



Selected topics in theoretical mathematics
by Deborah Gibson McAtee

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in
Mathematics

Montana State University

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Abstract:

Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact “math” and “mathematics” are as dissimilar as “spelling” and “creative writing” are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great mystery story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

And yet, in spite of all the excitement, people associate words like “boring”, “dry”, and “mechanical” with mathematics because they have never seen any of the beauty that the topic can display. They are led to believe that they can’t do mathematics because they had trouble with algebra or the multiplication tables— as if all great writers are perfect spellers. This thesis is an attempt to introduce people to some of the concepts of mathematics in a non-technical, and hopefully entertaining, way.

I have chosen some of the topics that I find the most interesting in classical mathematics (Cardinal and Ordinal Numbers, Geometry), as well as two new and flourishing fields (Fractals, Fuzzy Logic); I have also addressed the question “What is a Mathematical Proof?” In all of these essays, I have tried to present the ideas involved without getting side-tracked by the mechanics and technical details, and to show that mathematics really does involve ideas and concepts in much the same way that philosophy or theoretical physics does. This thesis is not intended to be a complete or technical dissertation, but rather an introduction to mathematics for the intelligent adult who is curious about what mathematicians do.

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

5/23/88
Date

Mary B
Chairperson, Graduate Committee

Approved for the Major Department

5/23/88
Date

KJ Fialant
Head, Major Department

Approved for the College of Graduate Studies

5/26/88
Date

Henry J Parsons
Graduate Dean

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ABSTRACT

Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact “math” and “mathematics” are as dissimilar as “spelling” and “creative writing” are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great mystery story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

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INTRODUCTION

Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact, “math” and “mathematics” are as dissimilar as “spelling” and “creative writing” are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

And yet, in spite of all the excitement, people associate words like “boring”, “dry”, and “mechanical” with mathematics because they have never seen any of the beauty that the topic can display. They are led to believe that they can’t do mathematics because they had trouble with algebra or the multiplication tables—as if all great writers are perfect spellers. Most people know much more mathematics than they realize and lack only the language in which to express it. For instance, everyone understands the basic idea behind different metrics: there are different ways to measure distance. Asked the distance to a favorite restaurant in town, almost everyone replies “Twelve blocks over and four up” instead of giving the distance “as the crow flies”; anyone who answers that way is using what mathematicians call the taxi-cab metric, while the distance as the crow flies is called the Euclidean metric. This is an example of an important topic in theo-

retical mathematics being used naturally in daily life, and it shows that no one is too stupid to understand at least some of the concepts involved.

Mathematics is rightly placed in the humanities, concerned as it is with ideas and concepts, but it also bridges the gap between the humanities and the sciences in its applications. Mathematics is at once brilliantly abstract and fundamentally applied; all sciences build on mathematics, but mathematics also soars above the applications to the high reaches inhabited by philosophy and art. Mathematics is much more than algebra or calculus; what follows is merely an introduction to some of the ideas to be found in the field.

FRACTALS

Fractals have attracted growing interest in the last couple of years, in part because computer graphics can produce fabulous full-color pictures of them. Fractals seem to be everywhere, from computer magazines to glossy books, from physics journals to economic conferences; the articles are almost always accompanied by illustrations and graphs, and frequently by full-color graphics designed to catch attention as much as to make a point. Fractals have even become an art form, with computers churning out ever more spectacular graphs whose appeal is aesthetic rather than scientific. But there is more to fractals than just pretty pictures.

Fractals are objects that have fractional dimensions, possess an attribute called self-similarity, and that are typically created by means of iteration. Fractals were so-named because of their fractional dimension, so let's look at this definition first.

Euclidean geometry teaches us that a point has zero dimension, a line has one dimension (length), a plane or flat surface has two (length and width), and space has three dimensions (length, width, and height). This seems simple enough—everything can easily be classified into one category or another—until we try to determine the dimension of a ball of string. From a great distance, the ball appears to be a point, and so has zero dimension; from closer, it appears as a ball and has three dimensions; closer yet, and the ball is seen to be made up of one long line of string and so must have one dimension; still closer, and the string can be seen to have width and height as well as length, and we're back to three dimensions.

So is the ball of string zero, one, or three dimensional? Obviously, it depends on how closely we look.

Now consider this line:

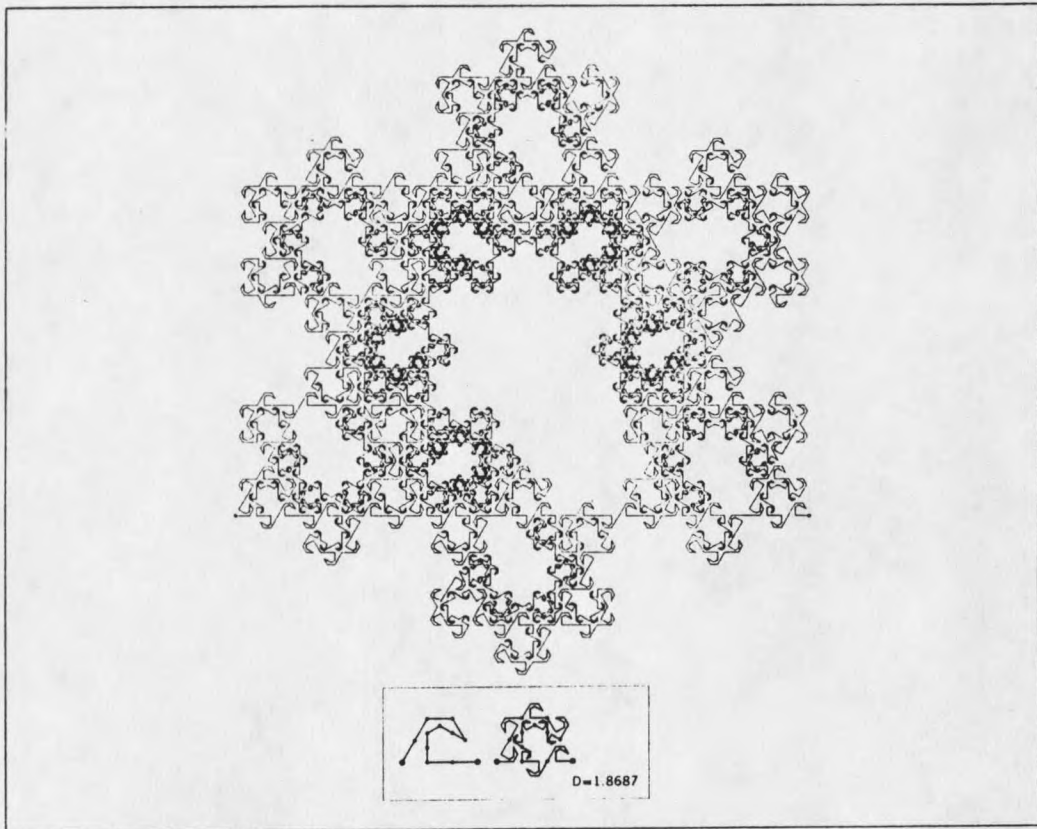


Figure 1. 1.87-dimensional line

The line itself is obviously one-dimensional, and lies in a two-dimensional plane. But in fractal dimensions, the line's dimension is between 1 and 2 (in fact, it is 1.8687) because, in Peter Sorenson's words, "... what follows will be an intuitive explanation rather than a technical one—the more complicated the wiggles get, the more the line's single dimension approaches the second dimension, until it could become infinitely wiggly, infinitely long, fill the plane, and be thoroughly

two-dimensional." In other words, the more the line fills the plane, the closer it gets to having the same dimension as the plane. The dimension is more than one but not yet two, so the line must have a dimension of one and a fraction—hence the name fractals.

Fractals can also have dimensions between zero and one or between two and three. Points that almost cover a line, such as something called the Cantor set, are an example of the former, and the radiator of your car provides an analogy for the latter. The sheet metal out of which your radiator was built was once a flat, two-dimensional surface; however, the manufacturing process folds it in such a way that it starts to fill the three-dimensional space in which it sits. So the radiator can be seen as having a dimension between the two of the plane and the three of the space.

Besides having fractional dimensions, most fractals have a characteristic known as self-similarity. Perhaps the best example from outside mathematics is Jonathan Swift's verse:

So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey;
And these have smaller Fleas to bite 'em;
And so proceed ad infinitum.

According to this verse, no matter how closely you look at the flea, you will see the same thing: fleas biting fleas. This idea, that the same thing appears at all scales, is called self-similarity. This similarity at different scales can be exact, with a precise duplication at each scale, as in Figure 2, or it can be a more approximate similarity. This latter type is usually called statistical or stochastic similarity, and

it means that views at different scales look “almost” alike, as in Figure 3; “stochastic’ is a learnedly elegant way of saying ‘random,’” as Paul Halmos commented.

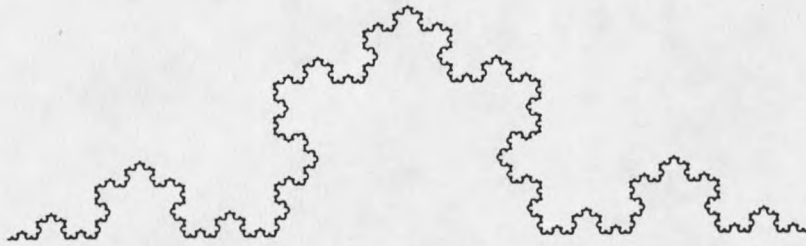


Figure 2. Koch snowflake

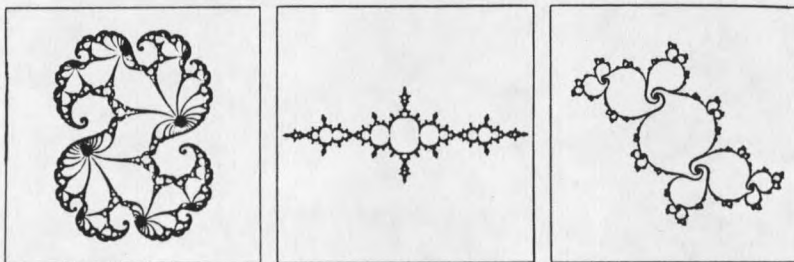


Figure 3. Julia sets

In Figure 2, it is easy to explain how to get the figure: Take a line and replace the middle third with an equilateral triangle;

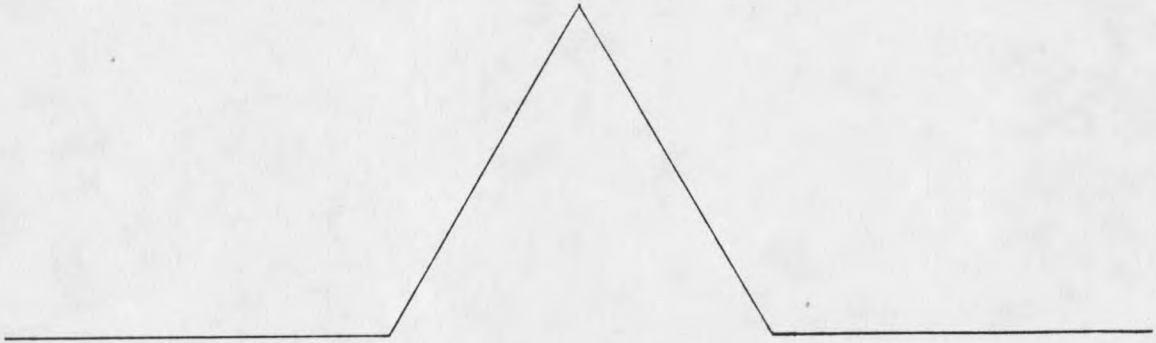


Figure 4. Koch snowflake

now take each straight line segment, remove the middle third, and replace it with a triangle;

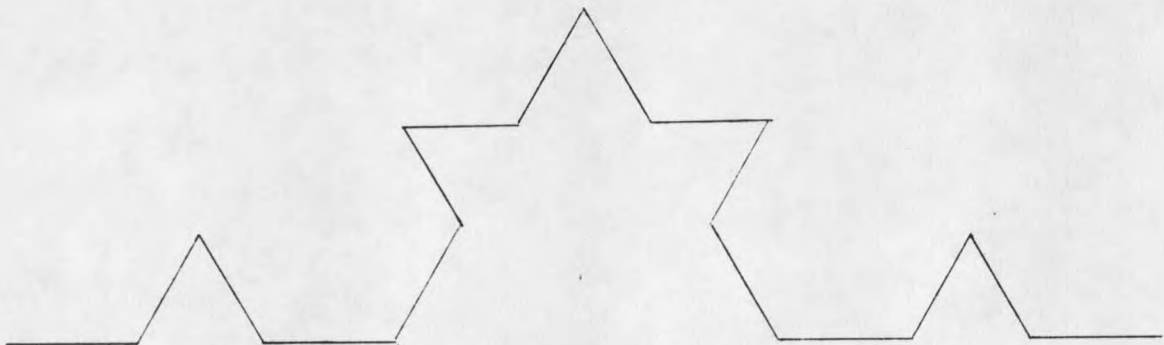


Figure 5. Koch snowflake

do the same thing again;

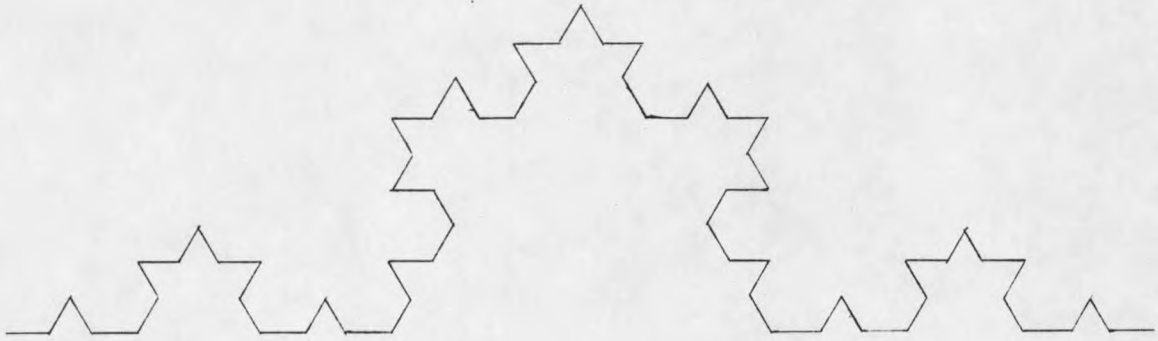


Figure 6. Koch snowflake

and so on. This process can go on as long as you like, limited only by your tolerance level for boredom. It takes a little work to translate these general directions into computer language, but once it's done, everything is known about the figure; there are no surprises. So it's not surprising that mathematicians quickly moved on to more intricate ways of drawing fractals.

