



Selection of parameters in differential games
by Thiagarajan Natarajan

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
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Abstract:

This research work is concerned, with optimal parameter selection in differential games. The emphasis is on deterministic two-person zero-sum differential games in which one or both the players are constrained to use feedback controls of specified form. The specified forms for the controls consist of weighted sums of state variables, the weighting factors being products of known time-varying functions and of constants to be determined in an optimal manner. For these types of constraints on feedback gains, necessary and sufficient conditions are examined for saddle points.

The general results are applied to linear-quadratic games, and for this class of games it has been shown that the optimal controller parameters for a given initial condition x_0 are also valid for initial conditions $\alpha x_0 - \infty < \alpha < \infty$. For the same class of linear-quadratic games, an additional development is given to obtain control law parameters that are independent of initial conditions, so that a saddle point with respect to the expected value of the performance index is obtained. In particular, it has been shown that, for a class of linear-quadratic games, the operation of taking the expected value of the necessary conditions corresponding to problems with known initial conditions is equivalent to finding the necessary conditions for optimality corresponding to the expected value of the cost functional. The approach and the results of this work are especially useful when only partial information about the possible initial states is available to the players and when on-line computational facilities are limited.

Necessary and sufficient conditions have also been examined for the existence of saddle points under the constraint of piecewise sub-optimal control laws, with consideration being given to the optimal choice of the gain-change points and also to partial information about the possible initial conditions.

Sufficient conditions for advantageous strategies for either player are examined for a class of linear-quadratic games in which the players are constrained to use suboptimal control laws. The concept of a bargaining matrix has been introduced for the case that players bargain to adopt constant feedback gains.

The results are applied to specific scalar and vector dynamic systems, and numerical solutions are presented. A Fortran Program for computing the optimal parameters for a specific vector case has been included. The results should broaden the class of practical problems to which differential game theory can be realistically applied. By restricting the form of feedback gains so as to reduce the "latent cost" of control implementation, it is possible in some cases to obtain control laws that are only nominally suboptimal, but are both computationally feasible and implementable in practice.

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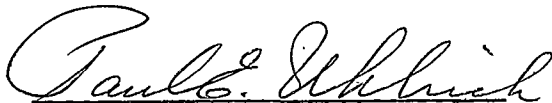
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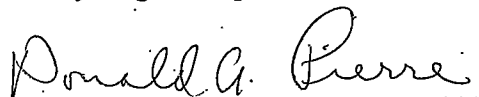
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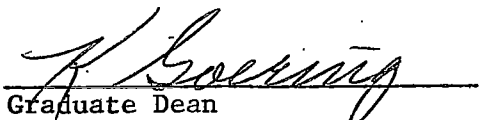
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LIST OF SYMBOLS AND CONVENTIONS

t	Time
t_0	Initial time
T	Terminal time
$\underline{x}, \underline{u}, \text{ etc.}$	All vectors are denoted by lower-case underscored letters
A	Matrices are denoted by upper-case letters
$A = [a_{ij}]$	Elements of a matrix are denoted by lower-case letters with double subscripts
$\underline{x}'$ or A'	Transpose of a vector or a matrix is denoted by the prime symbol. (')
$\underline{x} = [x_1 \ x_2 \ \dots \ x_n]'$	Elements of a vector are denoted by lower-case letters with single subscripts
$\text{tr}A$	Trace of matrix A
$i, j, k, \ell, \text{ and } s$	Indexing variables, e.g., $A\underline{x} = \sum_{k=1}^n a_{k\underline{x}} x_k$
$\dot{P}$	$\frac{dP}{dt}$
$\left. \frac{\partial L}{\partial \underline{x}} \right _{t=T}$ or $[\partial L / \partial \underline{x}]_{t=T}$	$\frac{\partial L}{\partial \underline{x}}$, evaluated at $t = T$
$A \geq 0$	Denotes that A is a nonnegative-definite symmetric matrix
$A \leq 0$	Denotes that A is a negative-semidefinite symmetric matrix
H	Scalar Hamiltonian
$H_{\underline{x}}$	$[\frac{\partial H}{\partial x_1} \ \frac{\partial H}{\partial x_2} \ \dots \ \frac{\partial H}{\partial x_n}]'$
$H_{\underline{xx}}$	Matrix with elements $\frac{\partial^2 H}{\partial x_i \partial x_j}$
$\frac{\partial}{\partial \underline{x}}(\underline{x}'A)$	A
$\frac{\partial}{\partial \underline{x}}(\underline{x}'A\underline{x})$	$A\underline{x} + A'\underline{x}$
$\partial H / \partial A$	$[\partial H / \partial a_1 \ \partial H / \partial a_2 \ \dots \ \partial H / \partial a_n]$ where the a_i 's are columns of A
H_{AA}	Matrix which consists of partitioned matrices $H_{a_i a_j}$
$\partial H / \partial a_k$	$(\partial \underline{u} / \partial a_k)' \partial H / \partial \underline{u}$, where H is a function of $\underline{u}$ and $\underline{u}$ is a function of $\underline{a}_k$

ABSTRACT

This research work is concerned with optimal parameter selection in differential games. The emphasis is on deterministic two-person zero-sum differential games in which one or both the players are constrained to use feedback controls of specified form. The specified forms for the controls consist of weighted sums of state variables, the weighting factors being products of known time-varying functions and of constants to be determined in an optimal manner. For these types of constraints on feedback gains, necessary and sufficient conditions are examined for saddle points.

The general results are applied to linear-quadratic games, and for this class of games it has been shown that the optimal controller parameters for a given initial condition x_0 are also valid for initial conditions αx_0 , $-\infty < \alpha < \infty$. For the same class of linear-quadratic games, an additional development is given to obtain control law parameters that are independent of initial conditions, so that a saddle point with respect to the expected value of the performance index is obtained. In particular, it has been shown that, for a class of linear-quadratic games, the operation of taking the expected value of the necessary conditions corresponding to problems with known initial conditions is equivalent to finding the necessary conditions for optimality corresponding to the expected value of the cost functional. The approach and the results of this work are especially useful when only partial information about the possible initial states is available to the players and when on-line computational facilities are limited.

Necessary and sufficient conditions have also been examined for the existence of saddle points under the constraint of piecewise suboptimal control laws, with consideration being given to the optimal choice of the gain-change points and also to partial information about the possible initial conditions.

Sufficient conditions for advantageous strategies for either player are examined for a class of linear-quadratic games in which the players are constrained to use suboptimal control laws. The concept of a bargaining matrix has been introduced for the case that players bargain to adopt constant feedback gains.

The results are applied to specific scalar and vector dynamic systems, and numerical solutions are presented. A Fortran Program for computing the optimal parameters for a specific vector case has been included. The results should broaden the class of practical problems to which differential game theory can be realistically applied. By restricting the form of feedback gains so as to reduce the "latent cost" of control implementation, it is possible in some cases to obtain control laws that are only nominally suboptimal, but are both computationally feasible and implementable in practice.

CHAPTER 1
INTRODUCTION

1.1 GENERAL

This research is concerned with optimal parameter selection in differential games. The emphasis is on deterministic two-person zero-sum differential games in which one player tries to minimize a certain given cost functional while the other player (his opponent) tries to maximize the same cost functional, and one or both are constrained to use feedback controls of specified form. The specified forms for the controls consist of weighted sums of state variables, the weighting factors being products of known time-varying functions and of constants to be determined in an optimal manner. The object is to determine the necessary and sufficient conditions for optimality for these types of feedback gains and to develop computational methods, for obtaining the optimal parameters.

In this chapter, a brief survey of differential games and contributions and merit of this thesis are presented.

1.2 A BRIEF SURVEY OF DIFFERENTIAL GAMES

Differential games are associated with competitive situations which evolve over time and are characterized by differential equations. Isaacs [21] established a base for study of such games.

Two-person zero-sum differential game can be described as follows. Let $\underline{x} = [x_1 \ x_2 \ \dots \ x_n]'$ denote a vector in real Euclidean n space and let R be a fixed region of $(\underline{x}, t)$ space, where t denotes time.

The evolution of $\underline{x}(t)$, known as the state of the system, is assumed to be determined by a set of first-order differential equations.

$$\frac{dx_i}{dt} = f_i(\underline{x}, \underline{u}, \underline{v}, t), \quad \underline{x}(t_0) = \underline{x}_0 \quad i = 1, 2, \dots, n \quad (1.1)$$

where $\underline{u} = [u_1 \ u_2 \ \dots \ u_p]^T$ is a vector chosen by the minimizing player and $\underline{v} = [v_1 \ v_2 \ \dots \ v_q]^T$ is a vector chosen by the maximizing player. The choices of $\underline{u}$ and $\underline{v}$ are governed by real-valued functions of $\underline{x}$ and time, defined over R , and are known as pure strategies. The vectors $\underline{u}$ and $\underline{v}$ are known as strategy vectors or control vectors.

In the game of perfect information the players know the state of the game at all times and they know how the game proceeds. Play begins at $(\underline{x}_0, t_0)$ and terminates at a previously defined surface ζ contained in the boundary of R .

A cost functional

$$J = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}, \underline{v}, t) dt \quad (1.2)$$

is specified, where T is the final time, $L(\underline{x}(T), T)$ is a real-valued function on ζ , and g is a real-valued function on $(\underline{x}, \underline{u}, \underline{v}, t)$ space.

The object of the players is to choose $\underline{u}^*(\underline{x}, t)$ and $\underline{v}^*(\underline{x}, t)$ which are optimal in the sense that for any other control vectors $\underline{u}$ and $\underline{v}$ there holds

$$J(\underline{u}^*, \underline{v}) \leq J(\underline{u}^*, \underline{v}^*) \leq J(\underline{u}, \underline{v}^*) \quad (1.3)$$

Such a pair of strategies $(\underline{u}^*, \underline{v}^*)$, if it exists, defines what is known as a saddle point. The various problems in zero-sum games are well documented by Berkovitz [5] and Simakova [42].

A rigorous and complete derivation of the necessary conditions for this two-person zero-sum game is presented by Berkovitz [3], by considering two related optimal control problems; namely, that

$\underline{u}^0(t) = \underline{u}^0(t, t_0, \underline{x}_0)$ is the minimizing solution of

$$\min_{\underline{u}} J_{\underline{u}} = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}, \underline{v}^*(\underline{x}, t)) dt \quad (1.4)$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{v}^*(\underline{x}, t), t), \quad \underline{x}(t_0) = \underline{x}_0 \quad (1.5)$$

and $\underline{v}^0(t)$ is the maximizing solution of

$$\max_{\underline{v}} J_{\underline{v}} = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}^*(\underline{x}, t), \underline{v}, t) dt \quad (1.6)$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}^*(\underline{x}, t), \underline{v}, t), \quad \underline{x}(t_0) = \underline{x}_0 \quad (1.7)$$

Berkovitz combines the two sets of necessary conditions for the above mentioned control problems, and his results are

$$H_{\underline{\lambda}} = \dot{\underline{x}} \quad (1.8)$$

$$H_{\underline{x}} = -\dot{\underline{\lambda}} \quad (1.9)$$

$$H_{\underline{u}} = 0 \quad (1.10)$$

$$H_{\underline{v}} = \underline{0} \quad (1.11)$$

and

$$L_{\underline{x}}(T) = \underline{\lambda}(T) \quad (1.12)$$

where $H_{\underline{x}}$ denotes $\partial H / \partial \underline{x}$, etc., $\underline{\lambda}$ is the Lagrange multiplier used to account for constraint (1.1), and where

$$H(\underline{x}, \underline{u}, \underline{v}, t) \triangleq g(\underline{x}, \underline{u}, \underline{v}, t) + \underline{\lambda}' f(\underline{x}, \underline{u}, \underline{v}, t) \quad (1.13)$$

In the proof Berkovitz shows that the multiplier $\underline{\lambda}_{\underline{u}}$ for the minimum problem is identical to the multiplier $\underline{\lambda}_{\underline{v}}$ for the maximum problem.

The related optimal control problems lead to two definitions of a conjugate point as in optimal control theory and two more sets of necessary conditions. Schmittendorf and Citron [40] have established the equivalence of the two definitions of a conjugate point and the equivalence of the two sets of necessary conditions through the accessory minimax problem. Their main result is that, along a trajectory resulting from a saddle point solution, there does not exist a conjugate point.

A variational approach is also adopted by Ho, Bryson, and Baron [17], and by Baron [1], in studying pursuit-evasion problems for linear-quadratic games (games with linear dynamics and quadratic performance indices). They initially seek open-loop solutions whereas Berkovitz [3] and Isaacs [21] seek closed-loop solutions.

Isaacs develops a two-player analogue of the Hamilton-Jacobi Bellman equation of one-sided optimal control theory and calls it "the main

equation", which is used as a heuristic tool to find solutions to a variety of examples. Isaacs proves a verification theorem by means of which he shows that if one obtains a sufficiently smooth solution to the main equation, it is indeed a solution to the game.

Isaacs assumes that the solution of a game J^* , u^* , v^* is regular everywhere except on a finite number of surfaces called singular surfaces. He also proposes a method for finding singular surfaces; the method is based on the qualitative behavior of optimal strategies in the neighborhood of these surfaces.

Problems on the convergence of solutions of multi-stage games to the solution of differential games and the existence of solutions of differential games are studied by Fleming [15] and Varaiya [46].

Rhodes [36], Porter [35], and Kirillowa [25] use functional analysis to study differential games. Kuo [28,29] uses geometric arguments to obtain necessary and sufficient conditions pertaining to differential games.

Schmittendorf [41] considers linear-quadratic games and the relationships governing the time intervals of existence of solutions for various problem formulations when one or both the players use open-loop control.

A gradient algorithm for solving time-invariant parameter minimax problems is presented by Heller, Cruz, and Medanic [16].

Nonzero-sum differential games in which the interests of the players are not directly opposite, and which admit no simple definition

of solution, are investigated by Case [10], Starr and Ho [43], and Rhodes [38].

Regade and Sarma [39] discuss some of the difficulties that can arise when the optimal control problem, in the presence of uncertainty in the plant, is formulated as a game between the uncertainty and the control variables. Nondeterministic differential games in which the controllers have available only noise-corrupted output measurements are considered by Behn and Ho [2], Rhodes and Luenberger [37], Rhodes [36], and Willman [49]. Ciletti [13] examines linear-quadratic games in which a single player has a time lag on the availability of the state vector.

Kervovskiy and Kunetsov [24] consider the case of the minimum principle in differential games for N players. Durato and Kastenbaum [52] apply game theory to sensitivity design of optimal systems. Extensions of some of the results of lumped parameter differential games to differential games which involve distributed systems are analyzed by Landy [30].

The above references could be embellished with additional ones listed in the bibliography of the recent first international conference [19] on the theory and applications of differential games.

1.3 FORMULATION OF THE PROBLEM

Consider a dynamic system given by (1.1) and the cost functional (1.2). $\underline{u}$ and $\underline{y}$ may be constrained by the equations

$$\underline{u} = \sum_{j=1}^{m_1} \alpha_j(t) A^j \underline{x} \quad (1.14)$$

$$\underline{v} = \sum_{j=1}^{m_2} \beta_j(t) B^j \underline{x} \quad (1.15)$$

where $\alpha_j(t)$'s and $\beta_j(t)$'s are known time functions, and A^j 's and B^j 's are constant matrices of appropriate dimensions. Let A denote the set $\{A^j\}$, and B denote the set $\{B^j\}$. The problem is to find or characterize A^0 and B^0 which are optimal in the sense that for any other constant matrices A and B there holds

$$J(A^0, B) \leq J(A^0, B^0) \leq J(A, B^0) \quad (1.16)$$

First the simplest problem of constant feedback gains is considered in Chapter 2, by assuming $m_1 = m_2 = 1$ and $\alpha_j(t) = \beta_j(t) = 1$. Then the results are extended in Chapters 4 and 6 to complex situations involving suboptimal control laws as specified by (1.14) and (1.15).

Following Baron [1] and Berkovitz [3] variational procedures are employed in Chapters 2, 4, and 6 to yield necessary conditions for optimality in terms of a two-point boundary-value problem and an integral equation. It is then shown that the sets of matrices A and B can be evaluated by following an iterative procedure. The optimal parameters found by the method suggested, will, in general, depend upon the initial conditions on $\underline{x}$. For a class of linear-quadratic games, a method is presented in Chapters 4 and 6 to find constant control parameters that are independent of initial conditions.

1.4 MERIT OF THE THESIS

It is evident from Section 1.2 that a number of papers have appeared on zero-sum and nonzero-sum differential games, since the appearance of Isaacs' book [21] on differential games. The results of all these works suffer because the implementation of time-varying control [26] is a difficult task for either player if truly optimal control is employed.

In differential game situations, one must be equipped with an automatic data-acquisition sensing system to know the present state of the system and a decision device to compute and regulate the control to be applied in accordance with the information available about the dynamics of the system. Herein lies a distinct advantage when constant feedback gains are employed. The constant feedback gains can be computed in advance, and the regulator is much simpler in construction than when optimal time-varying controls are employed. The storage facilities that are required when the control gains are to be computed at each instant are considerably higher than in the fixed gain case; hence, whenever space and costs have to be limited and are major considerations, constant feedback gains may be appropriate.

Moreover, as indicated by Kleinman and Athans [26] in the case of one-sided optimal control theory, the "on-line approach" of implementing the time-varying control is difficult. Even for linear-quadratic games, the optimal control laws depend on the solution of a matrix Riccati equation as shown in Chapter 2, and the solution is unstable in the

forward time direction. On the other hand, some engineering difficulties are inherent in the off-line approach. Multi-input, multi-output systems require the storage of a large number of time-varying signals. These signals must not only be implemented simultaneously but must be implemented in time synchronization with one another and in synchronization with real time. The circuitry demanded by high-order systems can become quite unwieldy. This difficulty can be circumvented by using suboptimal feedback laws, since only a smaller number of time-varying easily implementable functions have to be stored, or the time-varying functions for the suboptimal controls might even be chosen so as to be generated on line.

Another interesting point is seen in the scalar case considered in Chapter 3; namely, if cooperation is allowed between the players, both the players can use smaller feedback gains and still have the same saddle-point pay-off. It is also interesting to see that the suboptimal control laws can always be constant feedback laws if cooperation is allowed between the players and still the suboptimal cost and the optimal cost can exactly coincide. Starr and Ho [43] have indicated that in the case of nonzero-sum games both players could gain if cooperation were possible. In a similar fashion, cooperation from the point of view of ease in control implementation, can be advantageous to both the players. This conceptual and practical advantage of reducing the "latent" cost of control implementation is not apparent in the case of nonconstant-gain zero-sum games.

Finally, when choosing unpredictable parameters in an optimal fashion for one-sided optimal control problems, it is natural to attempt to formulate these parameters as random variables with associated probability distributions and then to minimize the expected value of a cost functional. But such a stochastic formulation might be difficult because very little might be known about the nature of the unknown parameters. In such a case, as a conservative approach, nature might be thought of as an opponent having conflicting interests with the designer and trying to maximize whatever the designer wants to minimize. This is one of the ways of replacing a stochastic problem by a deterministic one, and in such a case the approaches and results of this work might be useful to the designer.

1.5 CONTRIBUTIONS

The main contribution of this thesis is the derivation of the necessary and sufficient conditions for the existence of a saddle point for a general two-person zero-sum differential game (in addition to linear-quadratic games) when one or both the players use control laws as specified by equations (1.14) and (1.15). The approach adopted in this work is different from that adopted by Kleinman and Athans [26] for the one-sided optimal control theory and is more general and complete. The extension of the approach to nonzero-sum games is a matter for future investigation.

Partial information about the possible initial conditions have been considered through the "information matrix" in Chapters 4 and 6. The results of this work clearly indicate that by restricting the form of feedback gains, it is possible in some cases to obtain control laws that are only nominally suboptimal, but are both computationally feasible and implementable in practice.

Another contribution is the derivation of the sufficient conditions for advantageous strategies for either player for the case of linear-quadratic games. It is a natural extension of the results [26] available in one-sided optimal control theory. The concept of "equivalence point" introduced in Chapter 3 and the "bargaining matrix" in Chapter 5 can be used by players in deciding a mutually agreeable "negotiated" pay-off which is a function of the storage facilities of each player and technical knowhow, etc.

Another major contribution is the derivation of the necessary and sufficient conditions for the existence of a saddle point under the constraint of piecewise suboptimal control laws, with consideration being given to the optimal choice of the gain-change points in time and also to partial information about the possible initial conditions. The results are more complete than the existing results, even for the case of one-sided control theory [26], [27], [6].

1.6 ORGANIZATION OF THIS WORK

The organization of the remainder of the work is as follows. In Chapter 2, necessary and sufficient conditions are examined for the existence of saddle points for a general two-person zero-sum differential game when one or both players are constrained to use constant or piecewise-constant feedback gains. The general results are applied to linear-quadratic games and a solution for the constant feedback gains is obtained.

In Chapter 3, the results of Chapter 2 are applied to a specific scalar example and numerical solutions are presented. The equivalence point is introduced and it is shown that cooperation from the point of view of ease in control implementation can be advantageous to both the players.

In Chapter 4, necessary and sufficient conditions are derived for the existence of a saddle point for a general two-person zero-sum differential game when one or both the players use suboptimal control laws as specified by equations (1.14) and (1.15). For a class of linear-quadratic games an additional development is given to obtain control law parameters that are independent of initial conditions so that a saddle point with respect to the expected value of the performance index is obtained. The results are applied to specific scalar and vector dynamic systems and numerical solutions are presented.

In Chapter 5, sufficient conditions for advantageous strategies for either player are examined for linear-quadratic games. The concept of a "bargaining matrix" is introduced. Specific scalar and vector examples are included to illustrate the theory.

In Chapter 6, the procedure of Chapters 2 and 4 is extended to the case when the players use piecewise suboptimal control laws. Consideration is given to the case when partial information about the possible initial conditions is available to the players. The results are applied to specific scalar and vector dynamic systems.

In Chapter 7, the results of Chapters 2 through 6 are summarized and some open problems are discussed.

CHAPTER 2

OPTIMAL PARAMETER SELECTION

2.1 INTRODUCTION

In this chapter necessary and sufficient conditions for the existence of a saddle point for a general two-person, zero-sum differential game under the constraint of constant feedback gains is examined. The general analytical results are applied to a linear-quadratic game, and a solution for the constant feedback gains is obtained. A straight-forward iterative scheme for the evaluation of these constant feedback gains is suggested. This procedure is extended to the case where the players are restricted to use constant feedback gains during only specified intervals of time. The necessary conditions for the general two-person zero-sum differential game without constant-gain constraints is then derived from the necessary conditions for the constant-gain case.

The constant feedback gains found by the method suggested in this chapter depend in general upon the initial conditions of the system. The method of Kleinman and Athans [26] is extended in a subsequent chapter to compute feedback gains that are independent of initial conditions. The results of this chapter are applied to a particular scalar dynamic system in Chapter 3 in which detailed numerical solutions are given.

2.2 FORMULATION OF THE PROBLEM: CONSTANT GAIN CASE

Given is a dynamic system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{v}, t), \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.1)$$

where $\underline{x}$ is an $(n \times 1)$ state vector, $\underline{u}$ and $\underline{v}$ are the control vectors of dimensions p and q , respectively, and t_0 is the initial time.

A cost functional

$$J(\underline{u}, \underline{v}) = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}, \underline{v}, t) \, dt \quad (2.2)$$

is given with the final time T specified. The real-valued functions $\underline{f}$, L , and g are all assumed to be of class C^2 with respect to their arguments. The problem is to find or characterize, if they exist, particular controls $\underline{u}^0$ and $\underline{v}^0$ which are optimal in the sense that for any other control vectors $\underline{u}$ and $\underline{v}$, there holds

$$J(\underline{u}^0, \underline{v}) \leq J(\underline{u}^0, \underline{v}^0) \leq J(\underline{u}, \underline{v}^0) \quad (2.3)$$

subject to the conditions

$$\underline{u} = \underline{A}\underline{x} \quad (2.4)$$

and

$$\underline{v} = \underline{B}\underline{x} \quad (2.5)$$

where A and B are constant matrices of appropriate dimensions.

In the following, standard variational procedures are applied to yield necessary conditions for optimality in terms of a two-point boundary-value problem and an integral equation. It is then shown that the matrices A and B can be evaluated by following an iterative procedure.

The basic differential game problem without constraints (2.4) and (2.5) has been studied by various authors (Berkovitz [3], Baron [1], and Rhodes [36]) and the results for the control vectors $\underline{u}$ and $\underline{v}$ to be optimal in the minimax sense are presented in Section 2.6.

Following Baron and Rhodes open-loop solutions are obtained initially. Because conditions (2.4) and (2.5) are to be satisfied, the problem is more conveniently expressed as follows:

Given a dynamic system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t), \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.6)$$

and a cost functional

$$J(A, B) = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t) dt \quad (2.7)$$

find or characterize, if they exist, A^0 and B^0 which are optimal in the sense that for any other constant matrices A and B ,

$$J(A^0, B) \leq J(A^0, B^0) \leq J(A, B^0) \quad (2.8)$$

subject to the conditions

$$\dot{A} = 0 \quad (2.9a)$$

and

$$\dot{B} = 0 \quad (2.9b)$$

2.3 NECESSARY CONDITIONS

Vector Lagrange multiplier functions can be introduced to account for the constraints (2.6), (2.9a), and (2.9b). Because A is a $p \times n$ matrix and B is a $q \times n$ matrix, in general, rather than vectors, they can be partitioned in terms of columns: Let

$$A = [\underline{a}_1 \quad \underline{a}_2 \quad \dots \quad \underline{a}_n] \quad (2.10a)$$

and

$$B = [\underline{b}_1 \quad \underline{b}_2 \quad \dots \quad \underline{b}_n] \quad (2.10b)$$

where $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ and $\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n$ are constant vectors; hence, conditions (2.9a) and (2.9b) are rewritten as

$$\dot{\underline{a}}_k = \underline{0}, \quad k = 1, 2, \dots, n$$

$$\dot{\underline{b}}_k = \underline{0}, \quad k = 1, 2, \dots, n$$

Following Baron [1] and Berkovitz [3], two related control problems are considered: Find A to minimize $J(A, B^0)$ and find B to maximize $J(A^0, B)$. The second inequality of (2.8) suggests that A^0 is the minimizing solution for

$$\min_{A^0} J_1 = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{A}\underline{x}, B^0 \underline{x}, t) dt \quad (2.11a)$$

$$\dot{\underline{x}} = f(\underline{x}, A\underline{x}, B^0 \underline{x}, t) \quad , \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.11b)$$

$$\dot{A} = [0] \quad (2.11c)$$

The theory of one-sided optimal control theory is applicable to the above problems. The augmented performance index is

$$J_{1a} = L(\underline{x}(T), T) + \int_{t_0}^T [g(\underline{x}, A\underline{x}, B^0 \underline{x}, t) + \lambda_1' (f(\underline{x}, A\underline{x}, B^0 \underline{x}, t) - \dot{\underline{x}}) - \sum_{k=1}^n \eta_k' \dot{a}_k] dt \quad (2.12)$$

where $\lambda_1(t)$ and $\eta_k(t)$'s are Lagrange multipliers of appropriate dimensions. Integrating (2.12) by parts and considering only first-order variations

$$\delta J_{1a} = \left(\frac{\partial L}{\partial \underline{x}} \right) \Big|_{t=T} - \lambda_1(T)' \delta \underline{x}(T) + \lambda_1'(t_0) \delta \underline{x}(t_0) - \sum_{k=1}^n (\eta_k(T) - \eta_k(t_0))' \delta a_k + \int_{t_0}^T \left(\frac{\partial H}{\partial \underline{x}} + \lambda_1' \right) \delta \underline{x} dt + \int_{t_0}^T \sum_{k=1}^n \left(\frac{\partial H}{\partial a_k} + \eta_k' \right) \delta a_k \quad (2.13)$$

where

$$H_1 \triangleq g(\underline{x}, A\underline{x}, B^0 \underline{x}, t) + \lambda_1' f(\underline{x}, A\underline{x}, B^0 \underline{x}, t) \quad (2.14)$$

Necessary conditions for a minimum are found by equating first-order variations to zero. This is done by requiring

$$\eta_k(T) = \eta_k(t_0), \quad k = 1, 2, \dots, n \quad (2.15a)$$

$$\left(\frac{\partial L}{\partial \underline{x}} \right) \Big|_{t=T} = \lambda_1(T) \quad (2.15b)$$

$$\frac{\partial H_1}{\partial \underline{x}} = -\dot{\underline{\lambda}}_1 \quad (2.15c)$$

and

$$\frac{\partial H_1}{\partial \underline{a}_k} = -\dot{\underline{\eta}}_k, \quad k = 1, 2, \dots, n \quad (2.15d)$$

Conditions (2.15a) and (2.15d) give

$$\int_{t_0}^T \frac{\partial H_1}{\partial \underline{a}_k} dt = 0 \quad k = 1, 2, \dots, n \quad (2.15e)$$

Now the first inequality of (2.8) suggests that B^0 is the maximizing solution for

$$\max_{B^0} J_2 = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, A^0 \underline{x}, B \underline{x}, t) dt \quad (2.16a)$$

$$\dot{\underline{x}} = f(\underline{x}, A^0 \underline{x}, B \underline{x}, t), \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.16b)$$

$$B = [0] \quad (2.16c)$$

Paralleling the development adopted for the minimization problem, the necessary conditions for the related maximization problem can be shown to be

$$\frac{\partial H_2}{\partial \underline{x}} = -\dot{\underline{\lambda}}_2 \quad (2.17a)$$

$$\left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} = \underline{\lambda}_2(T) \quad (2.17b)$$

and

$$\int_{t_0}^T \frac{\partial H_2}{\partial \underline{b}_{-k}} dt = \underline{0} \quad k = 1, 2, \dots, n \quad (2.17c)$$

where

$$H_2 \triangleq g(\underline{x}, A^0 \underline{x}, B \underline{x}, t) + \underline{\lambda}'_2 f(\underline{x}, A^0 \underline{x}, B \underline{x}, t) \quad (2.17d)$$

From equations (2.15b), (2.17b), (2.15c), and (2.17a) it is evident that $\underline{\lambda}_2(t) = \underline{\lambda}_1(t) \triangleq \underline{\lambda}(t)$. Necessary conditions for optimality are, therefore,

$$\frac{\partial H}{\partial \underline{x}} = -\underline{\lambda} \quad (2.18a)$$

$$\left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} = \underline{\lambda}(T) \quad (2.18b)$$

$$\int_{t_0}^T \frac{\partial H}{\partial \underline{a}_{-k}} dt = \underline{0} \quad (2.18c)$$

and

$$\int_{t_0}^T \frac{\partial H}{\partial \underline{b}_{-k}} dt = \underline{0} \quad (2.18d)$$

ACCESSORY MINIMUM PROBLEM:

As in one-sided control theory, the accessory minimum problem for the related minimum problem is formulated by considering the second-order variations of J_{1a} in (2.12). The accessory minimum problem is

$$\min_{\delta \underline{a}'_k S} \bar{J}_1 = \frac{1}{2} \delta \underline{x}'(T) L_{\underline{x}\underline{x}}(T) \delta \underline{x}(T) + \frac{1}{2} \int_{t_0}^T [\delta \underline{x}' \delta \underline{a}'_1 \quad \delta \underline{a}'_2 \quad \dots \quad \delta \underline{a}'_n] \hat{H}_1 [\delta \underline{x}'_1 \quad \delta \underline{a}'_1 \quad \delta \underline{a}'_2 \quad \dots \quad \delta \underline{a}'_n]' dt \quad (2.19a)$$

$$\delta \dot{\underline{x}} = \underline{f}_{\underline{x}} \delta \underline{x} + \sum_{k=1}^n \underline{f}_{\underline{a}_k} \delta \underline{a}_k, \quad \delta \underline{x}(t_0) = \underline{0} \quad (2.19b)$$

$$\delta \dot{\underline{a}}_k = \underline{0} \quad (2.19c)$$

where

$$\hat{H}_1 = \begin{bmatrix} H_{\underline{x}\underline{x}} & H_{\underline{x}\underline{a}_1} & \dots & H_{\underline{x}\underline{a}_n} \\ H_{\underline{a}_1\underline{x}} & H_{\underline{a}_1\underline{a}_1} & \dots & H_{\underline{a}_1\underline{a}_n} \\ & & \ddots & \\ H_{\underline{a}_n\underline{x}} & H_{\underline{a}_n\underline{a}_1} & \dots & H_{\underline{a}_n\underline{a}_n} \end{bmatrix}$$

and where all the quantities $H_{\underline{x}\underline{x}}$, $H_{\underline{x}\underline{a}_k}$'s, $H_{\underline{a}_i\underline{a}_j}$'s, $\underline{f}_{\underline{x}}$, $\underline{f}_{\underline{a}_k}$'s, and $L_{\underline{x}\underline{x}}(T)$ are evaluated along the optimal path $\underline{x}^0(t)$ with $A = A^0$.

Equation (2.19a) can be rewritten as

$$\min_{\delta \underline{a}_k'} \bar{J}_1 = \frac{1}{2} \delta \underline{x}'(T) L_{\underline{x}\underline{x}}(T) \delta \underline{x}(T) + \frac{1}{2} \int_{t_0}^T \{ \delta \underline{x}' H_{\underline{x}\underline{x}} \delta \underline{x} + 2 \delta \underline{x}' \sum_{k=1}^n H_{\underline{x}\underline{a}_k} \delta \underline{a}_k + [\delta \underline{a}_1' \quad \delta \underline{a}_2' \quad \dots \quad \delta \underline{a}_n'] H_{AA} [\delta \underline{a}_1' \quad \delta \underline{a}_2' \quad \dots \quad \delta \underline{a}_n'] \} dt \quad (2.19e)$$

where H_{AA} is an $(np \times np)$ matrix which consists of partitioned matrices

$$H_{\underline{a}_i\underline{a}_j}$$

Necessary conditions (2.15a) - (2.15e) can be applied to the above problem. The Hamiltonian $\bar{H}_1$ for the accessory minimum problem is

$$\bar{H}_1 = \frac{1}{2} \{ \delta \underline{x}' H_{\underline{x}\underline{x}} \delta \underline{x} + 2 \delta \underline{x}' \sum_{k=1}^n H_{\underline{x}\underline{a}_k} \delta \underline{a}_k +$$

$$[\delta \underline{a}'_1 \quad \delta \underline{a}'_2 \quad \dots \quad \delta \underline{a}'_n] H_{AA} [\delta \underline{a}'_1 \quad \delta \underline{a}'_2 \quad \dots \quad \delta \underline{a}'_n]' + \underline{\alpha}'_1 (\underline{f}_{\underline{x}} \delta \underline{x} + \sum_{k=1}^n \underline{f}_{\underline{a}_k} \delta \underline{a}_k) \quad (2.20)$$

where $\underline{\alpha}_1(t)$ is a Lagrange multiplier of dimension $n \times 1$. Necessary condition (2.15e) implies that the constant parameters $\delta \underline{a}_k$'s are to be determined by minimizing the integral of Hamiltonian $\bar{H}_1$ with respect to the $\delta \underline{a}_k$'s, holding $\underline{x}$, $\underline{\alpha}_1$, and t fixed.

A necessary condition for a minimum of $\int_{t_0}^T \bar{H}_1 dt$ is

$$\frac{\partial^2}{\partial A^2} \int_{t_0}^T \bar{H}_1 dt \geq 0 \quad (2.21)$$

by which it is meant that the $(np \times np)$ matrix which consists of partitioned matrices $\frac{\partial^2}{\partial \underline{a}_i \partial \underline{a}_j} \int_{t_0}^T \bar{H}_1 dt$ must be positive semidefinite. It is evident from (2.20) and (2.21) that a necessary condition for the accessory minimum problem is

$$\int_{t_0}^T H_{AA} dt \geq 0 \quad (2.22)$$

Note that condition (2.22) is analogous to the Legendre-Clebsch condition in the nonconstant gain case.

Necessary condition (2.15e), when applied to the accessory minimum problem, reduces to

$$\int_{t_0}^T \{ H_{\underline{a}_k \underline{x}} \delta \underline{x} + \sum_{j=1}^n H_{\underline{a}_k \underline{a}_j} \delta \underline{a}_j + \underline{f}'_{\underline{a}_k} \underline{\alpha}_1 \} dt = 0 \quad k = 1, 2, \dots, n \quad (2.23a)$$

Condition (2.23a) can be rewritten as

$$\int_{t_0}^T H_{AA} dt \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \vdots \\ \delta a_n \end{bmatrix} + \int_{t_0}^T \begin{bmatrix} H_{a_1 x} \\ H_{a_2 x} \\ \vdots \\ H_{a_n x} \end{bmatrix} \delta x dt + \int_{t_0}^T \begin{bmatrix} f'_{a_1} \\ f'_{a_2} \\ \vdots \\ f'_{a_n} \end{bmatrix} \alpha_1 dt = 0 \quad (2.23b)$$

or

$$\begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \vdots \\ \delta a_n \end{bmatrix} = - \left[\int_{t_0}^T H_{AA} dt \right]^{-1} \left[\int_{t_0}^T \begin{bmatrix} H_{a_1 x} \\ H_{a_2 x} \\ \vdots \\ H_{a_n x} \end{bmatrix} \delta x dt + \int_{t_0}^T \begin{bmatrix} f'_{a_1} \\ f'_{a_2} \\ \vdots \\ f'_{a_n} \end{bmatrix} \alpha_1 dt \right] \quad (2.23c)$$

assuming that $\int_{t_0}^T H_{AA} dt$ is nonsingular. With the above assumption, and if condition (2.22) is satisfied,

$$\int_{t_0}^T H_{AA} dt > 0 \quad (2.24)$$

Note that condition (2.24) is analogous to the strengthened Legendre-Clebsch condition in the nonconstant gain case.

Necessary conditions (2.15b) and (2.15c) when applied to the accessory minimization problem reduce to

$$H_{\underline{x} \underline{x}} \delta \underline{x} + \sum_{k=1}^n H_{\underline{x} a_k} \delta a_k + \frac{f'_{\underline{x}}}{\underline{x}} \alpha_1 = -\dot{\alpha}_1 \quad (2.25a)$$

with the boundary condition

$$\underline{\alpha}_1(T) = L_{\underline{x} \underline{x}}(T) d\underline{x}(T) \quad (2.25b)$$

Assuming $\underline{\alpha}_1(t) = 2(Pd\underline{x} + \underline{\xi}_1)$, the necessary conditions equivalent to (2.25a) and (2.25b) are

$$\dot{P} + P \underline{f}_{\underline{x}} + \underline{f}'_x P + \frac{1}{2} H_{\underline{x} \underline{x}} = [0] \quad (2.25c)$$

with the boundary condition $P(T) = \frac{1}{2} L_{\underline{x} \underline{x}}(T)$. Also,

$$\dot{\underline{\xi}}_1 + \underline{f}'_x \underline{\xi}_1 + \frac{1}{2} \sum_{k=1}^n H_{\underline{x} \underline{a}_k} \delta \underline{a}_k + P \sum_{k=1}^n \underline{f}_{\underline{a}_k} \delta \underline{a}_k = 0 \quad (2.25d)$$

with the boundary condition $\underline{\xi}_1(T) = 0$.

The necessary conditions for the accessory minimum problem are satisfied by the trivial solution $\delta \underline{x}(t) \equiv 0$, $\delta \underline{a}_k = 0$, $k = 1, 2, \dots, n$, and $\underline{\alpha}_1(t) = 0$. The value of $\bar{J}_1$ obtained by using this trivial solution is

$$\bar{J}_1 = 0 \quad (2.26)$$

$$\delta \underline{a}_k's = 0$$

The following lemma proves that it is indeed a minimizing solution of $\bar{J}_1$.

