

MODELING SALINE FLUID FLOW IN SUBGLACIAL ICE-WALLED CHANNELS

by

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DEDICATION

This thesis is dedicated to Brian Blitz, my former advisor and mathematics professor at the University of Alaska Southeast. Thank you for believing in me and for all of the hours of support over the years. This thesis would not have been written without you.

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ABSTRACT

Subglacial hydrological systems have impacts on ice dynamics as well as nutrient and sediment transport. There has been an extensive effort to understand the dynamics of subglacial drainage through numerical modeling, however these models have focused on freshwater, neglecting the consideration of brine. Saline fluid can exist in cold-based glacier systems where freshwater cannot. Therefore, there exist subglacial hydrological systems where the only fluid is brine. Understanding the routing of saline fluid is important for understanding geochemical and microbiological processes in these saline cryospheric habitats. In this thesis, I present a model of channelized drainage from a hypersaline subglacial lake and highlight the impact of saline fluid on melt rates in an ice-walled channel. The model results show that channel walls grow more quickly when fluid contains higher salt concentrations, which results in greater peak discharge and faster drainage for a fixed lake volume. This model provides a framework to assess the relative impact of brine on discharge and drainage duration.

INTRODUCTION

A subglacial hydrological system is the drainage and storage network of water at the bed of a glacier. As meltwater and runoff accumulate at the bed, channels, cavities, and reservoirs form to store and move water beneath the glacier. Subglacial hydrology has important implications for glacier dynamics as water at the bed influences ice flow and glacier sliding [7]. There has been extensive work in modeling subglacial hydrology to understand the physical aspects of freshwater drainage [5, 14, 20, 29, 32, 34, 38].

Rothlisberger was the first to explore the physical theory behind subglacial drainage [32]. He hypothesized that water flows through ice-walled channels, now referred to as an R-channels, at the bed of glaciers and that these conduits either grow due to melting of the walls or close due to ice over-burden pressure, also known as creep closure. Using this theory, [29] built a model to describe drainage from a subaerial lake, which has been used in many subsequent models describing subglacial hydrology [37]. [36] built a very detailed model with more relevant physics describing the same phenomenon as [29]. [5] and [14] both adapted the work of [29] to explore different characteristics of subglacial drainage. [20] built on the work [29] and [14] to consider periodic flooding through ice-walled channels.

In the novel work of [32], he assumes water flows through a circular conduit at the bed of a glacier. He describes the flow of water through this conduit using an ordinary differential equation (ODE). An important part of the derivation of this system is the energy transferred to the ice-walled channel by water through heat. The energy loss per unit time as water flows along the channel is described as a differential by

$$dE = Qdp, \tag{1.1}$$

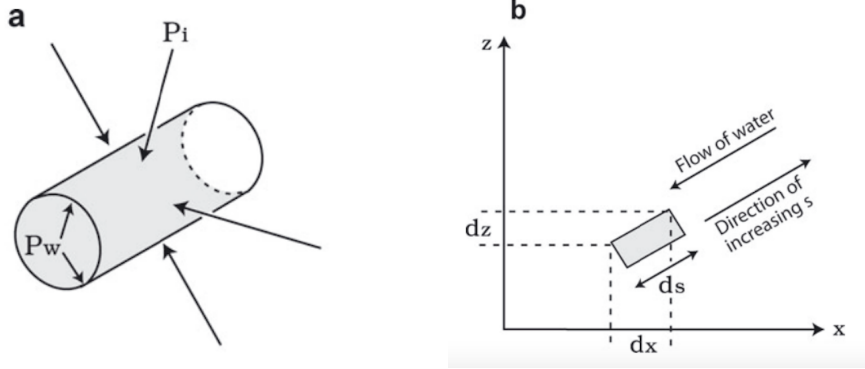


Figure 1.1: Schematic of a Rothlisberger channel. (a) The water pressure P_w acts to open the channel, while the ice overburden pressure P_i acts to close the channel. (b) The orientation of the channel and water flow is shown, where dz is the change in elevation, dx is the change in the horizontal distance, and ds is the distance along the channel length [37, Fig. 1].

where $E(x, t)$ is energy, $p(x, t)$ is water pressure, and Q is discharge. This equation relates the change in energy with respect to pressure to the discharge. The energy generated raises the water temperature to the pressure-dependent melting point and the rest of the energy is used to melt surrounding ice. This results in

$$dE_m = dE - dE_t, \quad (1.2)$$

where E_m and E_t are the energy lost to melting and temperature change respectively.

The change in the pressure melting point with pressure is $c_t = -7.5 \times 10^{-8} \text{ deg J}^{-1} \text{m}^3$ and the specific heat capacity of water is $c_w = -7.5 \times 10^{-8} \text{ J kg}^{-1} \text{ deg}^{-1}$. The density of water at the melting point is $\rho_w = 999.84 \text{ kg m}^{-3}$. This gives

$$dE_t = -c_t c_w \rho_w Q dp = 0.316 Q dp \quad (1.3)$$

and

$$dE_m = 0.684 Q dp = 0.684 dE. \quad (1.4)$$

This value for the change in energy E_m due to melting has been used in almost all subsequent models.

The closure rate of ice due to ice-overburden pressure, first discussed in [28], is given by

$$\frac{dr}{dt} = r \left(\frac{P - p}{nA} \right)^n, \quad (1.5)$$

where r is conduit radius, p is water pressure, P is the ice-overburden pressure, and n and A are ice flow parameters from [16]. Using this, [32] arrived at an ordinary differential equation (ODE) for the pressure gradient in a horizontal conduit,

$$\frac{dp}{dx} = B n'^{6/11} (nA)^{-8n/11} Q^{-2/11} (P - p)^{8n/11}, \quad (1.6)$$

where $B = 2^{12/11} \pi^{2/11} \left(\frac{c_m \rho_i}{0.684} \right) (\rho_w g)^{3/11}$ and g is acceleration due to gravity, c_m is the energy of fusion, Q is discharge, x is the horizontal distance along the channel, and n' is Manning's roughness coefficient related to channel roughness.

Nye built on the work of [32] by using R-channels to model the physical processes of drainage during outburst floods [29]. Outburst floods, also known as jökulhlaups, are floods that occur from subglacial or marginal glacial lakes when an ice dam fails and water rapidly flows from the lake subglacially to the glacier terminus. [29] coupled the equations for R-channels to equations describing the water pressure from the lake. He arrived at an equation for the total change of internal energy,

$$\delta E = S l \rho_w \sigma \frac{d\theta_w}{dt} \delta t + m l \delta t [\Delta u + \sigma(\theta_w - \theta_i)], \quad (1.7)$$

where σ is the specific heat capacity of water, Δu is the difference in the internal energy between water and ice at the melting point, S is the cross-sectional area of the channel, m is the melt rate, ρ_w is the density of water, θ_w is the temperature of water, θ_i is the melting

point of ice, and l is a short section of the conduit. Water is assumed to be incompressible.

Nye [29] arrives at the following five partial differential equations (PDEs) describing the geometry and flow of ice in a channel, continuity, flow of water, energy, and heat transfer.

Geometry and flow of ice:

$$\frac{\partial S}{\partial t} = \nu_i m - KS(p_i - p)^n \quad (1.8)$$

Continuity:

$$\frac{\partial S}{\partial t} = \nu_w m - \frac{\partial Q}{\partial s} \quad (1.9)$$

Flow of water:

$$\rho_w g_s - \frac{\partial p}{\partial s} = \frac{Q^2 \mathcal{N}}{S^{8/3}} \quad (1.10)$$

Energy:

$$Q \left(\rho_w g_s - \frac{\partial p}{\partial s} \right) = m\mathcal{L} + m\sigma(\theta_w - \theta_i) \quad (1.11)$$

Heat Transfer:

$$0.205 \left(\frac{2Q\rho_w}{\pi^{1/2} S^{1/2} \eta} \right)^{0.8} \kappa(\theta_w - \theta_i) = m\mathcal{L} + m\sigma(\theta_w - \theta_i), \quad (1.12)$$

where s is the distance from the lake along the channel, g_s is acceleration due to gravity in the s direction, ν_i and ν_w are the viscosities of ice and water respectively, p_i and p are the pressures at the seal due to ice overburden pressure and due to water depth respectively, and $\mathcal{N}$ is Manning's roughness coefficient which is a measure of channel roughness. K and n are constants from Glen's Flow Law [16], which describes the deformation rate of ice and $\mathcal{L}$ is the latent heat of fusion for water. The cross-sectional area of the conduit, $S(s, t)$, evolves by melting and creep closure, and consequently the discharge Q , effective pressure N (ice-overburden minus water pressure), and melt rate m vary temporally and spatially. [29]

assumes $d\theta_w/dt = 0$ and $\theta_w = \theta_i$ and thus Eq. (1.12), the heat transfer equation is not included in the model. This model has been built upon to understand subglacial drainage, as well as outburst floods.

Nye [29] assumes the temperature of water and ice are equal and at the pressure-dependent melting point and therefore eliminates the heat transfer equation from his model. The majority of subsequent work has followed in this assumption that heat transfer is negligible [14, 17, 26]. However, others have tried to address the heat transfer equation as it is important for modeling some subglacial systems [5, 6, 8, 11, 36].

In order to understand the impact of the temperature difference between the water and melting point of ice, [36] used a similar equation to [29] representing heat transfer from the water to the ice-walled channel. By neglecting the term $m\sigma$, they arrive at the heat transfer equation,

$$0.023\pi(Re)^{0.8}(Pr)^{0.4}\kappa(\theta_i - \theta_w) = m\mathcal{L}. \quad (1.13)$$

In this study, [36] test the effect of water temperature $\theta_w = 0, 1, 2, 4^\circ \text{ C}$ with constant ice temperature, $\theta_i = 0^\circ \text{ C}$ and find when the temperature of the water is warmer than the ice, peak discharge is higher Fig. 1.2. This is due to heat being transferred from the water to the ice and as a result, melting of the channel walls is favored over creep closure which acts to enlarge the channel more quickly. The study of [36] reveals the rate of change with respect to time of internal energy E is not negligible when modeling jökulhlaups.

Similar to [36], [5] also accounts for temperature difference, but only allows the water temperature to vary with time and not distance. He makes a number of simplifying assumptions to reduce the system of PDEs developed by [29] to a single equation which describes the rate of change over time of the conduit given by,

$$\frac{\partial S}{\partial t} = \frac{S^{4/3}(-\partial\phi/\partial s)^{3/2}}{\rho_i L' \mathcal{N}^{1/2}} + \frac{0.205 S^{2/3}}{\rho_i L'} \left(\frac{2\rho_w}{\eta} \right)^{4/5} \left[\frac{(-\partial\phi/\partial s)}{\pi \mathcal{N}} \right]^{2/5} k_w(\theta_0 - \theta_i) - KS(p_i - p)^n \quad (1.14)$$

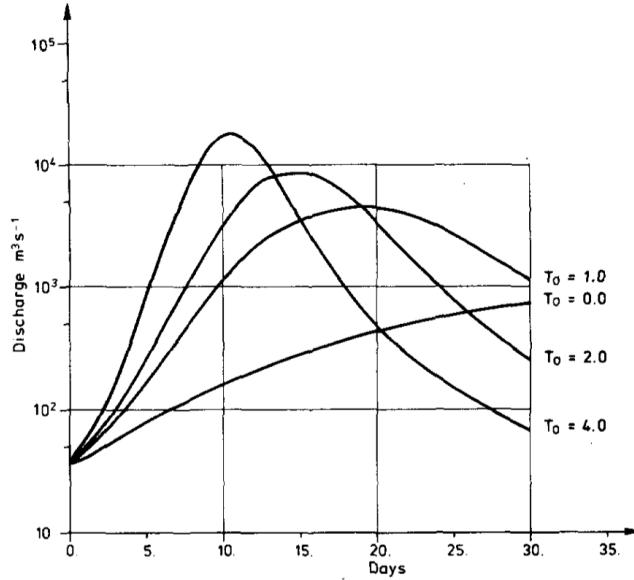


Figure 1.2: Modeled effects of temperature on discharge during jökulhlaups, where T_o is the initial water temperature at the lake [36, Fig. 14].

where ϕ is the fluid potential, L' is effective latent heat of fusion for water, θ_0 is the initial temperature of the water, and $\mathcal{N} = (4\pi)^{2/3}\rho_w g n'^2$. All other notation is the same as in [29]. [5] uses this simplified model to explore the influence of lake temperature and lake geometry on outburst flood magnitude. He expresses the difficulties in predicting outburst flood magnitude due to uncertainties such as flood initiation mechanisms.

