



A continuum mixture theory applied to stress waves in snow
by George Edward Austiguy

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in
Engineering Mechanics
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Abstract:

In avalanche control work the types of explosives and delivery methods used are primarily determined by trial and error. Understanding the propagation of stress waves in snow is a step towards eliminating some of this guesswork.

A continuum theory of mixtures is applied to model snow as a mixture of an elastic solid and an elastic fluid. Three wave types, two dilational and one rotational wave are shown to exist. Theoretical expressions are developed for the wave attenuation and propagation velocity of each of the wave types.

Numerical evaluation shows velocity and attenuation increasing with frequency for all three waves. Wave velocity increases with increasing density while attenuation decreases with increasing density for all three waves. The first dilational wave has a slow wave speed and is highly attenuated. This wave exhibits diffusive behavior at low frequencies and nondispersive behavior at high frequencies. The second dilation wave is the fastest of the three wave types and does not appreciably attenuate. Nondispersive wave behavior characterizes this wave at low and high frequencies. The rotational wave is the least attenuated of all three waves and propagates at velocities greater than that of the first dilational wave but less than that of the second dilational wave. The rotational wave exhibits nondispersive behavior at low and high frequencies. Wave velocities and attenuation show behavior that is in agreement with existing experimental data.

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency and is ready for submission to the College of Graduate Studies.

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ABSTRACT

In avalanche control work the types of explosives and delivery methods used are primarily determined by trial and error. Understanding the propagation of stress waves in snow is a step towards eliminating some of this guesswork.

A continuum theory of mixtures is applied to model snow as a mixture of an elastic solid and an elastic fluid. Three wave types, two dilational and one rotational wave are shown to exist. Theoretical expressions are developed for the wave attenuation and propagation velocity of each of the wave types.

Numerical evaluation shows velocity and attenuation increasing with frequency for all three waves. Wave velocity increases with increasing density while attenuation decreases with increasing density for all three waves. The first dilational wave has a slow wave speed and is highly attenuated. This wave exhibits diffusive behavior at low frequencies and nondispersive behavior at high frequencies. The second dilation wave is the fastest of the three wave types and does not appreciably attenuate. Nondispersive wave behavior characterizes this wave at low and high frequencies. The rotational wave is the least attenuated of all three waves and propagates at velocities greater than that of the first dilational wave but less than that of the second dilational wave. The rotational wave exhibits nondispersive behavior at low and high frequencies. Wave velocities and attenuation show behavior that is in agreement with existing experimental data.

CHAPTER 1

INTRODUCTION

Snow is a material which affects the activities of life in the temperate and polar latitudes causing significant and diverse effects on the environment and society. In its capacity as a water storage source, below normal snowpacks can result in drought while above normal snowpacks can bring on flooding. Blowing and drifting snow causes traveling hazards, transportation problems, and increased loading on buildings and other structures (McKay,1981). These effects of snow can lead to destruction of property, economic hardship and loss of life. In the winter of 1976 - 77 it was estimated that snow and cold weather reduced the United States GNP by \$20 billion (McKay,1981). In spite of these problems snow is an indispensable resource as a water supply for homes, livestock, wildlife, agricultural production and hydropower (McKay,1981). It is a recreational resource for activities such as skiing and snowmobiling and is used as a construction material in polar regions for roads and airstrips (McKay,1981). Avalanches are one of snow's more violent manifestations, producing powerful forces which can transport rock, soil and vegetation as well as ice and snow. When man's interests and avalanche terrain coincide, results can be devastating. Many instances are beyond human

control such as the massive, earthquake triggered avalanche which fell from the summit of North Huascarán, Peru in 1970 destroying the town of Yungay and killing 20,000 people (Perla and Martinelli, 1976). However many potential disasters can be avoided with proper planning and avalanche control measures. Due to increased recreation and development in mountain areas the recorded incidence of avalanches is becoming greater and the number of people affected by avalanche activity is increasing (SNHM, 1990). Avalanche activity is a significant hazard in the western United States having negative economic effects and resulting in destruction of life and property (SNHM, 1990). Present average annual mortality rates for snow avalanches exceed those due to earthquakes and all other forms of slope failure combined (SNHM, 1990). In North America and Europe, recreational, residential, and commercial use of alpine terrain has required an evolution of avalanche control techniques in order to limit or mitigate the damage caused by avalanches. These techniques consist of a variety of procedures. One of the more active measures involves the initiation of avalanches with explosives, thereby either avoiding large avalanches or eliminating the potential of the slope to avalanche unexpectedly. This technique is widely used in ski areas and over threatened highways. Over 100,000 explosive charges are detonated annually for avalanche control (Perla, 1978). The type of explosive, size of the charge and the style of delivery are primarily determined by trial and error. Information leading

to a more efficient determination of how to employ explosives in releasing avalanches is thus desirable. When an explosive charge is detonated in or closely above the snowpack, inelastic deformation in the immediate vicinity of the explosion takes place forming a crater and a region of cracks surrounding the crater. Outside of this region, inelastic deformation of the snow becomes insignificant but attenuation of the stress waves still occur through geometric spreading, porous structure effects and effects of ice structure/pore material interaction (Brown, 1980). It is the propagation of stress waves in the region outside of the crater which this investigation seeks to address.

The continuum theory of mixtures is a theory of mixtures based on rational thermodynamics. The fundamental premise is that the constituents of the mixture can be modeled as superimposed continua such that each point in the mixture is occupied by a material point of each constituent. Balance statements are then postulated for each constituent modeled on those of a single continua but contain additional supply terms to account for the interactions between the constituents. In the last twenty years much effort has been devoted to preserving the generality of the theory, in particular with respect to the constitutive equations. This generality has resulted in complex and cumbersome constitutive relations which inhibit the application process. Bedford and Drumheller [1983] cite this as one of the factors why applications of the theory have been so slow in

developing and that there is not a single comparison of theoretical results with experimental data in extensive reviews by Atkin and Craine [1976] and Bowen [1976]. While the main impetus is to study wave propagation in snow, it is felt that in light of the above statements the application process has merit in and of itself and is an appropriate and constructive course to pursue. As procedures used in applying the continuum theory of mixtures can be extended to mixtures other than snow, the application process has benefits outside the realm of snow mechanics.

Applications of the continuum theory of mixtures to model snow has seen relatively little attention. Decker and Brown [1983, 1985] used it to model a turbulent snow and air mixture. Adams and Brown [1989, 1990] applied a continuum mixture theory to model physical phenomena associated with snow metamorphism. There is no evidence of modern mixture theory being applied to the propagation of stress waves in snow. Bowen [1972, 1978, 1975] has investigated general wave fronts, shock waves and harmonic plane waves in binary mixtures of elastic materials, but these investigations have remained on the theoretical level. Atkins [1968] examined the propagation of small amplitude waves for a fictitious mixture of an isotropic elastic solid and an inviscid fluid with an early version of a continuum mixture theory. Brown [1980] studied plastic waves in snow but did not employ a mixture theory formulation in his work. Johnson [1978] in his Ph.D. thesis investigated elastic stress waves in snow using an early porous

media model developed by Biot. Biot's theory was specifically formulated to model the linear elastic behavior of a fluid saturated porous media rather than being a specialized case of a more general mixture theory. While his work was a significant contribution to porous media modeling there exists certain technical problems with it (Bowen and Wright, 1972). Specifically the constitutive relations for the momentum supplies do not satisfy certain thermodynamic constraints.

This investigation seeks to determine the propagating characteristics of stress waves in snow by using the continuum theory of mixtures to model snow as an isothermal mixture of an elastic ice frame and an inviscid fluid. Because the effects of heat conduction are most pronounced at high frequencies (Atkins, 1968) results for the low frequency regime are the most physically significant. A brief introduction to the continuum theory of mixtures is presented in Chapter 2. Because this is an isothermal model the thermodynamics of mixtures will not be addressed, although results of the second law will be used in implementing the theory. In Chapter 3 the specialized form of the mixture theory equations are presented for a binary mixture which is applicable to snow. The work which follows develops the relationships which govern propagation velocities and attenuation of a harmonic wave for one dimension in the low and high frequency regimes. Finally the results are presented in a graphical form.

CHAPTER 2

THE CONTINUUM THEORY OF MIXTURES

In this chapter the basic definitions and the development of the general continuum theory of mixtures is presented. For more detail on the development of mixture theory the reader is referred to Bowen [1976].

Tensors are denoted by boldface uppercase format while vectors are represented by lower case bold type and scalars as plain text.

Kinematics

A mixture is defined as a body $\mathcal{B}$ consisting of a combination of different materials. Each different material is considered a body in its own right and is denoted as $\mathcal{B}_a$; $a = 1, 2, \dots, N$, where N is the total number of materials in the mixture. Each body $\mathcal{B}_a$ is called a phase or constituent. For every phase $\mathcal{B}_a$ a fixed but otherwise arbitrary reference configuration and a motion can be assigned. In describing the motion of a continuum there are four formulations which are commonly used (Malvern, 1969). The material description, the Lagrangian description, the spatial description and the relative description. The Lagrangian and the spatial

descriptions are the ones commonly used in the theory of elasticity and thus will be used in this development. The Lagrangian formulation is in terms of the undeformed configuration which is customarily used as the reference configuration. The spatial formulation is in terms of the deformed configuration. In the reference position the particle X_a occupies the position X_a while in the deformed configuration the particle X_a at time t has the position:

$$x_a = \chi_a(X_a, t) \quad (1)$$

Where X_a is the coordinate in the reference configuration of a particle in the a th body and χ_a is the deformation function for the a th body. The deformation function χ_a is assumed invertible so that:

$$X_a = \chi_a^{-1}(x_a, t) \quad (2)$$

This relationship is shown in Figure 1

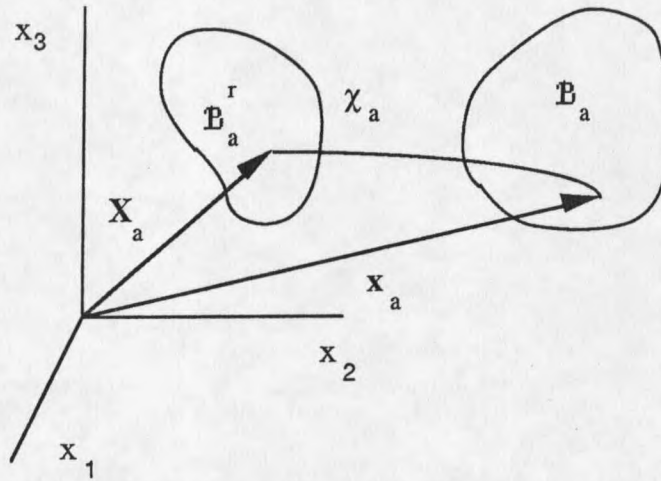


Figure 1: Mapping of the deformation function.

In Figure 1 the superscript r denotes the reference state. The velocity and acceleration of the particle X_a at time t are defined by:

$$\dot{\mathbf{x}}_a = \frac{\partial \chi_a(\mathbf{X}_a, t)}{\partial t} \quad (3)$$

$$\ddot{\mathbf{x}}_a = \frac{\partial^2 \chi_a(\mathbf{X}_a, t)}{\partial t^2} \quad (4)$$

The prime is indicative of the material derivative following the motion of the a th constituent. In a mixture of N constituents the bodies B_a , $a = 1, 2, \dots, N$ can occupy the same portions of space. It is assumed that each spatial position $\mathbf{x}$ is occupied by a particle from each constituent. This is shown for $N=2$ in Figure 2.

