



An investigation of prospective secondary mathematics teachers conceptions of proof and refutations
by Kathy Jo Riley

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Education
Montana State University

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Abstract:

This research study was conducted to investigate prospective secondary mathematics teachers' conceptions of proof and refutations as they were near completion of their preparation program. To research the primary question of the study, the researcher addressed two components of participants' conceptions of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. The researcher developed a questionnaire composed of two parts in order to assess the two components of proof. Both components focused on direct proof indirect proof and refutations. The sample for the study were 23 prospective secondary mathematics teachers that had completed an introduction to proof course, geometry course, and at least two calculus courses.

Results show that only 30% of the prospective teachers correctly answered 9 or more items, of 12 items, for the logical underpinnings of proof. The results show that participants have a weak understanding of the truth of a conditional statement and its related statements (e.g., converse, negation of conditional statement).

Examining prospective teachers' ability to complete mathematical proofs show that only 57% of the participants were able to write a valid direct proof of the Perpendicular Bisector Theorem, a proof common to the high school geometry curriculum. Only 39% of the participants were able to write a valid indirect proof about even integers. Results show that only 39% of the sample recognized and were able to refute a false conjecture about perimeter and area of rectangles.

Results of participants' overall performance on both parts of the questionnaire show that 52% of the sample scored 60% or less on both parts of the questionnaire. The vision of the MAA (1998) and the NCTM (2000) recommendations for teaching reasoning and proof to all students grades K - 12, and in all mathematics content areas, may not be attainable by all of the prospective secondary mathematics teachers in this study. The findings suggest that prospective teachers need more experiences in determining the true values of conjectures and that there is a correlation between an individual's understanding of the logical underpinnings of proof and ability to complete proofs.

AN INVESTIGATION OF PROSPECTIVE SECONDARY MATHEMATICS
TEACHERS' CONCEPTIONS OF PROOF AND REFUTATIONS

By

Kathy Jo Riley

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Education

MONTANA STATE UNIVERSITY - BOZEMAN
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Abstract

This research study was conducted to investigate prospective secondary mathematics teachers' conceptions of proof and refutations as they were near completion of their preparation program. To research the primary question of the study, the researcher addressed two components of participants' conceptions of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. The researcher developed a questionnaire composed of two parts in order to assess the two components of proof. Both components focused on direct proof, indirect proof, and refutations. The sample for the study were 23 prospective secondary mathematics teachers that had completed an introduction to proof course, geometry course, and at least two calculus courses.

Results show that only 30% of the prospective teachers correctly answered 9 or more items, of 12 items, for the logical underpinnings of proof. The results show that participants have a weak understanding of the truth of a conditional statement and its related statements (e.g., converse, negation of conditional statement).

Examining prospective teachers' ability to complete mathematical proofs show that only 57% of the participants were able to write a valid direct proof of the Perpendicular Bisector Theorem, a proof common to the high school geometry curriculum. Only 39% of the participants were able to write a valid indirect proof about even integers. Results show that only 39% of the sample recognized and were able to refute a false conjecture about perimeter and area of rectangles.

Results of participants' overall performance on both parts of the questionnaire show that 52% of the sample scored 60% or less on both parts of the questionnaire. The vision of the MAA (1998) and the NCTM (2000) recommendations for teaching reasoning and proof to *all* students grades K – 12, and in *all* mathematics content areas, may not be attainable by *all* of the prospective secondary mathematics teachers in this study. The findings suggest that prospective teachers need more experiences in determining the true values of conjectures and that there is a correlation between an individual's understanding of the logical underpinnings of proof and ability to complete proofs.

CHAPTER 1

STATEMENT OF THE PROBLEM

Current mathematics education reform efforts call for an increased emphasis in our school curricula on reasoning and proof as a stepping stone toward logical reasoning. The *Principles and Standards for School Mathematics* emphasized the need for opportunities in mathematical reasoning and proof for *all* students grades K – 12 and in *all* mathematics content areas (National Council of Teachers of Mathematics [NCTM], 2000).¹ The document deems these opportunities as essential to understanding mathematics. The NCTM added, "By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments" (2000, p. 56).

The NCTM further noted in the Teaching Principle of the *Standards* 2000 that students' understanding of mathematics, their ability to use it to solve problems, and their confidence and disposition toward mathematics are all shaped by the teaching they encounter in school. As a cornerstone to their reform vision, the NCTM emphasized that mathematics teachers must develop and retain the mathematical and pedagogical

¹ Herein the *Principles and Standards for School Mathematics* (NCTM, 2000) document is referred to as the *Standards* 2000, whereas the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) are referred to as the *Standards* 1989.

knowledge needed to effectively teach their students. With respect to proof and reasoning, the NCTM emphasized the need for teachers to understand proof and reasoning in order to help students develop critical thinking and reasoning skills.

To establish a backdrop for the study, Chapter One traces the history of proof and reasoning in the mathematics curriculum by examining the recommendations of selected reform efforts. When comparing current reform efforts with those of the past, important trends that emerge are the increased attention given to the preparation of teachers and the goal that proof and reasoning should be taught at all grade levels and in all mathematical content areas. Proof is a very difficult area for high school students—a fact that is noted by the NCTM (2000, p. 56) and supported by many research studies (see Chapter Two). Additionally, research suggests that teachers' content knowledge affects students' performance. The chapter concludes by raising the question of whether prospective secondary teachers are prepared to teach reasoning and proof, as measured by their understanding of these topics, as they embark on their teaching careers.

Proof and Reasoning in the Mathematics Curriculum

Prior Reform Efforts

In the 1930's, Fawcett addressed the need for reform within the mathematics curriculum (1938). Much like the mathematical reforms that would follow, Fawcett was concerned about the development of critical and reflective thought. According to Fawcett, the area of the curriculum most appropriate for addressing these concerns was geometry, since it was accepted as the primary course for acquainting students with the

nature of deductive thought. Fawcett described geometric proof as a means for cultivating critical and reflective thought—a convention that would become standard in the high school mathematics curriculum.

Fawcett's report indicated that the transfer of logical reasoning to situations outside of geometry was dependent on efficient instruction. Fawcett added, however, that most geometry teachers were not teaching in a manner that facilitated this transfer. The current reform efforts of the NCTM seem to reflect Fawcett's concern about the development of critical thinking skills and the need for improvement in teacher preparation at the collegiate level. One of the most significant recommendations of the *Standards 2000* was that proof and reasoning should be taught in all areas of school mathematics and not restricted to the geometry course.

In the late 1950's and early 1960's, efforts to reform school mathematics were referred to as the New Math movement. The New Math movement included the School Mathematics Study Group (MSG), which was founded in 1958 as a national organization devoted to the improvement of mathematics programs. The MSG's objectives included the development of an improved mathematics curriculum that would emphasize basic skills and promote a deeper understanding of basic concepts and the axiomatic structure of mathematics, logic, and methods of deductive proof. (Begle, 1968; Hanna, 1991). The New Math movement strove to align mathematics instruction in the public schools with the standards of practice by mathematicians. To establish their visions of an improved curriculum, the MSG wrote and published textbooks that they

deemed to be pedagogically and mathematically sound. These texts were to serve as models for what should be used in teaching school mathematics (Begle, 1968).

Unfortunately, the New Math movement and its emphasis on proof is widely considered a failure. Hanna and Jahnke (1993) attributed the demise of the New Math to exaggerated formalism, unsuccessful teaching, and a critical public, all of which eventually led to a critical reassessment of mathematics education. Likewise, Hirschi (1977) attributed the failure to too much emphasis on curriculum and too little on teacher preparation. Instead of considering the training of teachers, Hirshi maintains the New Math movement tried to establish reform through the introduction of textbooks:

If we had moved in a more cautious manner by a massive retraining of mathematics teachers at the inservice and college level, I believe we could have warded off most of our present difficulties (1977, p. 244).

More recent reform efforts are often traced to the publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The *Standards* 1989 set forth goals as a guideline for the learning and teaching of school mathematics. The document emphasized a dynamic form of literacy instead of the rigor of deductive proof (NCTM, 1989). "Making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics" (NCTM, 1989, p. 7). Tall (1992) noted that the *Standards* 1989 recommended increased attention should be given to deductive arguments expressed orally and in paragraph form and decreased attention to Euclidean geometry as a complete axiomatic system and two-column proofs. Many in the mathematics education and mathematics community, however, view the *Standards* 1989 as having a relaxed emphasis on deductive proof (Chazan, 1989).

Current Reform Efforts

A significant change from the *Standards* 1989 to the *Standards* 2000 document was the latter document's increased emphasis on reasoning and proof. In 1996, the NCTM's Commission on the Future of the *Standards* asked a group within the Mathematical Association of America (MAA) to respond to the role of proof and reasoning in school mathematics (Ross, 1998).² The MAA Task Force was asked what mathematical reasoning skills should be emphasized, how the *Standards* 2000 should address proof, and how to address topics within mathematical structure (Ross, 1998).

The Task Force emphasized that the foundation of mathematics is reasoning. They stated that the goals of school mathematics should include opportunities to learn logical reasoning, develop valid arguments or proofs, and criticize the arguments of others (Ross, 1998). According to the Task Force, students should understand that science verifies through observation and mathematics verifies through logical reasoning. The distinction among illustrations, conjectures, and proofs should be emphasized and students should understand that mathematical results are valid only after they have been carefully proved.

The MAA Task Force outlined seven key points of reasoning and proof that students should learn beginning in their eighth-grade mathematics course:

1. distinguish between inductive and deductive reasoning and explain when each is appropriate;
2. understand the meaning of logical implication, in particular, be able to identify the hypothesis and conclusion in a deduction;

² The group formed was referred to as the President's Task Force on the NCTM *Standards* herein referred to as the Task Force or MAA Task Force.

3. test an assertion with examples;
4. realize that one counterexample is enough to show that an assertion is false;
5. clearly recognize that the truth of an assertion in a few cases does not allow one to conclude that it is true in all cases;
6. recognize whether something is being proved or is merely being given a plausibility argument;
7. identify logical errors in chains of reasoning involving more than one step (Ross, 1998, p. 254).

The Task Force also recommended that the *Standards 2000* document include material on logic and mathematical language appropriate for the different grade levels. This material includes mathematical sentences with basic connectives in logic, quantifiers such as "all", and true and false statements, and strategies for justifying or refuting statements.³

In 2000, the NCTM released the *Principles and Standards for School Mathematics* after considering recommendations from the MAA. This document included a new Reasoning and Proof Standard, which provided a guideline for the education of students in pre-kindergarten through grade 12. The Reasoning and Proof Standard recommended that students in *all* grade levels should be able to establish the validity of a true conjecture and find a counterexample that demonstrates a conjecture is false (NCTM, 2000).

The explicit goals of NCTM 2000 Reasoning and Proof Standard for grades 9 – 12 recommended that students should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and methods of proof (2000, p. 342).

³ The five basic connectives in logic are: 1) not, 2) and, 3) or, 4) "if – then" statements, and 5) "if and only if" statements (Esty, 2000, p. 150).

In addition to these goals, the *Standards 2000* recommended that proof should be a focus of *all content areas* of grades 9 – 12, and that students should understand and be able to use proof techniques. These techniques include direct proof, indirect proof, and also the understanding that a counterexample refutes a false conjecture.

The *Standards 2000* state that opportunities to explore mathematical ideas and ponder questions such as "Why does that work?" help students develop conjectures. A natural way for students to justify their conjectures can be through mathematical reasoning and proof. According to Sowder and Harel (1998), if we are to promote mathematics as reasoning, students must be continuously exposed to the value of arguments that focus on the *why* of mathematics results and not just on the results themselves.

In summary, current reform efforts in mathematics education strongly support reasoning and proof for *all* students and in *all* mathematics courses. These recommendations appear to be "lofty goals", given results of prior research studies that have shown students' abilities to understand proof is inadequate. For example, Senk's (1985) study found that only 30% of a sample of 1,520 high school students in a full-year geometry class that emphasized traditional two-column proofs, achieved a 75% mastery level of proof. The new visions of mathematics classrooms bear little resemblance to the classrooms in which prospective teachers themselves were students. Therefore, prospective secondary teachers may need a more thorough, and possibly different, preparation if the NCTM vision is to be realized. The catalyst for building stronger understandings of reasoning and proof within a mathematics classroom is the teacher.

Teacher Preparation

The *Standards 2000* proposed students will better understand mathematics if they develop their mathematical reasoning and proof skills. The NCTM further note that teachers themselves need to understand proof and reasoning in order to help students develop critical thinking and reasoning skills. The NCTM's *Professional Standards for Teaching Mathematics* (1991) emphasized that mathematics teachers' experiences while learning mathematics have a powerful impact on the education they provide their students. The need for teachers to promote the development of their students' understanding of proof suggests that teachers themselves must have a robust understanding of proof. This robust understanding develops through opportunities to explore, conjecture, develop mathematical arguments, validate possible solutions, and recognize connections among mathematical ideas (NCTM, 1991).

The National Commission on Teaching and America's Future [Glenn Commission] published a report in which they reiterate the NCTM's concerns on the quality of mathematics and science teaching in the nation (2000).⁴ The report identified possible solutions, suggesting that students must improve their performance in science and mathematics and that the most direct route to improving mathematics achievement for all students is better teaching. The report stated, "...better teaching is the lever for change" (Glenn Commission, 2000, p. 14). It stressed that *better teaching* implied the

⁴ The report was entitled "*Before It's Too Late: A Report to the Nation from the National Commission on Mathematics and Science Teaching for the 21st Century*" (2000).

need for professional teacher development programs to *better prepare* mathematics teachers to meet the needs and demands in school mathematics. "A focused professional development experience led by qualified teachers, mentors, and colleagues is the indispensable foundation for competence and high-quality teaching" (Glenn Commission, 2000, p. 14).

In 2001 the Conference Board of the Mathematical Sciences (CBMS) published a report entitled *The Mathematical Education of Teachers*.⁵ The CBMS acknowledged that teachers must develop mathematical knowledge for teaching because this knowledge allows teachers to evaluate their students' work and recognize the sources of student understanding and errors. In the area of proof, the CBMS emphasized that prospective teachers at all grade levels need experience justifying conjectures and added, "Future high school teachers must develop a sound understanding of what it means to write a formal proof" (CBMS, 2001, p. 14).

Mathematics Teachers' Knowledge

The emphasis on teacher preparation raises the question of what knowledge teachers need to teach mathematics. "Teachers need to know and use 'mathematics for teaching' that combines mathematical knowledge and pedagogical knowledge" (NCTM, 2000, p. 370). Preparing teachers to enact recommendations set forth by the NCTM has been a focus of reform efforts in teacher preparation programs. In *Knowing and Learning Mathematics for Teaching*, the Mathematical Sciences Education Board

⁵ The CBMS is affiliated with the Mathematical Association of America.

(MSEB) state, "Many experts now agree that reforming teacher preparation in post-secondary institutions is central to sustaining and deepening efforts to provide quality mathematics education for all students" (National Research Council, 2001, p. 1). In steps toward reform of teacher preparation programs, the MSEB posed two questions: 1) "What is the mathematical knowledge teachers need to know to teach well?"; and 2) "How can teachers develop the mathematical knowledge they need to teach well?" (2001, p.2).

The concern of what knowledge is necessary to facilitate recommended changes within school mathematics has been the focus of many in mathematics education (Ball, 1989; Cooney, 1999; Fischbein, 1990; Galbraith, 1982; Shulman 1986). Ball (1989) stated that pre-service teachers are not likely to know and understand mathematics in ways that they will need in order to teach. She added that prospective teachers' schooling shapes their understanding of mathematics. "As this is the mathematics they will *teach*, what they have learned about the subject matter in elementary and high school turns out to be a significant component of their preparation for teaching" (Ball, 1989, p. 3).

Galbraith (1982) voiced similar concerns about the conceptions of secondary mathematics teachers and what these conceptions imply about their ability to teach school mathematics. He stated, "Concern has been expressed for the *recycling effect* induced when students lacking in some essential mathematical background, return to the education system as teachers" (1982, p. 91, emphasis added).

Concern about teachers' knowledge was also raised by Cooney (1999), who noted that previous conceptions of teachers' knowledge were based primarily on their

knowledge of mathematics, and that reforming mathematics teacher education basically meant requiring teachers to take more mathematics courses (1999). Mathematical knowledge is needed for teachers to be able to teach students. In addition, teachers must have the ability to transfer their knowledge to a form that students are able to understand and learn. Cooney (1999) and Shulman (1986) raised concerns that even though the pre-service teachers have a strong mathematical background, the translation of that mathematical knowledge into meaningful tasks for students is often suspect. Both Galbraith (1982) and Cooney (1999) emphasized the need for teacher preparation programs to help teachers make connections with the mathematics they eventually will be teaching.

Evidence supports the idea that a teacher's *sound* conceptual understanding of mathematics can influence students' learning in a positive way (Fennema & Franke, 1992). Researchers often characterize conceptual understanding in terms of two constructs: *concept definition* and *concept image* (Tall and Vinner, 1981). Chapter Two of this research study addresses these constructs in detail and their relationship to an individual's conceptual understanding of proof. Examining the skills that prospective teachers need leads to the question of what conceptions they possess as they complete their preparation program. In other words, are prospective teachers' conceptions of proof consistent with what they will need to teach reasoning and proof? This research study examines prospective secondary teachers' conceptions of proof and refutations.

Purpose of the Study

Many in the mathematics education community have emphasized the importance of teaching students logical reasoning and formal proofs, and providing opportunities to examine conjectures in all content areas within school mathematics (Hanna, 1997; MAA, 1998; NCTM, 2000; Schoenfeld, 1994; Sowder & Harel, 1998). The goals that the current reform effort sets forth for reasoning and proof are similar to goals of prior reforms, such as those from the New Math era. A key factor in the failure of the New Math movement was the lack of preparation of teachers (Hanna & Jahnke, 1993; Hirshi, 1977), a lesson not lost on subsequent reform efforts. The importance of preparing teachers to teach proof and reasoning has been widely recognized (Glenn Commission, 2000; MAA, 1998; MSEB, 2001; NCTM, 2000). One of the significant areas in the preparation of teachers is the teachers' mathematical content knowledge (Ball, 1989; Cooney, 1999; Galbraith, 1982; MSEB, 2001; NCTM, 2000; Shulman, 1986).

Studies have shown that students' inadequacies and misconceptions in the area of proof and reasoning are widespread at the secondary level (Balacheff, 1987; Bell, 1976; Burke, 1984; Chazan, 1989, 1993; Fischbein & Kedem, 1982; Galindo, 1997; Galindo, et. al., 1998; Healy & Hoyles, 1998; Schoenfeld, 1989; Senk, 1985, 1989; Williams, 1979). If professional development programs are to help prospective teachers change this cycle of inadequate understanding of proof, then educators of prospective teachers need to continually intervene in prospective teachers' experiences at the university level (Galbraith, 1982; Ball, 1989). Prospective teachers must understand and be able to teach

proof and reasoning in areas recommended by the NCTM (2000) *Standards*. Thus, the current reform recommendations in proof and reasoning, and the increased awareness of the preparation of teachers, establish a need for research to describe prospective secondary teachers' conceptions of proof as they are completing their teacher preparation program.

Many of the research studies in the area of proof and reasoning have focused on high school students or college students that are not majoring in education. Research that has focused primarily on elementary or secondary teachers' conceptions of proof is sparse and inconclusive. There has been little research in the area of prospective secondary teachers' conceptions of proof as they are completing their preparation programs, especially since the release of the *Standards* 2000. Also, very few studies exist that research prospective secondary teachers' cognitive understanding of proof by examining their understanding of the logical underpinnings of proof and their ability to complete proof.

The need for research examining prospective teachers' conceptions as they are completing their teaching training programs, is brought about by the recent reform efforts and the lofty goals of these efforts. This study intends to focus on prospective secondary teachers' conceptual knowledge about the nature of proof. To gain better insight into an individual's conceptions of proof, this study will investigate not only the ability to complete proofs but also an individual's understanding of the logical underpinnings of proof. An examination of prospective teachers' conceptions of proof helps the

mathematics education community evaluate the preparation of prospective teachers in this important area of mathematics education.

Statement of the Problem

This study investigates prospective secondary mathematics teachers' conceptions of proof and refutations. In keeping with the traditions of mathematics education research, conceptions are defined operationally as an individual's knowledge about the subject matter (Fennema & Franke, 1992; Knuth, 1999; Thompson, 1992).⁶

The fundamental purpose of proof, according to Layman (2002), is to show that the premises lead, by way of valid rules of inference, to the conclusion. To investigate the primary question of this study, the researcher addressed two criteria of conceptions of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. The logical underpinnings of proof are defined as the laws of logic that underlie proof. The research study first examined an individual's *understanding* of the logical underpinnings of proof of several common proof schemes emphasized by the NCTM *Standards* 2000 document—direct proof, indirect proof (proof by contradiction), and refutation of a false conjecture. Secondly, the study examined prospective secondary mathematics teachers' *abilities to complete* mathematical proofs that include methods that are emphasized by the *Standards* 2000—direct proof, indirect proof, and refutation of a false conjecture.

⁶ An overview of conceptions and logical underpinnings of proof is discussed in Chapter Two.

Research Questions

Primary Research Question

What are prospective secondary mathematics teachers' conceptions of proof and refutations?

Secondary Research Questions

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof and refutations?
 - a. What are their understandings of the logical underpinnings of direct proof?
 - b. What are their understandings of the logical underpinnings of indirect proof?
 - c. What are their understandings of the logical underpinnings of refutations?
 - d. Can they demonstrate an understanding of the laws of logic that underlie proof?
2. What are prospective secondary mathematics teachers' abilities to complete mathematical proofs?
 - a. What are their abilities to complete direct proof?
 - b. What are their abilities to complete indirect proof?
 - c. What are their abilities to refute a false conjecture?

Definition of Terms

For the purpose of this research, the following definitions were used in the study (Burke, 1984; Esty, 2000; Hershkowitz, 1990; Karush, 1962; Layman, 2002; Mish & Morse, 2000; Moore, 1990; Serra, 1993; Tall & Vinner, 1981; Williams, 1979).

Ability	Competence in doing; skill; natural aptitude or acquired proficiency.
Argument	A set of statements, one of which called the conclusion, is affirmed on the basis of other statements which are called premises.
Axiom (Postulate)	A mathematical proposition assumed to be true.
Complex Statement	A statement that is a simple (prime) statement (statement without connectives) or a statement containing one or more connectives; also called compound statement.
Concept Definition	A concept's formal mathematical definition with critical attributes and non-critical attributes; a criterion for classifying the structural characteristics of an object.
Concept Image	The total cognitive structure that an individual associates with the concept, which includes all the mental pictures, associated properties, and processes.
Concept Usage	The methods an individual operates with in generating or using examples or in completing a proof.
Conditional Statement	A statement written in symbolic form $P \rightarrow Q$ which means if you are given premise or statement P , then you can conclude Q ; written as <i>If P, then Q.</i>
Conjecture	A mathematical statement which has neither been proved nor refuted by a counterexample.
Connectives	The five connectives in logic are: 1) negation (not); 2) conjunction (and); 3) disjunction (or); 4) conditional (if H , then C); and 5) biconditional (if and only if).

Converse	The converse of the statement $P \text{ implies } Q$ where P and Q are statements, is the statement $Q \text{ implies } P$; written as <i>If Q, then P.</i>
Conversion	Reasoning from the converse is called conversion. This is an invalid pattern of reasoning whereby one concludes P when given $P \text{ implies } Q$ and Q .
Contradiction	A compound statement form which is false for all truth values of its components.
Contrapositive	The contrapositive of a statement $P \text{ implies } Q$ where P and Q are statements, is the statement $\text{not-}Q \text{ implies not-}P$; written as <i>If not-Q, then not-P.</i> A conditional statement and its contrapositive are logically equivalent.
Counterexample	An instance in mathematics that falsifies an assertion. A counterexample is used in proving the negation of a statement by finding at least one example in which the generalization is false. A counterexample demonstrates an argument form is invalid by showing the form can lead from true premises to a false conclusion.
Deductive Proof	A chain of reasoning based upon accepted assumptions called axioms (postulates), definitions, and/or previously proven propositions, which, provided the accepted rules of logic are followed, demonstrates that a conclusion is necessarily true if the postulates on which the argument is based are accepted as true.
Definition	A description of a new term by accepted undefined terms and previously defined terms.
Empirical Evidence	Evidence that is based on observation and experiments; the basis of inductive reasoning.
Generalization	A statement that asserts that something is always the case.
Indirect Proof	A proof that employs the Proof by Contradiction or the Proof by Contrapositive method to prove <i>If P, then Q.</i> The steps of an Indirect Proof use the negation of C (or the negation of a component of C) as a hypothesis.

Inductive Reasoning	Inductive reasoning is a process of observing data or empirical evidence, recognizing patterns, and making inferences about those observations.
Inference	In logic, inference is a single-step deduction from premises.
Inverse	The inverse of the statement <i>P implies Q</i> where <i>P</i> and <i>Q</i> are statements, is the statement <i>not P implies not Q</i> ; written as <i>If $\sim P$, then $\sim Q$</i> .
Logic	Logic is the study of methods for evaluating mathematical arguments.
Logically Equivalent	Two statements <i>P</i> and <i>Q</i> are logically equivalent if <i>P</i> and <i>Q</i> have the same truth values whenever all prime statements in one have the same values as corresponding prime statements in the other.
Logical Underpinnings of Proof	Laws of logic that underlie proof; the methods that underlie the basis of proof techniques and evaluation of mathematical arguments.
Modus Ponens	Modus Ponens is a valid rule of inference that allows an individual to conclude <i>Q</i> when given <i>P implies Q</i> and <i>P</i> .
Modus Tollens	Modus Tollens is a valid rule of inference that allows an individual to conclude <i>not P</i> when given <i>P implies Q</i> and <i>not Q</i> .
Premise (Hypothesis)	Statements on the basis on which a conclusion is affirmed. In logic, the proposition <i>P</i> in a conditional statement <i>If P, then Q</i> .
Proof (Valid Argument)	An argument that follows that if the premises are true, then the conclusion is true. The validity of an argument is guaranteed by its form and does not depend on the content of its subject matter.
Proof by Contradiction	To prove <i>If P, then Q</i> , the Proof by Contradiction method assumes that <i>P</i> is true and assumes the negation of <i>Q</i> is true, and then deduces any contradiction.

Proof by Contrapositive	To prove <i>If P, then Q</i> , the Proof by Contrapositive method consists of giving a direct proof of the contrapositive of the statement (<i>If not-Q, then not-P</i>).
Proof by Counterexample	A mathematical method of proving an assertion is false by the demonstration of the existence of a counterexample.
Proposition	A proposition (or statement) is a sentence that is either true or false.
Refutation	The process of proving a statement is false or wrong by argument or evidence.
Statement	A statement is a sentence that is either true or false; also called a proposition. The truth value of a statement is true if the statement is true and false if the statement is false.
Understanding	Comprehension; a mental grasp; to grasp the meaning of; knowing.