Outburst floods often initiate before the lake has reached flotation depth, the depth of the lake such that the ice-overburden pressure due to the glacier is equal to the water pressure at the outlet (Fig. 1.3). To investigate this phenomenon, [14] built on the work of [29] and allowed the water to flow into the lake by letting the basic hydraulic gradient become negative at the lake. An ice divide is the location along the channel where water above this divide flows into the lake and where water below flows in the down-glacier direction. [14] hypothesizes that the ice divide migrates toward the lake as the lake fills and floods initiate when the ice divide reaches the lake.

Clarke [6] was also interested in outburst floods, specifically the hydraulics of floods.

He used outburst flood data from three classic outburst flood lakes and compared the data to a model which he modified from the Spring–Hutter equations [36]. By comparing the observational and theoretical data, he claimed the hydraulic roughness coefficient $\mathcal{N}$ used in both [29] and [36] is inaccurate. He claimed this discrepancy is due to an assumption about the heat transfer equations in [36]. [29] ignores the change in temperature along the channel and underestimates the influence of temperature on the system, while the model of [36] suggested the impact of the change in temperature along the channel was greater than suggested by the data. [6] arrived at modeled exit water temperatures higher than observed when using the model equations and coefficients of [36].

Similar to [5], [26] reduced a system of PDEs to a single equation to explore the relationship between the peak magnitude of an outburst flood and the volume of water drained from the lake. They used the simplified model to explore the physical theory behind the Clague–Mathews relationship, a power law relationship between peak discharge and volume drained [4].

[11] built on the works of [6, 14, 29, 36] by modifying the equations and adapting model parameters to describe a subglacial lake. [11] modeled two lakes with significantly different storage capacities and explored a number of parameters including temperature. They claim periodic flooding is the typical drainage mechanism for sub-ice sheet lakes.

Numerous models assume subglacial drainage through an ice-walled channel. These are referred to as a channelized or efficient drainage systems. In less efficient systems, it is assumed there is a system of linked cavities which consist of a network of connected cavities, referred to as a distributed system (Fig. 1.4). Both channelized and distributed systems can exist beneath the same glacier or transition from one system to another over time [9]. Models which assume a distributed system account for this by assuming some thickness of water with an ice-roof which is the bed of the glacier. Research has investigated how channels develop from a distributed system [17].

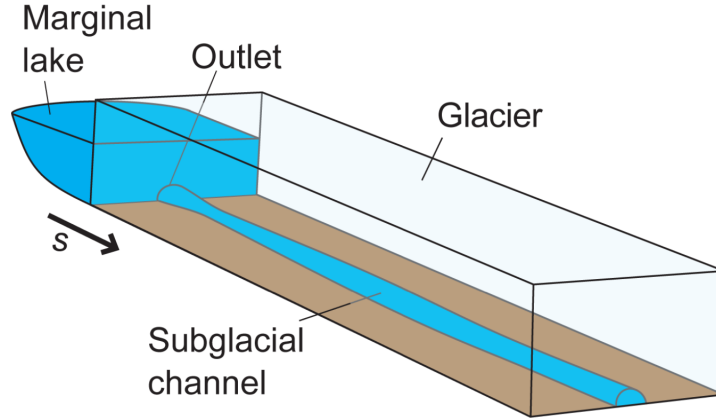


Figure 1.3: Schematic demonstrating glacier and basin geometry [19, Fig. 2.3]

Further research has been conducted to understand how cavities in a distributed system change over time. [8] uses a model of subglacial drainage to explore the impact of varying bed topography on the stability of cavity growth. They relate the melting rate of the ice roof to the temperature difference between the water and the melting point of the ice roof by

$$m\mathcal{L} = c(T - T_m) - q_0, \quad (1.15)$$

where T is the interior temperature of the water sheet, T_m is the melting temperature of ice, c is the heat transfer coefficient between ice and water, and q_0 is heat flux from geothermal heat or other heat sources/sinks. They find that protrusions at the bed reduce some of the ice-overburden pressure on the water. [8] investigate the impact of subglacial topography on drainage and include the heat transfer from the water to the ice.

Kingslake [20] builds on the work of [29] and [14] by assuming a semi-circular conduit at the bed which flows from a subaerial lake through the glacier to the glacier terminus. [20] assumes the ice divide can move along the channel length which allows water to flow into the lake from the channel. [20] assumes that the conduit at the terminus is open to the atmosphere and the lake is dammed by the margins, or sides, of a glacier (Fig. 1.4). With a

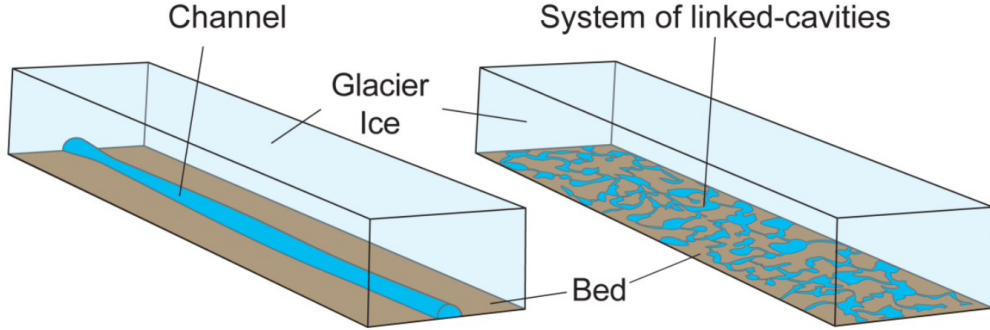


Figure 1.4: Schematic showing a channelized vs. distributed subglacial hydrologic system [19, Fig. 2.1].

number of simplifying assumptions, the model is reduced to a system of four equations that conserve mass, momentum, and energy as water flows through a conduit evolving with time. This subglacial hydrology model is used to describe outburst floods and periodicity of flood cycles.

The densities of ice and water are denoted as $\rho_i = 917 \text{ kg/m}^3$ and $\rho_w = 1000 \text{ kg/m}^3$, gravitational acceleration as g , and the latent heat of fusion for water as $\mathcal{L} = 3.34 \times 10^5 \text{ J/kg}$ [9]. The equation for the basic hydraulic gradient is

$$\psi(s, t) = \rho_w g \sin B - \frac{\partial P_i}{\partial s},$$

where B is the conduit slope assumed to be the same as the bed slope and constant along the channel, and s is the along-flow coordinate parallel to the bed (following [14]). The ice-overburden pressure P_i is given by $P_i = \rho_i g H$ where H is the glacier thickness.

The channel walls grow due to melt and close due to creep closure which governs the rate of change of conduit cross-sectional area,

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - K S N^n, \quad (1.16)$$

where $K = 2A/n^n$ [10]. Then $A = 2.4 \times 10^{-24} \text{Pa}^{-3} \text{s}^{-1}$ and $n = 3$ are the flow law parameter and exponent, respectively, in Glen's Flow Law [16]. Mass conservation relates the rate of change of conduit area to the spatial gradient in discharge, the production of water due to melt, and additional water added to the system along the conduit, such that

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{m}{\rho_w} + M, \quad (1.17)$$

where M is the additional water supplied to the conduit from other smaller channels with greater water pressure.

Manning's equation is used to describe conservation of momentum. This results in an equation relating the discharge and conduit cross-sectional area to the hydraulic gradient and effective pressure,

$$\psi + \frac{\partial N}{\partial s} = f \rho_w g \frac{Q|Q|}{S^{8/3}}, \quad (1.18)$$

where $f = (2(\pi + 2)^2/\pi)^{3/2} \mathcal{N}$ is a friction factor with $\mathcal{N}$, the hydraulic roughness. The conservation of energy gives,

$$m\mathcal{L} = Q \left(\psi + \frac{\partial N}{\partial s} \right). \quad (1.19)$$

[20] follows after [29] by neglecting the heat transfer equation which results in four fundamental equations. Two boundary conditions are imposed to solve this system. They set the effective pressure at the terminus to $N(L, t) = 0$ and at the conduit adjacent to the lake, the effective pressure is

$$N(0, t) = \rho_i g H - \rho_w g h(t), \quad (1.20)$$

where H is the ice thickness adjacent to the basin and h is water depth of the basin lake. Summarizing, the model equations are as follows:

Channel evolution:

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - KSN^n$$

Conservation of mass:

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{m}{\rho_w} + M$$

Conservation of momentum:

$$\psi + \frac{\partial N}{\partial s} = f\rho_w g \frac{Q|Q|}{S^{8/3}}$$

Conservation of energy:

$$m\mathcal{L} = Q \left(\psi + \frac{\partial N}{\partial s} \right)$$

The numerical methods to solve this system are described in full in [19].

Many of these models have been used to investigate periodic drainage of lakes through outburst floods. Although many subglacial lakes do not experience drainage via outburst floods, there have been observations of subglacial lakes draining with periodic behavior. For examples, the subglacial lakes located beneath Whillans and Mercer Ice Streams have been observed to drain periodically [9]. Modeling efforts also suggest the periodicity of subglacial lake drainage [11]. These models provide a framework to investigate subglacial drainage.

I aim to further the understanding of subglacial hydrology with saline fluid from a modeling perspective. I have adapted an existing system of equations describing freshwater subglacial drainage to account for saline fluid. I consider how channel geometry and discharge rates change when the fluid is hypersaline at the source and salinity changes along the channel length. The parameters I use are similar to the hypothesized subglacial system of the Devon Ice Cap.

MOTIVATION

Despite the extensive work on subglacial hydrological systems, some aspects of drainage have been neglected. For example, there has been no published research to the author's knowledge that investigates brine flow through ice-walled channels and the impacts of salinity on drainage. In models of subglacial drainage through ice-walled channels, it has been assumed that the fluid flowing through the system is at the pressure-dependent melting point and the fluid is not saline. These are realistic assumptions for most systems, but there are subglacial systems where this is not the case.

Subglacial hypersaline lakes can exist at temperatures below the pressure-melting point where a fresh water lake would not. In cold regions such as Antarctica and the Canadian Arctic, hypersaline lakes exist and are a source of fluid in the subglacial system at temperatures below the pressure-dependent freezing point of water. Hypersaline fluid flowing through ice-walled conduits has been observed in places such as Blood Falls [1].

The existence of a unique hydrological system in cold ice at the Taylor glacier has been proposed by [18] and [1]. The subglacial system consists of iron-rich brine which allows the fluid to remain liquid at temperatures as low as -18°C [1]. The brine content at the source is over 13% NaCl and this content decreases as brine flows away from the source, presumably through hydrofracture or subglacial channels. At the outflow at Blood Falls, the brine has a total iron concentration of ≈ 3.8 mM and similar Cl values to marine water that has experienced cryogenic concentration [25]. The source of the hypothesized brine location is in an overdeepened part of the bed more than a 1km wide and 75m deep and sits under 400 – 500m of ice [1]. Blood Falls is an outflow on the Taylor Glacier where the iron-rich brine dyes the ice as the iron oxidizes.