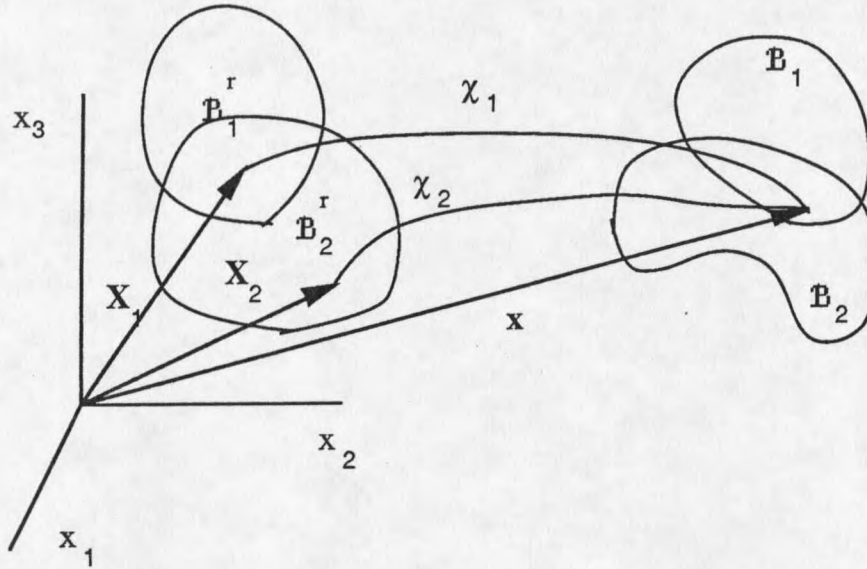


Figure 2: Constituent spatial position.

The density for the mixture is defined by:

$$\rho = \sum_{a=1}^N \rho_a \quad (5)$$

where ρ_a is the partial density with the definition:

$$\rho_a = \phi_a \gamma_a \quad (6)$$

ϕ_a is the volume fraction of the ath constituent and γ_a is the material density of the ath constituent. A more general

development considers the volume fraction of each constituent as an independent kinematic variable and generates additional balance equations to govern changes in the volume fraction. In the development presented here the volume fraction is considered constant. For the more general development see papers by Bowen [1982] or Passman, Nunziato, and Walsh [1984]. The velocity of the mixture is defined as the mass weighted average of the constituent velocities.

$$\rho \dot{\mathbf{x}} = \sum_{a=1}^N \rho_a \dot{\mathbf{x}}_a \quad (7)$$

The diffusion velocity of the ath constituent is defined as:

$$\mathbf{u}_a = \dot{\mathbf{x}}_a - \dot{\mathbf{x}} \quad (8)$$

In the Lagrangian coordinates the deformation is described in terms of the deformation gradient of $\mathbf{X}_a$ at time t and is given by:

$$\mathbf{F}_a = \mathbf{x}_a \overleftarrow{\nabla} = \chi_a(\mathbf{X}_a, t) \overleftarrow{\nabla} \quad (9)$$

$\overleftarrow{\nabla}$ is the gradient operator with respect to the Lagrangian coordinates $\mathbf{X}_a$. Eq.(9) has the inverse:

$$\mathbf{F}_a^{-1} = \chi_a^{-1}(\mathbf{x}_a, t) \hat{\nabla}_{\mathbf{x}} = \mathbf{X}_a \hat{\nabla}_{\mathbf{x}} \quad (10)$$

Where $\nabla_{\mathbf{x}}$ is the gradient operator with respect to the deformed coordinates $\mathbf{x}_a$. The velocity gradient for the a th constituent is defined in terms of the spatial coordinates as:

$$\mathbf{L}_a = \dot{\mathbf{x}}_a(\mathbf{x}, t) \hat{\nabla}_{\mathbf{x}} \quad (11)$$

This can be written in terms of the deformation gradient as:

$$\mathbf{L}_a = \dot{\mathbf{F}}_a \mathbf{F}_a^{-1} \quad (12)$$

Mass Balance Equation

In mixture theory balance principles follow what is known as the Mixture Balance Principle. This principle states that when the statement of balance for each constituent is summed over all the constituents, the sum must have the same form as the balance equation for a single constituent continuum. For a fixed spatial volume V the global balance of mass for the a th constituent is postulated as:

$$\frac{\partial}{\partial t} \left(\int_V \rho_a dV \right) = - \int_{\partial V} \rho_a \mathbf{x}_a \cdot \mathbf{n} ds + \int_V \hat{c}_a dV \quad (13)$$

This states that the rate of change in mass for the ath constituent contained in the region is due to the flux of mass across the boundary, plus the mass supply from the other constituents. In this balance statement the term $\hat{c}_a$ is called the mass supply term for the ath constituent and represents the rate at which the ath constituent is gaining mass from the other constituents. Thus if there are no chemical reactions or phase changes $\hat{c}_a = 0$.

The balance statement for the mixture is:

$$\frac{\partial}{\partial t} \left(\int_V \rho dV \right) = - \int_{\partial V} \rho \mathbf{x} \cdot \mathbf{n} ds \quad (14)$$

Note that Eq.(14) has the same form as that for a single constituent material. Physically it states that the rate of change of mass for the mixture is due to the mass flux across the boundary. Thus there is no net mass production in the mixture. By applying the divergence theorem the local form of the balance statements can be obtained for an arbitrary fixed volume. For the ath constituent this yields:

$$\frac{\partial \rho_a}{\partial t} + \vec{\nabla} \cdot (\rho_a \mathbf{x}_a) = \hat{c}_a \quad (15)$$

and for the mixture:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{x}) = 0 \quad (16)$$

Finally, when Eq.(14) is summed over N constituents and Eq.(5), Eq.(7) and Eq.(15) are considered, the mixture balance principle requires that:

$$\sum_{a=1}^N \hat{c}_a = 0 \quad (17)$$

Balance of Momentum

The balance of momentum equations take the form of balance statements for linear momentum and angular momentum for both the mixture and the constituents. For a mixture of two or more materials it is advantageous to derive the momentum balance statements on a given spatial domain (Malvern,1969). Thus consider the fixed volume V in space represented in Figure 3.

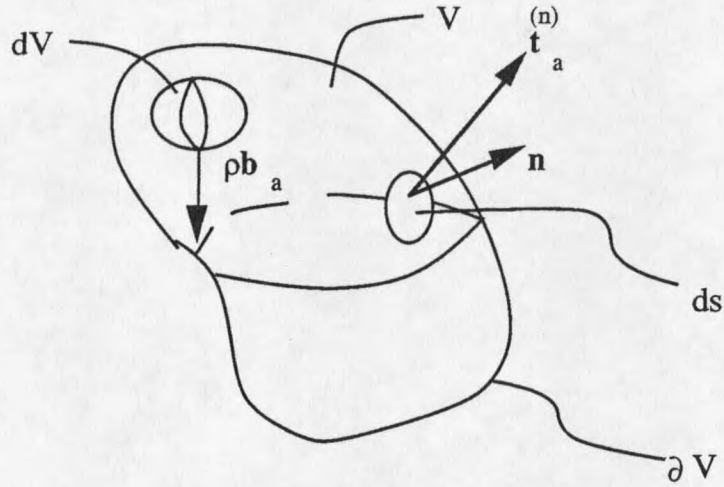


Figure 3: Fixed volume in space.

The balance of linear momentum for the ath constituent is postulated as:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_V \rho_a \dot{\mathbf{x}}_a dV \right) = & - \int_{\partial V} \{ \rho_a \dot{\mathbf{x}}_a (\dot{\mathbf{x}}_a \cdot \mathbf{n}) - \mathbf{t}_a^{(n)} \} ds \\ & + \int_V (\rho_a \mathbf{b}_a + \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \dot{\mathbf{x}}_a) dV \end{aligned} \quad (18)$$

The first term on the right hand side accounts for the momentum flux across the boundary ∂V plus the contact force (i.e. surface loading) on the ath constituent at the boundary ∂V , where $\mathbf{t}_a^{(n)}$ is the stress vector acting on the ath constituent. The second term on the right hand side accounts for the body forces acting on the ath constituent plus the complete local interaction force on the ath constituent from the other constituents in V . $\mathbf{b}_a$ is the body force

acting on the ath constituent, $\hat{\mathbf{p}}_a$ is called the momentum supply, which represents interactions with other phases and $\hat{\mathbf{c}}_a \dot{\mathbf{x}}_a$ represents the change in momentum due to a change in mass for the ath constituent. Substituting $\mathbf{t}_a^{(n)} = \mathbf{n} \cdot \mathbf{T}_a$ for the stress vector allows application of the divergence theorem and consideration of an arbitrary volume yields the local form of Eq.(18). Finally making use of Eq.(15) yields :

$$\rho_a \ddot{\mathbf{x}}_a = \vec{\nabla} \cdot \mathbf{T}_a + \hat{\mathbf{p}}_a + \rho_a \mathbf{b}_a \quad (19)$$

The linear momentum balance statement for the mixture is postulated as:

$$\rho \ddot{\mathbf{x}} = \vec{\nabla} \cdot \mathbf{T} + \rho \mathbf{b} \quad (20)$$

In consideration of Eq.(19) and Eq.(20) the summation rule of the mixture balance principle requires:

$$\mathbf{T} = \sum_{a=1}^N \{ \mathbf{T}_a - \rho_a (\mathbf{u}_a \mathbf{u}_a) \} \quad (21)$$

$$\sum_{a=1}^N \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \mathbf{u}_a = 0 \quad (22)$$

The balance of moment of momentum for the ath constituent is postulated as:

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \mathbf{x} \times \rho_a \dot{\mathbf{x}}_a dV = & - \int_{\partial V} (\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) (\dot{\mathbf{x}}_a \cdot \mathbf{n}) ds \\ & + \int_{\partial V} \mathbf{x} \times \mathbf{t}_a^{(n)} ds + \int_V \{ \mathbf{x} \times (\rho_a \mathbf{b} + \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \dot{\mathbf{x}}_a) + \hat{\mathbf{m}}_a \} dV \end{aligned} \quad (23)$$

The term $\hat{\mathbf{m}}_a$ is called the moment of momentum supply vector.

The term $\int_V \{ \mathbf{x} \times (\hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \dot{\mathbf{x}}_a) + \hat{\mathbf{m}}_a \} dV$ accounts for the moment due to the local interaction of the ath constituent with the other constituents in V . Using the divergence theorem the flux across the boundary can be written as:

$$- \int_{\partial V} (\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) (\dot{\mathbf{x}}_a \cdot \mathbf{n}) ds = - \int_V \{ [(\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) \dot{\mathbf{x}}_a] \cdot \hat{\mathbf{V}} \} dV \quad (24)$$

Where the definition of the tensor product has been used.

Applying the relationship $\mathbf{t}_a^{(n)} = \mathbf{n} \cdot \mathbf{T}_a$ and the tensor identities:

$$\mathbf{v} \cdot \mathbf{A} = \mathbf{A}^T \cdot \mathbf{v} \quad (25)$$

$$(\mathbf{x} \times \mathbf{A}) \cdot \mathbf{v} = (\mathbf{x} \times \mathbf{A} \cdot \mathbf{v}) \quad (26)$$

Where A is any tensor and v is any vector, the second surface integral in Eq.(23) can be written:

$$\int_{\partial V} (\mathbf{x} \times \mathbf{t}_a^{(n)}) ds = \int_{\partial V} \{(\mathbf{x} \times \mathbf{T}_a^T) \cdot \mathbf{n}\} ds \quad (27)$$

Then applying the divergence theorem transforms it to a volume integral yielding:

$$\int_{\partial V} \{(\mathbf{x} \times \mathbf{T}_a^T) \cdot \mathbf{n}\} ds = \int_V \{(\mathbf{x} \times \mathbf{T}_a^T) \cdot \hat{\nabla}\} dV \quad (28)$$

All terms in Eq.(23) can now be written within a volume integral. Since Eq.(23) is valid for an arbitrary volume the integral must vanish. Therefore:

$$\begin{aligned} \frac{\partial}{\partial t} (\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) = & -\{(\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) \cdot \dot{\mathbf{x}}_a\} \cdot \hat{\nabla} + (\mathbf{x} \times \mathbf{T}_a^T) \cdot \hat{\nabla} \\ & + \mathbf{x} \times (\rho_a \mathbf{b} + \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \dot{\mathbf{x}}_a) + \hat{\mathbf{m}}_a \end{aligned} \quad (29)$$

The second term on the right hand side can be rewritten using the identity:

$$(\mathbf{x} \times \mathbf{T}_a^T) \cdot \vec{\nabla} = \mathbf{x} \times (\mathbf{T}_a^T \cdot \vec{\nabla}) + \mathbf{t}_{aA} \quad (30)$$

The vector $\mathbf{t}_{aA}$ has the Cartesian form:

$$\mathbf{t}_{aA} = (T_{a23} - T_{a32})\mathbf{e}_1 + (T_{a31} - T_{a13})\mathbf{e}_2 + (T_{a12} - T_{a21})\mathbf{e}_3 \quad (31)$$

Where $\mathbf{e}_i$, $i=1,2,3$ are the unit vectors along the coordinate axes.