CHAPTER 2

LITERATURE REVIEW

Introduction

The focus of this study was to determine prospective secondary mathematics teachers' conceptions of proof and refutations. This chapter provides a synthesis of the literature on proof and refutations that are related to this study. The themes for this chapter are: the underpinnings of proof; the nature and role of proof; concept understanding; research on conceptions of proof; and theoretical framework of proof schemes. A summary of the results and methodological principles of the research studies conclude this chapter.

Underpinnings of Proof

According to Barnier and Feldman (1990) "A basic knowledge of logic is indispensable for analyzing and constructing proofs" (p. 1). Logic is the study of methods for evaluating mathematical arguments where arguments are defined as a set of statements, one of which, called the conclusion, is affirmed on the basis of the others, which are called the premises or hypotheses (Layman, 2002). The logical underpinnings of proof include propositional logic, which is the study of certain kinds of statements. Propositional logic involves forming complex statements from simple statements and

then determining the truth value of the complex statements from the truth values of the simple statements. According to Fendel and Resek (1990), the two primary ideas of formal mathematical logic are implication and negation. Implication is the causal relationship in a conditional statement between its hypothesis and conclusion, whereas negation provides a link between the two types of quantifiers (universal and existential), and is an essential element in understanding the connection between a conditional statement and a counterexample.

Logical underpinnings form the basis of proof techniques for the construction and analysis of proofs (Barnier & Feldman, 1990; Epp, in press). The basic idea of proof (valid argument) is to infer or derive the conclusion from the hypothesis through valid rules of inference (Nolt, Rohatyn, & Varzi, 1998). Therefore, a valid argument is one in which the truth of the conclusion is absolutely guaranteed, given the truth of the hypothesis. Rules of inference, also referred to as methods of proof, are used to construct proofs. Table 1 displays three common methods of proof of the conditional statement, *If P , then Q* (Fletcher & Patty, 1996, p. 27).

Table 1. Methods of Proof

Direct Proof	Contrapositive Proof	Contradiction Proof
Assume P .	Assume <i>not</i> Q .	Assume P and <i>not</i> Q .
⋮	⋮	⋮
(logical sequence of steps)	(logical sequence of steps)	(logical sequence of steps)
⋮	⋮	⋮
Conclude Q .	Conclude <i>not</i> P .	Conclude R and <i>not</i> R .

According to Barnier and Feldman (1990) *semantics* is a set of rules used to determine the truth or falsity of statements. An easy way to summarize semantic rules is

by using a *truth table*. Truth tables help distinguish if two statements are logically equivalent—have the same truth values whenever the main columns of their standard truth tables are identical. Table 2 represents the truth table for conditional statements (Fletcher & Patty, 1996, p. 4). Each row of the table represents a possible combination of the truth values of the simple statements and conditional statement.

Table 2. Truth Table for a Conditional Statement

Hypothesis H	Conclusion C	Conditional Statement $H \rightarrow C$
True	True	True
True	False	False
False	True	True
False	False	True

Barnier and Feldman (1990) suggest that to understand a proof one must know: 1) the goal of the proof; 2) the hypotheses; 3) the necessary facts and definitions of the content area; and 4) previously proved facts or laws of logic to be used in the proof. According to Epp (in press), goals for a first course in reasoning and proof for university students should include helping students appreciate the role of definitions in mathematical proof and reasoning and also develop an ability to evaluate the truth or falsity of mathematical statements. She further suggests that an introductory unit on the principles of logical reasoning provides a supportive framework in which students can draw from while learning various aspects of proof and disproof. Epp (in press) states that for future teachers to effectively guide the development of students' reasoning abilities, as

is expected by the NCTM 2000 *Standards*, they need to have an understanding of what constitutes a valid argument and what it means for statements of various forms to be true or false (2002, p. 8).

The Nature and Role of Proof

The conceptual framework for research in proof is addressed by examining the *nature* of proof and the *role* of proof (Borko, Peressini, & Romagnano, 2000). The *nature* of proof refers to an individual's explanations, justifications, and elaborations in order to make a conjecture more convincing (Lakatos, 1976; Borko, et.al 2000). The primary *role* of proof in mathematics is to validate propositions, but an additional role is to explain the meaning and the mathematical basis of the theorem being proved (Hanna, 1998).

The Nature of Proof

Fawcett (1938) stated that a pupil understands the nature of deductive proof when he or she understands:

1. The place and significance of undefined concepts in proving any conclusion.
2. The necessity for clearly defined terms and their effect on the conclusion.
3. The necessity for assumptions or unproved propositions.
4. That no demonstration proves anything that is not implied by the assumptions (p.10).

In addition to understanding these four ideas, it is assumed the student will also understand that the conclusions that are established can have universal validity only if the definitions and assumptions that imply these conclusions have universal validity.

Lakatos (1976) examined the nature of proof in *Proof and Refutations*, through a portrayal of a classroom debate among fictional students about the truth and meaning of a geometry theorem. Students presented arguments about what a proof of a theorem means in mathematics. Through reading the debate, one gains Lakatos's insight that mathematics develops by a process of "conscious guessing" about relationships.⁷ According to Lakatos, proof follows a "zig-zag" path that is prodded by counterexamples. Once counterexamples are discovered, proof moves from

... the naive conjecture to the premises and then turns back again to delete the naive conjecture and replace it by the theorem. Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end-product (Lakatos, 1976, p. 42).

An important part of Lakatos's book was the classroom discussion, in which students made conjectures and participated in a dialogue about the truth or falsity of the conjecture. Lakatos emphasized that students take a risk when they present their arguments and allow them to be evaluated by their peers.

Bell (1976) viewed the meaning of proof as having three senses: (a) *verification* or *justification*, which is concerned with the truth of a proposition; (b) *illumination*, which conveys insight into why a proposition is true; and (c) *systematisation*, the organization of results into a deductive system of axioms, major concepts and theorems.

Balacheff (1987) classified proofs as *pragmatic proofs* and *intellectual proofs*. Pragmatic proofs are naive empirical arguments or arguments based on crucial

⁷ Lakatos referred to conjectures as "conscious guesses" (1976).

experiments. Intellectual proofs assert the truth of a mathematical statement based on a process of reasoning that is usually in the making of a general argument.

To understand student' conceptions of the nature of mathematical proof, Harel and Sowder (1998) distinguished proof schemes. According to Harel and Sowder (1998) "A person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 244). Harel and Sowder (1998) distinguished between three proof categories or as they referred to as proof schemes—*external conviction proof scheme*, *empirical proof scheme*, and *analytical proof scheme*. They asserted that a person's evidence and justification of an observation is based on "...logical and deductive arguments, empirical evidence, intuitions, personal beliefs, an authority, ..., social conventions, or any other knowledge the person considers relevant to the truth of the observation" (p. 243).

The Role of Proof

In mathematics, one of the main functions of proof is to validate propositions (Alibert & Thomas, 1991; Balacheff, 1987; Bell, 1976; de Villiers, 1999; Fawcett, 1938; Hanna, 1998; Harel & Sowder, 1998; Schoenfeld, 1994). Schoenfeld (1994) stated that one of the most wonderful features of proof is that it yields certainty. "When you have a proof of something you know it *has to be* true, and why" (Schoenfeld, 1994, p. 74).

Mathematical proof also plays a role in convincing and explaining mathematical ideas (Hanna, 1998; Hersh, 1993; Schoenfeld, 1994). "The best proof, even in the eyes of practicing mathematicians, is one that not only establishes the truth of a theorem but

also helps understand it" (Hanna, 1998, p. 9). Proof provides insight into *why* the theorem is true.

Many researchers noted the importance of considering proof as both a cognitive (verification, convincing, or explanation) and a social process (Alibert & Thomas, 1991; Bell, 1976; de Villiers, 1999; Hanna, 1991, 1998; Harel & Sowder, 1998; Lampert, 1990; Simon & Blume, 1996; Vernaud, 1990; Yackel et. al, 2000). The role of proof as a social activity is when individuals determine the truth or falsity of mathematical statements to each other, promoting an understanding of *why* the statement is true or false. The manner that Alibert and Thomas (1991) described proof was as "...a means of convincing oneself whilst trying to convince others" (p. 215).

There are implications that an individual's concept image is influenced by the classroom social interactions as in Lakatos (1976) in which he created an imaginary classroom and narrated the social interactions amongst students as they tackle their concept image of proof. According to Lampert (1990), for students to expose their exploratory thinking to others with the hope that by engaging in an exchange of ideas in the classroom discussion, takes courage and modesty but students may end up with better ideas in end.

Bell (1976) also acknowledged the importance of a student's own activity of arriving at their conviction, of making a verification, and in communicating their convictions of results to others. In addition, Bell suggested the development of the proof process grows out of an internal testing of a mathematical idea that is either accepted or rejected, which in turn accompanies the development of a generalization. According to

Bell (1976), gradually the proof process becomes more externalized. This internal testing that Bell suggested is coined by others as a person's *concept image*.

Concept Understanding

Concept Image and Concept Definition

According to Piaget (1967) the last stage of cognitive development—Formal Operations—involves the transition from concrete to formal thinking, or what Piaget referred to as "hypothetico-deductive" thinking (p. 62). "Formal thought is 'hypothetico-deductive' in the sense that it permits one to draw conclusions from pure hypotheses and not merely from actual observations" (Piaget, 1967, p. 63). Piaget (1967) added that these conclusions have a validity independent of their factual truth which explains why formal thinking represents so much difficulty and so much more mental work than concrete thinking. In the formal stage of cognitive development, an individual constructs new operations and operations of propositional logic (Piaget, 1972).

While Piaget's cognitive stages of development were based on the maturation of the student, Pierre van Hiele and Dieke van Hiele-Geldof based their five levels of geometric thought on the experiences of the student (van Hiele, 1986). According to the van Hiele's (1986), the fourth level, Formal Logic, includes understanding the notions of mathematical definitions, axioms, and theorems and the ability to write formal proofs. The fifth and highest level of geometric reasoning is the Nature of Logical Laws, which is highly abstract. In this level, students can recognize the relationships between different axiomatic systems (e.g., Euclidean and non-Euclidean geometry).

To understand the cognitive aspects of proof in mathematics, we look outside of psychology. Fischbein (1990) suggested a particular system of concepts for use in mathematics and mathematics education was needed in addition to the general ones inspired by psychology. Fischbein stated "Even Piagetian stage theory and the respective findings concerning mathematical concepts (number, space, chance, function, etc.) cannot be translated directly in curriculum terms" (1990, p. 9). Two of the concepts that Fischbein referred to are: *concept image* and *concept definition*. According to Tall (1992), "Mathematical proof as a human activity requires not only an understanding of the concept definitions and the logical processes, but also insight into how and why it works" (p. 506). To understand the cognitive aspects of an individual's conceptual understanding of a mathematical idea, this research study examines the individual's *concept image* and the *concept definition*.

According to Tall and Vinner (1981), when comparing mathematics to other fields, mathematics is usually thought of as a subject of great precision where concepts can be defined accurately so as to provide a solid foundation of mathematical theory. Tall and Vinner suggested that the psychological reality of this idea rests instead on the fact that many concepts which are used in mathematics are not formally defined at all, but instead are learned by experience and usage in appropriate contexts. To understand how these processes work, Tall and Vinner formulated a distinction between mathematical concepts as they are formally defined and the cognitive processes by which they are learned (1981). Tall and Vinner stated,

We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures (p. 152).

As an example, they take the concept of subtraction. Subtraction is usually first introduced to children as a process that involves positive whole numbers. Children may observe that in this stage subtraction always reduces the answer. Tall and Vinner stated that this observation is part of the child's concept image and could possibly cause problems later on when subtraction of negative numbers is introduced. As in this example, this is the reason that all the mental attributes that are associated with a concept, conscious or unconscious, are included in the concept image which may consist of "seeds" of future conflict.

Vinner (1991) stated "A concept name when seen or when heard is a stimulus to our memory" (p. 68). Something in an individual's memory is evoked by the concept name and what is evoked is not the concept definition, but rather the "concept image" (Vinner, 1991). Whereas the concept image is the concept as it is reflected in an individual's mind, Hershkowitz (1990) described *concept definition* as the concept's mathematical definition that has critical attributes and non-critical attributes. The definition is considered as a criterion for classifying the concept; it is a structural characteristic of the object.

Tall (1995) addressed an individual's conceptual understanding of proof. He stated, "...the cognitive development of proof is dependent on the cognitive structure and representations available to the learner at the given time" (p.38). According to Tall,

students' difficulties with proof occur when the enactive or visual form of a proof does not suggest an obvious sequence of deductions to use for a formal proof (1995, p. 38).⁸ Tall added that the cognitive struggle to change from visual to formal is "huge" for the learner. He noted that the student seems to *know* that the theorem is true but has no method of proving it true (Tall, 1995). This helps mathematical educators understand a student's use of empirical evidence to attempt to justify mathematical arguments.

To understand the cognitive aspects of the students' conceptual understanding of a mathematics idea, Moore (1994) used a *concept-understanding scheme* that consisted of three components—concept image, concept definitions, and conceptual usage. According to Moore the *concept-understanding scheme* with its three components was the primary category in the data analysis and was useful in explaining the students' difficulties.

The importance of examining an individual's concept image in research studies is implied by Herschkowitz's statement that the aim of research is "...to follow the development of the concept image in the individual mind (or in a given population), where the concept provides the frame of reference against which this development is compared and examined" (1990, p. 81). Balacheff (1990) suggested that research should include studies on students' knowledge and the underlying difficulties they may have. According to Balacheff (1990), terms such as *misconception* are replaced by expressions

⁸ Tall's theory is in line with Herschkowitz (1990) as she suggested that the construction of a concept image is a mixture of visual and analytical processes.

like students' conception or *concept image*. Instead of exposing students' errors, Balacheff suggested a search for explanations of the origin of the errors.

Research on Conceptions of Proof

Research results on conceptions of proof are remarkably uniform. Some research studies focused on particular aspects of proof while other studies took a broader view focusing on students' conceptions of proof. Most results reveal that students' concept image shows their difficulties in not being able to distinguish between inductive and deductive reasoning, counterexamples, generalization aspects of proof, perceptual problems, and not being able to get started with proof. The current research study is focused on a broader view of proof. Although the present study is in the area of prospective teachers' conceptions of proof, some studies of adolescent and high school students are included because they address issues that are applicable across a wide range of age groups.

Research Studies of High School Students' Conceptions of Proof

Bell (1976) analyzed 14- to 15-year old students' attempts to construct proofs and their explanations to simple mathematical situations. He wished to observe how students' responses differed from mature mathematician's use of proof. He classified students' responses into two main categories—empirical and deductive. He then identified sub-categories for each of these categories. The sub-categories ranged from a complete lack

of understanding to the highest level—a complete explanation deriving the conclusion by valid, connected arguments.

Bell (1976) found that 48% of the students' explanations failed to generate correct examples or to comply to the given conditions on a problem based on elementary number concepts where there were a finite number of cases. Bell stated that 9% asserted the generalization after examining a few cases and 19% correctly checked the full set of 14 cases. The rest of the students asserted the truth of the generalization, partly by checking a small number of cases with a sense that this explanation was not complete. Another problem asked students to identify all the possible ways of producing triangles to meet given conditions. Results showed that 28% lacked an understanding between the conclusion and the details of the data. Bell attributed students' failure to a lack of knowledge about mathematical concepts related to the problems.

Williams (1979) researched 11th grade students' understanding of the nature and role of proof. The goal of the study was to identify, categorize, and describe students' subjective thinking processes in attempts to justify mathematical generalizations and conclusions. He administered a twelve-item questionnaire to 255 students in ten randomly selected classes. For each of the items, a categorization scheme was made to analyze each of the students responses to the "why" question. William's analysis of students' responses was based on a hierarchical ordering of the response categories for each item.

An overall summary of students' responses found that less than 30% of the sample exhibited any understanding of proof and those were students that were predominantly

classified as high achievers in comparison with their peers (Williams, 1979). Approximately 50% of the students did not see any need to prove a mathematical proposition that they consider to be intuitively obvious. At least 70% did not distinguish between inductive and deductive reasoning and therefore did not understand that induction was inadequate to support mathematical generalizations. Approximately 80% did not always realize the significance of the hypotheses and definitions in mathematical proof. Less than 20% understood the method of indirect proof and 80% did not understand the concept of counterexample. His results showed no evidence to suggest that students understand the logical equivalence of a mathematical argument and its contrapositive, or that a statement and its converse are not logically equivalent. Williams (1979) gave a proof problem that was generalizable for all triangles. Results showed that 20% of students view deductive proof as a proof for a single, isolated case, the case that is pictured in the given diagram. This is the generic example proof scheme. Students who held this belief did not understand the generic aspect for deductive proofs; they did not understand that the validity of the conclusion is meant to be generalizable to all figures that satisfy the given conditions.

A study by Fischbein and Kedem (1982) explored whether 397 high school students understood the conception of mathematical proof as a formal proof of a mathematical statement that excludes the need for any further checks. The answers of students in grades 10 to 12 were analyzed and were summarized in one of the five categories: 1) Consistently formal—understand correctly the nature of mathematical proof; 2) Basically inconsistent—demonstration of inconsistent behavior, accepting the

absolute validity of the proof, on the one hand, but not rejecting the need for additional checks on the other; 3) Consistently empirical—an empirical approach to mathematical proofs; students believed that additional checks of particular cases add support to the statement that was proved; the proof by itself does not guarantee the absolute validity of the statement; 4) Reject the proof—students in this category did not agree with proof; and 5) Others—irrelevant answers or did not answer various questions (Fischbein & Kedem, 1982).

Fischbein and Kedem (1982) found that fewer than 10% of the students were consistently formal and approximately 33% were basically inconsistent. Half of the students could not be classified according to their criteria. Many students did not grasp the understanding that mathematical proof requires no further empirical verification. Fischbein and Kedem state, "...students, after finding or learning a correct proof for a certain mathematical statement, will continue to consider that surprises are still possible, that further checks are desirable in order to render the respective statement more trustworthy" (p. 128). Students misunderstanding of empirical evidence as mathematical justification was also documented by a 1988 study by Balacheff in France.

Research studies by Chazan (1989) and Martin and McCrone (2001) explored high school students' beliefs about what constitutes a proof. Both studies used similar methodology in examining students' understanding of the logical underpinnings of what constitutes a proof. The similar methodology for the studies administered multiple-choice items in which students circled one of three answers—agree, neutral, or disagree. The Chazan study included items that tested students' propositional knowledge (i.e.,

logical underpinnings of proof) and items that asked students to decide whether an argument was a convincing argument or not and then were asked to explain their opinions. Martin and McCrone (2001) also explored students' ability to construct a proof.

The results of studies by Chazan (1989) and Martin and McCrone (2001) found that a sizable number of students viewed deductive geometry proofs as a proof for a single isolated case where the particular case in question was the one which was pictured in the associated diagram. Chazan also found that many students contended that measuring examples allows one to reach conclusions in mathematics that are safe from counterexample and applicable to sets that have an infinite number of members, such as the set of all triangles. Martin and McCrone also found that content knowledge was a major factor in student proof construction ability where students' showed great difficulty constructing proof unless key relationships were outlined for them.

Two large studies by Healy and Hoyles (1998, 2000) and Senk (1985, 1989) examined high school students' proof ability. In the results from the Healy and Hoyles study, 2,459 students from England and Wales, classified as high-attaining (top 20 to 25% of the student population), performed poorly in proof construction. According to Healy and Hoyles, empirical verification was the most popular form of argumentation used by students even though they were aware that the empirical arguments were not general. Problems in which empirical examples were not as easy to generate found the majority of students failed at engaging in the process of proving.

Senk (1985, 1989) conducted a large study of 1,520 United States high school students at the end of the school year. Senk's results were similar to results by Healy and

Hoyles (1998, 2000). According to Senk (1985) only 30% of high school students in a full-year geometry class could reach a 75% mastery level of proof. Senk (1989) suggested that proof achievement was attributed partly to a lack of the geometry curriculum for the elementary and middle grades and that the typical high school mathematics program does not provide opportunities for students to write proof in any other context outside of the geometry course. She stressed the need for greater attention to prerequisites for high school geometry (1989, p.320).

Burke (1984) investigated counterexample logic of 6th grade, 9th grade pre-algebra, geometry students that had taken pre-algebra, and geometry students that had not taken pre-algebra. Different tests were developed to examine students' counterexample logic. A computer test was developed that consisted of tasks in which the student was asked to evaluate three claims using three different lists of numbers stored in the computer. The claims had 0, 1, or 10 counterexamples for each of the three claims respectively.

Burke's results showed that sixth grade students were weak in all areas of logic while 9th graders exhibited weakness in most areas of counterexample logic. Results show that over 50% of ninth-graders' responses and approximately 67% of geometry students understood that a single counterexample falsified a claim. Geometry students that had not taken pre-algebra appeared to understand all the aspects of counterexample logic except that inductive evidence does not prove a claim. Geometry students that had taken pre-algebra appeared to understand some of the aspects of counterexample logic.

Research Studies of College Students' and Teachers' Conceptions of Proof

One the studies most similar to this research study was that of Galbraith's (1982) study which focused on the mathematical characteristics of prospective teachers and their understanding of mathematics. A questionnaire was given to 263 undergraduates and 116 postgraduates in three Australian states at the end of the 1980 academic year. The undergraduates were students enrolled in typical first-year mathematics courses. The postgraduates were students enrolled in the Diploma of Education teacher training courses having graduated with a major or minor in mathematics.

Galbraith used a multiple-choice format for the test where five possible answers were given for each of the 18 questions of which one and only one answer was correct. The items used were not designed to meet criteria for a particular course or to measure an algorithmic procedure. Instead, items were chose by Galbraith for the kinds of attributes that he felt mathematical students should possess. In fact Galbraith described the test as "a test of *mathematical vitality*" (1982, p. 91). Galbraith noted that the concepts and ideas in the questions are at the secondary level.

The following item was from Galibraith's study:

4. A theorem in geometry has been proved. Which one of the following statements is necessarily correct?

- (A) The converse is a theorem and does not need further proof.
- (B) The converse is a theorem and but needs further proof.
- (C) The converse is false and does not need disproof.
- (D) The converse is false but needs disproof.
- (E) None of the above (1982, p. 93).

The results for Problem 4 show that only 18% of the undergraduate and 37% of the post-graduates got this problem correct [answer (E)]. Galbraith also included a question (problem 15) that asked students if the evaluation of a statement was justified using the converse in which 24% and 34% of undergraduates and post-graduates respectively answered correctly. Galbraith stated the responses to the questions 4 and 15 show that at both levels students believed the converse to be true and that many students believed no proof was necessary if the original statement is known to be true.

In Galbraith's study, the following Problem 13 asked students to understand counterexample as a means of refuting a claim:

13. A statement (S) reads as follows:

S : A whole number is divisible by 6 if the sum of its digits is divisible by 6.
Select whichever of the following that you believe to be true.

- (A) The number 33 shows that S is false.
- (B) The number 30 shows that S is false.
- (C) The numbers 30 and 33 both show that S is false.
- (D) S is false but neither 30 or 33 is adequate to disprove it.
- (E) S is true (1982, p. 94).

The results for Problem 13 show that only 40% of the undergraduate and 59% of the post-graduates got this problem correct [answer (A)].

According to Galbraith the misconceptions and misunderstandings of high school students tend to remain and are resistant to correction as they eventually take advanced courses at the college level. He found that students have trouble understanding the use of counterexamples to evaluate the truth of a statement; a lack of understanding of conditions necessary for proof; and a lack of understanding of converse. He noted that

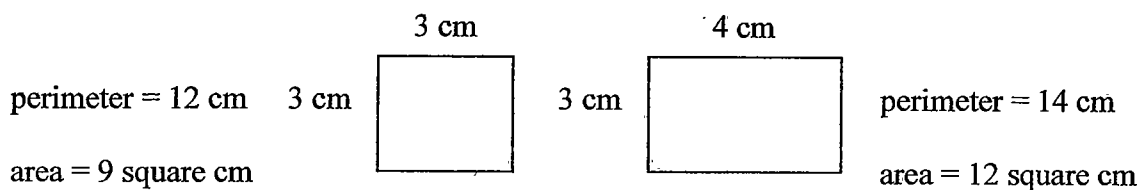
the general results of the study showed that mathematical vitality was not enhanced by taking more mathematics courses.

Galbraith emphasized that students need mathematics courses that ask more "Why is that the answer?" questions rather than "What is the answer?" type questions. What Galbraith described in 1982 is parallel with current reform efforts in the United States. He suggested without reform "...the wheel continues to turn" (p. 111). Galbraith's concern strengthens the need for this study because of the focus on pre-service mathematics teachers' conceptions of proof as they will eventually be faced with demands in current reform efforts in the area of reasoning and proof.

A research study by Ball and Wilson (1990) compared the mathematical understandings and pedagogical content knowledge of two groups of novice secondary school mathematics teachers. The participants were 22 undergraduate students preparing to teach mathematics with an emphasis in mathematics education (standard program), and a second group of 21 post baccalaureate mathematics majors that did not have an education emphasis (alternate program). The researchers challenged two common assumptions about the preparation of secondary school mathematics teachers. One assumption was that individuals that have a major in mathematics without an emphasis in education, are both more capable and knowledgeable than their mathematics education peers. The second assumption was that professional knowledge and pedagogical content knowledge are best acquired by practical experience as a full-time teacher. This implied that the university preparation makes few practical and significant contributions to what teachers need to know or be able to do.

A questionnaire was administered to the sample at repeated intervals and interviews were given to a smaller sample of students. Many of the questions were grounded in scenarios of classroom teaching with particular subject matter topics. Ball and Wilson (1990) reported on three questions from their study. The following is one of the three questions in which the participants were asked to assess the correctness of the mathematical ideas, and to explain and elaborate mathematical meanings that underlie some of the conventional procedures and ideas that they have learned.

1. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing.



One of the other questions in Ball and Wilson's study asked prospective teachers to come up with real-world situations or story problems to show the application of $1\frac{3}{4} \div \frac{1}{2}$. Another question asked participants how they would explain what 7 divided by 0 is.