Lemma 1: What ever may be the optimal path,

$$\min_{\delta \underline{a}_k's} \bar{J}_1 = 0 \quad (2.27)$$

Consider the identically zero integral

$$0 = \int_{t_0}^T \alpha_1' (\delta \dot{x} - \frac{f}{x} \delta x - \sum_{k=1}^n \frac{f}{a_k} \delta a_k) \quad (2.28)$$

Necessary condition (2.25b) gives

$$\alpha_1'(T) \delta x(T) = \delta x'(T) L_{\underline{x} \underline{x}}(T) \delta x(T) \quad (2.29)$$

From necessary condition (2.25a)

$$-\alpha_1' \delta x = \delta x' H_{\underline{x} \underline{x}} \delta x + \sum_{k=1}^n \delta a_k' H_{\underline{a}_k \underline{x}} \delta x + \alpha_1' f \delta x \quad (2.30)$$

From necessary condition (2.23a)

$$\begin{aligned} - \int_{t_0}^T \alpha_1' \sum_{k=1}^n \frac{f}{a_k} \delta a_k dt &= - \int_{t_0}^T \sum_{k=1}^n \delta a_k' f' \alpha_1 dt = \\ &= \int_{t_0}^T \left\{ \sum_{k=1}^n \delta a_k' H_{\underline{a}_k \underline{x}} \delta x + \delta a_1' \sum_{k=1}^n H_{\underline{a}_1 \underline{a}_k} \delta a_k + \right. \\ &\quad \left. \delta a_2' \sum_{k=1}^n H_{\underline{a}_2 \underline{a}_k} \delta a_k + \dots + \delta a_n' \sum_{k=1}^n H_{\underline{a}_n \underline{a}_k} \delta a_k \right\} dt \quad (2.31) \end{aligned}$$

Substituting (2.29), (2.30), and (2.31) in (2.28), it follows that

$$\begin{aligned} 0 &= \delta x'(T) L_{\underline{x} \underline{x}}(T) \delta x(T) + \int_{t_0}^T [\delta x' H_{\underline{x} \underline{x}} \delta x + 2 \sum_{k=1}^n \delta a_k' H_{\underline{a}_k \underline{x}} \delta x] dt + \\ &\quad \int_{t_0}^T (\delta a_1' \delta a_2' \dots \delta a_n') H_{AA} (\delta a_1' \delta a_2' \dots \delta a_n')' dt = 2\bar{J}_1 \end{aligned}$$

Therefore, $\min_{\delta a_k's} \bar{J}_1 = 0.$

Hence, it is evident that $\delta \underline{x}(t) = 0$, $\delta \underline{a}_k = 0$, $k = 1, 2, \dots, n$, and $\underline{\alpha}_1(t) = 0$ form an optimal solution for the accessory minimum problem.

With $\delta \underline{a}_k = 0$, necessary condition (2.25d) gives $\underline{\xi}_1(t) = 0$. It is evident that equation (2.25c) need be satisfied along an optimal path by A^0 and B^0 .

The above arguments also indicate that

$$\bar{J}_1(\delta A, \delta B^0) \geq 0 \quad (2.32a)$$

But for this accessory minimization problem B^0 is kept constant and is not a function of $\underline{x}$; and therefore $\delta B^0 = 0$, and

$$\bar{J}_1(\delta A, 0) \geq 0 \quad (2.32b)$$

ACCESSORY MAXIMUM PROBLEM:

The accessory maximum problem for the related maximum problem is

$$\max_{\delta \underline{b}_k, s} \bar{J}_2 = \frac{1}{2} \delta \underline{x}'(T) L_{\underline{x} \underline{x}}(T) \delta \underline{x}(T) \quad (2.33a)$$

$$+ \frac{1}{2} \int_{t_0}^T [\delta \underline{x}' \quad \delta \underline{b}_1' \quad \delta \underline{b}_2' \quad \dots \quad \delta \underline{b}_n'] \hat{H}_2 [\delta \underline{x}' \quad \delta \underline{b}_1' \quad \delta \underline{b}_2' \quad \dots \quad \delta \underline{b}_n']' dt \quad (2.33b)$$

where

$$\hat{H}_2 = \begin{bmatrix} H_{\underline{x} \underline{x}} & H_{\underline{x} b_1} & H_{\underline{x} b_2} & \dots & H_{\underline{x} b_n} \\ H_{b_1 \underline{x}} & H_{b_1 b_1} & H_{b_1 b_2} & \dots & H_{b_1 b_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{b_n \underline{x}} & H_{b_n b_1} & H_{b_n b_2} & \dots & H_{b_n b_n} \end{bmatrix} \quad (2.33c)$$

$$\dot{\underline{x}} = \underline{f}_{\underline{x}} \delta \underline{x} + \sum_{k=1}^n \underline{f}_{\underline{b}_k} \delta \underline{b}_k, \quad \underline{x}(t_0) = \underline{0} \quad (2.33d)$$

$$\delta \underline{b}_k = \underline{0} \quad (2.33e)$$

By following the procedure adopted for the accessory minimization problem, it can be shown that the necessary conditions for the accessory maximization problem are

$$\int_{t_0}^T H_{BB} dt \leq 0 \quad (2.34a)$$

$$\int_{t_0}^T \{ H_{\underline{b}_k \underline{x}} \delta \underline{x} + \sum_{j=1}^n H_{\underline{b}_k \underline{b}_j} \delta \underline{b}_j + \underline{f}'_{\underline{b}_k} \underline{\alpha}_2 \} dt = 0, \quad k = 1, 2, \dots, n \quad (2.34b)$$

where $\underline{\alpha}_2(t)$ is the Lagrange multiplier used to account for the constraint (2.33d). Thus,

$$\begin{bmatrix} \delta \underline{b}_1 \\ \delta \underline{b}_2 \\ \vdots \\ \delta \underline{b}_n \end{bmatrix} = - \left[\int_{t_0}^T H_{BB} dt \right]^{-1} \left[\int_{t_0}^T \begin{bmatrix} H_{\underline{b}_1 \underline{x}} \\ H_{\underline{b}_2 \underline{x}} \\ \vdots \\ H_{\underline{b}_n \underline{x}} \end{bmatrix} \delta \underline{x} dt + \int_{t_0}^T \begin{bmatrix} \underline{f}'_{\underline{b}_1} \\ \underline{f}'_{\underline{b}_2} \\ \vdots \\ \underline{f}'_{\underline{b}_n} \end{bmatrix} \underline{\alpha}_2 dt \right] \quad (2.34c)$$

assuming $\int_{t_0}^T H_{BB} dt$ is nonsingular, which, when combined with condition (2.34a) gives

$$\int_{t_0}^T H_{BB} dt < 0 \quad (2.34d)$$

$$H_{\underline{x} \underline{x}} \delta \underline{x} + \sum_{k=1}^n H_{\underline{x} b_k} \delta b_k + \frac{f'}{\underline{x}} \alpha_2 = -\dot{\alpha}_2 \quad (2.35a)$$

or, equivalently,

$$\dot{P} + P \frac{f'}{\underline{x}} + \frac{f'}{\underline{x}} P + \frac{1}{2} H_{\underline{x} \underline{x}} = [0] \quad (2.35b)$$

with the boundary condition $P(T) = \frac{1}{2} L_{\underline{x} \underline{x}}(T)$.

$$\dot{\xi}_2 + \frac{f'}{\underline{x}} \xi_2 + \frac{1}{2} \sum_{k=1}^n H_{\underline{x} b_k} \delta b_k + P \sum_{k=1}^n \frac{f_{b_k}}{\underline{x}} \delta b_k = 0 \quad (2.35c)$$

with the boundary condition $\xi_2(T) = 0$.

$$\max_{\delta b_k} \bar{J}_2 = 0 \quad (2.36a)$$

$$\bar{J}_2(0, \delta B) \leq 0 \quad (2.36b)$$

ACCESSORY MINIMAX PROBLEM:

Consider the differential game

$$J_2(\delta A, \delta B) = \frac{1}{2} \delta \underline{x}'(T) L_{\underline{x} \underline{x}}(T) \delta \underline{x}(T) + \frac{1}{2} \int_{t_0}^T (\delta \underline{x}' \delta \underline{a}_1' \dots \delta \underline{a}_n' \delta \underline{b}_n' \dots \delta \underline{b}_1' \hat{H}(\delta \underline{x}' \delta \underline{a}_1' \dots \delta \underline{a}_n' \delta \underline{b}_1' \dots \delta \underline{b}_n')) dt \quad (2.37a)$$

where

$$\hat{H} = \begin{bmatrix} H_{\underline{x}\underline{x}} & H_{\underline{x}a_1} & \dots & H_{\underline{x}a_n} & H_{\underline{x}b_1} & \dots & H_{\underline{x}b_n} \\ H_{a_1\underline{x}} & H_{a_1a_1} & \dots & H_{a_1a_n} & H_{a_1b_1} & \dots & H_{a_1b_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{a_n\underline{x}} & H_{a_na_1} & \dots & H_{a_na_n} & H_{a_nb_1} & \dots & H_{a_nb_n} \\ H_{b_1\underline{x}} & H_{b_1a_1} & \dots & H_{b_1a_n} & H_{b_1b_1} & \dots & H_{b_1b_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{b_n\underline{x}} & H_{b_na_1} & \dots & H_{b_na_n} & H_{b_nb_1} & \dots & H_{b_nb_n} \end{bmatrix} \quad (2.37b)$$

where $H_{\underline{a}_k\underline{b}_j} = H_{\underline{b}_k\underline{a}_j} = [0]$ if the Hamiltonian H is separable in A and B .

$$\delta \dot{\underline{x}} = \underline{f}_{\underline{x}} \delta \underline{x} + \sum_{k=1}^n \underline{f}_{\underline{a}_k} \delta \underline{a}_k + \sum_{k=1}^n \underline{f}_{\underline{b}_k} \delta \underline{b}_k, \quad \delta \underline{x}(t_0) = \underline{0} \quad (2.37c)$$

$$\delta \dot{\underline{a}}_k = \delta \dot{\underline{b}}_k = \underline{0}$$

Necessary conditions analogous to (2.18a) - (2.18d) are applicable to this accessory minimax problem.

From (2.32b) and (2.37a),

$$\bar{J}_1(\delta A, 0) = J_2(\delta A, 0) \geq 0 \quad (2.38a)$$

From (2.36b) and (2.37a),

$$\bar{J}_2(0, \delta B) = J_2(0, \delta B) \leq 0 \quad (2.38b)$$

It is also evident that

$$\bar{J}_1(0, 0) = \bar{J}_2(0, 0) = J_2(0, 0) = 0 \quad (2.38c)$$

Thus, if conditions (2.24) and (2.34d) are assumed, then there exists a unique saddle point solution (0,0) to the accessory minimax problem. It is evident, therefore, that (A^0, B^0) form a saddle point for the original problem if they satisfy the necessary conditions (2.18a) - (2.18d), (2.22), and (2.34a) and if there exists a $P(t)$ satisfying the differential equation (2.25c).

A Note on the Nonsingularity of $\int_{t_0}^T H_{AA} dt$ and $\int_{t_0}^T H_{BB} dt$:

$$\text{Because } \underline{u} = A\underline{x} = \sum_{k=1}^n \underline{a}_k x_k \quad (2.39a)$$

$$H_{\underline{a}_1} = x_1 H_{\underline{u}}^c$$

$$H_{\underline{a}_1 \underline{a}_1} = x_1^2 H_{\underline{u} \underline{u}}^c$$

$$H_{\underline{a}_i \underline{a}_j} = x_i x_j H_{\underline{u} \underline{u}}^c \quad (2.39b)$$

where the superscript c denotes constant gain case. Therefore,

$$H_{AA} = [x_1 I_p \ x_2 I_p \ \dots \ x_n I_p]' H_{\underline{u} \underline{u}}^c [x_1 I_p \ x_2 I_p \ \dots \ x_n I_p] \quad (2.39c)$$

Similarly,

$$H_{BB} = [x_1 I_p \ x_2 I_p \ \dots \ x_n I_p]' H_{\underline{v} \underline{v}}^c [x_1 I_p \ x_2 I_p \ \dots \ x_n I_p] \quad (2.39d)$$

Note that $H_{\underline{u} \underline{u}}^c$ and $H_{\underline{v} \underline{v}}^c$ are computed along the optimal path corresponding to constant gains.

Necessary conditions for the existence of a saddle point in the nonconstant gain case are

$$H_{\underline{u} \underline{u}}^{nc} \geq 0 \quad (2.40a)$$

and

$$H_{\underline{v} \underline{v}}^{nc} \leq 0 \quad (2.40b)$$

where $H_{\underline{u} \underline{u}}^{nc}$ and $H_{\underline{v} \underline{v}}^{nc}$ are to be computed along the optimal path corresponding to nonconstant gains.

Whenever $H_{\underline{u} \underline{u}}^{nc}$ and $H_{\underline{v} \underline{v}}^{nc}$ do not depend upon the optimal path (e.g., in linear-quadratic games),

$$H_{\underline{u} \underline{u}}^{nc} = H_{\underline{u} \underline{u}}^c$$

and

$$H_{\underline{v} \underline{v}}^{nc} = H_{\underline{v} \underline{v}}^c$$

Hence, for this class of differential games which are assumed to be well-posed in the sense that there exists a saddle point to the nonconstant game, it follows that

$$H_{AA} \geq 0 \quad (2.41a)$$

and

$$H_{BB} \leq 0 \quad (2.41b)$$

are also to be satisfied along the optimal path corresponding to constant gains. In cases where

$$H_{\underline{u}\underline{u}}^c > 0$$

$$H_{\underline{v}\underline{v}}^c < 0$$

$\int_{t_0}^T H_{AA} dt$ and $\int_{t_0}^T H_{BB} dt$ are nonsingular if and only if the components

$x_1, x_2, \dots, x_n$ of the state vector are linearly independent. A proof of this statement is given in the following section where general results of this section are applied to linear-quadratic games.

2.4 APPLICATION TO LINEAR QUADRATIC GAMES

Consider a linear system

$$\dot{\underline{x}} = F\underline{x} + G_1\underline{u} + G_2\underline{v} \quad (2.42)$$

where: $\underline{x}$ is an $n \times 1$ state vector,

F is an $n \times n$ matrix,

G_1 is an $n \times p$ matrix,

G_2 is an $n \times q$ matrix,

$\underline{u}$ is a $p \times 1$ control vector, and

$\underline{v}$ is a $q \times 1$ control vector.

The functions F , G_1 , and G_2 are assumed to be piecewise continuous functions of time with a finite number of jump discontinuities during the interval t_0 to T .

The cost function J is a quadratic function of the state vector and controls:

$$J = \underline{x}'(T)S\underline{x}(T) + \int_{t_0}^T (\underline{x}'Q\underline{x} + \underline{u}'R_1\underline{u} + \underline{v}'R_2\underline{v}) dt \quad (2.43)$$

where: S is an $n \times n$ symmetric constant nonnegative-definite matrix,

Q is an $n \times n$ symmetric nonnegative-definite matrix,

R_1 is a $p \times p$ symmetric positive-definite matrix,

R_2 is a $q \times q$ symmetric negative-definite matrix.

The entries of Q , R_1 , and R_2 are assumed to be piecewise continuous functions of time with a finite number of jump discontinuities during the interval t_0 to T . $\underline{u}$ and $\underline{v}$ are constrained by equations (2.4) and (2.5).

The problem is to find, if they exist, A^0 and B^0 which are optimal in the sense that for any other constant gain matrices A and B

$$J(A^0, B) \leq J(A^0, B^0) \leq J(A, B^0) \quad (2.44)$$

The results of Section 2.3 are now applied to the above problem.

The state equation is

$$\dot{\underline{x}} = F\underline{x} + G_1 A \underline{x} + G_2 B \underline{x} = (F + G_1 A + G_2 B) \underline{x} \quad (2.45)$$

The performance measure is

$$J = \underline{x}'(T)S\underline{x}(T) + \int_{t_0}^T (\underline{x}'A'R_1A\underline{x} + \underline{x}'B'R_2B\underline{x} + \underline{x}'Q\underline{x}) dt \quad (2.46)$$

By comparing (2.2) and (2.46),

$$L(\underline{x}(T), T) = \underline{x}'(T)S\underline{x}(T)$$

Condition (2.18b) therefore reduces to

$$\underline{\lambda}(T) = 2S\underline{x}(T) \quad (2.47)$$

Now

$$H = \underline{x}'A'R_1A\underline{x} + \underline{x}'B'R_2B\underline{x} + \underline{x}'Q\underline{x} + \underline{\lambda}'(F + G_1A + G_2B)\underline{x} \quad (2.48)$$

Condition (2.18a) becomes

$$\frac{\partial H}{\partial \underline{x}} = 2Q\underline{x} + 2A'R_1A\underline{x} + 2B'R_2B\underline{x} + (F + G_1A + G_2B)'\underline{\lambda} = -\dot{\underline{\lambda}} \quad (2.49)$$

with end condition defined by (2.47).

Assume that $\underline{\lambda}(t)$ and $\underline{x}(t)$ are related by

$$\underline{\lambda}(t) = 2P(t)\underline{x}(t) \quad (2.50)$$

where $P = P(t)$ is an $n \times n$ matrix to be determined.

Substituting (2.50) into (2.49) and using (2.45), it follows that

$$\dot{P} + P(F + G_1 A + G_2 B) + (F + G_1 A + G_2 B)'P + A'R_1 A + B'R_2 B + Q = [0] \quad (2.51)$$

with the boundary condition $P(T) = S$.

Now considering only the terms of H that depend on A ,

$$\frac{\partial H}{\partial a_1} = \frac{\partial}{\partial a_1} (\underline{x}' A' R_1 A \underline{x} + \lambda' G_1 A \underline{x}) \quad (2.52)$$

The first term within the parenthesis of (2.52) can be expanded, as follows:

$$\begin{aligned} \underline{x}' A' R_1 A \underline{x} &= [x_1 \quad x_2 \quad \dots \quad x_n] \\ &\begin{bmatrix} a_1' R_{1-1} a_1 & a_1' R_{1-2} a_2 & \dots & a_1' R_{1-n} a_n \\ a_2' R_{1-1} a_1 & a_2' R_{1-2} a_2 & \dots & a_2' R_{1-n} a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n' R_{1-1} a_1 & a_n' R_{1-2} a_2 & \dots & a_n' R_{1-n} a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= [x_1 \quad x_2 \quad \dots \quad x_n] \begin{bmatrix} a_1' R_{1-1} a_1 x_1 + a_1' R_{1-1} \sum_{k=2}^n a_k x_k \\ a_2' R_{1-2} a_2 x_2 + a_2' R_{1-1} \sum_{\substack{k=1 \\ k \neq 2}}^n a_k x_k \\ \vdots \\ a_n' R_{1-n} a_n x_n + a_n' R_{1-1} \sum_{k=1}^{n-1} a_k x_k \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \underline{a}_1' \underline{R}_1 \underline{a}_1 x_1^2 + \underline{a}_1' \underline{R}_1 x_1 \sum_{k=2}^n \underline{a}_k x_k + \\
&\quad \underline{a}_2' \underline{R}_1 \underline{a}_2 x_2^2 + \underline{a}_2' \underline{R}_1 x_2 \sum_{\substack{k=1 \\ k \neq 2}}^n \underline{a}_k x_k + \\
&\quad \vdots \\
&\quad \underline{a}_n' \underline{R}_1 \underline{a}_n x_n^2 + \underline{a}_n' \underline{R}_1 x_n \sum_{k=1}^{n-1} \underline{a}_k x_k
\end{aligned} \tag{2.53}$$

Therefore, after a little algebra,

$$\frac{\partial}{\partial \underline{a}_1} (\underline{x}' \underline{A}' \underline{R}_1 \underline{A} \underline{x}) = 2 \underline{R}_1 \underline{a}_1 x_1^2 + 2 \underline{R}_1 x_1 \sum_{k=2}^n \underline{a}_k x_k \tag{2.54}$$

and similarly

$$\frac{\partial}{\partial \underline{a}_j} (\underline{x}' \underline{A}' \underline{R}_1 \underline{A} \underline{x}) = 2 \underline{R}_1 \underline{a}_j x_j^2 + 2 \underline{R}_1 x_j \sum_{\substack{k=1 \\ k \neq j}}^n \underline{a}_k x_k, \quad j = 1, 2, \dots, n \tag{2.55}$$

Now

$$\underline{\lambda}' \underline{G}_1 \underline{A} \underline{x} = \underline{\lambda}' \underline{G}_1 (\underline{a}_1 x_1 + \underline{a}_2 x_2 + \dots + \underline{a}_n x_n) \tag{2.56}$$

Therefore

$$\frac{\partial}{\partial \underline{a}_1} (\underline{\lambda}' \underline{G}_1 \underline{A} \underline{x}) = x_1 \underline{G}_1' \underline{\lambda} = 2 x_1 \underline{G}_1' \underline{P} \underline{x} \tag{2.57}$$

and

$$\frac{\partial}{\partial \underline{a}_j} (\underline{\lambda}' \underline{G}_1 \underline{A} \underline{x}) = 2 x_j \underline{G}_1' \underline{P} \underline{x}, \quad j = 1, 2, \dots, n. \tag{2.58}$$

Conditions (2.18c) therefore reduce to

$$\begin{aligned}
 \int_{t_0}^T (R_{11} \dot{x}_1^2 + R_{11} x_1 \sum_{k=2}^n \dot{a}_k x_k) dt &= - \int_{t_0}^T x_1 G_1' P \underline{x} dt \\
 \int_{t_0}^T (R_{12} \dot{x}_1 \dot{x}_2 + R_{12} x_1 \sum_{k=1, k \neq 2}^n \dot{a}_k x_k) dt &= - \int_{t_0}^T x_2 G_1' P \underline{x} dt \\
 \int_{t_0}^T (R_{1n} \dot{x}_1 \dot{x}_n + R_{1n} x_1 \sum_{k=1}^n \dot{a}_k x_k) dt &= - \int_{t_0}^T x_n G_1' P \underline{x} dt
 \end{aligned} \tag{2.59a}$$

The above set of equations can be rewritten as

$$\int_{t_0}^T \begin{bmatrix} R_{11} \dot{x}_1^2 & R_{11} x_1 \dot{x}_2 & \dots & R_{11} x_1 \dot{x}_n \\ R_{12} \dot{x}_1 \dot{x}_2 & R_{12} \dot{x}_2^2 & \dots & R_{12} x_2 \dot{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} \dot{x}_1 \dot{x}_n & R_{1n} x_n \dot{x}_2 & \dots & R_{1n} \dot{x}_n^2 \end{bmatrix} dt \begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \vdots \\ \dot{a}_n \end{bmatrix} = - \int_{t_0}^T \begin{bmatrix} x_1 G_1' P \underline{x} \\ x_2 G_1' P \underline{x} \\ \vdots \\ x_n G_1' P \underline{x} \end{bmatrix} dt \tag{2.59b}$$

or equivalently,

$$\int_{t_0}^T \begin{bmatrix} x_1 I_p \\ x_2 I_p \\ \vdots \\ x_n I_p \end{bmatrix} R_1 \begin{bmatrix} x_1 I_p & x_2 I_p & \dots & x_n I_p \end{bmatrix} dt \begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \vdots \\ \dot{a}_n \end{bmatrix} = - \int_{t_0}^T \begin{bmatrix} x_1 I_p \\ x_2 I_p \\ \vdots \\ x_n I_p \end{bmatrix} G_1' P \underline{x} dt \tag{2.59c}$$

where I_p is the $p \times p$ identity matrix.

Let the $np \times np$ matrix $M_1(t_0, T)$ be defined by

$$M_1(t_0, T) \triangleq \int_{t_0}^T [x_1^I \ x_2^I \ \dots \ x_n^I]^T R_1 [x_1^I \ x_2^I \ \dots \ x_n^I] dt \quad (2.60a)$$

Therefore,

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_n \end{bmatrix} = -M_1^{-1}(t_0, T) \int_{t_0}^T \begin{bmatrix} x_1^I \\ x_2^I \\ \vdots \\ x_n^I \end{bmatrix} G_1^T P \underline{x} \, dt \quad (2.60b)$$

assuming that $M_1^{-1}(t_0, T)$ exists. Similar relations can be found for B , i.e.,

$$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_n \end{bmatrix} = -M_2^{-1}(t_0, T) \int_{t_0}^T \begin{bmatrix} x_1^I \\ x_2^I \\ \vdots \\ x_n^I \end{bmatrix} G_2^T P \underline{x} \, dt \quad (2.60c)$$

assuming that $M_2^{-1}(t_0, T)$ exists, and where $M_2(t_0, T)$ is defined by

$$M_2(t_0, T) \triangleq \int_{t_0}^T [x_1^I \ x_2^I \ \dots \ x_n^I]^T R_2 [x_1^I \ x_2^I \ \dots \ x_n^I] dt \quad (2.60d)$$

It is evident from (2.48), (2.59c), (2.60a), and (2.39c) that

$$\int_{t_0}^T H_{AA} \, dt = M_1(t_0, T) \geq 0$$

since R_1 is positive definite. In fact, H_{AA} is the integral of equation (2.60a) and hence $H_{AA} \geq 0$ is satisfied for the class of linear-quadratic games under consideration.

Note that $H_{uu} = R_1 > 0$. Similarly,

$$\int_{t_0}^T H_{BB} dt = M_2(t_0, T) \leq 0$$

since R_2 is negative definite and $H_{vv} = R_2 < 0$.

Equation (2.51) can be directly derived from equation (2.25c).

Hence, the existence of a $P(t)$ satisfying (2.51) is a sufficient condition for the existence of a saddle point.

Equations (2.60b) and (2.60c) are useful for the iterative technique to be discussed in Section 2.5. Another simplified form for the necessary conditions (2.18c) and (2.18d) can be derived from (2.59a).

The set of equations (2.59a) can be rewritten as

$$\int_{t_0}^T R_1 A \underline{x} \underline{x}' dt = - \int_{t_0}^T G_1' P \underline{x} \underline{x}' dt \quad (2.61a)$$

If R_1 is time invariant, (2.61a) reduces to

$$R_1 A \int_{t_0}^T \underline{x} \underline{x}' dt = - \int_{t_0}^T G_1' P \underline{x} \underline{x}' dt$$

Let the $(n \times n)$ matrix $\int_{t_0}^T \underline{x} \underline{x}' dt$ be denoted by $W(t_0, T)$. Then

$$A = -R_1^{-1} \left[\int_{t_0}^T G_1' P \underline{x} \underline{x}' dt \right] W^{-1}(t_0, T) \quad (2.61b)$$

assuming that $W^{-1}(t_0, T)$ exists.

Similar relations can be found for B; i.e.,

$$\int_{t_0}^T R_2 B \underline{x} \underline{x}' dt = - \int_{t_0}^T G_2' P \underline{x} \underline{x}' dt \quad (2.62a)$$

and if R_2 is time invariant,

$$B = -R_2^{-1} \left[\int_{t_0}^T G_2' P \underline{x} \underline{x}' dt \right] W^{-1}(t_0, T) \quad (2.62b)$$

Furthermore, if G_1 and G_2 are time-invariant also, equations (2.61b) and (2.62b) reduce to

$$A = -R_1^{-1} G_1' \xi(t_0, T) W^{-1}(t_0, T) \quad (2.63a)$$

and

$$B = -R_2^{-1} G_2' \xi(t_0, T) W^{-1}(t_0, T) \quad (2.63b)$$

where the $(n \times n)$ matrix $\xi(t_0, T)$ is defined by

$$\xi(t_0, T) \triangleq \int_{t_0}^T P \underline{x} \underline{x}' dt \quad (2.63c)$$

It can be observed that for a linear time-invariant system with constant feedback gains the $(n \times n)$ matrices $\underline{x} \underline{x}'$ and $P \underline{x} \underline{x}'$ satisfy the following linear vector matrix differential equations:

$$\frac{d}{dt}(\underline{x} \underline{x}') + (F + G_1 A + G_2 B)(\underline{x} \underline{x}') + (\underline{x} \underline{x}')(F + G_1 A + G_2 B)' = [0]$$

$$\frac{d}{dt}(P \underline{x} \underline{x}') - P \underline{x} \underline{x}'(F + G_1 A + G_2 B)' + (F + G_1 A + G_2 B)' P \underline{x} \underline{x}' +$$

$$(A'R_1 A + B'R_2 B + Q) \underline{x} \underline{x}' = [0]$$

The optimal cost of using the optimal constant feedback controls can be found by using the necessary conditions for optimality. The performance index

$$J = \underline{x}'(T)S\underline{x}(T) + \int_{t_0}^T (\underline{x}'A'R_1A\underline{x} + \underline{x}'B'R_2B\underline{x} + \underline{x}'Q\underline{x}) dt \quad (2.64a)$$

can be rewritten as

$$J = \underline{x}(t_0)P(t_0)\underline{x}(t_0) + \int_{t_0}^T [(\underline{x}'A'R_1A\underline{x} + \underline{x}'B'R_2B\underline{x} + \underline{x}'Q\underline{x}) + \frac{d}{dt}(\underline{x}'(t)P(t)\underline{x}(t))] dt \quad (2.64b)$$

where $P(t)$ satisfies equations (2.51) with the boundary condition $P(T) = S$. Using equations (2.45) and (2.51), equation (2.64b) can be shown to reduce to

$$J = \underline{x}'(t_0)P(t_0)\underline{x}(t_0) \quad (2.64c)$$

A Note on the Nonsingularity of M_1 , M_2 , and W .

The nonsingularity of W , M_1 , and M_2 depends on the independence of components of the state vector. Consider a set of vectors

$\underline{h}_1(t, t_0)$, $\underline{h}_2(t, t_0)$, ..., $\underline{h}_n(t, t_0)$ where

$$\underline{h}_j(t, t_0) = \begin{bmatrix} h_{1j}(t, t_0) \\ h_{2j}(t, t_0) \\ \vdots \\ h_{nj}(t, t_0) \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (2.65)$$

in which the elements $h_{ij}(t, t_0)$ are continuous functions of time t over the time interval $t_0 \leq t \leq T$. It is shown in [32] that the $\underline{h}_j$'s are linearly independent over the same time interval if and only if the Gram matrix $\int_{t_0}^T \underline{H}^*(t, t_0) \underline{H}(t, t_0) dt$ of the $m \times n$ matrix

$$\underline{H}(t, t_0) = [\underline{h}_1(t, t_0) \quad \underline{h}_2(t, t_0) \quad \dots \quad \underline{h}_n(t, t_0)] \quad (2.66)$$

is nonsingular ($\underline{H}^*$ denotes the conjugate transpose of $\underline{H}$). Now, the integrand of (2.60a) can be written as $\underline{H}^* \underline{H}$, as is shown below, and the columns of the corresponding $\underline{H}$ are shown to be linearly independent if the $\underline{x}_i$'s are linearly independent.

The integrand of (2.60a) is

$$[\underline{x}_1^T \underline{I}_p \quad \underline{x}_2^T \underline{I}_p \quad \dots \quad \underline{x}_n^T \underline{I}_p]^T \underline{R}_1 [\underline{x}_1^T \underline{I}_p \quad \underline{x}_2^T \underline{I}_p \quad \dots \quad \underline{x}_n^T \underline{I}_p]$$

It is well known [33], that if $\underline{R}_1$ is a positive-definite symmetric matrix, there exists a nonsingular matrix $\underline{C}$ such that

$$\underline{R}_1 = \underline{C} \underline{C}^T$$

It is evident, therefore, that the integrand of (2.60a) can be written as $\underline{H}^T \underline{H}$ where

$$\underline{H} = \underline{C}^T [\underline{x}_1^T \underline{I}_p \quad \underline{x}_2^T \underline{I}_p \quad \dots \quad \underline{x}_n^T \underline{I}_p]$$

Let $\underline{C}^T = [\underline{c}_1 \quad \underline{c}_2 \quad \dots \quad \underline{c}_p]$. Then

$$H = \begin{bmatrix} \underline{c}_1 x_1 & \underline{c}_2 x_1 & \dots & \underline{c}_p x_1 & \underline{c}_1 x_2 & \underline{c}_2 x_2 & \dots & \underline{c}_p x_2 & \dots & \underline{c}_1 x_n & \underline{c}_2 x_n & \dots \\ & & & & & & & & & & & \underline{c}_p x_n \end{bmatrix}$$

The columns of H are independent if and only if the linear combination

$$k_{11} \underline{c}_1 x_1 + k_{21} \underline{c}_2 x_1 + \dots + k_{p1} \underline{c}_p x_1 + \dots + k_{1n} \underline{c}_1 x_n + k_{2n} \underline{c}_2 x_n + \dots$$

$$k_{pn} \underline{c}_p x_n = \underline{0} \quad (2.67a)$$

is satisfied only when the constants

$$k_{ij} = 0, \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, n$$

Equation (2.67a) can be rewritten as

$$\begin{aligned} & [\underline{c}_1 \ \underline{c}_2 \ \dots \ \underline{c}_p] [k_{11} \ k_{21} \ \dots \ k_{p1}]' x_1 + [\underline{c}_1 \ \underline{c}_2 \ \dots \ \underline{c}_p] \\ & [k_{12} \ k_{22} \ \dots \ k_{p2}]' x_2 + \dots + [\underline{c}_1 \ \underline{c}_2 \ \dots \ \underline{c}_p] \\ & [k_{1n} \ k_{2n} \ \dots \ k_{pn}]' x_n = \underline{0} \end{aligned}$$

or

$$C' [\underline{k}_1 x_1 + \underline{k}_2 x_2 + \dots + \underline{k}_n x_n] = \underline{0} \quad (2.67b)$$

where

$$\underline{k}_i = [k_{1i} \ k_{2i} \ \dots \ k_{ni}]'$$

But, because $x_1, x_2, \dots, x_n$ are assumed to be linearly independent and because C' is nonsingular, equation (2.67b) is satisfied if and only if $\underline{k}_1 = \underline{k}_2 = \dots = \underline{k}_n = \underline{0}$.

Hence, M_1 and, similarly, M_2 and W are nonsingular if and only if $x_1, x_2, \dots, x_n$ are linearly independent.

Considering equations (2.39c) and (2.39d) and assuming $H_{uu}^T > 0$ and $H_{vv}^T < 0$, it is evident from the above discussion that $\int_{t_0}^T H_{AA} dt$ and $\int_{t_0}^T H_{BB} dt$ are nonsingular if and only if the components $x_1, x_2, \dots, x_n$ of the state vector are linearly independent.

2.5 SCALAR CASE

If $x, u,$ and v are all scalar functions of time, the necessary conditions for a saddle point become:

$$\dot{x} = (F + G_1 A + G_2 B)x, \quad x(t_0) = x_0 \quad (2.68)$$

$$\dot{P} + 2P(F + G_1 A + G_2 B) + A^2 R_1 + B^2 R_2 + Q = 0, \quad P(T) = S \quad (2.69)$$

$$A = - \frac{\int_{t_0}^T G_1 P x^2 dt}{\int_{t_0}^T R_1 x^2 dt} \quad (2.70)$$

and

$$B = - \frac{\int_{t_0}^T G_2 P x^2 dt}{\int_{t_0}^T R_2 x^2 dt} \quad (2.71)$$

The above equations suggest a procedure (even for the general case) for evaluating the constants A and B. First, assume some initial values for A and B and solve equations (2.68) and (2.69) for $\underline{x}(t)$ and $P(t)$, respectively. Next, evaluate equations (2.70) and (2.71) to obtain new values for A and B. Finally, repeat the procedure in the hope that A and B will eventually converge to a saddle point.

2.6 LINEAR DIFFERENTIAL GAME WITHOUT CONSTANT-GAIN CONSTRAINTS

The linear differential game without the constraints (2.4) and (2.5) has been considered by Baron [1] and Rhodes [36]; the results are

$$\underline{u}^0(t) = -R_1^{-1}(t)G_1'(t)P(t)\underline{x}(t) \quad (2.72)$$

$$\underline{v}^0(t) = -R_2^{-1}(t)G_2'(t)P(t)\underline{x}(t) \quad (2.73)$$

where $P(t)$ satisfies the matrix Ricatti equation

$$\dot{P} + F'P + PF - P(G_1R_1^{-1}G_1' + G_2R_2^{-1}G_2')P + Q = [0] \quad (2.74)$$

with the boundary condition $P(T) = S$. The cost from time t_0 to terminal time T of using these feedback controls is

$$J[\underline{u}^0, \underline{v}^0] = \underline{x}'(t_0)P(t_0)\underline{x}(t_0) \quad (2.75)$$

2.7 DIFFERENTIAL GAME WITH MIXED CONTROLS

2.7.1 FORMULATION OF THE PROBLEM

Consider the problem of finding $\min \max J$ when the players can use mixed controls; namely, each player $\underline{u}$ $\underline{v}$ is restricted to use constant feedback gains only during a certain specified interval of time, and the intervals of restriction may be different for each player. The control diagrams in Figures 2.1 indicate four different cases of mixed controls. In Figure 2.1(a), for example, the minimizing player is restricted to use constant feedback gains during the interval t_1 to T whereas the maximizing player is restricted during t_2 to T . It is evident that the minimizing player will choose different constant feedback gains for the intervals t_1 to t_2 and t_2 to T , and similarly the maximizing player will choose different controls for the intervals t_0 to t_1 and t_1 to t_2 .

Figure 2.1(a) is considered first for the derivation of the necessary conditions for a saddle point. As in Section 2.2, consider a dynamic system characterized by

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{v}, \underline{w}, A, C, B, t) , \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.76)$$

and a cost functional

$$J = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}, \underline{v}, \underline{w}, A, C, B, t) dt \quad (2.77)$$

with the initial time t_0 and final time T specified and where $\underline{f}$, L , and g have the properties given in Section 2.2.

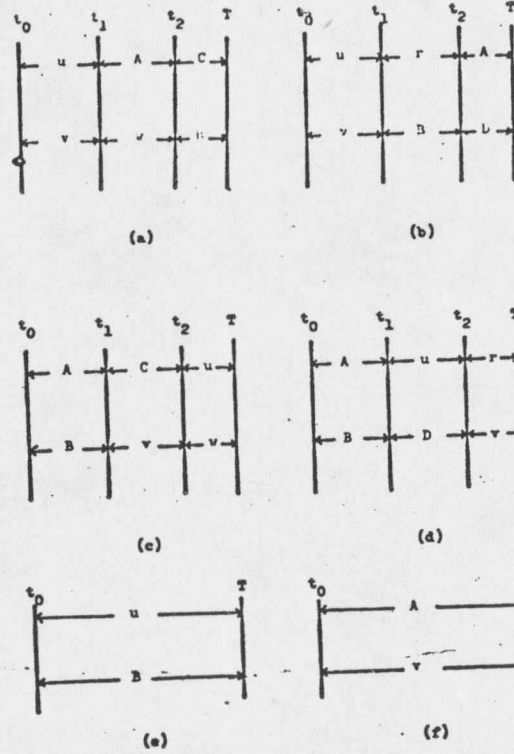


Figure 2.1. Control diagrams for mixed controls.

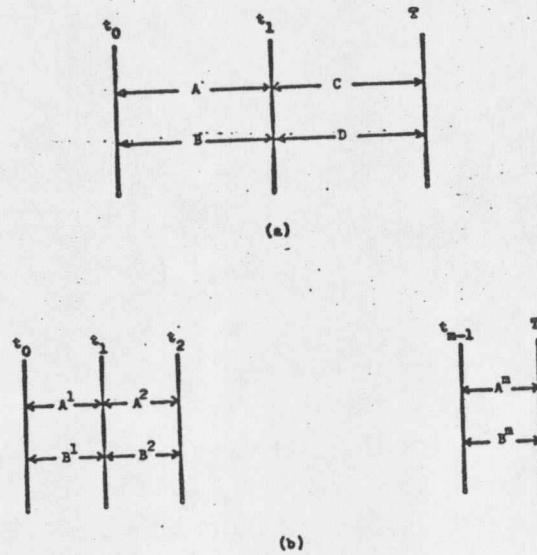


Figure 2.2. Control diagram for piecewise constant gains.