The other figures are called Julia sets, after the French mathematician who discovered them, and are some of the more complicated sets in fractal mathematics. In order to have any chance of even realizing its complexity, we have to start with the idea of iteration, or repeated actions. In an iterated function, the output of each calculation becomes the input of the next one. For instance, let $f(x) = x^2 + 1$ —that is, take a number, square it, and add 1—and let's start with 1 as our first number. Then squaring 1 and adding 1 gives us 2; now take 2 as the

number, square it and add 1, and we get 5; now take 5, and ... In mathematical notation, it looks like this:

$$f(1) = 1 + 1 = 2$$

$$f(2) = 4 + 1 = 5$$

$$f(5) = 25 + 1 = 26$$

...

or, on the number line, like this:

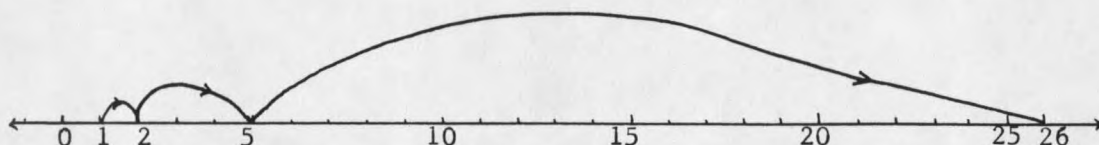


Figure 7. $f(x) = x^2$

with each new result getting farther from 0, or the origin.

While it might seem that new results always move away from the origin, we see something different if we let our function be $f = x^2$ and our first number be .25. Squaring .25 gives us .0625, which is closer to 0. Using this function, $f(x) = x^2$, some iterations will move away from 0 and others will move towards it; if the first number in the iteration is greater than 1, the iterations will move it farther from the origin, while if the first number is less than 1, the iterations will move closer to the origin. If the first number is 1 or -1, the iterations stay the same distance, 1, from the origin, although the iteration starting with -1 will move to 1 after the first iteration.

So far we have been looking at what mathematicians call a map from the line to the line, which just means taking a point on the line, applying a function to it, and looking at where the result ends up on another line. So in the picture below,

the function $f(x) = x^2$ maps the points on the top line to the points indicated on the bottom line.

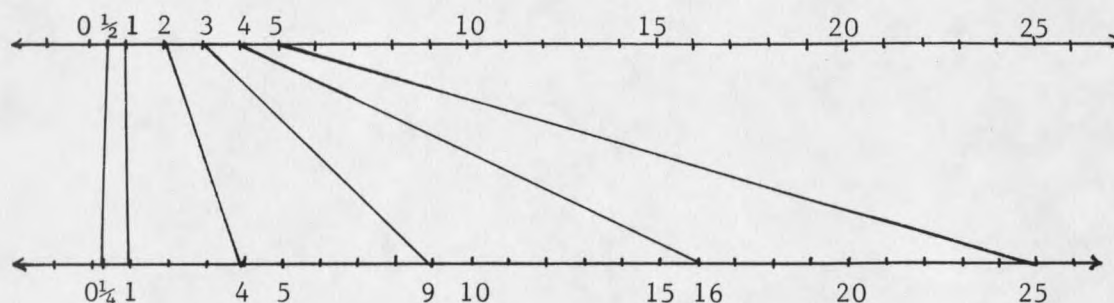


Figure 8. $f(x) = x^2$

Similarly, we can map from a plane to a plane using some function that takes a point on the plane and gives as a result another point on the plane. (Mappings can also be made from the plane to the line or vice-versa, but we needn't consider them here.)

Again using the squaring function, we see a familiar thing happening in the plane: some initial numbers give iterations that move away from the origin, others maintain a constant distance, and still others get closer to the origin. In computer-generated images of fractals such as the Julia set, colors are assigned to initial numbers according to how fast the iterations move away from the origin, with the iterations moving away from the origin the fastest colored red and the slowest colored violet. Iterations that do not move with respect to the distance from the origin are colored black; these are called repellers because other points appear to move away from them, and it is these points that make up the Julia set for our squaring function. For other functions, the repellers will be different, so the

Julia sets will be different for different functions. For instance, if the function is $f(x) = x^2$, any point that is one unit away from the origin will not move closer to or away from the origin, although it will probably move on the circle with a radius of one and its center at the origin; its Julia set looks like this:

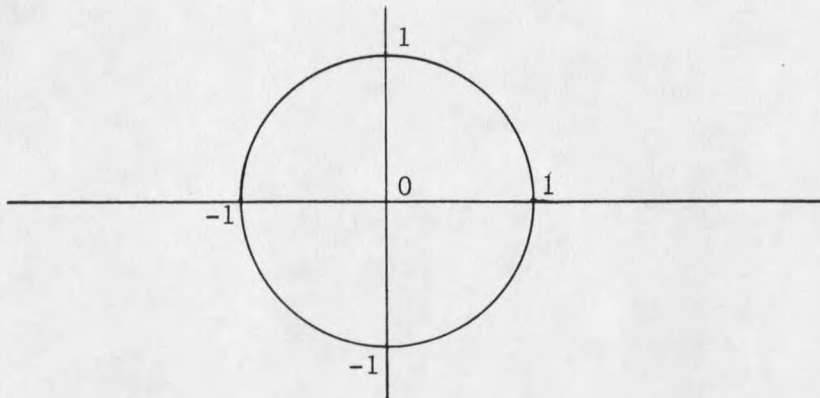


Figure 9. Julia set of $f(x) = x^2$

In this case, the ring of black dots that forms the Julia set is connected—that is, all the dots make one shape, with none floating around by themselves. For other functions, the Julia set is not so nicely connected, and the dots may form two or more shapes, or the dots may be completely scattered. It was in trying to classify which functions give rise to connected Julia sets that Benoit Mandelbrot created the Mandelbrot set, which is the set of all points in the plane that, when added to x^2 , result in a connected Julia set.

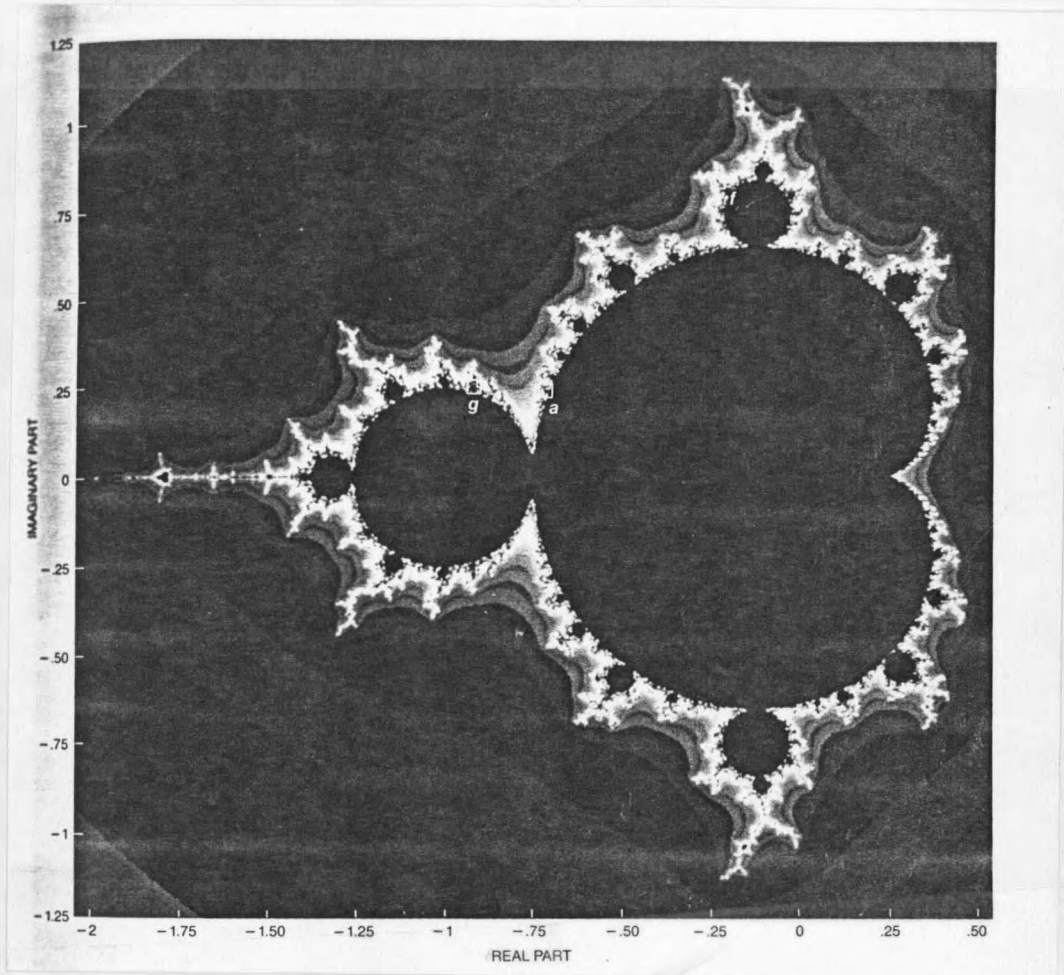


Figure 10. Mandelbrot set

This set is stochastically self-similar, and generates some of the most beautiful patterns in mathematical graphics. Each detail of the whole set is in its own right a complicated set, which has equally complicated subsets.

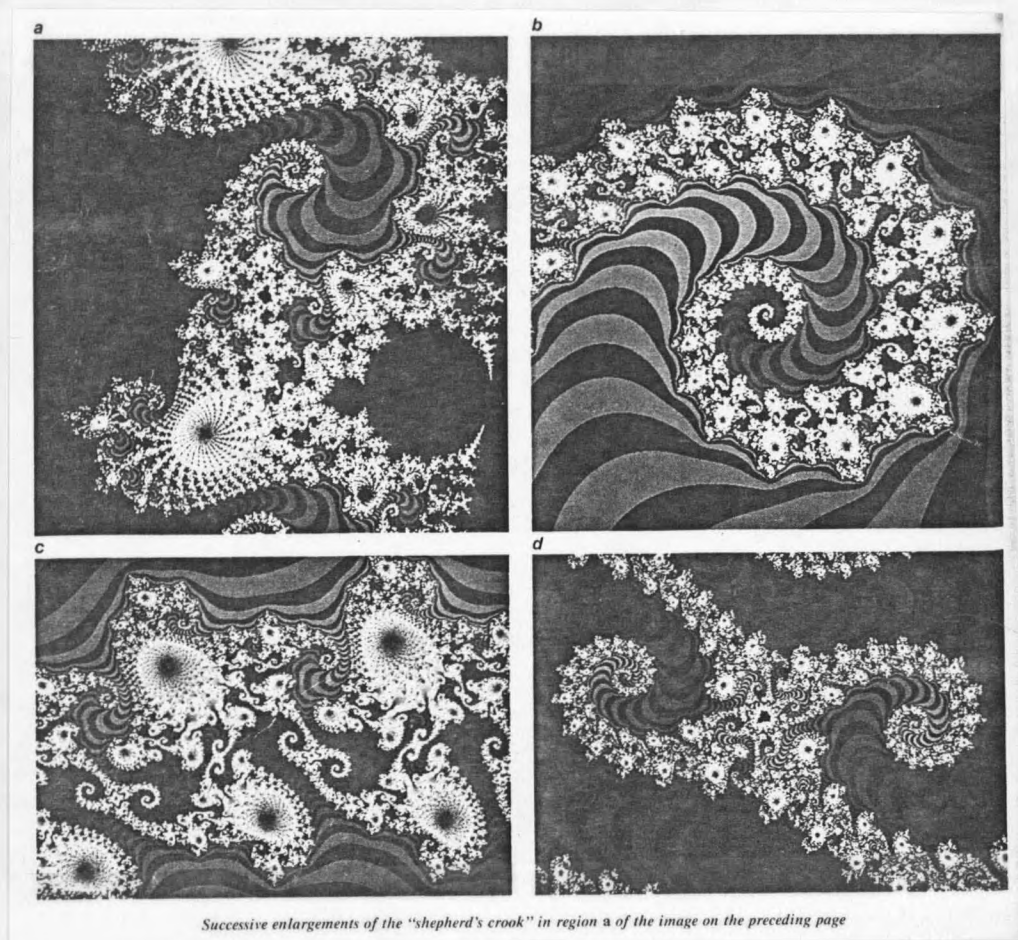


Figure 11. Details of the Mandelbrot set

These pictures show that mathematics really can be intriguing, beautiful, and even elegant (although elegance is what mathematicians see first). Sorenson found that, "When you talk to mathematicians and you ask them what it is about mathematics that interests them they say 'Well, mathematics is beautiful.'... And the thing about computer graphics [of fractals] is that you can actually... see that they really are beautiful." If fractals get people interested in mathematics, and if they show some of the beauty of higher mathematics, then fractals have done a great service to the mathematical community.