The three questions were given to students at entry into the program and at the completion of the program. Results showed that despite dramatic structural and philosophical differences in standard university-based and alternate programs, there were not many differences between the beginning teachers' responses. For the false conjecture problem, only 10 of the 19 novice teachers were able to see the false conjecture between the relationship between perimeter and area at the exit of the program. The authors'

stated, "We see problems inherent in assuming what people who have majored in mathematics should know" (1990, p. 10). Ball and Wilson also noted that the two groups of novice teachers' ability to offer mathematical explanations to students was discouraging. What was alarming about the results was that neither the teachers with an emphasis in education or the teachers without the emphasis were well-prepared to unpack the meanings of mathematical ideas (Ball & Wilson, 1990). According to Ball and Wilson, over 50% of the novice teachers were unable to generate an appropriate representation for the division of fractions and were unconcerned with issues of mathematical proof.

Martin and Harel (1989) conducted a study of 101 pre-service elementary students enrolled in a sophomore-level mathematics course. Their goal was to judge the mathematical correctness of the pre-service teachers' inductive and deductive verifications. Results found that 80% of the 101 prospective elementary teachers considered at least one of the inductive arguments to be mathematical proof. Approximately 38% accepted an incorrect deductive argument as being mathematically correct for familiar statements while 52% accepted an incorrect deductive argument as being mathematically correct for unfamiliar statements. Over 33% of students simultaneously accepted an inductive and a correct deductive argument as being mathematically valid. In Martin and Harel's (1989) study, the pre-service teachers were influenced by the appearance of the argument rather than the correctness of the argument (ritual proof scheme).

A study by Knuth and Elliot (1997) examined the nature of 9 pre-service secondary mathematics teachers' understanding of mathematical proof and their expectations for proof. Results showed that pre-service teachers' expectations for their future students' understanding of proof were characterized at the lowest level. Students justified by means of naive empiricism, crucial experiment, and generic example proof schemes. They generalized the assertion based on a number of randomly chosen cases "The results suggest that several of the pre-service teachers' interpretations of mathematical proof differed from what the mathematics community would consider as mathematically acceptable" (Knuth & Elliot, 1997, p. 550). The authors described the pre-service teachers understanding of mathematical proof to be inadequate and problematic.

A similar research study by Knuth (1999) explored 18 in-service secondary mathematics teachers' conception of proof. Data was collected by interviewing the in-service teachers and by analyzing their written responses to proof items to two different take-home assignments, referred to by Knuth as round one and two. The second round of data collection consisted of evaluating five student-generated proofs and completing an indirect proof.

Knuth analyzed his data by adapting the proof schemes by Harel and Sowder's (1998) and Balacheff's (1991). Results of Knuth's study suggested that it would be difficult for the in-service teachers to successfully enact reform recommendations based on their conceptions of proof. Many had limited conceptions of the nature and role of proof, inadequate understandings of what constitutes proof and different methods of

proof. Knuth's results suggested that teachers viewed proof as a mathematical experience for students enrolled in upper level high school mathematics classes, specifically geometry.

The results of the proofs that teachers attempted from the three problems in round one were—in proof one 11 teachers successfully completed a valid proof and 5 teachers' proofs were invalid because they were empirically based. Proof two only 4 of the 18 completed a valid proof and in proof three the majority of the teachers were able to successfully construct a proof. Knuth noted that 11 of 16 teachers were able to identify the starting assumption for a multiple choice item on direct proof. He added that only 4 of 14 teachers that attempted to complete an indirect proof were able to produce a valid justification. Knuth stated that teachers did not recognize proof as a means for promoting mathematical knowledge.

A study by Moore (1990) investigated 16 college students understanding of proof in a course designed to teach proof. He focused on students' cognitive difficulties in learning to read and do proof. Moore collected data through examination of students' tests and final exams, class observation, interviews with the professor of the class and students, and through outside tutoring of students. The data that Moore collected included both students' understanding of the logical underpinnings of proof and ability to complete proofs.

Moore observed seven major sources of students' difficulties when attempting proof:

1. The students did not know the definitions, that is, they were unable to state the definitions.
2. The students had little intuitive understanding of the concepts.
3. The students' concept images were inadequate for doing the proofs.
4. The students were unable, or unwilling, to generate and use their own examples.
5. The students did not know how to use definitions to obtain the overall structure of proofs.
6. The students were unable to understand and to use mathematical language and notation.
7. The students did not know how to begin proofs (p. 42).

Moore (1990) added that quantifiers within a statement were a source of difficulty. He emphasized that a very common problem was that students started proofs with the wrong assumption rather than focusing on the conclusion of the proposition they were trying to prove. Moore noted that when students did begin a proof correctly, a key step which usually was the discovery of a "trick" often times prevented their successful completion of the proof. One aspect of students' difficulty with proof was often a cognitive overload—students' struggle with understanding abstract concepts at the same time they were learning techniques of proof, how definitions are used in proofs, and how proofs are expressed in mathematical language and notation.

The need for students to understand conceptual components of proof in order to construct proofs was supported in a study by Selden and Selden (1995). The study researched 61 undergraduate students in a course that introduced proofs and mathematical reasoning. Data was collected from tests and final examinations from six courses that the authors had taught. Selden and Selden (1995) investigated undergraduate students' abilities to clarify the logical structure of mathematical statements and the ability to use such structures in the construction and validation of proofs. The results

showed that students that are unable to determine the logical structure of statements of theorems cannot be expected to determine the correctness of their proofs nor actually construct the proof.

Selden and Selden (1995) referred to students' ability to *unpack* the logical structure of an informal statement to a logically equivalent formal statement that includes the logical features. They give the example: the informal statement—"a function is continuous whenever it is differentiable" can be unpacked to the formal statement—"for every function f , if f is differentiable then f is continuous" (p. 128). According to Selden and Selden, students must be able to understand hidden quantifiers to rewrite the informal statement as a formal statement.

The results of Selden and Selden's (1995) study found that no students could consistently unpack informally written mathematical statements into an equivalent formal version, even though the majority were third- and fourth-year university students specializing in mathematics or secondary mathematical education. They found that the ability to unpack an informal statement was an essential part of knowing that the argument is a proof of a theorem (Selden & Selden, 1995). They added that it was not surprising that students have an inadequate conception of proof and they believe this is partially traceable to the absence of students constructing proofs in school mathematics.

Theoretical Framework for Proof Schemes

Many of the research studies assessed an individual's completion of proofs by characteristics or schemes (Balacheff, 1988; Bell, 1976; Chazan, 1989; Fischbein &

Kedem, 1982; Harel & Sowder, 1998; Healy & Hoyles, 1998; Knuth & Elliot, 1997; Knuth, 1999; Martin & Harel, 1989; Williams, 1979). To assess the completion of prospective secondary mathematics teachers' proof schemes, this research study used an adapted framework of Harel and Sowder (1998), Knuth (1999), and Tall (1995).

Proof Scheme Framework

According to Harel and Sowder (1998), an individual's proof scheme consists of whatever the individual uses to convince himself or herself and others. Categories of proof schemes represent the cognitive stages in the students' mathematical development. Harel and Sowder's proof schemes are divided into three categories with each category consisting of sub-categories. The three primary categories are: 1) *External Conviction Proof Scheme*; 2) *Empirical Proof Scheme*; and 3) *Analytical Proof Scheme*. Table 1 summarizes the proof schemes for the present study that were adapted from Harel and Sowder (1998), Knuth (1999), and Tall (1995).

Table 3. Proof Scheme Framework.

I. External Conviction Proof Scheme	II. Empirical Proof Schemes	III. Analytical Proof Schemes
A. Authoritarian Proof Scheme	A. Naive Empirical Proof Scheme	A. Transformational Proof Scheme 1. Visual Proof Scheme 2. Manipulative Proof Scheme
B. Ritual Proof Scheme	B. Crucial Experiment Proof Scheme	B. Axiomatic Proof Scheme
C. Symbolic Proof Scheme	C. Generic Example Proof Scheme	

The three primary categories and the sub-categories for the External Conviction Proof Schemes are the same as Harel & Sowder (1998) and Knuth (1999). The category, Empirical Proof Scheme, uses the same three sub-categories that are used by Knuth and Balacheff (1987). Knuth and Elliot (1998) noted "These levels [three sub-categories] represent a hierarchy through which students are expected to progress as their notions of mathematical justification develop..." (p. 714). The Analytic Proof Scheme and first sub-categories follows that of Harel and Sowder (1998) except for the second sub-categories that were suggested by Tall (1995). Descriptions on some of the sub-categories are supplemented with results from previous studies or examples to further elaborate each construct.

External Conviction Proof Schemes

Harel and Sowder (1998) described *external conviction proof schemes* as schemes from an outside source that removes doubt for the student individually and by what the student offers to persuade others. The outside sources in external conviction proof schemes are comprised of three sub-categories: (a) authoritarian proof scheme—the word of an authority; (b) ritual proof scheme—the form of the argument presentation; and (c) symbolic proof scheme—symbolic representation of the argument.

The Authoritarian Proof Scheme

Harel and Sowder (1998) stated that they believe that the reason many students lack an intellectual curiosity in wondering whether a theorem or formula is true is because the current mathematics curricula emphasizes truth instead of emphasizing the

reasons for truth. An authoritarian proof scheme is when the main source of a student's conviction is a statement appearing in a book or some mathematical relationship the teacher utters or writes on the board. Balacheff (1991b) suggested a student's means for validating a statement relied too much on the teacher. Students are too easily swayed to rely on an authority or textbook as Hersh (1993) implied, "In the classroom, convincing is no problem" (p. 396).

The Ritual Proof Scheme

A student that judges the validity of a proof by the form of the proof rather than by the correctness of the reasoning involved is exhibiting the ritual proof scheme. The NCTM *Standards* 2000 emphasize the need for different forms of proof than the two-column format such as flow chart and paragraph proofs. Schoenfeld (1988) stated that there is nothing sacred about the two-column format of proof but he adds, "In many high school classrooms, however, it has taken on nearly sacred status" (p. 158). The use of such form has been emphasized so much that students with this scheme feel it is needed for justification. Schoenfeld (1994) stated that a tremendous emphasis on *form* tends to override issues of substance of a proof. "Accepting false-proof verifications on the basis of their appearance is a severe deficiency in one's mathematical education, which is possibly attributable to the over-emphasis in schools on proof writing prior to and even in place of proof understanding, production, and appreciation" (Harel & Sowder, 1998, p. 246).

The Symbolic Proof Scheme

Students demonstrate a symbolic proof scheme when they use symbols without reference to the argument they are proving. Many times students carelessly manipulate the symbolic representations involved in the argument without comprehending the problem statement or its relationship to their manipulations. Harel and Sowder stated students treat symbols as if they possess a life of their own without comprehending or building a coherent image of the problem (1998). According to Harel and Sowder, when students are given a problem, they will read the problem one time and "haphazardly" begin manipulating the symbolic expressions in the problem with little regard to comprehending the problem statement. They believed that symbolic reasoning is a habit of mind students should acquire during their school years.

Empirical Proof Schemes

Empirical proof schemes are justifications made on the basis of inductive arguments. Balacheff (1991a) described three levels of empirical proof—1) naive empiricism, 2) crucial experiment, and 3) generic example. Balacheff's three levels represent a hierarchy through which students are expected to progress as their justification of mathematical arguments progress. Harel and Sowder (1998) suggested two types of empirical proof schemes, the *inductive proof scheme* and the *perceptual proof scheme*. An inductive proof scheme is justification of a conjecture by quantitatively evaluating their conjecture through one or more specific cases. The naive empirical proof scheme is similar to Harel and Sowder's inductive proof scheme; also the crucial

experiment and generic example proof schemes are similar to the perceptual proof scheme in that a student makes a conclusion based on their perceptions of a single drawing or occasionally several drawings.

Students classified in the empirical proof scheme may use such ideas as direct measurements of quantities, numerical computations, examples or figures, or substitutions of specific numbers in algebraic expressions to justify a mathematical statement. A common difficulty among students is the lack of understanding that once a formal proof of a mathematical statement has been presented it excludes a need for further checks of empirical evidence. (Fischbein & Kedem, 1982; Harel & Sowder, 1998; Martin & Harel, 1998). Research findings indicate the dominance of empirical proofs (Balacheff, 1988; Chazan, 1989; Fischbein & Kedem, 1982; Healy & Hoyles, 1998; Martin & Harel, 1989; Williams, 1979).

The empirical proof scheme is a dominant means of justifying for many students because the use of inductive evidence is very natural in everyday life. Harel and Sowder (1998) stressed that the dominance of inductive reasoning within college students is not surprising when students' previous mathematics instruction at the elementary and secondary levels is dominantly inductive at best and also suggested that many students experience authoritarian, ritual, and symbolic instruction at these levels.

Naive Empirical Proof Scheme

Naive empiricism proof scheme is when students' believe that empirical evidence is valid justification for mathematical arguments in which the domain of the set has an

infinite number of members. Students will often conclude that an assertion is valid based on one case or a small number of randomly chosen cases. The students reasoning is they believe that since their single example was randomly chosen and it conformed to the general statement, they conclude the statement must be true. Their single random sample is representative of all cases in their eyes. An example of a naive empirical proof is the following proof:

Prove that $\log (a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n) = \log a_1 + \log a_2 + \log a_3 + \dots + \log a_n$ for all positive integers n .

Justification:

$$\begin{array}{ll} \log (1 \cdot 2 \cdot 3) = \log 6 \approx 0.778 & \text{and} \quad \log 1 + \log 2 + \log 3 \approx 0.778 \\ \log (4 \cdot 3 \cdot 7) = \log 84 \approx 1.924 & \text{and} \quad \log 4 + \log 3 + \log 7 \approx 1.924 \\ \log (98 \cdot 99 \cdot 100) = \log 970,200 \approx 5.987 & \text{and} \quad \log 98 + \log 99 + \log 100 \approx 5.987 \end{array}$$

Since these all work, then it is true that

$$\log (a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n) = \log a_1 + \log a_2 + \log a_3 + \dots + \log a_n .$$

Therefore, by examining these cases where the mathematical conjecture is true, the student concludes it is true for all n .

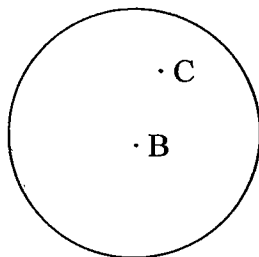
An interesting note that Harel and Sowder (1998) made was "Even when a counterexample to the statement is found, the statement still stands in the students' eyes, because the counterexample is just one exception to the general rule" (p. 254). A study by Fischbein and Kedem (1982) had similar findings as they stated, "Students, after finding or learning a correct proof for a certain mathematical statement, will continue to consider that surprises are still possible, that further checks are desirable in order to render the respective statement more trustworthy" (p. 128).

Crucial Experiment Proof Scheme

This proof scheme deals more explicitly with the question of generalization by examining an extreme case. Dreyfus (1991) stated "To generalize is to derive or induce from particulars, to identify commonalities, to expand domains of validity" (p. 35). Knuth (1999) noted "A defining feature of the crucial experiment proof scheme is the intentionality behind students' choices of cases to test. In other words, students recognize a need to check cases that aren't typical, cases that they feel would really test the validity of their assertions" (1999, p. 30). The following is an example of the crucial experiment proof scheme:

Present an argument to justify the following conjecture—given a circle B and a point C inside the circle, the products of the lengths of the two segments produced by all chords that pass through point C are equal.

Figure 1. Circle B



Students may draw an example or use dynamic geometry software and compare the segment product of the longest chord with the segment product to the smallest chord they can measure. When comparing these extreme cases, the students come to believe that they have justified this conjecture. In the naive empirical scheme the student comes to a conclusion by looking at one or more cases. The crucial experiment proof scheme is when the students "...feel the need to test particular cases, that is, extreme cases, chosen

because they can be used in generalizing to all cases" (Knuth & Elliot, 1998, p. 716). Students that use this scheme feel their assertion holds true because of the extreme cases they have looked at so it would necessarily be true for all the cases between the two extremes. Knuth and Elliot (1998) suggested that students at this proof level "...show a more sophisticated understanding of proof than students at the previous level" (p. 716).

Generic Example Proof Scheme

The generic example proof scheme is used when students base their arguments on an example representative of a class of objects—the perception of the case is the important factor. According to Knuth and Elliot (1998), this is the highest level that students can attain if they do not have formal experience in deductive proof. Even though a particular case is the focus, the student mistakenly perceives the case as being representative of all such cases. The perceptual scheme plagues many students. One example of this proof scheme is when students view a deductive proof as a proof for a single isolated case, the case that is pictured in the associated diagram. Students who hold this belief do not understand the generic aspect for deductive proofs; they do not understand that the validity of the conclusion is meant to be generalizable to all figures that satisfy the given conditions.

An example of this proof scheme is when a student was asked what quadrilateral is "ALWAYS" formed by the segments that connect the four midpoints of the four sides of a parallelogram (Harel & Sowder, 1998). A student drew a parallelogram that looked like a square and then reasoned based on her particular figure that the segments

connecting the four midpoints of the parallelogram always form a square. The student's reasoning was based on the particular figure she drew and not the possible variation of the figure (Harel & Sowder, 1998).

Analytic Proof Schemes

The analytic proof schemes are those in which a student validates conjectures by means of logical deductions (Harel and Sowder, 1998). In this scheme, justifications or arguments are qualified as a "formal mathematical proof." This raises the common question of what is a "formal mathematical proof" and how do we judge something as a proof. Tall (1995) stated, "Even at the formal level, the use of the single word "proof" disguises the fact that there are many different views of proof, dependent on different historical and cultural contexts" (p. 28). He adds, "By acknowledging that different standards and types of proofs exist even at a formal level, we can begin to appreciate that different forms of proof are likely to be appropriate in different contexts" (p. 28).

Harel and Sowder (1998) described analytic proof schemes by two sub-categories: the transformational proof scheme and the axiomatic proof scheme. In the sub-category transformational proof schemes, two further sub-categories adapted from Tall (1995) are used for the present study.

Transformational Proof Scheme

A transformational proof scheme is where students formulate general conjectures and reach conclusions by logical deduction. "Transformational observations involve

operations on objects and anticipations of the operation's results" (Harel and Sowder, 1998, p. 258). Harel and Sowder referred to these proofs as *transformational* because they involve the *transformations of images* by means of deduction. Sowder and Harel (1998) stated the general idea of this scheme is that a student's justification is concerned with the general aspects of the conjecture and reasoning is goal oriented and attempts to predict outcomes on the basis of general principles.

According to Sowder and Harel (1998), an example of a transformation proof scheme is in trying to predict the number of edges in an n -gonal prism. An argument might examine triangular, quadrilateral, and pentagonal prisms and notice that the number of edges of an n -gonal prism is $3n$. The person perceived the underlying structure behind the pattern. "The argument is general and involves reasoning rather than the counting and pattern finding of an empirical based proof" (Sowder & Harel, 1998, p. 673). Sowder and Harel stated, "We regard the transformational proof scheme as a necessary precedent to the last proof scheme, the axiomatic proof scheme" (1998, p. 674).

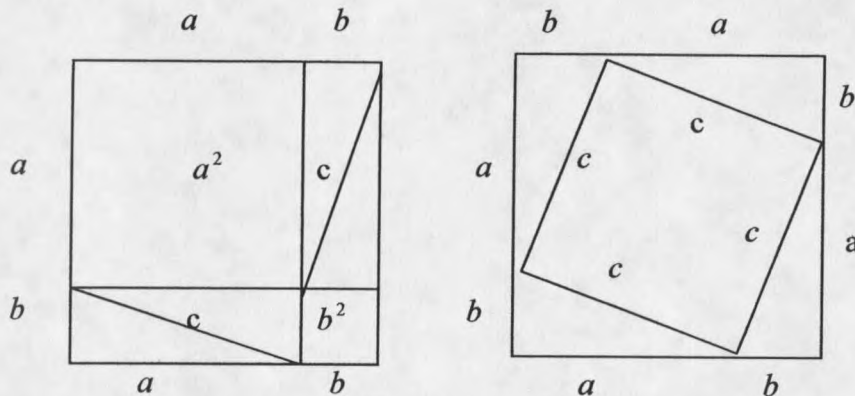
To further sub-divide Harel and Sowder's (1998) transformational proof scheme, this research study uses an adaptation of Tall's (1995) proofs. The two sub-categories of transformational proof schemes are: 1) visual proof scheme; and 2) manipulative proof scheme.

Visual Proof Scheme

Tall (1995) described a visual proof as one that has enactive elements and verbal support. The visual proof is a proof in which an observer may "see" the justification for a

mathematical idea. According to Tall (1995) a visual proof involves seeing the generalization of the mathematical idea. A common visual proof that Tall uses to demonstrate this idea is the proof of the Pythagorean Theorem (see Figure 2):

Figure 2. Pythagorean Theorem



Justification: In the square on the left, we see that the area of the square is equal to the area of the square on the right since the two squares are congruent. So,

$$\begin{aligned}(a+b)^2 &= 4\left(\frac{1}{2}ab\right) + c^2 \\ a^2 + 2ab + b^2 &= 2ab + c^2 \\ a^2 + b^2 &= c^2\end{aligned}$$

Therefore, the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. Tall (1995) noted that the diagram could be seen as a prototype or seeing the general in the specific.

Manipulative Proof Scheme

This proof scheme is common to show algebraic properties. For example, Tall (1995) stated that to show that $(a+b)(a-b) = a^2 - b^2$ all one needs to do is to multiply

out the parenthesis on the left side and cancel the terms ba and $-ab$ using basic axioms of the commutative property of multiplication.

Another example of a *manipulative* transformational proof scheme is the following example:

Prove: The sum, $S(n)$, of the first n positive integers is equal to $\frac{n(n+1)}{2}$.

$$\text{Proof: } S(n) = 1 + 2 + 3 + \dots + (n-1) + n$$

$$S(n) = n + (n-1) + (n-2) + \dots + 2 + 1$$

(Commutative Prop. of Add.)

Taking the sum of these two rows (Addition Prop. of Equality):

$$2S(n) = (1+n) + [2+(n-1)] + [3+(n-2)] + \dots + (n+1)$$

$$2S(n) = (n+1) + (n+1) + (n+1) + \dots + (n+1) \quad (\text{Associative Prop. of Add.})$$

$$2S(n) = n(n+1) \quad (\text{Arithmetic})$$

$$\text{Therefore, } S(n) = \frac{n(n+1)}{2} \quad (\text{Multiplication Prop. of Equality}).$$

This proof manipulates algebraically to find the sum.

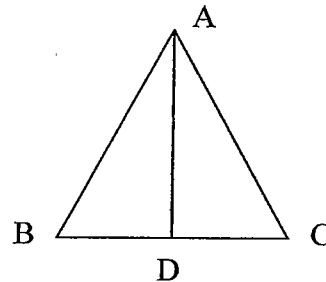
Axiomatic Proof Scheme

The axiomatic proof scheme is organized such that the student uses subsequent results that are logical consequences of preceding results. Harel and Sowder (1998) stated that the axiomatic proof scheme involves an awareness of an underlying formal development. The organization involves undefined terms, definitions, assumptions, and theorems. For example in proving triangles congruent students may use definitions, reflexive and symmetric properties, or theorems from Euclidean geometry (e.g. SAS congruence theorem). The axiomatic proof scheme notes special attention to the

formulation of undefined terms and accepted statements as the basis for the justification of a mathematical argument. The following is an axiomatic proof:

Figure 3. Triangle ABC

Prove: If two sides of a triangle are congruent, then the angles opposite them are congruent (Base Angles Theorem).



Justification: *Let segment AB be congruent to AC. Let D be the midpoint of segment BC and draw auxiliary line AD (ruler postulate and two points determine a line). Then segment BD is congruent to segment CD (definition of a midpoint) and segment AD is congruent to segment AD (reflexive property of congruence). Then triangle ABD is congruent to triangle ACD (SSS congruence postulate). Therefore, angle B is congruent to angle C (by CPCTC).*

Summary of Research Studies

Research studies included in the literature review on conceptions of proof examined particular aspects of proof (e.g., logical underpinnings of proof or counterexample logic), an individual's ability to complete proofs, or both—the logical underpinnings of proof and ability to complete proofs. Research studies that investigated both the logical underpinnings of proof and ability to complete proofs were Healy and Hoyles (1998), Martin and McCrone (2001), Moore (1990), Selden and Selden (1995),

and Williams (1979). Of these studies, the Moore (1990) and Selden and Selden (1995) study included prospective secondary mathematics teachers in a beginning proof course.

Summary of the Results

Results of research studies on individuals' conceptions of proof were uniform in many aspects. An overall summary of the results is broken down into two areas—understanding of the underpinnings of proof and ability to complete proofs.

Research studies of an individual's understanding of the underpinnings of proof included studies by Balacheff (1988), Bell (1976), Burke (1984), Chazan (1989), Fischbein and Kedem (1982), Galbraith (1982), Healy and Hoyles (1998, 2000), Martin and Harel (1989), Martin and McCrone (2001), Moore (1990), and Williams (1979).

Research that investigated underpinnings of proof similar to the present study were studies by Galbraith (1982) of 263 undergraduate and 116 post graduates in secondary mathematics and the Williams (1979) study of 255 high school students. Results of both studies show that very few students understand the logical equivalence of mathematical arguments and their contrapositive (35% of Galbraith's students and none of the students in the Williams study). Less than half the students in both studies understood that a statement and its converse are not logically equivalent. Approximately 10% of prospective teachers understood an item related to the use of axioms and definitions in the context of non-Euclidean argument—14% of the high school students could conclude some correct conclusions but none of the students could justify the conclusion.

Only 46% of the prospective teachers in Galbraith's (1982) study and 20% of the high school students in the Williams (1979) study understood a counterexample refutes a false conjecture. A research study by Burke (1984) showed much more promising results in that over 50% of ninth-graders responses and approximately 67% of geometry students understood that a single counterexample falsified a claim. One note for this discrepancy with high school students in the Williams study and the Burke study is the clarity of the items in the Burke study that provided subjects with clear choices for proving a claim is false (Burke, 1984).

A study of in-service mathematics teachers by Knuth (1999) found that 11 of 16 teachers were able to identify the starting assumption for a multiple choice item on indirect proof. Knuth notes that only 4 of 14 teachers that attempted to complete an indirect proof produced a valid justification. In the Williams study of high school students, only 20 % understood the method of indirect proof.

The study by Williams (1979) of 255 students in the 11th grade found that on one item at least 70% of the students did not distinguish between inductive and deductive reasoning and did not understand that induction is inadequate to support mathematical generalizations. Studies of high school students by Bell (1976), Chazan (1989), and Balacheff (1988) and a study by Martin and Harel (1989) of 101 pre-service elementary teachers found similar results.

Studies show that some high school students have a misunderstanding of deductive proof as a proof for a single, isolated case, the case that is pictured in the diagram. Also some high school students show a lack of understanding that formal proof

of a mathematical statement excludes the need for any further checks. Many of the studies of high school and university students show that students have difficulty with the definitions and the mathematical concepts of proofs. High school students show a lack of understanding of the domains of generalization that had been established in a proof.