The problem is to find or characterize, if they exist, $\underline{u}^0$, $\underline{v}^0$, $\underline{w}^0$, A^0 , C^0 , and B^0 which are optimal in the minimax sense.

The motivation for this problem stems from the fact that ordinarily the storage facilities for each player are limited and that the players actually want to switch their controls to constant feedback gains whenever possible, and each player may want to show, initially at least, to the other player that he is capable of implementing the nonconstant-gain controls. In this way the storage facilities for each player can be taken into account. As an illustration consider that the controls are applied as shown in Figure 2.1(a). The players may choose their intervals of using constant feedback gains according to an agreed prespecified law. For example, the players may choose

$$(T - t_1) = K_1 T$$

$$(T - t_2) = K_2 T$$

where K_1 and K_2 are proportionality factors that depend on storage capabilities and possibly other considerations such as technical knowhow. If the storage facilities of player A are extensive, he may choose K_1 to be small and vice versa. The player, say the minimizing one, with less storage facilities will obviously try to induce the other player to use constant controls, by making a side payment which may be a function of B and also of the storage facilities of the other (maximizing) player. The concept of bargaining

can also be introduced; namely, the minimizing player, for example, might tell his opponent that his side payment depends on the interval during which his opponent implements constant feedback gains. The element of cheating also enters the picture; for example, with a conservative approach, the first player might not announce his full storage capabilities and the other, honest player may in fact use constant feedback gains for a larger interval of time than is necessary. In such a case the optimal strategy is advantageous to the first player.

Again, controls as shown in Figures 2.1(b), 2.1(c), and 2.1(d) are to be considered to introduce flexibility for each player in his choice of intervals of constant or nonconstant feedback gains. Each player may try to confuse the other player by selecting his controls in a random manner depending upon his storage capabilities.

2.7.2 NECESSARY CONDITIONS

Consider the case of Figure 2.1(a). The cost functional is

$$\begin{aligned}
 J = L(\underline{x}(T), T) + \int_{t_0}^{t_1} g(\underline{x}, \underline{u}, \underline{v}, t) dt + \int_{t_1}^{t_2} g(\underline{x}, A\underline{x}, \underline{w}, t) dt \\
 + \int_{t_2}^T g(\underline{x}, C\underline{x}, B\underline{x}, t) dt \quad (2,78)
 \end{aligned}$$

The procedure of Section 2.3 is applicable. Let

$$H_1 = g(\underline{x}, \underline{u}, \underline{v}, t) + \underline{\lambda}' f(\underline{x}, \underline{u}, \underline{v}, t) \quad (2.79)$$

$$H_2 = g(\underline{x}, \underline{A}\underline{x}, \underline{w}, t) + \underline{\eta}' f(\underline{x}, \underline{A}\underline{x}, \underline{w}, t) \quad (2.80)$$

$$H_3 = g(\underline{x}, \underline{C}\underline{x}, \underline{B}\underline{x}, t) + \underline{\rho}' f(\underline{x}, \underline{C}\underline{x}, \underline{B}\underline{x}, t) \quad (2.81)$$

where $\underline{\lambda}$, $\underline{\eta}$, and $\underline{\rho}$ are the vector Lagrange multipliers used to account for constraint (2.76) in the intervals t_0 to t_1 , t_1 to t_2 , and t_2 to T , respectively.

Resulting necessary conditions are

$$[\partial L / \partial \underline{x}]_{t=T} = \underline{\rho}(T) \quad (2.82)$$

$$\underline{\eta}(t_2) = \underline{\rho}(t_2) \quad (2.83)$$

$$\underline{\lambda}(t_1) = \underline{\eta}(t_1) \quad (2.84)$$

In the interval t_0 to t_1 ,

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}, \underline{v}, t) \quad (2.85)$$

$$\partial H_1 / \partial \underline{u} = 0 \quad (2.86)$$

$$\partial H_1 / \partial \underline{v} = 0 \quad (2.87)$$

and

$$\partial H_1 / \partial \underline{x} = -\dot{\underline{\lambda}} \quad (2.88)$$

In the interval t_1 to t_2 ,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{Ax}, \underline{w}, t) \quad (2.89)$$

$$\frac{\partial H_2}{\partial \underline{x}} = -\dot{\underline{q}} \quad (2.90)$$

$$\frac{\partial H_2}{\partial \underline{w}} = \underline{0} \quad (2.91)$$

and

$$\int_{t_1}^{t_2} \frac{\partial H_2}{\partial \underline{a}_k} dt = \underline{0} \quad , \quad k = 1, 2, \dots, n \quad (2.92)$$

In the interval t_2 to T ,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{Cx}, \underline{Bx}, t) \quad (2.93)$$

$$\frac{\partial H_3}{\partial \underline{x}} = -\dot{\underline{p}} \quad (2.94)$$

$$\int_{t_2}^T \frac{\partial H_3}{\partial \underline{c}_k} dt = \underline{0} \quad , \quad k = 1, 2, \dots, n \quad (2.95)$$

and

$$\int_{t_2}^T \frac{\partial H_3}{\partial \underline{b}_k} dt = \underline{0} \quad , \quad k = 1, 2, \dots, n \quad (2.96)$$

Similar relations can be derived for the cases corresponding to Figures 2.1(b), 2.1(c), and 2.1(d).

2.8 APPLICATION TO LINEAR QUADRATIC GAMES

Consider a linear dynamic system given by (2.42) and a cost functional given by (2.43) with the associated weighting matrices having the properties described in Section 2.4 and the control diagram of Figure 2.1 (d). Following the development of the previous section, it can be shown that necessary conditions for a saddle point are as follows:

End-point conditions are

$$\frac{\partial L}{\partial \underline{x}} \Big|_{t=T} = \underline{\rho}(T) \quad (2.97)$$

$$\underline{n}(t_2) = \underline{\rho}(t_2) \quad (2.98)$$

$$\underline{\lambda}(t_1) = \underline{n}(t_1) \quad (2.99)$$

In the interval t_0 to t_1 ,

$$\dot{\underline{x}} = F\underline{x} + G_1 A \underline{x} + G_2 B \underline{x} \quad (2.100)$$

$$\frac{\partial H_1}{\partial \underline{x}} = -\dot{\underline{\lambda}} \quad (2.101)$$

$$\int_{t_0}^{t_1} \frac{\partial H_1}{\partial \underline{a}_k} dt = \underline{0}, \quad k = 1, 2, \dots, n \quad (2.102)$$

and

$$\int_{t_0}^{t_1} \frac{\partial H_1}{\partial \underline{b}_k} dt = \underline{0}, \quad k = 1, 2, \dots, n \quad (2.103)$$

In the interval t_1 to t_2 ,

$$\dot{\underline{x}} = \underline{F}\underline{x} + \underline{G}_1\underline{u} + \underline{G}_2\underline{D}\underline{x} \quad (2.104)$$

$$\frac{\partial H_2}{\partial \underline{x}} = -\dot{\underline{n}} \quad (2.105)$$

$$\frac{\partial H_2}{\partial \underline{u}} = \underline{0} \quad (2.106)$$

and

$$\int_{t_2}^T \frac{\partial H_2}{\partial D} dt = [0] \quad (2.107)$$

In the interval t_2 to T ,

$$\dot{\underline{x}} = \underline{F}\underline{x} + \underline{G}_1\underline{r} + \underline{G}_2\underline{v} \quad (2.108)$$

$$\frac{\partial H_3}{\partial \underline{x}} = -\dot{\underline{p}} \quad (2.109)$$

$$\frac{\partial H_3}{\partial \underline{r}} = \underline{0}, \text{ and} \quad (2.110)$$

$$\frac{\partial H_3}{\partial \underline{v}} = \underline{0} \quad (2.111)$$

where

$$L(\underline{x}(T), T) = \underline{x}'(T) \underline{S}\underline{x}(T) \quad (2.112)$$

$$H_1 = \underline{x}'\underline{Q}\underline{x} + \underline{x}'\underline{R}_1\underline{A}\underline{x} + \underline{x}'\underline{B}'\underline{R}_2 + \underline{B}\underline{x} + \underline{\lambda}'(\underline{F}\underline{x} + \underline{G}_1\underline{A}\underline{x} + \underline{G}_2\underline{B}\underline{x} - \dot{\underline{x}}) \quad (2.113)$$

$$H_2 = \underline{x}'\underline{Q}\underline{x} + \underline{u}'\underline{R}_1\underline{u} + \underline{x}'\underline{D}'\underline{R}_2\underline{D}\underline{x} + \underline{n}'(\underline{F}\underline{x} + \underline{G}_1\underline{u} + \underline{G}_2\underline{D}\underline{x} - \dot{\underline{x}}) \quad (2.114)$$

$$H_3 = \underline{x}'\underline{Q}\underline{x} + \underline{r}'\underline{R}_1\underline{r} + \underline{v}'\underline{R}_2\underline{v} + \underline{p}'(\underline{F}\underline{x} + \underline{G}_1\underline{r} + \underline{G}_2\underline{v}) \quad (2.115)$$

Using equations (2.112), (2.113), (2.114), and (2.115) the necessary conditions are rewritten below.

In the interval t_2 to T ,

$$\frac{\partial H_3}{\partial \underline{x}} = -\dot{\underline{p}}$$

or

$$2Q\underline{x} + F'\underline{p} = -\dot{\underline{p}} \quad (2.116)$$

Assuming

$$\underline{p} = 2P_\rho \underline{x} \quad (2.117)$$

and using equations (2.108), necessary condition (2.116) becomes

$$2R_1 \underline{x} = -G_1' \underline{p} = -2G_1' P_\rho \underline{x}$$

or

$$\underline{x} = -R_1^{-1} G_1' P_\rho \underline{x} \quad (2.118)$$

Similarly,

$$\underline{y} = -R_2^{-1} G_2' P_\rho \underline{x} \quad (2.119)$$

where P_ρ satisfies the matrix Riccati equation

$$\dot{P}_\rho + P_\rho F + F' P_\rho - P_\rho (G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') P_\rho + Q = [0] \quad (2.120)$$

with the boundary condition $P_\rho(T) = S$ and

$$\dot{\underline{x}} = \underline{F}\underline{x} - \underline{G}_1 \underline{R}_1^{-1} \underline{G}_1' \underline{P} \underline{x} - \underline{G}_2 \underline{R}_2^{-1} \underline{G}_2' \underline{P} \underline{x} \quad (2.121)$$

In the interval t_1 to t_2 , assume that

$$\underline{n}(t) = 2\underline{P}_n(t) \underline{x}(t) \quad (2.122)$$

The necessary condition (2.98) becomes

$$\underline{P}_n(t_2) = \underline{P}_\rho(t_2) \quad (2.123)$$

Condition (2.106) gives

$$\underline{u} = -\underline{R}_1^{-1} \underline{G}_1' \underline{P}_n \underline{x} \quad (2.124)$$

Differentiating (2.122) and using (2.104), (2.123), and (2.124), necessary condition (2.105) yields

$$\dot{\underline{P}}_n + \underline{P}_n(\underline{F} + \underline{G}_2 \underline{D}) + (\underline{F} + \underline{G}_2 \underline{D})' \underline{P}_n - \underline{P}_n \underline{G}_1 \underline{R}_1^{-1} \underline{G}_1' \underline{P}_n + \underline{D}' \underline{R}_2 \underline{D} + \underline{Q} = [0] \quad (2.125)$$

with the boundary condition $\underline{P}_n(t_2) = \underline{P}_\rho(t_2)$. Using an approach similar to that of Section 2.4, the necessary condition (2.107) yields

$$\begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \\ \vdots \\ \underline{d}_n \end{bmatrix} = -\underline{M}_n^{-1}(t_1, t_2) \int_{t_1}^{t_2} \begin{bmatrix} \underline{x}_1' \underline{I}_q \\ \underline{x}_2' \underline{I}_q \\ \vdots \\ \underline{x}_n' \underline{I}_q \end{bmatrix} \underline{G}_2' \underline{P}_n \underline{x} \, dt \quad (2.126)$$

where

$$M_n(t_1, t_2) \triangleq \int_{t_1}^{t_2} \begin{bmatrix} x_1^I q \\ x_2^I q \\ \vdots \\ x_n^I q \end{bmatrix} R_2 [x_1^I q \quad x_2^I q \quad \dots \quad x_n^I q] dt \quad (2.127)$$

In the interval t_0 to t_1 , assume that

$$\underline{\lambda}(t) = 2P_\lambda(t) \underline{x}(t). \quad (2.128)$$

The necessary condition (2.99) becomes

$$P_\lambda(t_1) = P_n(t_1) \quad (2.129)$$

Differentiating (2.128) and using equations (2.100), (2.128) and (2.129), the necessary condition (2.101) yields

$$\dot{P}_\lambda + P_\lambda(F + G_1 A + G_2 B) + (F + G_1 A + G_2 B)' P + A' R_1 A + B' R_2 B + Q = [0] \quad (2.130)$$

with the boundary conditions $P_\lambda(t_1) = P_n(t_1)$. Again, using an approach similar to that of Section 2.4, the necessary conditions (2.102) and (2.103) yield

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_n \end{bmatrix} = -(M_\lambda^1)^{-1} \int_{t_0}^{t_1} \begin{bmatrix} x_1^I p \\ x_2^I p \\ \vdots \\ x_n^I p \end{bmatrix} G_1' P_\lambda \underline{x} dt, \quad \text{and} \quad (2.131)$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = -(M_\lambda^2)^{-1} \int_{t_0}^{t_1} \begin{bmatrix} x_1^I q \\ x_2^I q \\ \vdots \\ x_n^I q \end{bmatrix} G_2' P_\lambda \underline{x} \, dt \quad (2.132)$$

where

$$M_\lambda^1(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} x_1^I p \\ x_2^I p \\ \vdots \\ x_n^I p \end{bmatrix} R_1 [x_1^I p \quad x_2^I p \quad \dots \quad x_n^I p] \, dt \quad (2.133)$$

and

$$M_\lambda^2(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} x_1^I q \\ x_2^I q \\ \vdots \\ x_n^I q \end{bmatrix} R_2 [x_1^I q \quad x_2^I q \quad \dots \quad x_n^I q] \, dt \quad (2.134)$$

The cost functional is

$$\begin{aligned} J = \underline{x}'(T) S \underline{x}(T) &+ \int_{t_0}^{t_1} (\underline{x}' A' R_1 A \underline{x} + \underline{x}' B' R_2 B \underline{x} + \underline{x}' Q \underline{x}) \, dt \\ &+ \int_{t_1}^{t_2} (\underline{u}' R_1 \underline{u} + \underline{x}' D' R_2 D \underline{x} + \underline{x}' Q \underline{x}) \, dt \\ &+ \int_{t_2}^T (\underline{r}' R_1 \underline{r} + \underline{v}' R_2 \underline{v} + \underline{x}' Q \underline{x}) \, dt \end{aligned} \quad (2.135)$$

This can be reduced to a simple form. Consider first

$$\underline{x}'(T)S\underline{x}(T) + \int_{t_2}^T (\underline{r}'R_1\underline{r} + \underline{v}'R_2\underline{v} + \underline{x}'Q\underline{x}) dt \quad (2.136)$$

$$= \underline{x}'(t_2)P_{\rho}(t_2)\underline{x}(t_2) + \int_{t_2}^T [(\underline{r}'R_1\underline{r} + \underline{v}'R_2\underline{v} + \underline{x}'Q\underline{x}) + \frac{d}{dt}(\underline{x}'P_{\rho}\underline{x})] dt \quad (2.137)$$

Now

$$\frac{d}{dt}(\underline{x}'P_{\rho}\underline{x}) = \dot{\underline{x}}'P_{\rho}\underline{x} + \underline{x}'\dot{P}_{\rho}\underline{x} + \underline{x}'P_{\rho}\dot{\underline{x}} \quad (2.138)$$

Substituting for $\dot{P}_{\rho}$ from equation (2.120) and for $\dot{\underline{x}}$ from equation (2.121), and after some algebraic manipulation, equation (2.137) yields

$$\underline{x}'(t_2)P_{\rho}(t_2)\underline{x}(t_2)$$

Now consider

$$\begin{aligned} & \underline{x}'(t_2)P_{\rho}(t_2)\underline{x}(t_2) + \int_{t_1}^{t_2} (\underline{u}'R_1\underline{u} + \underline{x}'D'R_2D\underline{x} + \underline{x}'Q\underline{x}) dt \\ &= \underline{x}'(t_1)P_{\eta}(t_1)\underline{x}(t_1) + \int_{t_1}^{t_2} (\underline{u}'R_1\underline{u} + \underline{x}'D'R_2D\underline{x} + \underline{x}'Q\underline{x}) + \frac{d}{dt}(\underline{x}'P_{\eta}\underline{x}) dt \end{aligned}$$

which by using equations (2.124) and (2.125) reduces to

$$\underline{x}'(t_1)P_{\eta}(t_1)\underline{x}(t_1)$$

Now it is evident that

$$J = \underline{x}'(t_1)P_{\eta}(t_1)\underline{x}(t_1) + \int_{t_0}^{t_1} (\underline{x}'A'R_1A\underline{x} + \underline{x}'B'R_2B\underline{x} + \underline{x}'Q\underline{x}) dt \quad (2.139)$$

By use of equation (2.130) and after some algebraic manipulations,
(2.139) reduces to

$$J = \underline{x}'(t_0) P_{\lambda}(t_0) \underline{x}(t_0) \quad (2.140)$$

2.9 SCALAR CASE

If x , u , and v are all scalar functions of time, the necessary conditions for a saddle point are simplified. In the interval t_2 to T ,

$$\dot{x} = Fx + G_1 r + G_2 v \quad (2.141)$$

$$r = -R_1^{-1} G_1' P_{\rho} x \quad (2.142)$$

and

$$v = -R_2^{-1} G_2' P_{\rho} x \quad (2.143)$$

where P_{ρ} satisfies the matrix Ricatti equation

$$\dot{P}_{\rho} + 2P_{\rho}F - P_{\rho}(G_1^2 R_1^{-1} + G_2^2 R_2^{-1}) + Q = 0 \quad (2.144)$$

with the boundary condition $P_{\rho}(T) = S$. In the interval t_1 to t_2 ,

$$\dot{x} = Fx + G_1 u + G_2 D x \quad (2.145)$$

$$u = -R_1^{-1} G_1' P_{\eta} x \quad (2.146)$$

and

$$D = - \frac{\int_{t_1}^{t_2} G_2 P_\eta x^2 dt}{\int_{t_1}^{t_2} R_2 x^2 dt} \quad (2.147)$$

where P_η satisfies the differential equation

$$\dot{P}_\eta + 2P_\eta (F + G_2 D) - P_\eta^2 G_1^2 R_1^{-1} + D^2 R_2 + Q = 0 \quad (2.148)$$

with the boundary condition $P_\eta(t_2) = P_\rho(t_2)$. In the interval t_0 to t_1 ,

$$\dot{x} = Fx + G_1 Ax + G_2 Bx \quad (2.149)$$

$$A = - \frac{\int_{t_0}^{t_1} G_1 P_\lambda x^2 dt}{\int_{t_0}^{t_1} R_1 x^2 dt} \quad (2.150)$$

and

$$B = - \frac{\int_{t_0}^{t_1} G_2 P_\lambda x^2 dt}{\int_{t_0}^{t_1} R_2 x^2 dt} \quad (2.151)$$

where P_λ satisfies the differential equation

$$\dot{P}_\lambda + 2P_\lambda (F + G_1 A + G_2 B) + A^2 R_1 + B^2 R_2 + Q = 0 \quad (2.152)$$

with the boundary condition $P_\lambda(t_1) = P_\eta(t_1)$.

The above equations suggest a procedure (even for the general case) for evaluating the controls u , r , and v and also the constants A , B , and D . The procedure is as follows: First, solve equation (2.144) and store $P_\rho(t)$. Then assume some initial value for D , and using $P_\rho(t_2)$, solve equation (2.148) and store $P_\eta(t)$. Then assume some initial values for A and B , and using $P_\eta(t_1)$, solve equation (2.152) and store $P_\lambda(t)$. Then solve for $x(t)$ for all the intervals $[t_0, t_1]$, $[t_1, t_2]$, and $[t_2, T]$. Next evaluate equations (2.147), (2.150), and (2.151) to obtain new values for D , A , and B . Finally, repeat the procedure until D , A , and B converge to a saddle point.

2.10 CONSTANT GAIN PLAYER VERSUS A VARIABLE GAIN PLAYER

Consider the differential game when one player is constrained to use constant feedback gains throughout the duration of interest while the other player is free to select his control at each instant. Figure 2.1(e) refers to the B constant case, and Figure 2.1(f) refers to A constant case.

Figure 2.1(e) is a particular case of Figure 2.1(d) when $t_0 = t_1$, $t_2 = T$, and $D = B$. Hence, the necessary conditions can be obtained by inspection.

Figure 2.1(f) is a particular case of Figure 2.1(a) when $t_0=t_1$, $t_2=T$, and $w=v$. Here also the necessary conditions can be obtained by inspection.

The results for the above two problems are applied to a scalar case in Chapter 3 in which detailed numerical results are given.

2.11 VARIABLE-GAIN CASE: DERIVATION BASED ON PIECEWISE-CONSTANT GAINS

By using the results of Section 2.7.2 necessary conditions for a saddle point for the unconstrained case when the players are free to choose their controls can be derived.

Consider a dynamic system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{v}, t) \quad , \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.153)$$

and a cost functional

$$J = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{u}, \underline{v}, t) \, dt \quad (2.154)$$

with the initial time t_0 and final time T specified and where $\underline{f}$, L , and g have the properties given in Section 2.2.

Consider the control diagram shown in Figure 2.2(a). From the nature of the results of Section 2.7.2, necessary conditions for a saddle point corresponding to controls implemented as shown in Figure 2.2(a) are

$$\int_{t_0}^{t_1} \frac{\partial H_1}{\partial \underline{a}_k} \, dt = 0 \quad (2.155)$$

$$\int_{t_1}^T \frac{\partial H_2}{\partial \underline{c}_k} dt = \underline{0} \quad (2.156)$$

$$\int_0^{t_1} \frac{\partial H_1}{\partial \underline{b}_k} dt = \underline{0} \quad (2.157)$$

$$\int_{t_1}^T \frac{\partial H_2}{\partial \underline{d}_k} dt = \underline{0} \quad (2.158)$$

$$\frac{\partial H_1}{\partial \underline{x}} = -\dot{\underline{\lambda}}^1 \quad (2.159)$$

$$\frac{\partial H_2}{\partial \underline{x}} = -\dot{\underline{\lambda}}^2 \quad (2.160)$$

and

$$\frac{\partial L}{\partial \underline{x}} \Big|_{t=T} = \underline{\lambda}^2(T) \quad (2.161)$$

$$\underline{\lambda}^2(t_1) = \underline{\lambda}^1(t_1) \quad (2.162)$$

where superscripted $\underline{\lambda}$'s denote distinct vector Lagrange multipliers, and

$$H_1 = g(\underline{x}, \underline{Ax}, \underline{Bx}, t) + (\underline{\lambda}^1)' (\underline{f}(\underline{x}, \underline{Ax}, \underline{Bx}, t) - \dot{\underline{x}}) \quad (2.163)$$

$$H_2 = g(\underline{x}, \underline{Cx}, \underline{Dx}, t) + (\underline{\lambda}^2)' (\underline{f}(\underline{x}, \underline{Cx}, \underline{Dx}, t) - \dot{\underline{x}}) \quad (2.164)$$

Similarly, if the duration of the game is divided into m parts as shown in the control diagram in Figure 2.2(b), necessary conditions for a saddle point are

$$\int_{t_{\ell-1}}^{t_{\ell}} \frac{\partial H_{\ell}}{\partial \underline{a}_k^{\ell}} dt = \underline{0}, \quad \begin{matrix} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{matrix} \quad (2.165)$$

$$\int_{t_{\ell-1}}^{t_{\ell}} \frac{\partial H_{\ell}}{\partial \underline{b}_k^{\ell}} dt = \underline{0}, \quad \begin{matrix} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{matrix} \quad (2.166)$$

$$\frac{\partial H_{\ell}}{\partial \underline{x}} = -\dot{\underline{\lambda}}^{\ell}, \quad \ell = 1, 2, \dots, m \quad (2.167)$$

$$\underline{\lambda}^m(T) = \left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} \quad (2.168)$$

$$\underline{\lambda}^{\ell}(t_{\ell-1}) = \underline{\lambda}^{\ell-1}(t_{\ell-1}), \quad \ell = 2, \dots, m \quad (2.169)$$

where

$$H_{\ell} = g(\underline{x}, A^{\ell} \underline{x}, B^{\ell} \underline{x}, t) + (\underline{\lambda}^{\ell})' (f(\underline{x}, A^{\ell} \underline{x}, B^{\ell} \underline{x}, t) - \dot{\underline{x}}) \quad (2.170)$$

Now if the interval $(t_{\ell} - t_{\ell-1})$, $\ell = 1, 2, \dots, m$, is made sufficiently small and writing $t_{\ell} = t_{\ell-1} + \Delta t$, the necessary condition (2.165) reduces to

$$\int_{t_{\ell-1}}^{t_{\ell-1} + \Delta t} \frac{\partial H_{\ell}}{\partial \underline{a}_k^{\ell}} dt = \underline{0}, \quad \begin{matrix} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{matrix} \quad (2.171)$$

If $\Delta t \rightarrow 0$ but not identically to zero the integrand may be assumed to be constant during Δt . Hence (2.171) reduces to

$$\left. \frac{\partial H_\ell}{\partial \underline{a}_k} \right|_{t_{\ell-1}} = \underline{0}, \quad \begin{array}{l} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{array} \quad (2.172)$$

From equation (2.170) it is evident that

$$\frac{\partial H_\ell}{\partial \underline{a}_k} = \left[\frac{\partial \underline{u}^\ell}{\partial \underline{a}_k} \right]' \frac{\partial H_\ell}{\partial \underline{u}}; \quad \begin{array}{l} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{array} \quad (2.173)$$

Therefore condition (2.172) reduces to

$$\left. \frac{\partial H_\ell}{\partial \underline{u}} x_k^\ell \right|_{t_{\ell-1}} = \underline{0}; \quad \begin{array}{l} \ell = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{array} \quad (2.174)$$

Since x_k^ℓ at $t_{\ell-1}$ is nonzero, in general, condition (2.174) reduces to

$$\left. \frac{\partial H_1}{\partial \underline{u}^1} \right|_{t_0} = \left. \frac{\partial H_1}{\partial \underline{u}^2} \right|_{t_1} = \dots = \left. \frac{\partial H_m}{\partial \underline{u}^m} \right|_{t_{m-1}} = \underline{0} \quad (2.175)$$

When $\Delta t \rightarrow 0$, equation (2.175) reduces to

$$\partial H / \partial \underline{u} = \underline{0} \quad (2.176)$$

Similar relationships hold for $\underline{v}$, i.e.,

$$\partial H / \partial \underline{v} = \underline{0} \quad (2.177)$$

When $\Delta t \rightarrow 0$, equations (2.167) and (2.168) become

$$\partial H / \partial \underline{x} = -\underline{\lambda} \quad (2.178)$$

with the boundary condition $\lambda(T) = [\partial L / \partial \underline{x}]_{t=T}$. Equations (2.176),

(2.177), and (2.178) are, in fact, necessary conditions for the unconstrained case.

SUMMARY OF THE RESULTS FOR THE PROBLEM POSED IN SECTION 2.2:

NECESSARY CONDITIONS FOR A SADDLE POINT

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t), \quad \underline{x}(t_0) = \underline{x}_0$$

$$J = L(\underline{x}(T), T) + \int_{t_0}^T g(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t) dt$$

$$\partial H / \partial \underline{x} = -\underline{\lambda}$$

$$[\partial L / \partial \underline{x}]_{t=T} = \underline{\lambda}(T)$$

$$\int_{t_0}^T \partial H / \partial A dt = [0]$$

$$\int_{t_0}^T \partial H / \partial B dt = [0]$$

$$\int_{t_0}^T H_{AA} dt \geq 0$$

$$\int_{t_0}^T H_{BB} dt \leq 0$$

where

$$H \triangleq g(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t) + \underline{\lambda}' \underline{f}(\underline{x}, \underline{A}\underline{x}, \underline{B}\underline{x}, t)$$

SUFFICIENT CONDITIONS

The existence of a continuous $p(t)$ satisfying

$$\dot{P} + P \underline{f}_{\underline{x}} + \underline{f}' P + \frac{1}{2} H_{\underline{xx}} = [0]$$

with the boundary condition $P(T) = \frac{1}{2} L_{\underline{xx}}(T)$, is a sufficient condition for

the existence of a saddle point. It is to be noted that $\underline{f}_{\underline{x}}$, $H_{\underline{xx}}$, and $L_{\underline{xx}}(T)$ are to be evaluated along the optimal path generated by (A^0, B^0) which satisfy the necessary conditions mentioned above.

CHAPTER 3

SCALAR QUADRATIC GAMES

3.1 INTRODUCTION

In this chapter, the results of Chapter 2 are first applied to a basic minimization problem for a scalar dynamic system and then to a two-person zero-sum differential game in which system response is characterized by a scalar dynamical equation. Numerical solutions are discussed in Section 3.4 where some interesting results are brought to light.

It is shown that cooperation is possible even in zero-sum games from the point of view of control implementation. This conceptual and practical advantage of reducing the "latent" cost of control implementation is not apparent in the case of nonconstant-gain zero-sum games.

3.2 A SCALAR MINIMIZATION PROBLEM

Consider the basic minimization problem for a linear scalar dynamic system

$$\dot{x} = -x + u, \quad x_0 = 1 \quad (3.1)$$

subject to the restriction

$$u = Ax \quad (3.2)$$

The object is to choose the parameter A so as to minimize the performance index J ,

$$J = x(T)^2 + \int_0^T u^2 dt \quad (3.3)$$

Now

$$\dot{x} = -x + u, \quad x(t_0) = 1$$

$$= (A-1)x$$

Therefore

$$x(t) = e^{(A-1)t} \quad (3.4)$$

and

$$x(T) = e^{(A-1)T} \quad (3.5)$$

Now

$$J = x(T)^2 + \int_0^T A^2 x^2 dt \quad (3.6)$$

By substituting (3.4) and (3.5) in (3.6) and simplifying,

$$J = e^{2(A-1)T} + \frac{A^2}{2(A-1)} (e^{2(A-1)T} - 1) \quad (3.7)$$

For an extremum of J ,

$$\frac{\partial J}{\partial A} = 0 \quad (3.8)$$

Differentiating equation (3.7) and simplifying, the necessary condition (3.8) is equivalent to

$$e^{2(A-1)T} [2T(A-1)(A^2 + 2A - 2) + (A^2 - 2A)] + (2A - A^2) = 0 \quad (3.9)$$

Another way to obtain the above result is to apply the results of Section 2.5 of Chapter 2, assuming $B = 0$, $F = -1$, $G_1 = 1$, $G_2 = 0$, and $R_1 = 1$. Equations (2.68), (2.69), (2.70), and (2.71) give

$$\dot{x} = (A-1)x, \quad x(t_0) = 1 \quad (3.10)$$

$$\dot{P} + 2P(A-1) + A^2 = 0, \quad P(T) = 1 \quad (3.11)$$

$$A = - \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} \quad (3.12)$$

and

$$B = 0 \quad (3.13)$$

To evaluate (3.12),

$$\int_0^T x^2 dt = \int_0^T e^{2(A-1)t} dt = \frac{(e^{2(A-1)T} - 1)}{2(A-1)}$$

The solution of (3.11) is

$$P(t) = \frac{(A^2 + 2A - 2)}{2(A-1)} e^{2(A-1)(T-t)} - \frac{A^2}{2(A-1)}$$

Let $S \triangleq (A^2 + 2A - 2)/2(A-1)$, and $W \triangleq A^2/2(A-1)$. Thus,

$$P(t) = Se^{2(A-1)(T-t)} - W$$

$$\int_0^T P x^2 dt = \int_0^T (Se^{2(A-1)T-W} e^{2(A-1)t}) dt$$

$$= STe^{2(A-1)T} - \frac{W}{2(A-1)}(e^{2(A-1)T} - 1)$$

$$A = - \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} = - \left(\frac{STe^{2(A-1)T} - W(e^{2(A-1)T} - 1)}{(e^{2(A-1)T} - 1)} \right)$$

and therefore,

$$A = - \frac{(2(A-1)STe^{2(A-1)T} - W(e^{2(A-1)T} - 1))}{(e^{2(A-1)T} - 1)}$$

By cross multiplying and simplifying,

$$e^{2(A-1)T}(2T(A-1)(A^2 + 2A - 2) + (A^2 - 2A)) + (2A - A^2) = 0$$

which checks with equation (3.9). The constant A can be evaluated by following the iterative procedure expressed in Section 2.5.

3.3 TWO-PERSON ZERO-SUM DIFFERENTIAL GAMES

3.3.1 CASE 1: CONSTANT FEEDBACK CASE

Consider the case of a linear time-invariant system governed by

$$\dot{x} = -0.5x + 1.25u + 1.5v, \quad x_0 = 2 \quad (3.14)$$

The quadratic cost functional is

$$J = x(T)^2 S + \int_0^T (u^2 - 4v^2 + 2x^2) dt \quad (3.15)$$

Equations (2.69), (2.70), and (2.71) of Section 2.5 give

$$\dot{P} + 2P(-0.5 + 1.25A - 1.5B) + A^2 - 4B^2 + 2 = 0, \quad P(T) = S \quad (3.16)$$

$$A = -1.25 \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} \quad (3.17)$$

$$B = 0.375 \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} \quad (3.18)$$

and

$$J_{\text{optimal}} = x_0^2 P(t_0) = 4P(0) \quad (3.19)$$

Different terminal times T are chosen, and for each T , the parameter S is varied from 0 to 10. For each S , the saddle point and the performance measure have been computed (Figures 3.1, 3.2, and 3.3).

3.3.2 MIXED CASES

The same dynamic system (3.14) and the same cost functional (3.15) are considered. Two cases depending on which player follows the constant feedback gains are considered.

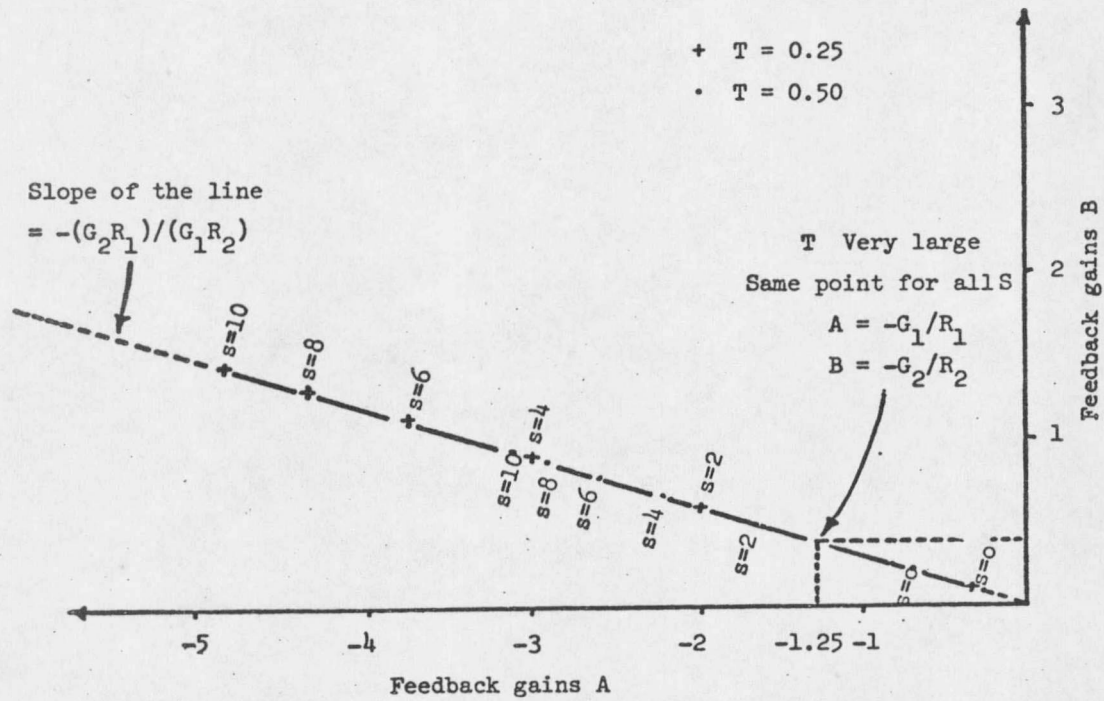
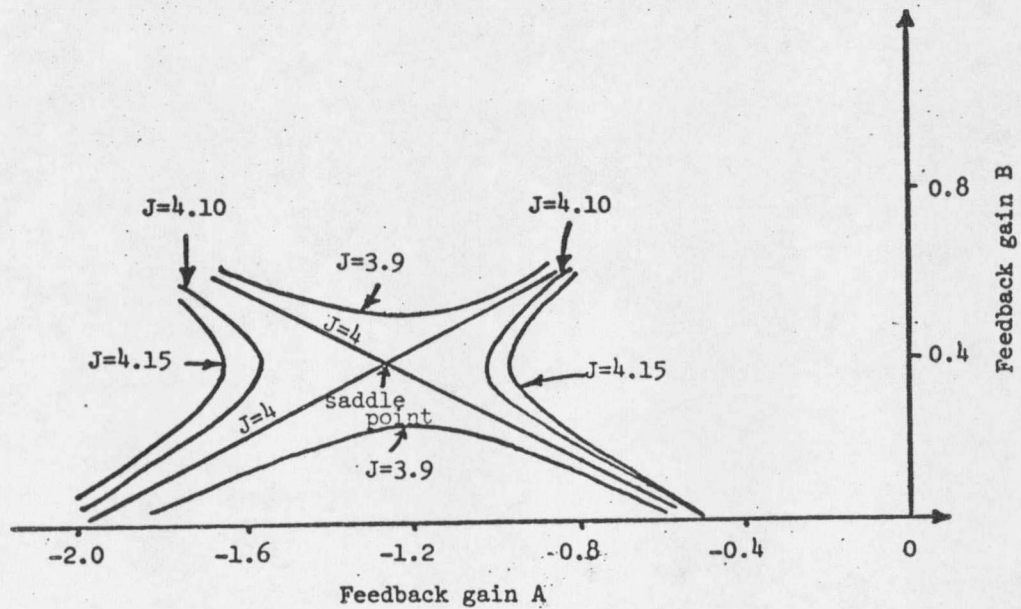


Figure 3.1. Variation of saddle point.

Figure 3.2. Locus of contours of constant J near the saddle point.