Then differentiating the first term in Eq.(29) and reorganizing yields:

$$\begin{aligned} & \mathbf{x} \times \{ \rho_a \ddot{\mathbf{x}}_a - \vec{\nabla} \cdot \mathbf{T}_a - \hat{\mathbf{p}}_a - \rho_a \mathbf{b} \} \\ & + \mathbf{x} \times \left\{ \frac{\partial \rho_a}{\partial t} \dot{\mathbf{x}}_a + (\vec{\nabla} \cdot \rho_a \dot{\mathbf{x}}_a) \dot{\mathbf{x}}_a - \hat{\mathbf{c}}_a \dot{\mathbf{x}}_a \right\} \\ & = \mathbf{t}_{aA} + \hat{\mathbf{m}}_a \end{aligned} \quad (32)$$

Considering Eq.(15) and Eq.(19) it is evident that the left hand side is zero leaving :

$$\mathbf{t}_{aA} + \hat{\mathbf{m}}_a = 0 \quad (33)$$

A more convenient form of (33) is:

$$\hat{\mathbf{M}}_a = \mathbf{T}_a - \mathbf{T}_a^T \quad (34)$$

In this expression $\widehat{\mathbf{M}}_a$ is a skew symmetric linear transformation with the components:

$$\begin{aligned}\widehat{M}_{a11} &= \widehat{M}_{a22} = \widehat{M}_{a33} = 0 \\ \widehat{M}_{a32} &= -\widehat{M}_{a23} = \widehat{m}_{a1} \\ \widehat{M}_{a13} &= -\widehat{M}_{a31} = \widehat{m}_{a2} \\ \widehat{M}_{a21} &= -\widehat{M}_{a12} = \widehat{m}_{a3}\end{aligned}\tag{35}$$

Eq.(34) is taken to be the axiom of the balance of moment of momentum for the ath constituent and shows that the stress tensor is not symmetric, unless of course, $\widehat{\mathbf{M}}_a$ is zero. Local representation of the moment of momentum balance statement for the mixture is given by:

$$\overline{\rho \mathbf{x} \times \dot{\mathbf{x}}} = \vec{\nabla} \cdot (\mathbf{x} \times \mathbf{T}) + \rho \mathbf{x} \times \mathbf{b}\tag{36}$$

Use of the same argument yields the axiom of the balance of the moment of momentum for the mixture:

$$\mathbf{T} = \mathbf{T}^T.\tag{37}$$

Thus the stress tensor for the mixture is symmetric even though the constituent stress tensor is not. The mixture balance principle requires the following summation rule.

$$\sum_{a=1}^N \widehat{\mathbf{M}}_a = 0 \quad (38)$$

Summary

As a summary and for ease of reference the field equations and summation constraints which result from the continuum theory of mixtures are collected and presented below.

Balance of Mass

$$\frac{\partial \rho_a}{\partial t} + \vec{\nabla} \cdot (\rho_a \dot{\mathbf{x}}_a) = \widehat{c}_a \quad \text{Constituent} \quad (39)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \dot{\mathbf{x}}) = 0 \quad \text{Mixture} \quad (40)$$

Balance of Linear Momentum

$$\rho_a \ddot{\mathbf{x}}_a = \vec{\nabla} \cdot \mathbf{T}_a + \widehat{\mathbf{p}}_a + \rho_a \mathbf{b}_a \quad \text{Constituent} \quad (41)$$

$$\rho \ddot{\mathbf{x}} = \vec{\nabla} \cdot \mathbf{T} + \rho \mathbf{b} \quad \text{Mixture} \quad (42)$$

Balance of Moment of Momentum

$$\widehat{\mathbf{M}}_a = \mathbf{T}_a - \mathbf{T}_a^T \quad \text{Constituent} \quad (43)$$

$$\mathbf{T} = \mathbf{T}^T \quad \text{Mixture} \quad (44)$$

Summation Constraints

$$\sum_{a=1}^N \hat{\mathbf{c}}_a = 0 \quad (45)$$

$$\mathbf{T} = \sum_{a=1}^N \{ \mathbf{T}_a - \rho_a (\mathbf{u}_a \mathbf{u}_a) \} \quad (46)$$

$$\sum_{a=1}^N \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a \mathbf{u}_a = 0 \quad (47)$$

$$\sum_{a=1}^N \widehat{\mathbf{M}}_a = 0 \quad (48)$$

CHAPTER 3

APPLICATION TO SNOW

Snow may be modeled as a binary mixture consisting of a solid phase and a fluid phase. The mixture is comprised of an ice matrix or skeleton in which the void spaces are filled with air. Thus the solid phase is represented by the ice matrix while the fluid phase is represented by the air. In general snow is considered to be an elastic-visco-plastic material. However when deformation occurs rapidly (where the yield stress is not exceeded) and then stops inelastic strains do not have time to form appreciably and thus are small enough to be negligible (private communication, R.L. Brown, 1991). Thus for high strain rates of short duration the snow responds in a linear elastic type manner. Langham [1981] assures us that snow may be deformed elastically when subjected to a small load applied over a short duration of time. Mellor [1977] further states that: "A rapidly generated pulse of stress or strain propagates as an elastic wave when the amplitude is less than the failure stress or the failure strain of the material." For snow this is not entirely true. While the material may respond in an elastic type manner when inelastic strains are negligible, the wave does attenuate due to porous structure effects and interaction of the ice and fluid phases (Brown, 1980). Thus

the wave is not a true elastic wave in the strictest sense. Mellor goes on to say that for the case of snow : "Maximum sustainable amplitude for an elastic wave probably corresponds quite closely to the uniaxial strength of snow for one dimensional propagation in a long bar and to the collapse strength of a wide layer for plane wave propagation". Thus it is apparent that snow can behave elastically under certain types of loading. It is desirable to have as simple a formulation as is reasonable. To this end the snow is assumed to be of a simple form, i.e. well sintered grains of equal size with homogeneous and isotropic properties. As a further simplification chemical reactions are omitted and the effects of body forces are neglected. It will be convenient to express the field equations in terms of the displacements. The displacement of the ath constituent is defined as:

$$\mathbf{w}_a = \mathbf{x}_a - \mathbf{X}_a \quad (49)$$

Using the standard linearization process from the theory of elasticity, the equations of motion for the ath constituent take the form:

$$\rho_{ak} \frac{\partial^2 \mathbf{w}_a}{\partial t^2} = \vec{\nabla} \cdot \mathbf{T}_a + \hat{\mathbf{p}}_a \quad (50)$$

Or in terms of the solid and gas phase:

$$\rho_{sR} \frac{\partial^2 \mathbf{w}_s}{\partial t^2} = \vec{\nabla} \cdot \mathbf{T}_s + \hat{\mathbf{p}}_s \quad (51)$$

$$\rho_{gR} \frac{\partial^2 \mathbf{w}_g}{\partial t^2} = \vec{\nabla} \cdot \mathbf{T}_g + \hat{\mathbf{p}}_g \quad (52)$$

Where the subscripts g and s are labels for the gas and the solid phase, respectively, and the subscript R denotes the reference state. Constitutive equations for the stress tensors and the momentum supply vector are required to put the equations of motion in a closed form. The derivations of the constitutive equations are both tedious and complex, adding little insight to the application process. Thus they will simply be presented. For a rigorous development of the constitutive equations presented here, the reader is referred to Bowen [1976]. The linearized version of the constitutive equations are:

$$\mathbf{T}_s = -\Pi_{sR} \mathbf{I} + (\sigma_{sg} + \lambda_{gs})(\text{tr} \mathbf{E}_g) \mathbf{I} + (\lambda_s - \sigma_{gs})(\text{tr} \mathbf{E}_s) \mathbf{I} + 2\mu_s \mathbf{E}_s \quad (53)$$

$$\mathbf{T}_g = -\Pi_{gR} \mathbf{I} - (\sigma_{sg} - \lambda_g)(\text{tr} \mathbf{E}_g) \mathbf{I} + (\sigma_{gs} + \lambda_{gs})(\text{tr} \mathbf{E}_s) \mathbf{I} \quad (54)$$

$$\hat{\mathbf{p}}_g = -\hat{\mathbf{p}}_s = \sigma_{sg} \vec{\nabla}(\text{tr} \mathbf{E}_g) - \sigma_{gs} \vec{\nabla}(\text{tr} \mathbf{E}_s) - \xi \left(\frac{\partial \mathbf{w}_g}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \quad (55)$$

Where E_s is the infinitesimal strain tensor for the solid and E_g is the infinitesimal strain tensor for the gas.

$$E_s = 1/2(\mathbf{w}_s \vec{\nabla} + \vec{\nabla} \mathbf{w}_s) \quad (56)$$

$$E_g = 1/2(\mathbf{w}_g \vec{\nabla} + \vec{\nabla} \mathbf{w}_g) \quad (57)$$

$\mathbf{I}$ is the identity tensor, Π_{gr} and Π_{sr} are constants which represent the pressure in each constituent when there is no strain measured relative to the reference state. $\sigma_{sg}, \sigma_{gs}, \lambda_{gs}, \mu_s, \lambda_g, \lambda_s$, are material constants. σ_{sg} and σ_{gs} are coupling coefficients which arise because of a dependence of the momentum supplies on the strain gradients. They account for the local interaction of the solid and the fluid that occurs even in static situations. The term $(\lambda_s - \sigma_{gs})$ is analogous to the lame parameter λ in elasticity. μ_s is the shear modulus for the solid phase. $(\sigma_{sg} + \lambda_{gs})$ accounts for the dependence of the stress on the solid due to the strain of the fluid. $(\sigma_{gs} + \lambda_{gs})$ accounts for the stress on the fluid due to the strain of the solid. The term $(\lambda_g - \sigma_{sg})$ is related to the modulus of elasticity for the fluid. ξ is the Stokes drag coefficient and arises due to the momentum supplies dependence on the relative velocities. ξ accounts for the drag force between the constituents due to the relative motion. It is a thermodynamic result that :

$$\xi > 0 \quad (58)$$

To insure that the isothermal strain energy of the mixture is positive definite (Bowen, 1976) $\lambda_g, \lambda_s, \lambda_{gs}$ and μ_s must satisfy the following inequalities:

$$\lambda_g > 0 \quad (59)$$

$$\mu_s > 0 \quad (60)$$

$$\lambda_g(\lambda_s + \frac{2}{3}\mu_s) > \lambda_{gs}^2 \quad (61)$$

Substituting the constituent equations and using the identity:

$$\text{tr} \mathbf{E}_a = \vec{\nabla} \cdot \mathbf{w}_a \quad (62)$$

yields the isothermal linear equations of motion for the solid and gas phases.

$$\begin{aligned} \rho_{SR} \frac{\partial^2 \mathbf{w}_s}{\partial t^2} &= (\lambda_s + \mu_s) \vec{\nabla} (\vec{\nabla} \cdot \mathbf{w}_s) + \mu_s \vec{\nabla} \cdot (\mathbf{w}_s \vec{\nabla}) \\ &+ \lambda_{gs} \vec{\nabla} (\vec{\nabla} \cdot \mathbf{w}_g) + \xi \left(\frac{\partial \mathbf{w}_g}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \end{aligned} \quad (63)$$

$$\begin{aligned} \rho_{gr} \frac{\partial^2 \mathbf{w}_g}{\partial t^2} = & \lambda_g \vec{\nabla}(\vec{\nabla} \cdot \mathbf{w}_g) \\ & + \lambda_{gs} \vec{\nabla}(\vec{\nabla} \cdot \mathbf{w}_s) - \xi \left(\frac{\partial \mathbf{w}_g}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \end{aligned} \quad (64)$$

Note that these equations reduce to the classical wave equations for a linear elastic solid and an elastic fluid if the coupling terms are neglected.