Research studies of an individual's ability to complete mathematical proofs included studies by Bell (1976), Senk (1985, 1989), Balacheff (1988), Martin and Harel (1989), Moore (1990), Ball and Wilson (1990), Selden and Selden (1995), Knuth and Elliot (1997), Knuth (1999), Healy and Hoyle (1998, 2000), and Martin and McCrone (2001).

Results of the ability to complete mathematical proofs found that the majority of students at the high school and university level are unable to construct valid proofs. The difficulty at both levels was attributed to students not knowing how to use definitions to obtain the overall structure of proof, a lack of understanding between the conclusion and the details of the data, and students' inability to intuitively understand the theorem or the concepts involved in a proof. Also many students at both levels were unable to even begin to construct a proof or began a proof with the wrong assumption. Students at the high school level were more apt to use empirical examples to justify mathematical arguments.

A study by Ball and Wilson (1990) showed that novice teachers lack understanding of a false conjecture. At the exit of the program study, only 10 of 19 novice teachers understood a false conjecture about a relationship between perimeter and area.

Studies by Moore (1990) and Selden and Selden (1995) both reported students having difficulty proving statements with universal quantifiers and a lack of ability to clarify the logical structure of mathematical statements. Moore (1990) noted that when students did begin a proof correctly, a key step which usually was the discovery of a "trick" often times prevented their successful completion of the proof. He added that students difficulty with proof was often a cognitive overload—students' struggle with understanding abstract concepts at the same time they were learning techniques of proof, how definitions are used in proofs, and how proofs are expressed in mathematical language and notation.

Summary of Methodological Principles

To examine an individual's logical underpinnings of proof, this study used multiple-choice items similar to the methodological strategies used by Chazan (1989), Healy and Hoyles (1998), Galbraith (1982), and Martin and McCrone (2001). To investigate an individual's ability to complete proofs, this study used open proof items that asked individuals to complete mathematical proofs. The use of open proof items to examine ability to complete proofs is similar to research methods from studies by Bell (1976), Healy and Hoyles (1998), Senk (1985, 1989), Ball and Wilson (1990), Knuth and Elliot (1997), Knuth (1999), Moore (1990), Martin and McCrone (2001) and Selden and Selden (1995).

Two studies that have greatly influenced this research study are the studies by Galbraith (1982) in which he investigated the mathematical vitality of prospective

secondary teachers that included their conceptions of the logical underpinnings of proof. This study emulates his methodology used to collect data. Another study that is significant is Moore's research (1990). In Moore's study he investigated mathematics and mathematical education undergraduates difficulties with proof in a semester long proof class that is generally students first proof class of their preparation program. Moore had an opportunity to collect data from students' homework, quizzes, and tests. In the present study, the examination of prospective secondary teachers as they are near the completion of their mathematical course work does not afford the opportunities that Moore had in his study because of the lack of a proof class as they are exiting the program. Instead the present study uses methodological principles that include examining individuals' logical underpinnings of proof by multiple choice items and examining their ability to complete proof. Also the present study is collecting data from three different universities where the Moore study was conducted in one course, at one university.

CHAPTER 3

RESEARCH METHODOLOGY

Introduction

This chapter contains the research questions and a discussion of the population, sample, research design, procedures, instrumentation, data collection, and data analysis that were used in this study. The discussion focuses primarily on four areas: the characteristics of the subjects, the data sources, data collection activities, and the nature of the data analysis.

Research Questions

Primary Question

What are prospective secondary mathematics teachers' conceptions of proof and refutations?

Secondary Questions

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof and refutations?
 - a. What are their understandings of the logical underpinnings of direct proof?
 - b. What are their understandings of the logical underpinnings of indirect proof?
 - c. What are their understandings of the logical underpinnings of refutations?

- d. Can they demonstrate an understanding of the laws of logic that underlie proof?
2. What are prospective secondary mathematics teachers' abilities to complete mathematical proofs?
 - a. What are their abilities to complete direct proof?
 - b. What are their abilities to complete indirect proof?
 - c. What are their abilities to refute a false conjecture?

The Participants

Population

The population for the research study were prospective secondary mathematics teachers in the state of Montana near the end of their teacher training programs, or students who had graduated but were completing teacher certification requirements for secondary mathematics. This population had completed, or had nearly completed, at least two calculus courses, an introductory proof class, and a college geometry course. Individuals had not completed their student teaching requirement.

Sample

A purposive sampling technique was used to select participants for the study. The purposive sampling technique allowed the researcher to select participants with specific criteria. A total of 23 participants met the requirements of the study and were enrolled in

a secondary mathematics education preparation program.⁹ Participants were enrolled at three of the six four-year public institutions in the state of Montana. Seven of the participants were from Montana State University-Bozeman, thirteen from the University of Montana-Missoula, and three from the University of Montana-Western located in Dillon, Montana.¹⁰ According to the Montana Office of Commission of Higher Education, the 2000 – 2001 enrollment of these three universities represented approximately 79% of the total four-year schools' enrollment in the state of Montana (2002).

The sample of participants were enrolled in one of the following courses: MATH 428C—Mathematical Modeling for Teachers (MSU), MATH 431—Euclidean and Non-Euclidean Geometry (U of MT), or MATH 331 College Geometry (Western). All of the participants had completed, or were near completion, of similar courses that included at least two calculus courses, an introductory proof class, and a college geometry course.¹¹ Mathematical concepts in these courses included techniques and methods of proof, the logical underpinnings of proof, and methods of completing proofs. None of the individuals had completed their student teaching requirement. Twelve of the total 23 participants of the study were female and 11 were male.

⁹ Originally, 28 individuals completed the information form and questionnaire but two individuals from MSU, two individuals from the U of MT, and one individual from Western did not meet the requirements (i.e., did not have an introductory proof class or the geometry course) and therefore were not considered as participants of the study.

¹⁰ Montana State University will be denoted as MSU, the University of Montana as U of MT, and University of Montana-Western as Western.

¹¹ Participants from the MATH 431 (U of MT) and MATH 331 (U of MT-Western) were near the completion of the course.

The participants were assumed to be a homogeneous sample as described by Gay and Airasian (2000). The confidentiality of the participants was maintained throughout the study. The research study was conducted during the Fall 2002 semester.

Course Descriptions

The following are descriptions of the courses from the university bulletins:

MATH 428C – Mathematical Modeling for Teachers—This senior capstone course includes using pre-college mathematics to explore a variety of application areas including the modeling process, technology, strategies to initiate modeling in a secondary classroom, and classroom assessment of modeling activities (Montana State University, 2000).

MATH 431 – Euclidean and Non-Euclidean Geometry—The study of Euclidean geometry is from a rigorous, axiomatic viewpoint and Non-Euclidean geometries chosen from Lobachevskian, projective, finite, and Riemannian (University of Montana, 2002).

MATH 331 – College Geometry—The content of this course includes the study of deductive and inductive reasoning and with axioms and theorems of Euclidean and Non-Euclidean geometries, performing geometric constructions, and proving geometric theorems (University of Montana—Western, 2002).

Data Collection Instruments

This quantitative, descriptive study investigated prospective secondary teachers' conceptual knowledge about the nature of proof and refutations. To gather data, an information form and a questionnaire were developed by the researcher.

Information Form

To gather demographic data, an information survey was designed for each university. The forms were the same for the universities except for the listing of mathematical courses, which were the specific courses available from each university. The Information Form (Appendix A) was given to the participants before they were administered the questionnaire on proof. It elicited data regarding the mathematics courses that participants had taken previously, gender, and class rank. Specifically, the researcher wanted to investigate whether prospective teachers had completed courses in mathematical proof. The individuals that had not completed the required courses were eliminated from the study.

The Questionnaire

A questionnaire was developed to study the main questions of the research study. The questionnaire was composed of two parts in order to assess two components of individuals' conceptions of proof—logical underpinnings of proof and completion of proof. Part I (Appendix B) addressed Secondary Research Question 1, that examined the participants understanding of the logical underpinnings of proof and refutations. The

logical underpinnings of proof include methods of evaluating mathematical arguments (i.e., evaluate the truth value of complex statements from the truth values of the simple statements). The study used two primary ideas of formal mathematical logic—implication and negation—as suggested by Fendel and Resek (1990). They stated that implication is the causal relationship in a conditional statement between its hypothesis and conclusion, whereas negation provides a link between two types of quantifiers (universal and existential) and is an essential element of understanding the connection between a conditional statement and a counterexample.

Part II of the questionnaire (Appendix C), addressed Secondary Research Question 2 and participants' ability to complete mathematical proofs. Both the logical underpinnings of proof and ability to complete proofs focused on the common schemes of direct proofs, indirect proofs, and refutations.

The two secondary research questions sought to answer the Primary Research Question: What are prospective secondary mathematics teachers' conceptions about the nature of proof and refutations? Tables 4 and 5 organized the data collection instrument with respect to the secondary research questions. Table 4 matched the research questions regarding prospective teachers' understanding of the logical underpinnings of proof with items from Part I of the questionnaire. Table 5 matched the research questions regarding prospective teachers' abilities to complete mathematical proofs with items from Part II.

Table 4. Secondary Research Question 1—Logical Underpinnings of Proof

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof?	Part I
a. What are their understandings of the logical underpinnings of direct proof?	# 1, 5, 10
b. What are their understandings of the logical underpinnings of indirect proof?	# 6, 8, 12
c. What are their understandings of the logical underpinnings of refutations?	# 4, 7, 11
d. Can they demonstrate an understanding of the laws of logic that underlie proof?	# 2, 3, 9

Table 5. Secondary Research Question 2—Completion of Proof

2. What are prospective secondary mathematics teachers' abilities to complete mathematical proofs?	Part II
a. What are their abilities to complete direct proof?	# 3
b. What are their abilities to complete indirect proof? ¹²	# 1
c. What are their abilities to refute a false conjecture?	# 2

The questionnaire was not designed to meet criteria for any particular course of study. Rather, items were chosen or constructed based on the common mathematical content that prospective teachers might encounter teaching proof at the secondary mathematics level. Items addressed the nature of proof as outlined by the MAA (1998) and the NCTM (2000) Reasoning and Proof Standard for grades 9 – 12.

¹² The goal of the researcher was to assess prospective teachers' abilities in completing an indirect proof by either the contradiction method or the contrapositive method of proof.

Many of the items were adapted from prior research studies of individuals' conceptions of proof. From Part I of the questionnaire, Items 1, 3, and 7 were adapted from Galbraith's (1982) study. Items 6 and 8 were similar to questions developed by Knuth's (1999) study. Item 2 of Part II was a false conjecture problem based in the content area of perimeter and proof. This item was adapted from Ball and Wilson's (1990) study. One constructed response item (Item 3) was adapted from a high school geometry text (Larson, Boswell, & Stiff, 1998). The other items were adapted from introductory proof texts, including those by Barnier and Feldman (1990), Esty (2000), and Fletcher and Patty (1996). The questionnaire was designed so that it could be completed within a traditional one-hour time frame of a university class.

Part I of the Questionnaire

Part I of the questionnaire addressed Secondary Research Question 1 with parts a, b, c, and d. It included 12 multiple choice items designed to measure prospective secondary teachers' understanding of the logical underpinnings of proof and refutations. It asked participants to select one response from a choice of five. For each item there was one and only one correct selection.

The evaluative criteria for Part I, logical underpinnings of proof, are the logical underpinnings of direct proof, indirect proof (proof by contradiction), refutation of a false conjecture, and laws of logic. The content of the items in Part I was based on standard conventions of the mathematical concepts that underlie proof. These standard conventions are addressed in the Underpinnings of Proof section of Chapter Two and also

in Appendix E, Content Validity. The data collection method for examining the logical underpinnings of proof was based upon the methodologies employed by Chazan (1989), Galbraith (1982), Healy and Hoyles (1998), and Martin and McCrone (2001).

Part II of the Questionnaire

Part II of the questionnaire addressed Secondary Research Question 2 with parts a, b, and c. The evaluative criteria for Part II included the ability to complete a direct proof, indirect proof, and refute a false conjecture. This portion of the questionnaire included three constructed response questions that required participants to produce their own logical justifications for mathematical ideas. According to Popham (2000), the use of the constructed response items requires the individual to *produce*, rather than merely recognize, a correct answer, thus rendering the constructed response questions a far more demanding kind of test item.

The content of the three constructed response items in Part II was based on standard conventions of the mathematical concepts of direct proof, indirect proof, and refutation of a false conjecture. Item 1 asked participants to complete a proof for which the most likely methods were the contradiction or contrapositive methods of proof. This item was based on the number and operations and algebra content areas. The false conjecture item (Item 2) was adapted from Ball and Wilson's (1990) study of novice secondary mathematics teachers. The content of the false conjecture included perimeter and area of rectangles. Item 3 was most likely to be completed via a direct proof. Item 3 was based on the Perpendicular Bisector Theorem that is found in traditional high school

geometry texts. The proof of this item did not require the drawing of auxiliary lines or other non-intuitive procedures (i.e., referred to as a trick by Moore (1990)). Rather the proof was based on common definitions, axioms, and theorems from Euclidean geometry.

The data collection method for examining the ability to complete mathematical proofs, was based upon methodologies employed by Balacheff (1988), Ball and Wilson (1990), Bell (1976), Healy and Hoyles (1998), Knuth (1999), Martin and Harel (1989), Martin and McCrone (2001), Moore (1990), Selden and Selden (1995), and Senk (1985, 1989). Both parts of the questionnaire collectively were designed to address the primary research question concerning an individual's conceptions of proof and refutations.

Procedure

The researcher administered the information form and the questionnaire to the participants during the 10th through the 12th week of the 15-week Fall 2002 semester. The participants were informed that the study focused on prospective teachers' conceptions of proof and refutations. They were also informed that the questionnaire was made up of two parts—12 multiple choice items where one and only one answer was correct and 3 constructed response items. Participants were paid ten dollars for taking part in the study. Participants took approximately 30 to 45 minutes to complete the questionnaire.

Validity and Reliability

Validity

The questionnaire was developed by the researcher to study the main questions of the research study. The researcher sought the advice of two mathematics education professors at MSU—Dr. Ted Hodgson and Dr. Maurice Burke—to help evaluate the content of the items. To establish the content validity of the research instrument, Dr. Warren W. Esty, a mathematics professor at MSU, was asked to examine the questionnaire to judge whether the test items measured mathematical proof knowledge. The content areas that defined the attributes for the secondary research questions judged the sub-areas of the items (Appendix E). Dr. Esty and the researcher agreed on 14 of the 15 items for an agreement rate of approximately 93 percent.¹³

To measure prospective teachers' conceptions about the nature of proof, two attributes of proof were analyzed: (1) the logical underpinnings of proof, and (2) the ability to complete mathematical proofs. Many of the items were adapted from previous research studies that focused on individuals' conceptions of proof. The pilot study confirmed that test directions were clear, consistent, and objective scoring methods were used. The researcher administered the questionnaire to the participants. According to Gay and Airasian (2000), these factors contribute to the validity of a test instrument.

¹³ Based on the advice of Dr. Esty, two multiple choice items were rewritten. The agreement rate for the revised questionnaire was still 93 percent.

The rationale for creating a questionnaire of multiple choice (selected-response) items and constructed response items was provided by Popham (2000). According to Popham, using multiple evaluative criteria to measure performance allows one to make an inference on an individual's level of achievement. In addition, Creswell (1998) suggests that using multiple choice items and constructed response items promotes methodological triangulation within a research study. Both types of items gather rich and varied data concerning the prospective teachers' conceptions of the nature of proof. The use of multiple choice items provides insight into prospective teachers' conceptual understanding of the underpinnings of proof. The constructed response items provide insight into their ability to write proofs. Measuring both the conceptual understanding of underpinnings of proof and ability to write proofs provides more insight into an individual's conceptions of the nature of proof rather than investigating only one facet of conceptual knowledge.

Reliability

The reliability of the questionnaire was established in the pilot study through a test/retest procedure that was conducted over a three-week period. The test-retest reliability coefficient was approximately 0.90.

A Kuder-Richardson level of 0.79 was calculated to estimate the internal consistency reliability of the multiple choice items in the questionnaire. The Kuder-Richardson 21 formula was used according to Popham (2000). According to Bernstein, Garbin, and Teng (1999), the test-retest method is not suitable for the constructed

response items since participants' familiarity with the content of the items is likely to influence the outcome of the second test. To address concerns about the reliability of scoring the constructed response items, an interjudge reliability was used, as suggested by Gay and Airasian (2000). Interjudge reliability was obtained by having two judges independently score the constructed response items. Scores were then compared by the two judges. The judges for this portion of the research study were Dr. Carol Lynn Hancock—former professor of mathematics education from Appalachian State University—and the researcher. The scores of Judge 1 were correlated with the scores of Judge 2. The judges had an agreement rate of 88 percent for the pilot study. When the scores on an item differed, the judges discussed their scoring schemes until they agreed upon a score. Interjudge reliability was also gathered for the three constructed response items for the study. Those reliabilities are presented in Chapter Four.

Data Analysis of the Research Questions

The results and analysis of the questionnaire focused on prospective teachers' responses collectively and individually. A frequency table was used to organize the data collectively per item and individually per sub-category. Data gathered from Part I of the questionnaire was organized in four sub-categories: (1) direct proof, (2) contradiction proof, (3) refutation, and (4) laws of logic. Data gathered from Part II was organized in three sub-categories: (1) direct proof, (2) indirect proof, and (3) refutation.

The analysis of the responses included identifying interesting trends or patterns in the responses, calculating the percentage of correct responses, calculating the mean

number of correct scores of the sub-categories of the research questions, and calculating the mean number of correct scores of the test as a whole. Interesting patterns of the participants' responses for some of the items was noted and compared to responses on similar items of the questionnaire. All calculations were rounded to the third decimal place. Following guidelines from Popham (2000), the researcher used quality levels to distinguish a participant's performance. The four levels of performance are the following: (1) competent, (2) some ability, (3) minimal ability, and (4) lack of ability. The distribution of data from each individual can be found in Appendix D.

Data Analysis of the Secondary Research Question 1

Analysis of Data for Each Multiple Choice Item

The Secondary Research Question 1 investigated prospective secondary teachers' understandings of the logical underpinnings of proof and refutations. To address this research question, a tally of the total number of participants that responded to each selection of the multiple-choice items was noted in a frequency table. Participants were given a score of 0 for an incorrect answer or a non-response and a score of 1 for a correct answer. The correct answer for each item was noted by an asterisk next to the total number of participants that responded with that selection. The Proportion Correct for each item was calculated as $\hat{p} = \frac{\text{count of successes in the sample}}{n}$. Also, the Total Proportion Correct for the sample was calculated by dividing the number of correct responses of the sample by $12n$ (the number of items multiplied by the number of

participants). The distribution of participants' responses for each item allowed the researcher to identify interesting patterns or trends.

Analysis of Sub-Category Data per Individual

Sub-questions 1a, 1b, 1c, and 1d investigated the participants' understanding of the logical underpinnings of direct proof, indirect proof, refutations, and understanding of the laws of logic that underlie proof, respectively. The individuals' responses and performance in the four sub-categories for the understanding of the logical underpinnings of proof and refutations were distributed in a table format and analyzed. The maximum possible score for the three items in each sub-category was 3 points. An analysis of the total number of correct responses of each participant was summarized by quality levels. These quality levels of the total correct responses in each of the four sub-categories are presented in Table 6.

Table 6. Quality Levels for Sub-Categories

Points (3 pts. possible)	Quality Levels for Sub-categories of the Logical Underpinnings of Proof
3 points	Competent understanding
2 points	Some understanding
1 point	Minimal understanding
0 points	Lack of understanding

The mean score per sub-category for participants collectively was calculated by dividing the total score per category by n . The mean score was analyzed to note trends and describe performance for the entire sample. The individuals' total points—total correct responses—where a total of 12 points were possible, was also calculated and

described by quality levels in Table 7. The description of the total correct responses was based on the percent correct. The number and percentage of participants within in each quality level was calculated and described.

Table 7. Quality Levels for Part I—Logical Underpinnings of Proof

# Correct Responses	Percent Correct	Quality Levels for Total Correct Responses
10 – 12 items correct	83 – 100 %	Competent understanding of the logical underpinnings of proof
7 – 9 items correct	58 – 75 %	Some understanding of the logical underpinnings of proof
4 – 6 items correct	33 – 50 %	Minimal understanding of the logical underpinnings of proof
0 – 3 items correct	0 – 25 %	Lack of understanding of the logical underpinnings of proof

Data Analysis of Secondary Research Question 2

Secondary Research Question 2 investigated prospective secondary teachers' ability to complete mathematical proofs. Part II of the questionnaire addressed this research question by allowing the researcher to analyze the prospective teachers' responses collectively and individually. Where it was appropriate, participants' responses were analyzed according to proof schemes (Table 8) that were adapted from studies by Harel and Sowder (1998), Knuth (1999), and Tall (1995).

Table 8. Proof Scheme Framework

I. External Conviction Proof Scheme	II. Empirical Proof Schemes	III. Analytical Proof Schemes
A. Authoritarian Proof Scheme	A. Naïve Empirical Proof Scheme	A. Transformational Proof Scheme 1. Visual Proof Scheme 2. Manipulative Proof Scheme
B. Ritual Proof Scheme	B. Crucial Experiment Proof Scheme	B. Axiomatic Proof Scheme
C. Symbolic Proof Scheme	C. Generic Example Proof Scheme	

Each of the responses was also analyzed by a numerical scoring scheme (Tables 9 and 10) that described the approach employed by the prospective teacher in attempting the mathematical proof. The researcher categorized the justifications based on guidelines adapted from Simon and Blume (1996), Williams (1979), and Senk (1985, 1989) and also the Educational Testing Services' (2002) Praxis Test on mathematical proof. Numerical scoring frameworks utilized in these studies and on the Proof Praxis Test included a hierarchical ordering of responses. Scores of 0 through 5 were assigned to each of the constructed response items, whereas a score of 4 or 5 represented a valid proof. Table 9 includes the scoring framework used to assess the participants' responses to the direct and indirect proof items. Table 10 includes the scheme used to score participants' responses to the refutation item.

Table 9. Numerical Scoring Framework for Direct and Indirect Proofs

Level #	Criteria for Direct and Indirect Proofs
Level 0	Responses identifying motivations that do <i>not</i> address justification; response restates the problem; no response or justifications; or response is invalid or useless deductions.
Level 1	Demonstrates a very limited understanding of the problem; may write a valid justification that does not lead to proof of the problem; may try to use invalid proof method (e.g., converse). May appeal to external authority—uses one of the External Conviction Proof Schemes (Authoritarian, Ritual, or Symbolic proof schemes).
Level 2	Demonstrates minimal progress toward a valid justification by writing at least one valid deduction and reason that will lead to a valid proof of the problem; uses an example or examples as demonstrations; Naive Empirical Proof Scheme.
Level 3	Response shows evidence of using a chain of reasoning by either completing at least half the logical steps correctly that will lead to a valid proof or by writing a sequence of statements that is invalid because it is based on faulty reasoning in the early steps. Response shows a lack of reasoning and/or lack of justification for more than one step. Response is justification that is expressed in terms of a particular instance or case; Crucial Experiment Proof Scheme; Generic Example Proof Scheme.
Level 4	Response is a justification in which steps follow a logical sequence but in which errors occur in notation, vocabulary, or names of theorems (e.g., may forget to state domain). Response shows a lack of reasoning and/or lack of justification for one step. It could be classified as one of the Analytical Proof Schemes with minor errors.
Level 5	Response is a valid justification. Clearly demonstrates a full understanding of the mathematical content and follows a logical order of justification; may contain a minor error in notation; can be classified as one of the Analytical Proof Schemes.

Table 10. Numerical Scoring Framework for Refutations

Level #	Criteria for Refutations
Level 0	Response is that conjecture is true but makes no attempt at justifying; response restates the problem; no response; or response is invalid or useless deductions.
Level 1	Response is that conjecture is true and attempts to prove it.
Level 2	Demonstrates minimal progress towards refuting the conjecture; may only state that conjecture is false; finds a counterexample that refutes but also tries to prove the conjecture is true.
Level 3	Response is a counterexample to disprove the conjecture, but the counterexample is not adequate at all. Demonstrates some understanding of the mathematical content.
Level 4	Response is a counterexample to disprove the conjecture, but the counterexample is not completely adequate; parts of counterexample would disprove, but contains errors; explanation is not clear or not complete. Demonstrates some understanding of the mathematical content.
Level 5	Response is a valid counterexample that refutes the false conjecture. Clearly demonstrates a full understanding of the mathematical content and methods to disprove a false conjecture.

Analysis of Sub-Category Data per Individual

Research sub-questions 2a, 2b, and 2c investigated participants' ability to complete a direct proof, indirect proof, and refute a false conjecture. To address the sub-questions, the responses on the constructed response problems were evaluated collectively based on a numerical scoring scheme that scored responses on the three items on a scale of 0 to 5 points. A frequency table was used to record the participants' scores on each of the proofs. The distribution of participants' responses per point value allowed

the researcher to identify interesting patterns or trends in relation to the participants' understanding of direct and indirect proofs, and the refutation item. For example, the researcher could examine how many participants scored 5 points—response is valid counterexample—on the refutation item. The mean score per proof item was also calculated and was used to identify trends and describe the performance.

The event that participants attempted proofs by an unusual or unexpected strategy was noted by the researcher. For example, participants attempting Item 3 (geometry content proof) with a method other than direct proof were noted. An analysis of the proof methods (proof by contradiction, contrapositive, or other) by which participants attempted to complete Item 1 was also noted in the same manner.

In addition to analyzing the participants' responses collectively, a table was used to record the responses and performance of each individual in the three sub-categories. This distribution allowed the researcher to evaluate the performance of the individuals in the research study. The mean score per item was calculated and used to address the research sub-questions 2a, 2b, and 2c.

The total score for each individual and the mean number of the total points for all the participants was calculated and analyzed for interesting patterns and trends. The researcher used quality levels that are similar to the quality levels for responses of the items in Part I. The percentage of participants within each quality level was calculated and described. The following are the quality levels for the constructed response problems:

Table 11. Quality Levels for Part II—Completing Proofs

Total points	Analysis of Ability to Complete Proofs
12 – 15	Competent in their ability to complete proofs
9 – 11	Some ability to complete proofs
6 – 8	Minimal ability to complete proofs
0 – 5	Lack of ability in completing a proof

Data Analysis of the Primary Research Question

To address the Primary Research Question—What are prospective teachers' conceptions of proof and refutations?—participants' responses from both parts of the questionnaire were examined. The analysis focused on patterns of responses in Part I of the questionnaire, the logical underpinnings of proof, and responses on the constructed response problems in Part II.