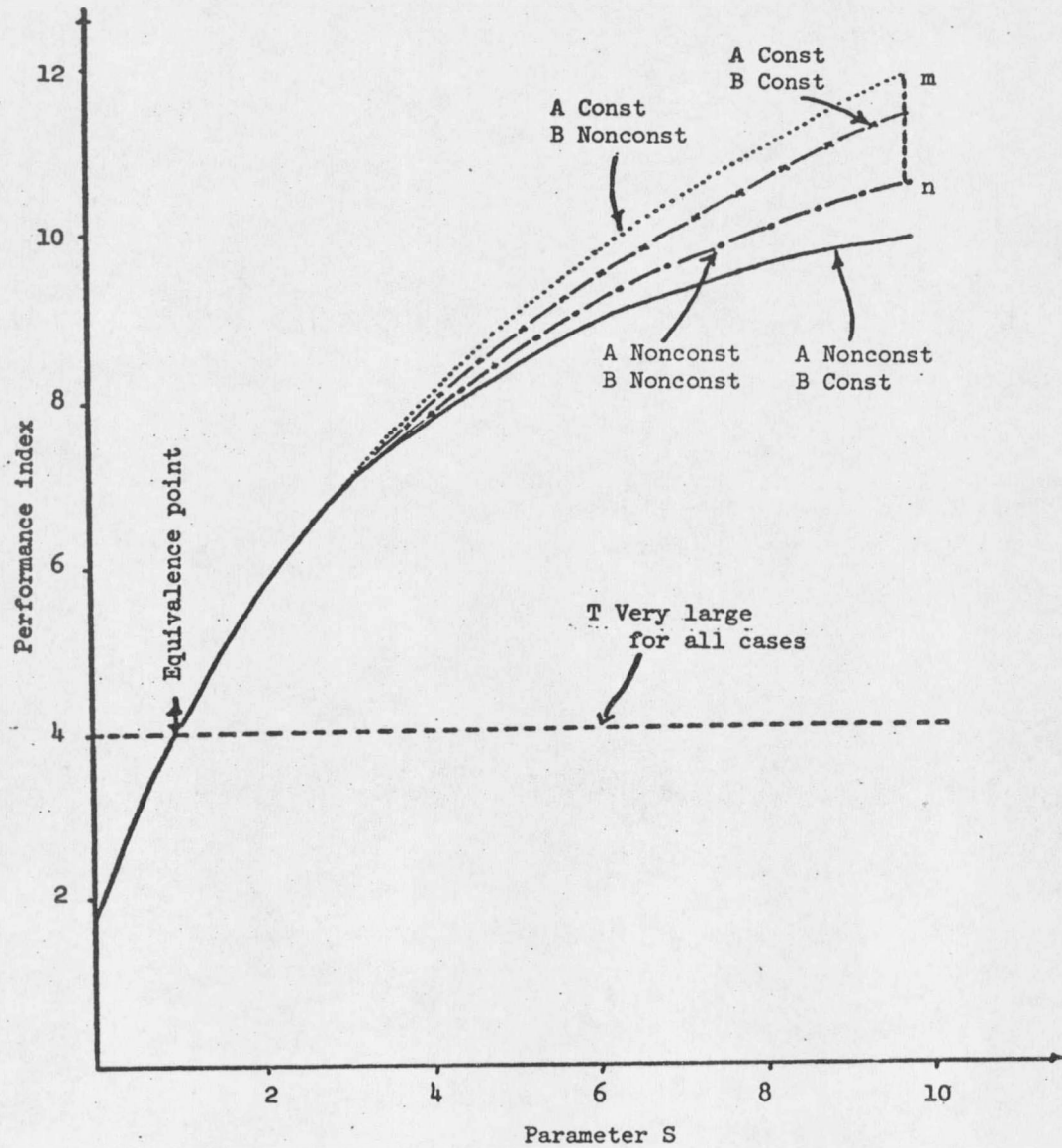


Figure 3.3. Variation of performance index.

CASE 2: A CONSTANT AND B NONCONSTANT

This case corresponds to the minimizing player using constant feedback gain while the maximizing player is using nonconstant control. Following Sections 2.10 and 2.11, the necessary conditions for optimality are

$$\dot{P} + 2P(-0.5 + 1.25A) + 0.5625P^2 + A^2 + 2 = 0, \quad P(T) = S \quad (3.20)$$

$$\dot{x} = (-0.5 + 1.25A + 0.5625P)x, \quad x_0 = 2 \quad (3.21)$$

$$A = -1.25 \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} \quad (3.22)$$

and

$$J_{\text{optimal}} = P(t_0)x_0^2 = 4P(0) \quad (3.23)$$

Different terminal times T are chosen, and for each T , the parameter S is varied from 0 to 10. For each S , the constant A and the performance measure have been computed (Figures 3.3 and 3.4) by following the iterative procedure suggested in Section 2.10.

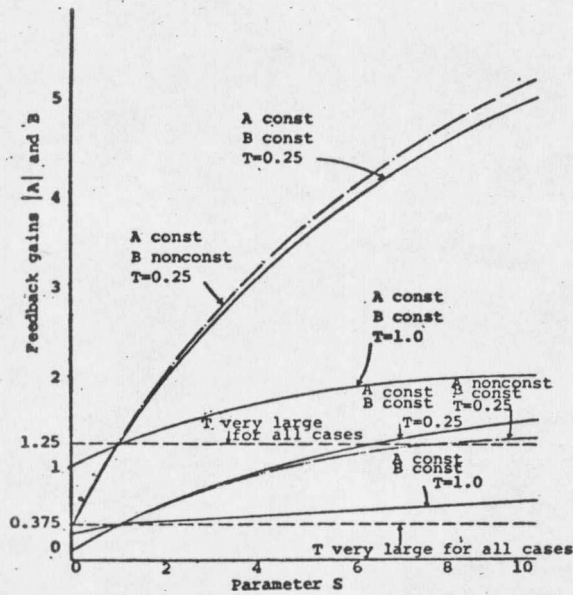


Figure 3.4. Variation of feedback gains.

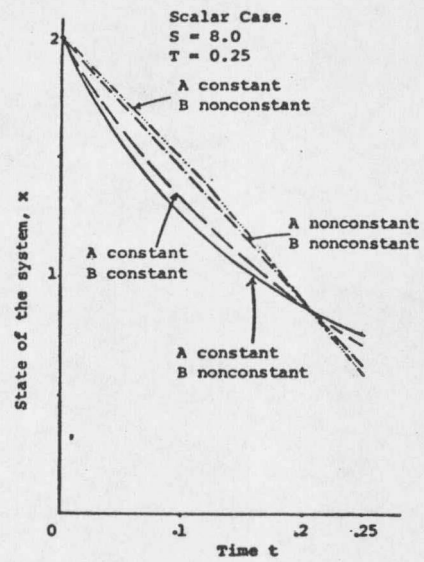


Figure 3.5. The state of the system.

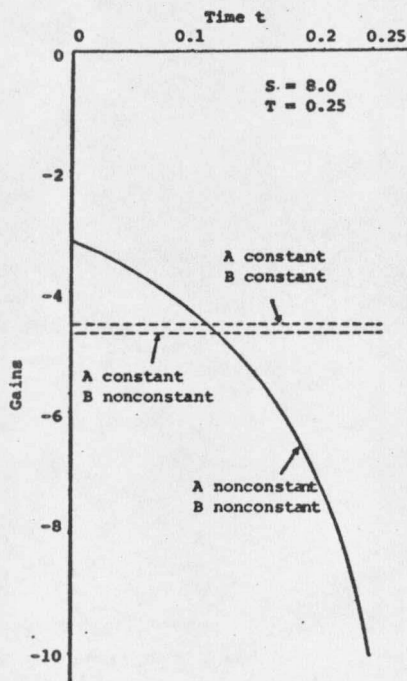


Figure 3.6. Gains for the minimizing player.

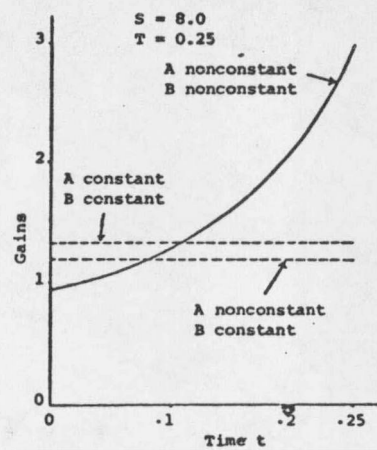


Figure 3.7. Gains for the maximizing player.

CASE 3: A NONCONSTANT AND B CONSTANT

This case corresponds to the maximizing player using constant feedback gain while the minimizing player is using variable gain. Following Sections 2.10 and 2.11, the necessary conditions are

$$\dot{P} + 2P(-0.5 + 1.5B) - 1.5625P^2 - 4B^2 + 2 = 0, \quad P(T) = S \quad (3.24)$$

$$\dot{x} = (-0.5 + 1.5B - 1.5625P)x, \quad x_0 = 2 \quad (3.25)$$

$$B = 0.375 \frac{\int_0^T P x^2 dt}{\int_0^T x^2 dt} \quad (3.26)$$

and

$$J_{\text{optimal}} = P(t_0)x_0^2 = 4P(0) \quad (3.27)$$

Different terminal times T are chosen, and for each T , the parameter S is varied from 0 to 10. For each S , the constant B and the performance measure have been computed (Figures 3.3 and 3.4) by following the iterative procedure suggested in Section 2.10.

3.3.3 CASE 4: DIFFERENTIAL GAME WITHOUT CONSTANT-GAIN CONSTRAINTS

This game is one in which both A and B uses nonconstant-gain controls. The same dynamic system and the same cost functional as for the previous cases are considered. Following Section 2.6, necessary conditions for a saddle point are

$$\dot{x} = -0.5 + 1.25u + 1.5v \quad (3.28)$$

$$\dot{P} - P - P^2 + 2 = 0, \quad P(T) = S \quad (3.29)$$

$$u^0 = -1.25Px \quad (3.30)$$

$$v^0 = 0.375Px \quad (3.31)$$

and

$$J_{\text{optimal}} = x(0)^2 P(t_0) = 4P(0) \quad (3.32)$$

Different terminal times T are chosen, and for each T , the parameter S is varied from 0 to 10. For each S , the performance measure has been computed.

The results for the above-mentioned four cases are given in Figures 3.1, 3.2, 3.3, and 3.4. Figure 3.1 illustrates the variation of saddle points A and B for Case 1 with respect to the parameter S for a fixed terminal time T .

Figure 3.2 illustrates the locus of contours of constant J near the saddle point for Case 1.

Figure 3.3 illustrates the variation of the performance measure with respect to S for all four cases for a fixed terminal time T .

Figure 3.4 illustrates the variation of constants A and B with respect to S for a fixed terminal time T for the first three cases.

3.4 DISCUSSION OF RESULTS

The steady-state value of the matrix Riccati equation (3.29), when the players are using nonconstant feedback gains, is equal to unity. Hence when the final time T is sufficiently large the nonconstant feedback gains essentially reduce to the constant feedback gains. Thus, when the final time T is very large, the saddle points A and B can be computed from equations (3.30) and (3.31), i.e.,

$$A = -R_1^{-1} G_1' P_{\text{steady-state}} = -1.25$$

$$B = -R_2^{-1} G_2' P_{\text{steady-state}} = 0.375$$

It is evident that this saddle point should be the same regardless of the value of S .

From equations (2.70) and (2.71) of Chapter 2, it is observed that for a scalar time-invariant case, A and B are related by the equation

$$B = -\frac{A R_1 G_2}{R_2 G_1}$$

From Figure 3.1 it is evident that they indeed follow this law; and as final time T and S are varied, the saddle points telescopically converge to the point $(-R_1^{-1} G_1', -R_2^{-1} G_2')$ as expected. As time $T \rightarrow \infty$, constant gain, mixed control gain, and nonconstant gain all become equivalent. Thus, whenever the duration of the game is sufficiently large, it seems appropriate to use constant gains.

Considering Figure 3.3, it can be concluded that whenever the player is restricted to use constant feedback gain he will be at a disadvantage, as expected. For the particular example under consideration, it is also seen from Figure 3.3 that it is advantageous for the maximizing player when both are using constant feedback gains. Under exactly what conditions constant gains are advantageous to one player or the other will be considered in a subsequent chapter.

It is interesting to note from Figure 3.3 that all the performance index curves for different terminal time T intersect at a common point, and the point has been named the "Equivalence Point", since at that point the performance index and control laws are the same regardless of the original design intent and regardless of the final time T . Consideration of Figure 3.4 reveals not only that the pay-off is the same at the equivalence point but that all feedback gains A are the same at the equivalence point, as also are gains B , whatever may be the final time and whatever may be the type of control being used. Whether such phenomena will occur in linear time-invariant vector cases and in linear time-varying cases is a matter for further investigation.

From Figure 3.3, when S is small and near the equivalence point, the performance index is almost the same as the nonconstant-gain performance index. Whenever the parameter S is near the equivalence point, therefore, it may be appropriate to use constant feedback gains.

Figure 3.3 suggests in fact that concepts such as negotiation, bargaining, and side payments between the players may be beneficial to both. Due to the lack of technical know-how and/or to the cost of implementing nonconstant gains, the minimizing player, for example, knowing his limitations, might induce player B to use constant gains by giving a fraction of pay-off mn as indicated in Figure 3.3, as a side payment. The players may agree on a negotiated pay-off where an intermediate agency, say the Government, may come into the picture; and any player who does not comply to the rules of agreement, may be heavily penalized. The rules of side payment, bargaining, etc., are matters for further investigation.

CHAPTER 4

SUBOPTIMAL CONTROL LAWS

4.1 INTRODUCTION

In this chapter necessary and sufficient conditions are derived for the existence of a saddle point for a general two-person zero-sum differential game when one or both players use control laws of specified form. The specified forms for the controls consist of weighted sums of the state variables, the weighting factors being products of known time-varying functions and of constants to be determined in an optimal manner. The general results are applied to linear-quadratic games (games with linear dynamics and quadratic performance indices). For this class of differential games, an additional development is given to obtain control law parameters that are independent of initial conditions so that a saddle point with respect to the expected value of the performance index is obtained.

The results are applied to specific scalar and vector dynamic systems, and numerical solutions are presented. The approach and the results of this chapter are especially useful when only partial information about possible initial states is available to the players and when on-line computational facilities are limited. The results should broaden the class of practical problems to which differential game theory can be realistically applied. By restricting the form of feedback gains, it is possible in some cases to obtain control laws that are only nominally suboptimal, but are both computationally feasible and implementable in practice.

In the case of one-sided optimal control theory, necessary conditions for computing constant feedback parameters that are independent of initial conditions have been derived by Kleinman and Athans [26], but their results are not directly applicable when only partial information about possible initial states is known. In addition, the approach adopted in this chapter for finding the optimal gains differ from that of Kleinman and Athans.

4.2 FORMULATION OF THE PROBLEM: BOTH PLAYERS CONSTRAINED

Consider a dynamic system given by (2.1) and a cost functional given by (2.2) with the associated weighting matrices having the properties described in Section 2.2. The problem is to find or characterize, if they exist, particular controls $\underline{u}^0$ and $\underline{v}^0$ which are optimal in the sense that for any control vectors, $\underline{u}$ and $\underline{v}$, there holds

$$J(\underline{u}^0, \underline{v}) \leq J(\underline{u}^0, \underline{v}^0) \leq J(\underline{u}, \underline{v}^0) \quad (4.1)$$

Controls $\underline{u}$ and $\underline{v}$ may be constrained to be of the form

$$\underline{u} = \sum_{j=1}^{m_1} \alpha_j(t) A^j \underline{x} \quad (4.2)$$

and

$$\underline{v} = \sum_{j=1}^{m_2} \beta_j(t) B^j \underline{x} \quad (4.3)$$

where

$$\dot{A}^j = [0] \quad j = 1, 2, \dots, m_1 \quad (4.4)$$

and

$$\dot{B}^j = [0] \quad j = 1, 2, \dots, m_2 \quad (4.5)$$

and where $\alpha_j(t)$'s and $\beta_j(t)$'s are scalar functions of specified form. A^j 's and B^j 's are constant matrices of appropriate dimensions and are to be determined to satisfy (4.1).

In general, the optimal A^j 's and B^j 's are functions of the initial state $\underline{x}(t_0)$. In Section 4.6, we consider the case where the players desire to form control laws (4.2) and (4.3) that are independent of the initial state: Each player assumes some a priori distribution of possible initial states and satisfies conditions for optimality only in an expected value sense.

4.3 NECESSARY CONDITIONS

Variational procedures are applied to yield the necessary conditions for optimality. For convenience the $(n \times p)$ matrix A^j and the $(n \times q)$ matrix B^j are partitioned in terms of columns: Let

$$A^j \triangleq [\underline{a}_1^j \quad \underline{a}_2^j \quad \dots \quad \underline{a}_n^j] \quad j = 1, 2, \dots, m_1$$

and

$$B^j \triangleq [\underline{b}_1^j \quad \underline{b}_2^j \quad \dots \quad \underline{b}_n^j] \quad j = 1, 2, \dots, m_2$$

where $\underline{a}_1^j, \underline{a}_2^j, \dots, \underline{a}_n^j$ and $\underline{b}_1^j, \underline{b}_2^j, \dots, \underline{b}_n^j$ are constant vectors, i.e.,

$$\dot{\underline{a}}_k^j = \underline{0} \quad \begin{matrix} j = 1, 2, \dots, m_1 \\ k = 1, 2, \dots, n_1 \end{matrix} \quad (4.6)$$

$$\dot{\underline{b}}_k^j = \underline{0} \quad \begin{matrix} j = 1, 2, \dots, m_2 \\ k = 1, 2, \dots, n_2 \end{matrix} \quad (4.7)$$

Let

$$g = g(\underline{x}, \sum_{j=1}^{m_1} \alpha_j^j A^j \underline{x}, \sum_{j=1}^{m_2} \beta_j^j B^j \underline{x}, t) \quad (4.8)$$

and

$$\underline{f} = \underline{f}(\underline{x}, \sum_{j=1}^{m_1} \alpha_j^j A^j \underline{x}, \sum_{j=1}^{m_2} \beta_j^j B^j \underline{x}, t) \quad (4.9)$$

The development of Section 2.3 is applicable. Let $\underline{n}_k^j(t)$'s and $\underline{\rho}_k^j(t)$'s denote the vector Lagrange multipliers that account for the constraints (4.6) and (4.7) respectively in the two related optimal control problems and $\underline{\lambda}(t)$ be the common Lagrange multiplier that account for the constraint (2.1). Also, let

$$H = g + \underline{\lambda}' \underline{f} \quad (4.10)$$

It can be easily seen that the necessary conditions are

$$\dot{\underline{\lambda}} = -\partial H / \partial \underline{x} \quad (4.11)$$

with the boundary conditions

$$\underline{\lambda}(T) = [\partial L / \partial \underline{x}]_{t=T} \quad (4.12)$$

$$\underline{n}_k^j(t_0) = \underline{n}_k^j(T) \quad (4.13)$$

$$\underline{\rho}_k^j(t_0) = \underline{\rho}_k^j(T) \quad (4.14)$$

$$\dot{\underline{H}}_{\underline{a}_k^j}^j = -\dot{\underline{n}}_k^j \quad (4.15)$$

$$\dot{H}_{\underline{b}_k^j} = - \dot{\underline{p}}_k^j \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_2 \end{array} \quad (4.16)$$

From (4.13), (4.14), (4.15), and (4.16),

$$\int_{t_0}^T \frac{\partial H}{\partial \underline{a}_k^j} dt = 0, \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_1 \end{array} \quad (4.17)$$

$$\int_{t_0}^T \frac{\partial H}{\partial \underline{b}_k^j} dt = 0, \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_2 \end{array} \quad (4.18)$$

Conditions (4.12), (4.17), and (4.18) are then necessary conditions for a saddle point.

4.4 SUFFICIENT CONDITIONS

The sufficient conditions derived in this chapter follow the procedure adopted by Rhodes [36] for the case of zero-sum games without gain constraints. Let A denote the set $\{A^j\}$, B denote the set $\{B^j\}$, η denote the set $\{\eta_k^j\}$, and ρ denote the set $\{\rho_k^j\}$.

Theorem 1: A sufficient condition that the A^0 and B^0 which satisfy equations (4.4), (4.5), and (4.11) to (4.18) are indeed optimal in the sense of a saddle point is that there exists a scalar function $I(A, B, \underline{x}, \eta, \rho)$ which satisfies the following five properties.

$$1) \quad \min_A S(\underline{x}, A, B^0, \eta, \rho) = 0 \quad (4.19)$$

$$2) \quad \text{The minimum in (4.19) occurs for } A = A^0 \quad (4.20)$$

$$3) \max_B S(\underline{x}, A^0, B, \eta, \rho) = 0 \quad (4.21)$$

$$4) \text{ The maximum in (4.21) occurs for } B = B^0 \quad (4.22)$$

$$5) I(A^0, B^0, \underline{x}(T), \eta(T), \rho(T)) = L(\underline{x}(T), T) \quad (4.23)$$

where $A^0, B^0, \eta(t), \rho(t)$ are determined by the necessary conditions (4.11) to (4.18), and where the scalar function S is defined by the equation

$$S(\underline{x}, A, B, \eta, \rho) = I_t + I'_x f + g \quad (4.24)$$

The prime in (4.24) denotes the transpose, and I_t and I'_x represent the partial derivatives with respect to t and $\underline{x}$, respectively.

Proof of the above theorem follows from that of Rhodes [36] and is given in Appendix A. The main difficulty in specific problems lies in the selection of the scalar function I . For linear-quadratic games with gain constraints, a suitable I function is given next.

4.5 APPLICATION TO LINEAR QUADRATIC GAMES

NECESSARY CONDITIONS

Consider a linear dynamic system given by (2.42) and a cost functional given by (2.43) with the associated weighting matrices having the properties described in Section 2.4. Following the procedure outlined in Section 2.4, it can be shown that the necessary conditions are

$$\int_{t_0}^T \alpha_k [R_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_1' P] \underline{x} \underline{x}' dt = [0], k = 1, 2, \dots, m_1 \quad (4.25)$$

$$\int_{t_0}^T \beta_k [R_2 \sum_{j=1}^{m_2} \beta_j B^j + G_2' P] \underline{x} \underline{x}' dt = [0], k = 1, 2, \dots, m_2 \quad (4.26)$$

$$\underline{n}_k^j(T) = \underline{n}_k^j(t_0) \quad \begin{matrix} j = 1, 2, \dots, m_1 \\ k = 1, 2, \dots, n_1 \end{matrix} \quad (4.27)$$

$$\underline{\rho}_k^j(T) = \underline{\rho}_k^j(t_0) \quad \begin{matrix} j = 1, 2, \dots, m_2 \\ k = 1, 2, \dots, n_2 \end{matrix} \quad (4.28)$$

$$\begin{aligned} P + P(F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j B^j) + (F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j B^j)' P + \\ (\sum_{j=1}^{m_1} \alpha_j A^j)' R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) + (\sum_{j=1}^{m_2} \beta_j B^j)' R_2 (\sum_{j=1}^{m_2} \beta_j B^j) + Q = [0] \end{aligned} \quad (4.29)$$

with the boundary condition $P(T) = S$. After some algebraic manipulations, the optimal cost J^0 can be shown to be equal to

$$J^0 = \underline{x}'(t_0) P(t_0) \underline{x}(t_0) \quad (4.30)$$

SUFFICIENT CONDITIONS

The sufficient conditions of Section 4.4 can now be applied to the above problem. Consider

$$\begin{aligned} I(A, B, \underline{x}, \eta, \rho) = \underline{x}'(t) P(t) \underline{x}(t) + \sum_{k=1}^n \sum_{j=1}^{m_1} \underline{n}_k^j (a_k^j - (a_k^j)^0) + \\ \sum_{k=1}^n \sum_{j=1}^{m_2} \underline{\rho}_k^j (b_k^j - (b_k^j)^0) \end{aligned} \quad (4.31)$$

Therefore,

$$\begin{aligned}
 S(\underline{x}, A, B^0, \eta, \rho) = & \underline{x}' \dot{P} \underline{x} + \sum_{k=1}^n \sum_{j=1}^{m_1} \dot{\bar{a}}_k^j (\underline{a}_k^j - (\underline{a}_k^j)^0) + \sum_{k=1}^n \sum_{j=1}^{m_2} \dot{\bar{b}}_k^j ((\underline{b}_k^j)^0 - \\
 & (\underline{b}_k^j)^0) + \underline{x}' P [F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j (B^j)^0] \underline{x} + \underline{x}' [F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + \\
 & G_2 \sum_{j=1}^{m_2} \beta_j (B^j)^0] ' P \underline{x} + \underline{x}' (\sum_{j=1}^{m_1} \alpha_j A^j)' R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) \underline{x} + \\
 & \underline{x}' (\sum_{j=1}^{m_2} \beta_j (B^j)^0)' R_2 (\sum_{j=1}^{m_2} \beta_j (B^j)^0) \underline{x} + \underline{x}' Q \underline{x}
 \end{aligned}$$

or

$$\begin{aligned}
 S(\underline{x}, A, B^0, \eta, \rho) = & \underline{x}' [\dot{P} + P(F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j (B^j)^0) + \\
 & (F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j (B^j)^0)' P + (\sum_{j=1}^{m_1} \alpha_j A^j)' R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) + \\
 & (\sum_{j=1}^{m_2} \beta_j (B^j)^0)' R_2 (\sum_{j=1}^{m_2} \beta_j (B^j)^0) + Q] \underline{x} + \sum_{k=1}^n \sum_{j=1}^{m_1} \dot{\bar{a}}_k^j (\underline{a}_k^j - (\underline{a}_k^j)^0) \quad (4.32)
 \end{aligned}$$

It is evident that the minimization of S with respect to $\underline{a}_k^j$ gives rise to the necessary condition (4.25); and from (4.29) and (4.32), the minimum of S is equal to zero.

Similarly it can be shown that the sufficient conditions (4.21) and (4.22) are also satisfied by the necessary conditions. Finally, the scalar function $I(\underline{x}, A^0, B^0, \eta, \rho)$ reduces to

$$I(\underline{x}, A^0, B^0, \eta, \rho) = \underline{x}' P \underline{x} \quad (4.33)$$

Sufficient condition (4.23) is then satisfied by the boundary condition $P(T) = S$.

Hence the existence of a $P(t)$ satisfying condition (4.29) is a sufficient condition for the existence of a saddle point.

4.6 AVERAGING OVER INITIAL CONDITIONS

Conditions (4.25) and (4.26) suggest a scheme for finding controller parameters that do not depend upon the initial conditions. The problem is to find or characterize, if they exist, A^* and B^* which are optimal in the sense that for any other sets of gain matrices A and B there holds

$$E_1[J(A^*, B)] \leq E_1[J(A^*, B^*)] \quad (4.34)$$

and

$$E_2[J(A^*, B^*)] \leq E_2[J(A, B^*)] \quad (4.35)$$

where the expectation operators E_1 and E_2 are taken over distributed initial states. The players will try to satisfy the conditions (4.25) and (4.26) in an average sense.

Now

$$\underline{x}(t) = \phi(t, t_0) \underline{x}(t_0) = \phi(t, t_0) \underline{x}_0 \quad (4.36)$$

where $\phi(t, t_0)$ is the transition matrix corresponding to

$$(F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j B^j).$$

Therefore conditions (4.25) and (4.26) reduce to

$$\int_{t_0}^T \alpha_k [R_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_1' P] \phi \underline{x}_0 \underline{x}_0' \phi' dt = [0] \quad (4.37)$$

$$\int_{t_0}^T \beta_k [R_2 \sum_{j=1}^{m_2} \beta_j B^j + G_2' P] \phi \underline{x}_0 \underline{x}_0' \phi' dt = [0] \quad (4.38)$$

It is evident from (4.37) and (4.38) that, in general, the constant gain matrices $(A^j)'$'s and $(B^j)'$'s do depend on the initial conditions.

Assume that $\underline{x}_0$ is a random vector and further assume that

$$E(\underline{x}_{i0} \underline{x}_{j0}') = K_{ij} E(\underline{x}_{10}^2) \quad (4.39)$$

If we then require that (4.37) and (4.38) be satisfied in an expected value sense,

$$\int_{t_0}^T \alpha_k [R_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_1' P] \phi K \phi' dt = [0] \quad (4.40)$$

and

$$\int_{t_0}^T \beta_k [R_2 \sum_{j=1}^{m_2} \beta_j B^j + G_2' P] \phi K \phi' dt = [0] \quad (4.41)$$

where K is the $(n \times n)$ matrix $[K_{ij}]$ and where the K_{ij} 's depend upon the advance information available to the players. For example, if uniformly distributed initial states apply, and if the components are independent, all K_{ii} 's are equal and all K_{ij} 's, $i \neq j$, are zero. But if $x_{10} = 4x_{20}$, for example, then $K_{22} = (1/16)K_{11}$ and $K_{12} = K_{21} = (1/4)K_{11}$. In this way partial information about the possible initial conditions can be taken into account in the design of the controller.

Note that if the ratios x_{10}/x_{i0} , x_{20}/x_{i0} , ..., x_{n0}/x_{i0} are known for some nonzero x_{i0} , equations (4.37) and (4.38) can be solved without knowledge of specific initial values.

A separate derivation of the necessary conditions (4.29), (4.40), and (4.41) starting with the problem of finding the expected value of the performance index is given in Appendix B.

Because equation (4.29) still has to be satisfied by the A^* 's and B^* 's which do not depend on the initial conditions, it is clear that the sufficient conditions of Section 4.4 are satisfied in an expected sense, the initial condition being treated as a random variable.

The value of J obtained by using these suboptimal A^* 's and B^* 's, after some algebraic manipulations is

$$J = \underline{x}_0^T P(t_0) \underline{x}_0 \quad (4.42)$$

and the expected value of J is

$$\hat{J} = \text{tr}(P(t_0)K) \quad (4.43)$$

where

$$K = E(\underline{x}(t_0)\underline{x}^T(t_0)) \quad (4.44)$$

Of course, each player could use a different K in the design of his controller, and the one coming closest to the true K would benefit in average performance. Equation (4.34) is the pessimistic viewpoint of the minimizing player and equation (4.35) is the pessimistic viewpoint of the maximizing player.

From equations (4.37) and (4.38), note that the optimal constant feedback gains for a scalar case never depend upon the initial state, even though the optimal performance index does depend upon the initial state; namely,

$$J_{\text{optimal}} = \underline{x}_0^T P(t_0) \underline{x}_0 \quad (4.45)$$

4.7. CONVERGENCE OF SUBOPTIMAL CONTROLS WHEN $m_1 \rightarrow \infty$ and $m_2 \rightarrow \infty$

Consider the Hilbert space $L_2[t_0, T]$ which consists of those real-valued functions of t in the interval $[t_0, T]$ for which $|f(t)|$ is Lebesgue integrable and in which the inner product is defined as

$$(x/y) \triangleq \int_{t_0}^T x(t) y(t) dt \quad (4.46)$$

and assume that there exists an $f(t) \in L_2[t_0, T]$ orthogonal to every $\alpha_j(t)$. This means

$$\int_{t_0}^T \alpha_j(t) f(t) dt = 0 \quad j = 1, 2, \dots, \infty \quad (4.47)$$

Assume that the $\alpha_j(t)$'s, $j = 1, 2, \dots$ are independent. By the Gram-Schmidt orthogonalization procedure [31], one can find an orthonormal sequence $\{e_j\}$ such that for each n , the space generated by the first n e_j 's is the same as the space generated by the first n α_j 's, i.e.,

$$e_k = \sum_{j=1}^k c_{kj} \alpha_j(t) \quad (4.48)$$

Consider

$$\int_{t_0}^T e_k(t) f(t) dt = \sum_{j=1}^k c_{kj} \int_{t_0}^T \alpha_j(t) f(t) dt = 0$$

i.e.,

$$\int_{t_0}^T e_k(t) f(t) dt = 0 \quad (4.49)$$

where e_k 's, $k = 1, 2, \dots$ form an orthonormal infinite sequence.

Assume further that the sequence $\{e_k(t)\}$ is complete.

Definition: An orthonormal sequence $\{e_i\}$ in a Hilbert space H is said to be complete if the closed subspace generated by the e_i 's is H .

It is well known [31] that an orthonormal sequence $\{e_k\}$ in a Hilbert space H is complete if and only if the only vector orthogonal to each of the e_i 's is the null vector, i.e., $f(t) = 0$ almost everywhere. If $f(t)$ is continuous, however, then

$$f(t) \equiv 0 \quad (4.50)$$

Considering the necessary conditions (4.37) and (4.38) or (4.40) and (4.41), it is evident that condition (4.50) is applicable component-wise. Therefore,

$$R_1 \sum_{j=1}^{\infty} \alpha_j A_j + G_1' P = [0] \quad (4.51)$$

and similarly

$$R_2 \sum_{j=1}^{\infty} \beta_j B_j + G_2' P = [0] \quad (4.52)$$

or

$$\sum_{j=1}^{\infty} \alpha_j A^j = -R_1^{-1} G_1' P \quad (4.53)$$

and

$$\sum_{j=1}^{\infty} \beta_j B^j = -R_2^{-1} G_2' P \quad (4.54)$$

Substituting (4.53) and (4.54) into (4.29) we obtain

$$\dot{P} + PF + F'P - P(G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2')P + Q = [0] \quad (4.55)$$

with the boundary condition $P(T) = S$. Therefore when $m_1 = \infty$ and $m_2 = \infty$, the problem solution essentially reduces to the unconstrained gain solution.

Consider

$$\int_{t_0}^T \alpha_k(t) f_{m_1}(t) dt = 0 \quad \begin{array}{l} k = 1, 2, \dots, m_1 \\ m_1 = 1, 2, 3, \dots \end{array} \quad (4.56)$$

where we have considered a finite sequence instead of the infinite sequence and let $m_1 \rightarrow \infty$. Assume that

$$\lim_{m_1 \rightarrow \infty} f_{m_1}(t) = g(t) \quad (4.57)$$

i.e., $f_{m_1}(t)$ uniformly converges to $g(t)$ as $m_1 \rightarrow \infty$. Then from (4.56)

$$\int_{t_0}^T \alpha_k(t) (f_{m_1}(t) - g(t)) dt = - \int_{t_0}^T \alpha_k(t) g(t) dt \quad (4.58)$$

$$k = 1, 2, \dots, m_1$$

For each k let $m_1 \rightarrow \infty$ to get

$$\int_{t_0}^T \alpha_k(t) \lim_{m_1 \rightarrow \infty} (f_{m_1}(t) - g(t)) dt = 0 = - \int_{t_0}^T \alpha_k(t) g(t) dt \quad (4.59)$$

$$k = 1, 2, \dots$$

Hence $g(t) = 0$ almost everywhere. Therefore, it is clear that if $m_1 \rightarrow \infty$ and $m_2 \rightarrow \infty$ the suboptimal control laws essentially converge to the truly optimal control laws in the linear-quadratic game.

4.8 u SUBOPTIMAL AND v TRULY OPTIMAL CASE

Variational procedures are employed in Section 2.7 to yield the necessary conditions when each player is restricted to use piecewise constant gains only during a certain specified interval of time. Similar procedures can be applied to the problem of this chapter. Consider, for example, the control diagram of Figure 4.1(a) which indicates that the minimizing player uses suboptimal control throughout the duration of play and the maximizing player uses truly optimal control.

Following the procedure outlined in Section 2.7, it can be shown that the necessary conditions are

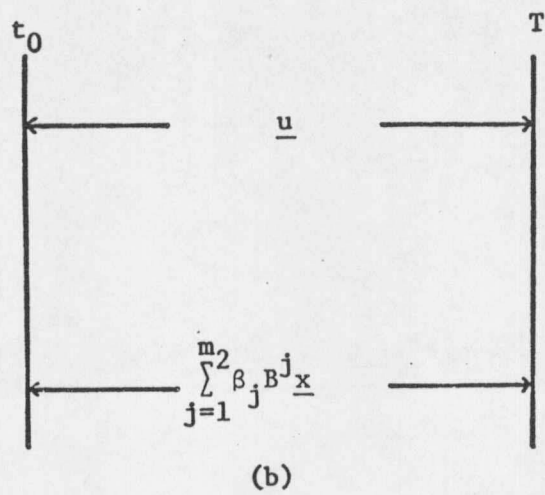
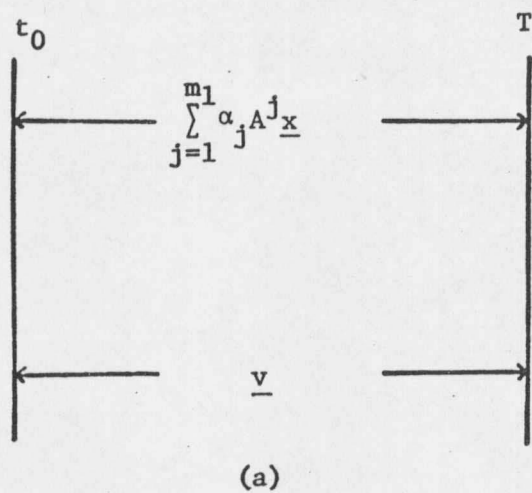


Figure 4.1. Control diagrams.

$$\int_{t_0}^T \frac{\partial H}{\partial a_k^j} dt = 0 \quad \begin{matrix} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_1 \end{matrix} \quad (4.60)$$

$$\frac{\partial H}{\partial v} = 0 \quad (4.61)$$

$$\underline{\lambda}(T) = \left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} \quad (4.62)$$

$$\underline{n}_k^j(T) = \underline{n}_k^j(t_0) \quad \begin{matrix} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_1 \end{matrix} \quad (4.63)$$

where $\underline{\lambda}$ and $\underline{n}_k^j(t)$'s are vector Lagrange multipliers that account for the constraints

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \sum_{j=1}^{m_1} \alpha_j A^j \underline{x}, \underline{v}, t) \quad (4.64)$$

and

$$\dot{\underline{a}}_k^j = 0 \quad \begin{matrix} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_1 \end{matrix} \quad (4.65)$$

Similar results apply to the case of Figure 4.1(b) where the maximizing player uses suboptimal control and the minimizing player uses truly optimal control.

APPLICATION TO LINEAR-QUADRATIC GAMES:

Consider a linear dynamic system given by (2.42) and the cost function (2.43) with the associated weighting matrices having the properties described in Section 2.4 and with the control diagram of Figure 4.1(a).

Following Sections 4.5 and 4.6, it can be shown that the necessary conditions are

$$\int_{t_0}^T \alpha_i [R_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_1' P] \underline{xx}' dt = [0] \quad i = 1, 2, \dots, m_1 \quad (4.66)$$

$$\underline{v} = -R_2^{-1} G_2' P \underline{x} \quad (4.67)$$

where $P(t)$ satisfies the differential equation

$$\begin{aligned} \dot{P} + P(F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j) + (F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j)' P - P G_2 R_2^{-1} G_2' P + \\ (\sum_{j=1}^{m_1} \alpha_j A^j)' R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) + Q = [0] \end{aligned} \quad (4.68)$$

with the boundary conditions $P(T) = S$.

If the first player is known to select his gains to be independent of initial conditions, condition (4.66) reduces to

$$\int_{t_0}^T \alpha_i [R_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_1' P] \phi K \phi' dt = [0] \quad i = 1, 2, \dots, m_1 \quad (4.69)$$

Again following the development of Section 4.5, it is evident that the existence of a P satisfying differential equation (4.68) with the boundary condition $P(T) = S$ is a sufficient condition for the existence of a saddle point.

4.9 NUMERICAL EXAMPLES

A. SCALAR CASE

Consider the case of a linear time-invariant system governed by

$$\dot{x} = -0.5x + 1.25u + 1.5v, \quad x_0 = 2 \quad (4.70)$$

The quadratic cost functional is

$$J = x(T)^2 S + \int_0^{0.25} (u^2 - 4v^2 + 2x^2) dt \quad (4.71)$$

By assuming different values to the $\alpha_j(t)$'s and $\beta_j(t)$'s, we can have different suboptimal controls for each player. Figure 4.2 shows the variation of performance index when various types of suboptimal controls are employed. It is seen that the variation of the performance index from the truly optimal saddle-point value is greatly reduced if we assume $u = (A_1 + A_2 t)x$ and $v = (B_1 + B_2 t)x$. The state of the system, the gains, and the controls for each player with different suboptimal controls are illustrated in Figures 4.3, 4.4, 4.5, 4.6, and 4.7.