Recently, two hypersaline lakes were discovered beneath the Devon Ice Cap (DIC) [33]. These lakes lie 700 meters beneath the ice surface at the bed. The fluid in the lakes is

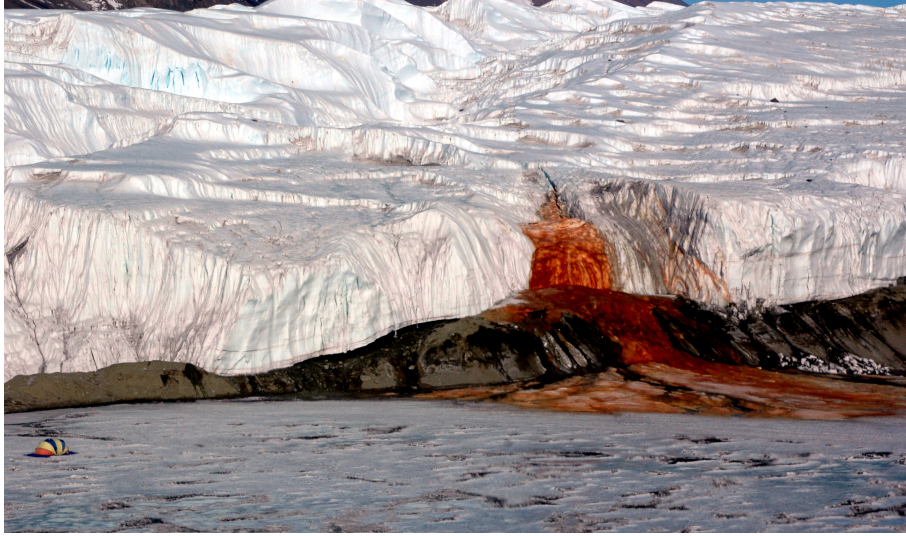


Figure 2.1: Brine outflow at Blood Falls on the Taylor Glacier [30]

estimated to be $-15.5^{\circ} \pm 3.5^{\circ}\text{C}$ with salinity of $140 - 160$ practical salinity units (psu). The ice temperature surrounding the lake is approximately -18.5°C . It is hypothesized that the hydrological system drains lakes to the north where fluid flows out of the terminus of the Sverdrup Glacier or to the south where fluid flows out of the terminus of the North or South Croker Bay Glaciers (Fig. 2.2) [33].

The study of brine in subglacial cold ice may have implications for other extreme terrestrial environments and for hydrological systems on icy moons [13]. This research is part of a larger NASA PSTAR (Planetary Science and Technology Through Analog Research) and ESCHER (Exploration of Saline Cryospheric Habitats with Europa Relevance) funded project which aims to investigate the DIC drainage system using ice-penetrating radar and by collecting samples from outflows at the terminus of different glaciers using submersible-borne instruments. Another aim of this modeling project is to help improve the understanding of the hydrological system beneath the DIC and to provide an initial exploration of channelized brine flow in this particular subglacial system.

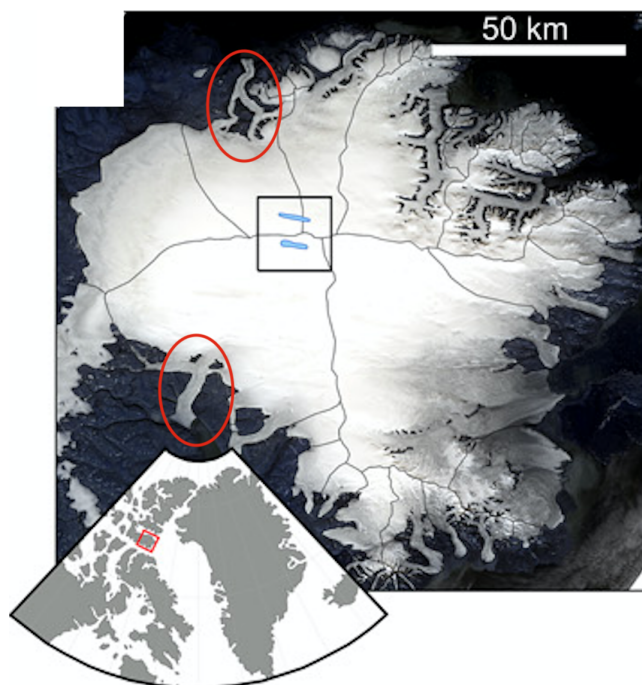


Figure 2.2: Devon Ice Cap with approximate subglacial lake locations in the square box and the hypothesized drainage outlets circled in red [33, Fig. 1a, modified].

MODEL DESCRIPTION

My research considering brine flow through cold ice addresses the following questions: (1) How significant is the effect of salinity on channel wall melt rates? (2) What flow rates are required for the channel to remain open? (3) How does the salt concentration change along the channel in response to the melting of the channel walls? I explore these questions by adapting the model of [19] to include saline fluid and by following the modeling approach of [11] for drainage from a subglacial lake. First I derive an equation to describe salt concentration along the channel in Chapter 3. Then, I adapt the model equations from [19] to couple the brine equation and then non-dimensionalize the system. I describe the numerical methods in Chapter 4 and then present and discuss the model results in Chapter 5. For a list of model parameters and values see Table 3.1. Unless otherwise specified, I adopt the notation from [19].

Consideration of brine

As the brine flows from the subglacial lake, the brine will be diluted as meltwater is added along the channel length. To account for this, I have developed a partial differential equation describing the concentration of salt β at position s and time t . The units of β is mass per volume $[\text{kg m}^{-3}]$.

A general conservation law for $\beta(s, t)$ is of the form,

$$\frac{d}{dt} \left(\int_a^b \beta(s, t) S(s, t) ds \right) = \phi(a, t) S(a, t) - \phi(b, t) S(b, t) + \int_a^b f(s, t) ds$$

where $\phi(s, t)$ is the flux and $f(s, t)$ is the source/sink term. This implies

$$\int_a^b \frac{\partial}{\partial t} (\beta(s, t) S(s, t)) ds = \int_a^b \frac{\partial}{\partial s} (-\phi(s, t) S(s, t)) ds + \int_a^b f(s, t) ds,$$

which gives

$$\frac{\partial}{\partial t} (\beta(s, t)S(s, t)) = \frac{\partial}{\partial s} (-\phi(s, t)S(s, t)) + f(s, t).$$

The fluid is moving along the channel at velocity $v(s, t)$ which gives

$$\phi(s, t) = v(s, t)\beta(s, t) - D\frac{\partial\beta(s, t)}{\partial s}$$

where D is the diffusion coefficient with units length squared per time [m^2s^{-1}]. In a turbulent flow regime, the mean velocity of the fluid at position s and time t is given by $v(s, t) = Q(s, t)/S(s, t)$ from [29]. Substituting this expression for v , the flux becomes

$$\phi(s, t) = \frac{Q(s, t)\beta(s, t)}{S(s, t)} - D\frac{\partial\beta(s, t)}{\partial s}.$$

Assuming there is no addition of brine added along the channel and there is no accretion of brine on the channel walls, there is no source or sink for salt and $f(s, t) = 0$.

With these flux and source terms, the PDE becomes

$$\frac{\partial}{\partial t} (\beta(s, t)S(s, t)) = \frac{\partial}{\partial s} \left(-Q(s, t)\beta(s, t) + D\frac{\partial\beta(s, t)}{\partial s}S(s, t) \right).$$

The boundary condition at the lake is the salt concentration of the fluid in the lake which is given by

$$\beta(0, t) = (0.05844)(1000)\alpha = 58.44\alpha,$$

where 0.05844 is the molar mass of salt [kg mol^{-1}] and α is the molarity of the brine at the lake which is estimated to be 2.4 – 2.7 M NaCl. I assume that the entire channel at time 0 is the same concentration as the brine in the lake because when the cross-sectional area of the

channel is small, the dilution of the brine is also minimal. This gives the initial condition

$$\beta(s, 0) = \beta(0, t) = 58.44\alpha.$$

I assume the salt concentration at the end of the channel is

$$\left. \frac{\partial}{\partial s} \beta(s, t) \right|_{s=L} = 0,$$

which means that eventually the change in salt concentration along the channel is zero. The molarity of the brine at position s and time t is given by

$$\alpha(s, t) = \frac{\beta(s, t)}{58.44}. \quad (3.1)$$

This salt concentration equation with the described boundary and initial conditions provides the salinity of the brine along the channel and in time.

Parameters involving brine in a cold system The properties of the brine which are dependent on salinity, will change as the concentration changes. With an equation describing the concentration of the salt along the channel, the density ρ_b , specific volume ν_b , specific heat σ_b , thermal conductivity κ_b , viscosity η_b can be calculated along the channel. The dimensionless Nusselt, Reynolds, and Prandtl numbers also vary over the length of the channel and in time as a result of the changing brine properties. See Appendix A for details on the empirical equations for these parameters as a function of salinity, brine temperature, and pressure.

The latent heat of fusion for water is 334 kJ and for sea water is 330 kJ. I assume the latent heat of fusion for brine is the same as sea water despite the changing brine salinity. For cold ice ($\approx -10^\circ \text{ C}$), I assume the Glen's flow law parameter, $A = 3.5 \times 10^{-25}$, $n = 3$,

and $K = 2An^n$ [9]. I outline the additional changes to the model of [20] and how the salt concentration equation is coupled to the model below.

Changes to existing model equations

In this model, I assume turbulent flow following [19]. The transition between laminar and turbulent flow occurs between Reynolds numbers of 2000 and 4000. Using Eq. A.10 with $\rho_b = 1130 \text{ kg m}^3$, $\eta_b = 0.0025$, and $r = 10 \text{ cm}$, the fluid velocity must be at least 0.022 m s^{-1} to be in a turbulent regime. For all scenarios in Chapter 5, this value is realistic. Therefore, using this justification and following [32], [29], and [14], I use the Gauckler-Manning Formula [23] instead of the Darcy-Weisbach Equation. This means that the empirical derivation of the conservation of momentum equation is unchanged.

The equations governing channel evolution, conservation of mass, and conservation of momentum from [19] before non-dimensionalization are unchanged. However, the conservation of energy and the heat transfer equations are influenced when considering saline fluid. Recall [29] derived the equation for the conservation of energy

$$Q \left(\rho_w g_s - \frac{\partial p}{\partial s} \right) - \rho_w \sigma S \frac{\partial \theta_w}{\partial s} = m\mathcal{L} + m\sigma(\theta_w - \theta_i).$$

[29] and therefore [19] assumed that the water temperature θ_w is always equal to the melting point of ice θ_i , that is $\theta_w = \theta_i$, which gives $m\sigma(\theta_w - \theta_i) = 0$. I do not assume this and I consider the changing freezing point of fluid as a function of salinity. The freezing point of saline fluid is

$$\hat{\theta}_b \approx \hat{\theta}_w - 0.056\beta,$$

where $\hat{\theta}_w$ is the pressure-dependent freezing point of water and β has units [psu] [27]. Note that the units $[\text{kg m}^3]$ is approximately equal to the units [psu], so I use these units interchangeably. The freezing point of water beneath 700 meters of ice is approximately

-0.5° C. The fluid/ice interface has the same salinity- and pressure-dependent freezing and melting points, which I will refer to as $\hat{\theta}$.

Therefore the salinity- and pressure-dependent melting point of ice is

$$\hat{\theta}(\beta) \approx \hat{\theta}_w - 0.056\beta,$$

where $\hat{\theta}_w = -0.5^\circ$ C and β is spatially and temporally varying. Note that the relationship between salinity and temperature is not linear and thus the melting point of ice is off for higher salt concentration. I assume the temperature of the brine θ_b is equal to the surrounding ice. I calculate the pressure and salinity-dependent melting point of ice from the concentration of the brine and set the ice and brine temperature equal to melting point at time zero. Additionally, I assume the brine temperature is constant along the channel such that

$$\frac{\partial \theta_b}{\partial s} = 0, \quad (3.2)$$

which is realistic for most systems [6]. This results in the following energy equation,

$$Q \left(\rho_w g_s - \frac{\partial p}{\partial s} \right) = m\mathcal{L} + m\sigma(\theta_b - \hat{\theta}) \quad (3.3)$$

which implies that the amount of energy available for melting the channel walls is a function of salinity.