The above system of equations can be decomposed into dilational and rotational components using a Helmholtz decomposition (Atkin, 1968; Fung, 1965).

$$\mathbf{w}_s = \phi_s \vec{\nabla} + \vec{\nabla} \times \psi_s \quad (65)$$

$$\mathbf{w}_g = \phi_g \vec{\nabla} + \vec{\nabla} \times \psi_g \quad (66)$$

Where ϕ_a is a scalar function of time and space:

$$\phi_a = \phi_a(\mathbf{x}, t); \quad a=s, g \quad (66)$$

And ψ_a is a vector function of time and space:

$$\psi_a = \psi_a(\mathbf{x}, t); \quad a=s, g \quad (67)$$

Physically the dilational part represented by ϕ_a corresponds to a change in volume (i.e. the displacement due to the normal stresses). Compression of a unit cube would be an example of this. The rotation part represented by ψ_a describes the displacement due to the shear stresses, thus the same unit cube would go through a rotation without undergoing a change in volume. In harmonic wave motion the displacement due to the dilational disturbance is parallel to the wave propagation direction, while the displacement due to the rotational disturbance occurs perpendicular to the direction of the wavefront motion. Dilational disturbances are also referred to as longitudinal waves. Rotational waves are also frequently called shear waves or transverse waves. There remains a necessary additional constraint to uniquely determine the three displacement components from the four components of ϕ_a and ψ_a . This constraint has the form:

$$\vec{\nabla} \cdot \psi_a = 0; \quad a=s,g \quad (68)$$

Substituting the above into Eq.(63) and Eq.(64) and using the identities

$$\vec{\nabla} \cdot \vec{\nabla} \phi = \nabla^2 \phi \quad (69)$$

$$\nabla^2(\vec{\nabla}\phi) = \vec{\nabla}(\nabla^2\phi) \quad (70)$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \psi = 0 \quad (71)$$

results in the following system of equations:

$$\rho_s \ddot{\phi}_s = \lambda_{gs} \nabla^2 \phi_g + (\lambda_s + 2\mu_s) \nabla^2 \phi_s + \xi(\dot{\phi}_g - \dot{\phi}_s) \quad (72)$$

$$\rho_s \ddot{\psi}_s = \mu_s \nabla^2 \psi_s + \xi(\dot{\psi}_g - \dot{\psi}_s) \quad (73)$$

$$\rho_g \ddot{\phi}_g = \lambda_{gs} \nabla^2 \phi_s + \lambda_g \nabla^2 \phi_g - \xi(\dot{\phi}_g - \dot{\phi}_s) \quad (75)$$

$$\rho_g \ddot{\psi}_g = -\xi(\dot{\psi}_g - \dot{\psi}_s) \quad (76)$$

Note that for the case $\lambda_{sg} = \xi = 0$ the above equations reduce to the traditional dilational and rotational wave equations for a linear elastic solid and an elastic fluid in terms of the scalar and vector potentials.

A solution to Eq.(63) and Eq.(64) can be found once the material parameters are known. Consideration of the governing equations shows that a total of six constants plus the diffusion coefficient and the partial densities must be determined. The

mixture density (i.e. snow density) can be determined by direct measurement. Eq.(5) and Eq.(6) yield:

$$\rho = \rho_s + \rho_g = \phi_s \gamma_s + \phi_g \gamma_g \quad (78)$$

Since this is a saturated binary mixture (no unoccupied regions):

$$\phi_s = 1 - \phi_g \quad (79)$$

The material density for the gas γ_g can be found in air tables (Keenan et al, 1983). The solid phase material density is modeled as polycrystalline ice. Material densities for polycrystalline ice have been tabulated by Hobbs [1974]. Thus for known values of ρ , γ_g and γ_s the volume fraction of the voids can be determined. Equations Eq.(78) and Eq.(79) yield:

$$\left(\frac{\rho - \gamma_s}{\gamma_g - \gamma_s} \right) = \phi_g = \phi \quad (80)$$

ϕ is the volume fraction of the interconnected void space also known in porous media models as the effective porosity. With this the partial densities are known for a given mixture density.

$$\rho_g = \phi \gamma_g \quad (81)$$

$$\rho_s = \rho - \rho_g \quad (82)$$

The values for λ_g , λ_s and μ_s can be determined by noting that the constitutive equations reduce to single phase constitutive equations when the coupling terms are omitted. The remaining coefficients must be equivalent to the coefficients in constitutive equations which are derived from a single phase continuum theory. The stress tensor for an ideal elastic fluid has the form:

$$\mathbf{T} = \Pi(\rho)\mathbf{I} \quad (83)$$

Where Π is the thermodynamic pressure and is a function of density only. An equation of state defines the thermodynamic pressure Π . From mixture theory the isothermal linear equation of state for an elastic fluid is (Bowen,1976) :

$$\Pi_g = \Pi_{gR} + (\sigma_{sg} - \lambda_g)(\text{tr}\mathbf{E}_g) - (\sigma_{gs} + \lambda_{gs})(\text{tr}\mathbf{E}_s) \quad (84)$$

Dropping the coupling terms yields:

$$\Pi_g = \Pi_{gR} - \lambda_g(\text{tr}\mathbf{E}_g) \quad (85)$$

For small departures from the reference state the volumetric dilation of the fluid may be written in terms of the density as:

$$\text{tr}E_g = \frac{\rho_{gR} - \rho_g}{\rho_{gR}} \quad (86)$$

Then the equation of state can be written in terms of the density as:

$$\Pi_g = \Pi_{gR} - \lambda_g \left(\frac{\rho_{gR} - \rho_g}{\rho_{gR}} \right) \quad (87)$$

For a fluid the bulk modulus of elasticity Y is defined as the ratio of pressure change to the relative change in volume (or density) (Saad, 1985).

$$Y = - \frac{d\Pi}{\frac{dv}{v}} = \rho \frac{d\Pi}{d\rho} \quad (88)$$

Using (87) yields:

$$Y = \lambda_g \frac{\rho_g}{\rho_{gR}} \quad (89)$$

For small departures from the reference state:

$$\frac{\rho_g}{\rho_{gr}} \approx 1 \quad (90)$$

Therefore:

$$Y = \lambda_g \quad (91)$$

Thus the coefficient λ_g is equivalent to the bulk modulus of elasticity for a linear isothermal elastic fluid. In the case of a gas, when the volume change (or density change) and pressure change take place during an isothermal process the modulus of elasticity is simply the reference pressure, and this can be obtained from air tables. To see this consider the definition of the modulus of elasticity Y of a volume change between states 1 and 2 (Schlichting, 1979):

$$\Delta P = - Y \frac{\Delta V}{V_1} \quad (92)$$

The ideal gas law can be written in the form:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \quad (93)$$

For an isothermal process this becomes:

$$P_1 V_1 = P_2 V_2 \quad (94)$$

Let the subscript 1 represent the reference state, then the departure from the reference state is given by:

$$P_1 + \Delta P = P_2 \quad (95)$$

and

$$V_1 + \Delta V = V_2 \quad (96)$$

The product $P_2 V_2$ can be written as:

$$P_2 V_2 = (P_1 + \Delta P)(V_1 + \Delta V) \quad (97)$$

By Eq.(94) this becomes:

$$P_1 V_1 = (P_1 + \Delta P)(V_1 + \Delta V) \quad (98)$$

Expanding Eq.(98) and noting that for small departures from the reference state $\Delta P \Delta V \approx 0$ the following expression is obtained:

$$\Delta P \approx - \frac{P_1 \Delta V}{V_1} \quad (99)$$

Comparison between this and Eq.(92) shows that:

$$Y = P_1 = \Pi_R \quad (100)$$

The constitutive equation for the stress tensor of a linear elastic, isotropic, isothermal solid is:

$$\mathbf{T} = \lambda(\text{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} \quad (101)$$

From mixture theory the constitutive equation for the solid phase stress tensor in the absence of coupling terms is:

$$\mathbf{T}_s = \lambda_s(\text{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_s\mathbf{E}_s \quad (102)$$

Where the residual stress - Π_{SR} is taken to be zero. Comparison of Eq.(101) and Eq.(102) shows that λ_s and μ_s are equivalent to the Lamé constant and the shear modulus for the solid. λ_s is related to young's modulus of elasticity Y_s by the relation:

$$\lambda_s = \frac{\mu_s(Y_s - 2\mu_s)}{(3\mu_s - Y_s)} \quad (94)$$

Dynamic elastic moduli for high density snow has been tabulated by Smith [1965], these values can be used to determine λ_s and μ_s .

Presently values for the coupling coefficients σ_{sg} and σ_{gs} are nonexistent and therefore it is necessary to consider the special case of $\sigma_{sg} = \sigma_{gs} = 0$. This constitutive assumption is known as an ideal mixture and corresponds to assuming that the equilibrium free energy of the i th constituent is independent of the deformation of the other constituents (Bowen, 1976, pg 99). Theoretical limits for λ_{gs} can be calculated from Eq.(59), Eq.(60) and Eq.(61). However it is a constraint of the ideal mixture assumption that when $\sigma_{sg} = \sigma_{gs} = 0$, $\lambda_{gs} = 0$ as well. The result of this is that the strain of the fluid does not affect the solid and the strain of the solid does not affect the fluid. Thus the formulation is reduced to the case where the coupling between phases is represented solely through the viscous friction. In spite of the lack of data for evaluating the coupling terms they will be retained as an effort to keep the analysis as complete as possible. It should be mentioned here that the thermodynamic inconsistency of Biot's theory arises over the constraint presented above (Bowen, 1976). Biot proposes constitutive equations which are similar to equations Eq.(53) - Eq.(55) when σ_{gs} and σ_{sg} are set to zero but retains the coupling coefficient Q , which is equivalent to λ_{gs} . Thus for Biot's theory to be consistent with the thermodynamics of mixtures Q must be set to zero. This however is not the case.

Tables one, two and three show the calculated values for the material constants. The fluid properties exhibit small changes with changes in density for the range considered in this study, and the

thermal effects are being ignored. Therefore the fluid properties are approximated as constants.

Table 1. Values for Material Constants at -5 °C

γ_g Fluid material density Kg/m ³	γ_s Solid material density 10 ³ Kg/m ³	λ_g Bulk modulus of elasticity 10 ³ Pa	μ Fluid viscosity 10 ⁻⁶ Pa-s	ν Kinematic viscosity 10 ⁻⁶ m ² /s
1.3175	0.9166	101.325	16.905	12.87

Table 2 Values for Material Constants at -5 °C.

ρ Mixture snow density Kg/m ³	ϕ Porosity	ρ_g Fluid partial density Kg/m ³	ρ_s Solid partial density Kg/m ³
410	0.554	0.730	409.270
440	0.521	0.6864	439.314
508	0.488	0.6430	507.357
551	0.401	0.5283	550.472
600	0.348	0.4585	599.541

Table 3 Values for Material Constants at -5 °C

ρ	λ_s	μ_s	λ_{gs}	
Mixture	Solid	Shear	Coupling	
snow density	Lame	modulus	coefficient	
Kg/m ³	constant	10 ⁹ Pa	10 ⁹ Pa	
			min	max
410	0.1761	0.1802	-0.0055	0.0055
440	0.2674	0.2529	-0.0066	0.0066
508	0.4955	0.5682	-0.0094	0.0094
551	1.0138	0.8205	-0.0126	0.0126
600	1.6827	1.1067	-0.0157	0.0157

It remains to determine values for the friction coefficient ξ . Biot (1956,II) writes this coefficient as:

$$\xi = bF(k) \quad (104)$$

where b is the ratio of the total frictional force between the fluid and the solid, per unit volume of bulk material, to the average fluid velocity relative to that of the solid in steady state flow (Poiseuille flow) (Deresiewicz and Rice,1962). $F(k)$ is a frequency dependent correction factor which is a measure of the deviation from poiseuille flow friction. Biot [1956, I] has determined that poiseuille flow breaks down when the quarter wavelength of the

boundary layer is of the order of the diameter of the pores.

Explicitly for porous material this threshold frequency has the form:

$$\omega_t = \frac{\pi^2 \nu}{2d^2} \quad (105)$$

Where ν is the kinematic viscosity of the fluid and d represents the diameter of the pores.

In the function $F(k)$ k is a nondimensional frequency parameter given by:

$$k = a\sqrt{\frac{\omega}{\nu}} \quad (106)$$

Where " a " is a representative or characteristic size of the pore crosssection. In a circular cylindrical pore " a " would be the radius of the pore. ω is the circular propagation frequency given by:

$$\omega = 2\pi f = 2\pi \frac{1}{T} \quad (107)$$

Where T is the period. ν is the kinematic viscosity of the fluid.