B. VECTOR CASE:

Consider the case of a linear time-invariant system governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v \quad (4.72)$$

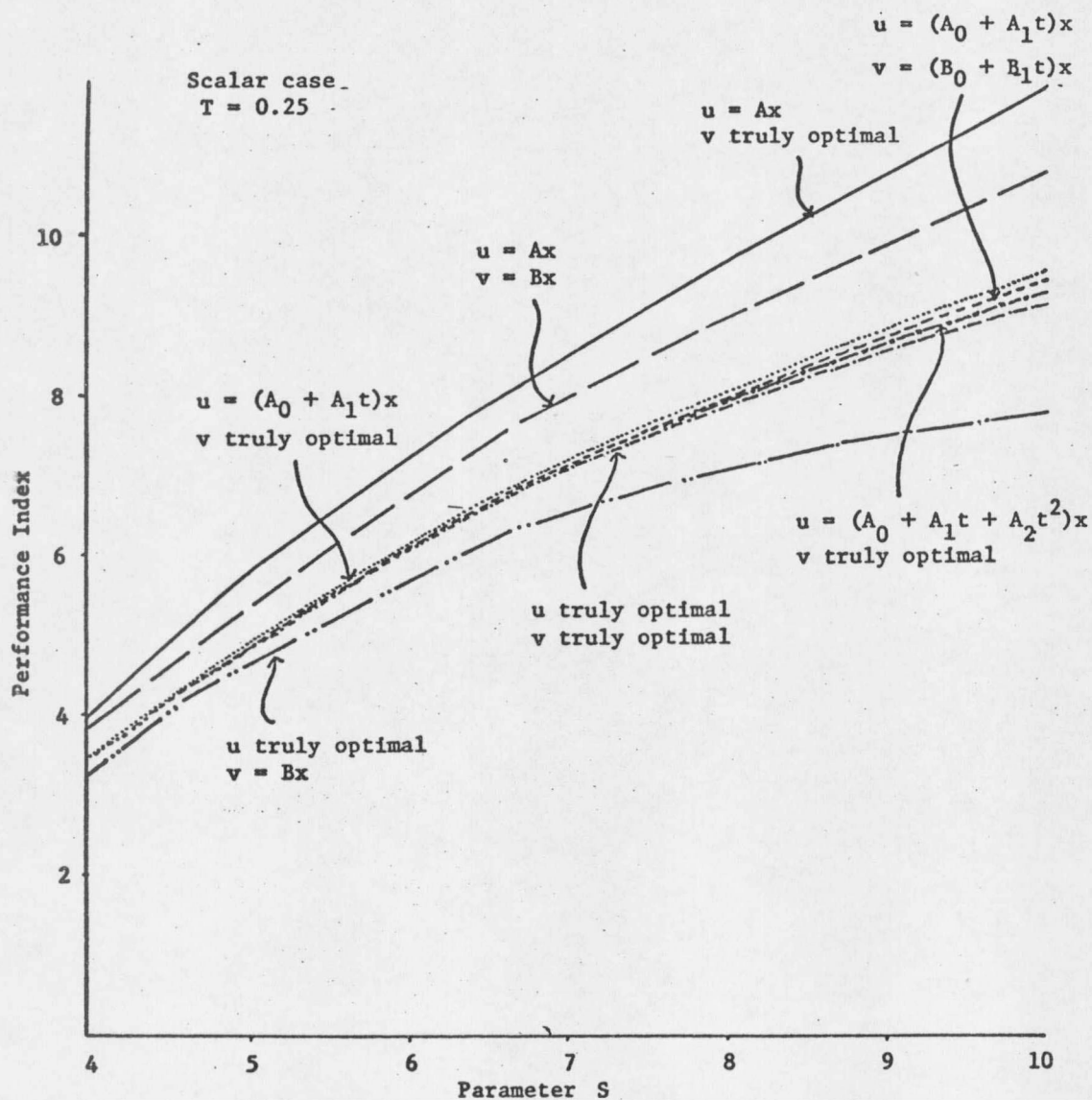


Figure 4.2. Variation of performance index.

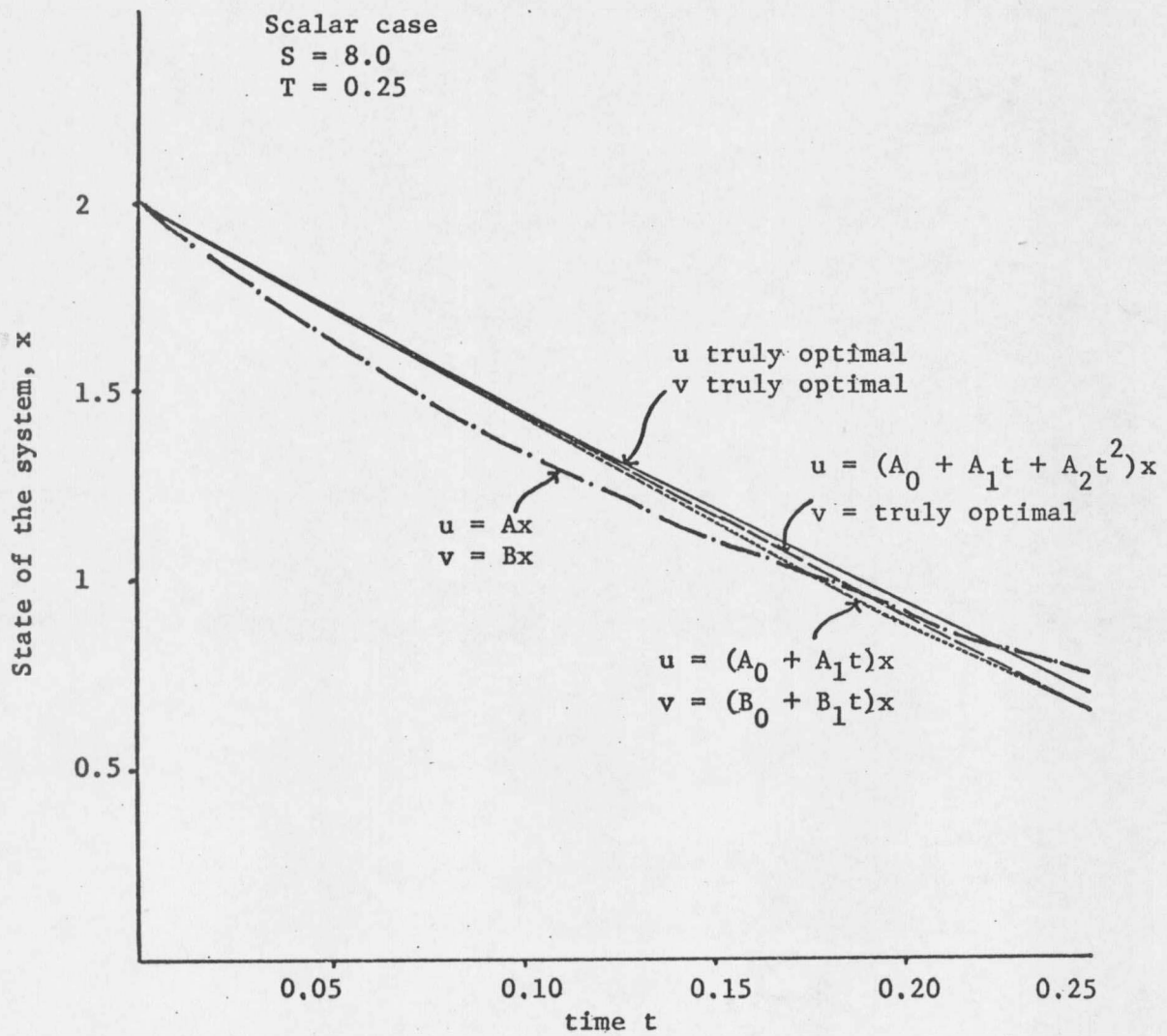


Figure 4.3. The state of the system.

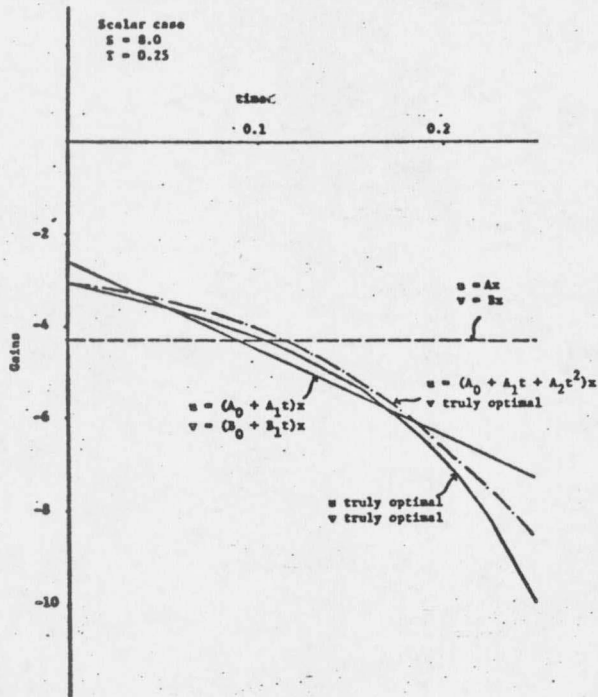


Figure 4.4 Gains for the minimizing player.

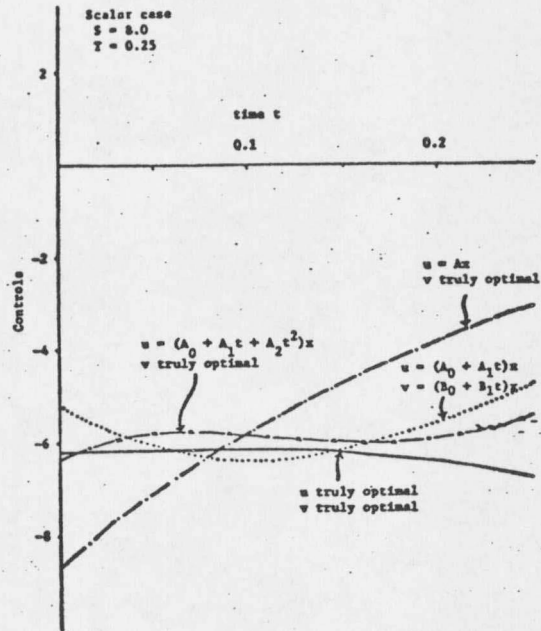


Figure 4.5 Controls for the minimizing player.

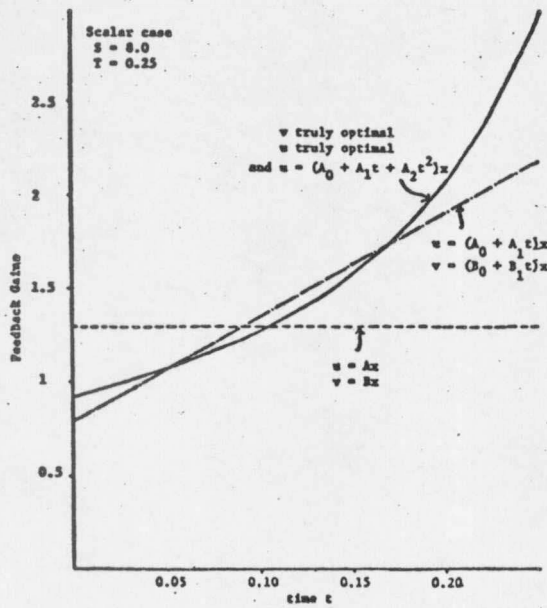


Figure 4.6 Gains for the maximizing player

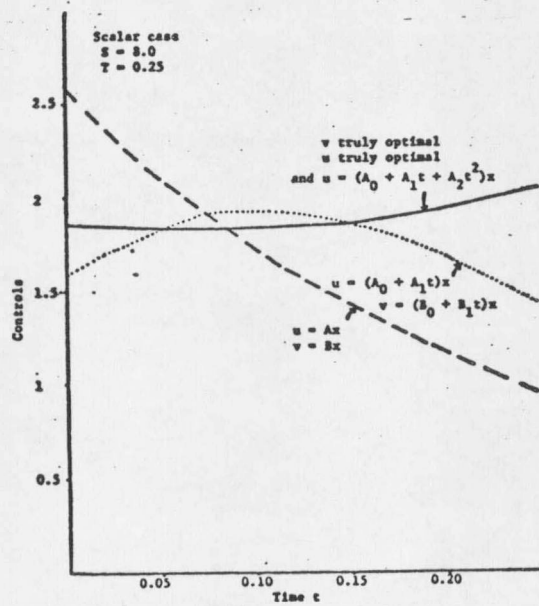


Figure 4.7 Controls for the maximizing player.

where u and v are scalar controls. The quadratic cost functional is

$$\underline{x}'(T) \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \underline{x}(T) + \int_0^T (x_1^2 + 0.5u^2 - 4v^2) dt \quad (4.73)$$

First it is assumed that the initial conditions are known to be

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 2.0 \\ 0.5 \end{bmatrix} \quad (4.74)$$

The optimal feedback gains and the performance index are computed.

The details of related computations are outlined in Appendix C.

Next the feedback gains that are independent of initial conditions are computed, as also are the average performance index and the actual performance index resulting from use of these feedback gains when the initial conditions are given by equation (4.74). In computing the above feedback gains, it is assumed that $E(x_{10}^2) = E(x_{20}^2)$ and that x_{10} and x_{20} are independent.

It is then assumed that partial information about the initial conditions are known--two cases are considered. In the first case it is assumed that $E(x_{10}^2) = 16E(x_{20}^2)$, and in the second case $x_{10} = 4x_{20}$.

It is evident that the matrix K will in the first case be equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$ and in the second case be equal to $\begin{bmatrix} 1 & 1/4 \\ 1/4 & 1/16 \end{bmatrix}$. It is

to be noted that the optimal parameters in the second case coincide with those when the specific initial conditions are known, and satisfy

$$x_{10} = 4x_{20}.$$

The above computational scheme is repeated for $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} -2.0 \\ 0.5 \end{bmatrix}$.

The results are tabulated in Table 1. It is to be noted that in calculating the average performance index from equation (4.43), K is normalized to have $E(x_{10}^2) = 1$. In other words, the normalized average performance index $\hat{J}_N$ given in Table 4-1 is to be multiplied by $E(x_{10}^2)$ to get the specific average performance index.

This example particularly illustrates the advantage of using the constant feedback gains since the variation of the performance index from the truly optimal saddle point value is nominal when constant feedback gains are employed. Table 4-1 clearly indicates the nature of the solutions.

4.10. CONCLUSION

Necessary and sufficient conditions for the existence of a saddle point when one or both the players use suboptimal controls have been examined. In the special case of linear-quadratic games, it has been shown that the optimal controller parameters for a given initial condition $\underline{x}_0$ are also valid for initial conditions $\alpha \underline{x}_0$, $-\infty < \alpha < \infty$. Consideration has also been given to the optimal choice of the parameters when only partial information about possible initial states is available to the players. It is evident that cooperation and negotiation are possible in zero-sum games from the point of view of control implementation.

TABLE 4-1

RESULTS FOR A VECTOR CASE
(Constant Feedback Gains)

Case #	Gain Conditions	Design Information	Actual Initial Conditions	Optimal Gains	Performance Index
1	u truly optimal v truly optimal	--	$x_{10} = 2.0$ $x_{20} = 0.5$	$\underline{A}(t) = -R_1^{-1}G_1'P(t)$ $\underline{B}(t) = -R_2^{-1}G_2'P(t)$ $P(t)$ from equation (4.55)	$J = 14.0195$ $\hat{J}_N = 3.5049$
2	$u = \underline{A}x$ $v = \underline{B}x$	Initial conditions included in the design	"	$\underline{A} = [-3.4825 \ -4.3371]$ $\underline{B} = [\ .8706 \ 1.0843]$	$J = 14.0867$ $\hat{J}_N = 3.5217$
3	$u = \underline{A}x$ $v = \underline{B}x$	$E(x_{10}^2) = E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	"	$\underline{A} = [-3.2786 \ -3.7776]$ $\underline{B} = [\ .8196 \ .9444]$	$J = 14.1215$ $\hat{J}_N = 4.4235$
4	$u = \underline{A}x$ $v = \underline{B}x$	$E(x_{10}^2) = 16E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	"	$\underline{A} = [-3.6166 \ -4.3788]$ $\underline{B} = [\ .9041 \ 1.0947]$	$J = 14.0988$ $\hat{J}_N = 2.6983$
5	$u = \underline{A}x$ $v = \underline{B}x$	$x_{10} = 4x_{20}$	"	Same as #2	Same as #2

(table continued)

TABLE 4-1 (continued)

Case #	Gain Conditions	Design Information	Actual Initial Conditions	Optimal Gains	Performance Index
6	u truly optimal v truly optimal	--	$x_{10} = -2.0$ $x_{20} = 0.5$	$\underline{A}(t) = -R_1^{-1} G_1' P(t)$ $\underline{B}(t) = -R_2^{-1} G_2' P(t)$ $P(t)$ from equation (4.55)	$J = 7.4371$ $\hat{J}_N = 1.8593$
7	$u = \underline{Ax}$ $v = \underline{Bx}$	Initial conditions included in the design	$x_{10} = -2.0$ $x_{20} = 0.5$	$\underline{A} = [-3.766 \ -4.3951]$ $\underline{B} = [\ .9415 \ 1.0988]$	$J = 7.4810$ $\hat{J}_N = 1.8702$
8	$u = \underline{Ax}$ $v = \underline{Bx}$	$E(x_{10}^2) = E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	"	Same as #3	$J = 7.5047$ $\hat{J}_N = 4.4235$
9	$u = \underline{Ax}$ $v = \underline{Bx}$	$E(x_{10}^2) = 16E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	"	Same as #4	$J = 7.4876$ $\hat{J}_N = 2.6983$
10	$u = \underline{Ax}$ $v = \underline{Bx}$	$x_{10} = -4x_{20}$	"	Same as #7	Same as #7

CHAPTER 5

SOME CONSIDERATIONS REGARDING ADVANTAGEOUS STRATEGIES

5.1 INTRODUCTION

In this chapter sufficient conditions for advantageous strategies for either player are examined for a linear-quadratic game in which the players are constrained to use suboptimal control laws as specified by equations (4.2) and (4.3). The concept of a "bargaining matrix" is introduced, and numerical results to illustrate the theory are included.

5.2 SUBOPTIMAL COSTS

It is shown in Chapters 2, 3, and 4 that the optimal performance index for linear-quadratic games depends on the initial state vector and on the initial value of the matrix P , which satisfies the matrix Riccati equation (4.55) for nonconstant feedback gains and (4.29) for suboptimal controls given by (4.2) and (4.3). In this section, an expression for the difference of $P(t_0)$'s is derived when the players use two different suboptimal controls. The procedure parallels the one adopted by Kleinman and Athans [26] for the corresponding one-sided optimal control problem.

Consider a linear dynamic system given by (2.42) and a cost functional given by (2.43) with the associated weighting matrices having the properties described in Section 2.4. The two suboptimal gains to be considered are defined as follows.

$$A_1 = \sum_{j=1}^{m_1} \alpha_j(t) A_1^j \quad (5.1)$$

$$A_2 = \sum_{j=1}^{s_1} v_j(t) A_2^j \quad (5.2)$$

$$B_1 = \sum_{j=1}^{m_2} \beta_j(t) B_1^j \quad (5.3)$$

$$B_2 = \sum_{j=1}^{s_2} \delta_j(t) B_2^j \quad (5.4)$$

Let

$$F_{N1} = F + G_1 A_1 + G_2 B_1 \quad (5.5)$$

$$F_{N2} = F + G_1 A_2 + G_2 B_2 \quad (5.6)$$

It is evident from the foregoing chapters that the following relations are to be satisfied for optimality.

$$\dot{P}_1 + P_1 F_{N1} + F_{N1}' P_1 + A_1' R_1 A_1 + B_1' R_2 B_1 + Q = [0] \quad (5.7)$$

with the boundary condition $P(T) = S$, and

$$\dot{P}_2 + P_2 F_{N2} + F_{N2}' P_2 + A_2' R_2 A_2 + B_2' R_2 B_2 + Q = [0] \quad (5.8)$$

with the boundary condition $P(T) = S$. Therefore,

$$\dot{P}_1 = -P_1 F_{N1} - F_{N1}' P_1 - A_1' R_1 A_1 - B_1' R_2 B_1 - Q \quad (5.9)$$

From equations (5.5) and (5.6),

$$F_{N1} = F_{N2} + G_1 (A_1 - A_2) + G_2 (B_1 - B_2)$$

$$F'_{N1} P_1 = ((A_1 - A_2)' G'_1 + (B_1 - B_2)' G'_2) P_1 + F'_{N2} P_1 \quad (5.10)$$

and

$$P_1 F'_{N1} = P_1 F'_{N2} + P_1 [G_1 (A_1 - A_2) + G_2 (B_1 - B_2)] \quad (5.11)$$

Substituting (5.10) and (5.11) in (5.9),

$$\begin{aligned} \dot{P}_1 = & -F'_{N2} P_1 - P_1 F'_{N2} - A'_1 R_1 A_1 - B'_1 R_2 B_1 - Q \\ & - [(A_1 - A_2)' G'_1 + (B_1 - B_2)' G'_2] P_1 \\ & - P_1 [G_1 (A_1 - A_2) + G_2 (B_1 - B_2)] \end{aligned} \quad (5.12)$$

From equation (5.8),

$$\dot{P}_2 = -F'_{N2} P_2 - P_2 F'_{N2} - A'_2 R_1 A_2 - B'_2 R_2 B_2 - Q \quad (5.13)$$

Subtracting (5.13) from (5.12),

$$\begin{aligned} \dot{P}_1 - \dot{P}_2 = \delta \dot{P}(t) = & -F'_{N2} (P_1 - P_2) - (P_1 - P_2) F'_{N2} + A'_2 R_1 A_2 - A'_1 R_1 A_1 \\ & + B'_2 R_2 B_2 - B'_1 R_2 B_1 - [(A_1 - A_2)' G'_1 + (B_1 - B_2)' G'_2] P_1 \\ & - P_1 [G_1 (A_1 - A_2) + G_2 (B_1 - B_2)] \end{aligned} \quad (5.14)$$

Equation (5.14) can be rewritten as

$$\begin{aligned} \delta \dot{P}(t) = & -F'_{N2} \delta P - \delta P F'_{N2} - (A_1 - A_2)' R_1 (A_1 - A_2) \\ & - (B_1 - B_2)' R_2 (B_1 - B_2) - (A_1 - A_2)' (G'_1 P_1 + R_1 A_2) + \end{aligned}$$

$$\begin{aligned}
& -(B_1 - B_2)'(G_2'P_1 + R_2B_2) - (G_1'P_1 + R_1A_2)'(A_1 - A_2) \\
& - (G_2'P_1 + R_2B_2)'(B_1 - B_2)
\end{aligned} \tag{5.15}$$

Because $P_1(T) = P_2(T) = S$,

$$\delta P(T) = [0] \tag{5.16}$$

It can be shown that the matrix differential equation

$$\dot{L} = -F'L - LF + GU \tag{5.17}$$

with boundary condition $L(T)$ has the solution

$$L(t) = \phi'(T, t)L(T)\phi(T, t) - \int_t^T \phi'(\tau, t)G(\tau)U(\tau)\phi(\tau, t) d\tau \tag{5.18}$$

where $\phi(t, t_0)$ is the transition matrix corresponding to $F(t)$. Therefore

$$\dot{\phi}(t, t_0) = F(t)\phi(t, t_0), \phi(t_0, t_0) = I \tag{5.19}$$

and

$$\dot{\phi}(t_0, t) = -\phi(t_0, t)F(t), \phi(t_0, t_0) = I \tag{5.20}$$

These results are used in equation (5.15) to obtain

$$\begin{aligned}
\delta P(t) = & \int_t^T \phi_2'(\tau, t) [(A_1 - A_2)'R_1(A_1 - A_2) \\
& + (B_1 - B_2)'R_2(B_1 - B_2) + (A_1 - A_2)'(G_1'P_1 + R_1A_2) \\
& + (B_1 - B_2)'(G_2'P_1 + R_2B_2) + (G_1'P_1 + R_1A_2)'(A_1 - A_2) \\
& + (G_2'P_1 + R_2B_2)'(B_1 - B_2)] \phi_2(\tau, t) d\tau
\end{aligned} \tag{5.21}$$

where $\phi_2(t, t_0)$ is the transition matrix corresponding to $F_{N2}(t)$.

Another expression for $\delta P(t)$ can be found by interchanging the subscripts in equation (5.21) and multiplying the resulting expression by -1.

$$\begin{aligned} \delta P(t) = & \int_t^T \phi_1'(\tau, t) [-(A_1 - A_2)' R_1 (A_1 - A_2) \\ & - (B_1 - B_2)' R_2 (B_1 - B_2) + (A_1 - A_2)' (G_1' P_2 + R_1 A_1) \\ & + (B_1 - B_2)' (G_2' P_2 + R_2 B_1) + (G_1' P_2 + R_1 A_1)' (A_1 - A_2) \\ & + (G_2' P_2 + R_2 B_1)' (B_1 - B_2)] \phi_1(\tau, t) d\tau \end{aligned} \quad (5.22)$$

Yet another form of $\delta P(t)$ which is used in Section 5.3 is obtained from the following identities:

$$G_1' P_2 + R_1 A_1 = (G_1' P_2 + R_1 A_2) + R_1 (A_1 - A_2)$$

$$G_2' P_2 + R_2 B_1 = (G_2' P_2 + R_2 B_2) + R_2 (B_1 - B_2)$$

Therefore,

$$(A_1 - A_2)' (G_1' P_2 + R_1 A_1) = (A_1 - A_2)' (G_1' P_2 + R_1 A_2) + (A_1 - A_2)' R_1 (A_1 - A_2) \quad (5.23)$$

$$(B_1 - B_2)' (G_2' P_2 + R_2 B_1) = (B_1 - B_2)' (G_2' P_2 + R_2 B_2) + (B_1 - B_2)' R_2 (B_1 - B_2) \quad (5.24)$$

$$(G_1' P_2 + R_1 A_1)' (A_1 - A_2) = (G_1' P_2 + R_1 A_2)' (A_1 - A_2) + (A_1 - A_2)' R_1 (A_1 - A_2) \quad (5.25)$$

$$(G_2'P_2 + R_2B_1)'(B_1 - B_2) = (G_2'P_2 + R_2B_2)'(B_1 - B_2) + (B_1 - B_2)'R_2(B_1 - B_2) \quad (5.26)$$

Substituting (5.23), (5.24), (5.25), and (5.26) in (5.22), the result is

$$\begin{aligned} \delta P(t) = & \int_t^T \phi_1'(\tau, t) [(A_1 - A_2)'R_1(A_1 - A_2) + (B_1 - B_2)'R_2(B_1 - B_2) \\ & + (A_1 - A_2)'(G_1'P_2 + R_1A_2) + (B_1 - B_2)'(G_2'P_2 + R_2B_2) + \\ & (G_1'P_2 + R_1A_2)'(A_1 - A_2) + (G_2'P_2 + R_2B_2)'(B_1 - B_2)] \phi_1(\tau, t) d\tau \end{aligned} \quad (5.27)$$

Equation (5.27) is a general expression for the difference in cost of implementing two different suboptimal gains. An expression for the difference between a suboptimal cost and the nonconstant saddle point pay-off is derived in the next section.

5.3 ADVANTAGEOUS STRATEGIES

If we assume that A_2 and B_2 correspond to optimal unconstrained controls and A_1 and B_1 correspond to suboptimal constrained-gain controls, then

$$A_2(t) = A_2^*(t) = -R_1^{-1}G_1'P_2 \quad (5.28)$$

$$B_2(t) = B_2^*(t) = -R_2^{-1}G_2'P_2 \quad (5.29)$$

where $P_2(t)$ satisfies (4.55).

Therefore,

$$G_1' P_2 + R_1 A_2 = [0] \quad (5.30)$$

$$G_2' P_2 + R_2 B_2 = [0] \quad (5.31)$$

with (5.30) and (5.31), (5.27) gives

$$\begin{aligned} \delta P(t) = & \int_t^T \phi_1'(\tau, t) [(A_1 - A_2^*)' R_1 (A_1 - A_2^*) \\ & + (B_1 - B_2^*)' R_2 (B_1 - B_2^*)] \phi_1(\tau, t) d\tau \end{aligned} \quad (5.33)$$

Whether the suboptimal controls A_1 and B_1 are advantageous to one player or the other with respect to the nonconstant saddle point pay-off is determined by (5.33): namely, if $\delta P(t_0)$ is greater than zero, then the suboptimal controls specified by equations (5.2) and (5.3) are advantageous to the maximizing player and vice versa.

It is evident from equation (5.33) that $\delta P(t_0) = 0$ if $A_1 = A_2^*$ and $B_1 = B_2^*$. But it is to be noted that because the integrand of equation (5.33) consists of a positive definite term and a negative definite term, the integral can become zero even if $A_1 \neq A_2^*$ and $B_1 \neq B_2^*$.

The above discussion can be illustrated by the following geometric argument. Let a denote the optimal saddle point pay-off, b the pay-off when the minimizing player uses suboptimal control while the maximizing

player uses optimal control, c the pay-off when the maximizing player uses suboptimal control while the minimizing player uses optimal control, and d the pay-off when both the players use sub-optimal control. Also, let $\underline{u}^*$ and $\underline{v}^*$ denote the truly optimal controls. Therefore,

$$J(\underline{u}^*, \underline{v}^*) = a$$

$$J(A_1 \underline{x}, \underline{v}^*) = b$$

$$J(\underline{u}^*, B_1 \underline{x}) = c$$

$$J(A_1 \underline{x}, B_1 \underline{x}) = d$$

It is evident from saddle point theory that

$$a \leq b$$

$$a \geq c$$

$$d \geq c$$

and

$$d \leq b$$

Therefore, d can lie anywhere between b and c and whether it is greater than a or less than a is actually determined by (5.33)

Sufficient conditions for advantageous strategies are derived next for the game with a time-invariant system and when the players use constant feedback gains.

TIME INVARIANT SYSTEM (CONSTANT FEEDBACK GAINS)Scalar Case

From Section 2.5 for a time-invariant system under the constraint of constant feedback gains,

$$A_1 = - \frac{G_1}{R_1} \frac{\int P_1 x^2 dt}{\int x^2 dt}$$

and

$$B_1 = - \frac{G_2}{R_2} \frac{\int P_1 x^2 dt}{\int x^2 dt}$$

from which

$$B_1 = \frac{A_1 G_2 R_1}{G_1 R_2} \quad (5.34)$$

Consider the term within brackets of (5.33):

$$\begin{aligned} & (A_1 - A_2^*)' R_1 (A_1 - A_2^*) + (B_1 - B_2^*)' R_2 (B_1 - B_2^*) \\ &= (A_1 - A_2^*)^2 R_1 + (B_1 - B_2^*)^2 R_2 \\ &= A_1^2 R_1 + B_1^2 R_2 - 2A_1 A_2^* R_1 - 2B_1 B_2^* R_2 + (A_2^*)^2 R_1 + (B_2^*)^2 R_2 \end{aligned} \quad (5.35)$$

in which

$$A_2^* = - \frac{G_1}{R_1} P_2$$

$$B_2^* = - \frac{G_2}{R_2} P_2$$

and the results of (5.34) are used to obtain

$$\frac{(G_1^2 R_2 + G_2^2 R_1)(AR_1 + P_2 G_1)^2}{G_1^2 R_1 R_2}$$

which is equivalent to the right-hand member of (5.35). Therefore,

$$\delta P(t_0) = \int_{t_0}^T \frac{\phi_1^2(\tau, t_0)(G_1^2 R_2 + G_2^2 R_1)(AR_1 + P_2 G_1)^2}{G_1^2 R_1 R_2} dt$$

Because $G_1^2 R_1 R_2$ is always negative, it is evident that constant feedback gains will be advantageous to the maximizing player if and only if $G_1^2 R_2 + G_2^2 R_1 < 0$ or $\frac{G_1^2}{R_1} + \frac{G_2^2}{R_2} > 0$ and vice versa for the minimizing player.

It is evident that the quantity $\frac{G_1^2}{R_1} + \frac{G_2^2}{R_2}$ would certainly be a deciding factor for the negotiated performance index in the case that the players bargain to adopt constant feedback gains.

VECTOR CASE

It is known from Sections 2.4 and 2.6 that for a time-invariant system

$$A_1 = -R_1^{-1} G_1' \xi(t_0, T) W^{-1}(t_0, T) \quad (5.36)$$

$$B_1 = -R_2^{-1} G_2' \xi(t_0, T) W^{-1}(t_0, T) \quad (5.37)$$

$$A_2^* = -R_1^{-1} G_1' P_2(t) \quad (5.38)$$

and

$$B_2^* = -R_2^{-1} G_2' P_2(t) \quad (5.39)$$

Substituting (5.36), (5.37), (5.38), and (5.39) in (5.33) and simplifying, it follows that

$$\delta P(t_0) = \int_{t_0}^T \phi_1'(\tau, t) (\xi W^{-1} - P_2)' (G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') (\xi W^{-1} - P_2) \phi_1(\tau, t) d\tau \quad (5.40)$$

Therefore, the maximizing player will be at an advantage if $(G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') > 0$ and vice versa for the minimizing player. If $(G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2')$ is indefinite, however, nothing can be said as to which player is benefited by the use of constant feedback gains unless the optimal gains are computed and substituted in equation (5.33). Conditions for which $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2'$ might become indefinite for a special case in which either $G_1 R_1^{-1} G_1'$ or $G_2 R_2^{-1} G_2'$ is nonsingular is given in the next section.

By restricting the information available to both players to be the same, partial information about the possible initial conditions can be accounted for by properly interpreting ξ and w matrices as done in Chapter 4.

It is clear that if $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2'$ equals zero, P_2 and P_1 satisfy the same differential equation

$$P + PF + F'P + Q = [0]$$

with the boundary condition $P(T) = S$ and the system equation reduces to

$$\dot{\underline{x}} = F\underline{x}$$

for the nonconstant case and also for the constant feedback case.

It is also clear that if the players decide to negotiate the use of constant feedback gains, the negotiations will certainly depend upon $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2'$, and hence the matrix $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2'$ may rightly be named as the bargaining matrix.

Care should be exercised in determining advantageous strategies, in regard to whether the problem is well posed in the saddle point sense; namely, whether there exists a closed-loop nonconstant saddle point solution. It is well known [1,41] that there exists a closed-loop saddle point solution for the unconstrained-gain linear-quadratic game under consideration if and only if

$$\det \left[I + \int_t^T c\psi(T,t) (G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') \psi'(T,t) c' dt \right] \neq 0 \quad (5.41)$$

$$t \in [t_0, T]$$

where $S = c'c$ and $\psi(t, t_0)$ is the transition matrix corresponding to F .

In other words, there exists a closed-loop saddle point solution if and only if $Q_i(t) \neq -1$ for all $t \in [t_0, T]$, where $Q_i(t)$ are the eigenvalues of the matrix

$$\int_t^T c\psi(T,t) (G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') \psi'(T,t) c' dt$$

If $(G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') > 0$, clearly each $Q_i(t) > 0$. Hence, a saddle point solution exists for the unconstrained-gain game whenever constant feedback gains are advantageous to the maximizing player. On the other hand, if $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2' < 0$, the pertinent saddle point solution exists only for a specified interval determined by (5.41). These facts are illustrated by numerical examples in Section 5.5.

5.4 INDEFINITENESS OF THE BARGAINING MATRIX

A test for the indefiniteness of the bargaining matrix is formulated for the special case when $G_1 R_1^{-1} G_1'$ or $G_2 R_2^{-1} G_2'$ is nonsingular. Without loss of generality, it is assumed that $G_1 R_1^{-1} G_1'$ is nonsingular. Then according to a theorem [33] concerned with the simultaneous reduction of two quadratic forms, there exists a nonsingular real matrix M such that

$$M' G_1 R_1^{-1} G_1' M = I \quad (5.42)$$

and

$$M' G_2 R_2^{-1} G_2' M = \text{diag}(r_1, r_2, \dots, r_n) \quad (5.43)$$

where for any choice of M , the quantities $r_1, r_2, \dots, r_n$ are necessarily the roots of the polynomial equation

$$|x G_1 R_1^{-1} G_1' - G_2 R_2^{-1} G_2'| = 0 \quad (5.44)$$

Equations (5.42) and (5.43) give

$$M' (G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2') M = \text{diag}(1 + r_1, 1 + r_2, \dots, 1 + r_n) \quad (5.45)$$

Because R_2 is negative definite, it is necessary that $r_i \leq 0$, $i = 1, 2, \dots, n$. Therefore, the negotiation matrix for the special case of $G_1 R_1^{-1} G_1'$ being nonsingular is indefinite if and only if there exist roots r_1, r_2 to the polynomial equation $|xG_1 R_1^{-1} G_1' - G_2 R_2^{-1} G_2'| = 0$ such that $|r_1| < 1 < |r_2|$.

5.5 NUMERICAL EXAMPLES

A. SCALAR CASE: ADVANTAGEOUS TO THE MAXIMIZING PLAYER

Consider the case of a linear time-invariant system governed by (4.70) and the cost functional (4.71). The bargaining matrix is

$$\frac{G_1^2}{R_1} + \frac{G_2^2}{R_2} = 1.0 > 0$$

Hence, for the specific scalar system considered, constant feedback gains are always advantageous to the maximizing player as is also evidenced by Figure 3.3. From Section 5.3, it is evident that unconstrained-gain saddle point solutions always exist for this specific game under consideration.

B. SCALAR CASE: ADVANTAGEOUS TO THE MINIMIZING PLAYER

Consider the case of a linear time-invariant system governed by

$$\dot{x} = -0.5x + u + 1.5v, \quad x_0 = 2.0 \quad (5.46)$$

The cost functional is

$$J = x^2(T) + \int_0^1 (u^2 - 2v^2 + x^2) dt \quad (5.47)$$

The bargaining matrix is

$$\frac{G_1^2}{R_1} + \frac{G_2^2}{R_2} = -.125 < 0$$

For the existence of an unconstrained saddle point solution we have from (5.41)

$$1 + \int_t^T \psi^2(T, t) \left(\frac{G_1^2}{R_1} + \frac{G_2^2}{R_2} \right) dt \neq 0 \quad \text{on } [t_0, T]$$

or

$$1 - .125 \int_t^T e^{-(T-t)} dt \neq 0 \quad \text{on } [t_0, T]$$

or

$$1 - .125(1 - e^{(t-T)}) \neq 0 \quad \text{on } [t_0, T] \quad (5.48)$$

Since $1 - e^{(t-T)} \geq 0$ and ≤ 1 , an unconstrained saddle point solution exists for all T . In this case constant feedback gains are advantageous to the minimizing player. The results are given below for $T = 1.0$.

Performance Index = 7.2727 without gain constraints;

Performance Index = 7.2532 with gain constraints;

Optimal Parameters: $A = -1.4939$ and $B = 1.1204$.

C. VECTOR CASE

Consider the case of a linear time-invariant system governed by equation (4.72) and the cost functional (4.73). The bargaining matrix is

$$G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2 \begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \geq 0$$

Hence, constant feedback gains for this specific example are always advantageous to the maximizing player as Table 4-1 clearly illustrates.

5.6 CONCLUSION

Consideration has been given to sufficient conditions for advantageous strategies in linear-quadratic games. This effort is by no means complete. It has been shown that for linear-quadratic time-invariant games, in general, a sufficient condition for constant feedback gains to be advantageous to the maximizing player is that the bargaining matrix $G_1 R_1^{-1} G_1' + G_2 R_2^{-1} G_2' > 0$. A matter for further investigation is the extension of the present approach to games in which partial information about the possible initial conditions is different for each player.

