dielectric properties of tissues," in *CRC Handbook of Biological Effects of Electromagnetic Fields*, C. Polk and E. Postow, Eds. Boca Raton, FL: CRC Press, 1986.
- [18] M. Miettinen, J. Mönkkönen, M.-R. Lahtinen, J. Nuutinen, and T. Lahtinen, "Measurement of oedema in irritant-exposed skin by a dielectric technique," *Skin Research and Technology*, vol. 12, pp. 235–240, 2006.
- [19] P.-O. Bagnaninchi, M. Dikeakos, T. Veres, and M. Tabrizian, "Complex permittivity measurement as a new noninvasive tool for monitoring *in vitro* tissue engineering and cell signature through the detection of cell proliferation, differentiation, and pretissue formation," *IEEE Transactions on NanoBioscience*, vol. 3, no. 4, pp. 243–250, 2004.
- [20] D.-S. Yoo, "The dielectric properties of cancerous tissues in a nude mouse xenograft model," *Bioelectromagnetics*, vol. 25, pp. 492–497, Oct. 2004.
- [21] A. J. Surowiec, S. S. Stuchly, J. R. Barr, and A. Swarup, "Dielectric properties of breast carcinoma and the surrounding tissues," *IEEE Transactions on Biomedical Engineering*, vol. 35, no. 4, Apr. 1988.
- [22] J. Yang, Y. Huang, X. Wang, X.-B. Wang, F. F. Becker, and P. R. C. Gascoyne, "Dielectric properties of human leukocyte subpopulations determined by electrorotation as a cell separation criterion," *Biophysical Journal*, vol. 76, pp. 3307–3314, Jun. 1999.
- [23] C. M. Harris, R. W. Todd, S. J. Bungard, R. W. Lovitt, J. G. Morris, and D. B. Kell, "Dielectric permittivity of microbial suspensions at radio frequencies: a novel method for the real-time estimation of microbial biomass," *Enzyme and Microbial Technology*, vol. 9, pp. 181–186, Mar. 1987.
- [24] L. Y. Zhang, C. B. M. du Puch, A. Lacroix, C. Dalmay, A. Pothier, C. Lautrette, S. Battu, F. Laloué, M.-O. Jauberteau, and P. Blondy, "Microwave biosensors for identifying cancer cell aggressiveness grade," in *2012 IEEE MTT-S International Microwave Symposium Digest (MTT)*, Montreal, QC, Canada, Jun. 17–22, 2012, pp. 1–3.
- [25] P. Marszalek, J. J. Zielinsky, M. Fikus, and T. Y. Tsong, "Determination of electric parameters of cell membranes by a dielectrophoresis method," *Biophysical Journal*, vol. 59, pp. 982–987, May 1991.

- [26] H. Morgan, T. Sun, D. Holmes, S. Gawad, and N. G. Green, "Single cell dielectric spectroscopy," *Journal of Physics D: Applied Physics*, vol. 40, pp. 61–70, 2007.
- [27] D. Holmes, D. Pettigrew, C. H. Reccius, J. D. Gwyer, C. van Berkel, J. Holloway, D. E. Davies, and H. Morgan, "Leukocyte analysis and differentiation using high speed microfluidic single cell impedance cytometry," *Lab on a Chip*, vol. 9, no. 20, pp. 2861–3024, Oct. 2009.
- [28] K. Asami, "Dielectric imaging of biological cells," *Colloid & Polymer Science*, vol. 273, no. 11, pp. 1095–1097, 1995.
- [29] G. R. Facer, D. A. Notterman, and L. L. Sohn, "Dielectric spectroscopy for bioanalysis: From 40 Hz to 26.5 GHz in a microfabricated wave guide," *Applied Physics Letters*, vol. 78, no. 7, pp. 996–998, Feb. 2001.
- [30] K. Grenier, D. Dubuc, P.-E. Poleni, M. Kumemura, H. Toshiyoshi, T. Fujii, and H. Fujita, "Integrated broadband microwave and microfluidic sensor dedicated to bioengineering," *IEEE Transactions on Microwave Theory and Techniques*, vol. 57, no. 12, pp. 3246–3253, Dec. 2009.
- [31] C. Dalmay, M. Cheray, A. Pothier, F. Lalloué, M.-O. Jauberteau, and P. Blondy, "Ultra sensitive biosensor based on impedance spectroscopy at microwave frequencies for cell scale analysis," *Sensors and Actuators A: Physical*, vol. 162, pp. 189–197, 2010.
- [32] K. Grenier, D. Dubuc, P. E. Poleni, M. Kumemura, H. Toshiyoshi, T. Fujii, and H. Fujita, "Resonant based microwave biosensor for biological cells discrimination," in *2010 IEEE Radio and Wireless Symposium (RWS)*, New Orleans, LA, Jan. 10–14, 2010, pp. 523–526.
- [33] K. C. Gupta, R. Garg, I. Bahl, and P. Bhartia, *Microstrip Lines and Slotlines*, 2nd ed. Norwood, MA: Artech House, 1996.
- [34] E. O. Hammerstad, "Equations for microstrip circuit design," in *Proceedings of the 5th European Microwave Conference*, 1975, pp. 268–272.
- [35] W. J. Chappell, M. P. Little, and L. P. B. Katehi, "High Q two dimensional defect resonators - measured and simulated," in *IEEE MTT-S International Microwave Symposium Digest*, vol. 3, 2000, pp. 1437–1440.
- [36] W. J. Chappell, X. Gong, and L. P. B. Katehi, "Narrow Ka bandpass filters using periodically loaded substrates," in *IEEE MTT-S International Microwave Symposium Digest*, vol. 3, 2003, pp. 1611–1614.
- [37] Rogers Corporation, "RT/Duroid 5870/5880 high frequency laminates," Datasheet, 2013. [Online]. Available: <http://www.rogerscorp.com/documents/606/acm/RT-duroid-5870-5880-Data-Sheet.pdf>

- [38] V. Kaajakari, *Practical MEMS*. Las Vegas, NV: Small Gear Publishing, 2009.
- [39] N. C. Knowles, "Finite element analysis," *Computer-Aided Design*, vol. 16, no. 3, pp. 134–140, 1984.
- [40] P. M. Kurowski, *Finite Element Analysis for Design Engineers*. London, United Kingdom: Society of Automotive Engineers, Inc., 2004.
- [41] E. Holzman, *Essentials of RF and Microwave Grounding*. Norwood, MA: Artech House, 2006.
- [42] D. Deslandes and K. Wu, "Integrated microstrip and rectangular waveguide in planar form," *Microwave and Wireless Components Letters, IEEE*, vol. 11, no. 2, pp. 68–70, February 2001.
- [43] K. Wu, D. Deslandes, and Y. Cassivi, "The substrate integrated circuits - a new concept for high-frequency electronics and optoelectronics," in *Telecommunications in Modern Satellite, Cable and Broadcasting Service, 2003. TELSIKS 2003. 6th International Conference on*, vol. 1, October 2003, pp. P – III–P–X vol.1.
- [44] D. Deslandes and K. Wu, "Design consideration and performance analysis of substrate integrated waveguide components," in *32nd European Microwave Conference*, 2002, pp. 1–4.
- [45] A. P. Gregory and R. N. Clarke, *Tables of the Complex Permittivity of Dielectric Reference Liquids at Frequencies up to 5 GHz*. National Physical Laboratory, 2001.
- [46] R. A. Waldron, "Perturbation theory of resonant cavities," *Proceedings of the Institution of Electrical Engineers*, vol. 107C, pp. 272–274, Apr. 1960.
- [47] L. F. Chen, C. K. Ong, C. P. Neo, V. V. Varadan, and V. K. Varadan, *Microwave Electronics: Measurement and Materials Characterization*. West Sussex, England: John Wiley & Sons, Inc., 2004.
- [48] K. T. Mathew, "Perturbation Theory," in *Encyclopedia of RF and Microwave Engineering*, K. Chang, Ed. Hoboken, New Jersey: John Wiley & Sons, 2005.
- [49] E. G. Spencer, R. C. LeCraw, and L. A. Ault, "Note on cavity perturbation theory," *Journal of Applied Physics*, vol. 28, no. 1, pp. 130–132, Jan. 1957.
- [50] S.-H. Chao, "Measurements of microwave conductivity and dielectric constant by the cavity perturbation method and their errors," *IEEE Transactions on Microwave Theory and Techniques*, vol. 33, no. 6, pp. 519–526, 1985.
- [51] V. Murthy and R. Raman, "A method for the evaluation of microwave dielectric and magnetic parameters using rectangular cavity perturbation technique," *Solid State Communications*, vol. 70, no. 8, pp. 847–850, 1989.

- [52] C. P. L. Rubinger and L. C. Costa, “Building a resonant cavity for the measurement of microwave dielectric permittivity of high loss materials,” *Microwave and Optical Technology Letters*, vol. 49, no. 7, pp. 1687–1690, Jul. 2007.
- [53] N. A. Campbell, B. Williamson, and R. J. Heyden, *Biology: Exploring Life*. Boston, Massachusetts: Pearson Prentice Hall, 2006.
- [54] R. C. Dorf and J. A. Svoboda, *Electric Circuits*, 8th ed. Hoboken, NJ: John Wiley & Sons, Inc., 2010.
- [55] P. R. Young, “Scattering parameters and circuit analysis,” in *Microwave Measurements*, 3rd ed., R. Collier and D. Skinner, Eds. London, United Kingdom: The Institution of Engineering and Technology, 2007.
- [56] S. A. Wartenberg, *RF Measurement of Die and Packages*. Norwood, MA: Artech House, 2002.
- [57] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations*, 6th ed. New York: John Wiley & Sons, Inc., 1997.

APPENDICES

APPENDIX A

SCATTERING PARAMETERS AND THE SOLT CALIBRATION METHOD

The concept of scattering parameters are reviewed, and the calibration method used in the measurement of the scattering parameters during the course of this research is provided.