The problem with fractals, as far as mathematicians are concerned, is that there are as yet no rigorous, or even non-rigorous, proofs of many of the things that are claimed for fractals, both about their mathematical nature and about their applicability; there isn't even a good, generally accepted definition of what a fractal is. Mathematicians have been trained to deal with things that can be proven, whether or not they are appealing, so fractals remain, for many in the mathematical community, primarily a curiosity; they are an interesting sidelight of an important field of mathematics known as dynamical systems, which studies new techniques for understanding models of complex physical phenomena. As more facts are discovered about fractals, they may well develop into a mathematical field of their own.

On the other hand, scientists have little need of rigorous mathematical proof if something works, and they quickly saw in fractals a combination of regularity and complexity that had the potential to be a useful model for complicated physical phenomena. The visual attractiveness of the Mandelbrot set, and other fractal patterns in full color, has contributed greatly to the appeal fractals have for many scientists; the graphics engage their sense of sight in a way that mathematics rarely does, allowing them to see what a phenomenon looks like. The graphics also allow scientists to consider huge amounts of information that would be impossible to make sense of in numerical form, and to see patterns in the information that are not readily apparent when the data is in tabular form.

Fractals have been applied to things as diverse as galactic clusters and protein molecules; other applications are turbulence in fluids, population rhythms, rainfall patterns and long-range weather forecasting, the Earth's surface, Jupiter's Red Spot, fault zones and earthquake patterns, swamp vegetation patterns, the way a flag flaps in a steady wind, and the shape and distribution of clouds. The fractal

models are useful, according to Ivars Peterson, because "some problems become very, very simple if you look at them in the right way. Now that fractals have come along, some things that were very difficult become easy. It gives you a language." That language has spawned hundreds of pages of articles, journals, and even books on fractals and the related topics of chaos and nonlinear systems.

One application of fractals that many people have seen, although they probably didn't realize it, is in the movies. Earth's landscape can be convincingly mimicked by computers using a fractal dimension of about 2.3; by changing the fractal dimension just a little, say to 2.4, landscapes can be produced that look realistically earthlike and yet a little too jagged to be quite familiar. In fact, George Lucas started a company named Pixar that does just that: the landscape of the Genesis planet in "Star Trek II: The Wrath of Khan" was created using fractals, as were the outlines of the Death Star in "Return of the Jedi".

Fractals, and the allied fields of chaos and nonlinear systems, appear to hold great promise for many areas of science. Sometimes they explain physical phenomena, such as why accurate long-term weather forecasting is impossible; more frequently, they allow analysts to see patterns in data that might not otherwise be evident. Once a scientist sees a pattern, he has a clue to follow, to explore; by providing a way to see new patterns, fractals give scientists a way to approach old problems and gain new insights into them. Fractals are also bringing together many disparate sciences and giving rise to a new field of interdisciplinary study which draws practitioners from biology, meteorology, medicine, and geology, as well as from physics and mathematics. If fractals' contribution were to be only the breaking down of the high walls between disciplines, they would already have provided a major advance; if they also expand our understanding of the world,

they will have fulfilled the promise of the glorious computer graphics which have attracted the interest of so many people.

GEOMETRY: EUCLID AND BEYOND

We now talk easily about fractal geometry, but for centuries, geometry meant only Euclidean geometry; not only was there only the one geometry, it was also completely unthinkable that more than one might exist. To Kant, the great philosopher, Euclidean geometry was inherent in our brains; if the very structure of our brains was Euclidean, how could we even conceive of any other geometries? It wasn't until 2100 years after Euclid wrote *The Elements* that his geometry had to be labeled "Euclidean" in order to distinguish it from the new geometries being developed.

Sometime around 300 B.C., Euclid codified what was then known about geometry in his massive work, *The Elements*. Although a large proportion of the work is not originally his, he organized what was known in such a way that all of the theorems, or "true statements", followed from his premises, which he called axioms or postulates (he made a subtle difference between the two terms—axioms stated facts from mathematics in general, postulates from geometry in particular—but today the terms are used interchangeably). These axioms were to state self-evident truths that all men would accept without argument; from this solid ground, all of the theorems of geometry were to be developed using logical rules to get from the axioms to the theorems. Although considerably modified over the years, this structure is still considered to characterize "Mathematics" by most people.

At the base of this grand edifice, Euclid placed five postulates, which are self-evident truths about geometry. The first four do, in fact, seem self-evident, and the Greeks and those who followed were happy to accept them as Euclid wrote

them; they are:

- 1) Between any two distinct points, there is exactly one straight line.
- 2) Every straight line can be continued indefinitely.
- 3) Given a point and a radius, a circle can always be drawn.
- 4) All right angles (90 degrees) are equal.

All of these statements seem to follow directly from our experience and are easily accepted as true. (Try drawing more than one straight line through two points on a flat sheet of paper, if you have any doubts.)

Unfortunately, Euclid's fifth postulate had neither the short, simple form nor the self-evident truth of the other postulates, and even during his time, mathematicians were trying to simplify it. The fifth postulate is:

- 5) If two straight lines lying in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

This is hardly immediately evident as true. The situation looks like this:

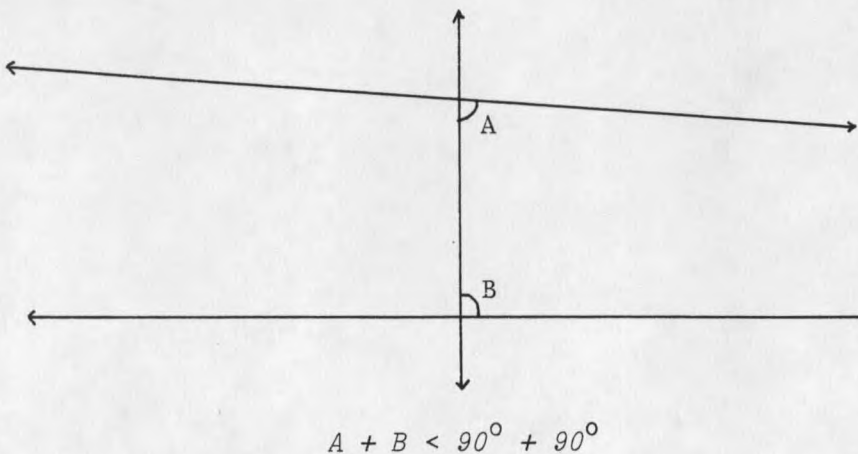


Figure 12. Euclid's fifth postulate

where the lines to the right are said to meet. The most common version today of this postulate is called Playfair's Axiom and says:

- 5) Given a line and a point not on a line, there is only one line through the point parallel to the given line.

While this version is easier to understand, it still lacks simplicity and immediately evident truth.

Euclid didn't like this postulate either, and put off using it as long as he could, first proving everything that could be proved without it. Other mathematicians attempted to prove it from the rest of the postulates and axioms, turning it into a theorem. The most "successful" tactic consisted of defining parallel lines in such a way that the postulate was assumed by the definition; after that was done, the postulate followed from the rest of the axioms and definitions (because it was already assumed). Down through the centuries, most of the energy directed at geometry went into trying to rid Euclid's postulates of this complicated monster.

One of the mathematicians who tried to prove the fifth postulate was Girolamo Saccheri, an Italian who lived from 1667 to 1733. The method that he used is known as proof by contradiction or "reductio ad absurdum" (literally, reduction to an absurdity), in which you deny an assumption and show that doing so results in a contradiction—therefore, the assumption must be true. If denying the fifth postulate resulted in a contradiction, then the postulate would be proven using the other axioms and would no longer be a problem.

Saccheri used an equivalent form of Euclid's fifth postulate called the hypothesis of the right angle (two statements are equivalent if, given either one, the other one can be derived from it). If Euclid's fifth postulate is true, then, given a quadrilateral $ABCD$, where AC and BD are the same length and A and

B are right angles, as in the following picture, then C and D will also be right angles.

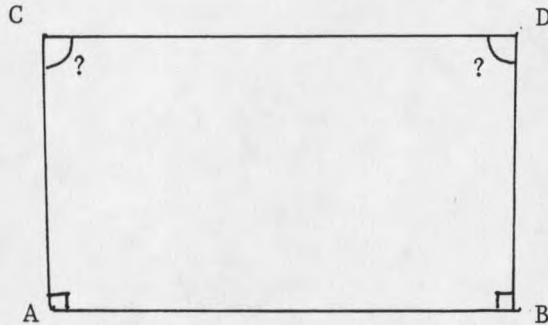


Figure 13. Saccheri's quadrilateral

However, if you don't assume the fifth postulate (and therefore the plane as the model for your geometry), then there are two other alternatives: the corners at the top of the figure could be greater than right angles (obtuse angles) or less than right angles (acute angles). Saccheri looked at what happened in each of the latter two cases; if they both led to contradictions, then the corners must be right angles (since an angle is either less than, equal to, or greater than a right angle—there are no other alternatives), and the fifth postulate must be true.

The first alternative, that the top corners are obtuse angles, leads to a contradiction with the postulate that a line can be extended indefinitely, and so Saccheri discarded it quickly. The other case, that the top corners are acute angles, leads to some odd results, results that seem to go against all common sense, but there is no true contradiction with Euclid's postulates. However, since geometry was regarded as the truth about the world, and since Saccheri was predisposed to

discard this second hypothesis also, these anomalies were sufficient to lead him to believe that he had discovered a contradiction, and to discard the second alternative also. Thus, as far as Saccheri was concerned, the hypothesis of the right angle, and consequently Euclid's fifth postulate, must be true; in fact, the title of the work that he wrote setting all this out was "Euclid Vindicated of All Fault".

It was not until the nineteenth century that mathematicians were ready to examine the anomalies that led Saccheri to discard the second case, and to develop new geometries. In the first half of the century, three great mathematicians—Gauss, Lobachevsky, and Bolyai—working independently, discovered that a consistent geometry could be developed without Euclid's fifth postulate. Lobachevsky was the first to publish his results and developed the ideas further than the others, and so this geometry, the first non-Euclidean geometry, is frequently known as Lobachevskian geometry.

In Lobachevskian geometry, which corresponds to the acute angle hypothesis that Saccheri discarded, the fifth postulate becomes:

- 5) Given a line and a point not on the line, there is more than one line through the point parallel to the given line.

The plane is no longer a model for this geometry; the most easily visualized surface that satisfies this postulate is called a pseudosphere, and looks like two infinitely long trumpets glued together at the bell ends.

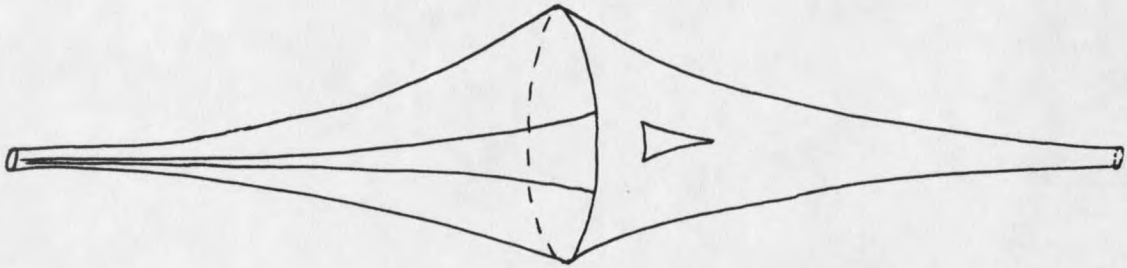


Figure 14. Pseudosphere

- Unfortunately, even on this surface, it is still difficult to visualize exactly what is happening with the parallel lines. The gist of the idea is that if the given line is AB and the point is C , then in the picture below, CD and CE are both parallel to AB , in the sense that neither CD nor CE will ever meet AB . (Of course, in the plane, this isn't true.)