CHAPTER 6

DIFFERENTIAL GAMES WITH PIECEWISE SUBOPTIMAL CONTROLS

6.1 INTRODUCTION

In this chapter necessary and sufficient conditions are derived for the existence of a saddle point for a general two-person zero-sum differential game when one or both the players use suboptimal control laws of specified form. The specified forms for the controls consist of weighted sums of the state variables, the weighting factors being products of known time varying functions and of piecewise-constant functions to be determined in an optimal manner. To simplify terminology, the controls which are formed in this way will hereafter be referred to as piecewise control laws. The time intervals associated with the piecewise control laws can be different for each player. The general results are applied to linear-quadratic games; and for this class of differential games, an additional development is given to obtain piecewise control law parameters that are independent of initial conditions, so that a saddle point with respect to the expected value of the performance index is obtained. Consideration has also been given to the problem of optimizing the gain change points.

The results are applied to scalar and vector dynamic systems; and numerical solutions are presented.

In the case of one-sided optimal control theory, necessary conditions for computing piecewise-constant feedback parameters that are independent of initial conditions have been derived by Kleinman, et al. [27], but they did not consider optimizing the gain change points.

An extension of the algorithm presented in [27] to the case where the gain-change points are also optimized has been presented by Bertsekas [6] for the case of one-sided optimal control theory, but neither all the necessary conditions nor piecewise suboptimal controls have been considered in [6]. Moreover, the results in both [27] and [6] are not directly applicable when partial information about possible initial states is known.

6.2 FORMULATION OF THE PROBLEM

Consider a dynamic system given by (2.1) and a cost functional given by (2.2) with the associated weighting matrices having the properties described in Section 2.2. The problem is to find or characterize, if they exist, particular controls $\underline{u}^0$ and $\underline{v}^0$ which are optimal in the sense that for any other control vectors, $\underline{u}$ and $\underline{v}$, there holds $J(\underline{u}^0, \underline{v}) \leq J(\underline{u}^0, \underline{v}^0) \leq J(\underline{u}, \underline{v}^0)$.

In Section 6.3, controls $\underline{u}$ and $\underline{v}$ are constrained to be of the form given in the control diagram of Figure 6.1(a). Results for the more general case, Figure 6.1(b), are readily obtained on this basis and are listed in Section 6.4.

Constraints include

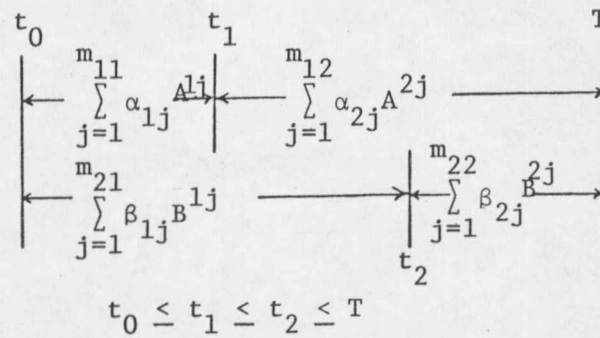
$$\dot{A}^{1j} = [0] \quad j = 1, 2, \dots, m_{11}$$

$$\dot{A}^{2j} = [0] \quad j = 1, 2, \dots, m_{12}$$

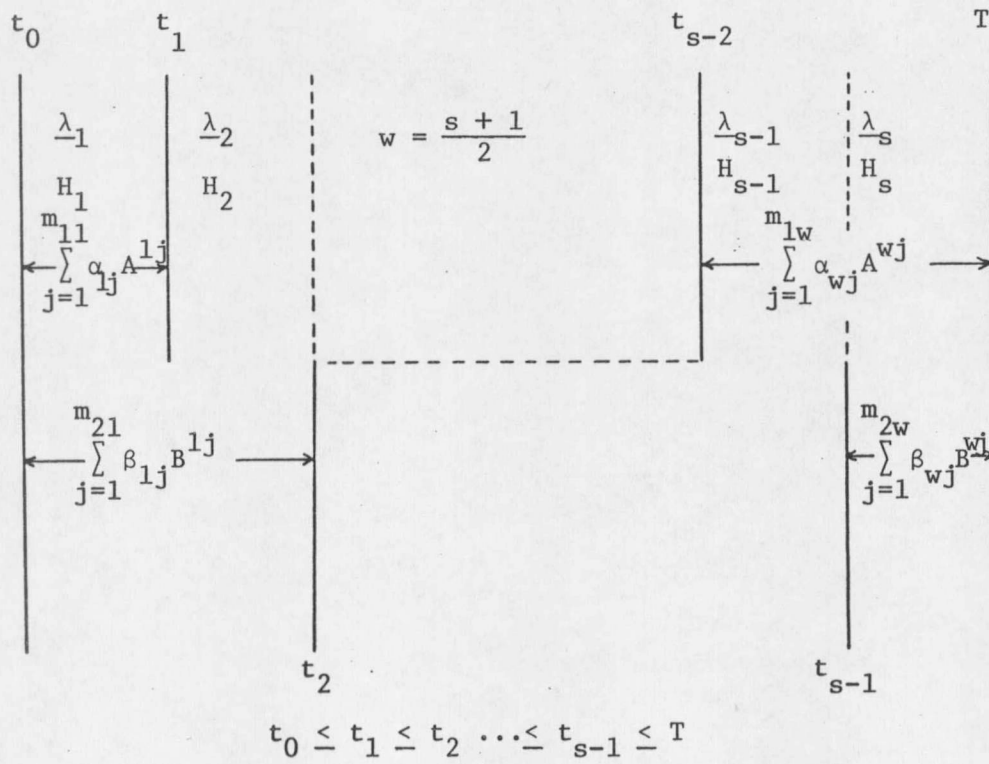
$$\dot{B}^{1j} = [0] \quad j = 1, 2, \dots, m_{21}$$

and

$$\dot{B}^{2j} = [0] \quad j = 1, 2, \dots, m_{22}$$



(a)



(b)

Figure 6.1. Control diagrams.

In general the optimal A^{1j}, s , A^{2j}, s , B^{1j}, s , B^{2j}, s are functions of the initial state $\underline{x}_0$. Section 6.6 is devoted to the case where the players desire to form control laws that are independent of the initial state: each player assumes some a priori distribution of possible initial states and satisfies conditions for optimality only in an expected value sense.

6.3 NECESSARY CONDITIONS

Variational procedures are applied to yield the necessary conditions for optimality. For convenience the $(p \times n)$ matrices A^{1j}, s , and A^{2j}, s and the $(q \times n)$ matrices B^{1j}, s and B^{2j}, s are partitioned in terms of columns: Let

$$A^{1j} = [\underline{a}_1^{1j} \quad \underline{a}_2^{1j} \quad \dots \quad \underline{a}_n^{1j}] \quad j = 1, 2, \dots, m_{11} \quad (6.2)$$

$$A^{2j} = [\underline{a}_1^{2j} \quad \underline{a}_2^{2j} \quad \dots \quad \underline{a}_n^{2j}] \quad j = 1, 2, \dots, m_{12} \quad (6.3)$$

$$B^{1j} = [\underline{b}_1^{1j} \quad \underline{b}_2^{1j} \quad \dots \quad \underline{b}_n^{1j}] \quad j = 1, 2, \dots, m_{21} \quad (6.4)$$

and

$$B^{2j} = [\underline{b}_1^{2j} \quad \underline{b}_2^{2j} \quad \dots \quad \underline{b}_n^{2j}] \quad j = 1, 2, \dots, m_{22} \quad (6.5)$$

where $\underline{a}_k^{ij}, s$ and $\underline{b}_k^{ij}, s$, $i = 1, 2$, are constant vectors.

For convenience let $\underline{x}_1$, $\underline{x}_2$, and $\underline{x}_3$ denote the state vector $\underline{x}$ of the system during the interval t_0 to t_1 , t_1 to t_2 , and t_2 to T , respectively, and let

$$g_1 = g(\underline{x}_1, \sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} \underline{x}_1, \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \underline{x}_2, t) \quad (6.6)$$

$$g_2 = g(\underline{x}_2, \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \underline{x}_2, \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \underline{x}_2, t) \quad (6.7)$$

$$g_3 = g(\underline{x}_3, \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \underline{x}_3, \sum_{j=1}^{m_{21}} \beta_{2j} B^{2j} \underline{x}_3, t) \quad (6.8)$$

Similar identities $\underline{f}_1, \underline{f}_2, \underline{f}_3$ are defined for $\underline{f}$. The augmented cost functional for the related minimum problem is

$$\begin{aligned} J_a = & L(\underline{x}(T), T) + \int_{t_0}^{t_1} [g_1 + \underline{\lambda}'_1(\underline{f}_1 - \dot{\underline{x}}_1) - \sum_{j=1}^{m_{11}} \sum_{k=1}^n \underline{n}_k^{1j} \dot{\underline{a}}_k^{1j}] dt + \\ & \int_{t_1}^{t_2} [g_2 + \underline{\lambda}'_2(\underline{f}_2 - \dot{\underline{x}}_2) - \sum_{j=1}^{m_{12}} \sum_{k=1}^n \underline{n}_k^{2j} \dot{\underline{a}}_k^{2j}] dt + \\ & \int_{t_2}^T [g_3 + \underline{\lambda}'_3(\underline{f}_3 - \dot{\underline{x}}_3) - \sum_{j=1}^{m_{12}} \sum_{k=1}^n \underline{n}_k^{2j} \dot{\underline{a}}_k^{2j}] dt \end{aligned} \quad (6.9)$$

where $\underline{f}_1, \underline{f}_2, \underline{f}_3, g_1, g_2,$ and g_3 are evaluated at the optimum B^{1j} 's and where $\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3, \underline{n}_k^{1j}$'s, $\underline{n}_k^{2j}$'s are vector Lagrange multipliers of appropriate dimensions. Integrating by parts and considering only first-order variations, it follows that

$$\begin{aligned} \delta J_a = & \left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} \delta \underline{x}(T) + \sum_{j=1}^{m_{11}} \sum_{k=1}^n \{ \underline{n}_k^{1j}(t_0) - \underline{n}_k^{1j}(t_1) \} \delta \underline{a}_k^{1j} + \\ & \sum_{j=1}^{m_{12}} \sum_{k=1}^n \{ \underline{n}_k^{2j}(t_1) - \underline{n}_k^{2j}(T) \} \delta \underline{a}_k^{2j} + (g_1 - g_2) \Big|_{t=t_1} dt_1 + \\ & \int_{t_0}^{t_1} \left[\frac{\partial H_1}{\partial \underline{x}} \delta \underline{x} - \underline{\lambda}'_1 \delta \dot{\underline{x}}_1 \right] dt + \end{aligned}$$

$$\begin{aligned}
& \int_{t_1}^{t_2} \left(\frac{\partial H_2}{\partial \underline{x}} \delta \underline{x} - \underline{\lambda}'_2 \delta \dot{\underline{x}}_2 \right) dt + \int_{t_2}^T \left(\frac{\partial H_3}{\partial \underline{x}} \delta \underline{x} - \underline{\lambda}'_3 \delta \dot{\underline{x}}_3 \right) dt + \int_{t_0}^{t_1} \left[\sum_{j=1}^{m_{11}} \sum_{k=1}^n \left(\frac{\partial H_1}{\partial \underline{a}_k} \right. \right. \\
& \left. \left. \dot{\underline{n}}_k^{1j} \right)' \delta \underline{a}_k^{1j} \right] dt + \int_{t_1}^{t_2} \left[\sum_{j=1}^{m_{12}} \sum_{k=1}^n \left(\frac{\partial H_2}{\partial \underline{a}_k} \right. \right. \\
& \left. \left. \dot{\underline{n}}_k^{2j} \right)' \delta \underline{a}_k^{2j} \right] dt + \\
& \int_{t_2}^T \left[\sum_{j=1}^{m_{12}} \sum_{k=1}^n \left(\frac{\partial H_3}{\partial \underline{a}_k} \right. \right. \\
& \left. \left. \dot{\underline{n}}_k^{2j} \right)' \delta \underline{a}_k^{2j} \right] dt \tag{6.10}
\end{aligned}$$

where

$$H_i = g_i + \underline{\lambda}'_i \underline{f}_i \quad i = 1, 2, 3 \tag{6.11}$$

A necessary condition for optimality is that $\delta J_a = 0$. Therefore, the terms that multiply the independent variations $\delta \underline{a}_k^{1j}$'s and $\delta \underline{a}_k^{2j}$'s must be zero. Thus:

$$\underline{n}_k^{1j}(t_0) = \underline{n}_k^{1j}(t_1) \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_{11} \end{array} \tag{6.12}$$

$$\underline{n}_k^{2j}(t_1) = \underline{n}_k^{2j}(T) \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_{12} \end{array} \tag{6.13}$$

$$\int_{t_0}^{t_1} \frac{\partial H_1}{\partial \underline{a}_k^{1j}} dt = 0 \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_{11} \end{array} \tag{6.14}$$

$$\int_{t_1}^{t_2} \frac{\partial H_2}{\partial \underline{a}_k^{2j}} dt + \int_{t_2}^T \frac{\partial H_3}{\partial \underline{a}_k^{2j}} dt = 0 \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_{12} \end{array} \tag{6.15}$$

Now consider $\int_{t_0}^{t_1} \underline{\lambda}'_1 \delta \dot{\underline{x}}_1 dt$. Integrating by parts gives

$$\int_{t_0}^{t_1} \underline{\lambda}'_1 \delta \dot{\underline{x}}_1 dt = \underline{\lambda}'_1 \delta \underline{x} \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{\underline{\lambda}}_1' \delta \underline{x} dt \tag{6.16}$$

But

$$\delta \underline{x} = d\underline{x} - \dot{\underline{x}}_1 dt_1 \quad (6.17)$$

Therefore,

$$\left. \lambda_1' \delta \underline{x} \right|_{t_0}^{t_1} = \lambda_1'(t_1) d\underline{x}(t_1) - \lambda_1'(t_1) \dot{\underline{x}}_1(t_1) dt_1$$

and

$$\int_{t_0}^{t_1} \lambda_1' \delta \dot{\underline{x}}_1 dt = \lambda_1'(t_1) d\underline{x}_1(t_1) - \lambda_1'(t_1) \dot{\underline{x}}_1(t_1) dt_1 - \int_{t_0}^{t_1} \lambda_1' \delta \underline{x} dt \quad (6.18)$$

Similarly,

$$\begin{aligned} \int_{t_1}^{t_2} \lambda_2' \delta \dot{\underline{x}}_2 dt &= \lambda_2'(t_2) d\underline{x}_2(t_2) - \lambda_2'(t_1) d\underline{x}_2(t_1) + \\ &\quad - \lambda_2'(t_1) \dot{\underline{x}}_2(t_1) dt_1 - \int_{t_1}^{t_2} \lambda_2' \delta \underline{x} dt \end{aligned} \quad (6.19)$$

$$\begin{aligned} \int_{t_2}^T \lambda_3' \delta \dot{\underline{x}}_3 dt &= \lambda_3'(T) d\underline{x}(T) - \lambda_3'(t_2) d\underline{x}_3(t_2) - \\ &\quad - \int_{t_2}^T \lambda_3' \delta \underline{x} dt \end{aligned} \quad (6.20)$$

Substituting (6.12) - (6.15) and (6.18) - (6.20) in (6.10) and assuming that $\underline{x}$ is continuous, it follows that

$$\begin{aligned}
\delta J_a = & \left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} d\underline{x}(T) - \underline{\lambda}_3'(T) d\underline{x}(T) + \{ \underline{\lambda}_2(t_1) - \underline{\lambda}_1(t_1) \}' d\underline{x}(t_1) + \\
& \{ \underline{\lambda}_3(t_2) - \underline{\lambda}_2(t_2) \}' d\underline{x}(t_2) + (H_1 - H_2) \Big|_{t=t_1} dt_1 + \\
& \int_{t_0}^{t_1} \left(\frac{\partial H_1}{\partial \underline{x}} + \underline{\lambda}_1 \right)' \delta \underline{x} dt + \\
& \int_{t_1}^{t_2} \left(\frac{\partial H_2}{\partial \underline{x}} + \underline{\lambda}_2 \right)' \delta \underline{x} dt + \int_{t_0}^{t_1} \left(\frac{\partial H_3}{\partial \underline{x}} + \underline{\lambda}_3 \right)' \delta \underline{x} dt
\end{aligned}$$

Therefore, for δJ_a to be zero, it is required that

$$H_1(t_1) = H_2(t_1)$$

$$\partial H_3 / \partial \underline{x} = -\underline{\lambda}_3$$

$$[\partial L / \partial \underline{x}]_{t=T} = \underline{\lambda}_3(T)$$

$$\partial H_2 / \partial \underline{x} = -\underline{\lambda}_2$$

$$\underline{\lambda}_2(t_2) = \underline{\lambda}_3(t_2)$$

$$\partial H_1 / \partial \underline{x} = -\underline{\lambda}_1$$

$$\underline{\lambda}_1(t_1) = \underline{\lambda}_2(t_1)$$

in addition to conditions (6.12) to (6.15).

A similar set of conditions can be found for the related maximum problem. The two sets of necessary conditions can be combined as was done in Section 2.3, and the necessary conditions for the problem formulated in Section 6.2 are

$$H_1(t_1) = H_2(t_1) \quad (6.21)$$

$$H_2(t_2) = H_3(t_2) \quad (6.22)$$

$$\partial H_3 / \partial \underline{x} = -\dot{\underline{\lambda}}_3 \quad (6.23)$$

$$[\partial L / \partial \underline{x}]_{t=T} = \underline{\lambda}_3(T) \quad (6.24)$$

$$\partial H_2 / \partial \underline{x} = -\dot{\underline{\lambda}}_2 \quad (6.25)$$

$$\underline{\lambda}_2(t_2) = \underline{\lambda}_3(t_2) \quad (6.26)$$

$$\partial H_1 / \partial \underline{x} = -\dot{\underline{\lambda}}_1 \quad (6.27)$$

$$\underline{\lambda}_1(t_1) = \underline{\lambda}_2(t_1) \quad (6.28)$$

$$\int_{t_0}^{t_1} \frac{\partial H}{\partial A^{1j}} dt = [0] \quad j = 1, 2, \dots, m_{11} \quad (6.29)$$

$$\int_{t_1}^{t_2} \frac{\partial H_2}{\partial A^{2j}} dt + \int_{t_2}^T \frac{\partial H_3}{\partial A^{2j}} dt = [0] \quad j = 1, 2, \dots, m_{12} \quad (6.30)$$

$$\int_{t_0}^{t_1} \frac{\partial H_1}{\partial B^{1j}} dt + \int_{t_1}^{t_2} \frac{\partial H_2}{\partial B^{1j}} dt = [0] \quad j = 1, 2, \dots, m_{21} \quad (6.31)$$

$$\int_{t_2}^T \frac{\partial H_2}{\partial B^{2j}} dt = [0] \quad j = 1, 2, \dots, m_{22} \quad (6.32)$$

It is evident from (6.21) and (6.22) that for optimality of gain-change points H has to be continuous at the gain-change points.

SUFFICIENT CONDITIONS

The sufficient condition for this problem follows that of Section 4.4 if A denotes the set $\{A^{1j}, A^{2j}, t_1\}$, B denotes the set $\{B^{1j}, B^{2j}, t_2\}$, η denotes the set $\{\eta_k^{1j}, \eta_k^{2j}\}$, and ρ denotes the set $\{\rho_{1k}^j, \rho_{2k}^j\}$.

6.4 NECESSARY AND SUFFICIENT CONDITIONS--GENERAL CASE

Consider the control diagram of Figure 6.1(b), which indicates that there are s distinct intervals of interest. It is assumed in Figure 6.1(b) that players alternate in changing their gain values; other assumptions are possible and related necessary conditions for optimality could be obtained. Note that s in Figure 6.1(b) is necessarily odd for any integer w .

Following the development of the previous section, it can be shown that necessary conditions are:

$$H_\ell(t_\ell) = H_{\ell+1}(t_\ell) \quad \ell = 1, 2, \dots, s-1 \quad (6.33)$$

$$\lambda_\ell(t_\ell) = \lambda_{\ell+1}(t_\ell) \quad \ell = 1, 2, \dots, s-1 \quad (6.34)$$

$$\left. \frac{\partial L}{\partial \underline{x}} \right|_{t=T} = \lambda_s(T)$$

$$\frac{\partial H_\ell}{\partial \underline{x}} = -\lambda_\ell \quad \ell = 1, 2, \dots, s \quad (6.35)$$

$$\sum_{\ell=i}^{i+1} \int_{t_{\ell-1}}^{t_\ell} \frac{\partial H_\ell}{\partial A \left(\frac{i}{2} + 1 \right)_j} dt = [0] \quad \begin{array}{l} i = 0, 2, 4, \dots, s-1 \\ j = 1, 2, \dots, m_{1, 1+(i/2)} \end{array} \quad (6.36)$$

where for $i = 0$, the first term in the summation is ignored; and

$$\sum_{\ell=i+1}^{i+2} \int_{t_{\ell-1}}^{t_{\ell}} \frac{\partial H_{\ell}}{\partial B \frac{(i+1)j}{2}} dt = [0] \quad \begin{array}{l} i = 0, 2, 4, \dots, s-1 \\ j = 1, 2, \dots, m_{2,1+(i/2)} \end{array} \quad (6.37)$$

where, for $i = s-1$, the last term in the summation is ignored.

The sufficient conditions of Section 6.3 apply for the general case if

A is the set $\{A^{1j}, A^{2j}, \dots, A^{\frac{s+1}{2}j}, t_1, t_3, \dots, t_{s-2}\}$
 B is the set $\{B^{1j}, B^{2j}, \dots, B^{\frac{s+1}{2}j}, t_2, t_4, \dots, t_{s-1}\}$
 η is the set $\{\eta^{1j}, \eta^{2j}, \dots, \eta^{\frac{s+1}{2}j}\}$, and
 ρ is the set $\{\rho^{1j}, \rho^{2j}, \dots, \rho^{\frac{s+1}{2}j}\}$.

6.5 APPLICATION TO LINEAR-QUADRATIC GAMES

NECESSARY CONDITIONS

Consider a linear dynamic system given by (2.42) and a cost functional given by (2.43) with the associated weighting matrices having the properties described in Section 2.4 and the control diagram of Figure 6.1(a). Following Section 4.5, it can be shown that the necessary conditions are

$$\int_{t_0}^{t_1} \alpha_{1k} [R_1 \sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} + G_1' P_1] \underline{x}_1 \underline{x}_1' dt = [0] \quad k = 1, 2, \dots, m_{11} \quad (6.38)$$

$$\int_{t_1}^{t_2} \alpha_{2k} [R_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_1' P_2] \underline{x}_2 \underline{x}_2' dt + \int_{t_2}^T \alpha_{2k} [R_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_1' P_3] \underline{x}_3 \underline{x}_3' dt = [0] \quad k = 1, 2, \dots, m_{12} \quad (6.39)$$

$$\int_{t_0}^{t_1} \beta_{1k} [R_2 \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} + G_2' P_1] \underline{x}_1 \underline{x}_1' dt + \int_{t_1}^{t_2} \beta_{1k} [R_2 \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} + G_2' P_2] \underline{x}_2 \underline{x}_2' dt = [0] \quad (6.40)$$

$k = 1, 2, \dots, m_{21}$

and

$$\int_{t_2}^T \beta_{2k} [R_2 \sum_{j=1}^{m_{22}} \beta_{2j} B^{2j} + G_2' P_3] \underline{x}_3 \underline{x}_3' dt = [0], \quad k = 1, 2, \dots, m_{22} \quad (6.41)$$

P_1 , P_2 , and P_3 satisfy the following differential equations:

$$\dot{P}_1 + P_1 F_1 + F_1' P_1 + C_1 = [0] \quad (6.42)$$

with the boundary condition $P_1(t_1) = P_2(t_1)$;

$$\dot{P}_2 + P_2 F_2 + F_2' P_2 + C_2 = [0] \quad (6.43)$$

with the boundary condition $P_2(t_2) = P_3(t_2)$; and

$$\dot{P}_3 + P_3 F_3 + F_3' P_3 + C_3 = [0] \quad (6.44)$$

with the boundary condition $P_3(T) = S$, where

$$F_1 = F + G_1 \sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} + G_2 \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \quad (6.45)$$

$$F_2 = F + G_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_2 \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \quad (6.46)$$

$$F_3 = F + G_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_2 \sum_{j=1}^{m_{22}} \beta_{2j} B^{2j} \quad (6.47)$$

$$C_1 = \left(\sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} \right)' R_1 \left(\sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} \right) + \left(\sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \right)' R_2 \left(\sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \right) + Q \quad (6.48)$$

$$C_2 = \left(\sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \right)' R_1 \left(\sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \right) + \left(\sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \right)' R_2 \left(\sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \right) + Q \quad (6.49)$$

and

$$C_3 = \left(\sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \right)' R_1 \left(\sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \right) + \left(\sum_{j=1}^{m_{22}} \beta_{2j} B^{2j} \right)' R_2 \left(\sum_{j=1}^{m_{22}} \beta_{2j} B^{2j} \right) + Q \quad (6.50)$$

For optimality of gain change points

$$H_1(t_1) = H_2(t_1) \quad (6.51)$$

$$H_2(t_2) = H_3(t_2) \quad (6.52)$$

Conditions (6.51) and (6.52) can be further simplified. Let

$$A_{1i} \triangleq \left. \sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} \right|_{t=t_i}, \quad i = 1, 2 \quad (6.53)$$

$$A_{2i} \triangleq \left. \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} \right|_{t=t_i}, \quad i = 1, 2 \quad (6.54)$$

$$B_{1i} \triangleq \left. \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} \right|_{t=t_i}, \quad i = 1, 2 \quad (6.55)$$

$$B_{2i} \triangleq \left. \sum_{j=1}^{m_{22}} \beta_{2j} B^{2j} \right|_{t=t_i}, \quad i = 1, 2 \quad (6.56)$$

Because $\underline{x}$ is continuous, condition (6.51) yields

$$\begin{aligned}
& \underline{x}'(t_1)(A_{11}'R_1(t_1)A_{11} + B_{11}'R_2(t_1)B_{11} + Q)\underline{x}(t_1) \\
& + 2\underline{x}'(t_1)P_1(t_1)(F + G_1A_{11} + G_2B_{11}) \Big|_{t=t_1} \underline{x}(t_1) \\
& = \underline{x}'(t_1)(A_{21}'R_1(t_1)A_{21} + B_{11}'R_2(t_1)B_{11} + Q)\underline{x}(t_1) \\
& + 2\underline{x}'(t_1)P_2(t_1)(F + G_1A_{21} + G_2B_{11}) \Big|_{t=t_1} \underline{x}(t_1) \quad (6.57)
\end{aligned}$$

Set

$$P(t_1) \stackrel{\Delta}{=} P_1(t_1) = P_2(t_1) \quad (6.58)$$

With this, equation (6.57) reduces to

$$\begin{aligned}
& \underline{x}'(t_1)[A_{11}'R_1(t_1)A_{11} + 2P(t_1)G_1(t_1)A_{11} \\
& - A_{21}'R_1(t_1)A_{21} - 2P(t_1)G_1(t_1)A_{21}]\underline{x}(t_1) = 0
\end{aligned}$$

or

$$\underline{x}'(t_1)[(A_{11} + A_{21})'R_1(t_1) + 2P(t_1)G_1(t_1)](A_{11} - A_{21})\underline{x}(t_1) = 0 \quad (6.59)$$

or

$$\begin{aligned}
& + \\
& \underline{x}'(t_1)(A_{11} - A_{21})'[R_1(t_1)(A_{11} + A_{21}) + 2G_1'(t_1)P(t_1)]\underline{x}(t_1) = 0 \quad (6.60)
\end{aligned}$$

Because $A_{11} \neq A_{21}$, in general, and because (6.60) must be satisfied by $\underline{x}(t_1)$, it follows that

$$R_1(t_1)(A_{11} + A_{21}) + 2G_1'(t_1)P(t_1) = [0] \quad (6.61)$$

Similarly, condition (6.52) yields

$$R_2(t_2)(B_{12} + B_{22}) + 2G_2'(t_2)P(t_2) = [0] \quad (6.62)$$

Conditions (6.61) and (6.62) are necessary conditions for optimality of gain change points and are to be satisfied in addition to (6.38)-(6.44).

After some algebraic manipulations, the optimal cost J^0 can be shown to be equal to

$$J^0 = \underline{x}'(t_0)P_1(t_0)\underline{x}(t_0) \quad (6.63)$$

SUFFICIENT CONDITIONS

The sufficient conditions of Section 6.3 can now be applied to the above problem. Consider

$$\begin{aligned} I(\underline{x}, A, B, n, \rho) = & \underline{x}'(t)P(t)\underline{x}(t) + \sum_{j=1}^{m_{11}} \sum_{k=1}^n \rho_k^{1j} (\underline{a}_k^{1j} - (\underline{a}_k^{1j})^0) + \\ & \sum_{j=1}^{m_{12}} \sum_{k=1}^n \rho_k^{2j} (\underline{a}_k^{2j} - (\underline{a}_k^{2j})^0) + t(t_1 - t_1^0)^2 + \\ & \sum_{j=1}^{m_{21}} \sum_{k=1}^n \rho_k^{1j} (\underline{b}_k^{1j} - (\underline{b}_k^{1j})^0) + \\ & \sum_{j=1}^{m_{22}} \sum_{k=1}^n \rho_k^{2j} (\underline{b}_k^{2j} - (\underline{b}_k^{2j})^0) - t(t_2 - t_2^0)^2 \end{aligned} \quad (6.64)$$

where $P(t)$ is defined by (6.42), (6.43), and (6.44) in the respective time intervals.

By paralleling the procedure outlined in Section 4.5, it can be shown that the sufficient conditions of Section 6.3 are satisfied

in view of the necessary conditions (6.38) - (6.52). Finally, the scalar function $I(\underline{x}, A^0, B^0, \eta, \rho)$ reduces to

$$I(\underline{x}, A^0, B^0, \eta, \rho) = \underline{x}' P \underline{x} \quad (6.65)$$

Hence the existence of $P(t)$ satisfying (6.42) - (6.44) in the respective time intervals is a sufficient condition for the existence of a saddle point.

6.6 AVERAGING OVER INITIAL CONDITIONS

Conditions (6.38) - (6.41) suggest a scheme for finding controller parameters that do not depend upon the initial conditions. The problem is to find or characterize, if they exist, A^* and B^* which are optimal in the sense that for any other sets of gains matrices A and B there holds

$$E_1[J(A^*, B)] \leq E_1[J(A^*, B^*)] \quad (6.66)$$

and

$$E_2[J(A^*, B^*)] \leq E_2[J(A, B^*)] \quad (6.67)$$

where the expectation operators E_1 and E_2 are taken over distributed initial states. The players will try to satisfy the necessary conditions in an average sense. By paralleling the procedure outlined in Appendix B, it can be shown that for the class of linear-quadratic

games under consideration, the operation of taking the expected values of the necessary conditions for optimality, conditions that correspond to known initial conditions, is equivalent to finding the necessary conditions for optimality corresponding to the expected value of J .

Now,

$$\underline{x}_i(t_i) = \phi_i(t_i, t_{i-1}) \underline{x}_i(t_{i-1}) \quad (6.68)$$

$$\phi_1 = F_1 \phi_1 ; \quad \phi_1(t_0, t_0) = I \quad (6.69)$$

$$\phi_2 = F_2 \phi_2 ; \quad \phi_2(t_1, t_1) = I \quad (6.70)$$

$$\phi_3 = F_3 \phi_3 ; \quad \phi_3(t_2, t_2) = I \quad (6.71)$$

Assume that $\underline{x}_0$ is a random vector and further assume that

$$E(\underline{x}_{i0} \underline{x}_{j0}') = K_{ij} E(\underline{x}_{10}^2) \quad (6.72)$$

and therefore,

$$E(\underline{x}_0 \underline{x}_0') = K \quad (6.73)$$

where the K_{ij} 's, the entries of K , depend upon the advance information available to the players as discussed in Section 4.6. Therefore,

$$E(\underline{x}_1 \underline{x}_1') = E[\phi_1(t, t_0) \underline{x}_0 \underline{x}_0' \phi_1'(t, t_0)] = \phi_1 K \phi_1' \quad (6.74)$$

Similarly,

$$E(\underline{x}_2 \underline{x}_2') = \phi_2(t, t_1) \phi_1(t_1, t_0) K \phi_1'(t_1, t_0) \phi_2'(t, t_1) \quad (6.75)$$

and

$$E(\underline{x}_3 \underline{x}_3') = \phi_3(t, t_2) \phi_2(t_2, t_1) \phi_1(t_1, t_0) K \phi_1'(t_1, t_0) \phi_2'(t_2, t_1) \phi_3'(t, t_2) \quad (6.76)$$

Let

$$\phi_i(t, t_0) = \phi_i(t, t_{i-1}) \phi_{i-1}(t_{i-1}, t_{i-2}) \dots \phi_2(t_2, t_1) \phi_1(t_1, t_0) \quad (6.77)$$

for all i . Equations (6.74), (6.75), and (6.76) therefore reduce to

$$E(\underline{x}_1 \underline{x}_1') = \Phi_1 K \Phi_1' \quad (6.78)$$

$$E(\underline{x}_2 \underline{x}_2') = \Phi_2 K \Phi_2' \quad (6.79)$$

$$E(\underline{x}_3 \underline{x}_3') = \Phi_3 K \Phi_3' \quad (6.80)$$

If it is then required that conditions (6.38) - (6.41) be satisfied in an expected value sense,

$$\int_{t_0}^T \alpha_{1k} [R_1 \sum_{j=1}^{m_{11}} \alpha_{1j} A^{1j} + G_1' P_1] \Phi_1 K \Phi_1' dt = [0] \quad k = 1, 2, \dots, m_{11} \quad (6.81)$$

$$\int_{t_1}^{t_2} \alpha_{2k} [R_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_1' P_2] \Phi_2 K \Phi_2' dt + \int_{t_2}^T \alpha_{2k} [R_1 \sum_{j=1}^{m_{12}} \alpha_{2j} A^{2j} + G_1' P_3] \Phi_3 K \Phi_3' dt = [0] \quad k = 1, 2, \dots, m_{12} \quad (6.82)$$

$$\int_{t_0}^{t_1} \beta_{1k} [R_2 \sum_{j=1}^{m_{21}} \beta_{1j} B^{1j} + G_2' P_1] \Phi_1 K \Phi_1' dt + \int_{t_1}^{t_2} \beta_{1k} [R_2 \sum_{j=1}^{m_{12}} \beta_{1j} B^{1j} + G_2' P_2] \Phi_2 K \Phi_2' dt = [0] \quad k = 1, 2, \dots, m_{21} \quad (6.83)$$

and

$$\int_{t_2}^T \beta_{2k} [R_{21}^{m_{22}} \beta_{2j} B^{2j} + G_2' P_3] \Phi_3 K \Phi_3' dt = [0]; k = 1, 2, \dots, m_{22} \quad (6.84)$$

Note that if the ratios x_{10}/x_{i0} , x_{20}/x_{i0} ... x_{n0}/x_{i0} are known for some nonzero x_{i0} , the necessary conditions can be solved without knowledge of the specific initial values.

By paralleling the procedure outlined in Sections 4.5 and 4.6, it can be shown that the sufficient conditions of Section 6.3 are satisfied by A^* 's and B^* 's in an expected sense, the initial condition being treated as a random variable.

The value of J obtained by using these suboptimal A^* 's and B^* 's, after some algebraic manipulations is

$$J = \underline{x}_0' P_1(t_0) \underline{x}_0 \quad (6.85)$$

and the expected value of J is

$$\hat{J} = \text{tr}(P(t_0)K) \quad (6.86)$$

Of course, each player could use a different K in the design of his controller and the one coming closest to the true K would benefit in average performance.

Note that whereas the constant feedback gains are here independent of the initial state, the optimal performance index (6.85) depends upon the initial state.

6.7 NUMERICAL EXAMPLES

A. SCALAR CASE: AC-B PROBLEM

Consider the case of a linear time-invariant system governed by

$$\dot{x} = -0.5x + 1.25u + 1.5v, \quad x_0 = 2 \quad (6.87)$$

The quadratic cost functional is

$$J = x(T)^2 S + \int_0^1 (u^2 - 4v^2 + 2x^2) dt \quad (6.88)$$

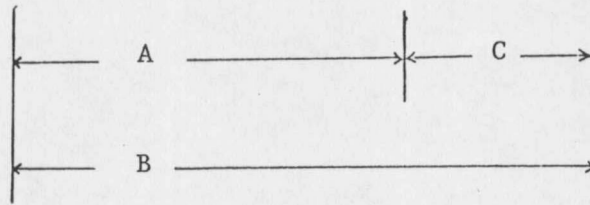
where $T = 1$, and $t_0 = 0$. Consider the control diagram of Figure 6.2(a).

The minimizing player can select two intervals of constant gains whereas the maximizing player is constrained to use the same constant gain throughout the duration of the play. Following Section 6.5, the optimal parameters for fixed gain intervals are

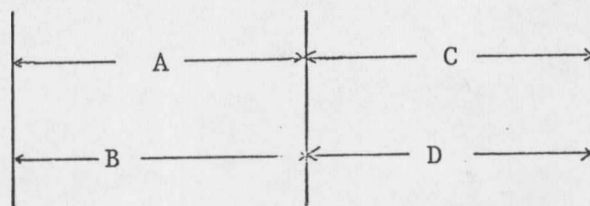
$$A = -1.25 \frac{\int_{t_0}^{t_1} P_1 x^2 dt}{\int_{t_0}^{t_1} x^2 dt} \quad (6.89)$$

$$C = -1.25 \frac{\int_{t_1}^T P_2 x^2 dt}{\int_{t_1}^T x^2 dt} \quad (6.90)$$

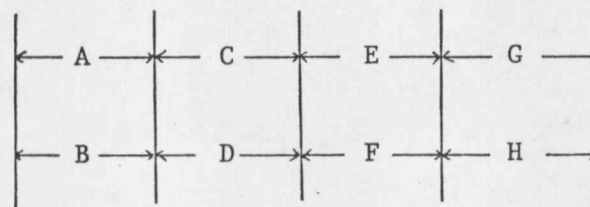
$$B = +0.375 \frac{\int_{t_0}^{t_1} P_1 x^2 dt + \int_{t_1}^T P_2 x^2 dt}{\int_{t_0}^{t_1} x^2 dt + \int_{t_1}^T x^2 dt} \quad (6.91)$$



(a)



(b)



(c)

Figure 6.2. Control diagrams.

where P_1 and P_2 satisfy (6.92) and (6.93), respectively:

$$P_2 + 2P_2(-0.5 + 1.25C + 1.5B) + C^2 - 4B^2 + 2 = 0 \quad (6.92)$$

with the boundary condition $P_2(T) = S$, and

$$P_1 + 2P_1(-0.5 + 1.25A + 1.5B) + A^2 - 4B^2 + 2 = 0 \quad (6.93)$$

with the boundary condition $P_1(t_1) = P_2(t_1)$.

The method of successive approximations is used to find the optimal gains. The details of related computations are outlined in Appendix C. Different gain-change times t_1 are chosen, and for each t_1 , the parameter S is varied from 4 to 10. Figure 6.3 shows the variation of performance index with t_1 . Note that the optimum t_1 occurs nearer to the end of the game than to the beginning. An analogy can be made here to the case of a runner who makes a final exerted effort near the end of a long race. It is evident from Figure 6.3 that with two gain-change intervals and with constant feedback gains, the minimizing player can force the difference between the saddle point value and the actual performance index to be nominal.