Comparisons of responses from both parts were organized according to the sub-categories direct proof, indirect proof, and refutation. For example, the score for the three items that related to refutations from Part I of the questionnaire were compared with participants' responses to the constructed response refutation item from Part II.

The total number of points assigned to participants' responses were calculated for both parts of the questionnaire. The total score possible in the four sub-categories from Part I was 12 points. The total score possible on the three constructed response problems from Part II was 15 points. The sums of the totals for both parts were examined for patterns and trends. Depending on the responses, participants were categorized into particular proof scheme (e.g., Empirical Proof Scheme) from the constructed response items in Part II.

Pilot Study

During the summer semester 2002, a pilot study at Montana State University—Bozeman (MSU) was conducted. Specifically, a draft version of the questionnaire was administered to eight graduate students from the Department of Mathematical Sciences. The goal of the pilot study was to assess the readability of the items, the time participants needed to complete the questionnaire, and to solicit suggestions for improvement.

Pilot study participants volunteered to complete the questionnaire and provide feedback. After completing the questionnaire, each participant was asked to provide comments about the readability of the items and suggestions for improvement. Feedback from the participants of the pilot study consisted of writing the words "false" and "true" in Items 5, 8, and 9 in bold print. The participants' comments and responses were noted, and revisions were made to the original questionnaire. Also, revisions that were suggested by Dr. Warren Esty while establishing the content validity were made to the questionnaire. The significant revisions suggested by Dr. Esty included rewriting Item 8 and the options a, c, and d for Item 2. The revised options for Item 2 assessed participants' knowledge of the negation of a conditional statement as a "H and (not C)" statement. The revised Item 8 was a conditional statement that asked participants to discern the assumptions to start a contradiction proof. The difference between the revised Item 8 and the old item was that the revised item was of the form "A implies B" rather than a "(A and B) implies C" statement.

From the pilot study it was determined that the time needed to complete the original questionnaire was problematic. The original questionnaire was composed of 13 multiple choice items and five constructed response items. Because of the extensive time needed to complete the questionnaire in the pilot study, two of the three constructed response items of proof in the geometry content area were eliminated. The geometry proof item that was retained for the questionnaire was chosen because the proof of the problem was based on familiar definitions, axioms, and theorems from Euclidean geometry. Moreover, the item was straight-forward in the sense that participants did not have to know a special trick to complete the proof. In Part I, one of the four items related to refutations was also eliminated, so that Part I of the questionnaire eventually consisted of three items for each of the four sub-categories. Eliminating these items reduced the questionnaire to 12 multiple choice items and 3 constructed response items.

Summary of Chapter Three

Chapter Three described the characteristics of the participants and discussed the development of the questionnaire that addressed research questions of this study. The methods for analyzing the results of the data collected from both parts of the questionnaire were also addressed. This chapter also included the validity and reliability of the research instrument and a description of the pilot study.

CHAPTER 4

RESULTS AND ANALYSIS OF THE DATA

Introduction

Chapter Three described the questionnaire that was designed to yield data about the primary and secondary research questions and outlined the procedures to be used to analyze the results. This chapter begins with a review of the data collected from 23 prospective teachers' responses to the 12 multiple choice items in Part I of the questionnaire and the three constructed response items in Part II. Tables are used to categorize and summarize the responses and note patterns that emerge from participants' responses.

Interjudge Reliability

The scoring procedures for the three constructed response items involved evaluations of prospective teachers' explanations and therefore introduced elements of subjectivity. Because of this subjectivity, interjudge reliability was gathered for the constructed response items. Scores for the constructed response items were based on the scoring rubric designed for specific types of proofs. The scoring rubric described in Chapter Three was used by the two judges, one of whom was the researcher. After the judges independently scored the items, their scores were compared. When the judges'

scores disagreed on the score of an item, the judges reevaluated their scoring of the item until they agreed upon a score. To determine the interjudge reliability, as suggested by Gay and Airasian (2000), the scores of Judge 1 were correlated with the scores of Judge 2. Table 12 shows the agreement rate, average difference, and correlation on each of the three constructed response items. A maximum difference of five was possible for each item. Overall, the two judges agreed on 56 of 69 total scores for an agreement rate of 81.2 percent.

Table 12. Interjudge Reliability

Item	Agreement (%)	Average Difference	Correlation
1. Indirect Proof	18/23 (78.3%)	1.2	$r = 0.96$
2. Refutation	19/23 (82.6%)	1.0	$r = 0.98$
3. Direct Proof	19/23 (82.6%)	1.0	$r = 0.98$

Data Analysis of the Research Questions

Data from Part I of the questionnaire were used to address Secondary Research Question 1 with parts a, b, c, and d. Data from Part II of the questionnaire were used to address Secondary Research Question 2 with parts a, b, and c. The data elicited by the secondary questions were analyzed collectively to address the primary research question. As described in Chapter Three, the distribution of the participants' responses per item allowed the researcher to identify interesting patterns or trends by examining responses collectively and individually on both parts of the questionnaire. Also, the data were analyzed per sub-category for the entire sample and individually. Data were recorded for

each individual from both parts of the questionnaire in Appendix D. Data were not analyzed until all the data were collected from the three campuses.

Data Analysis of the Secondary Research Question 1

Data gathered from responses to the multiple choice items (Part I of the questionnaire) were used to answer the following Secondary Research Question 1:

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof and refutations?

This question was addressed by asking the following sub-questions a, b, c, and d:

- a. What are their understandings of the logical underpinnings of direct proof?
- b. What are their understandings of the logical underpinnings of indirect proof?
- c. What are their understandings of the logical underpinnings of refutations?
- d. Can they demonstrate an understanding of the laws of logic that underlie proof?

The content of the 12 multiple choice items centered on concepts and ideas typically found at the secondary mathematics level. For each item, participants were directed to select one and only one answer from a choice of five responses. On the scoring of the items, participants were given a score of 0 for an incorrect answer or a non-response, and a score of 1 for a correct answer.

Sample Data for Each Multiple Choice Item

A frequency table (Table 13) presented the data for the collective responses from the participants that answered a, b, c, d, or e. The correct answer for each item was noted by an asterisk next to the total number of participants that chose the selection. The total

number of participants that responded was noted in the column n . Also, the Proportion Correct for each item and the Total Proportion Correct were calculated.

Table 13. Distribution of Data per Item—Logical Underpinnings of Proof

Item #	Category	Answer a	Answer b	Answer c	Answer d	Answer e	n	Proportion Correct
1	Direct	15*	2	3	2	1	23	0.652
2	Laws of Logic	3	4*	4	4	8	23	0.174
3	Laws of Logic	4	6	2	7	4*	23	0.174
4	Refutation	0	20*	3	0	0	23	0.870
5	Direct	1	1	2	17*	2	23	0.739
6	Indirect	1	6	3	10*	3	23	0.435
7	Refutation	15*	1	4	2	1	23	0.652
8	Indirect	5	3	11*	2	2	23	0.478
9	Laws of Logic	2	0	17*	2	2	23	0.739
10	Direct	0	22*	0	0	1	23	0.957
11	Refutation	6	0	0	17*	0	23	0.739
12	Indirect	1	3	18*	1	0	23	0.783
Total Proportion Correct							23	0.616

Sub-Category Data per Individual

Research sub-questions 1a, 1b, 1c, and 1d allowed the researcher to assess participants' understanding of logical underpinnings of proof in the following four categories—direct proof, indirect proof, refutations, and the laws of logic that underlie proof. Each of the four sub-categories consisted of three multiple choice items, therefore

the total possible correct in each sub-category was 3 points. The total items answered correctly for each individual in the four sub-categories were distributed in Table 14. Also, the mean per category and the individuals' total points were calculated.

Table 14. Distribution of Sub-Category Data per Individual—Underpinnings of Proof

Participants	Direct Proof Items #1, 5, 10	Indirect Proof Items #6, 8, 12	Refutation Items # 4, 7, 11	Laws of Logic Items #2, 3, 9	Total Points (%) (12 pts. Possible)
1	1	1	3	0	5 41.7 %
2	3	1	2	1	7 58.3 %
3	2	1	1	1	5 41.7 %
4	2	3	3	3	11 91.7 %
5	2	1	2	0	5 41.7 %
6	1	3	1	1	6 50 %
7	3	2	3	1	9 75 %
8	3	1	3	1	8 67 %
9	2	0	2	1	5 41.7 %
10	2	1	3	1	7 58.3 %
11	3	3	3	2	11 91.7 %
12	3	1	2	1	7 58.3 %
13	2	2	2	0	6 50 %
14	3	3	3	2	11 91.7 %
15	3	1	1	0	5 41.7 %
16	3	3	3	2	11 91.7 %
17	3	1	2	1	7 58.3 %
18	2	1	3	1	7 58.3 %
19	1	1	3	1	6 50 %
20	2	3	0	0	5 41.7 %
21	3	3	3	2	11 91.7 %
22	2	1	1	2	6 50 %
23	3	2	3	1	9 75 %
Total per Category	54	39	52	25	170
Mean per Category	2.349	1.696	2.261	1.087	7.391 61.6 %

Analysis of Sub-Question 1a—Direct Proof

From Table 13, data from Items 1, 5, and 10 investigated individuals' understanding of the logical underpinnings of direct proof. The items examined participants' understanding of the truth value of a conditional statement. A conditional statement is considered true if every substitution for its variables produces one of the following combinations of truth values for the hypothesis and conclusion: 1) hypothesis true, conclusion true; 2) hypothesis false, conclusion true; and 3) hypothesis false, conclusion false. To establish the truth of a conditional statement, an individual can also deduce it is true when the truth set of the hypothesis is a subset of the truth set of the conclusion.

Item 1, which was copied verbatim from Galbraith's (1982) study, pertained to an individual knowing when a conditional statement is true. Four of the five choices for this item used variables in the hypothesis and conclusion. In this research study, approximately 15 of 23 or 65.2% of the prospective teachers answered that part (a) If $a = 3$ then $a^2 = 9$, was a correct statement. The two participants that incorrectly selected answer (b)—If $a^2 = 9$, then $a = 3$ —seemingly had an incorrect concept image of knowing when a conditional statement is true. The truth set of the hypothesis, $\{3, -3\}$, was not a subset of the truth set of the conclusion $\{3\}$. For the same reason that (b) was incorrect, the 3 participants that chose (c), the biconditional, were also incorrect in their selection. For answer (d)—If $a \neq 3$, then $a^2 \neq 9$ —two participants misunderstood that the conclusion may or may not be true (i.e., $a^2 \neq 9$ is true for $a = 5$ and false if $a = -3$). The

conditional statement must hold for all a . A conditional statement is false only if values can be found for the variables that make the hypothesis true and the conclusion false.

The prospective teachers' performance on the direct proof Item 5 was slightly better than their performance on Item 1. Approximately 73.9% of the individuals correctly answered Item 5, which asked whether or not a conditional statement is considered false when the hypothesis is true and the conclusion is false. This item used single digit integers in an inequality sentence for the hypothesis and conclusion. Only choice (c) used a variable, $a = 2$, in the hypothesis. The results of both Items 1 and 5 show that 12 participants (52.2%) answered both items correctly and 3 participants (13%) missed both items.

Approximately 95.7% of the individuals correctly answered Item 10, which asked individuals to choose a true statement, based on two given conditional statements about finite geometry. In this item, participants seemed to understand the case that if the hypothesis of a true conditional statement was true, it implied that the conclusion was true.

From the data in Table 14, the participants' mean score for the three items was 2.349, which was the highest mean of all four sub-categories. Table 15 distributed the number of participants in each quality level of understanding logical underpinnings of direct proof by using the sum of the correct responses for Items 1, 5, and 10 from Table 14. Participants who did not select the correct response are said to have a concept image that does not match the concept definition for the logical form of a direct proof.

Table 15. Quality Levels for Sub-Category—Direct Proof

Points (3 pts. possible)	Quality Levels for Logical Underpinnings of Direct Proof	# of Participants per quality level (%)
3 points	Competent understanding	11 (47.8 %)
2 points	Some understanding	9 (39.1 %)
1 point	Minimal understanding	3 (13.0 %)
0 points	Lack of understanding	0 (0 %)

Analysis of Sub-Question 1b—Indirect Proof

Items 6, 8, and 12 focused on the logical underpinnings of indirect proof. These items focused on the contradiction method of proof, commonly referred to as an indirect proof. To prove “if H , then C ” by the contradiction method of proof, one assumes that H and the negation of C is true (i.e., assume C is false), and then deduces some contradiction.

Items 6 and 8 asked individuals what assumption should they begin with when using a proof by contradiction for a conditional statement. These items were based on the mechanics of negating a conditional statement with a quantifier. Both Items 6 and 8 were statements of the form “for all real numbers x , if $H(x)$, then $C(x)$.” Both items used the universal quantifier “for all”, therefore to negate this statement we write it as an existential statement—“there exists a real number x such that $H(x)$ is true and $C(x)$ is false.” As shown in Appendix D, only 9 of the participants (39.1%) answered both Items 6 and 8 correctly while 11 participants (47.8%) had incorrect responses for both items. Data from Table 13 show that 10 participants (43.5%) answered Item 6 correctly and 11 participants (47.8%) responded correctly to Item 8. On Item 6, six participants

incorrectly answered choice (b) which was of the form “there exists a real number x such that $H(x)$ is false and $C(x)$ is true.” On Item 8, five participants incorrectly answered choice (a) that was of the same incorrect form. The data presented in Appendix D, show that the five individuals that answered the incorrect form for Item 8 also answered the same incorrect form for Item 6.

The data also show that three individuals answered the same incorrect form for Item 6 and Item 8 by incorrectly answering choice (c) on Item 6 and choice (b) on Item 8. These choices were of the incorrect form “for all real numbers x , if $H(x)$, then $\sim C(x)$.”¹⁴ This form was incorrect because it does not negate the universal quantifier and it states that the negation of a conditional sentence is another conditional sentence. On both items, one participant answered the same incorrect form “for all real numbers x , if $\sim H(x)$, then $\sim C(x)$.” This was incorrect because it does not negate the universal quantifier, incorrectly expresses the negation of a conditional statement as a conditional statement, and negates the hypothesis.

Approximately 78.3% of the participants correctly answered Item 12. This item was an example of Dirichlet’s Pigeon Hole Principle. The item asked participants to recognize that the contradiction method of proof was used to prove that something does not happen. In this case, it was easier to assume that it does happen and reach a contradiction. Three of the participants misunderstood the application to be the direct proof method (answer (b)).

¹⁴ The notation for “not $C(x)$ ” is written $\sim C(x)$.

Table 16 lists the number of participants in each quality level of understanding logical underpinnings of indirect proof. The researcher used the sum of the correct responses for Items 6, 8, and 12 from Table 14. Only 30.4% got all three items correct. The mean score for the sample on this sub-category was 1.696 out of 3 possible points.

Table 16. Quality Levels for Sub-Category—Indirect Proof

Points (3 pts. possible)	Quality Levels for Logical Underpinnings of Indirect Proof	# of Participants per quality level (%)
3 points	Competent understanding	7 (30.4 %)
2 points	Some understanding	3 (13.0 %)
1 point	Minimal understanding	12 (52.2 %)
0 points	Lack of understanding	1 (4.3 %)

Analysis of Sub-Question 1c—Refutations

Items 4, 7, and 11 focused on falsifying a claim with a counterexample (i.e., recognize a counterexample and know it disproves a claim). The concept definition participants needed to understand for refutations was that a conditional sentence is considered false if there is a counterexample that refutes the claim. A counterexample is a choice of variables that makes the hypothesis of the conditional statement true and its conclusion false.

For Items 4 and 11, approximately 87% and 73.9% of the prospective teachers, respectively, answered the questions correctly. Both items were written in the form—"for all real numbers x , if $H(x)$, then $C(x)$." On both items, two choices gave an example and implied the statement was true and two choices gave an example and implied the statement was false. The data presented in Appendix D show that 17 participants

(73.9%) answered both of these items correctly. Three participants' (13%) response was incorrect on both items where they selected choice (c) on Item 4 and choice (a) on Item 11 that produced a true hypothesis and a true conclusion, therefore incorrectly implying the statement as true for all real numbers. These participants did not have an understanding that it must be true for all real numbers x and that one true example does not prove that the claim is true.

Item 7 was written in the traditional "if H , then C " form. Four choices gave an example(s) and implied the statement was false and one choice simply stated that the claim was true. Approximately 65.2% or 15 participants selected the correct choice (a) that made the hypothesis true and the conclusion false, therefore showing that the statement was indeed false. One individual selected (b), which contained a number that made the hypothesis false and conclusion true, and four participants selected choice (c), which contained two numbers that showed that the statement was false. The problem with this incorrect concept image was that even though the statement was indeed false, these examples did not provide an adequate counterexample to show it was false. Two individuals answered that they were not given adequate examples to disprove the claim and one individual believed the statement to be true.

Table 17 lists the number of participants in each quality level of understanding logical underpinnings of refutations. The researcher used the sum of the correct responses for Items 4, 7, and 11 from Table 14. The mean score for the sample was 2.261 points out of 3 possible points. Twelve participants got all three items correct,

which was more than any other sub-category, but 21.7% scored at most one correct answer, showing a minimal recognition that a counterexample refutes a claim.

Table 17. Quality Levels for Sub-Category—Refutations

Points (3 pts. possible)	Quality Levels for Logical Underpinnings of Refutations	# of Participants per quality level (%)	
3 points	Competent understanding	12	(52.2 %)
2 points	Some understanding	6	(26.1 %)
1 point	Minimal understanding	4	(17.4 %)
0 points	Lack of understanding	1	(4.3 %)

Analysis of Sub-Question 1d—Laws of Logic

Items 2, 3, and 9 focused on understanding the laws of logic that underlie proof. The three items covered general ideas associated with proof. Item 2 asked individuals to negate a conditional statement. "The negation of a conditional sentence is the assertion that the conditional sentence has a counterexample" (Fendel & Resek, 1990, p. 101). Therefore, Item 2 was similar to the items on indirect proof and refutations. The negation of a conditional statement is equivalent to asking what truth values for the simple statements make the conditional statement false— H true and C false. Only 17.4%, or 4 of the participants, understood that the negation of "if H , then C " is " H and $\sim C$." Participants' incorrect selections were varied, with 8 participants selecting (e), "None of the above." The 3 participants that selected choice (a) incorrectly indicated that the negation of "if H , then C " is a disjunction " $\sim H$ or C ." Four participants incorrectly selected (c), which was a conditional sentence "if H , then $\sim C$." Four participants selected (d), which was the converse of the given conditional sentence or "if C , then H ."

Item 3 asked individuals a question about what they could determine about the truth of the converse of a theorem if they knew the theorem had been proven. Participants needed to know the term “converse” and understand that the truth of a conditional does not imply that the converse is true. Only 4 of the prospective teachers or 17.4% answered the correct selection (e), “None of the above.” Four participants selected (a), which stated “the converse of the theorem is true and does not need further proof.” Six of the participants selected (b), which stated “the converse of the theorem is true and needs further proof.” Two participants selected (c), which stated “the converse of the theorem is false and does not need further disproof.” Seven of the participants selected (d), which stated “the converse of the theorem is false but needs disproof.”

Approximately 73.9% of participants answered Item 9 correctly. This item asked them about the laws of logic that underlie completing a proof when given three true statements—a simple statement, a conditional statement, and a disjunction (i.e., “or” statements). Individuals needed to understand the truth values of these three statements to answer the item correctly. Incorrect responses were varied: two participants incorrectly selected choice (a), two selected (d), and two selected (e).

Table 18 lists the number of participants in each quality level of understanding laws of logic by taking the sum of the correct responses for Items 2, 3, and 9 from Table 14. The mean score for the sample was 1.087 out of a possible 3 points, the lowest of all four sub-categories. Seventeen participants or 73.9% scored at most one correct answer of the three items, demonstrating a minimal understanding of general laws of logic of proof.

Table 18. Quality Levels for Sub-Category—Laws of Logic

Points (3 pts. possible)	Quality Levels for Laws of Logic	# of Participants per quality level (%)
3 points	Competent understanding	1 (4.3 %)
2 points	Some understanding	5 (21.7 %)
1 point	Minimal understanding	12 (52.2 %)
0 points	Lack of understanding	5 (21.7 %)

Analysis of Secondary Research Question 1—Logical Underpinnings of Proof

Overall the sample of 23 prospective secondary teachers answered 61.6% of the items correctly. There were only two items—4 and 10—that 80% or more of the participants answered correctly. On Items 2, 3, 6, and 8 fewer than 50% of the participants had correct answers.

In evaluating participants' responses to Items 2, 6, and 8, only 17.4% of the participants had a correct answer for Item 2, compared to 43.5% and 47.8% respectively for Items 6 and 8. Only 2 of 9 participants that answered both Items 6 and 8 correctly, also answered Item 2 correctly. Item 2 asked participants to negate a conditional statement, whereas Item 6 and 8 asked them what assumption one should start with to use the contradiction method of proof of a conditional statement. Items 6 and 8 asked participants to negate a quantifier and a conditional statement. The three problems were closely related in that they involved understanding what makes a conditional statement false. These three items were also closely related to the refutation items, which asked participants to recognize a counterexample and its power to disprove a claim. The

percentage of participants that answered correctly for the refutation Items, 4, 7, and 11 were 87%, 65.2%, and 73.9%, respectively.

Table 19. Quality Levels for Part I—Logical Underpinnings of Proof

# correct responses	Percent Correct	Quality levels for total correct responses	# of Participants per quality level (%)
10 – 12 items correct	83 – 100 %	Competent understanding of the logical underpinnings of proof	5 (21.7%)
7 – 9 items correct	58 – 75 %	Some understanding of the logical underpinnings of proof	8 (34.8%)
4 – 6 items correct	33 – 50 %	Minimal understanding of the logical underpinnings of proof	10 (43.5%)
0 – 3 items correct	0 – 25 %	Lack of understanding of the logical underpinnings of proof	0 (0%)

Table 19 lists the number of participants per quality level for the 12 multiple choice items. The number of correct responses for the 12 items ranged from a low of 5 to high of 11. The data show that five participants (21.7%) answered 11 of the 12 items correctly and were categorized as having a competent understanding of the logical underpinnings of proof. Approximately 43.5% were classified as having only a minimal understanding by correctly responding to 6 or fewer of the 12 problems. Data from Table 14 show that 16 of the 23 participants (69.6%) scored below 70% correct on the 12 multiple choice items.

Table 20. Quality Levels for Sub-Categories

Points (3 pts. possible)	Quality Levels for Sub- categories of the Logical Underpinnings of Proof	#Participants per quality level (%)			
		Direct Proof	Indirect	Refutation	Laws of Logic
3 points	Competent understanding	11 (47.8%)	7 (30.4%)	12 (52.2%)	1 (4.3%)
2 points	Some understanding	9 (39.1%)	3 (13.0%)	6 (26.1%)	5 (21.7%)
1 point	Minimal understanding	3 (13.0%)	12 (52.2%)	4 (17.4%)	12 (52.2%)
0 points	Lack of understanding	0 (0%)	1 (4.3%)	1 (4.3%)	5 (21.7%)

Table 20 lists the number of participants per quality level for the four sub-categories. Based upon this table, participants seemed to understand the logical underpinnings of direct proof and refutations, but scored poorly in the areas of logical underpinnings of indirect proof and laws of logic that underlie proof. Twenty participants (87%) answered at least 2 of the 3 items correctly for direct proof and 18 participants (78.3%) correctly answered the same number in the refutation category. Only 10 participants (43.5%) in the indirect sub-category, and 6 participants (26%) in the laws of logic sub-category answered at least two items correctly.

Data Analysis of Secondary Research Question 2

Data gathered from responses to the constructed response items (Part II of the questionnaire) were used to answer Secondary Research Question 2: What are prospective secondary mathematics teachers' abilities to complete mathematical proofs? This question was addressed through sub-questions a, b, and c:

- a. What are their abilities to complete direct proof?

- b. What are their abilities to complete indirect proof?
- c. What are their abilities to refute a false conjecture?

Item 1 asked participants to complete a proof for which the most likely methods were an indirect proof (i.e., proof by contradiction or proof by contrapositive). Item 2 was a false conjecture item that was adapted from the Ball and Wilson (1990) study of novice secondary mathematics teachers. Item 3 was most likely completed as a direct proof. Depending upon the responses to the open items, participants' proof schemes were analyzed according to the theoretical framework that was presented in Chapter Two.

Each of the responses was analyzed according to the proof schemes (Table 21) and also by a numerical scoring rubric (Tables 9 and 10 of Chapter Three). The level number was associated to the number of points (i.e., level 5 was scored as 5 points). The numerical scoring rubric assigned scores 0 through 5 points to each response of the constructed response items.

Scores of 4 and 5 points on the direct and indirect proof indicated that the response was a valid justification (5 points) or was valid but lacked one justification step (4 points). Only scores of 4 or 5 points were classified as an Analytic Proof. A score of 3 points indicated two or more steps missing, but the response showed some chain of reasoning. Two points indicated minimal progress with at least one valid deduction; one point demonstrated a lack of understanding, and zero points indicated a non-response, an invalid response, or pointless deductions.

Sample Data for Each Constructed Response Item

The total number of participants collectively that scored 0 to 5 points, as well as the mean score per proof item, is listed in Table 21.

Table 21. Distribution of Data per Item—Completion of Proof

Part II Constructed response Items Item # - Category	# Participants per Point Value						Total <i>n</i>	Mean score per item
	Invalid Proof				Valid Proof			
	0 pt.	1 pt.	2 pts.	3 pts.	4 pts.	5 pts.		
# 3 - Direct Proof	3	2	1	4	9	4	23	3.130
# 1 - Indirect Proof	2	6	3	3	3	6	23	2.739
# 2 - Refutation	7	6	0	1	1	8	23	2.304

Sub-Category Data per Individual

The individuals' scores in the three sub-categories for ability to complete mathematical proofs are listed by sub-categories in Table 22. The participants' total score as well as the mean in each sub-category are also listed.