A.1 Scattering Parameters

Voltage is defined as the amount of energy required to transfer a unit positive charge from one terminal to another [54]. In RF and microwave circuits, two clearly defined *terminals* are not always available (e.g., what would the terminals be for a waveguide composed of a single conductor such as a rectangular waveguide?). In fact, it can be shown that for transmission lines which do not support TEM waves, voltage and current cannot be *uniquely* defined [3, 55].

For RF and microwave systems, instead of measuring voltages and currents, scattering parameters are used to characterize the networks under study. Scattering parameters relate the magnitude and phase of the incident and reflected EM waves at the ports of a network. Consider the 2-port network shown in Figure A.1, where V_1^+ is the amplitude of the voltage incident on port 1, V_1^- is the amplitude of the voltage reflected from port 1, and similar definitions apply to V_2^+ and V_2^- for port 2.

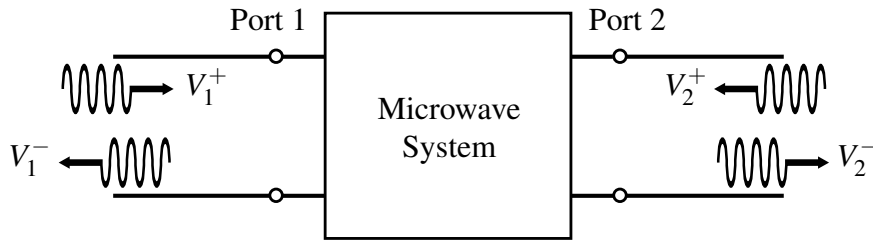


Figure A.1: 2-Port Network Scattering Parameters

The scattering parameters for such a network are defined as follows:

$$\begin{aligned} S_{11} &= \left. \frac{V_1^-}{V_1^+} \right|_{V_2^+=0} = 0, & S_{21} &= \left. \frac{V_2^-}{V_1^+} \right|_{V_2^+=0} = 0, \\ S_{22} &= \left. \frac{V_2^-}{V_2^+} \right|_{V_1^+=0} = 0, & S_{12} &= \left. \frac{V_1^-}{V_2^+} \right|_{V_1^+=0} = 0. \end{aligned}$$

- S_{11} is the ratio of the reflected voltage to the incident voltage at port 1 with port 2 terminated in a matched load (and thus, port 2 is able to absorb all of the power from the wave incident on it), effectively negating the presence of V_2^+ (thus $V_2^+ = 0$). S_{11} may also be viewed as the reflection coefficient at port 1.
- S_{21} is a measure of the amount of signal that is transmitted from port 1 to port 2; it is also the transmission coefficient of port 1 with port 2 terminated in a matched load.
- S_{22} and S_{21} are defined similarly as the reflection and transmission coefficients from port 2 to port 1.

Defining the scattering parameters as a *ratio* of incident and reflected wave magnitudes eschews the difficulty of uniquely defining a voltage for non-TEM waveguides.

A.2 SOLT Calibration Method

Measured data characterizing the SIW permittivity sensor's electrical characteristics, unloaded frequency response, and loaded response, were collected using a vector network analyzer (VNA). As with any test system, the VNA, as well as the components (e.g., coaxial cables and connectors) interfacing the sensor to the VNA, is a non-ideal device with inherent sources of error.

Calibration procedures are routinely used in RF and microwave measurements to quantify the errors associated with test systems. In such procedures, reference standards with known electrical properties are connected to the test system (in this case, the standards are

connected to the VNA via coaxial cables). Information collected by the VNA while measuring the reference standards is then used to develop a mathematical model of the error associated with the testing apparatus. This model is then used to correct for any systematic errors associated with the test system, a process called de-embedding. *Systematic* errors are repeatably and predictably measurable as distinguished from the irregular nature of *random* errors [56].

Many calibration techniques have been developed with different reference standards, measurement parameters, and algorithms for de-embedding test system errors, each with its own limitations and advantages. The method employed for the measurements reported in this research was a 2-port SOLT (short, open, load, thru) technique. The SOLT method uses well-characterized precision coaxial reference standards.

The ideal short completely reflects an incident signal in a manner that results in a reflection coefficient of $\Gamma = 1$ with 180° of phase shift. The ideal open also totally reflects an incident signal with $\Gamma = 1$ but with 0° of phase shift. The load is a terminal resistor connecting signal and ground, and is usually designed to match the test system's characteristic impedance (typically $Z_0 = 50 \Omega$). A thru line directly connects two ports of the VNA via a 50Ω line. Ideally, the thru line has no length at all. In actuality, none of these *ideal* reference standards can be manufactured. However, by characterizing the high frequency electrical characteristics of a reference standard, such as the parasitic inductance and capacitance associated with a short or an open, the non-idealities of the reference standards can be accounted for and included in the mathematical model of the test system error.

In the 2-port SOLT calibration, the VNA collects scattering parameter data while each reference standard (i.e., the short, open, and load) is individually connected to either port 1 or port 2. Only one standard is connected to the VNA during any one measurement (e.g., the short is connected to port 1 and nothing is connected to port 2). A thru line is also used to connect port 1 directly to port 2 while the VNA collects scattering parameter data. From

these seven measurements and the user-inputted data on the reference standard properties, a mathematical characterization of the test system's error is attained and used to correct for the system imperfections during measurements of the sensor or other devices of interest.

The calibration standards used for this research were included in the 8850Q model number calibration kit by Maury Microwave Corporation. This kit contains an open, a short, and load along with the electrical parameters of each component as characterized by the manufacturer.

APPENDIX B

PROPAGATION AND RESONANCE IN RECTANGULAR WAVEGUIDES AND
CAVITIES

A brief discussion of electromagnetic (EM) wave propagation in rectangular waveguides is given. Starting from Maxwell's equations, a derivation of the expressions for the instantaneous electric field $\mathbf{E}$ and the instantaneous magnetic field $\mathbf{H}$ are provided for the transverse electric (TE) and transverse magnetic (TM) modes that propagate in a rectangular waveguide. Other topics include cutoff frequencies and resonance in rectangular cavities.

B.1 TE and TM Modes in Lossless Rectangular Waveguides

In general, transmission lines and waveguides can be divided into two categories based on the characteristics of the EM waves they permit to propagate:

1. Those that can support transverse electromagnetic (TEM) fields and
2. those that cannot.¹

A TEM wave is characterized by having both electric field and magnetic field components that are entirely transverse (i.e., orthogonal or at 90°) to the direction of propagation. Conversely, TE and TM modes of wave propagation have one constituent of the EM wave (either the magnetic field or electric field) with a component along the direction of propagation (TE and TM modes cannot have both an electric *and* a magnetic field component in the direction of propagation; such a configuration is called a hybrid TM-TE mode). An EM wave for which $\mathbf{E}$ is transverse to the direction of propagation and $\mathbf{H}$ has a component along the propagation direction is called a TE mode; the analogous situation in which the roles of $\mathbf{E}$ and $\mathbf{H}$ are switched is referred to as a TM mode (see Figure B.1). Rectangular waveguides can support TE and TM modes, but not TEM waves [3].

¹There is another category for transmission lines that support so-called *quasi*-TEM fields. Microstrip lines support quasi-TEM fields (see the discussion on page 23).

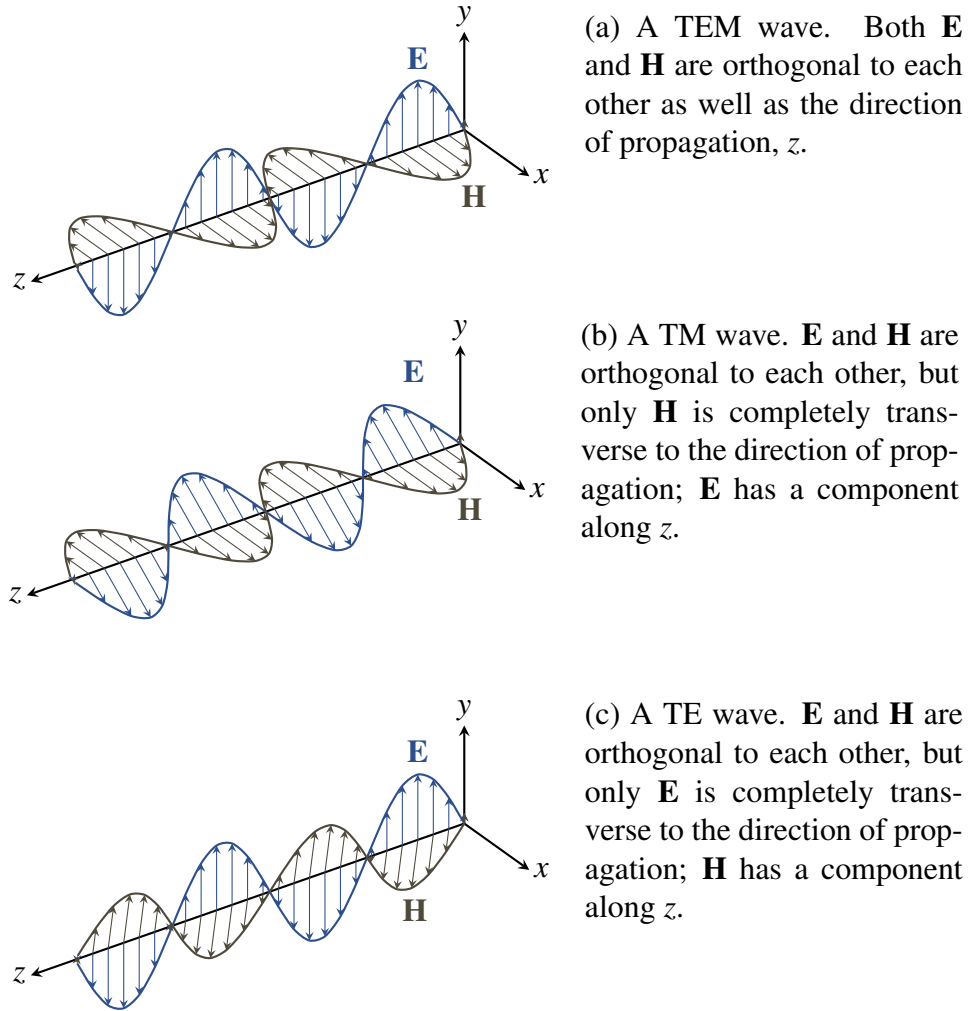


Figure B.1: EM wave classifications.