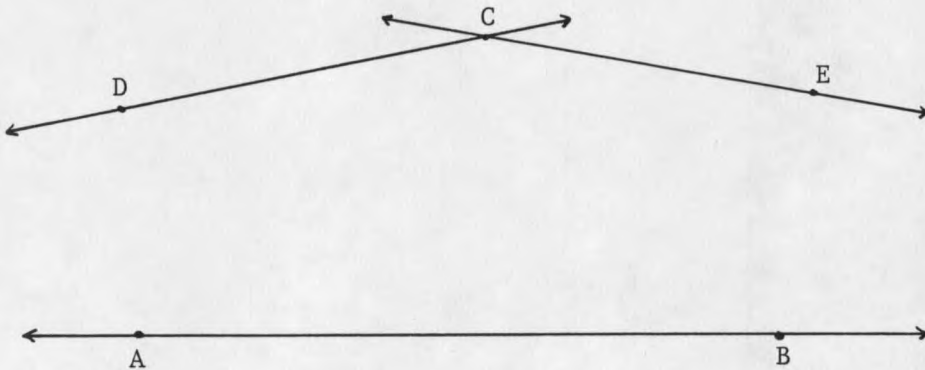


Figure 15. Parallel lines

Considering the difficulties of visualizing this, it isn't surprising that Saccheri and other mathematicians over the centuries rejected it as contradictory.

Once Euclid had been successfully defied, the door was open for all sorts of other geometries. The next non-Euclidean geometry to be developed is called Riemannian, or elliptical, geometry after the mathematician who was instrumental in creating it. In this geometry, corresponding to Saccheri's hypothesis of the obtuse angle, Playfair's Axiom becomes:

- 5) Given a line and a point not on the line, there are no lines through the point that are parallel to the line.

In addition, the second postulate, that a line can be extended indefinitely, is denied. It is in some ways surprising that this geometry was developed later than Lobachevskian, because mariners had been using it for centuries; Riemannian geometry is the geometry of great-circle sailing. For sailors, Euclidean geometry did not embody the one and only truth, because the ocean does not act as a flat plane, the Euclidean surface, except for very short voyages; a sailor navigating from the New World to the Old using Euclidean geometry would have taken a much longer route than was necessary, and he would probably have run out of food and water along the way. The difference between the two geometries can be easily seen on a map which has air or shipping routes drawn on it; on a flat map, these lines look curved rather than straight, but drawn on the globe, they are easily seen to be the shortest lines between two points.

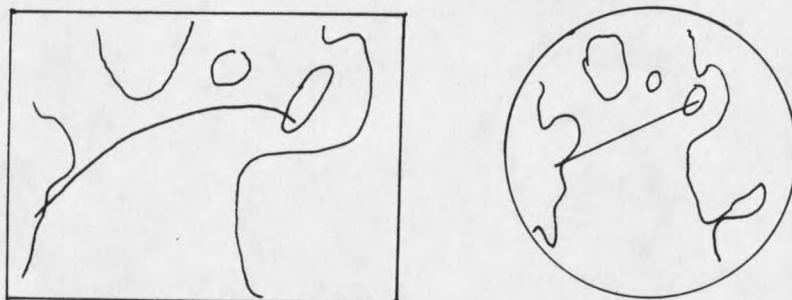


Figure 16. Maps

While Riemannian geometry has more than one model, the one that makes the most sense to us is a sphere like the globe. On the globe, the “lines” are great circles like the Equator and the lines of longitude; these lines can’t be extended indefinitely because they start repeating themselves. Calling great circles “lines” is not just an arbitrary definition; they correspond to straight lines in a very real way. On a flat plane, a straight line is the shortest route between two points; in fact, that is one way to define “straight line”. If we now let our surface be a sphere, then we can find our new lines by figuring out the shortest route between any two points; one way to do it is to stretch a piece of thread from one point to the other so that it is as taut as possible (without digging into the sphere). If this is done, the thread will trace part of a great circle; that is, it will follow the edge of a disk which goes through the two points and the center of the globe. This shortest route is called a geodesic, which is the concept generally used to define “lines” on surfaces other than a plane. Since great circles are geodesics on the surface of a sphere, they can legitimately be considered the counterpart of straight lines on a flat plane.

Using the definition of parallel lines that says that two lines are parallel if they never meet, it is easy to see that on a sphere there are no parallel lines. At the Equator, lines of longitude appear to be parallel, but they all meet at the Poles; the same thing happens for any pair of apparently parallel lines you choose.

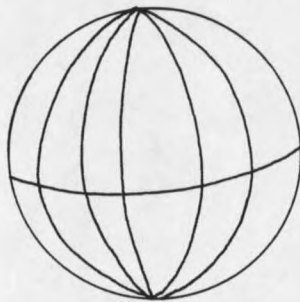


Figure 17. Sphere

Looking back at Saccheri's hypothesis, we can see that we can build his figure such that the top corners are obtuse angles, by letting the base be a great circle (the Equator, for instance) and the sides be parts of great circles that are perpendicular to the first one (lines of longitude). If we go up the sides the same distance and connect them with a "straight line", another great circle, we get obtuse angles at the top corners.

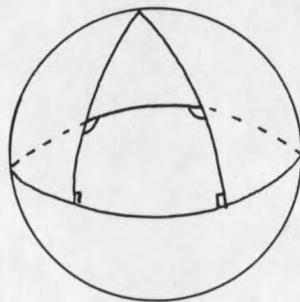


Figure 18. Quadrilateral with obtuse corners

This difference from Euclidean geometry shows up in many other places in spherical geometry. For instance, "everyone knows" that the sum of the angles of a triangle is 180 degrees, or, in Euclid's terms, equal to two right angles; on the sphere, the sum of the angles is always greater than 180° and can get as large as 360° .

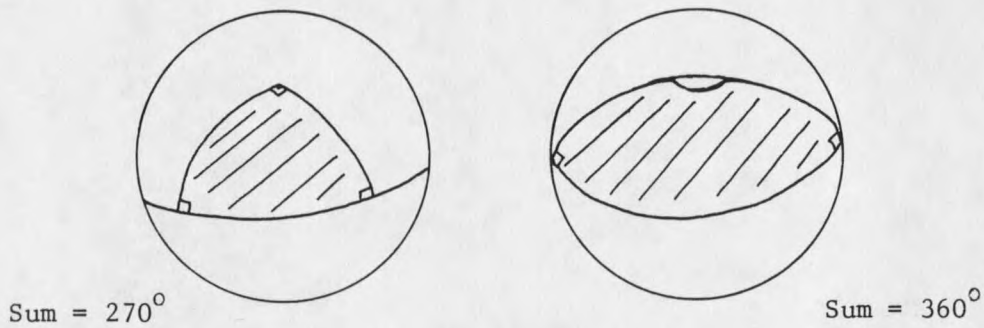


Figure 19. Spherical triangles

Using this fact, Gauss tried to determine which geometry is a better model of our universe by measuring a large triangle and determining whether the sum of the angles was less than 180° (Lobachevskian), equal to 180° (Euclidean), or greater than 180° (Riemannian). He chose three mountain peaks, located a good distance from each other, which formed a triangle, and measured the angles of the triangle. Unfortunately, while he did get a slight deviation from 180° , it was less than the experimental error and gave no information as to which geometry was correct.

Lobachevskian and Riemannian geometries are the two main challengers to Euclidean geometry, but a whole series of other geometries have been produced since mathematicians realized that they could create geometries rather than accept Euclid's ideal. Most of these different versions change an axiom slightly and then

see what happens to the theorems. And most of them are of interest only to mathematicians working in the field.

Although Lobachevski, Gauss, and Bolyai contributed a new geometry capable of explaining things that Euclid's cannot, their major contribution was to make mathematicians realize that Euclid's geometry was not the eternal truth handed down to mathematicians by a high priest. For decades, mathematicians resisted these strange new geometries, hoping to find some flaw in them that would once again vindicate Euclid from all fault. But when it was once proven that these new geometries were exactly as valid as Euclid's mathematicians began to realize that there is not one true body of mathematics, valid for all time and every purpose. Mathematics had been regarded as the perfect plan for the universe, laid down for all time by God, but now men were creating valid geometries that were heretical, that defied all common sense. Mathematics could no longer be regarded as a glorious search for *The Truth*.

If mathematics was no longer the truth behind the universe, the question immediately arose: Why does mathematics work so well to predict the world? And the answer to this is still that no one really knows. The confusing thing is that, while Euclidean geometry explains many things, the new geometries are more useful in some areas such as relativity, optics, and wave propagation; in fact, some physicists believe that the universe may be Lobachevskian rather than Euclidean. Physicists consistently turn to mathematics for models of the universe; while Euclidean geometry can no longer be considered *The Truth* about the world, it still holds a place of honor when it comes to explaining our world.

PROOFS

Ever since Euclid wrote *The Elements* in the axiom–theorem–proof format, mathematics has been widely known to involve proofs—in fact, some people would say that mathematics *is* proofs—but people, mathematicians included, seldom stop to think about what they are. The standard mathematician’s answer to the question is “I know a proof when I see one,” or “In *principle*, a proof should be...”, but he can rarely explain, in general terms, what is required for an argument to become a working proof. Adding to this confusion is the fact that the concept of what is an acceptable proof changes from field to field within mathematics as well as over time; what Euclid considered a good proof is seldom accepted by modern mathematicians, and a proof that is accepted in one field or by one school of mathematicians may not satisfy others. Yet, somehow, out of all this confusion, there comes a sort of consensus: a proof is an argument that convinces your colleagues.

Mathematics was developed well before the Greeks, but they were the first to bring a sense of scepticism to mathematics, to question its results and to begin building a body of proven mathematics. Euclid is the most famous of these mathematicians who believed that even the simplest idea should be proven, and his attempt to do that for geometry, *The Elements*, has influenced mathematicians over the centuries. Over time, many of the Greeks’ axioms have been proven as theorems using a new set of more basic axioms, but mathematicians have to start building somewhere, and so they cannot get rid of the unproven axioms altogether. Modern mathematics still rests, for the most part, on the foundations that the Greeks built, on questioning and proof; over the years, the foundations

have been dug deeper, but the Greek contribution is still apparent to anyone who has ever been exposed to a high school geometry proof.

For centuries, Greek mathematics was the mathematics of Europe; the concepts were elaborated upon, but the foundations remained unquestioned. Mathematicians believed that God had created the world according to an orderly plan, one that must be mathematical; the amazing correspondence between mathematical predictions and real world observations convinced them that mathematics must be true, that it represented God's plan of the universe. In the 19th century, this attitude changed after the discovery of non-Euclidean geometries showed that there were fields of mathematics which didn't correspond to nature, and a movement arose which attempted to base all mathematics on logic, and to prove that mathematics was consistent rather than true. During this period, mathematicians attempted to prove rigorously many of the profound developments of the previous century, such as calculus. These developments had come from the intuition of a handful of mathematical geniuses, and it was felt that it was time to prove them so that the ideas could be more fully developed and used with confidence, as they could no longer be assumed correct simply because they corresponded to nature. The 19th century mathematicians were the first to clarify the idea of proof, and in their monumentally unreadable opus, *Principia Mathematica*, Russell and Whitehead attempted to reduce all mathematics to pure logic, to prove that all mathematical results can be derived from a few axioms using only accepted rules of inference or reasoning. The quest to prove that all mathematics is consistent died in 1930, when Kurt Godel proved that any mathematics which is complex enough to be interesting cannot be proven to be consistent. But in spite of the fact that mathematics cannot be proven as the one and only truth,

mathematicians continue to prove their work since proofs are the surest weapon they have against error.

The best description of what a proof is comes from G.H. Hardy's article "Mathematical Proof", where he considers a mathematician as

an observer, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. ... When he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he points to it, either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the proof is finished.

Thus, a proof is an argument that convinces the reader that the result is correct; a proof may be written differently for different readers, but "the proof reader must experience the same mental image that the prover had."*

From this general concept of a proof there come several levels of proofs. The most rigorous and sure proof is a Formal Proof, one that goes back to the very foundations of mathematics and builds up results until the desired theorem is proved; it is "a finite sequence of statements, each statement a consequence of axioms and preceding statements through allowable rules of inference." However, these formal proofs quickly get incredibly long and tedious when the desired result isn't elementary. Also, another desired attribute of a good proof is that it increase

* All unattributed quotations in this section are from McAtee, *Survey of Montana State University Mathematics Professors*.

understanding of why the theorem is true, and a formal proof seldom does this. For these reasons, formal proofs are almost never seen in higher mathematics.

The most common type of proof in print is the informal proof. Although an informal proof has similarities to the formal proof, in that it starts with an axiom and ends with a theorem, it is defined as an argument which is "acceptable to other mathematicians working in the field" and generally uses a different method to get from axiom to theorem: in addition to the axioms of the formal proof, any generally accepted previous result known to the mathematicians in the field is allowed. The level of informality of the proof depends on how "the field" is defined. If the field is taken to be mathematicians in general, then the proof is fairly formal; if the field is a small group, then the proof is fairly informal. Except for the most general of these proofs, a problem quickly arises, since "in most areas there seems to be a large body of 'well-known' facts which are simply stated without substantiation. Many of these may not be known (or at all obvious) to non-specialists. This always complicates reading proofs outside your specific area, even in apparently closely related areas," and so most proofs can be validated only by mathematicians working in the same field. As Davis and Hersh point out,

The actual situation is this. On the one side, we have real mathematics, with proofs which are established by "consensus of the qualified." A real proof is not checkable by a machine, or even by any mathematician not privy to... the mode of thought of the particular field of mathematics in which the proof is located. Even to the "qualified reader," there are normally differences of opinion as to whether a real proof... is complete or correct. These doubts are resolved by communication and explanation, never by transcribing the proof into [formal language]. Once a proof is "accepted," the results of the proof are regarded as true (with very high probability).