I follow [14] and [19] in improving the energy equation formulated by [29]. The left hand side of Eq. (3.3) becomes

$$Q \left(\rho_w g_s - \frac{\partial p}{\partial s} \right) = Q \left(\psi + \frac{\partial N}{\partial s} \right), \quad (3.4)$$

where N is the effective pressure and ψ is the basic hydraulic gradient. The effective pressure

is the difference between the ice-overburden pressure and basal water pressure

$$N = P_i - p_w, \quad (3.5)$$

where p_w is the basal water pressure. The basic hydraulic gradient is given by

$$\psi = \rho_w g \sin(B) - \frac{\partial P_i}{\partial s}. \quad (3.6)$$

Thus the energy equation is

$$m\mathcal{L} + m\sigma(\theta_b - \hat{\theta}) = Q \left(\psi + \frac{\partial N}{\partial s} \right).$$

Following [29] and [19], I neglect the heat transfer equation. This is equivalent to assuming any heat generated from flow is instantaneously transferred to the channel walls.

Semi-circular channels are often assumed in subglacial hydrology models [14, 19]. Since a circular channel is observed at the brine outflow at terminus of the Taylor Glacier, I assume a circular channel for this model. I compare the effect of circular vs semi-circular channels in Sec. 5. The main difference between these assumptions is that in the semi-circular case, the water is flowing along the bed and therefore the roughness of the bed must be accounted for instead of the roughness of ice on one side. The friction factor in the conservation of energy equation changes depending on the roughness of the channel walls or the bed and the geometry. For a semi-circular channel, the roughness of channel when full is $n_f = n_i 0.611 + (1 - 0.611)n_b$ and the friction factor is $f = 6.6n_f^2$. For a circular channel, the roughness of channel when full is $n_f = n_i * 0.611$ and the friction factor is $f = 5.4n_f^2$, where $n_i = 0.06 \text{ m}^{-1/3}\text{s}$ is the roughness of ice wall and $n_b = 0.1629 \text{ m}^{-1/3}\text{s}$ is the roughness of bed material.

Location of the lake In this model, the lake is located under the ice at the bed, instead of on the margins of a glacier. I detail the resulting changes here. I assume there is no fluid flowing into the lake and the only fluid leaving is through the channel. Therefore the rate of change of lake volume V is given by,

$$\frac{dV}{dt} = -Q(0, t). \quad (3.7)$$

Following [19], I define the lake hypsometry by

$$\left(\frac{h}{h_i}\right)^p = \frac{V}{V_i} \quad (3.8)$$

where h is the depth of the lake, p is the dimensionless parameter depending on lake shape, h_i is the reference lake depth, and V_i is the reference lake volume.

Implicitly differentiating gives

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = \frac{pV_i}{h_i^p} h^{p-1} \frac{dh}{dt} \quad (3.9)$$

and by substitution, the lake depth evolves with time following

$$\frac{dh}{dt} = \frac{h_i^p}{pV_i h^{p-1}} (-Q(0, t)). \quad (3.10)$$

With the assumption of a marginal lake, [19] imposed the boundary conditions on the effective pressure at each end of the channel. The boundary condition at the lake was

$$N_L = N(0, t) = Hg\rho_i - hg\rho_w, \quad (3.11)$$

where H is the ice dam thickness (the thickness of the ice adjacent to the lake), h is the water depth of the lake, and ρ_i and ρ_w are the densities of ice and water respectively. The boundary

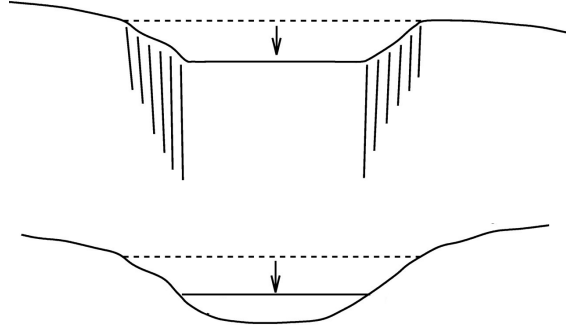


Figure 3.1: Schematic of ice above subglacial lake before and after drainage [11, Fig. 2, modified].

condition at the end of the conduit, which was assumed to be at the glacier terminus was given by

$$N_T = N(L, t) = 0. \quad (3.12)$$

Instead of a marginal lake, I consider a subglacial lake where the ice roof of the lake drops with the lake depth so the effective pressure is always zero (Fig. 3.1). The new boundary condition where the conduit meets the lake is

$$N_L = N(0, t) = 0 \quad (3.13)$$

following the 'Vostok model' in [11]. I impose a boundary condition at some distance L downstream from the lake where the end channel is not subaerial. Therefore, the effective pressure is not zero, but the boundary condition at the end of the channel is

$$\left. \frac{\partial}{\partial s} N(s, t) \right|_{s=L} = 0. \quad (3.14)$$

Dimensional model equations

The updated model equations coupled with the salt concentration equation and written in terms of the salinity-dependent parameters are listed below.

Channel evolution:

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - KSN^3$$

Conservation of mass:

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{m}{\rho_b} + M$$

Conservation of momentum:

$$\psi + \frac{\partial N}{\partial s} = f\rho_b g \frac{Q|Q|}{S^{8/3}}$$

Conservation of energy:

$$m\mathcal{L} + m\sigma_b(\theta_b - \hat{\theta}) = Q \left(\psi + \frac{\partial N}{\partial s} \right)$$

Salt Concentration:

$$\frac{\partial}{\partial t} (\beta S) = \frac{\partial}{\partial s} \left(-Q\beta + DS \frac{\partial \beta}{\partial s} \right)$$

Non-dimensionalization

The subscript 0 implies a variable is a scaling parameter, the subscript L refers to variables at the lake (when $s=0$), and the superscript $*$ denote the non-dimensionalized variables. I follow a similar procedure for non-dimensionalization as described in [19]. I let $s = s_0 s^*$, $S = S_0 S^*$, $N = N_0 N^*$, $Q = Q_0 Q^*$, $M = M_0 M^*$, $\psi = \psi_0 \psi^*$, $t = t_0 t^*$, $h = h_0 h^*$, $Q_{in} = Q_{in0} Q_{in}^*$, $m = m_0 m^*$, $\beta = \beta_0 \beta^*$, and $\theta = \theta_0 \theta^*$. Scaling for s_0 , Q_0 , and θ_0 are given by the maximum values for these parameters.

Table 3.1: List of model parameters. Values of constants are specified in brackets.

Variable	Description
ρ_i, ρ_w, ρ_b	densities of ice [917 kg m ⁻³], water [997 kg m ⁻³], and brine
g	gravitational acceleration [9.81 m s ⁻²]
$\mathcal{L}$	latent heat of fusion for brine [3.3 x 10 ⁵ J kg ⁻¹]
A, n	ice flow law parameter [3.5 x 10 ⁻²⁵] and exponent [3]
K	ice flow parameter for conduit closure
$f, \mathcal{N}$	friction factor and hydraulic roughness
x, s	horizontal and bed-parallel spatial coordinates
B, L, S, m	bed (conduit) slope, length, cross-sectional area, and melt rate
Q	discharge along the conduit
M	fresh water flux to the conduit per unit length
P_i, ψ	ice overburden pressure and basic hydraulic gradient
N, N_L	effective pressure and effective pressure at lake
h, h_i	lake depth and reference lake depth
H	ice thickness from surface to bed adjacent lake
p	coefficient that describe lake hypsometry
V, V_i	volume of lake and reference volume of lake
$\theta_i, \theta_w, \theta_b$	temperatures of basal ice, water, and brine
$\hat{\theta}_w, \hat{\theta}_b$	freezing point of water and brine at pressure
$\hat{\theta}$	salinity-dependent melting point of ice
β, α	salt concentration and molarity of brine

Replacing dimensional variables with the variables listed above gives the equation for cross-sectional area

$$\frac{S_0}{t_0} \frac{\partial S^*}{\partial t^*} = \frac{m_0 m^*}{\rho_i} - K S_0 N_0^3 S^* N^{*3}.$$

By equating the coefficients

$$\frac{S_0}{t_0} = \frac{m_0}{\rho_i} = K_0 S_0 N_0^3,$$

the scaling parameters N_0 and t_0 can be expressed as

$$N_0 = (K t_0)^{-1/3}$$

and

$$t_0 = \rho_i S_0 / m_0.$$

This gives the dimensionless equation for the evolution of the channel cross-sectional area,

$$\frac{\partial S^*}{\partial t^*} = m^* - S^* N^{*3}. \quad (3.15)$$

Now using a similar method for the conservation of mass equation, the equation is

$$\frac{S_0}{t_0} \frac{\partial S^*}{\partial t^*} + \frac{Q_0}{s_0} \frac{\partial Q^*}{\partial S^*} = \frac{m_0}{\rho_i} r m^* + M_0 M^*,$$

where r is the dimensionless parameter, $r = \rho_i / \rho_b$. Rearranging and equating coefficients gives

$$\epsilon \frac{\partial S^*}{\partial t^*} + \frac{\partial Q^*}{\partial S^*} = \epsilon r m^* + M^*, \quad (3.16)$$

where ϵ and M_0 are the dimensionless parameters,

$$\epsilon = \frac{s_0 m_0}{Q_0 \rho_i}$$

and

$$M_0 = \frac{Q_0}{s_0}.$$

Replacing the dimensional variables in the conservation of momentum equation results in

$$\psi_0 \psi^* + \frac{N_0}{s_0} \frac{\partial N^*}{\partial s^*} = f \rho_b g \frac{Q_0^2 Q^* |Q^*|}{S_0^{8/3} S^{*8/3}}.$$

Equating the coefficients requires that

$$\psi_0 = f \rho_b g \frac{Q_0^2}{S_0^{8/3}}$$

and the dimensionless conservation of momentum equation becomes

$$\psi^* + \delta \frac{\partial N^*}{\partial s^*} = \frac{Q^* |Q^*|}{S^{*8/3}}, \quad (3.17)$$

where

$$\delta = \frac{N_0}{s_0 \psi_0}.$$

After substituting the scaling parameters, the conservation of energy equation is

$$Q_0 Q^* \left(\psi_0 \psi^* + \frac{N_0}{s_0} \frac{\partial N^*}{\partial s^*} \right) = m_0 m^* \mathcal{L} + m_0 m^* \sigma_b \theta_0 (\theta_b^* - \hat{\theta}^*).$$

Repeating the procedure, the dimensionless equation for conservation of energy becomes

$$Q^* \left(\psi^* + \delta \frac{\partial N^*}{\partial s^*} \right) = m^* + \gamma m^* (\theta_b^* - \hat{\theta}^*), \quad (3.18)$$

where

$$m_0 = \frac{Q_0 \psi_0}{\mathcal{L}}$$

and

$$\gamma = \frac{\theta_0 \sigma_b}{\mathcal{L}}.$$

Notice that by substituting Eq. (3.17) into Eq. (3.18) and rearranging, the energy equation can be expressed as

$$m^* = \frac{Q^{*3}}{S^{*8/3}(1 + \gamma(\theta_b - \hat{\theta}))}. \quad (3.19)$$

This equation for m^* can be substituted into Eq. (3.15) to arrive at an updated dimensionless equation for cross-sectional area

$$\frac{\partial S^*}{\partial t^*} = \frac{Q^{*3}}{S^{*8/3}(1 + \gamma(\theta_b - \hat{\theta}))} - S^* N^{*3}, \quad (3.20)$$

which eliminates the need to solve the energy equation separately.