B. A VECTOR CASE

Consider the case of a linear time-invariant system governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v \quad (6.94)$$

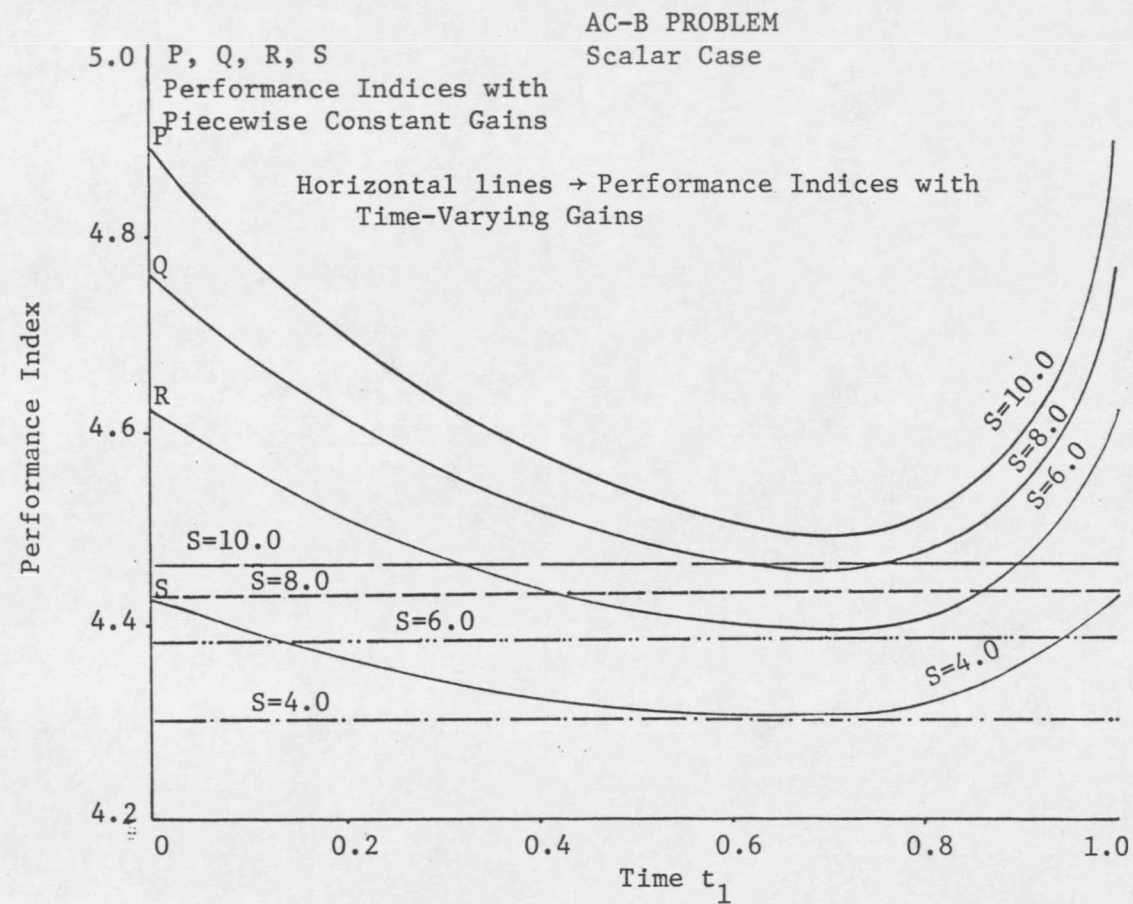


Figure 6.3. Variation of performance index with gain-change time.

where u and v are scalar controls. The quadratic cost functional is

$$\underline{x}'(T) \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \underline{x}(T) + \int_0^1 (x_1^2 + 0.5u^2 - 4v^2) dt \quad (6.95)$$

The players are restricted to use piecewise-constant feedback gains and four equal time intervals are chosen. The control diagram of Figure 6.2(c) illustrates the case. Following Section 6.5, the necessary conditions for the case of Figure 6.2(c) are

$$\underline{A} = -(0 \quad 2) \int_{t_0}^{t_1} P_1 \phi_1 K \phi_1' dt \left(\int_{t_0}^{t_1} \phi_1 K \phi_1' dt \right)^{-1} \quad (6.96)$$

$$\underline{B} = -\underline{A}/4 \quad (6.97)$$

$$\underline{C} = -(0 \quad 2) \int_{t_1}^{t_2} P_2 \phi_2 K_1 \phi_2' dt \left(\int_{t_1}^{t_2} \phi_2 K_1 \phi_2' dt \right)^{-1} \quad (6.98)$$

$$\underline{D} = -\underline{C}/4 \quad (6.99)$$

$$\underline{E} = -(0 \quad 2) \int_{t_2}^{t_3} P_3 \phi_3 K_2 \phi_3' dt \left(\int_{t_2}^{t_3} \phi_3 K_2 \phi_3' dt \right)^{-1} \quad (6.100)$$

$$\underline{F} = -\underline{E}/4 \quad (6.101)$$

$$\underline{G} = -(0 \quad 2) \left(\int_{t_3}^T P_4 \phi_4 K_3 \phi_4' dt \right) \left(\int_{t_3}^T \phi_4 K_3 \phi_4' dt \right)^{-1} \quad (6.102)$$

$$\underline{H} = -\underline{G}/4 \quad (6.103)$$

where it is assumed that $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, and $T = 1.0$ seconds, K , K_1 , K_2 , and K_3 are the expected value of the state at

t_0, t_1, t_2 , and t_3 , respectively, i.e.,

$$K_i = \phi_i(t_i, t_{i-1}) \phi_{i-1}(t_{i-1}, t_{i-2}) \dots \phi_2(t_2, t_1) \phi_1(t_1, t_0) K \phi_1'(t_1, t_0) \phi_2'(t_2, t_1) \dots \phi_i'(t_i, t_{i-1}) \quad (6.104)$$

and where

$$\phi_1 = (F + G_1 \underline{A} + G_2 \underline{B}) \phi_1 ; \phi_1(t_0, t_0) = I \quad (6.105)$$

$$\phi_2 = (F + G_1 \underline{C} + G_2 \underline{D}) \phi_2 ; \phi_2(t_1, t_1) = I \quad (6.106)$$

$$\phi_3 = (F + G_1 \underline{E} + G_2 \underline{F}) \phi_3 ; \phi_3(t_2, t_2) = I \quad (6.107)$$

$$\phi_4 = (F + G_1 \underline{G} + G_2 \underline{H}) \phi_4 ; \phi_4(t_3, t_3) = I \quad (6.108)$$

$$P_4 + P_4(F + G_1 \underline{G} + G_2 \underline{H}) + (F + G_1 \underline{G} + G_2 \underline{H})' P_4 + \underline{G}' R_1 \underline{G} + \underline{H}' R_2 \underline{H} + Q = [0] \quad (6.109)$$

with the boundary condition $P_4(T) = S$. Similarly,

$$P_3 + P_3(F + G_1 \underline{E} + G_2 \underline{F}) + (F + G_1 \underline{E} + G_2 \underline{F})' P_3 + \underline{E}' R_1 \underline{E} + \underline{F}' R_2 \underline{F} + Q = [0] \quad (6.110)$$

with the boundary condition $P_3(t_3) = P_4(t_3)$;

$$P_2 + P_2(F + G_1 \underline{C} + G_2 \underline{D}) + (F + G_1 \underline{C} + G_2 \underline{D})' P_2 + \underline{C}' R_1 \underline{C} + \underline{D}' R_2 \underline{D} + Q = [0] \quad (6.111)$$

with the boundary condition $P_2(t_2) = P_3(t_2)$; and

$$P_1 + P_1(F + G_1 A + G_2 B) + (F + G_1 A + G_2 B)' P_1 +$$

$$\underline{A}' R_1 \underline{A} + \underline{B}' R_2 \underline{B} + Q = [0] \quad (6.112)$$

with the boundary condition $P_1(t_1) = P_2(t_1)$.

The method of successive approximations is used to compute the optimal parameters. The details of the computations for this case are given in Appendix C.

In addition to the above results, a number of different constant-gain time intervals were chosen, and for each case, under different partial information about the initial conditions, the optimal parameters and the performance index were computed. Results are tabulated in Table C-2. The variation of optimal gains with time t for the case with four constant gain intervals is also illustrated in Figures 6.4 and 6.5. It is evident from Figures 6.4 and 6.5 that the optimal piecewise-constant gains approach the nonconstant feedback gains as the number of gain-change intervals increases. Table 6-1 clearly indicates the nature of the solutions.

6.8 CONCLUSION

Necessary and sufficient conditions for the existence of a saddle point when both players use suboptimal controls have been examined. Similar variational procedures can be applied to problems of finding

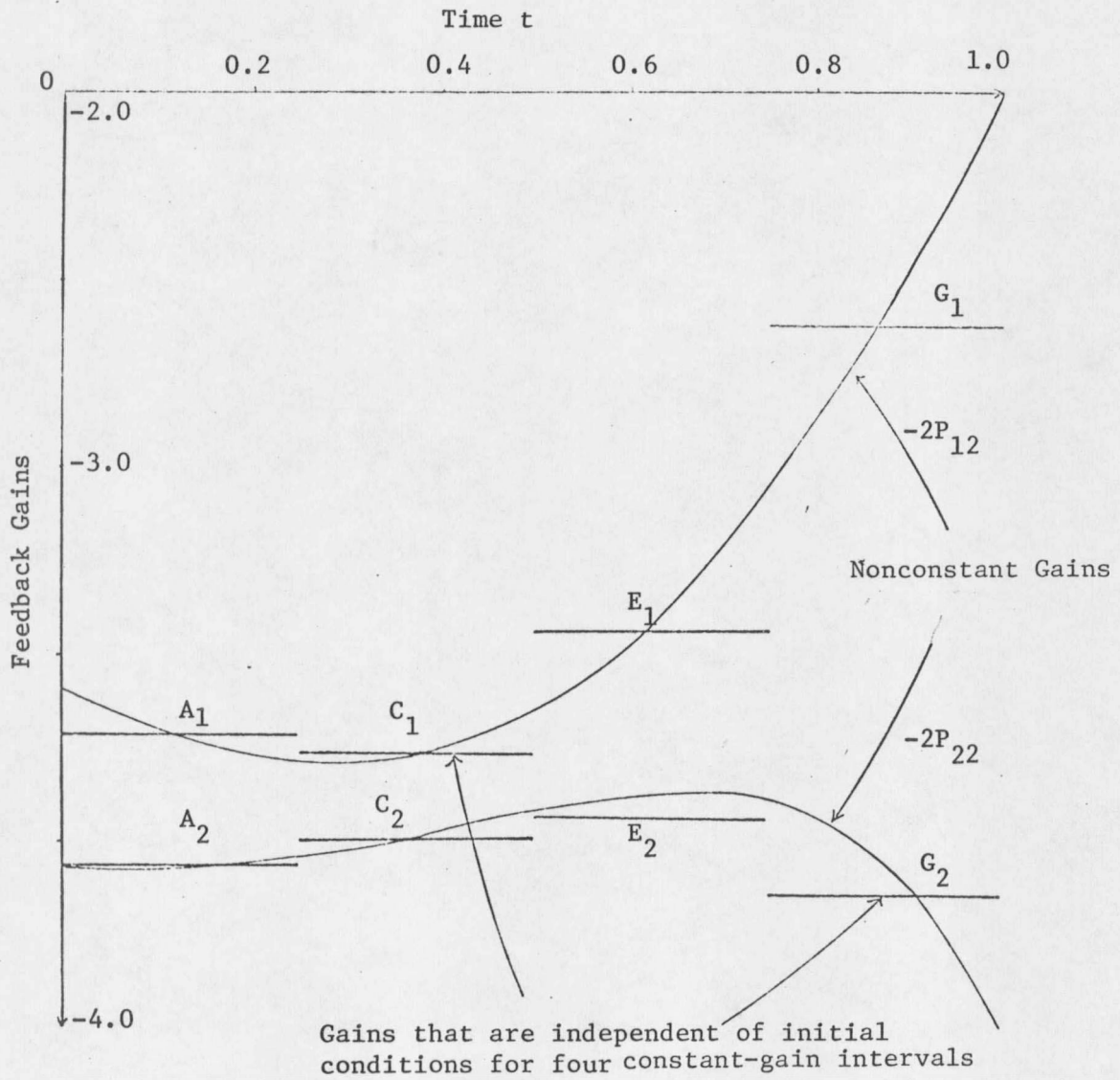
VECTOR CASE

Figure 6.4. Gains for the minimizing player. A_1 and A_2 are the entries of A , etc.

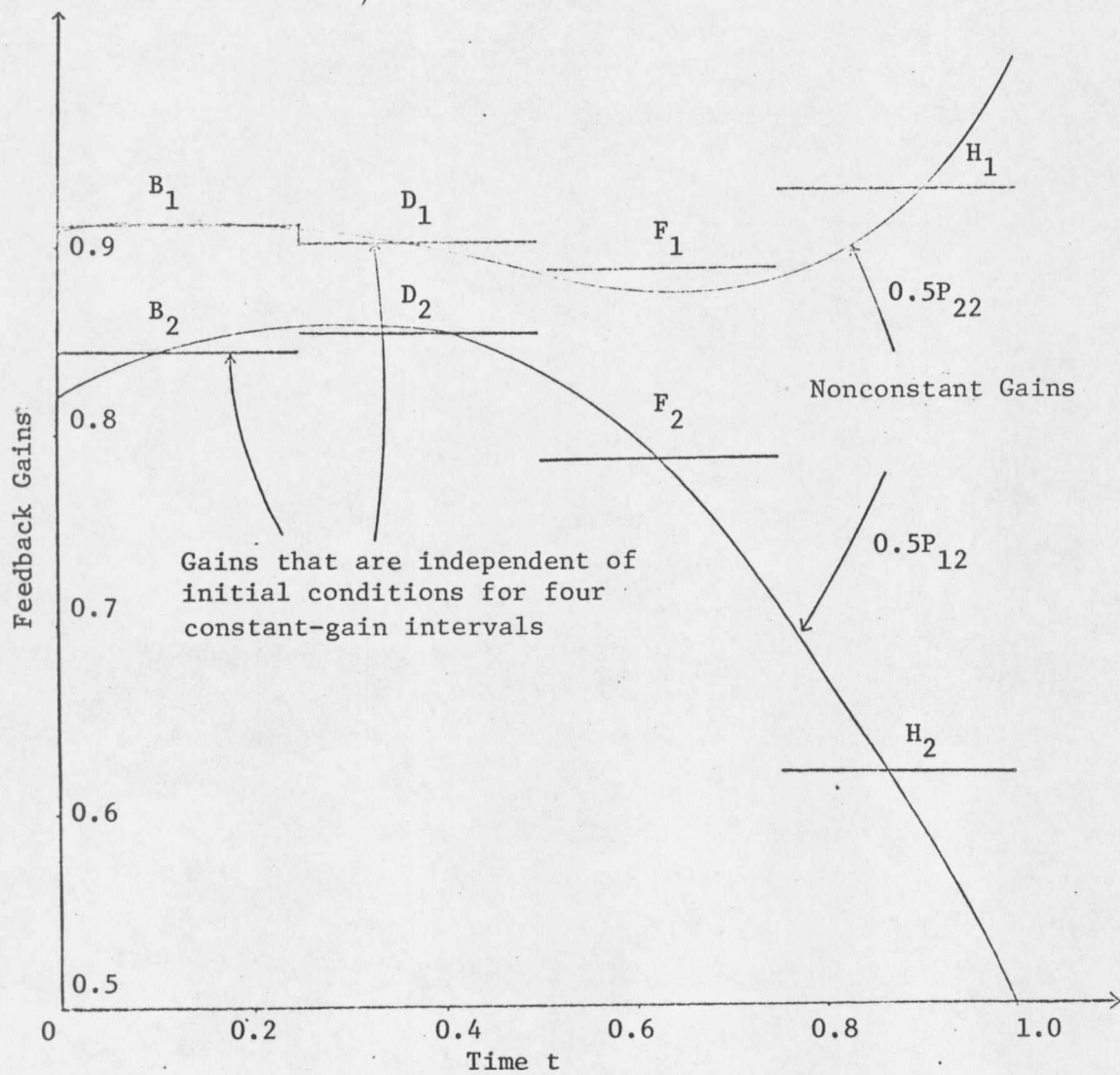
VECTOR CASE

Figure 6.5. Gains for the maximizing player. B_1 and B_2 are the entries of B , etc.

$\min_u \max_v J$ when the players can use mixed controls; namely, when each player is restricted to use suboptimal controls only during specified intervals of time. In the special case of linear-quadratic games, it has been shown that the piecewise optimal controller parameters for a given initial condition $\underline{x}_0$ are also valid for initial conditions $\alpha \underline{x}_0$, $-\infty < \alpha < \infty$. Consideration has been given to the optimal choice of gain-change times and also to the case when only partial information about possible initial states is available to the players.

TABLE 6-1
RESULTS FOR A VECTOR CASE
(Piecewise Constant Feedback Gains)

Case No.	Design Information	No. of Intervals	Actual Initial Conditions	Optimal Gains	Performance Index	$E(x_k)$ at Gain-Change Times
1	Initial conditions included in the design	1	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [\begin{smallmatrix} 3.766 & 4.3951 \end{smallmatrix}]$ $B = [\begin{smallmatrix} .9415 & 1.0988 \end{smallmatrix}]$	$J = 7.4810$ $\hat{J}_H = 1.8702$	$E(x_0 x_0') = \begin{bmatrix} 1 & -1/4 \\ -1/4 & 1/16 \end{bmatrix}$
	Initial conditions included in the design	2	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.2753 \quad -3.3717]$ $C = [-2.9827 \quad -3.2012]$ $B = [\begin{smallmatrix} .8188 & .8429 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .7457 & .8003 \end{smallmatrix}]$	$J = 7.5302$ $\hat{J}_H = 1.8825$	$E(x_0 x_0') = \begin{bmatrix} 1 & -1/4 \\ -1/4 & 1/16 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .5907 & -.4578 \\ -.4578 & .3547 \end{bmatrix}$
	Initial conditions included in the design	4	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.1514 \quad -3.0863]$ $C = [-3.6183 \quad -3.8957]$ $E = [-5.2147 \quad -5.8809]$ $G = [-12.5685 \quad -13.4328]$ $B = [\begin{smallmatrix} .7879 & .7716 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .9046 & .9739 \end{smallmatrix}]$ $F = [\begin{smallmatrix} .7877 & .8861 \end{smallmatrix}]$ $H = [\begin{smallmatrix} .6232 & .9282 \end{smallmatrix}]$	$J = 7.4386$ $\hat{J}_H = 1.8596$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .8206 & -.4368 \\ -.4368 & .2325 \end{bmatrix}$ $E(x(t_2) x'(t_2)) = \begin{bmatrix} .5910 & -.4575 \\ -.4575 & .3542 \end{bmatrix}$ $E(x(t_3) x'(t_3)) = \begin{bmatrix} .3808 & -.3710 \\ -.3710 & .3615 \end{bmatrix}$
2	$E(x_{10}^2) = E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	1	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.2786 \quad -3.7776]$ $B = [\begin{smallmatrix} .8196 & .9444 \end{smallmatrix}]$	$J = 7.5047$ $\hat{J}_H = 4.4235$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
	$E(x_{10}^2) = E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	2	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.4351 \quad -3.6456]$ $C = [-3.4160 \quad -3.6207]$ $B = [\begin{smallmatrix} .8588 & .9114 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .854 & .9052 \end{smallmatrix}]$	$J = 7.5264$ $\hat{J}_H = 4.4345$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .8045 & -.3549 \\ -.3549 & .3573 \end{bmatrix}$
	$E(x_{10}^2) = E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	4	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.3783 \quad -3.6520]$ $C = [-3.4185 \quad -3.6002]$ $E = [-3.1507 \quad -3.5445]$ $G = [-2.4929 \quad -3.7126]$ $B = [\begin{smallmatrix} .8446 & .9130 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .8546 & .9001 \end{smallmatrix}]$ $F = [\begin{smallmatrix} .7877 & .8861 \end{smallmatrix}]$ $H = [\begin{smallmatrix} .6232 & .9282 \end{smallmatrix}]$	$J = 7.4504$ $\hat{J}_H = 4.4005$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .9505 & -.2008 \\ -.2008 & .4646 \end{bmatrix}$ $E(x(t_2) x'(t_2)) = \begin{bmatrix} .8078 & -.3528 \\ -.3528 & .3561 \end{bmatrix}$ $E(x(t_3) x'(t_3)) = \begin{bmatrix} .6206 & -.3819 \\ -.3819 & .3434 \end{bmatrix}$
3	$E(x_{10}^2) = 16E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	1	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.6166 \quad -4.3788]$ $B = [\begin{smallmatrix} .9041 & 1.0947 \end{smallmatrix}]$	$J = 7.4876$ $\hat{J}_H = 2.6983$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$
	$E(x_{10}^2) = 16E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	2	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.3970 \quad -3.5479]$ $C = [-3.2652 \quad -3.4468]$ $B = [\begin{smallmatrix} .8493 & .8870 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .8163 & .8617 \end{smallmatrix}]$	$J = 7.5335$ $\hat{J}_H = 2.7130$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .7182 & -.4370 \\ -.4370 & .2807 \end{bmatrix}$
	$E(x_{10}^2) = 16E(x_{20}^2)$ $E(x_{10} x_{20}) = 0$	4	$x_{10} = -2.0$ $x_{20} = 0.5$	$A = [-3.3647 \quad -3.5819]$ $C = [-3.4410 \quad -3.6458]$ $E = [-3.3872 \quad -3.8679]$ $G = [-2.9981 \quad -4.2787]$ $B = [\begin{smallmatrix} .8412 & .8955 \end{smallmatrix}]$ $D = [\begin{smallmatrix} .8603 & .9115 \end{smallmatrix}]$ $F = [\begin{smallmatrix} .8468 & .9670 \end{smallmatrix}]$ $H = [\begin{smallmatrix} .7495 & 1.0697 \end{smallmatrix}]$	$J = 7.4500$ $\hat{J}_H = 2.6859$	$E(x_0 x_0') = \begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$ $E(x(t_1) x'(t_1)) = \begin{bmatrix} .9142 & -.3107 \\ -.3107 & .1335 \end{bmatrix}$ $E(x(t_2) x'(t_2)) = \begin{bmatrix} .7202 & -.4357 \\ -.4357 & .2778 \end{bmatrix}$ $E(x(t_3) x'(t_3)) = \begin{bmatrix} .5054 & -.4083 \\ -.4083 & .3375 \end{bmatrix}$
4	$x_{10} = -4x_{20}$	1	$x_{10} = -2.0$ $x_{20} = 0.5$	Same as #1	Same as #1	Same as #1
		2	$x_{10} = -2.0$ $x_{20} = 0.5$	Same as #1	Same as #1	Same as #1
		4	$x_{10} = -2.0$ $x_{20} = 0.5$	Same as #1	Same as #1	Same as #1

CHAPTER 7
CONCLUSIONS

7.1 REPRISE

This thesis has been concerned with the analysis of differential games with constraints on feedback gains. Continuous two-person zero-sum differential games with perfect information have been considered. The analysis with the simple constraint of constant feedback gains paved the way for the analysis of games with more complex situations involving suboptimal and piecewise suboptimal control laws. Under these constraints on feedback gains, necessary and sufficient conditions are examined for the existence of saddle points in general and also in linear-quadratic games. For a class of linear-quadratic games a way is indicated to obtain control-law parameters that are independent of initial conditions, so that a saddle point with respect to the expected value of the performance index is obtained. Partial information about the possible initial states has also been considered. For the class of linear-quadratic games, sufficient conditions for advantageous strategies for either player have been examined.

The general results were applied to specific scalar and dynamic systems, and numerical solutions were presented. The method of successive approximation was used to compute the optimal parameters.

7.2 CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER RESEARCH

The results of this thesis can be extended to cover nonzero-sum differential games. For the most part, these extensions are straight-

forward. All of the existing solution concepts (e.g., nash-equilibrium strategies, minimax strategies, and noninferior sets of strategies) can all be analyzed with feedback gain constraints. Noting that constant feedback gain strategies have added additional features, such as negotiation and bargaining, to the originally considered zero-sum games, it is natural to suspect that they will yield additional interesting features in nonzero-sum differential games.

One such feature is apparent at the outset. Starr and Ho [43] have shown that for nonzero-sum differential games, closed-loop solutions, in general, are different from open-loop solutions. It is evident, however, that closed-loop and open-loop solutions will be the same if constant feedback gain strategies are adopted. The effect of adding constraints on feedback gains to the existing solution concepts of nonzero-sum differential games is a matter for further investigation.

The extension of constant feedback gain strategies to stochastic games of imperfect information is also a matter worthy of further investigation. For example, consider the case of a linear-dynamic system with a quadratic cost functional and with independent white zero-mean Gaussian noise additively corrupting the output measurements of each controller. A possibility for the form of control strategies is

$$\underline{u}^0(t) = \hat{A}\underline{x}_1(t/t)$$

and

$$\underline{v}^0(t) = B\underline{\hat{x}}_2(t/t)$$

where

$$\underline{\hat{x}}_i(t/t) = E\{\underline{x}(t)/Y_i(t)\} \quad i = 1, 2$$

where $Y_i(t)$ is the measured output function for the i^{th} player up to time t , and includes a priori information, and where A and B are constant matrices of appropriate dimensions. The object may be to find or characterize, if they exist, A^0 and B^0 which are optimal in the sense that for any other matrices A and B ,

$$E\{J(A, B^0)/Y_1(T)\} \leq E\{J(A^0, B^0)/Y_1(T)\}$$

$$E\{J(A^0, B)/Y_2(T)\} \leq E\{J(A, B^0)/Y_2(T)\}$$

It is suggested that the problem of differential games with constant feedback gains might be reformulated in a Hilbert space and be analyzed using functional analysis techniques. Such a reformulation could simultaneously consider such system types as linear discrete-time systems and linear distributed-parameter systems.

It is also suggested that attention be given to sensitivity considerations in differential games. For example, one would like to know the effect of variations in the weighting matrices of the dynamical system and of the cost functional on the saddle point and

on the performance index. It is suggested that the approach and the results of Chapter 5 might be useful in this respect.

Also, worthy of further investigation are the sufficient conditions for advantageous strategies for the case of linear time-varying systems and when partial information about the possible initial states is different for each player.

The development of efficient computational algorithms for the computation of optimal parameters is another area for further research. In particular, the method of successive approximations, used in this thesis to compute the optimal parameters, assumes that the process will eventually converge. An attempt can be made to prove the convergence of this process by the use of the contraction mapping theorem. This is an open area even for the case of one-sided optimal control theory.

More consideration should also be given to the choice and the number of time functions to be generated for a satisfactory approximation to truly optimal control laws and to the speed of convergence of the suboptimal control laws to the truly optimal case.

Finally, all the investigations suggested above can be effected with other constraints imposed on the system, e.g., terminal-time constraints, energy and amplitude constraints, and also systems with inaccessible states.

APPENDIX A

PROOF FOR THE SUFFICIENT CONDITIONS OF SECTION 4.4

In this appendix a proof is presented for the sufficient conditions of Section 4.4, for the existence of a saddle point for a general two-person zero-sum differential game when the players use control laws as specified by equations (4.2) and (4.3).

Consider equation (4.19)

$$\min_A S(\underline{x}, A, B^0, \eta, \rho) = 0$$

Therefore, for all A,

$$S(\underline{x}, A, B^0, \eta, \rho) \geq 0 \quad (A.1)$$

or

$$(I_t + I_{\underline{x}}' \underline{f} + g) \Big|_{B=B^0} \geq 0 \quad (A.2)$$

or

$$g \Big|_{B^0} \geq -(I_t + I_{\underline{x}}' \underline{f}) \Big|_{B=B^0} \quad (A.3)$$

Thus,

$$\begin{aligned} \int_{t_0}^T g \Big|_{B^0} dt &\geq - \int_{t_0}^T (I_t + I_{\underline{x}}' \underline{f}) \Big|_{B=B^0} dt \\ &\geq - \int_{t_0}^T \frac{dI}{dt} \Big|_{B=B^0} dt \end{aligned} \quad (A.4)$$

It follows that

$$\begin{aligned} \int_{t_0}^T g \Big|_{B^0} dt &\geq -I \Big|_{t=T, B=B^0} + I \Big|_{t=t_0, B=B^0} \\ &\geq -L(\underline{x}(T), T) + I \Big|_{t=t_0, B=B^0} \end{aligned} \quad (A.5)$$

or

$$L(\underline{x}(T), T) + \int_{t_0}^T g \Big|_{B^0} dt \geq I \Big|_{t=t_0, B=B^0} \quad (A.6)$$

It is evident that equality holds throughout the above development for $A = A^0$, in which case

$$L(\underline{x}(T), T) + \int_{t_0}^T g \Big|_{(A^0, B^0)} dt = I \Big|_{t=t_0, A=A^0, B=B^0} \quad (A.7)$$

From equations (A.6) and (A.7) it follows that

$$I \Big|_{t=t_0, A=A^0, B=B^0} = J(A^0, B^0) \leq J(A, B^0) \quad (A.8)$$

It can similarly be shown that

$$I \Big|_{t=t_0, A=A^0, B=B^0} = J(A^0, B^0) \geq J(A^0, B) \quad (A.9)$$

APPENDIX B

DERIVATION OF THE NECESSARY CONDITIONS FOR THE EXPECTED VALUE OF J

B.1 PRELIMINARY REMARKS

It is indicated in Section 4.6 that for the linear-quadratic games a saddle point with respect to the expected value of the performance index, the expectation being taken over distributed initial states, will be obtained if the players satisfy necessary conditions (4.25) and (4.26) in an average sense. In this section a separate derivation of the necessary conditions (4.29), (4.40), and (4.41) starting with the problem of finding the expected value of the performance index is presented. The results of this section clearly indicate that for the class of linear-quadratic games under consideration, the operation of taking the expected value of the necessary conditions for optimality corresponding to J with known initial conditions is equivalent to finding the necessary conditions for optimality corresponding to the expected value of J .

B.2 DERIVATION OF THE NECESSARY CONDITIONS FOR THE EXPECTED VALUE OF J

Given is

$$J = \underline{x}'(T)S\underline{x}(T) + \int_{t_0}^T (\underline{u}'R_1\underline{u} + \underline{v}'R_2\underline{v} + \underline{x}'Q\underline{x}) dt$$

Let

$$C \equiv \left(\sum_{j=1}^{m_1} \alpha_j A^j \right)' R_1 \left(\sum_{j=1}^{m_1} \alpha_j A^j \right) + \left(\sum_{j=1}^{m_2} \beta_j B^j \right)' R_2 \left(\sum_{j=1}^{m_2} \beta_j B^j \right) + Q \quad (B.1)$$

and

$$F_N \equiv F + G_1 \sum_{j=1}^{m_1} \alpha_j A^j + G_2 \sum_{j=1}^{m_2} \beta_j B^j \quad (B.2)$$

Therefore,

$$J = \underline{x}'(T) S \underline{x}(T) + \int_{t_0}^T \underline{x}' C \underline{x} dt = \underline{x}_0' [\phi'(T, t_0) S \phi(T, t_0) + \int_{t_0}^T \phi'(t, t_0) C \phi(t, t_0) dt] \underline{x}_0 \quad (B.3)$$

where $\phi(t, t_0)$ is the transition matrix corresponding to F_N . Therefore,

$$E(J) = \text{tr}\{[\phi'(T, t_0) S \phi(T, t_0) + \int_{t_0}^T \phi' C \phi dt] K\} \quad (B.4)$$

where E is the expected-value operator, tr denotes the matrix trace operator, and $K = E[\underline{x}_0 \underline{x}_0']$.

The object is to find the $\min_{A^j} \max_{B^j} E(J)$ subject to the conditions

$$\dot{\phi} = F_N \phi, \quad \phi(t_0, t_0) = I \quad (B.5)$$

$$\dot{\underline{a}}_k^j = 0 \quad (B.6)$$

and

$$\dot{\underline{b}}_k^j = 0 \quad (B.7)$$

For convenience the $(n \times n)$ matrix ϕ is partitioned in terms of its

columns, i.e.,

$$\phi = [\underline{\phi}_1 \quad \underline{\phi}_2 \quad \dots \quad \underline{\phi}_n] \quad (\text{B.8})$$

where $\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n$ are $(n \times 1)$ vectors. The development of Section 2.3 is applicable. Let $\underline{\eta}_k^j(t)$'s and $\underline{\rho}_k^j(t)$'s denote vector Lagrange multipliers that account for constraints (B.6) and (B.7), respectively, in the two related optimal control problems and $\underline{\lambda}(t)$ be the common Lagrange multiplier to account for constraint (B.8). Also, let

$$H = \text{tr}(\phi' C \phi K) + \sum_{k=1}^n \underline{\lambda}'_k F_N \underline{\phi}_k \quad (\text{B.9})$$

It can be easily seen that the necessary conditions are

$$\underline{\eta}_k^j(t_0) = \underline{\eta}_k^j(T) \quad \begin{array}{l} j = 1, 2, \dots, m_1 \\ k = 1, 2, \dots, n \end{array} \quad (\text{B.10})$$

$$\underline{\rho}_k^j(t_0) = \underline{\rho}_k^j(T) \quad \begin{array}{l} j = 1, 2, \dots, m_2 \\ k = 1, 2, \dots, n \end{array} \quad (\text{B.11})$$

$$\partial H / \partial \underline{\phi}_k = -\underline{\lambda}_k \quad k = 1, 2, \dots, n \quad (\text{B.12})$$

$$\partial / \partial \underline{\phi}_k \{ \text{tr}(\phi' S \phi K) \}_{t=T} = \underline{\lambda}_k(T) \quad k = 1, 2, \dots, n \quad (\text{B.13})$$

$$\int_{t_0}^T \partial H / \partial \underline{a}_k^j dt = 0 \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_1 \end{array} \quad (\text{B.14})$$

$$\int_{t_0}^T \partial H / \partial \underline{b}_k^j dt = 0 \quad \begin{array}{l} k = 1, 2, \dots, n \\ j = 1, 2, \dots, m_2 \end{array} \quad (\text{B.15})$$

It is shown in the following that the above necessary conditions actually reduce to equations (4.29), (4.40), and (4.41).

Some useful relations are in order:

$$\frac{\partial}{\partial A^j} () = \left[\frac{\partial}{\partial a_1^j} () \quad \frac{\partial}{\partial a_2^j} () \quad \dots \quad \frac{\partial}{\partial a_n^j} () \right] \quad (B.16)$$

and

$$\frac{\partial}{\partial \phi_j} \text{tr}(\phi' S \phi K) = 2S \phi_k \quad (B.17)$$

where

$$K = [\underline{k}_1 \quad \underline{k}_2 \quad \dots \quad \underline{k}_n] \quad (B.18)$$

Also,

$$\frac{\partial}{\partial A^1} \phi_1' (A^1)' R_1 A^1 \phi_1 = 2R_1 A^1 \phi_1 \phi_1' \quad (B.19)$$

$$\frac{\partial}{\partial A^1} \phi_1' (A^1)' R_1 A^1 \phi_2 = R_1 A^1 \phi_1 \phi_2' + R_1 A^1 \phi_2 \phi_1' \quad (B.20)$$

and

$$\frac{\partial}{\partial A^1} (\underline{z}' A^1 \underline{x}) = \underline{z} \underline{x}' \quad (B.21)$$

in which $\underline{z}$ is a $(p \times 1)$ vector and $\underline{x}$ is an $(n \times 1)$ vector.

In view of (B.17), condition (B.13) reduces to

$$2S \phi(T) \underline{k}_j = \underline{\lambda}_k(T) \quad k = 1, 2, \dots, n \quad (B.22)$$

or

$$2S\phi(T) [\underline{k}_1 \quad \underline{k}_2 \quad \dots \quad \underline{k}_n] = [\underline{\lambda}_1(T) \quad \underline{\lambda}_2(T) \quad \dots \quad \underline{\lambda}_n(T)] \quad (B.23)$$

Denoting

$$\lambda(t) \triangleq [\underline{\lambda}_1(t) \quad \underline{\lambda}_2(t) \quad \dots \quad \underline{\lambda}_n(t)] \quad (B.24)$$

It follows that

$$2S\phi(T) K = \lambda(T) \quad (B.25)$$

Condition (B.12) becomes

$$2C\phi \underline{k}_{-j} + F_N' \lambda_{-j} = -\underline{\lambda}_j \quad j = 1, 2, \dots, n \quad (B.26)$$

or

$$2C\phi K + F_N' \lambda = -\lambda \quad (B.27)$$

Assume that $\lambda(t)$ and $\phi(t)$ are related by

$$\lambda(t) = 2P(t)\phi(t)K \quad (B.28)$$

with the boundary condition $P(T) = S$. Substituting (B.28) into (B.27)

and using (B.5) it follows that

$$\dot{P} + PF_N + F_N'P + C = [0] \quad (B.29)$$

with the boundary condition $P(T) = S$. Now,

$$\frac{\partial}{\partial A^1} \text{tr}(\phi' C \phi K) = \frac{\partial}{\partial A^1} \text{tr}[\phi' (\sum_{j=1}^{m_1} \alpha_j A^j) 'R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) \phi K] =$$

$$\frac{\partial}{\partial A^1} \sum_{i=1}^n \sum_{\ell=1}^n \phi_i' (\sum_{j=1}^{m_1} \alpha_j A^j) 'R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) \phi_{\ell} K_{\ell i} =$$

$$\sum_{i=1}^n \sum_{\ell=1}^n \frac{\partial}{\partial A^1} \phi_i' (\sum_{j=1}^{m_1} \alpha_j A^j) 'R_1 (\sum_{j=1}^{m_1} \alpha_j A^j) \phi_{\ell} K_{\ell i} =$$

$$\sum_{i=1}^n \sum_{\ell=1}^n [\frac{\partial}{\partial A^1} \phi_i' \alpha_1^2 (A^1) 'R_1 A^1 \phi_{\ell} K_{\ell i} + \frac{\partial}{\partial A^1} \phi_i' (\sum_{j=2}^{m_1} \alpha_j A^j) 'R_1 \alpha_1 A^1 \phi_{\ell} K_{\ell i} +$$

$$\frac{\partial}{\partial A^1} \phi_i' \alpha_1 (A^1) 'R_1 (\sum_{j=2}^{m_1} \alpha_j A^j) \phi_{\ell} K_{\ell i}] \quad (B.30)$$

Equation (B.30) can be evaluated term by term: The first term is

$$\sum_{i=1}^n \sum_{\ell=1}^n \frac{\partial}{\partial A^1} \phi_i' \alpha_1^2 (A^1) 'R_1 A^1 \phi_{\ell} K_{\ell i} =$$

$$\sum_{i=1}^n \sum_{\ell=1}^n \alpha_1^2 (R_1 A^1 \phi_{\ell} K_{\ell i} \phi_i' + R_1 A^1 \phi_i K_{\ell i} \phi_{\ell}') =$$

$$\alpha_1^2 (R_1 A^1 \phi K' \phi' + R_1 A^1 \phi K \phi') = 2 \alpha_1^2 R_1 A^1 \phi K \phi' \quad (B.31)$$

in which the last step is possible because K is an (n x n) symmetric matrix. The second term of equation (B.30) is

$$\sum_{i=1}^n \sum_{\ell=1}^n \phi_i' (\sum_{j=2}^{m_1} \alpha_j A^j) 'R_1 \alpha_1 A^1 \phi_{\ell} K_{\ell i} = \sum_{i=1}^n \sum_{\ell=1}^n \alpha_1 R_1 (\sum_{j=2}^{m_1} \alpha_j A^j) \phi_i' K_{\ell i} \phi_{\ell}' =$$

$$\alpha_1 R_1 (\sum_{j=2}^{m_1} \alpha_j A^j) \phi K \phi' \quad (B.32)$$

The third term of equation (B.30) can similarly be shown to reduce to

$$\alpha_1 R_1 \left(\sum_{j=2}^{m_1} \alpha_j A^j \right) \phi K \phi'$$

The net result is

$$\frac{\partial}{\partial A^1} \text{tr}(\phi' C \phi K) = 2\alpha_1 R_1 \sum_{j=1}^{m_1} \alpha_j A^j \phi K \phi' \quad (\text{B.33})$$

Consider also

$$\frac{\partial}{\partial A^1} \left(\sum_{\ell=1}^n \lambda_\ell' F_{N-1} \phi_\ell \right) = \sum_{\ell=1}^n \frac{\partial}{\partial A^1} \lambda_\ell' G_{11} \alpha_1 A^1 \phi_\ell = \sum_{\ell=1}^n \alpha_1 G_{11} \lambda_\ell \phi_\ell' = \alpha_1 G_{11}' \lambda \phi' =$$

$$2\alpha_1 G_{11}' P \phi K \phi'$$

Condition (B.14), therefore, reduces to condition (4.40) of Chapter 4.