Thus, a proof is judged correct because mathematicians agree that it's correct.

The most informal proof is probably the most common, although it never appears in the journals: a proof which depends upon "techniques and results known to those working on precisely the problems in question; they are often communicated verbally and by diagrams and drawings." This type of proof is the everyday working material of the mathematician, allowing him to try new ideas and discuss them with others working on the same problems; when he arrives at a result that he wants to publish, he writes it up in a more formal style.

Proofs are important to mathematicians because they allow mathematicians to build upon previous results with confidence in the tools they are using. But there is a strongly felt aesthetic component involved in theoretical mathematics—"the real appeal of mathematics lies in the certainty of its results"—as well as a feeling that "proof is ritual, and a celebration of the power of pure reason," as Davis and Hersh point out. And yet, in spite of the vaunted certainty of mathematics, human error is always a factor in writing any proof. "Many 'proofs' (not most) turn out to be wrong, although the result is correct. . . . Ultimately, as far as [mathematical] Journals are concerned, proofs are subjective." This is why Davis and Hersh said that proofs are accepted as being true "with very high probability," rather than absolutely correct. Reason may be perfect, but humans aren't, and, in spite of the rumors, mathematicians are only human.

This problem of human error isn't a problem when a proof is short enough for other mathematicians to check and correct, but as proofs pass 400 pages, the problem gets worse quickly. Theoretically, any proof could be reduced to formal language and thereby be checked; practically, a 400 page proof would probably be well over 1000 pages if transcribed, and no one is likely to take on the task—even if they did, there is still the possibility of an error in the transcription. Currently, large proofs are frequently broken up into sections, each one of which is checked

by a separate person; however, this always leaves the possibility that there could be an assumption that makes sense in each section but is false when applied to the whole proof, making the proof incorrect even though each section is correct. Even in a narrow field of specialization, mathematicians can have trouble checking proofs for errors. Gina Bari Kolata tells of a case where “one investigator came up with a proof of a statement and another came up with proof of its negation. Both proofs were long and very complicated, hence the two investigators exchanged proofs to check each other’s work. Neither could find a mistake in his colleague’s proof.” A logical proof can theoretically be perfect, but the limitations of the human mind decree that this will only happen with short, simple proofs, or those that are repeated over and over in the classroom.

This ever-present possibility of error has opened up a whole new field of proofs: probabilistic proofs, or proofs by computer. Because the certainty of reason is so aesthetically pleasing to mathematicians, computer proofs strike many as an abomination to be avoided in real (i.e. theoretical) mathematics. And yet, computers can solve some complicated problems with “very high probability”—in fact, they can calculate the probability of an error in the proof, or solve a problem to within a predetermined range of error. Computer proofs are gaining a following because logical proofs are becoming so long and complicated that no one can check their validity; as proofs reach 400 pages and more, the probability of human error quickly becomes greater than the probability of computer error.

Another challenge to the preeminent position of rigorous proof in mathematics comes from the new fields of nonlinear studies, fractals and chaos. Mathematicians and scientists working in these fields use what they call experimental mathematics; rather than proving their results in the traditional way, they check their proposed solutions by running them on a computer and seeing if the result matches the

data, much the way a physicist "proves" his theories. This type of computer check hardly constitutes a proof, and yet it gives the mathematician insight into the nature of the problem on which he is working, just as a good proof does. And, as these mathematicians point out, the results can still be proven rigorously if they are ever in doubt.

A good computer proof, or even a computer check of a solution to a problem, certainly seems to satisfy the common definition of a proof: an argument that convinces your colleagues. It can hardly be less aesthetically pleasing than a 400 page proof that no one can understand, and the former will frequently have a higher probability of being right than the latter. As proofs become longer and more complicated, the definition of a proof may have to be widened to accept computer proofs and other methods of proving that something is "probably true" if mathematicians want to make the maximum progress rather than spend their time trying to check other people's proofs. Thus, the idea of a proof will, in all likelihood, grow to accommodate new and powerful methods in the continuing effort to arrive at the truth.

CARDINAL NUMBERS

Infinity! The word conjures up visions of vast reaches of outer space, of an uncountable number of stars stretching out forever. Infinity is the thing that cannot be counted, that cannot be delineated or defined; to pin it down would be to destroy its very essence. And yet, mathematicians have managed to find a way to pin it down enough so that they can work with it; rather than destroying the mystery of the infinite, the concepts involved have raised new questions and demonstrated how infinite infinity really is.

For a look at some of the unusual properties of infinity, Hilbert's Hotel is a good place to start. David Hilbert was a mathematician around the turn of the century who worked extensively with the developing ideas of infinity, and the Hotel was one model that he devised in order to demonstrate some of the properties of infinity. The unusual thing about this hotel is that it has an infinite number of rooms. If you go into an ordinary hotel and the manager tells you that it is full, you're not likely to get a room—unless, of course, you can bribe the manager. On the other hand, if you go to Hilbert's Hotel, you will always be able to get a room, without a bribe, even if the hotel was full before you arrived. The manager can accomplish this seemingly impossible task by moving the person in Room 1 into Room 2, the person in Room 2 into Room 3, and so on down the line, with the person in Room " N " being moved into Room " $N + 1$ ". This leaves Room 1 open for you; since there will always be a Room " $N + 1$ ", there are an infinite number of rooms and no one is left without a room. Obviously, if you arrive with a large party of, say, 100 people, the manager can find room for all of you by moving the person in Room 1 into Room 101, the person in Room 2 into Room 102, and so

on down the line. Again, this leaves the first 100 rooms open for your group and no one is left without a room.

Now, to take things a step further, suppose that you arrive at the hotel with an infinite number of people. Since anyone counting you would never finish, the manager can hardly move everyone already in the hotel over a certain, finite number of rooms in order to make room for your group. A different strategy is called for. This time, the manager moves the person in Room 1 into Room 2, the person in Room 2 into Room 4, ad nauseam, so that the person in Room " N " goes into Room " $2N$ ". Since there are an infinite number of even numbers, and thus even-numbered rooms, every guest already at the hotel has a room; there are an infinite number of odd-numbered rooms left vacant by all this shifting, and so everyone in your party gets a room.

The ability of the manager to fit an infinite number of people into his already full hotel demonstrates the property that is often considered to be the definition of an infinite set (a set with an infinite number of elements): the odd fact that an infinite set can be "equal" to a part of itself. This goes against our intuition, since everyone knows that the whole is greater than the part, and this was, in fact, the barrier that long prevented mathematicians from accepting what are now called transfinite numbers, numbers that give mathematicians a way to deal with the infinite. In a finite set, our intuition is correct: there is no way to pair up elements in a part of the set with the elements in the whole set in such a way that no elements from either side are left over. For instance, if the set is considered to be (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) and the part of the set, or subset, to be (1, 2, 3, 4, 5, 6), then there will be four elements from the set left over which cannot be paired with elements of the subset.

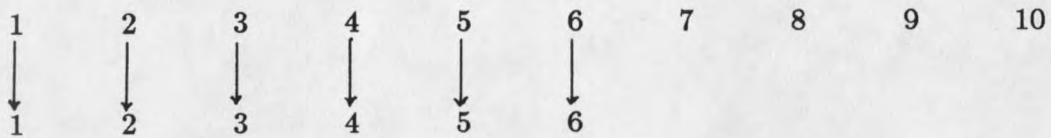


Figure 20. Set to subset map

In other words, there is no one-to-one correspondence between the elements of the part and the elements of the whole, since the set will always have more elements than the subset.

However, no one ever said that infinite sets would behave nicely, and things work quite differently among them. Infinite sets have the counterintuitive property that there can be a one-to-one correspondence between the elements of a set and a subset. For instance, every counting number can be paired with an even number in the same way that the hotel manager found an infinite number of empty rooms. If any counting number is multiplied by 2, the result is an even number; working in the other direction, if any even number is divided by 2, the result is a counting number. These facts give us the correspondence illustrated below.

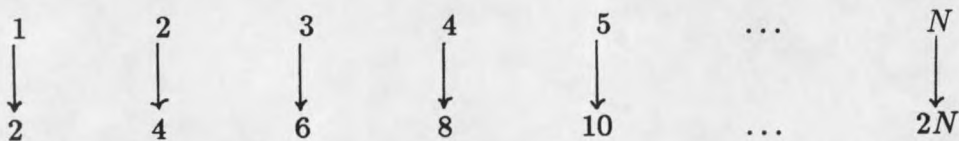


Figure 21. Set to subset map

In this correspondence, every counting number is paired with one even number and no even numbers are missed; this tells us that there are just as many even numbers as there are numbers both odd and even, or that the part is equal to the whole. This equality can be more easily seen by analogy: if you have a large

stadium full of seats and a large crowd of people, one easy way to tell if you have the same number of people and seats is to let everyone sit down in a seat. If, after the pandemonium dies down, there are empty seats left over, then you have more seats than people; if instead there are people standing in the aisles—no sitting on laps is permitted—then you have more people than seats. If no seats are left over and there are no people left standing, then you have the same number of people and seats; in this way, you can establish equality of two sets without having to do any tedious counting. Equality among infinite sets is determined the same way: if the elements of two sets can be paired off, with no elements of either set left over, then a one-to-one correspondence has been set up and the sets are said to be equivalent. In some sense, the sets have the same number of elements. In the example above, the counting numbers are equivalent to the even numbers because every counting number, N , can be paired with an even number, $2N$, and no even numbers will be skipped. All this means that infinity is so big that even after you take half of it away, it is still infinite.

The fact that an infinite set can be “equal” to one of its subsets was an idea that took a long time to be accepted by any but crackpots and mystics. Another barrier to the study of transfinite numbers was the concept of infinity that was handed down from Aristotle. He believed that there were two kinds of infinity, potential and actual. The potentially infinite is usually symbolized by ∞ , the “lazy eight” found in most mathematics and physics; it is the process of counting 1, 2, 3, 4, 5, 6... for an infinite length of time. Cantor called it the improper infinity and defined it as “a magnitude [number] which... increases above all limits...but always remains finite.” In calculus, ∞ is often read as “increasing without bound” or “increasing without end”; the potentially infinite is an ongoing process symbolized by ∞ rather than a number labeled by it.

The actually infinite can be thought of as a noun, an object that can be dealt with in the same way that a number is. It is a "fixed, constant magnitude lying beyond all finite magnitudes," and is represented by ω , which is read "little omega" and is the last letter of the Greek alphabet. It represents the number of positive integers, or the counting numbers, whereas ∞ represents the process of counting. At least for most of us, the actually infinite is outside of our daily experience and so is harder to accept than the potentially infinite; in fact, Aristotle believed that only the latter existed. Georg Cantor was the first person to believe that the actually infinite existed and to develop the idea into a system.

Georg Cantor was born in St. Petersburg, Russia, in 1845, but his family soon moved to Berlin, where he spent most of his life. At the university there, he studied under Weierstrass, one of the great mathematicians of the nineteenth century; it was there that he started the studies that were to lead him to the development of his theory of transfinite numbers. He worked on the theory periodically throughout the 1870's and 80's, and he wrote the major papers that carry the results of his work in 1895 and 1897. Reaction to transfinite numbers was immediate and highly divided. Some mathematicians saw its usefulness at once and in 1897, it was already being used in analysis; it soon spread to other fields of mathematics. Bertrand Russell, a prominent mathematician of the time, called it "probably the greatest [work] of which the age can boast." On the other side of what became, at times, a vicious debate were mathematicians such as Kronecker, who had once been one of Cantor's teachers. Since Kronecker did not even believe in the existence of irrational numbers such as $\sqrt{2}$, transfinite numbers were an abomination to be destroyed, and he led violent and sometimes personal attacks on Cantor. Other leading mathematicians were also critical of the theory; one called it "a fog on a fog" and dismissed it entirely. Although transfinite numbers

are currently recognized as useful and even necessary for some mathematics, there are still some mathematicians who reject them entirely and continue to work only with "real" numbers.

In developing his theory of transfinite numbers, Cantor made use of two properties of numbers: ordinality and cardinality. An ordinal number is one which counts off order, such as 1st, 2nd, 3rd, and so on; a cardinal number tells how many elements there are in a set: 1, 2, 3, etc. In a finite set, we find the cardinal number of the set by ordering the elements and choosing the largest ordinal number as the cardinal number; if the elements in a set can be ordered 1st, 2nd, 3rd, 4th, then the cardinal number of the set is 4—that is, there are 4 elements in it. This idea can be developed so that a set is finite only if its ordinal numbers produce its cardinal number. By now, it is probably not too surprising to find that infinite sets play by different rules; in fact, there are many infinite sets with the same cardinal number but different ordinal numbers.