Time-varying electric and magnetic fields (as well as their sources, the charge density ρ , the current density $\mathbf{J}$, and the fictitious magnetic current density $\mathbf{M}$) are, in general, functions of the spatial variables x , y , and z as well as time t . It is useful to consider the situation in which the electric and magnetic fields and their sources vary sinusoidally with time because the sinusoidal case can be extended to other cases in which the fields vary by some arbitrary periodic function via the Fourier transform or to situations in which the fields vary in a nonperiodic fashion by way of Fourier integrals. If only sinusoidal steady-state solutions to $\mathbf{E}$ and $\mathbf{H}$ are of interest, then adopting phasor notation simplifies

the calculations; let²

$$\mathbf{E}(x, y, z, t) = \Re \{ \tilde{\mathbf{E}}(x, y, z) e^{j\omega t} \}, \quad (\text{B.1a})$$

$$\mathbf{H}(x, y, z, t) = \Re \{ \tilde{\mathbf{H}}(x, y, z) e^{j\omega t} \}. \quad (\text{B.1b})$$

The phasor vector $\tilde{\mathbf{E}}$ of equation B.1a (and similarly for $\tilde{\mathbf{H}}$ of equation B.1b) may be represented in expanded form as

$$\tilde{\mathbf{E}} = \hat{\mathbf{x}}\tilde{E}_x + \hat{\mathbf{y}}\tilde{E}_y + \hat{\mathbf{z}}\tilde{E}_z, \quad (\text{B.2})$$

where $\tilde{E}_x$, $\tilde{E}_y$, and $\tilde{E}_z$ are the components of $\tilde{\mathbf{E}}$ along the x -, y -, and z -directions respectively.

To obtain equations describing the propagation of EM waves in a rectangular waveguide, the direction of propagation will be assumed to be in the $+\hat{\mathbf{z}}$ direction. The geometry of a rectangular waveguide is shown in Figure B.2 along with the assumed propagation direction.

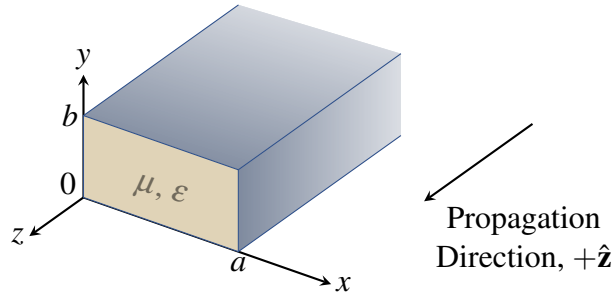


Figure B.2: Rectangular waveguide geometry and assumed propagation direction. The waveguide is filled with a material characterized by μ and ϵ ; the outer walls are metal. By convention, the waveguide is oriented to have its longest side along the x -axis so that $a > b$.

The material filling the inside of a rectangular waveguide is usually either air or some dielectric substance. In the derivation to follow, it will be *assumed* that the metal walls are perfect conductors and the dielectric material inside of the waveguide is

- linear: ϵ and μ do not depend on the magnitude of $\mathbf{E}$ or $\mathbf{H}$,

²Cosine-based phasors have been chosen as the standard for phasor notation here.

- isotropic: ϵ and μ are independent of the direction of EM wave propagation,
- homogenous: ϵ and μ take on a unique constant value throughout the material,
- nondispersive: ϵ and μ are not a function of the frequency of the propagating EM wave,
- devoid of any excess charge and lossless (ϵ and μ are purely real).³

Thus, the charge density is $\rho = 0$ since there is no unneutralized free charge, and likewise $\mathbf{J} = 0$ and $\mathbf{M} = 0$. Maxwell's equations describing the EM waves in such a medium free of source charges may be written in phasor form as

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}, \quad (\text{B.3a})$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}}, \quad (\text{B.3b})$$

$$\nabla \cdot \tilde{\mathbf{E}} = 0, \quad (\text{B.3c})$$

$$\nabla \cdot \tilde{\mathbf{H}} = 0. \quad (\text{B.3d})$$

Each component of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in equations B.3a–B.3d may in general be a function of the spatial variables x , y , and z . However, since only steady-state solutions (which must be sinusoidal) are being sought and the propagation direction is assumed to be in the $+\hat{\mathbf{z}}$ direction, the electric and magnetic fields should be described by a sinusoidal wave travelling in the $+\hat{\mathbf{z}}$ direction. Therefore, it shall be assumed that the component vectors of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ have a sinusoidal dependence on the variable z of the form $e^{-\gamma z}$ where γ is a yet unknown parameter called the propagation constant. In general, γ may be complex such that

$$\gamma = \alpha + j\beta, \quad (\text{B.4})$$

³In general, most materials violate one or all of these conditions to some degree, and thus the equations to follow are in fact approximate rather than exact. However, for many situations, only small deviations from these supposed constraints are observed and the results to follow provide negligible error for a broad range of situations.

where α is the attenuation constant, a parameter which describes the decay of the magnitude of a travelling wave due to loss mechanisms, and β is the phase constant or wavenumber (in the time domain $e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}$ becomes $e^{-\alpha z} \cos(\omega t - \beta z)$).

The functional form that describes the dependence on x and y for the vector components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ will be determined through application of equations B.3a–B.3d. The following summarizes the adopted notation that will be used and makes one final notational definition to make the spatial dependence of the vector components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ explicit; let

$$\mathbf{E}(x, y, z, t) = \Re \{ \tilde{\mathbf{E}}(x, y, z) e^{j\omega t} \}, \quad (\text{B.5a})$$

$$\mathbf{H}(x, y, z, t) = \Re \{ \tilde{\mathbf{H}}(x, y, z) e^{j\omega t} \}, \quad (\text{B.5b})$$

$$\tilde{\mathbf{E}}(x, y, z) = \hat{\mathbf{x}}\tilde{E}_x + \hat{\mathbf{y}}\tilde{E}_y + \hat{\mathbf{z}}\tilde{E}_z, \quad (\text{B.6a})$$

$$\tilde{\mathbf{H}}(x, y, z) = \hat{\mathbf{x}}\tilde{H}_x + \hat{\mathbf{y}}\tilde{H}_y + \hat{\mathbf{z}}\tilde{H}_z, \quad (\text{B.6b})$$

$$\tilde{E}_x(x, y, z) = \tilde{e}_x(x, y) e^{-\gamma z}, \quad (\text{B.7a})$$

$$\tilde{E}_y(x, y, z) = \tilde{e}_y(x, y) e^{-\gamma z}, \quad (\text{B.7b})$$

$$\tilde{E}_z(x, y, z) = \tilde{e}_z(x, y) e^{-\gamma z}, \quad (\text{B.7c})$$

$$\tilde{H}_x(x, y, z) = \tilde{h}_x(x, y) e^{-\gamma z}, \quad (\text{B.8a})$$

$$\tilde{H}_y(x, y, z) = \tilde{h}_y(x, y) e^{-\gamma z}, \quad (\text{B.8b})$$

$$\tilde{H}_z(x, y, z) = \tilde{h}_z(x, y) e^{-\gamma z}. \quad (\text{B.8c})$$

This notation makes it clear that $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ may depend on x , y , and z , but $\tilde{e}_x$, $\tilde{e}_y$, $\tilde{e}_z$, $\tilde{h}_x$, $\tilde{h}_y$, and $\tilde{h}_z$, may only depend on x and y (the z dependence of $\tilde{E}_x$, $\tilde{E}_y$, $\tilde{E}_z$, $\tilde{H}_x$, $\tilde{H}_y$, and $\tilde{H}_z$ is contained in the $e^{-\gamma z}$ expression). Combining equations B.7a–B.7c and equation B.6a (and similarly equations B.8a–B.8c and equation B.6b) yields

$$\tilde{\mathbf{E}} = (\hat{\mathbf{x}}\tilde{e}_x + \hat{\mathbf{y}}\tilde{e}_y + \hat{\mathbf{z}}\tilde{e}_z) e^{-\gamma z}, \quad (\text{B.9a})$$

$$\tilde{\mathbf{H}} = (\hat{\mathbf{x}}\tilde{h}_x + \hat{\mathbf{y}}\tilde{h}_y + \hat{\mathbf{z}}\tilde{h}_z) e^{-\gamma z}. \quad (\text{B.9b})$$

The curl of a general vector $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ can be shown to be

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right).$$

Also recall from elementary calculus the product rule:

$$\text{The derivative of } u(x)v(x) = \frac{d}{dx}(u(x)v(x)) = u(x)\frac{d\{v(x)\}}{dx} + v(x)\frac{d\{u(x)\}}{dx}.$$