Condition (B.15) can similarly be shown to be equivalent to condition (4.41) of Chapter 4.

APPENDIX C

FORTRAN PROGRAM FOR COMPUTING OPTIMAL PARAMETERS FOR A
VECTOR CASE WITH THE CONTROL DIAGRAM OF FIGURE 6.2(c)

C.1 GENERAL

The particular Fortran program to be described below can be used to solve two-point boundary-value problems occurring in optimal control theory and differential games. The program has been written for the computation of the optimal parameters for the vector case with four constant gain intervals corresponding to the control diagram of Figure 6.2(c). The constant gain intervals need not be equal and partial information about the possible initial conditions can be taken into account by merely changing specific cards. The flow diagram of digital computer program is given in Figure C.1. Results for a vector case with four gain-change points is given in Table C.2. The seventh and eighth iteration gave identical results.

C.2 SUBROUTINE RKAM

The computation of the optimal parameters require extensive solution of differential equations. For this purpose, the main computer program frequently calls RKAM(Y,DY,YD,VF,UY) which is a Fortran IV subroutine [53] used on the XDS Sigma 7 Computer for solving any set of coupled, first-order, ordinary differential equations of order 201 or less. It is stored in the MSU-EE Fortran Library.

The subroutine allows the programmer to choose from the following integration methods:

- 1) Fourth-order Runga-Kutta;
- 2) Adams-Moulton Predictor Corrector with relative error check;
- 3) Adams-Moulton Predictor Corrector with absolute error check.

The programmer must supply:

- 1) A short main program which calls RKAM;
- 2) A subroutine, DERFUN, which contains the system of n simultaneous first-order equations to be solved; and
- 3) A set of data cards which gives all the appropriate initial values, etc.

The data cards contain the initial conditions, the integration method to be used, the point interval desired and the desired number and type of graphical output. For a detailed discussion of the data cards see [53]. A fourth-order Runga-Kutta integration was used in the Fortran program given in Table C.1.

C.3 DETAILS OF THE FORTRAN PROGRAM

Statements 17 and 18 are to be changed when the information available to the players about the possible initial states is to be changed. In the Fortran Program indicated, $A = 1.0$ and $B = 0.0$.

Hence, $E(\underline{x}_0 \underline{x}_0') = K = \begin{bmatrix} 1 & B \\ B & A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This corresponds to the case when the initial conditions are not included in the design.

If $A = 1/16$ and $B = -1/14$, $E(\underline{x}_0 \underline{x}_0') = K = \begin{bmatrix} 1 & B \\ B & A \end{bmatrix} = \begin{bmatrix} 1 & -1/14 \\ -1/14 & 1/16 \end{bmatrix}$. Because the actual initial conditions are $x_{10} = -2.0$ and $x_{20} = 0.5$, it is evident that, in this case, the players are including the initial conditions in computing their optimal parameters. It is to be noted that the scalars A and B are used here to denote entries of K during the first constant-gain time interval and should not be confused with the parameters $\underline{A}$ and $\underline{B}$. Because the matrix K accounts for a priori information available to the players, it can rightly be called the information matrix for the first time interval.

Statements 72 - 77 are for calculating $E(\underline{x}(t_1)\underline{x}'(t_1)) = \begin{bmatrix} SP1 & SQ1 \\ SQ1 & SR1 \end{bmatrix}$. Therefore $K_1 = \begin{bmatrix} 1 & SQ1/SP1 \\ SQ1/SP1 & SR1/SP1 \end{bmatrix} \underline{\Delta} \begin{bmatrix} 1 & D \\ D & C \end{bmatrix}$ is used by players as the information matrix for the second constant-gain time interval. Statements 84 - 89 are for calculating the information matrix $K_2 = \begin{bmatrix} 1 & F \\ F & E \end{bmatrix}$ for the third time interval and statements 99 - 104 are for calculating the information matrix $K_3 = \begin{bmatrix} 1 & H \\ H & G \end{bmatrix}$ for the fourth time interval.

SUBROUTINE DERFUN

Statements 52 - 55 are for modifying the initial conditions from what has been read from the data cards to the computed value of $P_4(t_3)$. Statements 76 - 79 and statements 100 - 103 are for

changing the initial conditions to $P_3(t_2)$ and $P_2(t_1)$, respectively. The flow diagram clearly illustrates the major features of the program.

A NOTE ON THE STEP SIZE USED FOR INTEGRATION:

A step size of 0.001 was selected on the basis that successive reduction in step size did not alter the results appreciably. A very small step size might lead to undesirable round-off errors during computation and so, in general, the step size should not be too small or too large.

Consider the scalar dynamic system governed by (4.70) and the cost functional (4.71) with $S = 6$ and consider the case with constant feedback gains for both the players.

The percentage errors in the performance index with constant gains, the performance index with nonconstant gains, and the parameter A for a step size of 0.005 are about .2%, 0%, and 4.5%, respectively, when compared with the results for step size 0.001. The percentage errors for the same quantities for a step size of 0.002 are .026%, 0%, and 1.2%, respectively, when compared with the results for step size 0.001.

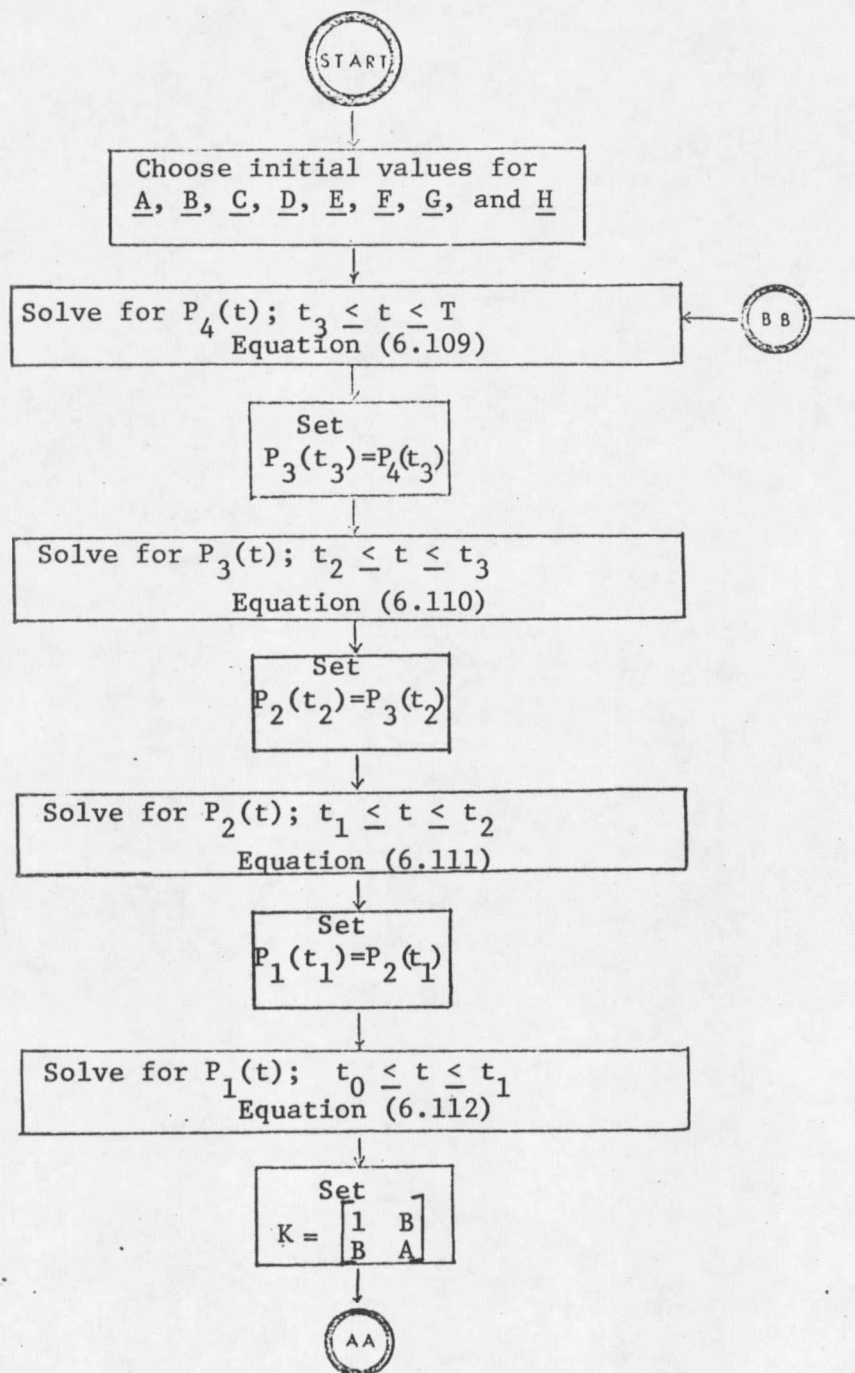


Figure C.1. Digital computer flow diagram.

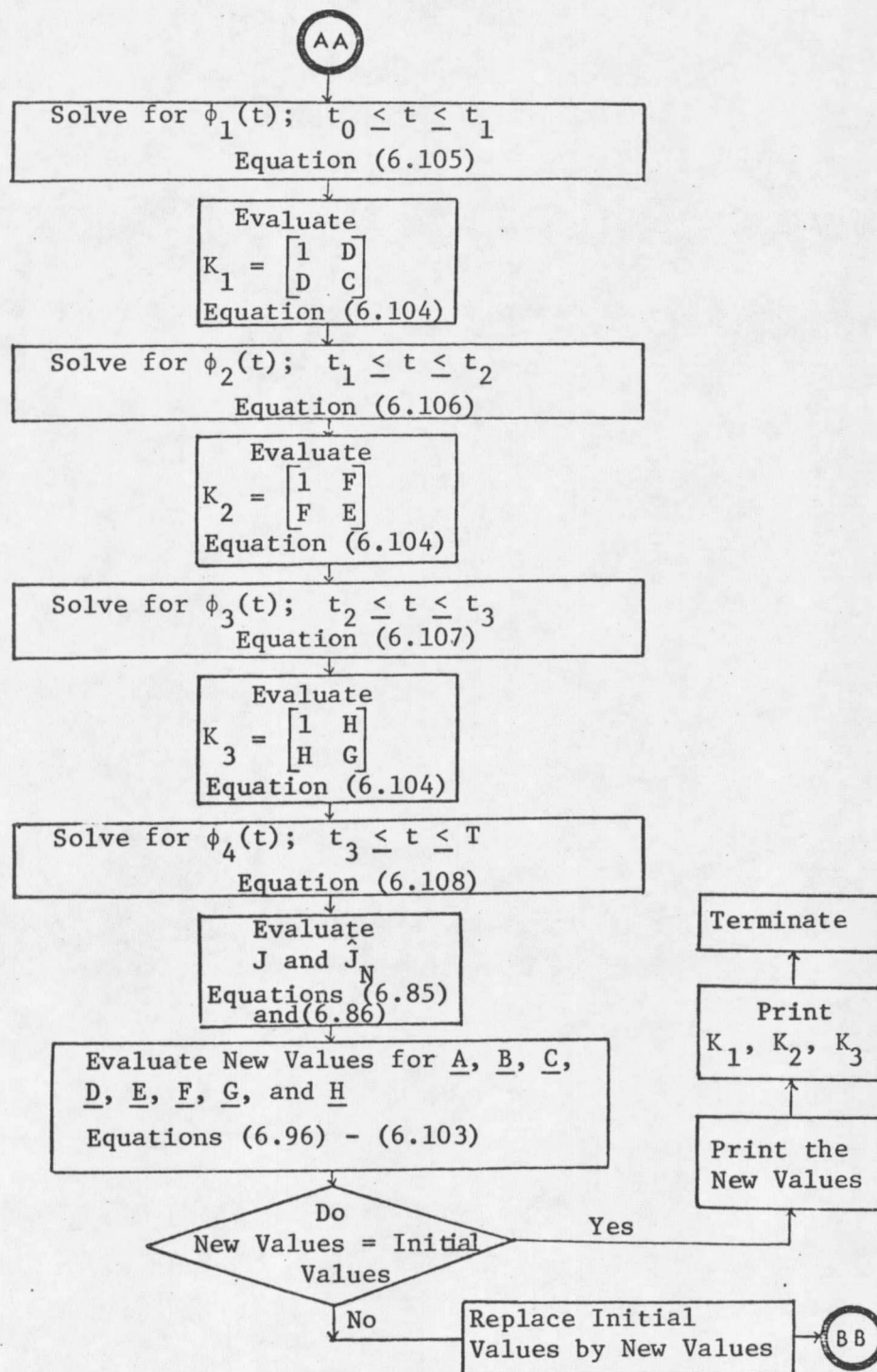


Figure C.1. Digital computer flow diagram (continued).

TABLE C-1. FORTRAN PROGRAM FOR A VECTOR CASE WITH FOUR GAIN CHANGE POINTS

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DOUBLE PRECISION UY,AY,BY,CY,RY,RY,ZY,RY,RY,RY,RY
DIMENSION Y(9),DY(9),YD(36),VF(45),UY(45)
EQUIVALENCE (UY(1),AY),(UY(2),BY),(UY(3),CY),(UY(4),RY)
#,(UY(5),RY),(UY(6),ZY),(UY(7),PY),(UY(8),QY),(UY(9),TY)
COMMON/NAH1/AY,BY,CY,RY,RY,ZY,RY,RY,RY
COMMON I,J,T1,T2,N,A,B,C,D,J1,J2,E,F,G,H
COMMON A1,B1,A2,B2,C1,D1,C2,D2,J3,J4
COMMON E1,E2,F1,F2,G1,G2,H1,H2
COMMON Y1(300),Y2(300),Y3(300),Y4(300),Y5(300),Y6(300),Y7(300),
#Y8(300),Y9(300),Y10(300),Y11(300),Y12(300),Y13(300),Y14(300),
#Y15(300),Y16(300),Y17(300),Y18(300),Y19(300),Y20(300),Y21(300),
#Y22(300),Y23(300),Y24(300)
COMMON Y25(300),Y26(300),Y27(300),Y28(300),Y29(300),Y30(300),
#Y31(300),Y32(300),Y33(300),Y34(300),Y35(300),Y36(300),Y37(300),
#Y38(300),Y39(300),Y40(300),Y41(300),Y42(300),Y43(300),Y44(300)
#Y45(300),Y46(300),Y47(300),Y48(300)
C ELEMENTS OF K OR E(X8*X8)
A=1.0
B=0.0
C INITIAL VALUES FOR THE PARAMETERS
A1=-3.6
A2=-4.0
C1=-3.2
C2=-2.8
B1=0.9
B2=1.0
D1=0.8
D2=0.7
E1=-2.8
E2=-3.2
F1=0.7
F2=0.8
G1=-2.0
G2=-3.2
H1=0.5
H2=0.8
C STEP SIZE
T2=.001
C (INTERVAL OF CONSTANT GAIN/STEP SIZE)+1
J1=251
J2=251
J3=251
J4=251
C ITERATIVE PROCEDURE BEGINS
DO 25 N=1,5
C SOLUTION FOR P4(T)
J=1
I=J4
T1=.99901
CALL RK4M (Y,DY,YD,VF,UY)
C SOLUTION FOR P3(T)
J=2
I=J3
T1=.74901
CALL RK4M (Y,DY,YD,VF,UY)
C SOLUTION FOR P2(T)
J=3

```

TABLE C-1. (continued)

I=J2	58
T1=.49901	59
CALL RKAM (Y,DY,YD,VF,UY)	60
C SOLUTION FOR F1(T)	61
J=4	62
I=J1	63
T1=.24901	64
CALL RKAM (Y,DY,YD,VF,UY)	65
C SOLUTION FOR PHI1(T)	66
J=5	67
I=1	68
T1=.00099	69
CALL RKAM (Y,DY,YD,VF,UY)	70
C COMPUTATION OF K1 OR E(X(T1)*X(T1))	71
SP1=(Y13(J1)**2)+(A*(Y14(J1)**2))+(2.0*Y13(J1)*Y14(J1)*B)	72
SQ1=(Y13(J1)*Y15(J1))+(Y14(J1)*Y16(J1)*A)+(Y13(J1)*Y16(J1)*B)+	73
*(Y14(J1)*Y15(J1)*B)	74
SR1=(Y15(J1)**2)+(A*(Y16(J1)**2))+(2.0*Y15(J1)*Y16(J1)*B)	75
C=SR1/SP1	76
D=SQ1/SP1	77
C SOLUTION FOR PHI2(T)	78
J=6	79
I=1	80
T1=.25099	81
CALL RKAM (Y,DY,YD,VF,UY)	82
C COMPUTATION OF K2 OR E(X(T2)*X(T2))	83
SP2=(Y22(J2)**2)+(C*(Y23(J2)**2))+(2.0*Y22(J2)*Y23(J2)*D)	84
SQ2=(Y22(J2)*Y24(J2))+(Y23(J2)*Y25(J2)*C)+(Y22(J2)*Y25(J2)*D)	85
*(Y23(J2)*Y24(J2)*D)	86
SR2=(Y24(J2)**2)+(C*(Y25(J2)**2))+(2.0*Y24(J2)*Y25(J2)*D)	87
E=SR2/SP2	88
F=SQ2/SP2	89
SP2N=(SP2*SP1)	90
SQ2N=(SQ2*SP1)	91
SR2N=(SR2*SP1)	92
C SOLUTION FOR PHI3(T)	93
J=7	94
I=1	95
T1=.50099	96
CALL RKAM (Y,DY,YD,VF,UY)	97
C COMPUTATION OF K3 OR E(X(T3)*X(T3))	98
SP3=(Y31(J3)**2)+(E*(Y32(J3)**2))+(2.0*Y31(J3)*Y32(J3)*F)	99
SQ3=(Y31(J3)*Y33(J3))+(Y32(J3)*Y34(J3)*E)+(Y31(J3)*Y34(J3)*F)+	100
*(Y32(J3)*Y33(J3)*F)	101
SR3=(Y33(J3)**2)+(E*(Y34(J3)**2))+(2.0*Y33(J3)*Y34(J3)*F)	102
G=SR3/SP3	103
H=SQ3/SP3	104
SP3N=(SP3*SP2N)	105
SQ3N=(SQ3*SP2N)	106
SR3N=(SR3*SP2N)	107
C SOLUTION FOR PHI4(T)	108
J=8	109
I=1	110
T1=.75099	111
CALL RKAM (Y,DY,YD,VF,UY)	112
C ACTUAL INITIAL CONDITIONS	113
X19=.2.0	114

TABLE C-1. (continued)

	X28=0.5	115
C	ELEMENTS OF S	116
	S1=4.0	117
	S2=1.0	118
	S4=2.0	119
	TRPB=Y10(1)+(A*Y12(1))+(2.0*Y11(1)*B)	120
	WRITE (108,170) TRPB	121
170	FORMAT (1H0,15X,1THE AVERAGE PERFORMANCE INDEX IS:',E15.8)	122
	PFBR=((X18**2)+Y10(1))+(2.0*X19-X28*Y11(1))+((X28**2)*Y12(1))	123
	WRITE (108,150) PFBR	124
150	FORMAT (1H0,15X,1THE CONSTANT PERFORMANCE INDEX BY FORMULA	125
	# IS:',E15.8)	126
C	NEW APPROXIMATION FOR THE PARAMETERS	127
	SA1=Y17(J1)	128
	SB1=Y18(J1)	129
	SC1=Y19(J1)	130
	SD1=Y20(J1)	131
	SE1=Y21(J1)	132
	DET1=(SA1*SC1)-(SB1**2)	133
	A1=-2.0*((SD1*SC1)-(SB1*SE1))/DET1	134
	A2=-2.0*((SA1*SE1)-(SB1*SD1))/DET1	135
	B1=-A1/4.0	136
	B2=-A2/4.0	137
	SA2=Y26(J2)	138
	SB2=Y27(J2)	139
	SC2=Y28(J2)	140
	SD2=Y29(J2)	141
	SE2=Y30(J2)	142
	DET2=(SA2*SC2)-(SB2**2)	143
	C1=-2.0*((SD2*SC2)-(SB2*SE2))/DET2	144
	C2=-2.0*((SA2*SE2)-(SB2*SD2))/DET2	145
	D1=-C1/4.0	146
	D2=-C2/4.0	147
	SA3=Y35(J3)	148
	SB3=Y36(J3)	149
	SC3=Y37(J3)	150
	SD3=Y38(J3)	151
	SE3=Y39(J3)	152
	DET3=(SA3*SC3)-(SB3**2)	153
	E1=-2.0*((SD3*SC3)-(SB3*SE3))/DET3	154
	E2=-2.0*((SA3*SE3)-(SB3*SD3))/DET3	155
	F1=-E1/4.0	156
	F2=-E2/4.0	157
	SA4=Y44(J4)	158
	SB4=Y45(J4)	159
	SC4=Y46(J4)	160
	SD4=Y47(J4)	161
	SE4=Y48(J4)	162
	DET4=(SA4*SC4)-(SB4**2)	163
	G1=-2.0*((SD4*SC4)-(SB4*SE4))/DET4	164
	G2=-2.0*((SA4*SE4)-(SB4*SD4))/DET4	165
	H1=-G1/4.0	166
	H2=-G2/4.0	167
	WRITE (108,551) A1,A2,C1,C2	168
	WRITE (108,552) B1,B2,D1,D2	169
	WRITE (108,553) E1,E2,F1,F2	170
	WRITE (108,554) G1,G2,H1,H2	171

TABLE C-1. (continued)

	WRITE (108,559) SP1,SQ1,SR1	172
	WRITE (108,560) SP2N,SQ2N,SR2N	173
	WRITE (108,561) SP3N,SQ3N,SR3N	174
551	FORMAT (1H0,5X,'AC PARAMETER VALUES ARE',3X,4(F8.5,3X))	175
552	FORMAT (1H0,5X,'BD PARAMETER VALUES ARE',3X,4(F8.5,3X))	176
553	FORMAT (1H0,5X,'EF PARAMETER VALUES ARE',3X,4(F8.5,3X))	177
554	FORMAT (1H0,5X,'GH PARAMETER VALUES ARE',3X,4(F10.5,3X))	178
559	FORMAT (1H0,5X,'E(XXT) AT T1 IS',3X,3(F8.5,3X))	179
560	FORMAT (1H0,5X,'E(XXT) AT T2 IS',3X,3(F8.5,3X))	180
561	FORMAT (1H0,5X,'E(XXT) AT T3 IS',3X,3(F8.5,3X))	181
25	CONTINUE	182
	END	183
	SUBROUTINE DERFUN(Y,DY,T)	1
	DOUBLE PRECISION CY,AY,BY,CY,RY,RY,ZY,PY,QY,TY	2
	DIMENSION Y(1),DY(1)	3
	EQUIVALENCE (UY(1),AY),(UY(2),BY),(UY(3),CY),(UY(4),RY)	4
	*,(UY(5),BY),(UY(6),ZY),(UY(7),PY),(UY(8),QY),(UY(9),TY)	5
	COMMON/NAM1/AY,BY,CY,RY,RY,ZY,PY,QY,TY	6
	COMMON I,J,T1,T2,N,A,B,C,D,J1,J2,E,F,G,H	7
	COMMON A1,B1,A2,B2,C1,D1,C2,D2,J3,J4	8
	COMMON E1,E2,F1,F2,G1,G2,H1,H2	9
	COMMON Y1(300),Y2(300),Y3(300),Y4(300),Y5(300),Y6(300),Y7(300),	10
	#Y8(300),Y9(300),Y10(300),Y11(300),Y12(300),Y13(300),Y14(300),	11
	#Y15(300),Y16(300),Y17(300),Y18(300),Y19(300),Y20(300),Y21(300),	12
	#Y22(300),Y23(300),Y24(300)	13
	COMMON Y25(300),Y26(300),Y27(300),Y28(300),Y29(300),Y30(300),	14
	#Y31(300),Y32(300),Y33(300),Y34(300),Y35(300),Y36(300),Y37(300),	15
	#Y38(300),Y39(300),Y40(300),Y41(300),Y42(300),Y43(300),Y44(300)	16
	#Y45(300),Y46(300),Y47(300),Y48(300)	17
	J5=J4-3	18
	J6=J3-3	19
	J7=J2-3	20
	J8=J1-3	21
	IF (J.EQ.2) GO TO 9	22
	IF (J.EQ.3) GO TO 19	23
	IF (J.EQ.4) GO TO 29	24
	IF (J.EQ.5) GO TO 55	25
	IF (J.EQ.6) GO TO 65	26
	IF (J.EQ.7) GO TO 75	27
	IF (J.EQ.8) GO TO 85	28
	IF (J.EQ.9) GO TO 12	29
	IF (J.EQ.10) GO TO 22	30
	IF (J.EQ.11) GO TO 32	31
	IF (T.GT.T1) GO TO 5	32
	IF (I.GT.J5) GO TO 4	33
	L=1+3	34
	Y1(L)=AY	35
	Y2(L)=BY	36
	Y3(L)=CY	37
4	I=I-1	38
	T1=T1-T2	39
5	M=1	40
	IF (I.LE.0) M=1	41
	TQ1=G1+(2.0*H1)	42
	TC2=G2+(2.0*H2)	43

TABLE C-1. (continued)

	ZQ1=(4.0*(H1**2))-(0.5*(S1**2))-1.0	44
	ZQ2=(4.0*(H1*H2))-(0.5*(G1*G2))	45
	ZQ3=(4.0*(H2**2))-(0.5*(G2**2))	46
	DY(1)=-(2.0*ZQ1*Y(2))+ZQ1	47
	DY(2)=-Y(1)-(TQ2*Y(2))-(TQ1*Y(3))+ZQ2	48
	DY(3)=-(2.0*Y(2))-(2.0*TQ2*Y(3))+ZQ3	49
	RETURN	50
C	MODIFYING INITIAL CONDITIONS FOR THE THIRD TIME INTERVAL	51
9	Y(1)=Y1(1)	52
	Y(2)=Y2(1)	53
	Y(3)=Y3(1)	54
	J=9	55
12	IF (T.GT.T1) GO TO 8	56
	IF (1.GT.J6) GO TO 7	57
	L=I+3	58
	Y4(L)=AY	59
	Y5(L)=BY	60
	Y6(L)=CY	61
7	I=I-1	62
	T1=T1-T2	63
8	M=1	64
	IF (1.LE.0) M=1	65
	TQ3=E1+(2.0*F1)	66
	TQ4=E2+(2.0*F2)	67
	ZQ4=(4.0*(F1**2))-(0.5*(E1**2))-1.0	68
	ZQ5=(4.0*F1*F2)-(0.5*E1*E2)	69
	ZQ6=(4.0*(F2**2))-(0.5*(E2**2))	70
	DY(1)=-(2.0*ZQ3*Y(2))+ZQ4	71
	DY(2)=-Y(1)-(TQ4*Y(2))-(TQ3*Y(3))+ZQ5	72
	DY(3)=-(2.0*Y(2))-(2.0*TQ4*Y(3))+ZQ6	73
	RETURN	74
C	MODIFYING INITIAL CONDITIONS FOR THE SECOND TIME INTERVAL	75
19	Y(1)=Y4(1)	76
	Y(2)=Y5(1)	77
	Y(3)=Y6(1)	78
	J=10	79
22	IF (T.GT.T1) GO TO 18	80
	IF (1.GT.J7) GO TO 17	81
	L=I+3	82
	Y7(L)=AY	83
	Y8(L)=BY	84
	Y9(L)=CY	85
17	I=I-1	86
	T1=T1-T2	87
18	M=1	88
	IF (1.LE.0) M=1	89
	TQ5=C1+(2.0*D1)	90
	TQ6=C2+(2.0*D2)	91
	ZQ7=(4.0*(D1**2))-(0.5*(C1**2))-1.0	92
	ZQ8=(4.0*C1*D2)-(0.5*C1*C2)	93
	ZQ9=(4.0*(D2**2))-(0.5*(C2**2))	94
	DY(1)=-(2.0*TQ5*Y(2))+ZQ7	95
	DY(2)=-Y(1)-(TQ6*Y(2))-(TQ5*Y(3))+ZQ8	96
	DY(3)=-(2.0*Y(2))-(2.0*TQ6*Y(3))+ZQ9	97
	RETURN	98
C	MODIFYING INITIAL CONDITIONS FOR THE FIRST TIME INTERVAL	99
29	Y(1)=Y7(1)	100
	Y(2)=Y8(1)	101
	Y(3)=Y9(1)	102
	J=11	103

TABLE C-1. (continued)

32	IF (T.GT.T1) GO TO 38	104
	IF (I.GT.J8) GO TO 37	105
	L=I+3	106
	Y10(L)=AY	107
	Y11(L)=BY	108
	Y12(L)=CY	109
37	I=I-1	110
	T1=T1-T2	111
38	M=I	112
	IF (I.LE.O) M=1	113
	TC7=A1+(2.O*B1)	114
	TC8=A2+(2.O*B2)	115
	ZCF=(4.O*(B1**2))-(0.5*(A1**2))-1.0	116
	ZCG=(4.O*B1*B2)-(0.5*A1*A2)	117
	ZCH=(4.O*(B2**2))-(0.5*(A2**2))	118
	DY(1)=-(2.O*TC7*Y(2))+ZCF	119
	DY(2)=-Y(1)-(TC8*Y(2))-(TC7*Y(3))+ZCG	120
	DY(3)=-(2.O*Y(2))-(2.O*TC8*Y(3))+ZCH	121
	RETURN	122
55	IF (T.LT.T1) GO TO 50	123
	IF (I.LT.4) GO TO 48	124
	M=I-3	125
	Y13(M)=AY	126
	Y14(M)=BY	127
	Y15(M)=CY	128
	Y16(M)=RY	129
	Y17(M)=BY	130
	Y18(M)=ZY	131
	Y19(M)=PY	132
	Y20(M)=QY	133
	Y21(M)=TY	134
48	I=I+1	135
	T1=T1+T2	136
50	CONTINUE	137
	TQ7=A1+(2.O*B1)	138
	TQ8=A2+(2.O*B2)	139
	DY(1)=Y(3)	140
	DY(2)=Y(4)	141
	DY(3)=(TQ7*Y(1))+(TQ8*Y(3))	142
	DY(4)=(TQ7*Y(2))+(TQ8*Y(4))	143
	DY(5)=(Y(1)**2)+(A*(Y(2)**2))+(2.O*Y(1)*Y(2)*B)	144
	DY(6)=(Y(1)*Y(3))+(A*Y(2)*Y(4))+(Y(1)*Y(4)*B)+(Y(2)*Y(3)*B)	145
	DY(7)=(Y(3)**2)+(A*(Y(4)**2))+(2.O*Y(3)*Y(4)*B)	146
	DY(8)=(Y11(M)+DY(5))+(Y12(M)+DY(6))	147
	DY(9)=(Y11(M)+DY(6))+(Y12(M)+DY(7))	148
	RETURN	149
65	IF (T.LT.T1) GO TO 60	150
	IF (I.LT.4) GO TO 58	151
	M=I-3	152
	Y22(M)=AY	153
	Y23(M)=BY	154
	Y24(M)=CY	155
	Y25(M)=RY	156
	Y26(M)=BY	157
	Y27(M)=ZY	158
	Y28(M)=PY	159
	Y29(M)=QY	160
	Y30(M)=TY	161
58	I=I+1	162
	T1=T1+T2	163
60	CONTINUE	164
	TQ5=C1+(2.O*D1)	165
	TC6=C2+(2.O*D2)	166
	DY(1)=Y(3)	167

TABLE C-1. (continued)

	DY(2)=Y(4)	168
	DY(3)=(TQ5*Y(1))+(TQ6*Y(3))	169
	DY(4)=(TQ5*Y(2))+(TQ6*Y(4))	170
	DY(5)=(Y(1)**2)+(C*(Y(2)**2))+(2.0*Y(1)*Y(2)*D)	171
	DY(6)=(Y(1)*Y(3))+(C*Y(2)*Y(4))+(Y(1)*Y(4)*D)+(Y(2)*Y(3)*D)	172
	DY(7)=(Y(3)**2)+(C*(Y(4)**2))+(2.0*Y(3)*Y(4)*D)	173
	DY(8)=(Y8(M)*DY(5))+(Y9(M)*DY(6))	174
	DY(9)=(Y8(M)*DY(6))+(Y9(M)*DY(7))	175
	RETURN	176
75	IF (T.LT.T1) GO TO 70	177
	IF (I.LT.4) GO TO 68	178
	M=I-3	179
	Y31(M)=AY	180
	Y32(M)=BY	181
	Y33(M)=CY	182
	Y34(M)=RY	183
	Y35(M)=BY	184
	Y36(M)=ZY	185
	Y37(M)=PY	186
	Y38(M)=QY	187
	Y39(M)=TY	188
68	I=I+1	189
	T1=T1+T2	190
70	CONTINUE	191
	TQ3=E1+(2.0*F1)	192
	TQ4=E2+(2.0*F2)	193
	DY(1)=Y(3)	194
	DY(2)=Y(4)	195
	DY(3)=(TQ3*Y(1))+(TQ4*Y(3))	196
	DY(4)=(TQ3*Y(2))+(TQ4*Y(4))	197
	DY(5)=(Y(1)**2)+(E*(Y(2)**2))+(2.0*Y(1)*Y(2)*F)	198
	DY(6)=(Y(1)*Y(3))+(E*Y(2)*Y(4))+(Y(1)*Y(4)*F)+(Y(2)*Y(3)*F)	199
	DY(7)=(Y(3)**2)+(E*(Y(4)**2))+(2.0*Y(3)*Y(4)*F)	200
	DY(8)=(Y5(M)*DY(5))+(Y6(M)*CY(6))	201
	DY(9)=(Y5(M)*CY(6))+(Y6(M)*DY(7))	202
	RETURN	203
85	IF (T.LT.T1) GO TO 80	204
	IF (I.LT.4) GO TO 78	205
	M=I-3	206
	Y40(M)=AY	207
	Y41(M)=BY	208
	Y42(M)=CY	209
	Y43(M)=RY	210
	Y44(M)=BY	211
	Y45(M)=ZY	212
	Y46(M)=PY	213
	Y47(M)=QY	214
	Y48(M)=TY	215
78	I=I+1	216
	T1=T1+T2	217
80	CONTINUE	218
	TQ1=G1+(2.0*H1)	219
	TQ2=G2+(2.0*H2)	220
	DY(1)=Y(3)	221
	DY(2)=Y(4)	222
	DY(3)=(TQ1*Y(1))+(TQ2*Y(3))	223
	DY(4)=(TQ1*Y(2))+(TQ2*Y(4))	224
	DY(5)=(Y(1)**2)+(G*(Y(2)**2))+(2.0*Y(1)*Y(2)*H)	225
	DY(6)=(Y(1)*Y(3))+(G*Y(2)*Y(4))+(Y(1)*Y(4)*H)+(Y(2)*Y(3)*H)	226
	DY(7)=(Y(3)**2)+(G*(Y(4)**2))+(2.0*Y(3)*Y(4)*H)	227
	DY(8)=(Y2(M)*DY(5))+(Y3(M)*DY(6))	228
	DY(9)=(Y2(M)*DY(6))+(Y3(M)*DY(7))	229
	RETURN	230
	END	231

TABLE C-2.

RESULTS FOR A VECTOR CASE--FOUR GAIN CHANGE POINTS
 (Initial Conditions Not Included in the Design)
 (Actual Initial Conditions-- $x_{10} = -2.0$, $x_{20} = 0.5$)

Number of Iterations	0	1	7
Parameter <u>A</u>	[-3.6, -4.0]	[-3.3836 -3.7036]	[-3.3783 -3.6520]
<u>B</u>	[0.9, 1.0]	[.8459 .9259]	[.8446 .9130]
<u>C</u>	[-3.2, -2.8]	[-3.4397 -3.6520]	[-3.4185 -3.6002]
<u>D</u>	[0.8, 0.7]	[.8599 .9130]	[.8546 .9000]
<u>E</u>	[-2.8, -3.2]	[-3.1654 -3.5583]	[-3.1507 -3.5445]
<u>F</u>	[0.7, 0.8]	[.7913 .8896]	[.7877 .8861]
<u>G</u>	[-2.0, -3.2]	[-2.4891 -3.7098]	[-2.4929 -3.7126]
<u>H</u>	[0.5, 0.8]	[.6223 .9275]	[.6232 .9282]
J	7.4730	7.4506	7.4504
$\hat{J}_N$	4.4248	4.007	4.005
$E(x(t_1)x'(t_1))$	$\begin{bmatrix} .9445 & -.2217 \\ -.2217 & .4415 \end{bmatrix}$	$\begin{bmatrix} .9503 & -.2015 \\ -.2015 & .4596 \end{bmatrix}$	$\begin{bmatrix} .9505 & -.2008 \\ -.2008 & .4646 \end{bmatrix}$
$E(x(t_2)x'(t_2))$	$\begin{bmatrix} .7938 & -.3644 \\ -.3644 & .3974 \end{bmatrix}$	$\begin{bmatrix} .8070 & -.3543 \\ -.3543 & .3526 \end{bmatrix}$	$\begin{bmatrix} .8078 & -.3528 \\ -.3528 & .3561 \end{bmatrix}$
$E(x(t_3)x'(t_3))$	$\begin{bmatrix} .6084 & -.3678 \\ -.3678 & .3573 \end{bmatrix}$	$\begin{bmatrix} .6190 & -.3833 \\ -.3833 & .3429 \end{bmatrix}$	$\begin{bmatrix} .6206 & -.3819 \\ -.3819 & .3434 \end{bmatrix}$

DATA CARDS FOR ONE ITERATION

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

3.	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	1.0	.746							
	4.0	1.0	2.0						
	.75	-.001	.746						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

3.	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.75	.496							
	4.0	1.0	2.0						
	.50	-.001	.496						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

3.	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.50	.246							
	4.0	1.0	2.0						
	.25	-.001	.246						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

3.	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.25	-.004							
	4.0	1.0	2.0						
	0.0	-.001	-.004						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

9	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	0.0	.254							
	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
	0.0								
	.25	.001	.254						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

9	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.25	.504							
	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
	0.0								
	.50	.001	.504						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

9	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.50	.754							
	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
	0.0								
	.75	.001	.754						

PARAMETER OPTIMIZATION IN DIFFERENTIAL GAMES

9	1	1	2	1	0	0	0	1.0E-06	.5
	.001	1.0E-04	1.0E-07	1.0E00	1.0E-06				
	.75	1.004							
	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
	0.0								
	1.0	.001	1.004						

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