Table 22. Distribution of Sub-Category Data per Individual—Completion of Proof

Participant	Part II – Constructed Response Proof Items			Total Points (15 points possible - %)	
	Direct Proof item #3 5 pts. possible	Indirect Proof item #1 5 pts. possible	Refutation item #2 5 pts. possible		
1	3	1	1	5	33.3%
2	4	3	1	8	53.3%
3	3	1	4	8	53.3%
4	5	3	5	13	86.7%
5	4	0	1	5	33.3%
6	0	2	0	2	13.3%
7	4	2	5	11	73.3%
8	4	4	0	8	53.3%
9	1	5	5	11	73.3%
10	0	2	1	3	20%
11	3	5	3	11	73.3%
12	3	1	5	9	60%
13	0	3	0	3	20%
14	2	4	0	6	40%
15	4	5	1	10	66.7%
16	4	5	5	14	93.3%
17	5	1	0	6	40%
18	5	1	5	11	73.3%
19	4	5	0	9	60%
20	1	0	5	6	40%
21	4	5	0	9	60%
22	4	4	1	9	60%
23	5	1	5	11	73.3%
Total	72	63	53	188	
Mean (Total/n)	3.130	2.739	2.304	8.174	54.5%

Analysis of Sub-Question 2a—Direct Proof

Item 3 asked participants to prove that if a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment. This conditional

statement is commonly referred to as the Perpendicular Bisector Theorem. This problem was adapted from a traditional high school geometry textbook.¹⁵ To prove a conditional sentence “if H , then C ” by the direct method of proof, one assumes that the hypothesis H is true and, based on this assumption, proceeds through a logical sequence of steps to deduce that the conclusion C is true. One way to prove this item involved the use of common geometry definitions (e.g., definition of perpendicular bisector, perpendicular, right angles) and properties (e.g., Reflexive Property of Congruence) to reason that two triangles are congruent by the Side-Angle-Side (SAS) Congruence Theorem. Then one can apply the definition of congruence—corresponding parts of congruent triangles are congruent (CPCTC). Another approach involved the use of similar geometry definitions and properties and the Pythagorean Theorem.

The data from Table 21 show that 17 participants (73.9%) constructed a chain of reasoning (i.e., scored 3, 4, or 5 points) that *could* lead to a valid proof, however, only 13 of those 17 participants (56.5% of the sample) completed a valid proof. Four participants (17.4%) completed a valid justification, scoring 5 points. Nine (39.1%) completed a valid justification, but missed at most one reasoning or justification step, scoring 4 points. These 13 participants (56.5%) were classified as using an Analytic Proof Scheme—Axiomatic Proof.¹⁶ Four participants responded with evidence of a chain of reasoning that could lead to a valid proof, but failed to adequately justify their steps (score 3

¹⁵ Item #3 was adapted from *Geometry: An Integrated Approach* by Larson, Boswell, and Stiff (1998).

¹⁶ According to Harel and Sowder (1998), an individual’s proof scheme consists of whatever an individual uses to convince himself or herself and others.

points). Six participants (26.1%) did not construct a chain of reasoning that would lead to a valid justification (scored 0 to 2 points). Table 22 displays the mean score for the direct proof item as 3.130 out of 5 possible points. This was the highest mean of the 3 sub-categories.

Analysis of Sub-Question 2b—Indirect Proof

Item 1 required an understanding of the definitions of even and odd integers. Table 21 shows that 12 participants (52.2%) constructed a chain of reasoning that could lead to a valid proof, however, only 9 of those 12 participants (39.1% of the sample) completed a valid proof. The 9 participants that completed a valid proof or a valid proof with minor errors were classified as using an Analytic Proof Scheme—Axiomatic Proof. Four of the 9 individuals that were classified as using an Analytic Proof Scheme (scored 4 or 5 points), employed the contradiction method of proof, and five participants employed the contrapositive method of proof.

The responses of 14 participants (60.8%) were scored 0 to 3 points and judged to be invalid proofs. Three of these 14 participants scored 3 points by responding with a chain of reasoning that could lead to a valid proof, however, their responses lacked appropriate justification. The responses of 11 participants (47.8%) lacked any chain of reasoning that could lead to a valid proof (score 0 – 2 points). Analyzing the responses of the 11 invalid proofs reveal that 3 participants attempted a direct proof; 3 participants attempted to prove the converse; 3 participants' response was completely invalid or a non-response; and 2 participants used examples as justification, exhibiting an Empirical Proof Scheme. The data show that the mean for this category is 2.739 out of 5 points.

Analysis of Sub-Question 2c—Refutations

Item 2 was the false conjecture: *As the perimeter of a rectangle increases, then the area of it also increases.* Below the conjecture, two rectangles were drawn—one was 3 centimeters by 3 centimeters and the other was 3 centimeters by 4 centimeters. The perimeter and area of both rectangles were given below the drawings of the rectangle.

Participants were asked to reply to this conjecture. They should have questioned the truth of this conjecture and established that the conjecture was false. To disprove this conjecture, participants needed to respond with a counterexample. Table 21 show that 10 participants (43.5%) recognized that the conjecture was false, however, only 9 of the 10 participants (39.1% of the sample) were able to refute it with a counterexample. Eight of those 9 participants completed a valid justification (score 5 points) and 1 participant responded with a partially correct counterexample and was given a score of 4 points. The one participant that scored 3 points implied that the conjecture was false by stating “try a real skinny rectangle.” Thirteen participants or 56.5% responded that the conjecture was true. Six of the 13 attempted to prove it was true and 7 participants responded by restating the conjecture or by an invalid response. The data from Table 22 show the mean for this category as 2.304 out of 5 points, the lowest mean of the 3 categories.

Analysis of Secondary Research Question 2—Completion of Proof

An Analytic Proof was the classification for responses that scored 4 or 5 points; these responses were considered to be valid justifications, or a valid justification that lacked at most one justification step. From Table 21, the number of responses considered as evidence of an Analytic Proof Scheme varied. For the direct proof item, 13

participants' responses (56.5%) were classified as Analytic Proof. Only 9 participants (39.1%) responded with an Analytic Proof for both the indirect proof and the refutation item. Participants that did not respond with a valid justification scored 0 to 3 points. For direct proof, 10 participants (43.5%) did not complete a valid justification. Fourteen participants (60.9%) did not complete a valid justification for the indirect proof item or the refutation item. From Table 22, the mean total points for the sample was 8.174 out of 15 points or 54.5%.

Table 23. Quality Levels for Part II—Completing Proofs

Total points for 3 constructed response items	Quality Levels for Ability to Complete Mathematical Proofs	# of Participants per quality level (%)
12 – 15	Competent in their ability to complete proofs	2 (8.7%)
9 – 11	Some ability to complete proofs	10 (43.5%)
6 – 8	Minimal ability to complete proofs	6 (26.1%)
0 – 5	Lack of ability to complete a proof	5 (21.7%)

Table 23 presents the samples' responses per quality level for ability to complete mathematical proofs. Only 2 participants were classified as being competent on each of the three proofs whereas 11 participants (47.8%) were classified as having minimal or lacking ability to complete proofs. Data from Table 22 and 23 show that only one participant responded with a valid proof (scored a 4 or 5) on all three items. Six participants (26.1%) responded with a valid proof for the direct proof and indirect proof items. Only one of these 6 individuals recognized that the false conjecture item was indeed false.

Data Analysis of the Primary Research Question

The data in Table 24 were used to address the Primary Research Question: What are prospective secondary mathematics teachers' conceptions of proof and refutations?

Table 24. Distribution of Data from Part I & II

Participant	Direct Proof		Indirect Proof		Refutation		Laws of Logic	Total Points	
	Part I #correct items #1,5,10	Part II item #3 5 pts. possible	Part I #correct items #6, 8, 12	Part II item #1 5 pts. possible	Part I #correct items #4,7, 11	Part II item #4 5 pts. possible	Part I #correct items #2,3,9	Part I #correct 12 pts. Possible (%)	Part II 15 pts. Possible (%)
1	1	3	1	1	3	1	0	5 41.7%	5 33.3%
2	3	4	1	3	2	1	1	7 58.3%	8 53.3%
3	2	3	1	1	1	4	1	5 41.7%	8 53.3%
4	2	5	3	3	3	5	3	11 91.7%	13 86.7%
5	2	4	1	0	2	1	0	5 41.7%	5 33.3%
6	1	0	3	2	1	0	1	6 50%	2 13.3%
7	3	4	2	2	3	5	1	9 75 %	11 73.3%
8	3	4	1	4	3	0	1	8 67 %	8 53.3%
9	2	1	0	5	2	5	1	5 41.7%	11 73.3%
10	2	0	1	2	3	1	1	7 58.3%	3 20%
11	3	3	3	5	3	3	2	11 91.7%	11 73.3%
12	3	3	1	1	2	5	1	7 58.3%	9 60%
13	2	0	2	3	2	0	0	6 50 %	3 20%
14	3	2	3	4	3	0	2	11 91.7%	6 40%
15	3	4	1	5	1	1	0	5 41.7%	10 66.7%
16	3	4	3	5	3	5	2	11 91.7%	14 93.3%
17	3	5	1	1	2	0	1	7 58.3%	6 40%
18	2	5	1	1	3	5	1	7 58.3%	11 73.3%
19	1	4	1	5	3	0	1	6 50 %	9 60%
20	2	1	3	0	0	5	0	5 41.7%	6 40%
21	3	4	3	5	3	0	2	11 91.7%	9 60%
22	2	4	1	4	1	1	2	6 50 %	9 60%
23	3	5	2	1	3	5	1	9 75 %	11 73.3%
Total	54	72	39	63	52	53	25	170	188
Mean Total/n	2.349	3.130	1.696	2.739	2.261	2.304	1.087	7.391 61.6%	8.174 54.5%

Data from Part I and II of the questionnaire were used to address the secondary research questions. Comparisons of responses were distributed according to the sub-categories of direct proof, indirect proof, refutation, and laws of logic. Participants' total scores were also examined. The mean total score for the participants on the multiple choice items was 7.391 correct out of 12 points possible for a percentage of 61.6% correct. For the constructed response items, the mean total score was 8.174 out of 15 points or 54.5% correct.

From Table 19, only 5 participants were classified as competent in their understanding of the logical underpinnings of proof by responding to at least 10 of the 12 items correctly. Table 23 shows that two of these five participants were classified as competent in their ability to complete proofs.

Table 24 shows 14 participants (60.9% of the sample) scored 70% or lower on both the multiple choice items and the constructed proof items. Eighteen participants (78.2% of the sample), scored 75% or lower on both the multiple choice items and the constructed proof items.

Data in Appendix D were used to compare how the participants performed on both parts of the questionnaire in the sub-categories of direct proof (Table 25), indirect proof (Table 26), and refutation (Table 27).

Table 25. Performance on Direct Proof Items

Direct Proof Items (# participants per performance)		Participants' Performance on the Logical Underpinnings of Direct Proof			
		3/3 items correct	2/3 items correct	1/3 items correct	0/3 items correct
Valid Direct Proof – Score 4 or 5 points		8	4	1	0
Invalid Proof	3 points	2	1	1	0
	2 points	1	0	0	0
	1 point	0	2	0	0
	0 points	0	2	1	0

Table 25 shows that 13 participants completed a valid direct proof. Eight of these 13 participants also correctly answered all 3 items for the logical underpinnings of direct proof. Four of these 13 participants answered 2 of the 3 items correct for logical underpinnings of direct proof. Data also show that 2 participants scored 3 points out of 5 points (i.e., constructed a chain of reasoning in completing a direct proof) and answered all 3 items correct for the logical underpinnings of direct proof. This data suggests that the better a person is at understanding the logical underpinnings of direct proof, the more likely the person is to construct a valid direct proof.

Table 26. Performance on Indirect Proof Items

Indirect Proof Items (# participants per performance)		Participants' Performance on the Logical Underpinnings of Indirect Proof			
		3/3 items correct	2/3 items correct	1/3 items correct	0/3 items correct
Valid Indirect Proof – Score 4 or 5 points		4*	0	4**	1***
Invalid Proof	3 points	1****	1	1	0
	2 points	1	1	1	0
	1 point	0	1	5	0
	0 points	1	0	1	0

* Three of these 4 participants used the contradiction method of proof; one used contrapositive method.

**Three of these 4 participants used the contrapositive method of proof; one used contradiction method.

***This participant used the contrapositive method of proof.

****This participant attempted a contradiction method of proof.

Table 26 shows that 9 participants completed a valid indirect proof. Four of these 9 participants answered all 3 items for the logical underpinnings of indirect proof correctly.¹⁷ Three of these 4 participants used the contradiction method of proof; one participant used the contrapositive method of proof. Another 4 of these 9 participants answered only 1 of the 3 items correct for the logical underpinnings of indirect proof. Three of these 4 participants used the contrapositive method of proof and one participant used the contradiction method of proof. One of the 9 participants correctly used the contrapositive method of proof, but did not answer any of the 3 logical underpinnings of indirect proof correct. The data shows that one participant answered all 3 items correct on the logical underpinnings of indirect proof, but scored 3 points (out of 5 points) on completion of an indirect proof. This participant attempted to prove by using the contradiction method of proof.

¹⁷ The three items for logical underpinnings of indirect proof asked participants to understand logical underpinnings of contradiction proof.

Table 27. Performance on Refutation Items

Refutation Items (# participants per performance)		Participants' Performance on the Logical Underpinnings of Refutation			
		3/3 items correct	2/3 items correct	1/3 items correct	0/3 items correct
Valid Disproof – Score 4 or 5 points		5	2	1	1
Invalid Disproof	3 points	1	0	0	0
	2 points	0	0	0	0
	1 point	2	2	2	0
	0 points	4	2	1	0

Table 27 shows that 5 of the 9 participants that disproved the false conjecture item also answered all 3 items correctly for the logical underpinnings of refutation. In total, 12 participants scored all 3 items correct for the logical underpinnings of refutation. Six of these 12 participants scored 1 or 0 points (out of 5) correct for disproving the false conjecture. It seemed that the better an individual is at recognizing a counterexample and knowing it disproves a claim (i.e., the multiple choice items) does not have much influence on a person's ability to recognize a false conjecture.

Table 28 presents the overall performance of the participants according to their total score from the logical underpinnings of proof and completion of proof items.

Table 28. Distribution of Participants' Total Score on Both Parts of the Questionnaire

COMPLETION OF PROOF														
	Competent	15pt 100%												
		14pts 93%										#16		
		13pts 87%										# 4		
		12pt 80%												
	Some	11pts 73%				#9		#18		#7 #23		#11		
		10pts 67%				#15								
		9pt 60 %					#19 #22	#12				#21		
	Minimal	8pts 53%				#3		#2	#8					
		7pts 47%												
		6pt 40 %				#20		#17				#14		
	Lack	5pts 33%				#1 #5								
		4pts 27%												
		3pt 20 %					#13	#10						
		2pts 13%					#6							
		1pts 7 %												
			1pt 8 %	2pts 17%	3pts. 25%	4 pts 33%.	5 pts 42%	6 pts 50%	7pts 58%	8 pts 67%	9pts 75%	10pts 83%	11pts 92%	12pts 100%
			Lacking			Minimal			Some			Competent		
LOGICAL UNDERPINNINGS OF PROOF														

There were 12 points possible for the logical underpinnings of proof and 15 points possible for completion of proof. From Table 28 it is easy observe the overall performance of the participants individually and as a whole group. For example, participant #16 scored 14 of the 15 points for completion of proof and 11 of the 12 points

for logical underpinnings of proof. This participant was categorized as being competent in both areas. At the other end of the spectrum, participant #6 scored only 2 of the 15 points for completion of proof and 6 of the 12 points for logical underpinnings of proof. This participant was categorized as possessing minimal understanding of the logical underpinnings of proof and lacking in ability to complete proof. As noted by Popham (2000), the three constructed response problems that asked participants to complete a proof are judged to be more difficult items than the multiple choice items.

Summary of Chapter Four

Chapter Four presented the data and results from both parts of the questionnaire that was administered to the 23 participants in this study. It was divided into four sections. The first section discussed the interjudge reliability for the evaluations of the constructed response items. The next three sections discussed the two secondary research questions and their parts and the primary research question. The analysis of the data described the prospective teachers' conceptions in logical underpinnings of proof and ability to complete mathematical proofs.

CHAPTER 5

CONCLUSIONS

Introduction

Chapter Four presented the data and results from both parts of the questionnaire. This chapter contains interpretations and conclusions based on the results of the data and findings from the study. Additionally, the relevance of these results to previous research, notes on the limitations of the study, and recommendations for future research are discussed.

Overview of the Study

The purpose of this research study was to investigate prospective secondary mathematics teachers' conceptions of proof and refutations. The significance of this study was two-fold. First, many in the mathematics education community have emphasized the importance of teaching students logical reasoning and formal proofs and the need to provide opportunities for students to examine conjectures in *all* content areas within *all* levels of school mathematics (MAA, 1998; NCTM, 2000). Secondly, the importance of preparing teachers to teach proof and reasoning, which is dependent upon their mathematical content knowledge of the nature of proof, was recognized by many organizations including the NCTM (2000), MAA (1998), and the MSEB (2001).

To research the primary question of this study, the researcher addressed two components of participants' conceptions of proof—1) understanding the logical underpinnings of proof, and 2) ability to complete mathematical proofs. Both components focused on the common proof schemes that were emphasized by the MAA (1998) and the NCTM *Standards* (2000)—direct proofs, indirect proofs, and refutations. A questionnaire was developed and administered to 23 prospective secondary mathematics teachers near the end of their teacher training programs during the Fall 2002 Semester.

Participants of the study were enrolled at three of Montana's six four-year public institutions. The 2000 – 2001 enrollment of these three universities represented approximately 79% of the total four-year school enrollment in the state of Montana. The questionnaire was made up of two parts in order to assess two components of individuals' conceptions of proof and refutations—logical underpinnings of proof and completion of proof. Data gathered from responses to the multiple choice items (Part I of the questionnaire) addressed Secondary Research Question 1 and its sub-questions 1a, 1b, 1c, and 1d. Data gathered from the responses to the constructed response items addressed Secondary Research Question 2 and its sub-questions 2a, 2b, and 2c. The data analysis of the secondary questions was used to address the Primary Research Question.

Secondary Research Question 1: Conclusions

Based on the results described in Chapter Four, conclusions were reached concerning the Secondary Research Question 1 and its sub-questions:

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof and refutations?

- a. What are their understandings of the logical underpinnings of direct proof?
- b. What are their understandings of the logical underpinnings of indirect proof?
- c. What are their understandings of the logical underpinnings of refutations?
- d. Can they demonstrate an understanding of the laws of logic that underlie proof?

Secondary Research Question 1a Conclusions

Question 1a assessed the participants' understandings of the logical underpinnings of direct proof. Data for this conclusion was gathered from Items 1, 5, and 10.

Conclusion 1: Participants generally demonstrated an understanding of the logical underpinnings of direct proof, but approximately one-third of the sample had difficulty evaluating the truth values of a conditional statement.

Evidence of this conclusion is supported by data presented in Table 15 that shows that 11 participants (47.8%) answered all three items in this category correctly. Approximately 96% of the sample recognized the conclusion given a conditional statement and hypothesis (Item 10). However, approximately one-third of the sample had difficulty evaluating the truth values of a conditional statement (Items 1 and 5). On Item 1, only 65.2% had correct responses whereas 73.9% of the participants answered Item 5 correctly. To understand the discrepancy of these responses, the researcher examined all three items.

Items 1 and 5 assessed participants' understanding of the truth values of a conditional statement. Item 1 (65.2% correct) asked when a conditional statement was

true and Item 5 (73.9% correct) asked when a conditional statement was false. Only 52.2% answered both Items 1 and 5 correctly. Three participants responded incorrectly on both items. Participants' incorrect answers were varied on both items. The researcher did not find any pattern of the incorrect responses for Items 1 and 5.

Approximately 95.7% of the participants answered Item 10 correctly. This item asked participants to reason with the principles of the logical underpinnings of direct proof, in a finite geometry problem involving parallel and intersecting lines. Participants were given two true conditional statements and asked what conclusion could be deduced. This item used the following Modus Ponens form of proof:

$$\begin{array}{ll} H \rightarrow C & \text{(given the conditional statement was true)} \\ H & \\ \text{therefore } C. & \end{array}$$

The improved performance on this item compared to the other direct proof items was perhaps best explained by the fact that Item 10 assessed their understanding of logical reasoning, whereas Items 1 and 5 were formal logic questions.¹⁸ In Item 10, participants were given a conditional statement was true and supplied with examples that made the hypothesis true. Therefore, it was easy for them to deduce the conclusion. Items 1 and 5 were more difficult for participants in that they had to determine whether a conditional statement was true (or false) by discerning the truth of the simple statements (i.e., the hypothesis and conclusion). The varied responses imply that approximately one-

¹⁸ Formal logic was described in Chapter Two. The primary ideas of formal mathematical logic are implication, negation, and quantification (Fendel & Resek, 1990).

third of the sample had difficulty evaluating the truth values of conditional statements because they were formal logic questions.

Secondary Research Question 1b Conclusions

Question 1b assessed participants' understandings of the logical underpinnings of indirect (contradiction) proof. Evidence for this conclusion is based on data gathered from Items 6, 8, and 12.

Conclusion 2: Participants demonstrated an understanding of the logical underpinnings of contradiction proof in so far as they recognized the logical form of proof by contradiction; however, participants demonstrated difficulty with the mechanics of contradiction proof.

Data show that approximately 78% of the sample recognized the logical form of proof by contradiction (Item 12). The conclusion was also based on the results from Items 6 and 8, in which only 45% of the sample demonstrated an ability to negate a universal quantified conditional statement, a logical underpinning of contradiction proof. The percentage of participants with correct responses to Items 6 and 8 were 43.5% and 47.8%, respectively. Data presented in Table 16 that show 13 participants (56.5%) answered at most 1 item correctly.

To examine why approximately 57% of the sample responded incorrectly to Items 6 and 8, the researcher examined the items and the incorrect responses. Both Items 6 and 8 were formal logic questions that asked participants to identify the assumption they should begin with when proving a conditional statement by contradiction. These items were based on the mechanics of negating a conditional statement with a quantifier. Both items were written in the form, "for all x in U , if $H(x)$, then $C(x)$ ", where $H(x)$ and $C(x)$

were open sentences. Therefore, to negate “for all x in U , if $H(x)$, then $C(x)$ ” one must find a substitution for a variable that makes $H(x)$ true and $C(x)$ false. Therefore, the negation of this statement is the existential statement “there exists an x in U such that $H(x)$ true and $C(x)$ false.”

Appendix D shows that only 9 participants (39.1%) answered both Items 6 and 8 correctly, while 11 participants (47.8%) had incorrect responses for both items. The 14 incorrect responses on these two items show some interesting patterns. Five participants incorrectly selected choice (b) on Item 6 and choice (a) on Item 8—both were of the form “there exists a real number x such that $H(x)$ is false and $C(x)$ true.” These individuals correctly negated the universal quantifier with an existential quantifier, but did not correctly negate the conditional sentence.

The data also show that 3 participants chose choice (c) on Item 6 and choice (b) on Item 8. This choice was incorrect because it implies that the negation of a conditional statement with a quantifier is the form “for all real numbers x , if $H(x)$, then $\sim C(x)$.” This response is incorrect because the participant fails to recognize the negation of the universal quantifier and also incorrectly concludes that the negation of the conditional sentence is another conditional sentence. One participant answered the same incorrect form “for all real numbers x , if $\sim H(x)$, then $\sim C(x)$ ” on both items (choice (a) on Item 6 and choice (d) on Item 8). This was an incorrect form because the participant did not negate the universal quantifier and incorrectly expressed the negation of a conditional statement as the inverse of the conditional statement. These results show that

approximately 57% of the participants have a weak understanding of the mechanics of negating a quantified conditional statement.

For Item 12, approximately 78.3% of the participants recognized that the objective of the contradiction method of proof is to prove that something does not happen. This item was an example of Dirichlet's Pigeon Hole Principle. To prove that something does not happen, it is easier to assume that it does happen and reach a contradiction. Three of the five individuals that answered this item incorrectly selected choice (b), which incorrectly identified the example as an application of the direct proof method.

Secondary Research Question 1c Conclusions

Question 1c assessed participants' understandings of the logical underpinnings of refutations.

Conclusion 3: Participants generally understood the logical underpinnings of refutations, recognized counterexamples, and understand that a counterexample disproves a claim.

Approximately three-fourths of the participants demonstrated an understanding of the logical underpinnings of refutations. Evidence of this conclusion was based on the data from Items 4, 7, and 11, which 87%, 73.9%, and 65.2% of the participants, respectively, answered correctly. The data show that 18 participants (78.3%) correctly answered at least two of the three items correctly. Twelve participants (52.2%) answered all three items correctly, which was more than any other sub-category. Unfortunately 21.7% scored at most one correct answer, showing a minimal recognition that the counterexample refuted the claim.

On Item 4, 20 participants (87%) had correct responses, whereas 17 participants (73.9%) correctly answered Item 11. There were some interesting patterns of the responses for Items 4 and 11. Both items were very similar in the manner that they were written and in their list of "solutions." Both were of the form—"for all real numbers x , if $H(x)$, then $C(x)$." On both items, two choices gave an example and implied the statement was true, and two choices gave an example and implied the statement was false. The data presented in Appendix D show that the 6 participants that answered incorrectly for Item 11, all selected choice (a). Three of these 6 individuals also responded incorrectly on Item 4 by selecting choice (c). These 3 participants (13%) selected the response (choice (c) on Item 4 and choice (a) on Item 11) representing a true hypothesis and a true conclusion, therefore incorrectly implying the statement as true for all real numbers. These participants did not have an understanding of the generalization for all real numbers x and that one true example does not prove the claim is true.

Another possible explanation of why the participants responded incorrectly was perhaps the wording of the choices. When writing Items 4 and 11, the researcher used the same wording format as the Galbraith (1982) study. For example, Item 4 asked participants to determine the truth of a statement R (a false statement). Three participants selected the incorrect choice (c): "The numbers $x = 1$ and $y = 2$ show that R is true." The word "show" was used, but the word "prove" might have provided a better assessment of the participants' understanding that if the statement R is true, it must be true for all real

numbers x and y . The word “show” may have been misunderstood to mean “provides supportive evidence” to say that the statement is true.¹⁹

Item 7 was written as a conditional statement. On this item, approximately 65.2% of the 23 prospective teachers selected an example that made the hypothesis true and the conclusion false, therefore showing that the statement was indeed false. Interesting patterns of incorrect responses on this item revealed 1 participant selected choice (b) and 4 participants selected choice (c). Both choice (b) and (c) presented an example that made the hypothesis false and conclusion true. Two individuals responded that they were not given adequate examples to disprove the claim and one individual believed the statement to be true.

Secondary Research Question 1d Conclusions

Question 1d assessed participants’ understandings of the laws of logic that underlie proof. Data that addressed this question were from Items 2, 3, and 9.

Conclusion 4: Participants demonstrated an understanding of the laws of logic in so far as they were able to apply logical principles such as Modus Tollens and disjunction in a reasoning task; however, they demonstrated a weak understanding of formal logic.