Finally, making similar substitutions into the salt concentration equation yields

$$\frac{\beta_0 S_0}{t_0} \frac{\partial}{\partial t^*} (\beta^* S^*) = \frac{\beta_0}{s_0} \frac{\partial}{\partial s^*} \left(-\beta^* Q_0 Q^* + \frac{D S_0 S^*}{s_0} \frac{\partial \beta^*}{\partial s^*} \right).$$

Rearranging gives

$$\frac{S_0 s_0}{t_0 Q_0} \frac{\partial}{\partial t^*} (\beta^* S^*) = \frac{\partial}{\partial s^*} \left(-\beta^* Q^* + \frac{D S_0 S^*}{s_0 Q_0} \frac{\partial \beta^*}{\partial s^*} \right).$$

By letting

$$\lambda = \frac{S_0 s_0}{t_0 Q_0}$$

and

$$\Lambda = \frac{D S_0}{s_0 Q_0},$$

the dimensionless equation for salt concentration becomes,

$$\lambda \frac{\partial}{\partial t^*} (\beta^* S^*) = \frac{\partial}{\partial s^*} \left(-\beta^* Q^* + \Lambda S^* \frac{\partial \beta^*}{\partial s^*} \right). \quad (3.21)$$

I non-dimensionalize the lake depth evolution equation

$$\frac{h_0}{t_0} \frac{dh^*}{dt^*} = \frac{h_i^p}{pV_i h_0 h^{*(p-1)}} (-Q_0 Q^*(0, t))$$

and arrive at,

$$\frac{dh^*}{dt^*} = \frac{\zeta}{h^{*(p-1)}} Q^*(0, t),$$

where

$$\zeta = \frac{t_0 h_i^p Q_0}{pV_i h_0^p}.$$

The dimensionless boundary condition at the lake becomes,

$$N^*(0, t^*) = N_L^* = 0$$

and the dimensionless boundary condition at distance L from the lake is,

$$\left. \frac{\partial}{\partial s^*} N^*(s, t) \right|_{s^*=1} = 0.$$

Dimensionless model equations

Dropping the superscripts, the dimensionless model equations, boundary conditions, and parameters are as follows.

Channel Evolution:

$$\frac{\partial S}{\partial t} = \frac{Q^3}{S^{8/3}(1 + \gamma(\theta_b - \hat{\theta}))} - SN^3 \quad (3.22)$$

Conservation of Mass:

$$\frac{\partial Q}{\partial s} = \epsilon(r-1)\frac{|Q|^3}{S^{8/3}} + \epsilon SN^3 + M \quad (3.23)$$

Conservation of Momentum:

$$\frac{\partial N}{\partial s} = \frac{1}{\delta} \left(\frac{Q|Q|}{S^{8/3}} - \psi \right) \quad (3.24)$$

Salt Concentration:

$$\lambda \frac{\partial}{\partial t} (\beta S) = \frac{\partial}{\partial s} \left(-\beta Q + \Lambda S \frac{\partial \beta}{\partial s} \right) \quad (3.25)$$

Boundary Conditions:

$$N_L = 0 \quad (3.26)$$

$$\left. \frac{\partial}{\partial s} N(s, t) \right|_{s=1} = 0 \quad (3.27)$$

$$\frac{dh}{dt} = \frac{\zeta}{h^{p-1}} Q(0, t) \quad (3.28)$$

Model and Scaling Parameters:

$$\begin{aligned} N_0 &= (Kt_0)^{-1/3}, \quad m_0 = \frac{Q_0\psi_0}{\mathcal{L}}, \quad S_0 = \left(\frac{f\rho_b g Q_0^2}{\psi_0} \right)^{3/8}, \quad t_0 = \frac{\rho_i S_0}{m_0}, \\ M_0 &= \frac{Q_0}{s_0}, \quad r = \frac{\rho_i}{\rho_b}, \quad \epsilon = \frac{s_0 m_0}{Q_0 \rho_i}, \quad \delta = \frac{N_0}{s_0 \psi_0}, \quad \gamma = \frac{\theta_0 \sigma_b}{\mathcal{L}}, \quad \lambda = \frac{S_0 s_0}{t_0 Q_0}, \\ \Lambda &= \frac{DS_0}{s_0 Q_0}, \quad \zeta = \frac{t_0 h_i^p Q_0}{p V_i h_0^p} \end{aligned}$$

NUMERICAL METHODS

I use the subscripts $j = 1, 2, \dots, n$ to denote the grid points along the channel which are separated by Δs and the superscripts $i = 1, 2, \dots, m$ denote time steps separated by Δt . To solve Eq. (3.28), I follow [19] in using the Euler Method to evolve the lake depth forward in time.

$$h^{i+1} = h^i + \frac{\Delta t \zeta}{(h^i)^{p-1}} Q_1^i.$$

Similarly, I solve Eq. (3.22) using the same method. The channel cross-sectional area S is moved forward in time at all grid points by

$$S_j^{i+1} = S_j^i + \Delta t \left[\frac{(Q_j^i)^3}{(S_j^i)^{8/3}} \frac{1}{1 + \gamma(\theta_b - \hat{\theta}_j^i)} - S_j^i (N_j^i)^3 \right]$$

for $j = 1, 2, \dots, n$.

To evolve these two equations forward in time, the discharge, effective pressure, and salt concentration (to calculate $\hat{\theta}_j^i$) at time step i is needed. These variables can be found simultaneously solving the mass and momentum equations with what [19] calls the Boundary Layer Method.

I follow [19] in assuming ϵ is small enough to neglect the terms containing ϵ in Eq. (3.23). With parameter values, $m_0 = 0.15 \text{ kg m}^{-1} \text{ s}^{-1}$, $s_0 = 10000 \text{ m}$, $\rho_i = 917 \text{ kg m}^{-3}$, and $Q_0 = 100 \text{ m}^3$, ϵ is on the order of 10^{-2} and thus I neglect these terms. This simplifies Eq. (3.23) to

$$\frac{\partial Q}{\partial s} = M \tag{4.1}$$

where M is constant along the channel and in time.

Combining Eq. (3.24) and Eq. (3.27) and rearranging gives a boundary condition for

discharge at the end of the channel

$$Q(s = 1, t) = \sqrt{S(s = 1, t)^{8/3}\psi(s = 1)}. \quad (4.2)$$

Integrating Eq. (4.1) from s to the end of the channel

$$\int_{Q(s,t)}^{Q(1,t)} 1 dQ = M \int_s^1 1 ds, \quad (4.3)$$

gives

$$Q(s, t) = Q(1, t) - M(1 - s).$$

By substituting Eq. (4.2), I arrive at the following equation for discharge as a function of time and space

$$Q(s, t) = \sqrt{S(s = 1, t)^{8/3}\psi(s = 1) - M(1 - s)}.$$

Using the Boundary Layer Method [19], I solve this equation by calculating the discharge profile at each grid point by

$$Q_j^i = \sqrt{(S_n^i)^{8/3}\psi_n - M[1 - s_j]} \quad (4.4)$$

To calculate the effective pressure along the channel, I start with the boundary condition at the lake and then integrate Eq. (3.26) by iterating

$$N_{j+1}^i = N_j^i + \frac{\Delta s}{\delta} \left(\frac{(Q_j^i)^2}{(S_j^i)^{8/3}} - \psi_j \right) \quad (4.5)$$

from $j = 1, 2, \dots n$.

When solving the brine equation, I neglect the diffusion term. The Péclet number (Pe) is used to determine the relative influence of advection and diffusion on fluid flow. The

equation is

$$(Pe) = \frac{s_0 v}{D} = \frac{s_0 Q}{SD}, \quad (4.6)$$

where D is the diffusion coefficient for salt in water. For the parameter values, $s_0 = 10,000$ m, $v = Q/S = 0.035 - 1.5$ m s⁻¹, and $D = 1.5 \times 10^{-9}$ m² s⁻¹, the Péclet number is $(Pe) = 5.25 \times 10^{11} - 2.25 \times 10^{13}$ which suggests that the diffusion term is negligible.

The salt concentration without diffusion is,

$$\lambda \frac{\partial}{\partial t^*} (\beta^* S^*) = \frac{\partial}{\partial s^*} (-\beta^* Q^*).$$

To solve the simplified dimensionless brine equation Eq. (3.25), I use an upwind difference scheme such that

$$\beta_j^{i+1} = \beta_j^i - \Delta t \left[\frac{\beta_j^i}{S_j^i} \frac{\partial S_j^i}{\partial t} + \frac{Q_j^i}{\lambda S_j^i} \left(\frac{\beta_{j+1}^i - \beta_j^i}{\Delta s} \right) + \frac{\beta_j^i M}{\lambda S_j^i} \right].$$

Using the brine equation at time $i + 1$, the updated salinity-dependent melting point of the ice can be calculated using

$$\hat{\theta}_j^{i+1} = \theta_w^{hat} - 0.056 \beta_j^{i+1},$$

for $i = 1, 2, \dots m$.

I use a constant time step in dimensionless units of $dt = 0.0005$ which is about 80 s and a constant grid spacing in dimensionless units of $ds = s/40$, which is a step size of 250 m when the channel length is 10000 m.

RESULTS

Idealized model simulations were run to investigate the impact of brine on discharge rates, channel height, and the duration of lake drainage. The simulations were run with parameters ranges motivated by the lakes beneath the Devon Ice Cap from [33].

Unless otherwise specified, the parameters used for the following scenarios are as follows: lake hypsometry parameter $p = 1$, lake volume $V_i = 5 \times 10^6 \text{ m}^3$, lake depth $h_i = 10 \text{ m}$ (assumed to be full at time 0), channel length $s = 10000 \text{ m}$, initial cross-sectional area $S = 0.001 \text{ m}^2$, bed and conduit slope $B = 3^\circ$, additional water added along channel $M = 0 \text{ m}^2 \text{ s}^{-1}$, and circular channel geometry. In these scenarios, I assume the temperature of the brine is equal to the ice temperature for all time. Both the brine and ice temperature is equal to the pressure and salinity-dependent melting point of the ice at the various salt concentrations at time zero (Table 5.1). Each model simulation is run independently with prescribed initial conditions and boundary conditions. The simulations run until open channel flow occurs at which point, the model run ends. I refer to this time as when the lake has drained, even though there is some amount of fluid (1 – 5 m) remaining in the lake.

To investigate the effect of saline fluid, I run five scenarios with $\beta = \{0, 1.25, 2.5, 3.75, 5\}$ to explore the range of possible outcomes. The discharge rates are greater for fluid with higher salt concentrations (Fig. 5.1). For all salt concentrations, the channel roof drops below the channel height at a similar lake depth of $h \approx 5 \text{ m}$ (Fig. 5.1a). Higher salt concentrations increased the peak velocity reached and decreased the amount of time to reach peak velocity (Fig. 5.1b). The peak velocity and drainage duration change almost linearly with increased salt concentrations. The channel height increases slightly along the channel for all scenarios (Fig. 5.1c). Towards the end of the simulations (day 20), the channel height is greatest for scenarios with higher salt concentrations.