The function $F(k)$ is written as

$$F(k) = \frac{1}{4} \frac{k^2 T(k)}{k + 2iT(k)} \quad (108)$$

In which:

$$T(k) = \frac{\text{ber}'k + i\text{bei}'k}{\text{ber}k + i\text{bei}k} \quad (109)$$

Here $\text{ber } k$ and $\text{bei } k$ denote the real part and the imaginary part of the Bessel-Kelvin function of the first kind and order zero. And:

$$\text{ber}'k = \frac{d\text{ber}k}{dk} \quad (110)$$

$$\text{bei}'k = \frac{d\text{bei}k}{dk} \quad (111)$$

For pores which behave like circular tubes of diameter d , b may be written as (Biot, 1956, I):

$$b = \frac{32\mu\phi}{d^2} \quad (112)$$

Where μ is the fluid viscosity and ϕ is the effective porosity. For small values of k ($k \ll 1$) $F(k)$ reduces to (Deresiewicz and Rice, 1962):

$$F_R \approx 1 + \frac{k^4}{1552} \quad (113)$$

$$F_I \approx \frac{k^2}{24} - \frac{k^6}{34560} \quad (114)$$

F_R and F_I denote the real and imaginary part respectively of the function $F(k)$. Eq.(106) shows that the above approximations are appropriate for low frequencies, a highly viscous fluid or a porous medium with small pore sizes. Note that as $k \Rightarrow 0$, $F(k) \Rightarrow 1$. For large values of k , $F(k)$ takes on the approximation (Deresiewicz and Rice, 1962):

$$F_R \approx \frac{3}{8} + \frac{k\sqrt{2}}{8} + \frac{15\sqrt{2}}{64k} - \frac{135\sqrt{2}}{1024k^3} \quad (115)$$

$$F_I \approx \frac{k\sqrt{2}}{8} - \frac{15\sqrt{2}}{64} - \frac{15}{32k} - \frac{135\sqrt{2}}{1024k^2} \quad (116)$$

This approximation would be valid for high frequencies or an inviscid fluid ($k \gg 1$).

In order to determine the size of the pore space, data from Edens [1989] is plotted in Figure 4. The values shown as void space diameter correspond to Eden's mean pore length λ . Note that in the density range being considered the difference in pore size is

not appreciable, thus it is reasonable to assume a constant pore space diameter of 0.3 mm.

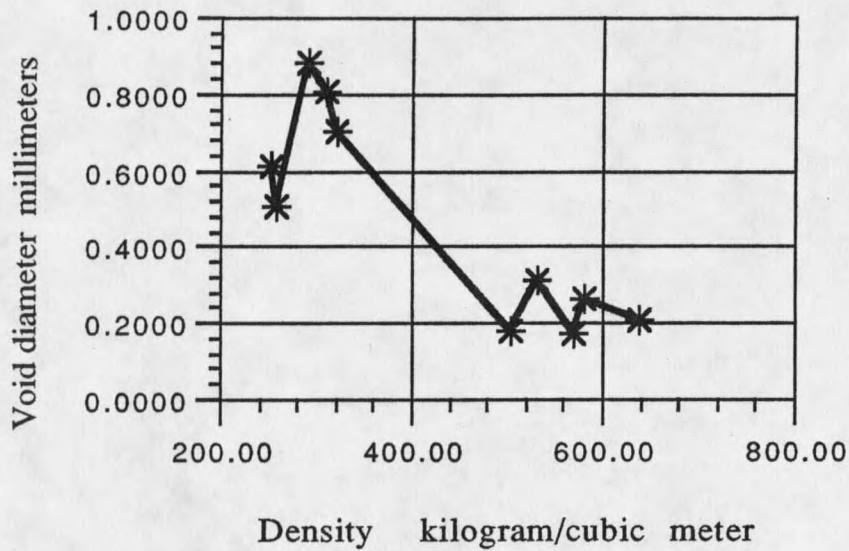


Figure 4: Pore size variation as a function of snow density (from Edens, 1989).

Dilation Waves

For the purposes of application a harmonic propagating disturbance is considered. For a dilation wave this is represented by:

$$\phi_a = \tilde{\phi}_a \exp\{i(\eta \mathbf{n} \cdot \mathbf{x} - \omega t)\}; a = s, g \quad (117)$$

Where :

$\mathbf{n}$ is the unit propagation vector

η is the circular wave number

ω is the circular frequency

In general η and ω are complex and dependent on properties from both constituents. However this investigation is concerned with waves of assigned wavelength only and thus ω is taken to be real. The wave number can be written as:

$$\eta = \eta_R + i\eta_I \quad (118)$$

Where η_R is the real part and η_I is the imaginary part.

Substituting Eq.(118) in Eq.(117) yields:

$$\phi_a = \tilde{\phi}_a \exp\{-\eta_I \mathbf{n} \cdot \mathbf{x}\} \exp\{i\eta_R(\mathbf{n} \cdot \mathbf{x} - \frac{\omega}{\eta_R} t)\}; \quad a = s, g \quad (119)$$

This shows that the amplitude of the wave attenuates, and that the attenuation is proportional to the imaginary part of the wave number. Thus η_I is called the attenuation coefficient. In addition Eq.(107) indicates the propagation velocity is given by:

$$c = \frac{\omega}{\eta_R} \quad (120)$$

A dispersive media is one in which the wave propagation velocity is a function of frequency. The relationship which yields this frequency dependent velocity function is termed the dispersion relationship. Thus determination of η (the dispersion relationship) is required in order to evaluate the attenuation and the propagation velocity. To this end the one dimensional forms of Eq.(72) and Eq.(75) are considered.

$$\rho_{sr}\ddot{\phi}_s = (\lambda_s + \mu_s)\frac{\partial^2 \phi_s}{\partial x^2} + \lambda_{gs}\frac{\partial^2 \phi_g}{\partial x^2} + \xi(\dot{\phi}_g - \dot{\phi}_s) \quad (121)$$

$$\rho_{gr}\ddot{\phi}_g = \lambda_{gs}\frac{\partial^2 \phi_s}{\partial x^2} + \lambda_g\frac{\partial^2 \phi_g}{\partial x^2} - \xi(\dot{\phi}_g - \dot{\phi}_s) \quad (122)$$

The one dimensional form of Eq.(117) is :

$$\phi_j = \tilde{\phi}_j \exp\{i(\eta x - \omega t)\} \quad (123)$$

Substitution of Eq.(123) into Eq.(122) and Eq.(121) results in the following system of equations:

$$\begin{bmatrix} -\rho_s\omega^2 + (\lambda_s + 2\mu_s)\eta^2 - i\xi\omega & \lambda_{gs}\eta^2 + i\xi\omega \\ \lambda_{gs}\eta^2 + i\xi\omega & -\rho_g\omega^2 + \lambda_g\eta^2 - i\xi\omega \end{bmatrix} \begin{bmatrix} \phi_s \\ \phi_g \end{bmatrix} = 0 \quad (124)$$

For a nontrivial solution it is a necessary and sufficient condition for the determinant of the coefficients to vanish. Carrying out this operation yields the characteristic equation:

$$\begin{aligned} & \left[\lambda_g(\lambda_s + 2\mu_s) - \lambda_{gs}^2 \right] \eta^4 \\ & + \left[(-\rho_s \lambda_g - \rho_g(\lambda_s + 2\mu_s)) \omega^2 - \xi(\lambda_s + 2\mu_s + 2\lambda_{gs} + \lambda_g) \omega \right] \eta^2 \\ & + \rho_s \rho_g \omega^4 + i\xi(\rho_s + \rho_g) \omega^3 = 0 \end{aligned} \quad (125)$$

The roots of this characteristic equation yield the dispersion relationship. Eq.(125) is a biquadratic equation and has roots $\pm\eta_1$ and $\pm\eta_2$ thus two of the four roots are just mirror images of the other two, in the negative realm. Substitution of $\eta^2 = \beta$ allows use of the quadratic equation and results in the following expression for the square of the dispersion relation:

$$\begin{aligned} \eta^2 = \beta &= \frac{A\omega^2 + i\xi C\omega}{2B} \\ &\pm \frac{\sqrt{A^2\omega^4 + 2i\xi AC\omega^3 - \xi^2 C^2\omega^2 - 4B(\rho_s \rho_g \omega^4 + i\xi \rho \omega^3)}}{2B} \end{aligned} \quad (126)$$

Where:

$$A = \rho_s \lambda_g - \rho_g(\lambda_s + 2\mu_s) \quad (127)$$

$$B = \lambda_g(\lambda_s + 2\mu_s) - \lambda_{gs}^2 \quad (128)$$

$$C = \lambda_s + 2\mu_s + 2\lambda_{gs} + \lambda_g \quad (129)$$

In order to investigate the dispersion relationship Eq.(111) in more detail the quantity under the radical sign is expanded with a binomial expansion. By factoring out the term $-\omega^2 \xi^2 C^2$ and noting that $i = \sqrt{-1}$ the radical term can be written as:

$$i\omega\xi C \sqrt{1 + \left(\frac{4B\rho_s\rho_g - A^2}{\xi^2 C^2}\right)\omega^2 + 2i\xi\left(\frac{2B\rho - AC}{\xi^2 C^2}\right)\omega} \quad (130)$$

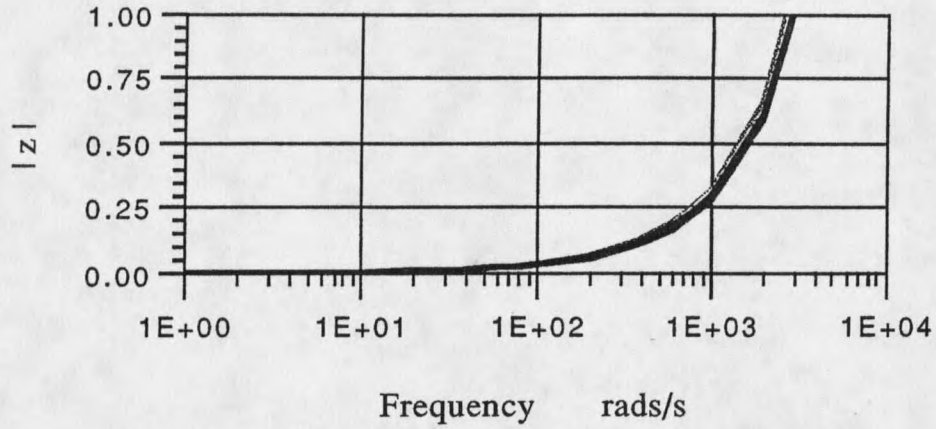
where $\rho = \rho_s + \rho_g$. The term under the radical can now be expanded in a binomial expansion of the form:

$$(1+z)^s = 1 + sz + \frac{s(s-1)}{2!} z^2 + \frac{s(s-1)(s-2)}{3!} z^3 + \dots \quad (131)$$

which converges for $|z| < 1$, where z is:

$$z = \left(\frac{4B\rho_s\rho_g - A^2}{\xi^2 C^2}\right)\omega^2 + 2i\xi\left(\frac{2B\rho - AC}{\xi^2 C^2}\right)\omega \quad (132)$$

Figure 5 shows the range of frequencies with $\lambda_{gs} = 0$ and $\xi = b$ for which this form of z converges for the density values being considered.



— $\rho = 410$ - - - $\rho = 440$ — · — $\rho = 508$ — — — $\rho = 551$

Figure 5: Frequency range for which the first binomial expansion is convergent.

Thus the above formulation is appropriate for low frequencies.