Evidence of this conclusion was based on the data from Table 20 that shows 17 participants (73.9%) answered at most one item correctly. Also, this conclusion was based on the data from the results of Items 2, 3, and 9, which were answered correctly by 17.4%, 17.4%, and 73.9% of the participants, respectively. Participants demonstrated an

¹⁹ One of the limitations of this study was the fact that insight into why participants responded incorrectly was not assessed through follow-up interviews.

understanding of the laws of logic when those principles were applied in a real world problem (Item 9). However, participants had great difficulty with the formal logic in Items 2 and 3. These items asked participants to make connections between the truth values of a conditional statement and its related forms (i.e., negation of a conditional and the converse of a conditional).

Results from Item 2 show that only 17% of the sample were able to negate a conditional statement. Only 4 participants (17.4%), correctly understood that the negation of "if H , then C " is " H and $\sim C$." Participants' incorrect selections were varied, with 8 participants selecting option (e), "None of the above." The 8 participants that selected "None of the above" perhaps thought a choice should have been just the negation of the conclusion (i.e., $\sim C$) or possibly thought the answer should have been the inverse—"If $\sim H$, then $\sim C$." It is difficult to understand why these participants selected this choice without an interview, but it is clear that they lack an understanding of the negation of a conditional statement. The patterns for the other incorrect responses for Item 2 show that three participants selected choice (a); these participants incorrectly stated that the negation of "if H , then C " is " $\sim H$ or C ." Four participants selected (c), which was written as a conditional sentence "if H , then $\sim C$." Four participants selected (d), which was the converse of the given conditional sentence.

The other formal logic problem was Item 3 in which only 4 of the prospective teachers (17.4%) correctly selected option (e), none of the above. In this item, participants were told that a theorem from geometry had been proven. They were asked what could be determined about the truth of the converse of the theorem. Participants

needed to know the term "converse" and understand that the truth of a conditional statement does not imply that the converse is true. The incorrect responses were varied. Four participants selected (a), which stated "the converse of the theorem is true and does not need further proof." Six participants selected (b), which states "the converse of the theorem is true and needs further proof." Two participants selected (c), which stated "the converse of the theorem is false and does not need further disproof." Seven participants selected (d), which stated "the converse of the theorem is false but needs disproof."

Approximately 82.6% of the participants seemed to have an inadequate concept image of what can be implied about the truth of the converse of a theorem. Four possible explanations of the poor performance on this item are the following: (1) participants do not understand the term "converse"; (2) participants don't know that every theorem has a converse; (3) participants mistook the word "converse" for the word "contrapositive"; or (4) participants do not understand that the truth value of a conditional statement does not imply the truth of the converse (i.e., participants have a poor understanding of the relationship between a conditional statement and its converse).

Approximately 73.9% of participants answered Item 9 correctly. This item asked participants to draw correct conclusions from three true statements set in a concrete situation. The three statements were a simple statement, a disjunction (i.e., "or" statement), and a conditional statement. Participants needed to use Modus Tollens and reason with disjunction. Incorrect responses varied, with two participants incorrectly selecting choice (a), two participants selecting (d), and two participants selecting (e).

Secondary Research Question 1 Conclusions

The Secondary Research Question 1 investigated the prospective teachers' understandings of the logical underpinnings of proof and refutations.

Conclusion 5: Participants demonstrated some understanding of the logical underpinnings of direct proof and refutations; however, at a formal level they generally showed a lack of understanding of the laws of logic that underlie proof and the mechanics of proof by contradiction.

Evidence for this conclusion was supported by the results presented in Table 20 that show 20 participants (87%) answered at least 2 of the 3 items correctly for direct proof and 18 participants (78.3%) answered the same number in the refutation category. Only 10 participants (43.5%) in the indirect sub-category, and 6 participants (26%) in the laws of logic sub-category answered at least two items correctly.

Overall, participants answered 61.6% of the 12 multiple choice items correctly. The distribution of participants per quality levels in Table 19 indicated that only 5 participants (21.7%) were classified as competent in their understanding of the logical underpinnings of proof. Eight participants were classified as showing some understanding. What is perhaps the most disturbing result in Table 19, is that 10 participants (43.5%) correctly responded to at most 6 of the 12 items. In other words, 44% of the sample correctly responded to 50% or fewer of logical underpinnings of proof items.²⁰ These 10 participants were classified as showing a minimal understanding of the logical underpinnings of proof.

²⁰ Data from Table 22 show that only 7 participants (30% of the sample) correctly responded to more than 70% of the 12 items of logical underpinnings of proof.

The data show that participants have a weak understanding of the truth of a conditional statement and its related statements at a formal level. This is supported by participants' poor performance on Item 1 (65.2% correct), Item 2 (17.4% correct), Item 3 (17.4% correct), Item 6 (43.5% correct), and Item 8 (47.8% correct)—all formal logic questions. Participants seemed to have trouble with the language of formal logic and lacked an understanding of the principles of formal logic, including the truth values of a conditional statement and forms related to a conditional statement (e.g., negation of a conditional statement, converse of a conditional statement, and conditional statements with universal quantifiers).

The NCTM *Standards* 2000 recommended that by the end of high school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypothesis. The NCTM added that students should have opportunities to explore with conjectures, converse, and indirect proof that includes proof by contradiction (p. 58 and p. 345). Students should develop confidence in their reasoning abilities and be able to question others' mathematical arguments by relying on logic to determine the soundness of the argument (NCTM, 2000, p. 346). The MAA recommended that students have opportunities in logic that include the basic connectives, quantifiers, true and false statements, and strategies for justifying or refuting statements (Ross, 1998). Results from this study suggest that some prospective teachers may have a difficult time implementing recommendations by the MAA (1998) and the NCTM (2000) in the area of the logical underpinnings of proof.

Secondary Research Question 2: Conclusions

Based on the results described in Chapter Four, conclusions were made concerning Secondary Question 2 and its sub-questions:

2. What are prospective secondary mathematics teachers' abilities to complete mathematical proofs?
 - a. What are their abilities to complete direct proof?
 - b. What are their abilities to complete indirect proof?
 - c. What are their abilities to refute a false conjecture?

Secondary Research Question 2a Conclusions

Question 2a evaluated the participants' ability to complete a direct proof. The direct proof item asked participants to prove the Perpendicular Bisector Theorem that is found in most traditional high school geometry texts. One method of proving this theorem requires the use of common definitions and theorems, such as perpendicular bisector, Side-Angle-Side Congruency Theorem, and the fact that corresponding parts of the triangles are congruent. Another approach involved the use of coordinate geometry and the Pythagorean Theorem.

Conclusion 6: Seventeen participants (73.9%) constructed a chain of reasoning that could lead to a valid direct proof, however, only 13 of those 17 participants (56.5% of the sample) completed a valid proof. Six participants (26.1%) did not construct a chain of reasoning that would lead to a valid proof.

Support for this conclusion is provided by the data in Table 21 and 22. Seventeen participants scored 3, 4, or 5 points because they constructed a chain of reasoning that

could lead to a valid proof. Only 13 of these 17 individuals, however, completed a valid proof. Four participants (17.4%) provided a complete valid justification and received a score of 5 points. The 9 participants (39.1%) that received a score of 4 points, constructed a valid justification but missed at most one reasoning or justification step. The 13 participants' (56.5%) that received a score of 4 or 5 points (valid proof) used an Analytic Proof Scheme—Axiomatic Proof. The 4 participants that received scores of 3 points, provided evidence of a chain of reasoning, but their responses lacked appropriate justification. Six participants (26.1%) failed to construct a chain of reasoning that would lead toward a valid justification and received scores of 0, 1, or 2 points.

The 10 responses that were scored 0 to 3 points were judged as invalid proofs. Four of these 10 participants responded with a chain of reasoning that would lead to a valid proof, but did not justify their reasoning. They might be capable of completing a valid proof, but were unable to write a formal proof or did not understand what was being asked of them. Of greater concern are the 6 participants (26.1%) that provided no chain of reasoning. This result is especially discouraging when one considers that the proof of the Perpendicular Bisector Theorem asks prospective secondary teachers to understand basic geometry concepts that are common in a secondary mathematics curriculum. Even though all the participants of this study have completed a college geometry course, the question arises of how well these participants will be able to teach high school geometry and proofs to their prospective students.

Secondary Research Question 2b Conclusions

Question 2b evaluated participants' ability to complete an indirect proof. The item asked participants to prove that if x^3 is even, then x is even, where x is an integer.

The content of this item involved the definitions of even and odd integers.

Conclusion 7: Twelve participants (52.2%) constructed a chain of reasoning that could lead to a valid indirect proof, however, only 9 of those 12 participants (39.1% of the sample) completed a valid proof. Eleven participants (47.8%) were unable to construct a chain of reasoning that would lead to a valid proof.

Conclusion 7 was supported by data presented in Table 21 and 22. Twelve participants scored 3, 4, or 5 points because they constructed a chain of reasoning that could lead to a valid proof. Only 9 of those 12 participants, however, completed a valid proof. Six participants (26.1%) completed a valid justification and received a score of 5 points. The 3 participants (13%) that received a score of 4 points, constructed a valid justification but missed at most one reasoning or justification step. The 9 participants (39.1%) that received a score of 4 or 5 points (valid proofs), used an Analytic Proof Scheme—Axiomatic Proof. Four of these 9 participants used a contradiction method of proof whereas 5 participants used a contrapositive method of proof.

Fourteen responses (60.8%) that were scored 0 to 3 points were judged as invalid proofs. Three of these 14 participants responded with a chain of reasoning that could lead to a valid proof, but their responses lacked mathematical justification and were scored 3 points. It is discouraging that 11 of the 14 invalid proof responses (47.8% of the sample) provided no chain of reasoning that could lead to a valid proof (i.e., responses were scored 0 – 2 points). Analyzing the responses of the 11 invalid proofs, reveal that 3

participants attempted a direct proof, some by taking the cube root of x^3 ; 3 participants attempted to prove the converse; 3 responses were completely invalid or a non-response; and 2 participants used examples as justification and therefore employed an Empirical Proof Scheme. Even though all the participants of the study had completed an introduction to proof class that would include methods of indirect proof, a question arises of how well they will be able to teach proofs in *all* courses, as recommended by the NCTM *Standards* 2000 and MAA (1998).

Secondary Research Question 2c Conclusions

Question 2c evaluated participants' ability to refute the following false conjecture:

As the perimeter of a rectangle increases, then the area of it also increases.

Conclusion 8: Only 10 participants (43.5%) recognized a false conjecture; 9 of the 10 participants (39.1% of the sample) were able to refute it with a counterexample.

This conclusion was supported by data from Table 21 and Table 22 which show that 9 participants (39.1%) exhibited an Analytic Proof Scheme—Axiomatic Proof. Eight of those 9 participants completed a valid justification (score 5 points) and 1 participant completed a valid justification, but omitted at most one reasoning or justification step (score 4 points). Approximately 56.5% (13 participants) responded incorrectly stating that the conjecture was true. It seems highly doubtful that participants responded incorrectly due to the content of the item—area and perimeter of a rectangle. The reason why 56.5% of the participants believed this was true was most likely their lack of experience in determining whether a conjecture is true or false, referred to as verification bias (i.e., they did not consider that the conjecture may have been false).

Another possible explanation is that participants did not understand the generality of the mathematical statement (for all cases) and instead viewed this conjecture as being a specific case.

This item was adapted from Ball and Wilson's (1990) study where they sketched a picture of two rectangles (3 by 3 cm. and 3 by 4 cm.) and stated that a student claimed to have discovered a theory—if the perimeter of a closed figure increases, the area also increases.²¹ In this study, a similar conjecture and a sketch of two rectangles was included using the same dimensions as the item in the Ball and Wilson (1990) study. Even though the conjecture was stated—*as the perimeter of a rectangle increases, then its area also increases*—it is possible (because of the associated diagram) that participants viewed this item as the case where one side of the rectangle needed to remain a fixed length. This response is referred to as a Generic Example Proof (Chapter Two). In this proof scheme, one demonstrates a misunderstanding of the generic aspect of deductive proofs; that the validity of the conclusion is meant to be generalizable to all figures that satisfy the given conditions, not a single isolated case. These results are similar to Ball and Wilson's (1990) study in which they stated that the students' claim of discovering a theory and offering a sketch of a single case does not establish the truth of a generalization in mathematics.

The NCTM (2000) has suggested that teachers can help students develop and justify general conjectures and refute conjectures through asking questions such as “Does

²¹ In the Ball and Wilson (1990) study, 47.4% of a sample of novice teachers responded that the conjecture was true.

this always work?", "Does it work some of the time?", "Never?", and "Why does it work?" (p. 58). The Reasoning and Proof Standard stated that this extension to general cases requires a more sophisticated mathematical knowledge that students should build over their years of schooling (NCTM, 2000, p. 58). The refutation item is an example of a general case item that asked the prospective teachers to understand the generality of perimeter and area of a rectangle. Even so, the results from this refutation item raise some suspicion to the validity of this item because it is possible that the context of the item (a conjecture developed from a dynamic geometry software program) may have misled some of the participants. Some of these participants may have interpreted the refutation item as a rectangle with one side a fixed length, which in turn would mean that the conjecture was true. This is a limitation of the study because interviews and gathering more data from multiple refutation items would have been helpful in determining whether participants' errors were due to misconceptions regarding the refutation process or an inability to construct appropriate counterexamples.

Secondary Research Question 2 Conclusions

Question 2 investigated participants' ability to complete mathematical proofs.

Conclusion 9: Only 39.1% of the sample were able to construct a chain of reasoning that would lead to a valid proof for both the direct and indirect proof items; however, only 6 participants (26.1% of the sample) completed a valid proof for both items. Only 43.5% of the participants recognized a false conjecture and 39% were able to refute it with a false conjecture.

Conclusion 9 was based on the data from Tables 23 and 24. On the direct proof item, approximately 74% of the participants constructed a chain of reasoning that could

lead to a valid proof, however, only 57% of the participants completed a valid proof. Likewise, on the indirect proof item approximately 52% of the participants constructed a chain of reasoning that could lead to a valid proof, but only 39% of the sample completed a valid proof.

Data presented in Table 24 show that 15 participants (65% of the sample) scored 60% or less on the three constructed response items (Completion of Proof). Only 7 participants (30% of the sample) scored more than 70% on Completion of Proof. The results show that only one participant (#16) responded with a valid proof (scored a 4 or 5) on all three proof items. Only 2 participants were classified as competent in their ability to complete proofs and 10 participants were classified as having some ability to complete proofs. Eleven participants (47.8%) were classified as having minimal ability or lacking in their abilities to complete proofs.

Data from Table 24 show that only 6 participants (26.1%) completed a valid proof (i.e., score of 4 or 5 points) for both the direct proof and indirect proof items. It was interesting to note that of these 6 individuals that did so well on these proof items, only 1 of the 6 participants recognized that the false conjecture problem was indeed false. This data suggests that the participants lacked experiences in determining the truth or falsity of conjectures. Data also show that only 5 participants (21.7%) responded with a valid proof for the direct proof and the refutation items, and only 2 (8.7%) participants completed a valid proof for the indirect proof and refutation items.

Perhaps what is most disturbing was participants' poor performance in completing a valid proof, in light of the mathematical content level of the three proof items (i.e.,

perpendicular bisector, triangle congruency theorems, even integers, and area and perimeter of rectangles). The proof of the Perpendicular Bisector Theorem is not complicated. Prospective teachers are likely to teach proofs of this difficulty level or greater, if they teach a high school geometry class. Likewise, the proof of the even/odd integers is not a complicated indirect proof and proofs of this nature are recommended for high school students by the *Standards 2000* and the MAA (1998). In the *Standards 2000*, the NCTM recommends that "High School students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians" (2000, p. 58). This data suggests that some prospective teachers may have difficulty teaching proof and reasoning effectively, as is outlined by the NCTM *Standards 2000* and the MAA (1998).

Primary Research Question: Conclusion

The primary question of the study was the following: What are prospective secondary mathematics teachers' conceptions of proof and refutations?

Conclusion 10: In the area of proof and refutations, approximately 25% of the prospective secondary mathematics teachers' conceptions are adequate; approximately 30% of the participants' conceptions are inadequate; and approximately 45% of the participants' conceptions are varied.

This conclusion was based on the data from both parts of the questionnaire. The 23 prospective teachers' average for the multiple choice items was 61.6% correct (mean total of 7.391 correct out of 12 possible points). The samples' average for the constructed response items was 54.5% (mean score of 8.174 of 15 possible points).

Table 29 (based on Table 28 from Chapter 4), presents the overall performance of the participants according to their total score from both parts of the questionnaire—logical underpinnings of proof and completion of proof. The table presents three distinct groups within the sample, denoted as Group A, B, and C.

Table 29. Distribution of Groups

COMPLETION OF PROOF														
	Competent	15pt 100%												
		14pts 93%									#16			
		13pts 87%								Group A	# 4			
		12pt 80%												
	Some	11pts 73%				#9		#18		#7 #23		#11		
		10pts 67%				#15		Group B						
		9pt 60 %					#19 #22	#12				#21		
	Minimal	8pts 53%				#3		#2	#8					
		7pts 47%												
		6pt 40 %				#20		#17				#14		
	Lacking	5pts 33%				#1 #5								
		4pts 27%						Group C						
		3pt 20 %					#13	#10						
		2pts 13%					#6							
		1pts 7 %												
			1pt 8 %	2pts 17%	3pts. 25%	4 pts 33%.	5 pts 42%	6 pts 50%	7pts 58%	8 pts 67%	9pts 75%	10pts 83%	11pts 92%	12pts 100%
			Lacking			Minimal			Some			Competent		
			LOGICAL UNDERPINNINGS OF PROOF											

The data in Table 29 show that 6 participants (26.1%) scored 9 points or better on both the logical underpinnings of proof and completion of proof items—demonstrating their conceptions of proof and refutations are adequate (Group A). At the other end of the spectrum, Table 29 shows that 7 participants (30.4%) scored 7 points, or less, out of 12 points on the logical underpinnings of proof, and 6 points, or less, out of 15 points on ability to complete proof (Group C). In other words, approximately 30% of the participants scored less than 60 % of the items correct in both areas—demonstrating their conceptions of proof and refutations are inadequate.²² Nine participants' (39.1%) overall performance on both parts of the questionnaire fell into a middle area (Group B)—showing some understanding to minimal understanding of the logical underpinnings of proof and ability to complete proofs. Only one participant's (#14) results seemed to be an outlier. This participant scored 11 of the 12 points in logical underpinnings of proof, but then only scored 6 of the 15 points for completion of proof.

Table 29 also shows that 12 participants (52% of the sample) scored 60% or less on both parts of the questionnaire—understanding of logical underpinnings of proof and ability to complete proofs. Fourteen participants (61% of the sample) scored 70% or less on both parts of the questionnaire—understanding of logical underpinnings of proof and ability to complete proofs.

The results from the study show some areas of concern. One concern is for the 30% of the participants (Group C) that scored less than 60% of the items correct on

²² Seven or less out of 12 points is less than 58% correct; 6 or less correct out of 15 points is 40% or less correct.

logical underpinnings of proof and less than 40% correct on the completion of proofs. It raises concern for how well these individuals will be able to teach reasoning and proof, as recommended by the NCTM (2000) and MAA (1998). Another area of concern is that the results show that only 39% of the participants were able to complete a chain of reasoning for both the direct and indirect proof items; 26.1% completed a valid proof for both these items. Given the difficulty of the items on both parts of the questionnaire, these findings are somewhat disturbing.

Another interesting pattern of the data is found when comparing the participants' score from both parts of the questionnaire. The data from Table 25 in Chapter Four show that 12 of the 13 participants that completed a valid direct proof correctly answered at least 2 of the 3 items for logical underpinnings of direct proof. This suggests the better a participant is at understanding the logical underpinnings of direct proof, the more likely the person is able to construct a valid direct proof.

This pattern is not as prevalent in the indirect proof items. A possible explanation lies in the fact that the 3 items for logical underpinnings of indirect proof asked participants to understand the logical underpinnings of contradiction proof. In this light, the data in Table 26 show 3 of the 4 participants that answered all 3 items correct for logical underpinnings of indirect proof, also completed a valid indirect proof by using the contradiction method of proof (the other participant used the contrapositive method of proof). One participant completed a valid indirect proof by using the contradiction method of proof, but only answered 1 of the 3 logical underpinnings of indirect proof correctly. Five of the 9 participants that completed a valid indirect proof used the

contrapositive method of proof. The data show that 4 of these 5 participants only answered at most 1 of the 3 items correct on the logical underpinnings of indirect (contradiction) proof. This suggests that these participants have some understanding of contrapositive method of proof.

Table 27 shows that 18 participants answered at least 2 of the 3 items correctly for the logical underpinnings of refutation. Only 8 of these 18 participants recognized the false conjecture and therefore scored 3, 4, or 5 points (7 of the 8 participants were able to refute it with a counterexample). The other 10 participants scored 0 or 1 point by responding that the conjecture was true. It seems highly doubtful that participants responded incorrectly due to the content of the item—area and perimeter of a rectangle. A more probable explanation is that participants lack experience in determining whether a conjecture is true or false.

In Table 30, a positive trend can be seen in the rank order of the quality levels for both parts of the questionnaire. An oval figure in Table 30 represents a positive trend that suggests there is some correlation between an individual's understanding of the logical underpinnings of proof and ability to complete mathematical proofs. It seems that if an individual's understanding of the logical underpinnings of proof is adequate, then they can complete mathematical proofs. For example, 7 participants (#16, #4, #11, #7, #23, #21, and #14) answered at least 9 of the 12 items for logical underpinnings of proof correct. Six of these 7 individuals scored 9 points or better on ability to complete mathematical proof. After examination, the responses of participant (#14) appear to fall out of this trend. Based on participant #14's responses it suggests that the individual

rushed through the completion of the constructed response items without care of completing a valid proof or disproof.²³

Table 30. Overall Comparison of the Data

COMPLETION OF PROOF														
COMPLETION OF PROOF	Competent	15pt 100%												
		14pts 93%										#16		
		13pts 87%										# 4		
		12pt 80%												
	Some	11pts 73%				#9		#18		#7 #23		#11		
		10pts 67%				#15								
		9pt 60 %					#19 #22	#12				#21		
	Minimal	8pts 53%				#3		#2	#8					
		7pts 47%												
		6pt 40 %				#20		#17				#14		
	Lack	5pts 33%				#1 #5								
		4pts 27%												
		3pt 20 %					#13	#10						
		2pts 13%					#6							
		1pts 7 %												
			1pt 8 %	2pts 17%	3pts. 25%	4 pts 33%.	5 pts 42%	6 pts 50%	7pts 58%	8 pts 67%	9pts 75%	10pts 83%	11pts 92%	12pts 100%
			Lacking			Minimal			Some			Competent		
	LOGICAL UNDERPINNINGS OF PROOF													

²³

The researcher based this opinion on the participant's response for the direct proof item and the refutation item. For example on the direct proof item, the participant wrote "Side-Angle-Side Theorem" and drew an isosceles triangle with a perpendicular bisector drawn.

Relevance of the Study

The results of this study are particularly relevant to those interested in the mathematical preparation of secondary teachers. Many concerned about individuals' understanding of proof recommended that individuals should receive explicit instruction in the logical principles that underlie basic forms of proof and proof techniques (Barnier & Felman, 1990; Epp, in press; Moore, 1990; Williams, 1979). Few studies exist that research prospective teachers' cognitive understanding of the logical underpinnings of proof and their ability to complete proofs. Also, few studies exist that research prospective teachers' cognitive understanding of proof at the end of the teacher preparation experience, especially since the release of the NCTM (2000) *Standards*. Therefore, the data gathered in this study contributes to the research of the preparation of teachers in the area of proof and reasoning.

Some of the items were adapted from previous research studies. In Part I of the questionnaire, Items 1, 3, and 7 were adapted from Galbraith's (1982) study. The participants in Galbraith's study were undergraduate students (first-year mathematics students) and students enrolled in the Diploma of Education teacher training courses (prospective mathematics teachers). In Part II, Item 2 was adapted from the Ball and Wilson (1990) study of novice secondary mathematics teachers. Data from these studies was compared to the data in the present study.

Item 1, which was copied verbatim from Galbraith's (1982) study, pertained to an individual knowing when a conditional statement is true. Galbraith used this question as

the first question of 18 multiple choice items in his study. In Galbraith's study, 113 of 116 or 97.4% of the prospective teachers answered the item correctly (1982). In this research study, approximately 15 of 23 or 65.2% of the prospective teachers answered correctly.

Item 3 asked individuals a question about what they could determine about the truth of the converse of a theorem if they knew the theorem had been proven. This item was adapted with a slight change in wording from Galbraith's (1982) study. A summary of this item from Galbraith's 1982 study can be found on page 37 in Chapter Two. Only 4 of the prospective teachers or 17.4% answered correctly in this study compared to 43 of 116 or 37.1% from Galbraith's study.

Item 7 was adapted from Galbraith's (1982) study by writing the conditional sentence in the form "if H , then C ." A summary of this item from Galbraith's 1982 study can be found on page 38 in Chapter Two. Galbraith's question stated the conditional statement in a " C if H " form. In his study, approximately 59% of the prospective teachers (68 of 116 participants) answered the item correctly. In this study, approximately 65% of the prospective teachers (15 of 23 participants) selected the choice that made the hypothesis true and the conclusion false, therefore showing that the statement was indeed false.

Item 2 of Part II—Completion of Proof—was adapted from the Ball and Wilson (1990) study of novice secondary teachers. The description of this item can be found on page 40 of Chapter Two. The Ball and Wilson study did not describe the geometric figure as a rectangle, but instead said it was a "closed figure" even though their diagrams

were two rectangles. The responses of the participants in this study were similar to those in the Ball and Wilson (1990) study. Approximately 56.5% of the participants in this study responded that the conjecture was true whereas 47.4% responded that the conjecture was true in the Ball and Wilson study.

The results of this study are puzzling, especially in light of the results elicited by similar items in the Galbraith study (1982) of prospective teachers, and the Ball and Wilson study (1990) of novice teachers. Comparing data on these 4 items to data from previous studies show that in the present study participants fared better on only one item—Item 7. This causes concern especially since the 3 items from the Galbraith study and the open ended refutation problem from the Ball and Wilson study were adapted with slight changes that clarified the wording of the items.

Many of results of this study were similar to results from previous research studies. Less than half of the prospective teachers in Galbraith's (1982) study and high school students in the Williams (1979) study understood that a statement and its converse are not logically equivalent. In this study, one item (Item 3) assessed this understanding. Only 17% understood that the truth of a conditional statement does not imply the converse is true. Only 46% of the prospective teachers in Galbraith's (1982) study, 20% of the high school students in the Williams (1979) study, and 67% of the high school geometry students in the Burke (1984) study understood a counterexample refutes a false conjecture. Participants in this study did somewhat better with correct responses from 87%, 65%, and 74% of the participants on the three refutation multiple choice items (Items 4, 7, and 11). Even though participants in this study showed some ability to

recognize that a counterexample refutes a false conjecture in the multiple choice items, only 9 participants (39.1%) correctly identified a false conjecture and provided a counterexample to disprove the claim. These results were similar to those of Ball and Wilson's (1990) in which only 52.6% responded correctly.