Early in the simulations (days 0 - 10), the discharge and cross-sectional area are nearly

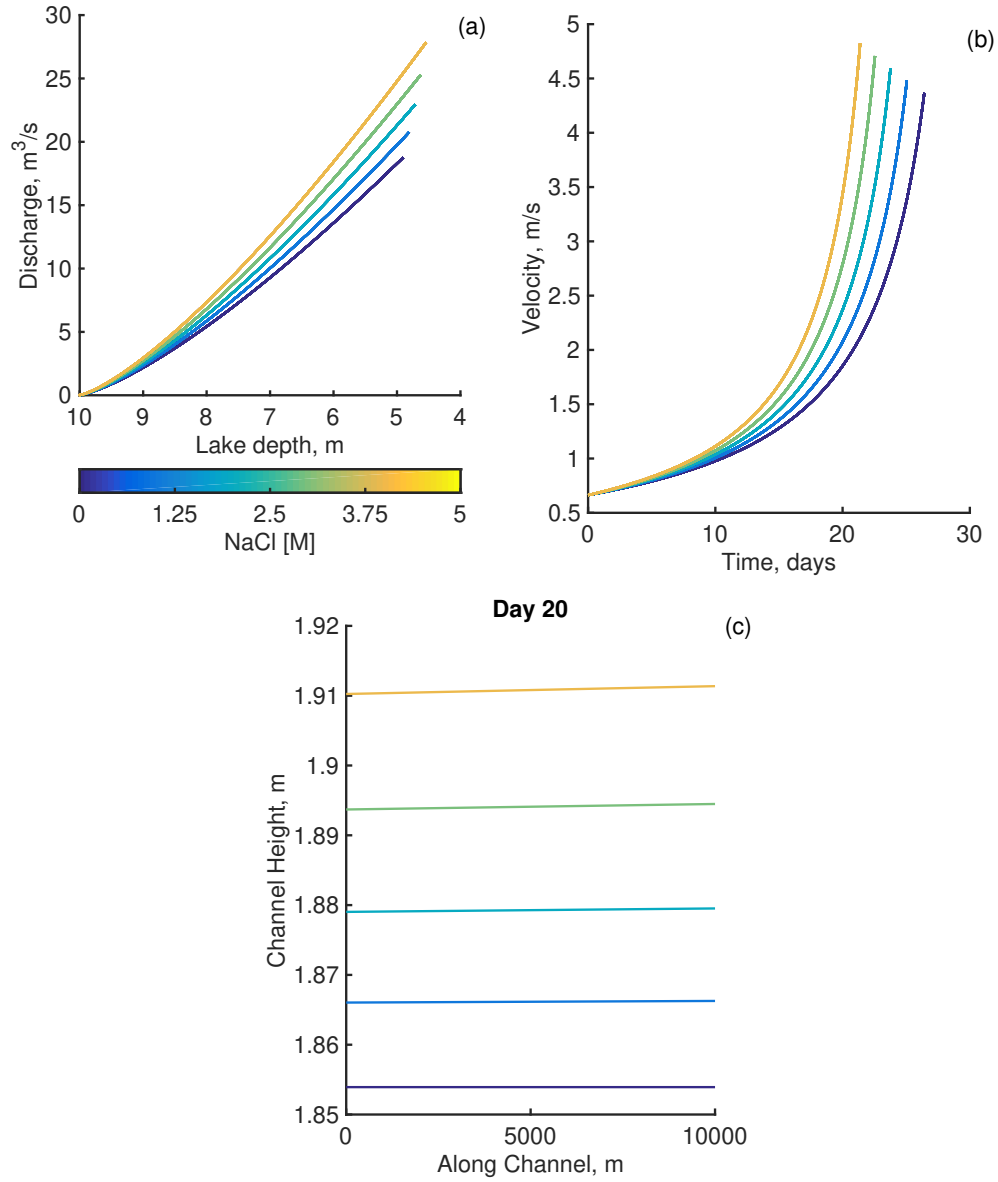


Figure 5.1: The impact of brine on discharge, velocity, and channel height for various initial salt concentrations, shown by the color bar in (a). (a) Discharge at the lake outlet vs. lake depth as the lake drains, (b) velocity at the lake outlet over time in days, (c) channel height along channel at day 20 in the simulations.

Table 5.1: Temperatures of brine and ice for all five salinity scenarios.

Salinity (NaCl)	Brine and Ice Temp.
0M	-0.5°C
1.25M	-4.56°C
2.5M	-8.65°C
3.75M	-12.74°C
5M	-16.83°C

equal for the scenarios with no brine and brine with a salinity of 2.5 M at the lake (Fig. 5.2). At day 20, there is a noticeable difference in cross-sectional area and discharge rates between the two scenarios. The differences in cross-sectional area and discharge at day 20 between the simulations is greater for higher salt concentrations.

I systematically vary parameters to explore the sensitivity of the model. I test the impact of channel geometry, lake volume, and initial cross-sectional area on discharge and the duration of drainage (Fig. 5.3). The channel geometry (circular vs. semi-circular) changes the time until lake drainage as well as the peak discharge. I also test the impact of channel geometry for freshwater and for a lake with salinity 2.5 M (Fig. 5.3a). For a semi-circular channel, the difference in time until lake drainage is more significant than the difference in peak discharge between the freshwater and brine scenarios. For the circular channel, the difference in peak discharge is greater than the difference in the duration of time until lake drainage. In both scenarios, the circular channel drains the lake in less than half the time than for a semi-circular channel and the peak discharge is about twice as high when the channels are circular. The volume of the lake impacts the discharge and the timing of drainage by extending the amount of time the model is run (Fig. 5.3b). The

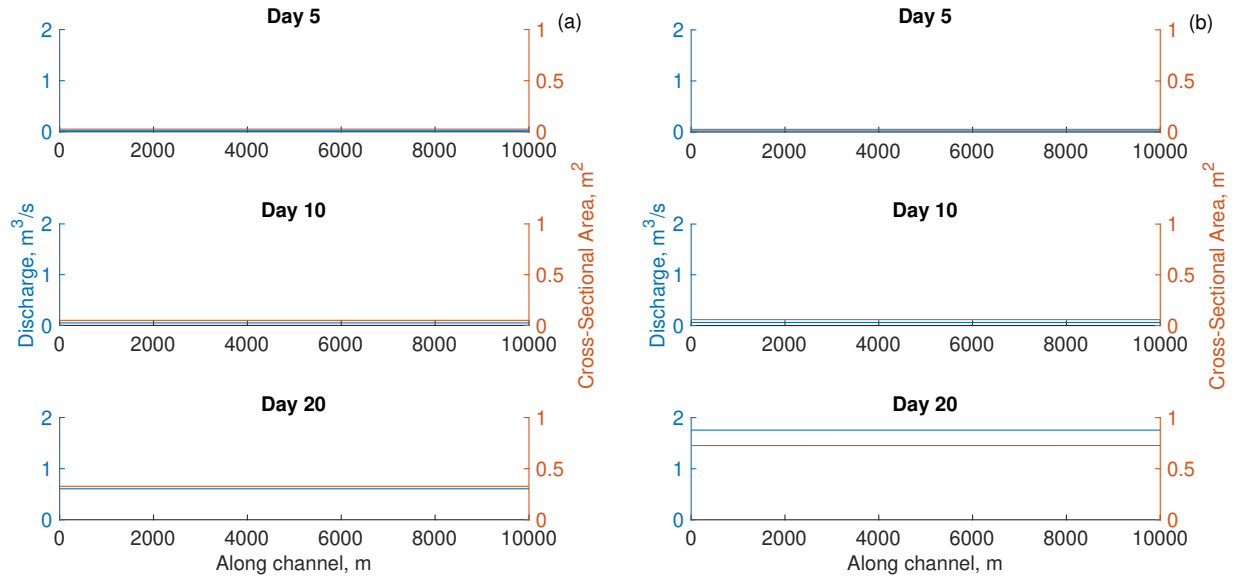


Figure 5.2: Time series of discharge and cross-sectional area along-channel profiles for subglacial lake without brine (a) and for subglacial lake with salinity of 2.5 M (b) from day 5 to day 20.

Table 5.2: Prescribed model parameter variables and descriptions with baseline simulation values and ranges explored.

Variable	Description	Baseline	Range
s	length of channel	10000 m	[10000 m]
B	bed (conduit) slope	3°	[$1 - 5^\circ$]
S	initial cross-sectional area	0.015 m^2	[$0.0015 - 0.15 \text{ m}^2$]
h	initial lake depth	10 m	[10 – 70 m]
h_i	reference lake depth	10 m	[10 – 70 m]
H	ice thickness above bed and channel	700 m	[700 m]
p	lake hypsometry parameter	1	[1, 2, 3]
V_i	reference volume of lake	$5 \times 10^6 \text{ m}^3$	[$1 \times 10^6 - 1 \times 10^7 \text{ m}^3$]
α	salt concentration of brine in lake	0, 2.5 M	[0 – 5 M]

discharge curves for all lake volumes follows the same curve until the volume of the smaller lake drains. As the model is run for longer for greater lake volumes, the peak discharge increases by an order of magnitude when comparing $V_i = 1 \times 10^6$ and $V_i = 1 \times 10^7$. The initial cross-sectional area impacts the amount of time until the lake drains but does not change the peak discharge reached, where the smaller the initial cross-sectional area takes longer to reach peak discharge (Fig. 5.3c). For a list of parameters and the range of values explored, see Table 5.2.

I also explore the impact of bed slope, lake depth, the effective pressure at the lake, ice thickness, and brine temperature on discharge. Increasing the bed slope and as a result the conduit slope increases the peak discharge. Assuming a 3° slope and a conduit length of 10000 m means that from the lake to the end of the conduit, there is a difference in elevation of $z = \sin(3)10000 \approx 523 \text{ m}$, which is reasonable for the DIC. If the initial fluid level is equal to the depth of the lake, then different lake depths do not impact the peak discharge or the

duration of drainage. The model is not sensitive to the effective pressure boundary condition at the lake. Decreasing or increasing the channel length by an order of magnitude leads to numerical instabilities in the salt concentration equation so I do not explore the impact of channel length. I explored ice thickness ranges of 400 – 1000 m and found that the thickness of ice above the system does not impact the discharge or timing of drainage. Varying brine temperatures between 0° to 2° C above the salinity-dependent freezing point of brine has little impact on discharge rates and timing. I assume a circular channel in these simulation because this has been observed in hypersaline systems such as Blood Falls. I do not account for any brine accumulated from the bed rock or deposits in the ice, which would tend to increase the discharge.

As the walls of the channel melt and the brine is diluted, the properties of the brine change. At day 20 for a subglacial lake with salinity 5 M, the salt concentration at the lake is equal to the concentration in the lake and decreases along the channel. When neglecting to account for the change in brine properties, the brine concentration is slightly lower than when the changes in brine properties are considered (Fig. 5.4a). At the farthest end of the channel (9000 – 10000 m) where the most meltwater has been added to the system, the difference in the salt concentration of the brine is still marginal, but is greater than at the beginning of the channel when the cumulative meltwater added is less (Fig. 5.4b).

The salinity and pressure-dependent melting points of ice for different salt concentrations ($\alpha = 0$ M and $\alpha = 2.5$ M) are shown in Fig. 5.5. I assume the temperature of the brine is constant throughout the simulation and along the channel and is equal to the initial melting point of the ice for respective simulations (Fig. 5.5). The melting point of ice depends on the salinity of the brine and thus varies between different scenarios and throughout simulations as the brine is diluted by meltwater.