Using the binomial expansion, the dispersion relationship can now be written as:

$$\begin{aligned}
 \eta_1^2 = & \frac{(4B\rho_g\rho_s - A^2)(2B\rho - AC)}{4\xi^2 BC^3} \omega^4 + \left\{ \frac{A}{B} - \frac{\rho}{C} \right\} \omega^2 \\
 & + i \left\{ \frac{-(4B\rho_g\rho_s - A^2)^2}{16\xi^3 BC^3} \omega^5 + \frac{\xi C}{B} \omega \right\} \\
 & + i \left\{ \frac{(4B\rho_g\rho_s - A^2)}{4\xi BC} + \frac{(2B\rho - AC)^2}{4\xi BC^3} \right\} \omega^3
 \end{aligned} \tag{133}$$

$$\begin{aligned}
\eta_2^2 = & \frac{-(4B\rho_g\rho_s-A^2)(2B\rho-AC)}{4\xi^2BC^3}\omega^4 + \frac{\rho}{C}\omega^2 \\
& + i\left\{\frac{(4B\rho_g\rho_s-A^2)^2}{16\xi^3BC^3}\omega^5\right\} \\
& - i\left\{\frac{(4B\rho_g\rho_s-A^2)}{4\xi BC} + \frac{(2B\rho-AC)^2}{4\xi BC^3}\right\}\omega^3
\end{aligned} \tag{134}$$

To investigate the low frequency limit consider Eq.(133) and Eq.(134) as $\omega \Rightarrow 0$. This yields:

$$\eta_1 \approx \sqrt{\frac{i\xi C}{B}}\omega \tag{135}$$

$$\eta_2 \approx \omega\sqrt{\frac{\rho}{C}} \tag{136}$$

These low frequency limits show that the dilation mode associated with the roots $\pm \eta_1$ exhibits a diffusive behavior and is not a true wave while the dilation mode associated with the roots $\pm \eta_2$ is wave like in nature with the propagation velocity given by:

$$c_2 = \sqrt{\frac{\lambda_g + \lambda_s + 2\mu_s + 2\lambda_{gs}}{\rho_s + \rho_g}} \tag{137}$$

Note that this second mode exhibits a nondispersive character in the low frequency limit and thus the velocity approaches a low frequency asymptote. This behavior in the low frequency regime is in agreement with other porous media models such as Atkins [1967] and Biot [1956,I]. To investigate the high frequency behavior it is necessary to utilize the binomial expansion in terms of inverse powers of frequency. Returning to Eq.(126) the radical term can be expressed in inverse powers of ω as:

$$\omega^2 \sqrt{A^2 - 4B\rho_g\rho_s} \sqrt{1 + \frac{2i\xi(AC - 2\rho B)}{A^2 - 4B\rho_g\rho_s} \omega^{-1} - \frac{\xi^2 C^2}{A^2 - 4B\rho_g\rho_s} \omega^{-2}} \quad (138)$$

With this the binomial expansion can be utilized to remove the radical. Since this is the high frequency regime the assumption of poiseuille flow is no longer valid and it is necessary to include the correction function $F(k)$ for the high frequency regime (Eq.(115) and Eq.(116)). Thus the friction coefficient is expressed as:

$$\xi = \xi_R + i\xi_I = b(F_R(k) + iF_I(k)) \quad (139)$$

Substitution of Eq.(139) into Eq.(138) yields the expression for z :

$$z = -\frac{2\xi_I(AC-2\rho B)}{A^2-4B\rho_g\rho_s}\omega^{-1} - \frac{(\xi_R^2-\xi_I^2)C^2}{A^2-4B\rho_g\rho_s}\omega^{-2} + 2i\left\{\frac{\xi_R\xi_IC^2\omega^{-2} + \xi_R(AC-2\rho B)\omega^{-1}}{A^2-4B\rho_g\rho_s}\right\} \quad (140)$$

With $\lambda_{gs} = 0$ Figure 6 shows the convergent frequency range for the considered densities.

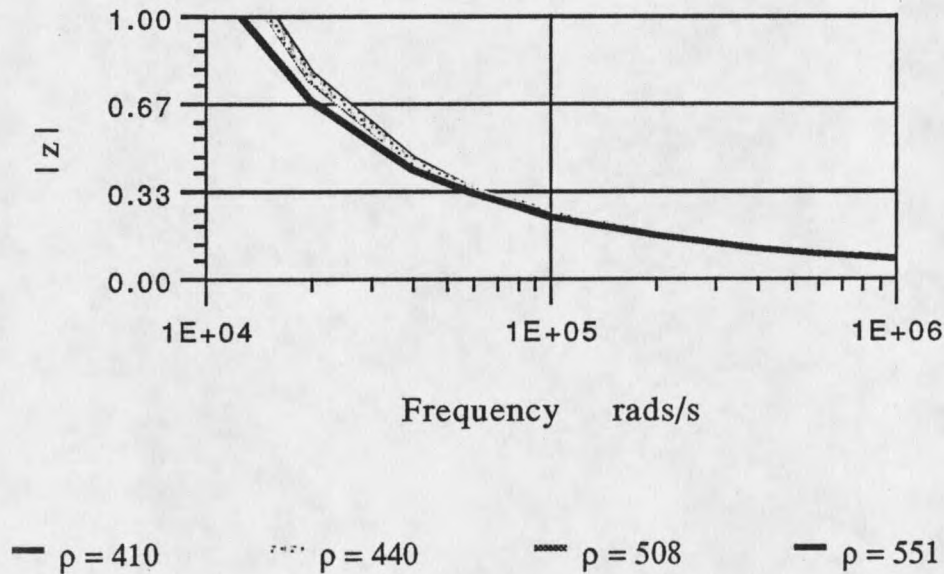


Figure 6: Frequency range for which the second binomial expansion is convergent.

Figures 5 and 6 indicate that the transition from the low to high frequency regimes occurs sharply. Using the second form of z in

the binomial expansion yields the following dispersion relationships for the high frequency regime:

$$\begin{aligned}
 \eta_1^2 = & (\xi_R^4 - 6\xi_R^2\xi_I^2 + \xi_I^4) \frac{C^4}{16B\Lambda_1^{3/2}} \omega^{-2} \\
 & + \left(\frac{A}{2B} + \frac{\Lambda_1^{1/2}}{2B} \right) \omega^2 + \frac{(\xi_R^2 - \xi_I^2)}{4B\Lambda_1^{1/2}} \left(\frac{\Lambda_2^2}{\Lambda_1} - C^2 \right) \\
 & - \frac{\xi_I}{2B} \left(C + \frac{\Lambda_2}{\Lambda_1^{1/2}} \right) \omega - (3\xi_R^2\xi_I - \xi_I^3) \left(\frac{C^2\Lambda_2}{4B\Lambda_1^{3/2}} \right) \omega^{-1} \\
 & - i(\xi_R^3\xi_I - \xi_R\xi_I^3) \left(\frac{C^4}{4B\Lambda_1^{3/2}} \right) \omega^{-2} + i \frac{\xi_R\xi_I}{2B\Lambda_1^{1/2}} \left(\frac{\Lambda_2^2}{\Lambda_1} - C^2 \right) \\
 & + i \frac{\xi_R}{2B} \left(C + \frac{\Lambda_2}{\Lambda_1^{1/2}} \right) \omega + i(\xi_R^3 - 3\xi_R\xi_I^2) \left(\frac{C^2\Lambda_2}{4B\Lambda_1^{3/2}} \right) \omega^{-1}
 \end{aligned} \tag{141}$$

$$\begin{aligned}
 \eta_2^2 = & (\xi_R^4 - 6\xi_R^2\xi_I^2 + \xi_I^4) \frac{C^4}{16B\Lambda_1^{3/2}} \omega^{-2} \\
 & + \left(\frac{A}{2B} - \frac{\Lambda_1^{1/2}}{2B} \right) \omega^2 + \frac{(\xi_R^2 - \xi_I^2)}{4B\Lambda_1^{1/2}} \left(C^2 - \frac{\Lambda_2^2}{\Lambda_1} \right) \\
 & - \xi_I \left(\frac{C}{2B} - \frac{\Lambda_2}{2B\Lambda_1^{1/2}} \right) \omega + (3\xi_R^2\xi_I - \xi_I^3) \left(\frac{C^2\Lambda_2}{4B\Lambda_1^{3/2}} \right) \omega^{-1} \\
 & + i(\xi_R^3\xi_I - \xi_R\xi_I^3) \left(\frac{C^4}{4B\Lambda_1^{3/2}} \right) \omega^{-2} + i \frac{\xi_R\xi_I}{2B\Lambda_1^{1/2}} \left(C^2 - \frac{\Lambda_2^2}{\Lambda_1} \right) \\
 & + i \frac{\xi_R}{2B} \left(C - \frac{\Lambda_2}{\Lambda_1^{1/2}} \right) \omega - i(\xi_R^3 - 3\xi_R\xi_I^2) \left(\frac{C^2\Lambda_2}{4B\Lambda_1^{3/2}} \right) \omega^{-1}
 \end{aligned} \tag{142}$$

Where:

$$\Lambda_1 = A^2 - 4\rho_s\rho_g B \quad (143)$$

$$\Lambda_2 = AC - 2\rho B \quad (144)$$

To investigate the high frequency limit consider Eq.(141) and Eq.(142) as $\omega \Rightarrow \infty$. This leads to:

$$\pm \eta_1 \approx \omega \sqrt{\frac{A + \sqrt{A^2 + 4\rho_s\rho_g\lambda_{gs}^2}}{2B}} \quad (145)$$

$$\pm \eta_2 \approx \omega \sqrt{\frac{A - \sqrt{A^2 + 4\rho_s\rho_g\lambda_{gs}^2}}{2B}} \quad (146)$$

It can be seen that both dilational modes propagate as true waves in the high frequency limit. In fact for the case $\lambda_{gs} = 0$ the first and second dilation mode dispersion relationships reduce to:

$$\pm \eta_1 = \omega \sqrt{\frac{\rho_g}{\lambda_g}} \quad (147)$$

$$\pm \eta_2 = \omega \sqrt{\frac{\rho_s}{\lambda_s + 2\mu_s}} \quad (148)$$

These limiting cases correspond to the propagation velocities of a harmonic disturbance in an elastic fluid and a linear elastic solid respectively. Thus it can be seen that the two dilational modes propagate as dispersionless waves in the high frequency limit where the first mode is dominated by the properties of the interstitial fluid and the second mode is dominated by the properties of the solid constituent. These results are in agreement with an earlier mixture theory by Atkins [1968] and a porous media theory formulated by Biot [1956,II].

It remains to eliminate the radical from Eq.(133), Eq.(134), Eq.(141) and Eq.(142) in order to evaluate the velocity and attenuation expressions in the low and high frequency regimes. Note that these equations have the form:

$$\eta_k^2 = \zeta_k + i \vartheta_k ; k = 1, 2 \quad (149)$$

or

$$\pm \eta_k = \sqrt{\zeta_k + i \vartheta_k} ; k = 1, 2 \quad (150)$$

Where ζ is the real part of η^2 and ϑ is the imaginary part. This is suggestive of a polar representation for η_k . Recall that a complex number s can be represented in polar form as:

$$s = \zeta + i\vartheta = r(\cos \theta + i \sin \theta) \quad (151)$$

Where:

$$r^2 = \zeta^2 + \vartheta^2 \quad (152)$$

$$\tan \theta = \frac{\vartheta}{\zeta} \quad (153)$$

With this polar representation in mind the following relationships can be defined:

$$\gamma_k^4 = \zeta_k^2 + \vartheta_k^2 \quad (154)$$

$$\tan 2\beta_k = \frac{\vartheta_k}{\zeta_k} \quad (155)$$

Then using the trigonometric identities :

$$\cos 2\beta = \cos^2 \beta - \sin^2 \beta \quad (156)$$

$$\sin 2\beta = 2\cos\beta \sin\beta \quad (157)$$

The dispersion relationship can be written as:

$$\pm \eta_k = \gamma_k(\cos \beta_k + i \sin \beta_k) \quad ; k = 1, 2 \quad (158)$$

Thus the phase velocity for the dilation wave is:

$$c_k = \frac{\omega}{\eta_{kR}} = \frac{\omega}{\gamma_k \cos \beta_k} ; k = 1,2 \quad (159)$$

and the attenuation coefficient is given by the imaginary part of the wave number:

$$\eta_{kI} = \gamma_k \sin \beta_k ; k=1,2 \quad (160)$$

Shear Waves

Analysis of the shear waves is analogous to that of the dilation waves. The rotational disturbance is assumed to be of the form:

$$\psi_j = \tilde{\psi}_j \exp\{i(\eta \mathbf{n} \cdot \mathbf{x}) - \omega t\} ; j = s, g \quad (161)$$

substituting the one dimensional form of Eq.(161) into the one dimensional form of equations Eq.(73) and Eq.(76) and setting the determinant of the resulting coefficient matrix equal to zero yields the characteristic equation for the shear waves which is:

$$(\rho_g \mu_s \omega^2 + i \mu_s \xi \omega) \eta^2 - \rho_s \rho_g \omega^4 - i \xi \rho \omega^3 = 0 \quad (162)$$