Multiplying out equation B.9a, recalling that $\tilde{e}_x$, $\tilde{e}_y$, and $\tilde{e}_z$ only depend on variables x and y (thus $\frac{\partial \tilde{e}_x}{\partial z} = \frac{\partial \tilde{e}_y}{\partial z} = \frac{\partial \tilde{e}_z}{\partial z} = 0$), and taking the curl of $\tilde{\mathbf{E}}$ gives

$$\begin{aligned} \tilde{\mathbf{E}} &= \hat{\mathbf{x}}(\tilde{e}_x e^{-\gamma z}) + \hat{\mathbf{y}}(\tilde{e}_y e^{-\gamma z}) + \hat{\mathbf{z}}(\tilde{e}_z e^{-\gamma z}), \\ \nabla \times \tilde{\mathbf{E}} &= \hat{\mathbf{x}} \left(\frac{\partial \{\tilde{e}_z e^{-\gamma z}\}}{\partial y} - \frac{\partial \{\tilde{e}_y e^{-\gamma z}\}}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial \{\tilde{e}_x e^{-\gamma z}\}}{\partial z} - \frac{\partial \{\tilde{e}_z e^{-\gamma z}\}}{\partial x} \right) \\ &\quad + \hat{\mathbf{z}} \left(\frac{\partial \{\tilde{e}_y e^{-\gamma z}\}}{\partial x} - \frac{\partial \{\tilde{e}_x e^{-\gamma z}\}}{\partial y} \right), \\ \nabla \times \tilde{\mathbf{E}} &= \hat{\mathbf{x}} \left(\tilde{e}_z \frac{\partial \{e^{-\gamma z}\}}{\partial y} + e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial y} - \tilde{e}_y \frac{\partial \{e^{-\gamma z}\}}{\partial z} - e^{-\gamma z} \frac{\partial \tilde{e}_y}{\partial z} \right) \\ &\quad + \hat{\mathbf{y}} \left(\tilde{e}_x \frac{\partial \{e^{-\gamma z}\}}{\partial z} + e^{-\gamma z} \frac{\partial \tilde{e}_x}{\partial z} - \tilde{e}_z \frac{\partial \{e^{-\gamma z}\}}{\partial x} - e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial x} \right) \\ &\quad + \hat{\mathbf{z}} \left(\tilde{e}_y \frac{\partial \{e^{-\gamma z}\}}{\partial x} + e^{-\gamma z} \frac{\partial \tilde{e}_y}{\partial x} - \tilde{e}_x \frac{\partial \{e^{-\gamma z}\}}{\partial y} - e^{-\gamma z} \frac{\partial \tilde{e}_x}{\partial y} \right), \\ \nabla \times \tilde{\mathbf{E}} &= \hat{\mathbf{x}} \left(\tilde{e}_z \cdot 0 + e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial y} + \gamma \tilde{e}_y e^{-\gamma z} - e^{-\gamma z} \cdot 0 \right) \\ &\quad + \hat{\mathbf{y}} \left(-\gamma \tilde{e}_x e^{-\gamma z} + e^{-\gamma z} \cdot 0 - \tilde{e}_z \cdot 0 - e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial x} \right) \\ &\quad + \hat{\mathbf{z}} \left(\tilde{e}_y \cdot 0 + e^{-\gamma z} \frac{\partial \tilde{e}_y}{\partial x} - \tilde{e}_x \cdot 0 - e^{-\gamma z} \frac{\partial \tilde{e}_x}{\partial y} \right), \\ \nabla \times \tilde{\mathbf{E}} &= \hat{\mathbf{x}} \left(e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial y} + \gamma \tilde{e}_y e^{-\gamma z} \right) + \hat{\mathbf{y}} \left(-\gamma \tilde{e}_x e^{-\gamma z} - e^{-\gamma z} \frac{\partial \tilde{e}_z}{\partial x} \right) \\ &\quad + \hat{\mathbf{z}} \left(e^{-\gamma z} \frac{\partial \tilde{e}_y}{\partial x} - e^{-\gamma z} \frac{\partial \tilde{e}_x}{\partial y} \right), \end{aligned}$$

and finally

$$\nabla \times \tilde{\mathbf{E}} = \left\{ \hat{\mathbf{x}} \left(\frac{\partial \tilde{e}_z}{\partial y} + \gamma \tilde{e}_y \right) - \hat{\mathbf{y}} \left(\frac{\partial \tilde{e}_z}{\partial x} + \gamma \tilde{e}_x \right) + \hat{\mathbf{z}} \left(\frac{\partial \tilde{e}_y}{\partial x} - \frac{\partial \tilde{e}_x}{\partial y} \right) \right\} e^{-\gamma z}. \quad (\text{B.10})$$

From equation B.3a and equation B.9b

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}} = -j\omega\mu (\hat{\mathbf{x}}\tilde{h}_x + \hat{\mathbf{y}}\tilde{h}_y + \hat{\mathbf{z}}\tilde{h}_z) e^{-\gamma z}. \quad (\text{B.11})$$

Setting the right-hand side of equation B.10 equal to the right-hand side of equation B.11 gives

$$\begin{aligned} \left\{ \hat{\mathbf{x}} \left(\frac{\partial \tilde{e}_z}{\partial y} + \gamma \tilde{e}_y \right) - \hat{\mathbf{y}} \left(\frac{\partial \tilde{e}_z}{\partial x} + \gamma \tilde{e}_x \right) + \hat{\mathbf{z}} \left(\frac{\partial \tilde{e}_y}{\partial x} - \frac{\partial \tilde{e}_x}{\partial y} \right) \right\} e^{-\gamma z} \\ = -j\omega\mu (\hat{\mathbf{x}}\tilde{h}_x + \hat{\mathbf{y}}\tilde{h}_y + \hat{\mathbf{z}}\tilde{h}_z) e^{-\gamma z}. \end{aligned}$$

Notice that the $e^{-\gamma z}$ terms cancel. Then, equating the coefficients of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ yields the following relationships

$$\frac{\partial \tilde{e}_z}{\partial y} + \gamma \tilde{e}_y = -j\omega\mu\tilde{h}_x, \quad (\text{B.12a})$$

$$\frac{\partial \tilde{e}_z}{\partial x} + \gamma \tilde{e}_x = j\omega\mu\tilde{h}_y, \quad (\text{B.12b})$$

$$\frac{\partial \tilde{e}_y}{\partial x} - \frac{\partial \tilde{e}_x}{\partial y} = -j\omega\mu\tilde{h}_z. \quad (\text{B.12c})$$

Performing similar mathematical manipulations by substituting equation B.9a for $\tilde{\mathbf{E}}$ in equation B.3b and expanding $\nabla \times \tilde{\mathbf{H}}$ by use of equation B.9b, the following relationships are obtained

$$\frac{\partial \tilde{h}_z}{\partial y} + \gamma \tilde{h}_y = j\omega\epsilon\tilde{e}_x, \quad (\text{B.13a})$$

$$\frac{\partial \tilde{h}_z}{\partial x} + \gamma \tilde{h}_x = -j\omega\epsilon\tilde{e}_y, \quad (\text{B.13b})$$

$$\frac{\partial \tilde{h}_y}{\partial x} - \frac{\partial \tilde{h}_x}{\partial y} = j\omega\epsilon\tilde{e}_z. \quad (\text{B.13c})$$

Next, using equations B.12a–B.13c, the goal is to obtain expressions for $\tilde{e}_x$, $\tilde{e}_y$, $\tilde{h}_x$, and $\tilde{h}_y$ in terms of $\tilde{e}_z$ and $\tilde{h}_z$. This can be achieved by algebraically rearranging equations B.12a through B.13c. As an example, $\tilde{e}_x$ is found:

From equation B.12b:

$$\begin{aligned} \frac{\partial \tilde{e}_z}{\partial x} + \gamma \tilde{e}_x &= j\omega\mu\tilde{h}_y, \\ \tilde{e}_x &= \frac{j\omega\mu\tilde{h}_y - \frac{\partial \tilde{e}_z}{\partial x}}{\gamma}. \end{aligned} \quad (\text{B.14})$$

From equation B.13a:

$$\begin{aligned} \frac{\partial \tilde{h}_z}{\partial y} + \gamma \tilde{h}_y &= j\omega\epsilon\tilde{e}_x, \\ \tilde{h}_y &= \frac{j\omega\epsilon\tilde{e}_x - \frac{\partial \tilde{h}_z}{\partial y}}{\gamma}. \end{aligned} \quad (\text{B.15})$$

Substitute equation B.15 into equation B.14:

$$\begin{aligned} \tilde{e}_x &= \frac{j\omega\mu}{\gamma} \left(\frac{j\omega\epsilon\tilde{e}_x - \frac{\partial \tilde{h}_z}{\partial y}}{\gamma} - \frac{1}{\gamma} \frac{\partial \tilde{e}_z}{\partial x} \right), \\ \tilde{e}_x &= -\frac{\omega^2\mu\epsilon}{\gamma^2} \tilde{e}_x - j\frac{\omega\mu}{\gamma^2} \frac{\partial \tilde{h}_z}{\partial y} - \frac{1}{\gamma} \frac{\partial \tilde{e}_z}{\partial x}, \\ \tilde{e}_x \left(\frac{\omega^2\mu\epsilon}{\gamma^2} + 1 \right) &= -j\frac{\omega\mu}{\gamma^2} \frac{\partial \tilde{h}_z}{\partial y} - \frac{1}{\gamma} \frac{\partial \tilde{e}_z}{\partial x}, \\ \tilde{e}_x \left(\frac{\omega^2\mu\epsilon + \gamma^2}{\gamma^2} \right) &= -j\frac{\omega\mu}{\gamma^2} \frac{\partial \tilde{h}_z}{\partial y} - \frac{1}{\gamma} \frac{\partial \tilde{e}_z}{\partial x}, \\ \tilde{e}_x &= \frac{-1}{\omega^2\mu\epsilon + \gamma^2} \left(\gamma \frac{\partial \tilde{e}_z}{\partial x} + j\omega\mu \frac{\partial \tilde{h}_z}{\partial y} \right). \end{aligned}$$

Following similar procedures to find $\tilde{e}_y$, $\tilde{h}_x$, and $\tilde{h}_y$ yields

$$\tilde{e}_x = \frac{-1}{\omega^2\mu\epsilon + \gamma^2} \left(\gamma \frac{\partial \tilde{e}_z}{\partial x} + j\omega\mu \frac{\partial \tilde{h}_z}{\partial y} \right), \quad (\text{B.16a})$$

$$\tilde{e}_y = \frac{1}{\omega^2\mu\epsilon + \gamma^2} \left(-\gamma \frac{\partial \tilde{e}_z}{\partial y} + j\omega\mu \frac{\partial \tilde{h}_z}{\partial x} \right), \quad (\text{B.16b})$$

$$\tilde{h}_x = \frac{1}{\omega^2\mu\epsilon + \gamma^2} \left(j\omega\epsilon \frac{\partial \tilde{e}_z}{\partial y} - \gamma \frac{\partial \tilde{h}_z}{\partial x} \right), \quad (\text{B.16c})$$

$$\tilde{h}_y = \frac{-1}{\omega^2\mu\epsilon + \gamma^2} \left(j\omega\epsilon \frac{\partial \tilde{e}_z}{\partial x} + \gamma \frac{\partial \tilde{h}_z}{\partial y} \right). \quad (\text{B.16d})$$