I explore the model's response to a perturbation in the initial cross-sectional area along the channel at the beginning of the simulation with $\alpha = 2.5$ M (Fig. 5.6). The simulation

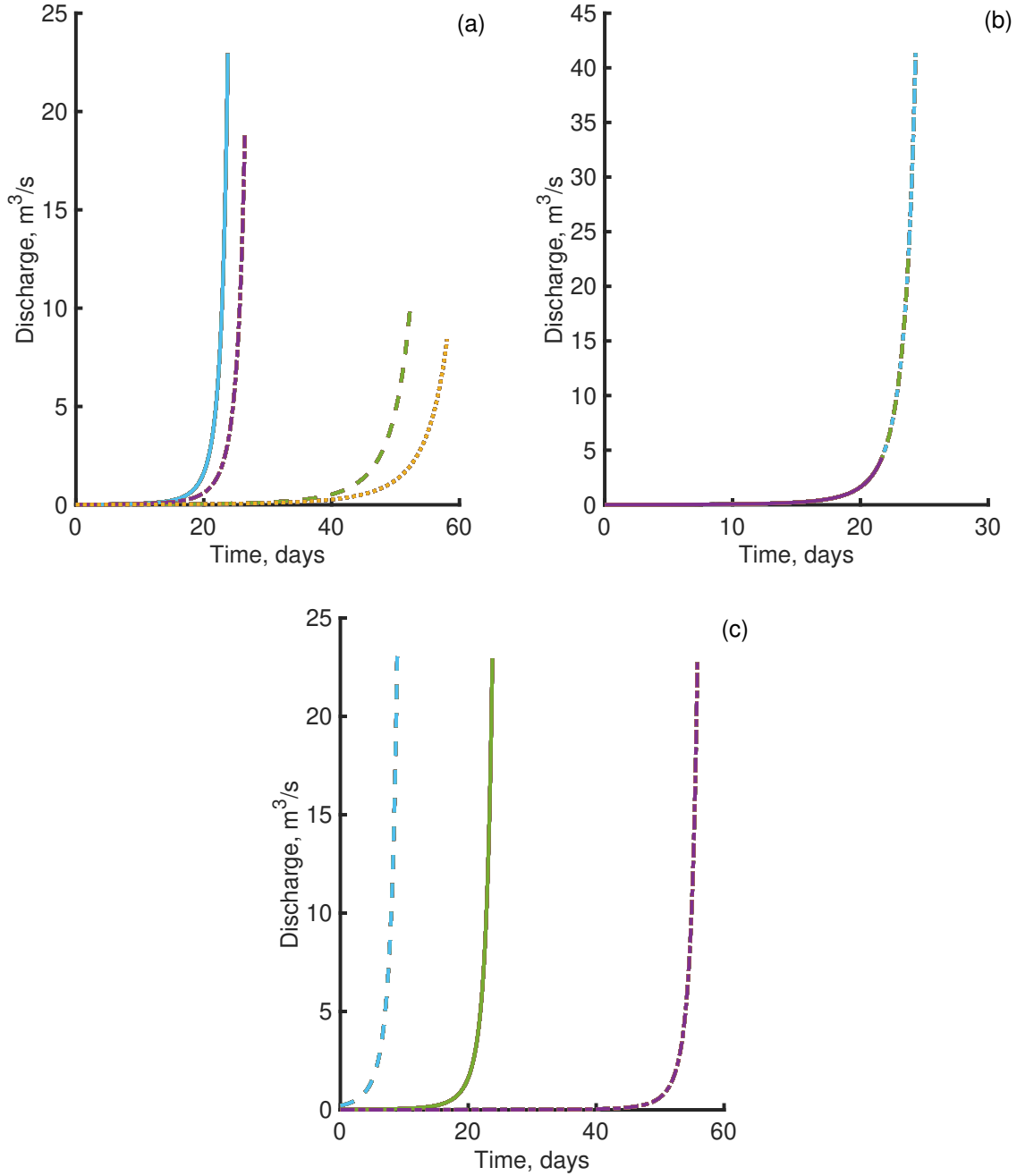


Figure 5.3: Discharge over time for different channel geometries, lake volumes, and initial cross-sectional areas. (a) A channel with semi-circular geometry for a lake with salinity 2.5 M is the green dashed line and for a lake with salinity 0 M is the yellow dotted line. A circular channel geometry for a lake with salinity 2.5 M is shown by the solid blue line and for a lake with salinity 0 M is given by the purple dashed-dotted line. (b) Lake volumes for $V_i = 1 \times 10^6 \text{ m}^3$ (blue dot-dashed line), $V_i = 5 \times 10^6 \text{ m}^3$ (dashed green line), and $V_i = 1 \times 10^7 \text{ m}^3$ (solid purple line) are plotted over top one another for salinity of 2.5 M. (c) The initial cross-sectional area is varied such that the blue dashed line is $S = 0.15 \text{ m}^2$, the green solid line is $S = 0.015 \text{ m}^2$, and the purple dot-dashed line is $S = 0.0015 \text{ m}^2$.

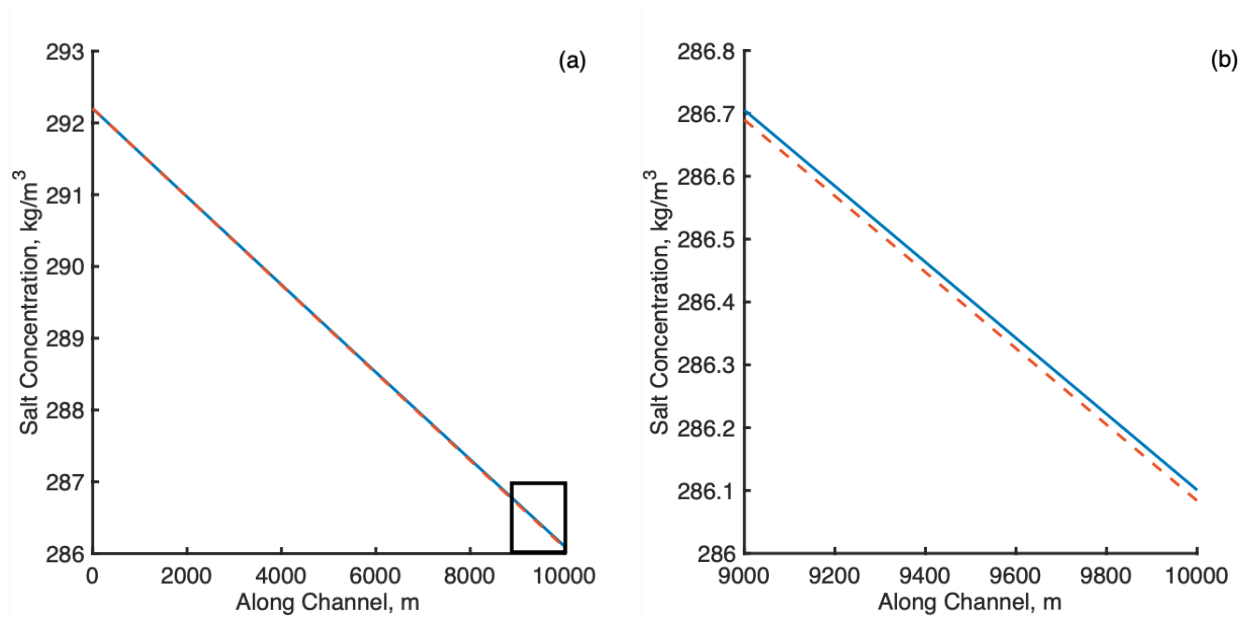


Figure 5.4: Along channel profile of salt concentration with constant brine properties (blue solid line) and properties as a function of changing salinity (red dashed line) as the brine is diluted for brine with salinity 5 M at day 20. (a) Full channel is shown. (b) The portion of the channel referenced by the black box in (a) is enlarged to highlight the difference.

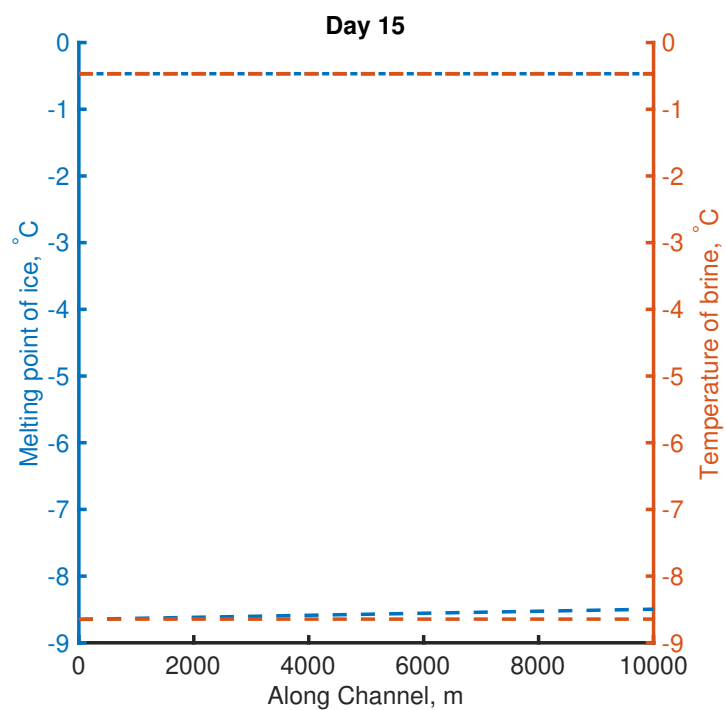


Figure 5.5: The salinity and pressure-dependent melting point of ice and the temperature of the fluid for the scenario with no salinity (dashed-dotted lines) and the scenario with a salinity of 2.5 M (dotted lines) under 700 m of ice.

with no perturbation has an initial cross-sectional area for the entire length of the channel, $S = 0.015 \text{ m}^2$. Two simulations with positive and negative perturbations of initial cross-sectional area are run. For these positive and negative perturbations, the channel between 1000 m and 1750 m has an initial channel cross-sectional area of $S = 0.0225 \text{ m}^2$ and $S = 0.0075 \text{ m}^2$, respectively (Fig. 5.6a). Early in the simulation (day 5), the cross-sectional area away from the perturbed section is almost constant for all three simulations, but the negative perturbation simulation is slightly less along the channel. The cross-sectional area inside the perturbed part of the channel grows more rapidly for the positive than the negative perturbation (Fig. 5.6b). At day 10, the same pattern continues except the difference between the perturbed section and the rest of the channel grows smaller (Fig. 5.6c). Towards the end of the simulation (day 15), the cross-sectional area linearly increases for the entire length of the channel and there is almost no distinction between where the initial cross-sectional area was perturbed and the rest of the channel (Fig. 5.6d). With the prescribed parameters, the model responds to a perturbation in the cross-sectional area differently if the initial channel is larger or smaller than the uniform channel. By the end of the simulations, the channel smooths along the length which suggests that over time, the channel returns to a linearly increasing cross-sectional area in response to a perturbation. Further exploration is needed to understand the robustness of this result and the implications this has for moving between channelized and distributed systems.