As before the roots of this equation yield the dispersion relationship. In terms of the complex friction coefficient ξ Eq.(162) takes on the form:

$$\eta^2 = \frac{\rho_s \rho_g \mu_s \omega^4 - \xi_I \mu_s \rho_g (\rho + \rho_s) \omega^3 + \rho \mu_s (\xi_R^2 + \xi_I^2) \omega^2}{\rho_g^2 \mu_s^2 \omega^2 - 2 \rho_g \mu_s^2 \xi_I \omega + \mu_s^2 (\xi_R^2 + \xi_I^2)} + i \left\{ \frac{\xi_R \rho_g^2 \mu_s}{\rho_g^2 \mu_s^2 \omega^2 - 2 \rho_g \mu_s^2 \xi_I \omega + \mu_s^2 (\xi_R^2 + \xi_I^2)} \right\} \omega^3 \quad (163)$$

For the low frequency limit consider Eq.(163) as $\omega \Rightarrow 0$. This yields:

$$\pm \eta \approx \omega \sqrt{\frac{\rho}{\mu_s}} \quad (164)$$

This corresponds to a positive propagation velocity of:

$$c_s = \sqrt{\frac{\mu_s}{\rho}} \quad (165)$$

For the high frequency limit consider Eq.(163) as $\omega \Rightarrow \infty$, which gives:

$$\pm \eta \approx \omega \sqrt{\frac{\rho_s}{\mu_s}} \quad (166)$$

Which corresponds to a propagation velocity given by:

$$c_s = \sqrt{\frac{\mu_s}{\rho_s}} \quad (167)$$

Eq.(163) shows that there is only one positive propagation mode. This mode behaves as a wave which is both damped and dispersed. However in the low and high frequency limit the wave takes on a nondispersive character. Furthermore since $\rho = \rho_s + \rho_g$ the high and low frequency asymptotes differ only by the value of ρ_g with the high frequency having the greater velocity value. In the limit as $\rho_g \Rightarrow 0$ (in the absence of the fluid constituent) Eq.(163) takes on an undamped, nondispersed character which is in agreement with classical elastic wave theory. If on the other hand the limit as $\mu_s \Rightarrow 0$ is considered, $\eta^2 \Rightarrow \infty$ which indicates that the fluid phase cannot support a shear disturbance in the absence of the solid constituent. In a manner analogous to the dilational modes the dispersion relation for the shear waves can be expressed in polar form as:

$$\pm \eta = \gamma(\cos \beta + i \sin \beta) \quad (168)$$

Where γ and β have the same definitions as in Eq.(154) but with respect to the squared shear wave dispersion relation.

CHAPTER 4

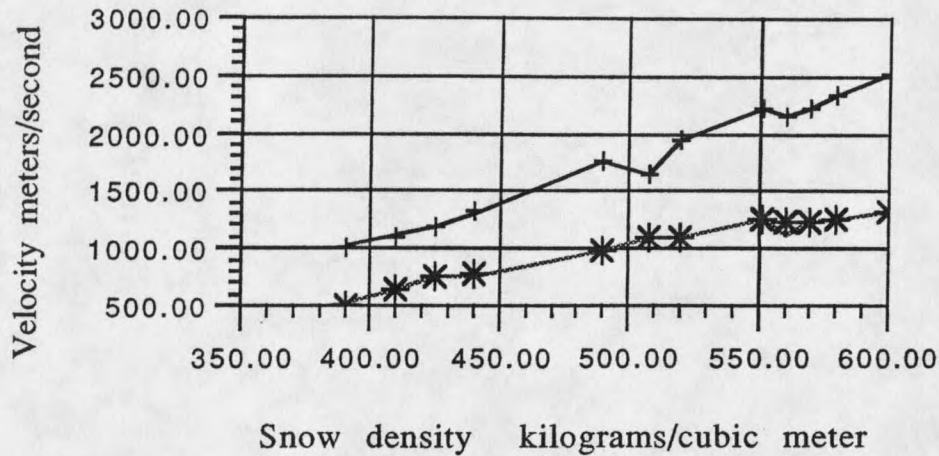
RESULTS

Equations developed in the last chapter are plotted using data from Table 1, Table 2 and Table 3.

Table 4 and Figure 7 show a good comparison between wave velocity values from J.L. Smith [1965] and theoretically calculated propagation velocities. Smith's data for the elastic moduli of snow was determined by sonic wave techniques which correspond to low amplitude high frequency waves. To make a comparison, Smith's elastic moduli were used in the high frequency limit of the theoretical wave speeds. These values were then compared with the known sonic wave velocities. Note here that the first dilation mode wave does not show up in the experimental data, this is due to the fact that the first dilation mode is highly attenuated (which will be shown in the following figures) and therefore hard to detect with measurement techniques.

Table 4: Comparison of Experimental Data with Theoretical Velocities.

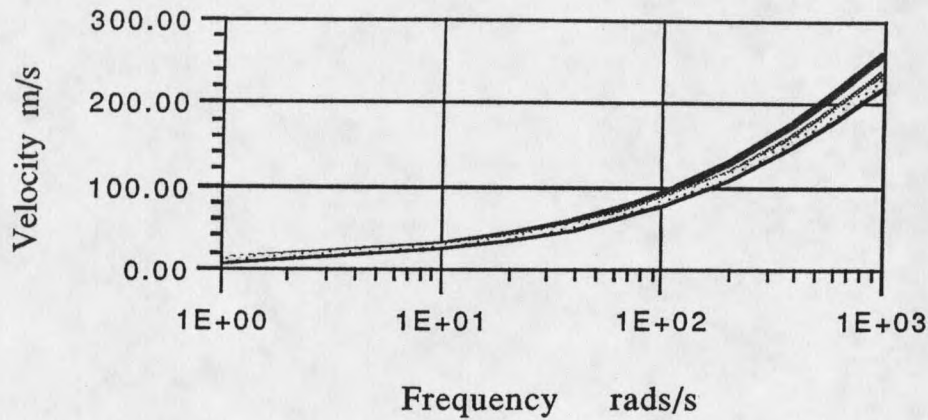
Mixture snow density Kg/m ³	Experimental		Theoretical	
	Longitudinal wave velocity 10 ³ m/s	Shear wave velocity 10 ³ m/s	Second mode Longitudinal wave velocity 10 ³ m/s	Shear wave velocity 10 ³ m/s
390	1.0031	0.5089	1.0235	0.5267
410	1.0972	0.6420	1.1083	0.6483
425	1.1827	0.7562	1.1781	0.7559
440	1.3153	0.7643	1.3122	0.7577
490	1.7638	0.9846	1.7566	0.9860
508	1.6455	1.0913	1.6368	1.0851
520	1.9385	1.0910	1.9370	1.0872
551	2.2234	1.2638	2.2250	1.2649
560	2.1498	1.2262	2.1528	1.1296
570	2.2300	1.2411	2.2272	1.2311
580	2.3382	1.2615	2.3445	1.2530
600	2.5210	1.3166	2.5180	1.3213



- + experimental longitudinal wave velocity
- theoretical longitudinal wave velocity
- * experimental shear wave velocity
- theoretical shear wave velocity

Figure 7: Experimental and theoretical propagation velocities from Table 4.

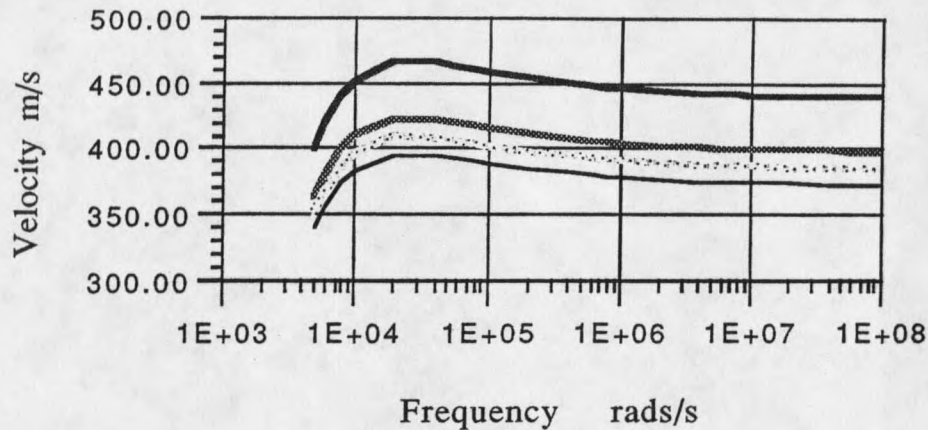
Figures 8 through Figure 31 show velocities and attenuation coefficients as a function of frequency for the range of densities considered in the low and high frequency regimes. For the first dilation mode, Figure 8 and Figure 9 indicate that the propagation velocity increases with increasing frequency and increasing density.



— $\rho = 410$ - - - $\rho = 440$ - · - $\rho = 508$ — $\rho = 551$

Figure 8: Propagation velocity for the first dilation mode in the low frequency regime

The asymptotic behavior accurately corresponds to that predicted by the limiting cases of low and high frequency considered earlier. Figure 9 shows that the velocity for the first dilation mode approaches its high frequency asymptote from above. This behavior is characteristic of partial differential equations which describe a coupling of scalar or vector wave fields (Atkins, 1968)



— $\rho = 410$ ···· $\rho = 440$ - - - $\rho = 508$ - - - $\rho = 551$

Figure 9: Propagation velocity for the first dilation mode in the high frequency regime.

Attenuation coefficients for the first dilation mode are plotted in Figure 10 and Figure 11. These plots indicate that the attenuation increases with increasing frequency and decreases with increasing density. Acoustic attenuation measurements by Lang [1976], in natural blocks of snow with a mean density of 300 kg/m³ show attenuation increasing with an increase in frequency. Thus it is encouraging to see that the theoretical attenuation coefficients exhibit the same behavior.

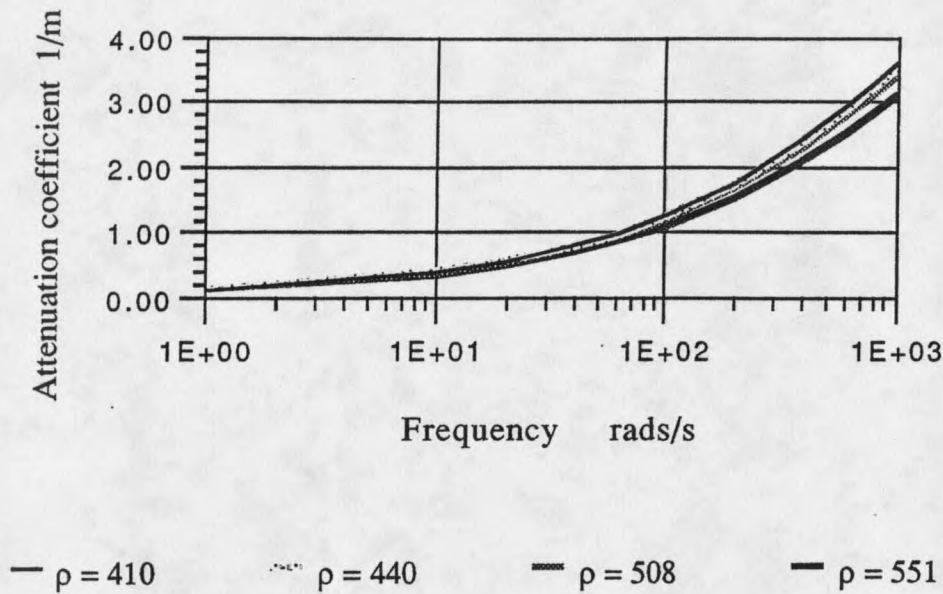


Figure 10: Attenuation coefficient for the first dilation mode in the low frequency regime.