By defining $\tilde{e}_x$, $\tilde{e}_y$, $\tilde{h}_x$, and $\tilde{h}_y$ in terms of $\tilde{e}_z$ and $\tilde{h}_z$, six unknowns have been reduced to two. To solve for $\tilde{e}_z$, take the curl of both sides of equation B.3a:

$$\nabla \times (\nabla \times \tilde{\mathbf{E}}) = -j\omega\mu (\nabla \times \tilde{\mathbf{H}}). \quad (\text{B.17})$$

Substitute equation B.3b for $\nabla \times \tilde{\mathbf{H}}$ in equation B.17:

$$\nabla \times (\nabla \times \tilde{\mathbf{E}}) = -j\omega\mu (j\omega\epsilon\tilde{\mathbf{E}}) = \omega^2\mu\epsilon\tilde{\mathbf{E}}. \quad (\text{B.18})$$

Employing the following identity to the left-hand side of equation B.18

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A},$$

where ∇^2 is the Laplacian:

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{A},$$

the following is obtained:

$$\nabla (\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}} = \omega^2\mu\epsilon\tilde{\mathbf{E}}. \quad (\text{B.19})$$

Substitute equation B.3c for $\nabla \cdot \tilde{\mathbf{E}}$ in equation B.19:

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2\mu\epsilon\tilde{\mathbf{E}} = 0. \quad (\text{B.20})$$

Following a similar procedure but starting by taking the curl of both sides of equation B.3b yields

$$\nabla^2 \tilde{\mathbf{H}} + \omega^2\mu\epsilon\tilde{\mathbf{H}} = 0. \quad (\text{B.21})$$

Equations B.20 and B.21 are known as wave equations, or the Helmholtz equations, for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$.

To satisfy equation B.20, each component of equation B.20 must be equal to zero. That is

$$\hat{\mathbf{x}} (\nabla^2 \tilde{E}_x + \omega^2\mu\epsilon\tilde{E}_x) + \hat{\mathbf{y}} (\nabla^2 \tilde{E}_y + \omega^2\mu\epsilon\tilde{E}_y) + \hat{\mathbf{z}} (\nabla^2 \tilde{E}_z + \omega^2\mu\epsilon\tilde{E}_z) = 0,$$

which is only fulfilled if

$$\hat{\mathbf{x}} (\nabla^2 \tilde{E}_x + \omega^2 \mu \varepsilon \tilde{E}_x) = 0,$$

$$\hat{\mathbf{y}} (\nabla^2 \tilde{E}_y + \omega^2 \mu \varepsilon \tilde{E}_y) = 0,$$

$$\hat{\mathbf{z}} (\nabla^2 \tilde{E}_z + \omega^2 \mu \varepsilon \tilde{E}_z) = 0.$$

Since the x and y components of $\tilde{\mathbf{E}}$ have been found in terms of the z components, the equation of interest now is

$$\begin{aligned} \nabla^2 \tilde{E}_z + \omega^2 \mu \varepsilon \tilde{E}_z &= 0, \\ \nabla^2 (\tilde{e}_z e^{-\gamma z}) + \omega^2 \mu \varepsilon \tilde{e}_z e^{-\gamma z} &= 0, \\ \frac{\partial^2 \{\tilde{e}_z e^{-\gamma z}\}}{\partial x^2} + \frac{\partial^2 \{\tilde{e}_z e^{-\gamma z}\}}{\partial y^2} + \frac{\partial^2 \{\tilde{e}_z e^{-\gamma z}\}}{\partial z^2} + \omega^2 \mu \varepsilon \tilde{e}_z e^{-\gamma z} &= 0, \\ \frac{\partial^2 \tilde{e}_z}{\partial x^2} e^{-\gamma z} + \frac{\partial^2 \tilde{e}_z}{\partial y^2} e^{-\gamma z} + \gamma^2 \tilde{e}_z e^{-\gamma z} + \omega^2 \mu \varepsilon \tilde{e}_z e^{-\gamma z} &= 0, \\ \frac{\partial^2 \tilde{e}_z}{\partial x^2} + \frac{\partial^2 \tilde{e}_z}{\partial y^2} + (\omega^2 \mu \varepsilon + \gamma^2) \tilde{e}_z &= 0. \end{aligned} \tag{B.22}$$

To solve equation B.22, use the trial solution

$$\tilde{e}_z = X(x)Y(y), \tag{B.23}$$

in which $X(x)$ is only a function of variable x and $Y(y)$ is only a function of y . Substituting equation B.23 into equation B.22 gives

$$\begin{aligned} \frac{\partial^2 \{X(x)Y(y)\}}{\partial x^2} + \frac{\partial^2 \{X(x)Y(y)\}}{\partial y^2} + (\omega^2 \mu \varepsilon + \gamma^2) X(x)Y(y) &= 0, \\ Y(y) \frac{d^2 X(x)}{dx^2} + X(x) \frac{d^2 Y(y)}{dy^2} + (\omega^2 \mu \varepsilon + \gamma^2) X(x)Y(y) &= 0. \end{aligned} \tag{B.24}$$

It is appropriate to use the single variable differential operator instead of the partial differential operator in equation B.24 because $X(x)$ and $Y(y)$ are both functions of a single

variable. Divide both sides of equation B.24 by $X(x)Y(y)$ to obtain

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + (\omega^2 \mu \varepsilon + \gamma^2) = 0. \quad (\text{B.25})$$

The last term in equation B.25 ($\omega^2 \mu \varepsilon + \gamma^2$) is a constant; none of these parameters change (recall the assumptions made about ε and μ). As such, let the constant k_c be defined as

$$k_c^2 = \omega^2 \mu \varepsilon + \gamma^2. \quad (\text{B.26})$$

Substituting equation B.26 into equation B.25 yields

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_c^2 = \text{a constant}. \quad (\text{B.27})$$

The significance of equation B.27 is that

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \text{a constant, and} \quad (\text{B.28a})$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = \text{a constant}. \quad (\text{B.28b})$$

To check the validity of equations B.28a and B.28b, consider equation B.27; at some value of x , $\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}$ will be some number, say g . For equation B.27 to be true, $\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}$ at this value of x must be a number equal to $-k_c^2 - g$ (a constant). If x is held constant while y varies, then for equation B.27 to remain true, $\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}$ must always equal $-k_c^2 - g$ for all values of y , and thus, $\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}$ must be a constant. A similar argument can be made for $\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}$ to see that this term must also be a constant.

Now define the constants k_x and k_y as

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2 \quad \longrightarrow \quad \frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0, \quad (\text{B.29a})$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2 \quad \longrightarrow \quad \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0. \quad (\text{B.29b})$$

Substituting equations B.29a and B.29b into equation B.27 gives

$$k_x^2 + k_y^2 = k_c^2. \quad (\text{B.30})$$

Equations B.29a and B.29b are ordinary second order differential equations. It can be shown that the most general solution to equations of the form

$$r \frac{d^2 y(t)}{dt^2} + s \frac{dy(t)}{dt} + v y(t) = 0, \quad (\text{B.31})$$

(where y is a function of t , and r , s , and v are arbitrary constants) is

$$y(t) = c_1 e^{\chi t} \cos(\psi t) + c_2 e^{\chi t} \sin(\psi t), \quad (\text{B.32})$$

where

$$\chi \pm j\psi = \frac{-s \pm \sqrt{s^2 - 4rv}}{2r},$$

and c_1 and c_2 are constants which can be determined from initial or boundary conditions [57]. Comparing equations B.29a and B.29b to equations B.31 and B.32 gives the solutions to $X(x)$, $Y(y)$, and $\tilde{e}_z$ as

$$X(x) = A \cos(k_x x) + B \sin(k_x x), \quad (\text{B.33a})$$

$$Y(y) = C \cos(k_y y) + D \sin(k_y y), \quad (\text{B.33b})$$

$$\tilde{e}_z = X(x)Y(y) = (A \cos(k_x x) + B \sin(k_x x)) (C \cos(k_y y) + D \sin(k_y y)). \quad (\text{B.33c})$$

To solve for the unknown coefficients A , B , C , and D of equation B.33c, boundary conditions from the geometry of the rectangular waveguide are needed (see Figure B.2). The electric field inside of a perfect conductor is zero (if it were nonzero, an electric force would exist inside the conductor which would push all free charges as far away as possible from each other, which results in all of the charge density being located on the conductor surface and zero electric field inside the conductor). Thus (assuming the waveguide walls are perfect electrical conductors), $\mathbf{E} = 0$ at $x = 0$, $x = a$, $y = 0$, and $y = b$. That is

$$\tilde{e}_z(x = 0, y) = \tilde{e}_z(x = a, y) = \tilde{e}_z(x, y = 0) = \tilde{e}_z(x, y = b) = 0. \quad (\text{B.34})$$

For $\tilde{e}_z(x=0, y) = 0$,

$$\tilde{e}_z(x=0, y) = (A \cos(k_x \cdot 0) + B \sin(k_x \cdot 0)) (C \cos(k_y y) + D \sin(k_y y)) = 0,$$

$$= (A \cos(0) + B \sin(0)) (C \cos(k_y y) + D \sin(k_y y)) = 0,$$

$$= A (C \cos(k_y y) + D \sin(k_y y)) = 0,$$

$$A = 0.$$

For $\tilde{e}_z(x, y=0) = 0$,

$$\tilde{e}_z(x, y=0) = (B \sin(k_x x)) (C \cos(k_y \cdot 0) + D \sin(k_y \cdot 0)) = 0,$$

$$= (B \sin(k_x x)) (C \cos(0) + D \sin(0)) = 0,$$

$$= (B \sin(k_x x)) C = 0,$$

$$C = 0.$$

For $\tilde{e}_z(x=a, y) = 0$,

$$\tilde{e}_z(x=a, y) = B \sin(k_x a) D \sin(k_y y) = 0,$$

$$\sin(k_x a) = 0,$$

$$k_x a = m\pi, \quad \text{for } m = \text{an integer},^*$$

$$k_x = \frac{m\pi}{a}. \tag{B.35}$$

* A discussion of the possible values for m is deferred for now but will be addressed momentarily. Presently, it seems that $m = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$ should be valid; a closer inspection will find far-reaching consequences.