As reported in Chapter Two, research has discovered that the majority of students at the high school and university level are unable to construct valid proofs. The difficulty at both levels was attributed to participants' inability to use definitions to establish the overall structure of the proof, a lack of understanding of the relevant theorem, or mathematical concepts addressed in the proof. Some participants were unable to even begin to construct a proof or began with the wrong assumption. Moore (1990) found that some of the difficulties of undergraduate students enrolled in an Introduction to Proof class could be attributed to a cognitive overload (i.e., students' struggle with understanding abstract concepts at the same time they were learning techniques of proof). Moore (1990) found that undergraduates demonstrated an inability of how to use definitions in proofs and showed difficulty with the mathematical language and notation of proofs. Participants in this study struggled with some of the same difficulties—knowing how to start a proof, how definitions are used in proof, the mathematical language, and the techniques of proof.

The MAA (1998) and the NCTM (2000) recommended that students in grades 9 – 12 should understand and be able to use proof techniques that include direct and indirect proofs, and should be able to recognize a false conjecture and find a counterexample that refutes the conjecture. The NCTM (2000) stated "By the end of secondary school,

students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments” (p. 56). High school students should be able to write mathematical proofs in a form that would be acceptable to a professional mathematician (NCTM, 2000, p. 58). The vision of the MAA (1998) and the NCTM (2000) recommendations of reasoning and proof for *all* students grades K – 12, and in *all* mathematics content areas, may not be attainable by *all* of the prospective secondary teachers in this study.

The results of this study raise a concern about the 30% of the participants who scored 60% or less on the items from both parts of the questionnaire. The question arises of how well these prospective teachers will be able to teach proof and reasoning. Even though all the participants had completed an introduction to proof course and a geometry course, the results of the study imply that some participants have not retained their knowledge of proof, or that their knowledge is insufficient.

Data from Table 27 show that 10 of the 18 participants that answered at least 2 of the 3 items correctly for the logical underpinnings of refutation only scored 0 or 1 point on the refutation constructed response item (perimeter and area of a rectangle item). In other words, these 10 participants believed that the conjecture was true. These results suggest that participants need more experiences in determining whether a conjecture is true or false.

A trend did emerge that suggests there is a correlation between understanding the logical underpinnings of proof and ability to complete mathematical proof. This trend

seems to be in line with ideas that Epp expressed regarding the role of logic in the teaching of proof in an Introduction to Proof course. Epp (in press) agreed with L.S. Vygotsky's idea of the *zone of proximal development*. Epp stated:

I believe that these words [zone of proximal development] articulate a profound truth about education, namely that at any given point in the learning process, the insight and intuitions the learner has previously developed provide a basis that defines and limits the amount that can be accomplished in the next stage of instruction (in press, p. 8).

Epp stated that before she introduces her undergraduates to the concepts of proof and disproof, she first broadens their zone of proximal development by beginning the course with basic notions of elementary logic and by working with the language of logical connectives and quantifiers and true and false statements (in press).

As Galbraith (1982) suggested, there is a concern about the *recycling effect* that seems likely, when students that are lacking in proof understanding, return to the school system as teachers. A seemingly monumental challenge of mathematics educators is to prepare *all* prospective teachers to meet the expectations of the NCTM and MAA in the area of reasoning and proof. If a goal for our prospective teachers is to ask more *why* questions in their future classrooms, then mathematics preparation programs must help our future teachers understand *why* mathematical relationships are so. To meet this challenge, mathematics preparation programs need to continually evaluate their programs and seek ways that they can help all prospective teachers learn proof and reasoning skills. The MAA (1998) and the NCTM (2000) *Standards* have suggested that proof be emphasized in all mathematics content areas at the secondary level to help secondary students learn proof. Maybe a possible solution to helping *all* the prospective teachers in

the area of proof and refutations at the collegiate level, is to establish that attention be given to proof in *all* their mathematics preparation courses.

Implications for the Preparation of Mathematics Teachers

The examination of prospective teachers' conceptions in proof and refutations is useful to the mathematics community in providing information concerning the preparation of teachers to teach reasoning and proof as outlined by the NCTM 2000 *Standards* and MAA (1998). Results from this study imply that we need to continue to build prospective teachers understanding of the logical underpinnings of proof and in their ability to complete mathematical proofs.

Data show that many of the responses were varied in the logical underpinnings of proof. Participants seem to grasp some of the logical underpinnings of direct proof and refutations but were lacking in their understanding of the formal logic of contradiction proof and laws of logic that underlie proof. Concepts related to understanding the truth values of conditional statements and also negating a conditional statement should be given attention. Also, the preparation of mathematics teachers needs to include opportunities to develop an understanding of the logic and proof techniques for indirect proofs. Mathematics educators need to help students understand the converse of a conditional statement.

Instruction in how to use definitions in building a proof needs attention as well as continued instruction in students' proof techniques for direct and indirect proofs. Instructors should consider giving students true and false conjectures and then ask them

to determine the truth or falsity of the conjecture. Students at all levels and in all content areas of mathematics should have opportunities to disprove false conjectures with counterexamples.

The Reasoning and Proof Standard suggests that students K – 12 should have opportunities to work with other students to formulate and explore conjectures and to make presentations of their mathematical reasoning (NCTM, 2000). Modeling the expectations of the NCTM and asking prospective teachers to present their mathematical arguments to the class is suggested. The Standard also recommends that high school students should be able to complete mathematical arguments that are acceptable to professional mathematicians and develop logical reasoning to determine the soundness of mathematical argument (NCTM, 2000, p. 58 and p. 346). To attain these lofty goals, those that teach prospective mathematics teachers should see proof as a strand within the preparation program. In addition to including some proofs in introductory courses, perhaps requiring all prospective teachers to take abstract algebra and courses such as advanced calculus would help prospective teachers develop more proof knowledge. But also courses that help prospective teachers learn how to teach basic geometry and number theory proofs should be included in their preparation program.

Limitations

A limitation of this study was the amount of data that was gathered from the participants of the study. The researcher gathered data from one visit at each of the campuses in the study. Part of this was because of the time that the professors could give

up from their own course schedules. Data for participants' ability to complete mathematical proofs was only gathered from one item in the three sub-categories. Several items that assessed the three sub-categories would provide more data for conclusions to be drawn.

A second limitation was the lack of items that assessed an individual's understanding of the logical underpinnings of the contrapositive method of proof. For example, items could have addressed whether individuals understood the logical equivalence between a conditional statement and its contrapositive.

Another limitation was the design of the study was quantitative. A smaller sample size and the use of both quantitative and qualitative designs would have allowed for more data to be collected and would have been helpful in understanding the prospective teachers' abilities and thought processes as they completed mathematical proofs. For example, in the present study follow-up interviews after the questionnaire would have allowed the researcher to assess if participants had trouble with the wording of the problem (e.g, Items 4, 7, and 11 where the word "show" was used implying "prove"). Interviews with the participants would have added to understanding why they took certain approaches at justifying or why they considered the false conjecture as a true conjecture.

Suggestions for Further Research

There are a number of relevant issues the study addressed in the conclusions and limitations of the study. Some of these issues are recommended for further research.

There is a need to replicate the ideas of the study by gathering information quantitatively and qualitatively. A smaller size sample would allow a researcher to gather data through interviews. Also, gathering data by collecting quizzes, tests, or multiple questionnaires would allow data to be gathered on several items of criteria.

A longitudinal study of prospective teachers' conceptions of proof and refutations in an introduction proof course compared with their conceptions as they are completing their teacher training would be useful to evaluate preparation programs for teachers. A study such as this would trace the development of conceptions of proof and help mathematics teacher programs assess their own preparation programs and establish areas of concern. Research that follows prospective teachers through their preparation program and into their teaching careers would help in examining the link between teachers' knowledge and teaching ability.

Few studies exist that examine prospective teachers at the end of their training programs. Therefore the mathematics community needs more research of prospective teachers' knowledge in each of the areas—direct proof, indirect proof, and refutations as they are completing their teacher training program.

The mathematics community needs research that investigates an individual's ability to recognize whether a conjecture is true or false and also their ability to prove the conjecture. Research such as this could explore how using technology such as the graphing calculator or a dynamic geometry software program, effects an individuals' ability to determine the truth or falsehood of conjectures and also their ability to complete a valid mathematical proof or disprove the conjecture.

Summary of Chapter Five

Chapter Five presented conclusions of the study and provided relevant implications for mathematics educators. The limitations of the study were described as well as suggestions for further research. The conclusion of the primary research question was that 25% of the prospective secondary mathematics teachers' conceptions of proof and refutations are adequate, and approximately 30% of the participants' conceptions are inadequate. This causes concern within the mathematics community particularly if our new generation of teachers try to meet the goals set by the NCTM's new Reason and Proof Standard (2000).

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APPENDICES

APPENDIX A

INFORMATION FORM

MSU--INFORMATION FORM—Please complete the following information as accurately as possible. **All information will be confidential.** Thank you for participating in this research study!

Name: _____ Gender: _____ Male or _____ Female

Classification level: _____ Freshman _____ Sophomore _____ Junior _____ Senior

Please check ($\checkmark$) which degree you seek in your preparation as a high school mathematics teacher:

☐ Major degree

☐ Minor degree

☐ Neither: My major degree is: _____

Please check ($\checkmark$) courses (or comparable transfer courses) you are **currently taking** or have **completed**:

☐ MATH 181 Calculus & Analytical Geometry I

☐ MATH 182 Calculus & Analytical Geometry II

☐ MATH 221 Matrix Theory

☐ MATH 224 Calculus of Functions of Several Variables

☐ MATH 225 Introduction to Differential Equations

☐ MATH 256 Foundations of Higher Mathematics (Introduction to Logic & Proofs)

☐ MATH 328 Discrete Mathematics

☐ MATH 329 Modern Geometry

☐ MATH 416 Modern Algebra (Abstract Algebra)

☐ MATH 428 Mathematical Modeling for Teachers

Other MATH courses you have taken: _____

U of MT--INFORMATION FORM—Please complete the following information as accurately as possible. **All information will be confidential.** Thank you for participating in this research study!

Name: _____ Male: _____ or Female: _____

Classification level: _____ Freshman _____ Sophomore _____ Junior _____ Senior

Please check ($\checkmark$) which degree you seek in your preparation as a high school mathematics teacher:

- ☐ Major degree
☐ Minor degree
☐ Neither: My major degree is: _____

Please check ($\checkmark$) courses (or comparable transfer courses) you are **currently taking** or have **completed**:

- ☐ MATH 152 Calculus I
☐ MATH 153 Calculus II
☐ MATH 221 Linear Algebra
☐ MATH 225 Discrete Mathematics
☐ MATH 301 Mathematics with Technology for Teachers
☐ MATH 305 Introduction to Abstract Mathematics
☐ MATH 326 Elementary Number Theory
☐ MATH 341 Introduction to Probability and Statistics
☐ MATH 406 History of Mathematics
☐ MATH 421 Abstract Algebra I
☐ MATH 431 Euclidean and Non-Euclidean Geometry
☐ MATH 382 Linear Optimization (optional)

Other MATH courses you have taken: _____

U of MT-Western--INFORMATION FORM—Please complete the following information as accurately as possible. All information will be confidential. Thank you for participating in this research study!

Name: _____ Male: _____ or Female: _____

Classification level: _____ Freshman _____ Sophomore _____ Junior _____ Senior

Please check ($\checkmark$) which degree you seek in your preparation as a high school mathematics teacher:

- ☐ Major degree
- ☐ Minor degree
- ☐ Neither: My major degree is: _____

Please check ($\checkmark$) courses (or comparable transfer courses) you are **currently taking** or have **completed**:

- ☐ MATH 110 Probability & Linear Mathematics
- ☐ MATH 151 Precalculus
- ☐ MATH 152 Calculus & Analytic Geometry I
- ☐ MATH 153 Calculus & Analytic Geometry II
- ☐ MATH 210 Computer Mathematics
- ☐ MATH 232 Fundamentals of Statistics
- ☐ MATH 251 Calculus & Analytic Geometry III
- ☐ MATH 331 College Geometry
- ☐ MATH 351 Methods & Materials in Math
- ☐ MATH 361 Introduction to Abstract Algebra
- ☐ MATH 363 Linear Algebra
- ☐ MATH 460 History of Mathematics

Other MATH courses you have taken: _____

APPENDIX B

QUESTIONNAIRE PART I

QUESTIONNAIRE

NAME: _____

PART I—Please complete the following items by circling the letter of your answer.
For each item 1 through 12, one and only one answer is correct.

Item 1 Which of the following statements is correct?

- (a) If $a = 3$ then $a^2 = 9$
- (b) If $a^2 = 9$ then $a = 3$
- (c) $a = 3$ if and only if $a^2 = 9$
- (d) If $a \neq 3$ then $a^2 \neq 9$
- (e) None of the above.

Item 2 Statement K : If $2 \neq 5$, then $f(2) \neq f(5)$.

Which of the following is a negation of the conditional Statement K ?

- (a) $2 = 5$ or $f(2) \neq f(5)$.
- (b) $2 \neq 5$ and $f(2) = f(5)$.
- (c) If $2 \neq 5$, then $f(2) = f(5)$.
- (d) If $f(2) \neq f(5)$, then $2 \neq 5$.
- (e) None of the above.

Item 3 A theorem in geometry has been proved. Which one of the following statements is necessarily correct?

- (a) The converse of the theorem is true and does not need further proof.
- (b) The converse of the theorem is true but needs further proof.
- (c) The converse of the theorem is false and does not need further disproof.
- (d) The converse of the theorem is false but needs disproof.
- (e) None of the above.

Item 4 Statement R reads as follows:

For all real numbers x and y , if $x < y$, then $x^2 < y^2$.

Which of the following is correct?

- (a) The numbers $x = \frac{1}{4}$ and $y = \frac{1}{2}$ show that R is false.
- (b) The numbers $x = -3$ and $y = 2$ show that R is false.
- (c) The numbers $x = 1$ and $y = 2$ show that R is true.
- (d) The numbers $x = -1$ and $y = -2$ show that R is true.
- (e) None of the above.

Item 5 Which of the following conditional statements is **false**?

- (a) If $2 > 7$, then $1 > 3$.
- (b) If $2 < 7$, then $1 < 3$.
- (c) If $a = 2$, then $1 < 5$.
- (d) If $5 > 3$, then $2 > 4$.
- (e) None of the above.

Item 6 When using a contradiction method to prove the following conditional statement is **true**, what assumption should you begin with?

Statement: For all real numbers x and y , if x and y are positive, then $\frac{x}{y} + \frac{y}{x} \geq 2$.

- (a) For all real numbers x and y , if x and y are negative, then $\frac{x}{y} + \frac{y}{x} < 2$.
- (b) There exists an x and y that are negative real numbers and $\frac{x}{y} + \frac{y}{x} \geq 2$.
- (c) For all real numbers x and y , if x and y are positive, then $\frac{x}{y} + \frac{y}{x} < 2$.
- (d) There exists an x and y that are positive real numbers and $\frac{x}{y} + \frac{y}{x} < 2$.
- (e) None of the above.

Item 7 Statement S reads as follows:

If the sum of the digits of a whole number is divisible by 6, then the whole number is divisible by 6.

Which of the following is correct?

- (a) The number 33 shows that S is false.
- (b) The number 30 shows that S is false.
- (c) The numbers 30 and 33 both show that S is false.
- (d) S is false but neither 30 or 33 is adequate to disprove it.
- (e) S is true.

Item 8 When using a contradiction method to prove the following conditional statement is **true**, what assumption should you begin with?

Statement: For all real numbers x , if x is positive, then $\frac{x}{x+1} < \frac{x+1}{x+2}$.

- (a) There exists an x that is a negative real number and $\frac{x}{x+1} < \frac{x+1}{x+2}$.
- (b) For all real numbers x , if x is positive, then $\frac{x}{x+1} \geq \frac{x+1}{x+2}$.
- (c) There exists an x that is a positive real number and $\frac{x}{x+1} \geq \frac{x+1}{x+2}$.
- (d) For all real numbers x , if x is negative, then $\frac{x}{x+1} \geq \frac{x+1}{x+2}$.
- (e) None of the above.

Item 9 Suppose each of the following three statements is **true**.

Statement 1: Cole is smart.

Statement 2: Cole or Hannah is nine years old.

Statement 3: If Hannah is nine years old, then Cole is not smart.

Which one of the following statements is necessarily **true**?

- (a) We cannot determine Cole or Hannah's age.
- (b) Neither Cole nor Hannah is nine years old.
- (c) Hannah is nine years old.
- (d) Cole is nine years old.
- (e) None of the above.

Item 10 In a certain finite geometry, there are exactly four points and six lines. Every line contains exactly two points. The points are A, B, C, and D, and the lines are {A, B}, {A, C}, {A, D}, {B, C}, {B, D}, and {C, D}.

In this finite geometry, the following two conditional statements are true:

Statement 1: If two lines have a point in common, then they intersect.

Statement 2: If two lines have no points in common, then they are parallel.

From that information, which one of the following is necessarily **true** in this finite geometry?

- (a) {A,C} and {B,D} intersect.
- (b) {A,C} and {B,D} are parallel.
- (c) {B,C} and {C,D} are parallel.
- (d) {A,D} and {B,C} intersect.
- (e) None of the above.

Item 11 *Statement K* reads as follows:

For all real numbers y , if $0 < y$, then $\sqrt{y} \leq y$.

Which of the following is correct?

- (a) The number $y = 4$ shows that K is true.
- (b) The number $y = 0$ shows that K is true.
- (c) The number $y = -4$ shows that K is false.
- (d) The number $y = \frac{1}{9}$ shows that K is false.
- (e) None of the above.

Item 12 One of your students went to a large party where there were 367 people in attendance. A prize was given to the person who could prove that at least two people at the party have birthdays that fall on the same day of the year. Your student thought of how he could possibly do this and came up with the following plan:

Start by assuming that no two peoples' birthdays fall on the same day of the year, or in other words, that everyone's birthday falls on a different day of the year. Assign numbers to each partygoer in such a way that the person with the earliest birthday of the year receives a #1, the person with the next earliest birthday receives a #2, and so on. Since each person's number must occur on different days, then the birthday #37 must occur at least one day later than the person with birthday #36, and so on. Therefore, the birthday of the person whose number is 367 must occur at least 366 days after the person whose number is 1. But a year has at most 366 days, so this is impossible. Thus, there must be at least two people with birthdays on the same day.

The argument is an application of which proof method?

- (a) If P then Q method
- (b) Proof by Contradiction
- (c) Direct Proof
- (d) Proof by Mathematical Induction
- (e) None of the above.

APPENDIX C

QUESTIONNAIRE PART II

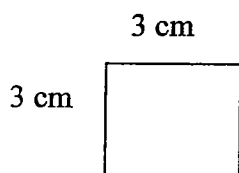
PART II Directions: Please answer the questions in the space provided; be sure to show all your work.

1. Prove the following:

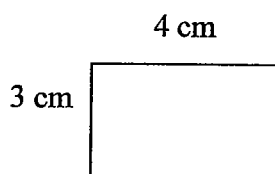
Prove: Let x be an integer. If x^3 is even, then x is even.

2. Imagine that one of your students comes to your geometry class very excited. He tells you that he has figured out a theory that you never mentioned in class. He explains that he has discovered that as the perimeter of a rectangle increases, the area also increases. He shows you this conjecture and a print out from his computer dynamic geometry software program:

Conjecture: As the perimeter of a rectangle increases, then its area also increases.



perimeter = 12 cm
area = 9 square cm



perimeter = 14 cm
area = 12 square cm

How would you reply to your mathematics student's theory?

3. Prove: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

APPENDIX D

DATA FOR INDIVIDUALS

PART I-Logical Underpinnings of Proof			PARTICIPANTS' SCORE & INCORRECT ANSWER							
Item #	Sub-Category	Corr Ans.	1	2	3	4	5	6	7	8
1	Direct	a	0 - c	1	0 - d	1	0 - c	0 - c	1	1
2	Laws of Logic	b	0 - a	0 - a	0 - c	1	0 - e	0 - d	0 - c	0 - d
3	Laws of Logic	e	0 - b	0 - b	0 - b	1	0 - d	0 - d	0 - a	0 - b
4	Refutation	b	1	1	1	1	1	1	1	1
5	Direct	d	0 - a	1	1	0 - e	1	0 - e	1	1
6	Indirect	d	0 - c	0 - b	0 - b	1	1	1	1	0 - b
7	Refutation	a	1	0 - d	0 - e	1	1	0 - b	1	1
8	Indirect	c	0 - b	0 - a	0 - a	1	0 - d	1	1	0 - a
9	Laws of Logic	c	0 - d	1	1	1	0 - e	1	1	1
10	Direct	b	1	1	1	1	1	1	1	1
11	Refutation	d	1	1	0 - a	1	0 - a	0 - a	1	1
12	Indirect	c	1	1	1	1	0 - a	1	0 - b	1
Total (12 points possible)			5	7	5	11	5	6	9	8
% for 12 items			41.7%	58.3%	41.7%	91.7%	41.7%	50%	75%	66.7%
PART II-Completion of Proofs			PARTICIPANTS' SCORE ON MATHEMATICAL PROOF							
1	Indirect Proof		1	3	1	3	0	2	2	4
2	Refutation		1	1	4	5	1	0	5	0
3	Direct Proof		3	4	3	5	4	0	4	4
Total (15 points possible)			5	8	8	13	5	2	11	8
% for #1, 2, 3			33.3%	53.3%	53.3%	86.7%	33.3%	13.3%	73.3%	53.3%

PART I-Logical Underpinnings of Proof			PARTICIPANTS' SCORE & INCORRECT ANSWER							
Item #	Sub-Category	Corr Ans.	9	10	11	12	13	14	15	16
1	Direct	a	1	1	1	1	0 - b	1	1	1
2	Laws of Logic	b	0 - e	0 - e	0 - c	0 - e	0 - a	0 - e	0 - e	0 - e
3	Laws of Logic	e	0 - d	0 - d	1	0 - b	0 - d	1	0 - d	1
4	Refutation	b	1	1	1	1	1	1	0 - c	1
5	Direct	d	0 - b	1	1	1	1	1	1	1
6	Indirect	d	0 - e	0 - a	1	0 - c	1	1	0 - b	1
7	Refutation	a	0 - c	1	1	0 - c	0 - c	1	1	1
8	Indirect	c	0 - e	0 - d	1	0 - b	1	1	0 - a	1
9	Laws of Logic	c	1	1	1	1	0 - a	1	0 - d	1
10	Direct	b	1	0 - e	1	1	1	1	1	1
11	Refutation	d	1	1	1	1	1	1	0 - a	1
12	Indirect	c	0 - d	1	1	1	0 - b	1	1	1
Total (12 points possible)			5	7	11	7	6	11	5	11
% for 12 items			41.7%	58.3%	91.7%	58.3%	50%	91.7%	41.7%	91.7%
PART II-Completion of Proofs			PARTICIPANTS' SCORE ON MATHEMATICAL PROOF							
1	Indirect Proof		5	2	5	1	3	4	5	5
2	Refutation		5	1	3	5	0	0	1	5
3	Direct Proof		1	0	3	3	0	2	4	4
Total (15 points possible)			11	3	11	9	3	6	10	14
% for #1, 2, 3			73.3%	20%	73.3%	60%	20%	40%	66.7%	93.3%

APPENDIX E

CONTENT VALIDITY

CONTENT VALIDITY

To establish the content validity of the multiple choice items from Part I of the questionnaire, the following are content areas that define the attributes for the secondary research questions 1a, 1b, 1c, 1d.

1a. Attribute: logical underpinnings of direct proof.

The logical underpinnings of direct proof includes the content area truth value of a conditional statement (the instance where the hypothesis implies the conclusion).

The truth table for the conditional "A implies B"

A	B	A $\rightarrow$ B
True	True	True
True	False	False
False	True	True
False	False	True

1b. Attribute: logical underpinnings of indirect proof.

The logical underpinnings of indirect proof includes the following content areas

1. Discerning the assumptions to start a contradiction proof--assume the hypothesis and the negation of the conclusion. To prove by the contradiction method, the correct assumption for the conditional statement "If A, then B" is:

Assume A and (not B)

2. When trying to show that "A implies B", another valuable use for the contradiction method is when statement B contains the quantifier "there is" (Solow, 1990). Instead of trying to show that there is an object with the certain property such that something happens, instead proceed from the assumption that there is no such object? Then you use this information to reach some kind of contradiction. An example is the pigeonhole principle.

1c. Attribute: logical underpinnings of refutations.

The logical underpinnings of refutation items includes the following content area:

If $P(x)$ is a propositional function with variable x , then a counterexample to $[(\forall x) P(x)]$ is an object k in the set of meanings such that $P(k)$ is false (Fletcher & Patty, 1996, p. 22).

A counterexample to the generalization "For all x , If $H(x)$, then $C(x)$ " is an example of x such that $H(x)$ is true and $C(x)$ is false (Esty, 2000, p. 244).

1d. Attribute: what constitutes a proof.

The "what constitutes a proof" items include these content areas:

1. Negation of a generalization-- $P(x)$ is a propositional function (Fletcher & Patty, 1996)

a) $\neg[(\forall x)P(x)]$ is logically equivalent to $(\exists x)\neg[P(x)]$

b) $\neg[(\exists x)P(x)]$ is logically equivalent to $(\forall x)\neg[P(x)]$

2. The negation of a conditional statement not (H implies C) is logically equivalent to " H and (not C)" (Esty, 2000).

3. Converse is not logically equivalent to a statement;

4. Disjunction (A or B) truth values:

A	B	A or B
True	True	True
True	False	True
False	True	True
False	False	False

5. Conjunction (A and B) truth values:

A	B	A and B
True	True	True
True	False	False
False	True	False
False	False	False

To establish the content validity of the open proof items from Part II of the questionnaire, the following are content areas that define the attributes for the secondary research questions 2a, 2b, 2c.

2a. Attribute: ability to complete direct proof.

The "ability to complete direct proof" item is the problem where the most likely method to prove the problem is by using the direct proof method.

2b. Attribute: ability to complete indirect proof.

The "ability to complete indirect proof" item is the problem where the most likely method to prove the problem is by using the contradiction method of proof or contrapositive method of proof.

2a. Attribute: ability to refute a false conjecture.

The "ability to refute a false conjecture" item is the false conjecture problem--to refute the false conjecture an individual will find a counterexample.

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