The cardinal number of a set indicates how many elements there are in the set. For instance, the set of all eggs in a full egg carton has a cardinality of 12. Every finite number is a cardinal number, but the transfinite cardinal numbers have special names; they are designated by the first letter of the Hebrew alphabet, aleph or \aleph , with a subscript indicating where they belong in the hierarchy of transfinite cardinals. For instance, the first transfinite cardinal is \aleph_0 , read "aleph nought", and is defined to be the number of elements in the set containing all the counting numbers; in other words, if you have a set, $(1, 2, 3, 4, \dots)$, where the dots indicate "keep counting", then there are \aleph_0 elements in the set. There are many infinite sets that have a cardinality of \aleph_0 ; every set that can be put into a one-to-one correspondence with the counting numbers has the same number of elements as the counting numbers and therefore has the same cardinality, \aleph_0 .

We have already seen that the even numbers are equivalent to the counting numbers, and so the cardinality of the set of all even numbers is \aleph_0 . With a little more work, we can show that the set of all fractions has the same cardinality. To show this, we need to show that all the positive fractions can be counted, that they can be lined up in such a way that none of them will be missed. The trick, first used by Cantor, is to list all of them in a triangular format:

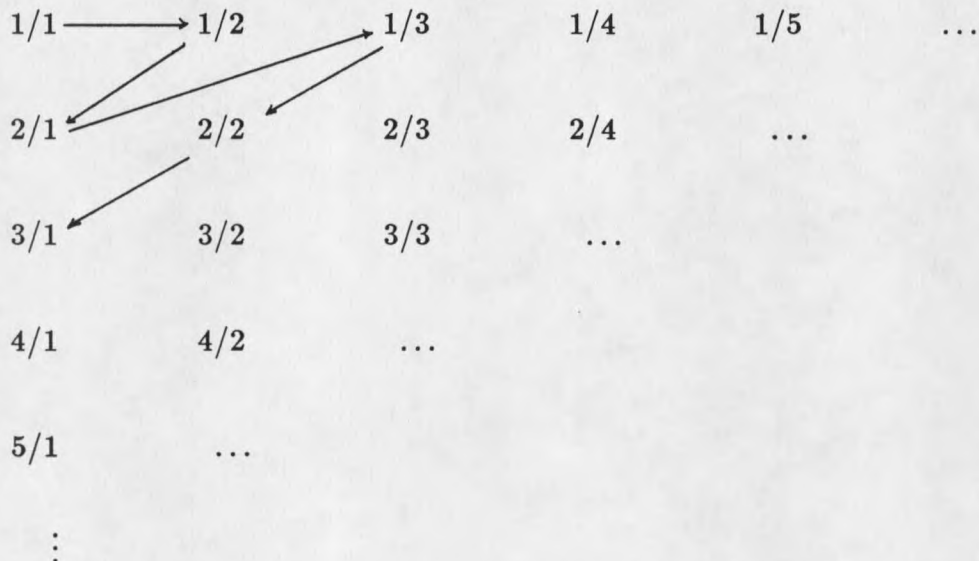


Figure 22. Cantor diagonalization of fractions

Every fraction can be found in this format: if the fraction is a/b , then it will be found in row a and column b . In order to count all of them, without missing any, start counting at $1/1$; next go to $1/2$ and down its diagonal; then $1/3$ and its diagonal, and so on. In this way, every fraction will be counted at least once; if you wanted to, you could skip every fraction that had already been counted under a different name. Since the fractions can, by this method, be put into one-to-one

correspondence with the counting numbers, they are equivalent and the fractions have the same cardinality as the counting numbers, \aleph_0 .

Every set with cardinality \aleph_0 is called countably infinite, a name which immediately brings up the question of whether there are any uncountably infinite sets. In order to answer this, we can look at the real line, or the continuum, as shown below.



Figure 23. The Real Line

Since this line is continuous, it has the property that between any two fractions, or rational numbers, you can find at least one irrational number, and vice-versa. An irrational number is one like $\sqrt{2}$ which cannot be expressed as a fraction; in decimal format, they are infinite decimals which never terminate or repeat. In contrast, rational numbers in decimal form either terminate after a certain number of places, like .5 ($1/2$) or .125 ($1/8$), or repeat, like .666666... ($2/3$) or .03030303... ($1/33$). It turns out that there are many more irrational numbers than there are rational numbers; in fact, there are an uncountable number of irrationals.

In order to show this, I will show that there are an uncountable number of real numbers between 0 and 1 on the real line. If the real numbers are uncountably infinite, then we can take away the countably infinite set of rational numbers and we will be left with an uncountable number of irrationals. Now every such real

number can be expressed as an infinite decimal such as $.475869701\dots$. If we try to list all of them; we'll get a list that looks something like this:

$.1234657687899\dots$

$.2375648596857\dots$

$.3459475764893\dots$

$.3920847472828\dots$

$.0039483837354\dots$

This certainly looks countable; after all, you can assign the first one to 1, the second to 2, and so on down the list. The problem is that no matter how carefully you try to make up a complete list, there is always one more number that you missed. To find one of these missing numbers, we can build it by the following method devised by Cantor: take the first digit of the first number, the second digit of the second number, and so on forever—remember, these are infinitely long decimals. This will give you a new number, $.13504\dots$; now change each digit by adding 1 to it, but let 8 and 9 become 0. This will give you $.24615\dots$, which is different from every number in the list by at least one digit. Since this number is not on the list, the list must be incomplete. This contradiction of the assumption that the list was complete means that the irrationals simply cannot be counted. The reals are not countable, and so they give us an example of an uncountably infinite set.

Uncountably infinite sets have many of the same qualities as countable sets. For instance, a line segment has just as many points as one twice its length. If we let AB be the first line segment and CD be the second, we can draw the figure below. By drawing a line from O through any point on AB , we will cross CD at some point, or we can draw a line from a point on CD to O and cross AB at some

point. Either way, we can put the points on AB into a one-to-one correspondence with the points on CD .

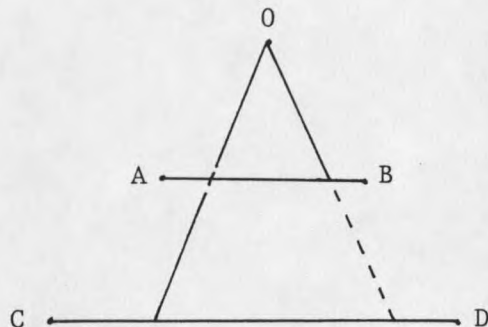


Figure 24. Line to line correspondence

If the picture is not convincing, we can do it arithmetically. Let AB be the line segment from 0 to 1 and CD be the line segment from 0 to 2. Then any point on AB can be given a decimal expansion; this decimal can be multiplied by 2 to give a new decimal between 0 and 2. For instance, $.765000\dots$ becomes $1.5900\dots$. In this way, every point on AB can be paired with a point on CD . Now we need to check that we haven't skipped any points on CD : any point on CD can be divided by 2 to give a number between 0 and 1, so every point on CD is paired with one on AB and we have a one-to-one correspondence between AB and CD . Thus there are exactly the same number of points on a short line segment as on a long line segment.

However, it can be shown that there are as many points in a line segment as there are on the entire real line, a feature which has no analogy in countably infinite sets. Geometrically, this is done by taking each point on a line segment AB to a point on a semi-circle; a line is then drawn from the center of the circle

through the points on the semi-circle until it intersects a point on the real line, as in the drawing below.

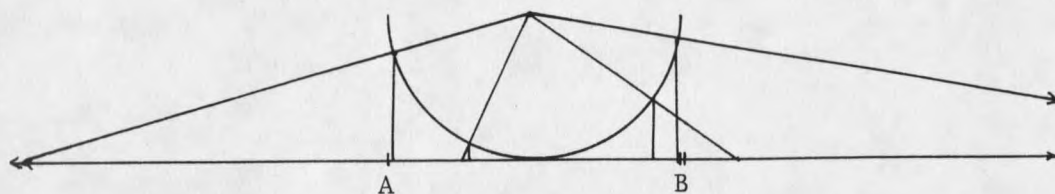


Figure 25. Segment to real line correspondence

Through arithmetic arguments, it can also be shown that there are as many points on a line as there are points in a plane or, by extension, in three-dimensional space.

The cardinality of the irrationals, which is the same as the cardinality of the real line, is labeled c , for continuum. It is intuitively obvious that c is bigger than \aleph_0 , since the real line contains all of the rational numbers as well as the irrationals, and, in fact, c is equal to 2^{\aleph_0} , which is called the power set of \aleph_0 and which can be thought of as consisting of all the possible subsets of \aleph_0 . Then the question occurs, is there anything between \aleph_0 and 2^{\aleph_0} ? In other words, if \aleph_1 represents the second transfinite cardinal, is \aleph_1 equal to c , or is there something that has a cardinality greater than the rational numbers but less than the reals? This question is called the Continuum Hypothesis, and the answer has been argued over ever since Cantor first introduced c and \aleph_0 . Cantor showed that 2^{\aleph} is never equivalent to \aleph , for any transfinite cardinal \aleph (just as 2^8 is not equal to 8), and that $c = 2^{\aleph_0}$, but no one was able to show that c is the second transfinite cardinal.

Finally, in 1938 and 1963, Kurt Godel and Paul Cohen respectively showed that the hypothesis could neither be proven nor disproven. Godel showed that if

the axioms of set theory are consistent—that is, if they don't lead to a pair of contradictory theorems—then adding the Continuum Hypothesis will not result in an inconsistent system; the hypothesis cannot be disproven because its use will not lead to contradictions. And Cohen showed that adding the negation of the hypothesis to set theory axioms will not result in contradictions, so it cannot be proven because using its negation doesn't lead to contradictions either. It is independent of the rules of set theory, of which the theory of transfinite numbers is part, and it is equally valid to accept or reject the hypothesis.

In other words, not only can the question not be answered, it doesn't even matter what the answer might be anymore, and mathematicians are free to choose according to their own biases. Generally, mathematicians accept the Continuum Hypothesis when it is needed, they accept a negation when it is useful, and in areas which do not require the Continuum Hypothesis it is left completely out of the theory. Much like Euclid's Fifth Postulate, it has become an axiom that can be accepted or not, depending on whether or not it is needed for the theory on hand. This freedom to choose leads to arguments over the ultimate "truth" of the hypothesis; Bernard Cohen, who has done much of the work in the field, believes that it will be considered "false" by future mathematicians, in much the same way Euclidean geometry is "false" (i.e. not the Truth). These judgements usually involve aesthetic opinions rather than rigorous mathematics, which just goes to show that mathematicians are human too.

So as you can see, mathematicians haven't really solved anything by defining infinity; they are still asking the age-old question, "How big is 'really big'?" They have just managed to move the question to a more esoteric level, which means that infinity, like most areas of science and mathematics, is becoming the domain of the specialist. In the process of this advance, they have raised some intriguing

questions that require no more than curiosity and the willingness to think to be explored. And it may just turn out that the questions about infinity arise from our inability to comprehend the infinite rather than from our definition of it, and the resolution to the problems may come from an area completely outside of mathematics which teaches us how to comprehend the eternal and the infinite.

ORDINAL NUMBERS

Imagine a game in which there is an inexhaustible supply of ping-pong balls and a bottomless pit containing no ping-pong balls. The object of the game is to transfer as many ping-pong balls as possible into the pit before noon, but there is a demonic creature in the pit which is determined to keep the ping-pong balls out of the pit. You are allowed to throw balls into the pit ten at a time, while the creature can only throw out one ball at a time. More precisely, after you throw ten balls in, the creature throws one out, and you throw ten more in. Now here's the catch—you are allowed $1/2$ minute to throw the first ten balls in, the creature has $1/4$ minute to throw his first ball out, you have $1/8$ minute to throw in the next ten, the creature has $1/16$ minute to throw out the next one, and so on, and so on. The time allowed for throwing balls is diminished by half each turn; if the game is started at 11:59, an infinite number of tosses will take place before noon. If the creature gets all the balls out of the pit by noon, it wins; otherwise, you win. So who wins?

To understand this problem, we need the idea of well-ordering of a set, which ties into ordinal numbers. In contrast to the cardinal numbers, the ordinal numbers are used to characterize an ordering of a set rather than to count the number of elements in the set; everyday examples of ordinal numbers include first, second, and thirty-third. Ordinals are interesting because their arithmetic is so different from what we expect since the commutative laws fail; for finite numbers $ab = ba$,

but for transfinite ordinals that is not usually true. This counterintuitive behavior comes from the definition of the ordinal numbers and the properties of infinite sets.

Ordinal numbers indicate the order of the elements of a set; for infinite sets, this primarily means that they indicate what kind of, and how many, infinitely long gaps there are in a set and where they are placed. ω , omega, is the symbol for the actually infinite, but it is also defined as the ordinal number of the counting numbers, $(1, 2, 3, 4, 5, \dots)$, where the integers are in standard order and an infinitely long, but still countable, gap is indicated by "...". (There can also be uncountably long gaps, with a new corresponding ordinal number.) From our arithmetic experience, we would expect that $\omega + 1 = 1 + \omega$, but that is not the case.

$1 + \omega$ is defined to be $(1; 1, 2, 3, 4, \dots)$, which is equal to ω because the gap is in the same place as it is in the counting numbers, at the end; in more technical terms, there is an order-preserving one-to-one correspondence between $1 + \omega$ and ω . On the other hand, $\omega + 1 = (1, 2, 3, 4, \dots; 1)$, which is not the same as ω because there is a number after the gap, and so $1 + \omega$ is not equal to $\omega + 1$. Similarly, $\omega + 2 = (1, 2, 3, \dots; 1, 2)$, $\omega + 3 = (1, 2, 3, \dots; 1, 2, 3)$, and so on; none of these are equal to each other or to ω because the gap appears in a different place in each one.

