Assignment 4: Evaluation of LU with Partial Pivoting

Name:

Instructions. The write-up below is an AI-generated solution. Your task is to check the reasoning and accuracy of the solution. **This is independent work without devices unless otherwise instructed.**

Problem

Perform an LU decomposition **with partial pivoting** of

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Solution

Step 1. First pivot (column 1)

Column 1 is $(0,4,7)^\top$. The largest $|\cdot|$ is 7 in row 3. Swap $R_1 \leftrightarrow R_3$.

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A^{(1)} = P_1 A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 0 & 2 & 3 \end{pmatrix}.$$

Eliminate below the pivot $a_{11}^{(1)} = 7$:

$$m_{21} = \tfrac{4}{7}, \quad m_{31} = \tfrac{0}{7} = 0.$$

Row operations:

$$R_2 \leftarrow R_2 - \tfrac{4}{7} R_1, \qquad R_3 \leftarrow R_3 - 0 \cdot R_1.$$

This gives

$$A^{(2)} = \begin{pmatrix} 7 & 8 & 9 \\ 0 & \frac{3}{7} & \frac{6}{7} \\ 0 & 2 & 3 \end{pmatrix}.$$

(Record in L: $\ell_{21} = \frac{4}{7}$, $\ell_{31} = 0$.)

Step 2. Second pivot (column 2)

Consider the subcolumn $\left(\frac{3}{7}, 2\right)^\top$. The largest $|\cdot|$ is 2 in row 3. Swap $R_2 \leftrightarrow R_3$.

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^{(3)} = P_2 A^{(2)} = \begin{pmatrix} 7 & 8 & 9 \\ 0 & 2 & 3 \\ 0 & \frac{3}{7} & \frac{6}{7} \end{pmatrix}.$$

Eliminate below the new pivot $a_{22}^{(3)} = 2$:

$$m_{32} = \frac{\frac{3}{7}}{2} = \frac{3}{14}.$$

Row operation:

$$R_3 \leftarrow R_3 - \tfrac{3}{14} R_2 = \begin{pmatrix} 0 & \frac{3}{7} & \frac{6}{7} \end{pmatrix} - \tfrac{3}{14} \begin{pmatrix} 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{3}{14} \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 7 & 8 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{3}{14} \end{pmatrix}.$$

(Record in L: $\ell_{32} = \frac{3}{14}$.)

Step 3. Assemble P, L, U

Overall permutation:

$$P = P_2 P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{7} & 1 & 0 \\ 0 & \frac{3}{14} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 7 & 8 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{3}{14} \end{pmatrix}.$$

So $PA = LU$.

Question

Assess the accuracy of the solution and of the steps leading to the solution. What steps are correct (if any) and what are incorrect (if any)? If you suspect the final solution to be correct or incorrect, but cannot rigorously explain why, provide your intuition for partial credit.