For $\tilde{e}_z(x, y = b) = 0$,

$$\tilde{e}_z(x, y = b) = B \sin\left(\frac{m\pi}{a}x\right) D \sin(k_y b) = 0,$$

$$\sin(k_y b) = 0,$$

$$k_y b = n\pi, \quad \text{for } n = \text{an integer},^\dagger$$

$$k_y = \frac{n\pi}{b}. \quad (\text{B.36})$$

[†]As with the condition following from $\tilde{e}_z(x = a, y) = 0$, a discussion of the acceptable values for n will be taken up shortly.

Thus,

$$\begin{aligned} \tilde{e}_z(x, y) &= B \sin\left(\frac{m\pi}{a}x\right) D \sin\left(\frac{n\pi}{b}y\right), \\ &= BD \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right). \end{aligned}$$

Letting $BD = E_0$,

$$\tilde{e}_z(x, y) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right). \quad (\text{B.37})$$

where E_0 is the coefficient that describes the magnitude of the electric field intensity in the waveguide.

Instead of starting from B.20 to solve for $\tilde{e}_z$, applying a similar procedure starting with equation B.21 to solve for $\tilde{h}_z$ yields

$$\tilde{h}_z(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), \quad (\text{B.38})$$

where H_0 is an arbitrary constant associated with the amplitude of the intensity of the magnetic field in the waveguide.

With $\tilde{e}_z(x, y)$ and $\tilde{h}_z(x, y)$ now known, all components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ can be found using equations B.16a–B.16d since these equations only depend on $\tilde{e}_z$ and $\tilde{h}_z$. Substituting equations B.37 and B.38 into equations B.16a–B.16d yields

$$\tilde{e}_x(x, y) = \frac{-1}{k_c^2} \left(\gamma E_0 \frac{m\pi}{a} - j\omega\mu H_0 \frac{n\pi}{b} \right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), \quad (\text{B.39a})$$

$$\tilde{e}_y(x, y) = \frac{-1}{k_c^2} \left(\gamma E_0 \frac{n\pi}{b} + j\omega\mu H_0 \frac{m\pi}{a} \right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), \quad (\text{B.39b})$$

$$\tilde{e}_z(x, y) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), \quad (\text{B.39c})$$

$$\tilde{h}_x(x, y) = \frac{1}{k_c^2} \left(j\omega\varepsilon E_0 \frac{n\pi}{b} + \gamma H_0 \frac{m\pi}{a} \right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), \quad (\text{B.39d})$$

$$\tilde{h}_y(x, y) = \frac{-1}{k_c^2} \left(j\omega\varepsilon E_0 \frac{m\pi}{a} - \gamma H_0 \frac{n\pi}{b} \right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), \quad (\text{B.39e})$$

$$\tilde{h}_z(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), \quad (\text{B.39f})$$

where equation B.26 has been substituted for the $\omega^2\mu\varepsilon + \gamma^2$ terms in equations B.16a–B.16d.

Substituting equations B.39a–B.39f into equations B.9a and B.9b gives

$$\tilde{E}_x(x, y, z) = \frac{-1}{k_c^2} \left(\gamma E_0 \frac{m\pi}{a} - j\omega\mu H_0 \frac{n\pi}{b} \right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-\gamma z}, \quad (\text{B.40a})$$

$$\tilde{E}_y(x, y, z) = \frac{-1}{k_c^2} \left(\gamma E_0 \frac{n\pi}{b} + j\omega\mu H_0 \frac{m\pi}{a} \right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-\gamma z}, \quad (\text{B.40b})$$

$$\tilde{E}_z(x, y, z) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-\gamma z}, \quad (\text{B.40c})$$

$$\tilde{H}_x(x, y, z) = \frac{1}{k_c^2} \left(j\omega\varepsilon E_0 \frac{n\pi}{b} + \gamma H_0 \frac{m\pi}{a} \right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-\gamma z}, \quad (\text{B.40d})$$

$$\tilde{H}_y(x, y, z) = \frac{-1}{k_c^2} \left(j\omega\varepsilon E_0 \frac{m\pi}{a} - \gamma H_0 \frac{n\pi}{b} \right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-\gamma z}, \quad (\text{B.40e})$$

$$\tilde{H}_z(x, y, z) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-\gamma z}. \quad (\text{B.40f})$$

Recall that the values on the left-hand side of equations B.40a–B.40f represent the magnitude of the components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in the x -, y -, and z -directions. From the discussion of the classification of electromagnetic waves, there are three categories: TEM, TE, and TM waves. For the TEM case, the electric and magnetic field components transverse to the direction of propagation ($\hat{\mathbf{z}}$ in the present case) are zero, meaning both $\tilde{E}_z = 0$ and $\tilde{H}_z = 0$. From equations B.40c and B.40f it is seen that the only way to satisfy $\tilde{E}_z = \tilde{H}_z = 0$ is if $E_0 = 0$ and $H_0 = 0$. Substituting $E_0 = 0$ and $H_0 = 0$ into equations B.40a–B.40f results in the situation where $\tilde{E}_x = \tilde{E}_y = \tilde{E}_z = \tilde{H}_x = \tilde{H}_y = \tilde{H}_z = 0$, and thus *no EM wave exists at all*. This shows that rectangular waveguides cannot support TEM waves.

Now consider the TE case in which $\tilde{E}_z = 0$ and $\tilde{H}_z \neq 0$. For $\tilde{E}_z = 0$, equation B.40c indicates that E_0 must be 0. Substituting $E_0 = 0$ into equations B.40a–B.40f results in

$$\tilde{E}_x(x, y, z) = \frac{j\omega\mu}{k_c^2} \left(\frac{n\pi}{b} \right) H_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.41a})$$

$$\tilde{E}_y(x, y, z) = -\frac{j\omega\mu}{k_c^2} \left(\frac{m\pi}{a} \right) H_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.41b})$$

$$\tilde{E}_z(x, y, z) = 0, \quad (\text{B.41c})$$

$$\tilde{H}_x(x, y, z) = \frac{\gamma}{k_c^2} \left(\frac{m\pi}{a} \right) H_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.41d})$$

$$\tilde{H}_y(x, y, z) = \frac{\gamma}{k_c^2} \left(\frac{n\pi}{b} \right) H_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.41e})$$

$$\tilde{H}_z(x, y, z) = H_0 \cos \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right) e^{-\gamma z}. \quad (\text{B.41f})$$

Similarly, for the TM case in which $\tilde{H}_z = 0$ (and thus $H_0 = 0$) and $\tilde{E}_z \neq 0$, substituting $H_0 = 0$ into equations B.40a–B.40f results in

$$\tilde{E}_x(x, y, z) = -\frac{\gamma}{k_c^2} \left(\frac{m\pi}{a} \right) E_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.42a})$$

$$\tilde{E}_y(x, y, z) = -\frac{\gamma}{k_c^2} \left(\frac{n\pi}{b} \right) E_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.42b})$$

$$\tilde{E}_z(x, y, z) = E_0 \sin \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.42c})$$

$$\tilde{H}_x(x, y, z) = \frac{j\omega\epsilon}{k_c^2} \left(\frac{n\pi}{b} \right) E_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.42d})$$

$$\tilde{H}_y(x, y, z) = -\frac{j\omega\epsilon}{k_c^2} \left(\frac{m\pi}{a} \right) E_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right) e^{-\gamma z}, \quad (\text{B.42e})$$

$$\tilde{H}_z(x, y, z) = 0. \quad (\text{B.42f})$$

Thus, a TEM wave cannot exist in a rectangular waveguide, but TE and TM waves can as described by equations B.41a–B.41f and B.42a–B.42f respectively.

The only unknowns now are the integers m and n and the propagation constant γ . First consider m and n . Recall that m and n appeared because of the boundary conditions placed on $\tilde{E}_z$ and $\tilde{H}_z$ due to the finite geometry of the rectangular waveguide. For both parameters the constraints are $\sin \left(\frac{m\pi}{a} \right) = 0$ and $\sin \left(\frac{n\pi}{b} \right) = 0$, which are satisfied as long as m and n are integers. Physically, it is of no consequence whether $m = -1$ or $m = +1$ as both values of m result in $\sin \left(\frac{m\pi}{a} \right) = 0$ (and similarly for n). To simplify matters, the convention is to restrict m and n to positive values such that $m = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$. However, there is one more caveat to address. Consider the field expressions for the TE case in equations B.41a–B.41f. If $m = n = 0$ then each component of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are zero and no electromagnetic wave exists. All other combinations of m and n give nonzero values for at least some of the fields and thus produce physical TE waves. Each distinct combination of m and n describe what

are referred to as modes. The notation TE_{mn} is used to refer to each mode, for instance a TE_{10} mode or a TE_{32} mode (it was just shown that the TE_{00} mode does not exist).

Taking similar considerations for the field expressions for the TM case in equations B.42a–B.42f, if $m = n = 0$, $m = 1$ and $n = 0$, or $m = 0$ and $n = 1$, then all of the fields vanish. All other combinations for m and n result in valid nonzero fields leading to TM modes.