Now consider $\omega + \omega$; it equals $(1, 2, 3, \dots; 1, 2, 3, \dots)$, which is very different from all the preceding sums because the infinitely long gaps and the strings of integers are in a different order. By the definition of multiplication that we learned in grade school, $3 \circ 4$ means "take 3 four times", which has always been the same as "take 4 three times", or $4 \circ 3$. $\omega + \omega$ means take ω and then take ω , or take ω two times, which we write $\omega \circ 2$. This looks funny, but the order is important because $\omega \circ 2$ is not the same as $2 \circ \omega$; $2 \circ \omega$ means take 2 "omega" times and

equals $(1, 2; 1, 2; 1, 2; \dots)$, which equals ω because of the placement of the gap. $\omega \circ 2$ has two gaps and so looks entirely different from $2 \circ \omega$ with its single gap. All this means that the ordinal number of $2 \circ \omega$ is ω , whereas the ordinal number of $\omega \circ 2$ is just $\omega \circ 2$; while both sets have the same number of elements, and therefore the same cardinal number— \aleph_0 —the order of the elements is different and so the ordinal number is different. From this, it can be seen that while two infinite sets with the same ordinal number always have the same cardinal number, the reverse is not necessarily true: sets of the same cardinality can have different ordinal numbers. This is a situation that never occurs in the well-behaved finite sets, where the cardinality of a set is determined by ordering the elements in terms of first, second, etc., and taking the largest ordinal number encountered as the cardinality. A similar distinction between cardinals and ordinals does occur in everyday life: every child knows that the difference between being chosen first and last out of 15 kids for a baseball team is tremendously more important than the fact that 15 kids are chosen.

The concern with how sets are ordered leads to the concept of well-ordering, which is an important property in theoretical mathematics; for us, it is important because we can't solve the puzzle without it. A set is well-ordered if any subset of it has a smallest element contained in the subset. For instance, the counting numbers are well-ordered because if you take any set of them you can always find one of them that is smaller than all the others; on the other hand, all the integers are not well-ordered because if you take the set of all negative integers, there is no one number that is smaller than all the others. More accurately, these statements are true if the integers are ordered in the usual way, with the familiar meanings of less than and greater than; other ways of ordering the counting numbers, with ordinal numbers not equal to ω , can cause them to cease being well-ordered.

One nice characteristic of a well-ordered set is that you can always pick the next number in a sequence without worrying if there are any numbers in between. For instance, to find the integer that comes after 3, you look at the set of all integers greater than three and choose the smallest element of that set, 4. In sets that lack well-ordering, this can be impossible; in the rational numbers (the numbers that can be written as fractions), choosing the next number after 3 is impossible because the set of all rational numbers greater than 3 doesn't have a smallest number—give me any fraction close to 3 and I can give you one even closer. It is this characteristic of well-ordered sets that allows us to solve the problem at the beginning of the section.

The first time the game is played, the demonic creature loses because each time its turn comes, it stupidly throws out one of the ten balls that were just thrown in. At any given time, there are nine balls in the pit for every one that it has thrown out, so by noon there are an infinite number of balls in the pit.

Before the second game the creature takes a quick refresher course in mathematics, and it wins this game. To do this, the creature stacks the balls thrown in into pyramids of ten balls each and keeps track of the order in which the groups of ten balls were thrown into the pit. Now the balls are ordered by arrival time, and the creature always throws out the next ball in this line-up when his turn comes, throwing out the ten from the first pyramid on the first ten turns, then the balls from the second pyramid, and so on. In this way, every ball will be thrown out on one of the creature's turns, and no balls are missed (remember, there are an infinite number of turns taken by noon). At noon, there are no balls left in the pit and the creature wins the game.

This already looks like another of the paradoxes that we come to expect when dealing with infinite numbers, but it gets more confusing when you consider that in the second case, after any finite number, k , of turns, there are $9k$ balls left in the pit, regardless of how the creature has ordered them. It is only after an infinite number of turns that the creature can get all of the balls out of the pit. (If this still seems implausible, it works on the same principle that pairs up every number, n , with $10n$; see the section on cardinal numbers.) And if the creature gets really devious, it can arrange to have any finite number of balls in the pit at noon; can you see how?

MATHEMATICAL PARADOXES

Considering how badly our intuition is mistreated when we are dealing with infinite sets, it is probably not surprising that infinite sets lead to some contradictions, or paradoxes. Zeno, a Greek mathematician around 450 B.C., was, as far as we know, the first to ask questions about the problems that the infinite creates. One of his paradoxes is called Achilles and the Tortoise, and says that, in a foot race, if the tortoise, a very slow beast indeed, is given a head start, then Achilles, the fleetest runner in all of Greece, can never win the race; he can't even catch up to the tortoise, because first he has to get to where the tortoise was when the race started. But the tortoise isn't there anymore, so Achilles has to reach the place where the tortoise was when Achilles reached the spot where the tortoise was when the race started; and on it goes, *ad nauseam*. Zeno said that since Achilles was always catching up to the tortoise, he could never win the race, but everyone, including Zeno, knows that Achilles would win the race if it were actually run.

Another of Zeno's paradoxes, similar to Achilles and the Tortoise and called the Dichotomy, says that you can't even get started on the race, much less win it—or, as a professor once put it, you can't leave the classroom at the end of the hour. The problem is that in order to move, say, eight feet, first you have to get to four feet; in order to move four feet, first you have to move two feet; but first you have to get to one foot, but before that you have to get to six inches, and before that you have to move three inches.... Suffice it to say that before you can move any distance, you must first move half that distance, and half of that distance,

and so on; since you never get to a "first" distance to move, you are effectively stuck where you stand.

Both of these paradoxes appear to refute the idea that distance can be divided indefinitely, the idea that the number line contains an uncountable number of distinct points. Not content with this, Zeno also came up with two paradoxes that appear to refute the opposite idea, that the number line contains only a countable number of points. One of these, called the Stadium paradox, is hard to follow, but the other one, the Arrow, makes the point nicely. Since time is made up of some sort of indivisible unit, let's call them moments; now, consider an arrow in flight at some moment. The arrow must be stationary at that moment, since if it were moving the moment must be divisible, which it is not (by definition). In any such moment, the arrow is not moving, so when we add up all the moments of the flight of the arrow, we find that it hasn't moved. So much for target practice.

These paradoxes baffled the Greeks and they decided to get rid of infinity altogether in order to get rid of the paradoxes—rather a case of throwing out the baby with the bath water—and that attitude persisted down to Cantor's time. The paradox has been resolved in modern times by pointing out that Zeno phrased his paradoxes in discrete terms while the things involved are continuous; a similar problem would be saying that you cannot draw the real line because first you have to fill in every rational number.

Continuous: _____

Discrete:

Figure 26. Continuous versus discrete

So the problem as stated is meaningless and can safely be ignored.

Another paradox, this one with a theological bent, was proposed by a tenth century Babylonian named Saadia Gaon, who used it to prove that the world was created by a god. His idea was that if the world had no beginning, then time would be infinite. But infinite time cannot pass, so today could never come into being. Therefore, the world had a beginning and, by theological arguments, must have been created. As with Zeno's paradoxes, this is another case of stating a problem about a continuous thing, in this case time, as if it could be broken down into a finite number of distinct units.

Unfortunately, several modern paradoxes can't be disposed of so neatly. The most famous one is called Russell's Paradox after the man who first proposed it, Bertrand Russell. In 1902, Russell pointed out a simple but thorny problem posed by an intuitive definition of sets. As an example, consider words like "short" or "polysyllabic" which are examples of the properties that they name: "short" is a short word. Words of this kind can be called homological, while words like "long" or "monosyllabic", which are not examples of the properties that they name, can be called heterological. The problem comes with the word "heterological". If it is not an example of the property that it names, then it is heterological and so is an example of the property that it names. On the other hand, if it is an example of the property that it names, then it is homological and so is not heterological.

This paradox, similar to the one that states "This sentence is false," winds up creating a tangled mess in the mind, but it can be resolved by a careful definition of terms and an awareness of when you are using a meta-language. A meta-language is a language used to speak about another language: in a French literature class conducted in English, English becomes the meta-language used to discuss French. Self-referential statements like "This sentence is false" lose their paradox when it is realized that "this sentence" is part of the meta-language, while "is false" is

part of the language being used. As a meta-language statement, "This sentence is false" makes sense, because "this sentence" refers to something in the language under discussion; for instance, if a line in a French text is wrong, declaring that "This sentence is false" is hardly paradoxical.

A similar attempt was made to eliminate Russell's paradox in set theory by building a theory of types. In set theory, Russell's paradox considers the set of all sets which are not members of themselves. Is this set a member of itself? A set of bananas is clearly not a banana, so this set is not a member of itself; on the other hand, a list of lists is a list and so belongs to itself. So if Russell's set is a member of itself, then by definition it must have the characteristic that it is not a member of itself; if Russell's set is not a member of itself, then it has the required characteristic and belongs in itself. The theory of types is an attempt to deal with this paradox in the same manner that meta-languages deal with self-referential statements. The details are complicated, but the idea is that in order to talk about something at one level, we must go to the next level up. In this case, an object like a banana might be of type 1, while a set of bananas would be of type 2; a list would be of type 2, while a list of lists would be type 3, and so the list of lists cannot belong to itself because it is made up of type 2 members while it is itself of type 3. Similarly, Russell's set is of higher type than the sets in it, and so it cannot be a member of itself, and there is no more paradox. Alternatively, talking about sets that are not members of themselves makes no sense in the theory of types, and so Russell's paradox becomes meaningless. While it clears up the paradox, the theory of types unfortunately runs into difficulties of its own, and so it has sunk into semi-obscurity.

Other paradoxes have arisen since Cantor wrote his works on transfinite numbers. For instance, what is the cardinal number of the set of cardinal numbers?

Say you call it C , so that there are C elements in the set; but now C is a cardinal number and so is a member of the set of all cardinal numbers. Then the set has one more element and so the cardinal number must be $C + 1$. And so it goes. In ordinals, a similar problem arises: the sequence of ordinal numbers must have an ordinal number that is greater than all the ordinal numbers. But if it is an ordinal number, then it should be in the sequence and so cannot be larger than all of the ordinal numbers. Like most paradoxes, especially Russell's paradox, these are resolved by saying that they don't make sense, that there is no such thing as a set of cardinal numbers; and that if we define things carefully enough, the paradoxes will go away and leave us in peace.

Although frustrating to work with, paradoxes have a very useful place in mathematics, pushing mathematicians to dig further into their subject, denying the easy answers and forcing a deeper understanding of the problems. Each new paradox, which at first appears to block the route of progress, in the end leads to new fields of study and new ways of looking at things.

FUZZY LOGIC

Many paradoxes present situations that appear to deny one of the two fundamental laws of western logic: the law that says that everything either has or does not have a certain attribute, or the one that says that something cannot both have and not have an attribute. In other words, either an object is a rock or it isn't a rock; one of those two phrases must apply to the object, even if we lack the knowledge to decide which one applies. Similarly, an object cannot be both a rock and a non-rock at the same time. These two laws are the basis of our familiar Aristotelian logic, the one that science has been so successfully built upon, the one that philosophy departments teach in their classes called Logic. They define the western approach, which feels that all questions have an answer, that every statement is either true or false even if we can't decide which at the moment, which likes the world laid out in black and white without ambiguities. This approach has given us mathematics and the sciences, building blocks for a more advanced society, but it has its limitations.

These limitations surface most frequently when dealing with people. For instance, we would all agree that an infant is young, and that an octogenarian is not young, but what about a 20-year-old? A 30-year-old? A 40-year-old? A 50-year-old? Where is the line that makes someone young on one side of it and old on the other? If it's at, say 50, is someone who is 50 really that much older than someone who is 49? It is in cases like this that Aristotelian logic breaks down, because although there is a distinct line between rock and not-rock, there is no similar line between young and not-young, or between tall and not-tall, or between bald and not-bald. It was to provide some way of working with cases

like this, where the border line between a characteristic and its opposite is fuzzy, that fuzzy logic was developed.

In Aristotelian logic, something either belongs to a set or it doesn't; if it is in a given set, called the characteristic set, then it is assigned the value 1, and if it isn't in the set it is assigned the value 0. For instance, if the characteristic set is the set of all round objects, a beachball would be assigned 1 while a building block would be assigned 0. In other words, logic maps all objects to the set $\{0,1\}$, where the mapping depends on the characteristic set being used. If the characteristic set is the even numbers, then the mapping would look like this:

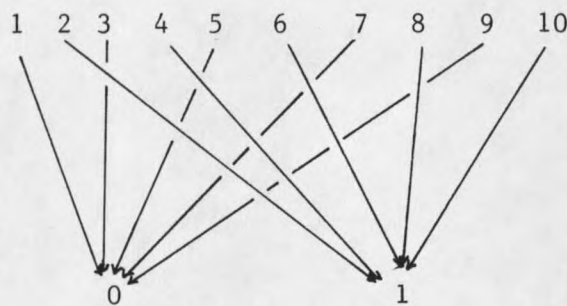


Figure 27. Aristotelian mapping

Every object is assigned either 1 or 0, and none are assigned both; in this way, the two basic laws of Aristotelian logic are put into mathematical terms that are easier to manipulate.