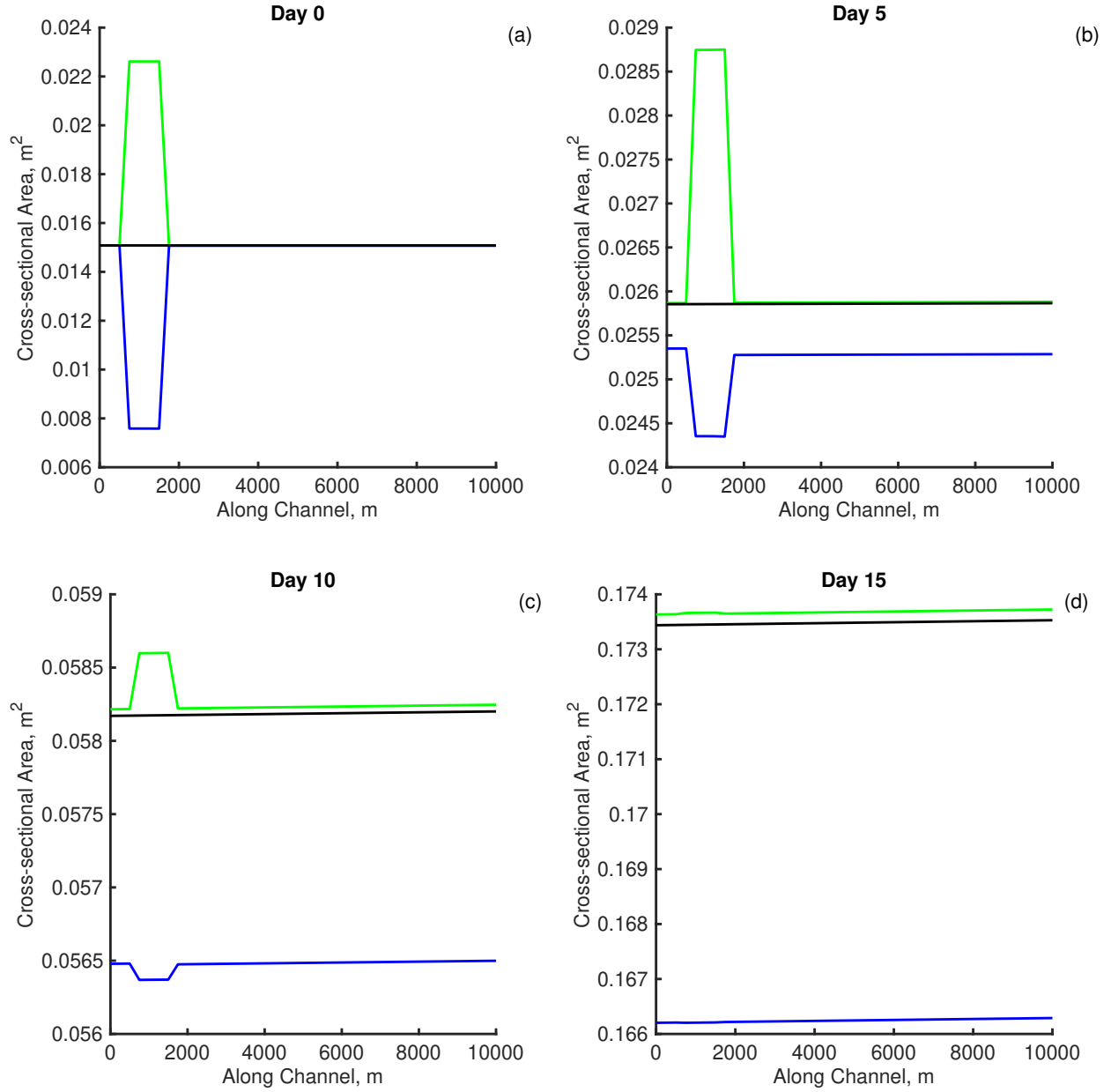


Figure 5.6: Time series of cross-sectional area along-channel profile with perturbations in the initial cross-sectional area. The simulation with the initial radius of $r \approx 7$ cm along channel is plotted with two simulations with the same initial cross-sectional area for most of the channel, but with perturbations in the initial cross-sectional area between 1000 – 1750 m where $r \approx 5$ cm (blue line) and $r \approx 8.5$ cm (green line) for the two simulations, respectively. The cross-sectional area along the channel for the three simulations are plotted at (a) day 0, (b) day 5, (c) day 10, and (d) day 15.

DISCUSSION

The results of this model suggest that the consideration of brine in relevant glacial systems may be important for understanding the dynamics of drainage through ice-walled channels (Fig. 5.1). The impact of brine is greater for larger cross-sectional areas and higher discharge rates (Fig. 5.2). This implies for systems with lower fluid velocities, the consideration of brine may be less important. However, the feedback of melting when the cross-sectional area is greater indicates that for higher discharge and greater cross-sectional area, the consideration of brine may be necessary to fully capture the dynamics of the system when the fluid is saline.

These considerations are necessary for modeling systems where the brine is present. In particular, if the hypersaline subglacial lakes at the bed of DIC contain brine and drain, then these modeling results may give insight to the drainage from this subglacial hydrological system. In Fig. 5.1 c, the channel height is greater for saline systems which implies the presence of brine increases the drainage from a subglacial system.

Blood Falls is an episodic saline outflow of the Taylor Glacier that drains into Lake Bonney, a hypersaline lake in the McMurdo Dry Valleys. The subglacial system that evolves to route the brine from the groundwater system to Lake Bonney is not yet understood. Some have hypothesized hydraulic fracturing as a mechanism of drainage while others have hypothesized injection of brine into crevasses [1, 2].

Along with the impact of fluid at the bed on ice flow, the flow of brine increases nutrient transport and has implications for microbial life [24]. In the case of Lake Bonney where the outflow is relatively accessible, understanding current brine outflow drainage mechanisms may provide insights to paleoclimates [21].

In addition to the DIC and Blood Falls, there are other examples of saline subglacial discharge. Some instances of this are found in polythermal systems where water is stored

subglacially beyond the melt season. Water stored in these glacier systems over the winter has been observed to accumulate more solutes. For example, subglacial discharge has been observed during the winter in areas of the Greenland Ice Sheet [31]. The geochemistry of these outflows contain solutes from the dissolution of minerals which suggests contact with bedrock/sediments. This implies that subglacial reservoirs that store water for longer than a summer melt season may change the geochemistry of water. In some locations where salts are present in the bedrock, water may become saline over time [35].

These processes may be occurring on a number of polythermal glaciers, but many of these glaciers are not observed in the winter season. Winter discharge is evident from some glaciers in the form of icings. Icings are ice structures formed by subglacial discharge or groundwater at the terminus of polythermal glaciers in subzero temperatures [22]. More research is needed on winter discharge and the implications for solute accumulation.

CONCLUSION

In this thesis, I have presented a subglacial hydrology model which includes the consideration of saline fluid. The results of this model suggest that brine increases the peak discharge and decreases the duration of time for a fixed volume to drain from a subglacial lake. The duration of drainage is most sensitive to initial cross-sectional area and channel geometry and peak discharge is most sensitive to lake volume and channel geometry. I make a number of simplifying assumptions in the model such as (i) consideration of heat transfer is neglected, (ii) the temperature of the brine is assumed to be constant along the channel, (iii) brine is not added along the channel, (iv) diffusion of salt is negligible, and (v) the ice thickness is constant along the channel, and (vi) the system is channelized. In this light, the model presented here should be viewed as an initial exploration of the impact of brine on the dynamics of a subglacial hydrological system. Additional modeling efforts are needed to include the heat transfer equation and provide a thorough sensitivity and stability analysis. Further research is required to understand the periodicity of drainage from hypersaline subglacial lakes and to understand the initiation of drainage in cold environments.

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APPENDIX A

EQUATIONS FOR NON-CONSTANT BRINE PROPERTIES

The ice-overburden pressure is given by $P_i = g\rho_i H$ with units $[\text{N m}^{-1}]$ where g is the acceleration due to gravity, $\rho_i = 917 \text{ kg m}^{-3}$ is the density of ice, and H is the ice thickness above the channel. Note for the following calculations, I convert P_i from N m^{-1} to bars.

The secant bulk modulus of fluid K is used to calculate density. As a function of salt concentration β with units [psu] and temperature θ_b with units $[\text{° C}]$ at atmospheric pressure, the secant bulk modulus from [15] is given by

$$\begin{aligned} K(\beta, \theta_b, 0) \approx & 19652.21 + 148.42\theta_b - 2.34\theta_b^2 + 1.36 * 10^{-2}\theta_b^3 \\ & - 5.16 * 10^{-5}\theta_b^4 + \beta(54.67 - 0.60\theta_b + 1.10 * 10^{-2} * \theta_b^7 - 6.17 * 10^{-5}\theta_b^3) + \\ & \beta^{3/2}(7.94 * 10^{-2} + 1.65 * 10^{-2}\theta_b - 5.30 * 10^{-4}\theta_b^2). \end{aligned} \quad (\text{A.1})$$

The secant bulk modulus as a function of salt concentration β , temperature θ_b , and ice-overburden pressure P_i from [15] is empirically calculated using

$$\begin{aligned} K(\beta, \theta_b, P_i) \approx & K(\beta, \theta_b, 0) + P_i(3.24 + 1.44 * 10^{-3}\theta_b + 1.16 * 10^{-4}\theta_b^2 - 5.78 * 10^{-7}\theta_b^3) \\ & + P_i\beta(2.28 * 10^{-3} - 1.10 * 10^{-5}\theta_b - 1.61 * 10^{-6}\theta_b^2) + 1.91 * 10^{-4}P_i\beta^{3/2} \\ & + P_i^2(8.51 * 10^{-5} - 6.12 * 10^{-6}\theta_b + 5.28 * 10^{-8}\theta_b^2) \\ & + P_i^2\beta(-9.93 * 10^{-7} + 2.08 * 10^{-8}\theta_b + 9.17 * 10^{-10}\theta_b^2). \end{aligned} \quad (\text{A.2})$$

The density of brine at atmospheric pressure as a function of salt concentration and temperature from [15] is given by,

$$\begin{aligned} \rho_b(\beta, \theta_b, 0) \approx & 999.84 + 6.79 * 10^{-2}\theta_b - 9.10 * 10^{-3}\theta_b^2 + 10^{-4}\theta_b^3 - 1.12 * 10^{-6}\theta_b^4 + \\ & 6.54 * 10^{-9}\theta_b^5 + \beta(0.82 - 4.09 * 10^{-3}\theta_b + 7.64 * 10^{-5}\theta_b^2 - 8.25 * 10^{-7}\theta_b^3 + \\ & 5.39 * 10^{-9}\theta_b^4) + \beta^{3/2}(-5.72 * 10^{-3} + 1.02 * 10^{-4}\theta_b - 1.65 * 10^{-6}\theta_b^2) + \\ & 4.83 * 10^{-4}\beta^2. \end{aligned} \quad (\text{A.3})$$

The density of the brine as a function of salt concentration β , temperature θ_b , and ice-overburden pressure P_i from [15] results in

$$\rho_b(\beta, \theta_b, P_i) = \frac{\rho_b(\beta, \theta_b, 0)}{1 - P_i/K(\beta, \theta_b, P_i)}. \quad (\text{A.4})$$

The specific heat of brine as a function of salt concentration and temperature based on empirical data from [15] is

$$\begin{aligned} \sigma_b(\beta, \theta_b) \approx & 4217.4 - 3.72\theta_b + 0.14\theta_b^2 - 2.65 * 10^{-3}\theta_b^3 + 2.09 * 10^{-5}\theta_b^4 \\ & + \beta(-7.64 + 0.11\theta_b - 1.3810^{-3}\theta_b^2) \\ & + \beta^{3/2}(0.18 - 4.08 * 10^{-3}\theta_b + 5.35 * 10^{-5}\theta_b^2) \end{aligned} \quad (\text{A.5})$$

where I assume that the change in specific heat due to pressure is negligible.

The thermal conductivity of brine as a function of temperature, and pressure from [3] is

$$\kappa_b(\theta_b, P_i) \approx 0.55 + 3.40 * 10^{-4}(P_i * 0.1) + 1.84 * 10^{-3}\theta_b - 3.31 * 10^{-7}\theta_b^3. \quad (\text{A.6})$$

The viscosity of the brine as a function of concentration and temperature from [12] is given by

$$\eta_b(\beta, \theta_b) \approx \eta_w(1 + (1.54 + 2.0 * 10^{-2}\theta_b - 9.52 * 10^{-5}\theta_b^2)(0.001\beta) + ((7.97 - 7.56 * 10^{-2}\theta_b + 4.72 * 10^{-4}\theta_b^2)(0.001\beta)^2). \quad (\text{A.7})$$

The Nusselt number as a function of salt concentration is

$$(Nu_b) = \frac{2r\hat{h}}{\kappa_b}, \quad (\text{A.8})$$

where $\hat{h}$ is the heat transfer coefficient and r is the radius of the channel.

The Prandtl number as a function of salt concentration is given by

$$(Pr_b) = \frac{\eta_b\sigma_b}{\kappa_b}. \quad (\text{A.9})$$

The Reynolds number as a function of salt concentration is

$$(Re_b) = \frac{2rv\rho_b}{\eta_b}, \quad (\text{A.10})$$

where v is the mean velocity of the fluid.