The final parameter to discuss is the propagation constant γ which describes the wave $e^{-\gamma z}$. From equations B.4, B.26, B.30, B.35, and B.36

$$\begin{aligned}\gamma &= \sqrt{k_c^2 - \omega^2 \mu \epsilon}, \\ \gamma &= \sqrt{k_x^2 + k_y^2 - \omega^2 \mu \epsilon}, \\ \alpha + j\beta &= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \epsilon}.\end{aligned}\tag{B.43}$$

Observe that each term under the square root in equation B.43 is a positive real number (the lossless case is being considered: ϵ and μ are real). Consider the two possibilities:

$$\begin{aligned}\text{if } \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 &> \omega^2 \mu \epsilon, \text{ then } \gamma = \alpha \text{ (some real number) and } j\beta = 0, \\ \text{if } \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 &< \omega^2 \mu \epsilon, \text{ then } \gamma = j\beta \text{ (some imaginary number) and } \alpha = 0.\end{aligned}$$

Therefore, γ is either purely real or purely imaginary. If γ is real, then $e^{-\gamma z}$ becomes $e^{-\alpha z}$ which describes an evanescent wave; that is, a wave with an amplitude that rapidly decays with z . If γ is imaginary, then $e^{-\gamma z} = e^{-j\beta z}$ which describes a wave travelling in the $+z$ -direction. This means that the only values for m and n which produce values for γ that correspond to propagating electromagnetic waves are those that adhere to the following constraint (evanescent waves do not propagate, they die out quickly)

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 < \omega^2 \mu \epsilon.\tag{B.44}$$

Equation B.44 indicates that there is a minimum frequency, corresponding to each mode, below which waves will not propagate. Rearranging equation B.44 and solving for the minimum ω gives

$$\omega = \frac{\pi}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \quad (\text{B.45})$$

This frequency is known as the cutoff frequency for a rectangular waveguide and is usually expressed in Hz:

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \quad (\text{B.46})$$

As can be seen from equation B.46, as the values from m and n increase, so does the value for the cutoff frequency. The mode with the lowest propagation frequency is called the dominant mode. The dominant mode for a TE wave is the TE_{10} mode⁴; for a TM wave, the dominant mode is the TM_{11} mode.

Table B.1 summarizes the results for sinusoidal steady-state wave propagation in a rectangular waveguide. The direction of wave propagation is assumed to be in the $+z$ -direction as shown, along with waveguide dimensions, in Figure B.2. The instantaneous electric and magnetic fields are defined as $\mathbf{E}(x, y, z, t) = \Re\{\tilde{\mathbf{E}}(x, y, z)e^{j\omega t}\}$ and $\mathbf{H}(x, y, z, t) = \Re\{\tilde{\mathbf{H}}(x, y, z)e^{j\omega t}\}$. The electric and magnetic field phasors are defined as $\tilde{\mathbf{E}}(x, y, z) = \hat{\mathbf{x}}\tilde{E}_x(x, y, z) + \hat{\mathbf{y}}\tilde{E}_y(x, y, z) + \hat{\mathbf{z}}\tilde{E}_z(x, y, z)$ and $\tilde{\mathbf{H}}(x, y, z) = \hat{\mathbf{x}}\tilde{H}_x(x, y, z) + \hat{\mathbf{y}}\tilde{H}_y(x, y, z) + \hat{\mathbf{z}}\tilde{H}_z(x, y, z)$.

⁴By convention, rectangular waveguide dimensions are normally defined such that $a > b$, and thus, from equation B.46, the TE_{10} mode will always have a lower cutoff frequency than the TE_{01} mode if $a > b$ (see Figure B.2).

Table B.1: Summary of EM Wave Propagation in Lossless Rectangular Waveguides

Quantity	TE _{mn} Mode	TM _{mn} Mode
$\tilde{E}_x(x, y, z)$	$\frac{j\omega\mu}{k_c^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$	$-\frac{j\beta_{mn}}{k_c^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$
$\tilde{E}_y(x, y, z)$	$-\frac{j\omega\mu}{k_c^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$	$-\frac{j\beta_{mn}}{k_c^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$
$\tilde{E}_z(x, y, z)$	0	$E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$
$\tilde{H}_x(x, y, z)$	$\frac{j\beta_{mn}}{k_c^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$	$\frac{j\omega\varepsilon}{k_c^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$
$\tilde{H}_y(x, y, z)$	$\frac{j\beta_{mn}}{k_c^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$	$-\frac{j\omega\varepsilon}{k_c^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$
$\tilde{H}_z(x, y, z)$	$H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{mn}z}$	0
m	0, 1, 2, 3, ...	1, 2, 3, ...
n	0, 1, 2, 3, ...	1, 2, 3, ...
	$m = n \neq 0$	
Dominant mode	TE ₁₀	TM ₁₁
Cutoff Frequency	$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$	
Propagation Constant	$\beta_{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$	
k_c^2	$k_c^2 = \omega^2\mu\varepsilon - \beta^2$	

B.2 Lossless Rectangular Waveguide Cavity Resonators

A rectangular waveguide cavity can be constructed by adding two metal walls to the rectangular waveguide shown in Figure B.2 to fully enclose the waveguide. Adding apertures to accomodate input and output signal probes that introduce and measure electromagnetic waves inside the cavity allows for the coupling of the cavity to other circuit elements. Such an enclosed cavity behaves much like an *RLC* resonant circuit which can be used as a filter or oscillator in a microwave device.

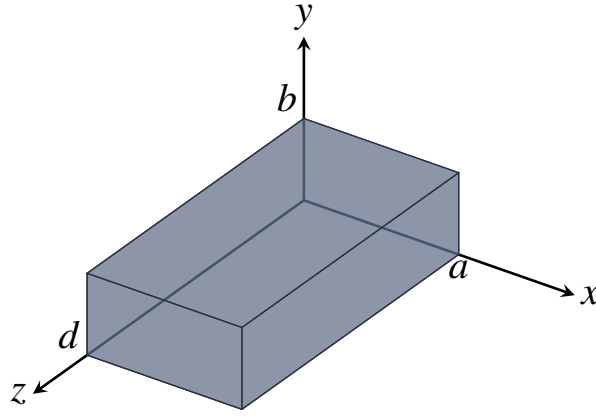


Figure B.3: Enclosed rectangular cavity resonator geometry.

In section B.1, the Helmholtz equation for the electric field (equation B.20) was solved for the $\hat{z}$ component of $\tilde{\mathbf{E}}$ by the method of separation of variables and the subsequent application of boundary conditions to find $\tilde{E}_z = E_0 \sin\left(\frac{m\pi}{a}\right) \sin\left(\frac{n\pi}{b}\right) e^{-j\beta_{mn}z}$. There were no boundary conditions for the z -dimension for the open-ended rectangular waveguide. Also, because the open-ended waveguide lacked any bounding structure to cause reflections in the z -direction, there was only one travelling wave propagating in the $+z$ -direction. In the present situation, the two extra conductive walls at $z = 0$ and $z = d$ present two more boundary conditions; namely, the transverse components of $\mathbf{E}$ ($\tilde{E}_x$ and $\tilde{E}_y$) at $z = 0$ and $z = d$ must be zero. Furthermore, the metal walls create the possibility for travelling waves

to reflect off of them creating waves simultaneously travelling in the $+z$ - and $-z$ -directions. As with any situation in which two travelling waves oscillating at the same frequency exist on an arbitrary transmission line, the result of the constructive and destructive interference of the travelling waves is a standing wave. Observe that this is also an adequate description of what occurs in the x - and y -directions for both the electric and magnetic fields in the open-ended rectangular waveguide as considered in section B.1: in the x - and y -directions, the electric and magnetic fields are described by standing waves because travelling waves reflect off of the walls of the waveguide, whereas in the z -direction the fields are characterized by a travelling wave moving in one direction. When the ends of a rectangular waveguide get shorted to the other four walls with some conductive material, the travelling waves propagating along $\hat{z}$ reflect off the additional walls and create a standing wave.

To characterize the travelling waves moving in the z -direction, the equations obtained in section B.1 for the $\hat{z}$ component of the electric and magnetic fields (equations B.41f and B.42c) require modification; the $e^{-j\beta_{mn}z}$ terms must be replaced by $A^+e^{-j\beta z} + A^-e^{j\beta z}$. For example

$$\tilde{E}_y = -\frac{j\beta}{k_c^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \left(A^+e^{-j\beta z} + A^-e^{j\beta z}\right) \quad (\text{for the TM mode}); \quad (\text{B.47})$$

$$\tilde{H}_z = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \left(B^+e^{-j\beta z} + B^-e^{j\beta z}\right) \quad (\text{for the TE mode}), \quad (\text{B.48})$$

where the $e^{-j\beta z}$ terms represent waves travelling in the $+z$ -direction and the $e^{j\beta z}$ terms represent waves travelling in the $-z$ -direction; the constants A^+ , A^- , B^+ , and B^- describe the amplitudes of the forward and backward travelling waves.

The additional conductive plates capping the ends of the rectangular waveguide introduce the boundary conditions $\tilde{E}_x(z=0) = \tilde{E}_x(z=d) = \tilde{E}_y(z=0) = \tilde{E}_y(z=d) = 0$ and $\tilde{H}_z(z=0) = \tilde{H}_z(z=d) = 0$. Applying boundary conditions $\tilde{E}_y(z=0) = \tilde{E}_y(z=d) = 0$ for

the TM case to equation B.47 yields

$$\tilde{E}_y(x, y, z = 0) = -\frac{j\beta}{k_c^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) (A^+ e^0 + A^- e^0) = 0,$$

$$A^+ + A^- = 0,$$

$$A^+ = -A^-, \quad (\text{B.49})$$

and

$$\tilde{E}_y(x, y, z = d) = -\frac{j\beta}{k_c^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) (A^+ e^{-j\beta d} - A^+ e^{j\beta d}) = 0,$$

$$A^+ (e^{-j\beta d} - e^{j\beta d}) = 0,$$

$$e^{-j\beta d} - e^{j\beta d} = 0. \quad (\text{B.50})$$

Notice that equation B.50 can be equated to a sinusoid as a consequence of Euler's formula:

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j},$$

$$2j \sin(x) = e^{jx} - e^{-jx},$$

$$-2j \sin(x) = e^{-jx} - e^{jx},$$

and thus,

$$e^{-j\beta d} - e^{j\beta d} = -2j \sin(\beta d), \quad (\text{B.51})$$

$$-2j \sin(\beta d) = 0,$$

$$\sin(\beta d) = 0,$$

$$\beta d = p\pi, \quad \text{for } p = 1, 2, 3, \dots, \frac{\ddagger}{2}$$

$$\beta = \frac{p\pi}{d}. \quad (\text{B.52})$$

[‡] As previously discussed for the variables m and n , negative values of p are not considered because such values would, in essence, duplicate the information contained in positive values for p of equal magnitude; the value $p = 0$ is excluded because this results in a trivial solution in which both $\tilde{E}_z = 0$ and $\tilde{H}_z = 0$ (i.e., a TEM mode), which for a rectangular has shown to result in the situation where all components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are zero.