Fuzzy logic is also a mapping from all objects to a set; but this time, instead of a set consisting of two values, we have the set $[0,1]$, which consists of all the values between 0 and 1 as well as 0 and 1 themselves. Just as in Aristotelian logic, if an object is assigned the value 0, it is not in the characteristic set at all; if it

is assigned the value 1, it is completely in the set. However, an object can also be assigned values between 0 and 1; an object just barely in the characteristic set might be given a value of .3, while an object that is mostly in the set might have a value of .8. These values indicate the degree of membership of the object in the set. For instance, if the characteristic set is the set of all tall adults, someone who is 4' must be assigned 0, someone who is 5'4" might be assigned .3, someone who is 6' might be assigned .8, and someone who is 7'2" would be assigned 1. A graph of this mapping would look like this:

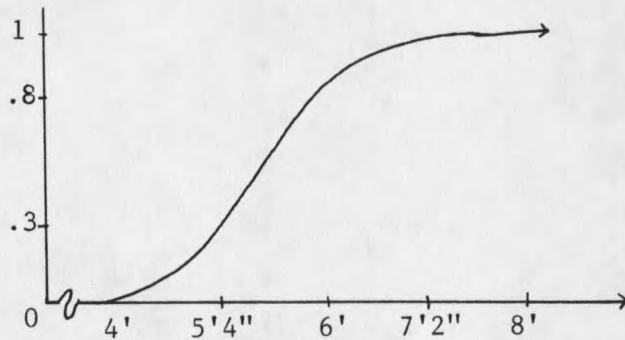


Figure 28. Tall adults

It is easy to see that fuzzy logic is more applicable to this situation than Aristotelian logic, where anyone above, say, 5'8" would be tall and anyone under that would be considered short.

Once elements have been assigned a value that indicates their membership in a set, operations can be performed on these elements just as in regular set theory. The two most common operations are union and intersection, which correspond to "or" and "and" respectively. An element is in the union of two sets if it is in one OR the other, while an element is in the intersection of two sets if it is in both

sets, one AND the other. For instance, if the two sets are tall men and bearded men, then the union of the two sets is the set of all men who are bearded or tall, and the intersection is the set of all tall, bearded men. Abraham Lincoln would belong in both the intersection and the union, Fidel Castro would belong in the union but not in the intersection, and Marilyn Monroe would belong in neither the union nor the intersection.

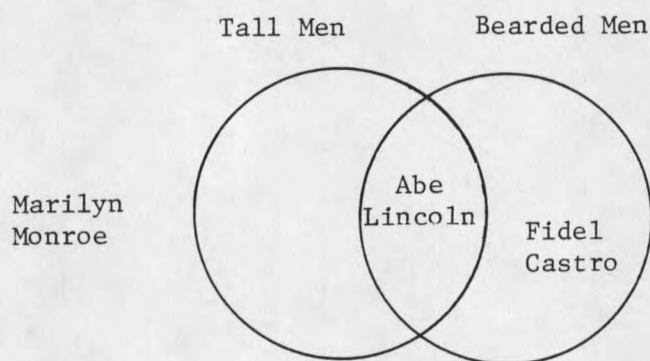


Figure 29. Union and intersection

In fuzzy set theory, these two operations still exist, but since membership in a set can be partial, the definitions have been reworded somewhat. However, just as fuzzy logic contains Aristotelian logic, fuzzy set theory contains regular set theory, at least for these two operations. The degree of membership in the union of two sets is found by taking the maximum of the degree of membership in the two sets, while the degree of membership in the intersection is found by taking the minimum. If an element has a degree of membership of .7 in set A and .3 in set B , it has a degree of membership of .7 in the union of A and B and of .3 in the intersection of A and B . In other words, if Harold is pretty tall and not very

fat (tall corresponds to set A , fat to set B); then he is pretty tall-or-fat, and not very tall-and-fat. These operations, along with others, allow us to work with degrees of properties, of shades of difference, and to draw conclusions from them.

One of the major advantages of fuzzy logic is that it allows terms such as "mostly", "somewhat", and "not very" to be quantified in a remarkably consistent way; although the assignment of values to terms like these seems to be arbitrary and highly dependent on the individual doing the assigning, people are actually quite consistent as to what value they give various terms. It is because of this consistency that fuzzy logic can be applied at all, and the usefulness of quantifying these terms has led to many attempts to apply fuzzy logic to such diverse fields as pattern recognition in artificial intelligence and production scheduling in manufacturing; some of these attempts may not work, but in many fields, fuzzy logic seems to be a useful new tool.

One of the fields most interested in the possibilities of fuzzy logic is artificial intelligence, the attempt to make machines think like humans. Machines, primarily computers, have been hard to program to think like people because they make only yes/no decisions and are incapable of recognizing shades of gray; fuzzy logic appears as one way to teach them to see gray. Fuzzy logic has had some success in the area of pattern recognition in programming computers to read handwriting. Handwriting is a particular problem because people all write so differently, yet people seldom have trouble reading each other's writing; if computers could be taught to read handwriting, it would vastly expand their versatility.

An important practical application of fuzzy logic is to the control of furnaces and kilns used in industrial processes. These furnaces have long been the province of skilled operators who knew, with an instinct developed by years of practice and training, when the temperature in a furnace needed to be turned up, or when

more raw material was needed. Computers had been tried, but they created abrupt changes of temperature, much the way a thermostat will let a house cool down, then suddenly turn the furnace on and heat the house up, rather than maintaining a constant temperature all day. Now a Danish company has developed a program that uses fuzzy logic to mimic the operator; using fuzzy logic, the program can make decisions based on several factors, weighing them and deciding whether or not to add more fuel to the furnace, or to add more raw material. The new program can avoid the abrupt changes that can damage the product and place excess strain on the equipment. A similar program is being developed in Japan to run trains; theoretically, the program will choose the shortest route, save fuel, and provide a smoother ride by using fuzzy logic.

AT&T is using fuzzy logic in "expert systems", computer programs designed to help in decision making, and they are already starting to advertise the program. In a recent ad, they refer to "'associative' memories for computers, [which are] further enabling the machines to work with incomplete, imprecise, or even contradictory information," which corresponds closely to a new chip that they have developed. The new chip uses fuzzy logic to provide a few "rough and ready" rules, which take up much less space than having to store huge tables of information in the computer's memory. This chip could be used in robotics, system design, and control of industrial processes such as furnaces.

Other fields in which fuzzy logic is being applied are:

- Computer backgammon programs

- Data display

- Water resources planning

- Communications

- Earthquake engineering and research

Medical diagnosis
Chemistry, biology (nervous systems)
Market research
Production scheduling
Psychology
Linguistics
Social sciences
Political sciences

With so many people working in so many fields, it is likely that more applications for fuzzy logic will turn up soon. While some of the applications could undoubtedly be done better with other tools, it seems likely that fuzzy logic has a legitimate place in the mathematical toolbox.

Fuzzy logic does have its detractors, who claim that all the things being done with fuzzy logic could be done with some other, established method instead, that fuzzy logicians have found a new toy in fuzzy set theory. In particular, probability is held up as an established system that works for the problems that fuzzy logic is being used to solve. Also, many critics say that fuzzy logic exchanges the rigidity of the yes/no system for an equally rigid system of membership in sets. Some mathematicians claim that if fuzzy logic didn't have such a catchy name, if it had a more serious name like "Degree of Membership in Set Theory", there would be very little interest in it.

Fuzzy logicians are playing with fuzzy logic, trying to see what happens when various problems are tackled with it, but that is a common phenomenon when a new technique is developed; the same thing happened when non-Euclidean geometries were introduced. It is a way of finding out both the powers and limitations of a new technique; once this is accomplished, the technique can be used to solve

new problems that have resisted the older methods. Like fractals, fuzzy logic is a new subject; it may fade away next year, or it may prove to have undreamed-of powers as a tool to solve difficult problems. As a technique deliberately developed to mimic the way humans think, fuzzy logic has both appeal and promise; it could end up being a useful tool for scientists in many different fields.

CONCLUSION

Mathematics is a far from stagnant discipline; the burgeoning growth in fuzzy logic and in the fields associated with fractals belie that misconception. Even the basis of modern mathematics, the proof, is changing in ways unheard-of twenty years ago. It is in this forefront that the spirit of discovery and the challenge of new ideas are found. It is here that mathematics takes on its most intellectually compelling form, here that different ideas and ways of thought are brought into conjunction to produce new theorems and new ways of looking at the world. It is, unfortunately, a place that few people ever see or even know exists. This thesis has been an attempt to rectify that situation, to show that mathematics goes far beyond the 200- or 300-year-old math that is taught in most high school and college classes. The topics discussed are only a small part of the large and varied discipline that is modern mathematics; there is far more awaiting anyone with the curiosity to explore farther.

REFERENCES CITED

General

- Bell, E.T. *Men of Mathematics*. New York: Simon and Schuster, 1937.
- Davis, Philip J., and Reuben Hersh. *The Mathematical Experience*. Boston: Houghton Mifflin Company, 1981.
- Guillen, Michael. *Bridges to Infinity: The Human Side of Mathematics*. Los Angeles: Jeremy P. Tarcher, Inc., 1983.
- Kline, Morris. *Mathematics: The Loss of Certainty*.

Fractals

- Dewdney, A.K. "Computer Recreations", *Scientific American*, August 1985.
- Edelson, Edward. "The Ubiquity of Nonlinearity", *Mosaic*, v. 17/3.
- Gleik, James. *Chaos: Making a New Science*. Viking Press, 1987.
- Kadanoff, Leo P. "Fractal: Where's the Physics?" *Physics Today*, February 1986.
- La Breque, Mort. "Fractal Applications", *Mosaic*, Winter 1986-87.
- Mandelbrot, Benoit B. *Fractals: Form, Chance, and Dimension*. San Francisco: W.H. Freeman and Company, 1977.
- Peterson, Ivers. "Ants in Labyrinths and Other Fractal Excursions", *Science News*, 21 January 1984.
- Sorenson, Peter R. "Fractals", *Byte*, September 1984.
- Thomsen, D.E. "Fractals: Magical Fun or Revolutionary Science?" *Science News*, 21 March 1987.

Geometry: Euclid and Beyond

Kulozycki, Stéfan. *Non-Euclidean Geometry*. Translated by Stanislaw Knapowski. New York: Pergamon Press, 1961.

Lobachevski, Nicholai. *Geometrical Researches on The Theory of Parallels*. Berlin, 1840. Translated by George Bruce Halsted. Chicago: Open Court Publishing Company, 1914.

Rucker, Rudolf v.B. *Geometry, Relativity and the Fourth Dimension*. New York: Dover Publications Inc., 1977.

Proofs

Edelson, Edward. "The Ubiquity of Nonlinearity", *Mosaic*, v. 17/3.

Hardy, G.H. "Mathematical Proof", *Mind*, January 1929.

Kolata, Gina Bari. "Mathematical Proofs: The Genesis of Reasonable Doubt." *Science*, 4 June 1976.

McAtee, Deborah. "Survey of MSU Mathematics Professors." Unpublished.

Cardinal Numbers and Ordinal Numbers

Cantor, Georg. *Contributions to the Founding of the Theory of Transfinite Numbers*. Translation, introduction and notes by Phillip E.B. Jourdain. New York: Dover Publications, Inc.; 1955 reprint of 1915 edition by Open Court Publishing Company.

Cohen, Paul J., and Reuben Hersh. "Non-Cantorian Set Theory", *Scientific American*, December 1967.

Dauben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Cambridge, Mass.: Harvard University Press, 1979.

Dauben, Joseph Warren. "Georg Cantor and the Origins of Transfinite Set Theory", *Scientific American*, June 1983.

Drake, Frank R. *Set Theory: An Introduction to Large Cardinals*. *Studies in Logic and the Foundations of Mathematics Series*. Amsterdam: North-Holland Publishing Company, 1974.

Gillette, Dick. "Ping-pong Ball Paradox". Private communication.

Rotman, B., and G.T. Kneebone. *The Theory of Sets and Transfinite Numbers*. New York: American Elsevier Publishing Company, Inc., 1966.

Sierpinski, Waclaw. *Cardinal and Ordinal Numbers*. New York: Hafner Publishing Company, 1958.

Vilenkin, N. Ya. *Stories About Sets*. Translated by Scripta Technica. New York: Academic Press, 1968.

Mathematical Paradoxes

See General listings.

Fuzzy Logic

King, John F. "Fuzzy Logic Provides New Way to Deal with Uncertainty", *Electronics Week*, 3 June 1985.

Lerner, Eric J. "AT&T researchers use fuzzy logic to put an expert system on a computer chip", *Research & Development*, March 1986.

McKean, Kevin, and Tom Dworetzky. "Computers and Logic: Fuzzy Means to Logical Ends", *Discover*, February 1985.

Schmucker, Kurt J. *Fuzzy Sets, Natural Language Computations, and Risk Analysis*, Chapter 2. Computer Science Press, 1984.

Yager, Ronald R., editor. *Fuzzy Set and Possibility Theory: Recent Developments*. Pergamon Press, 1982.

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