Substituting the results from application of the boundary conditions (equations B.51 and B.52) into equation B.47 gives

$$\tilde{E}_y(x, y, z) = -2A^+ E_0 \left(\frac{\beta n \pi}{k_c^2 b} \right) \sin \left(\frac{m \pi}{a} x \right) \cos \left(\frac{n \pi}{b} y \right) \sin \left(\frac{p \pi}{d} z \right). \quad (\text{B.53})$$

Maxwell's equations still hold for the rectangular cavity, as does the Helmholtz equation for the electric field:

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu \epsilon \tilde{\mathbf{E}} = 0.$$

Recall equations B.35 and B.36, and let a similar definition exist for the term k_z :

$$\begin{aligned} k_x &= \frac{m \pi}{a}, \\ k_y &= \frac{n \pi}{b}, \\ k_z &= \frac{p \pi}{d}. \end{aligned} \quad (\text{B.54})$$

Substituting equations B.35, B.36, and B.54 into equation B.53 gives

$$\tilde{E}_y(x, y, z) = -2A^+ E_0 \left(\frac{\beta n \pi}{k_c^2 b} \right) \sin(k_x x) \cos(k_y y) \sin(k_z z). \quad (\text{B.55})$$

Now plug equation B.55 into equation B.20:

$$2A^+ E_0 \left(\frac{\beta n \pi}{k_c^2 b} \right) \sin(k_x x) \cos(k_y y) \sin(k_z z) \{k_x^2 + k_y^2 + k_z^2 - \omega^2 \mu \epsilon\} = 0.$$

The $2A^+E_0 \left(\frac{\beta n \pi}{k_c^2 b} \right) \sin(k_x x) \cos(k_y y) \sin(k_z z)$ term drops out to give

$$k_x^2 + k_y^2 + k_z^2 - \omega^2 \mu \epsilon = 0,$$

$$\omega^2 \mu \epsilon = k_x^2 + k_y^2 + k_z^2,$$

$$\omega = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{k_x^2 + k_y^2 + k_z^2}. \quad (\text{B.56})$$

Substituting the relationships for k_x^2 , k_y^2 , and k_z^2 from equations B.35, B.36, and B.54 into equation B.56 and using the relationship $\omega = 2\pi f$,

$$f_{0_{mnp}} = \frac{1}{2\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}. \quad (\text{B.57})$$

Equation B.57 is the equation for the resonant frequency of the rectangular cavity. Contrast the relationships shown in equation B.44 for a rectangular waveguide and equation B.56 for a rectangular cavity. Equation B.44 indicates that a rectangular waveguide will support EM waves of *any* frequency as long as the frequency is above a certain cutoff value. Thus, a rectangular waveguide behaves like a high pass filter. Conversely, equation B.56 for the rectangular cavity indicates that there is only *one* frequency corresponding to each mode for which an EM wave will exist inside a resonant cavity.

B.3 Loss Considerations in Rectangular Waveguides and Cavities

The relations derived in Sections B.1 and B.2 were arrived at assuming rectangular waveguides and cavities composed of *perfect* electrical conductors surrounding *lossless* dielectric materials. This theoretical situation suggests that a rectangular waveguide acts as a perfect “brick wall” high-pass filter. In reality, there is some loss associated with the dielectric material and the walls of a rectangular waveguide have finite conductivity. As discussed in Chapter 1, to account for finite conductivity and a lossy dielectric material,

the permittivity is represented as a complex number $\epsilon = \epsilon' - j\epsilon''$ and the loss tangent is defined as $\tan \delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'}$. If ϵ is no longer purely real, as was assumed in Section B.1, then the term under the square root in equation B.43 is not necessarily purely real or purely imaginary but can take on complex values. The significance of this result is that there is no longer a clearly defined cutoff frequency above which EM waves may propagate and below which they may not. Rather, the frequency response appears more like that of a typical lumped element *RC* or *RL* high-pass filter, exhibiting a gradual rolling off of the passband as opposed to a sharp corner (see Figure B.4). It is common practice to adopt the convention of defining the cutoff frequency f_c as the frequency at which the magnitude of transmission drops 3 dB from the maximum.

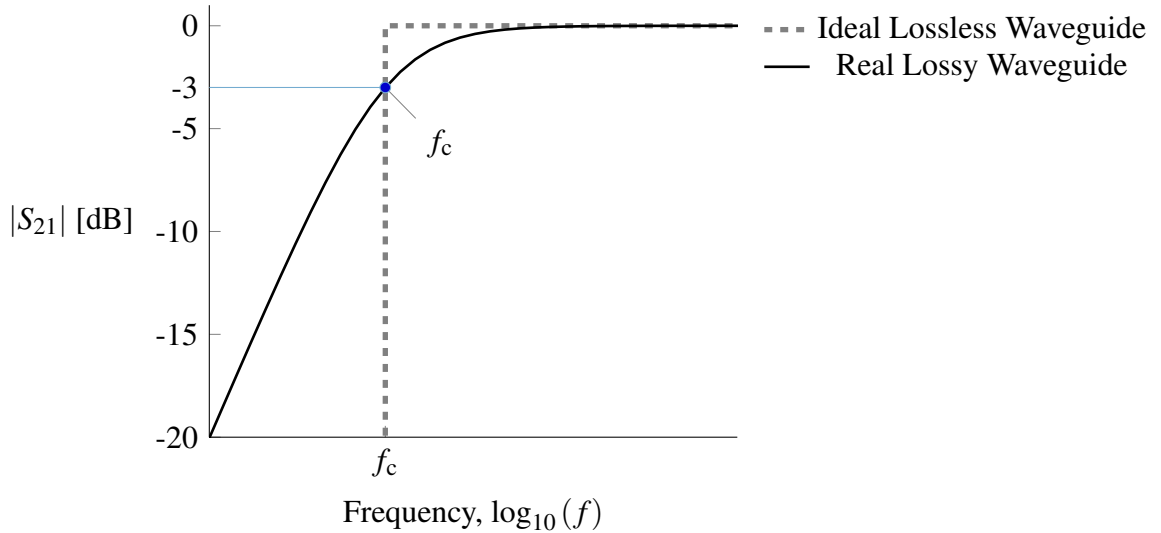


Figure B.4: The frequency response for a lossless and a lossy rectangular waveguide.

A similar situation applies to rectangular cavities. The derivation of equation B.57 implies that, for each mode, only one possible frequency of EM wave can be sustained. This suggests a frequency response with a single spike at each mode. In practice, the frequency response of a rectangular cavity is more like that of the lossy case shown in

Figure B.5. The bandwidth BW of such a response is defined as

$$BW = f_2 - f_1,$$

where f_1 is the frequency at which the amplitude is $1/\sqrt{2}$ of the amplitude at f_0 such that $f_1 < f_0$ and f_2 is the frequency at which the amplitude is $1/\sqrt{2}$ of the amplitude at f_0 such that $f_2 > f_0$ (f_0 is the resonant frequency of the rectangular cavity, given *approximately* by equation B.57).⁵ Converting $\frac{1}{\sqrt{2}}$ to decibels gives $-20\log\left(\frac{1}{\sqrt{2}}\right) \approx -3$ dB. The quality factor Q is defined as

$$Q = \frac{BW}{f_0}.$$

For the ideal lossless rectangular cavity $Q = \infty$ and the resonant frequency is $f_{0_{mp}}$ as given by equation B.57. For physical situations, Q is finite and is a measure of the loss of the resonator; high values of Q correspond to lower loss [3].

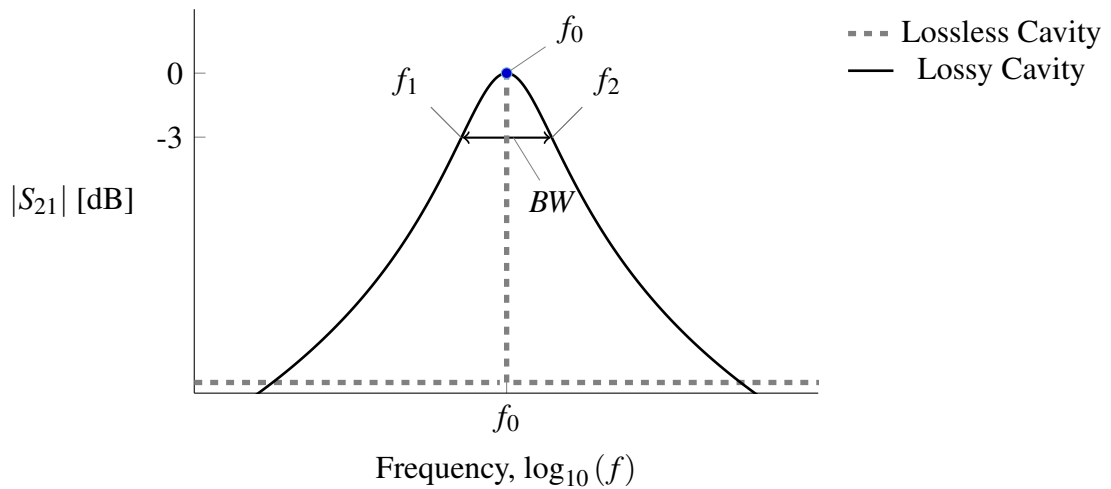


Figure B.5: The frequency response for a lossless and a lossy rectangular cavity.

⁵This definition of bandwidth is commonly referred to as the *half-power bandwidth* as it can be shown that, in a lumped-element circuit at DC, $\frac{1}{\sqrt{2}}$ of the maximum voltage occurs when the power has fallen to half of the maximum power $\left(P_{\max} = \frac{V_{\max}^2}{R}, P_{\frac{1}{2}} = \frac{\left(\frac{1}{\sqrt{2}}V_{\max}\right)^2}{R} = \frac{1}{2}P_{\max}\right)$.