If the attenuation coefficient in the high frequency regime is calculated without the correction factor for deviation from poiseuille flow (thus assuming poiseuille flow to be valid at high frequencies) it approaches a constant value. This behavior is also seen in the porous media model formulated by Biot [1956,II]. Results for the second dilation mode are shown in Figures 12 through Figure 21.

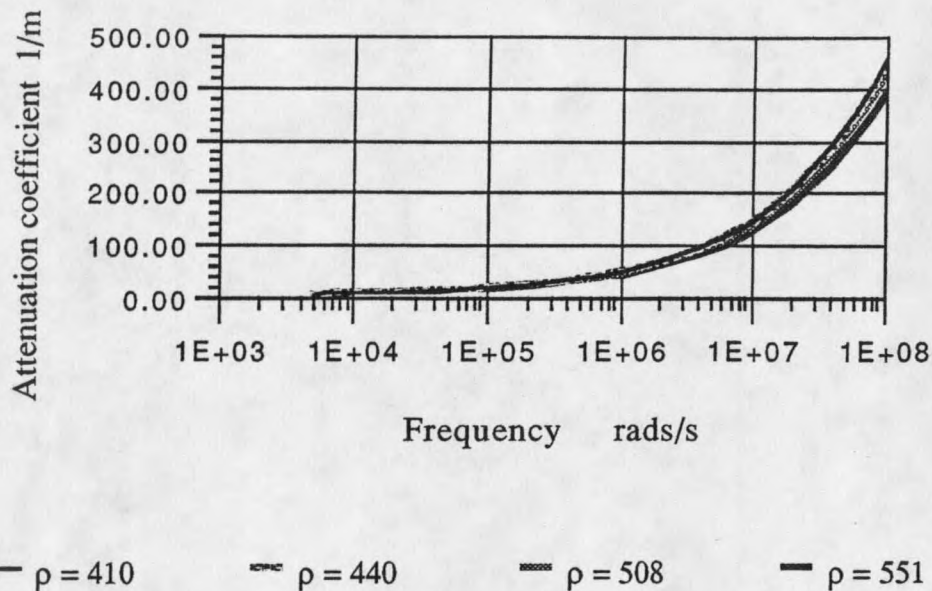


Figure 11: Attenuation coefficient for the first dilation mode in the high frequency regime.

There are two separate constraints which are both functions of frequency that define the transition range (the frequency range between low and high frequency). One constraint is that required for the convergence of the binomial expansions (for the dilation waves). The other constraint is the departure from poiseuille flow. Reasonable behavior for the second dilation mode breaks down as these constraint boundaries are approached. One reason for this may be that the effective range of the function which corrects the deviation from poiseuille flow (recall that expressions for $F(k)$ are effective for $k \ll 1$ and $k \gg 1$) does not correspond to exactly the same range where poiseuille flow is an invalid assumption. The equations which govern the first dilation wave and the shear wave

exhibit a more stable behavior in this "boundary " region. The result of this is that the transition range for the second dilation mode is larger than that which exists for the first dilation wave and the shear wave. Velocities plotted in Figures 12 through Figure 19 show that the propagation velocities increase with increasing frequency and increasing density. Furthermore they accurately approach the asymptotic behavior predicted by the limiting cases. Also, the velocity values for the second mode fall within the range of experimental velocities determined by Yamada, et al [1974] (this paper is in Japanese and the comparison was made by noting the velocity ranges on the plots which were labeled in English).

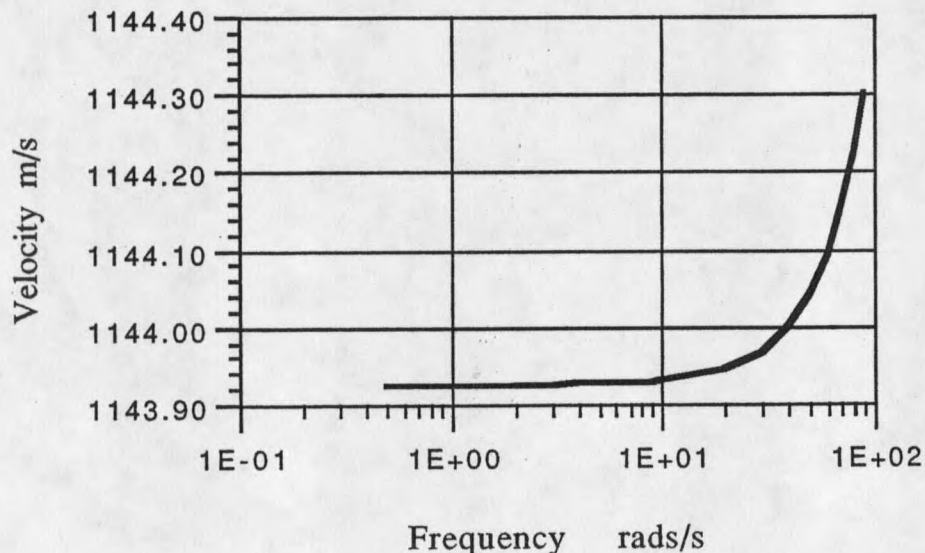


Figure 12: Propagation velocity for the second dilation mode in the low frequency regime with a density of 410 kg/m^3 .

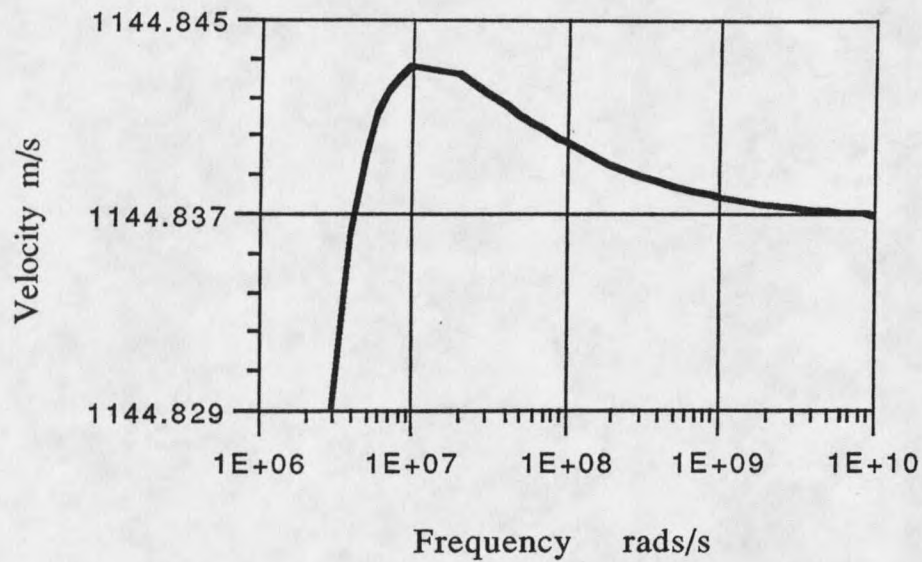


Figure 13: Propagation velocity for the second dilation mode in the high frequency regime with a density of 410 kg/m³.

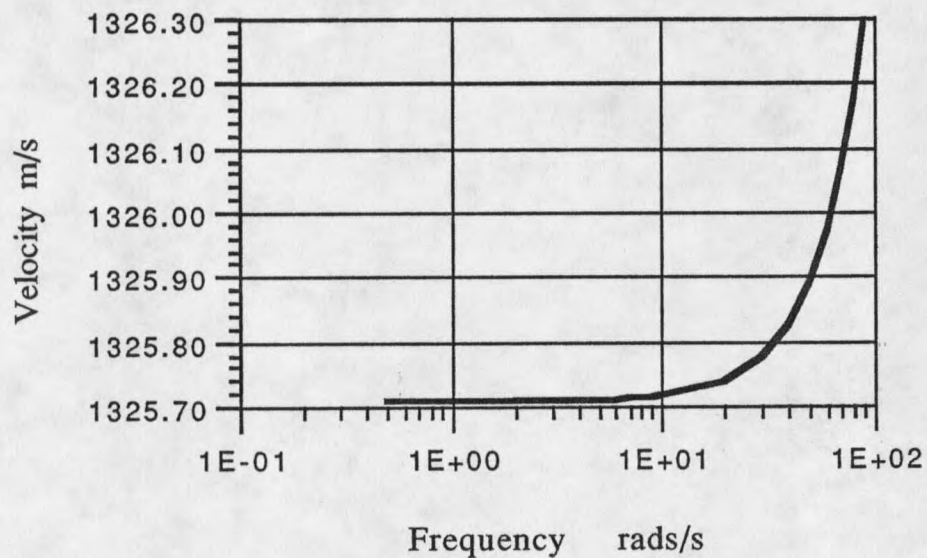


Figure 14: Propagation velocity for the second dilation mode in the low frequency regime with a density of 440 kg/m³.

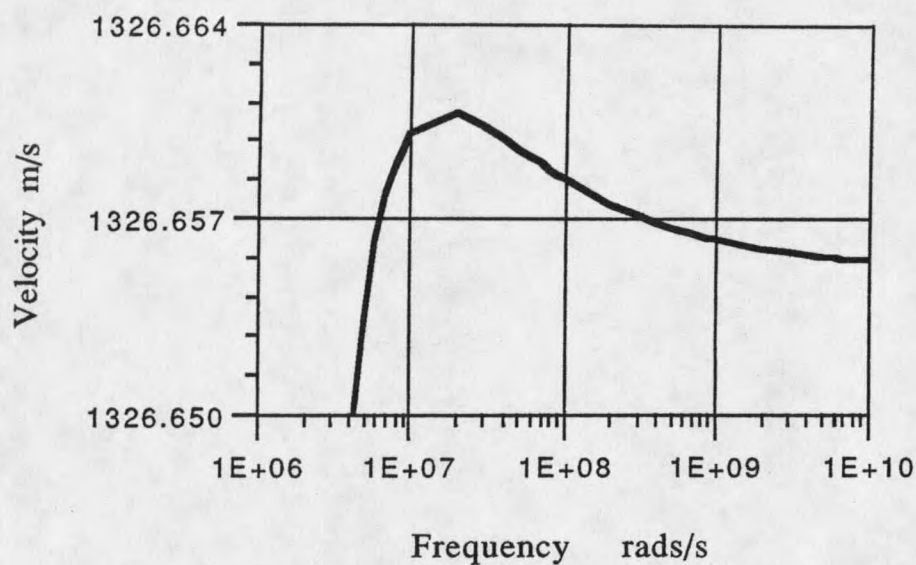


Figure 15: Propagation velocity for the second dilation mode in the high frequency regime with a density of 440 kg/m³.

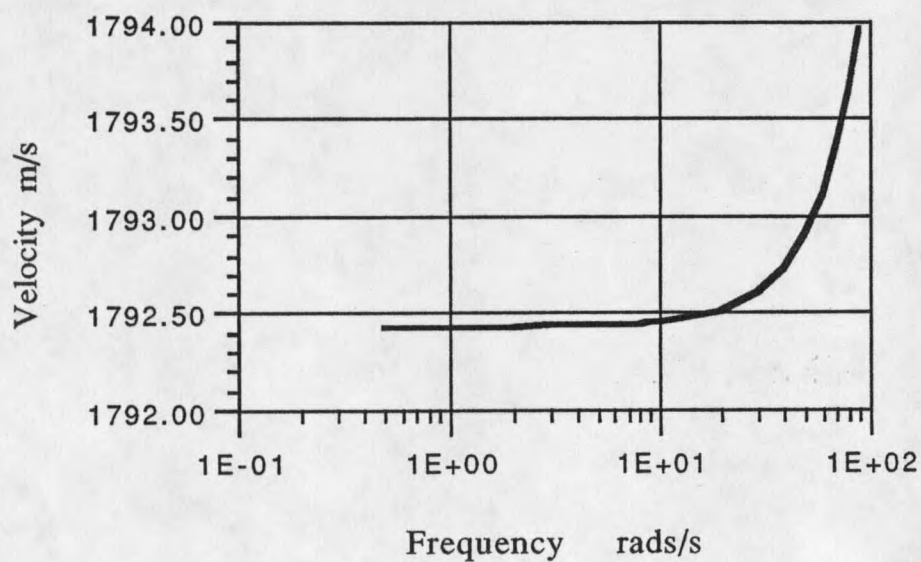


Figure 16: Propagation velocity for the second dilation mode in the low frequency regime with a density of 508 kg/m³.

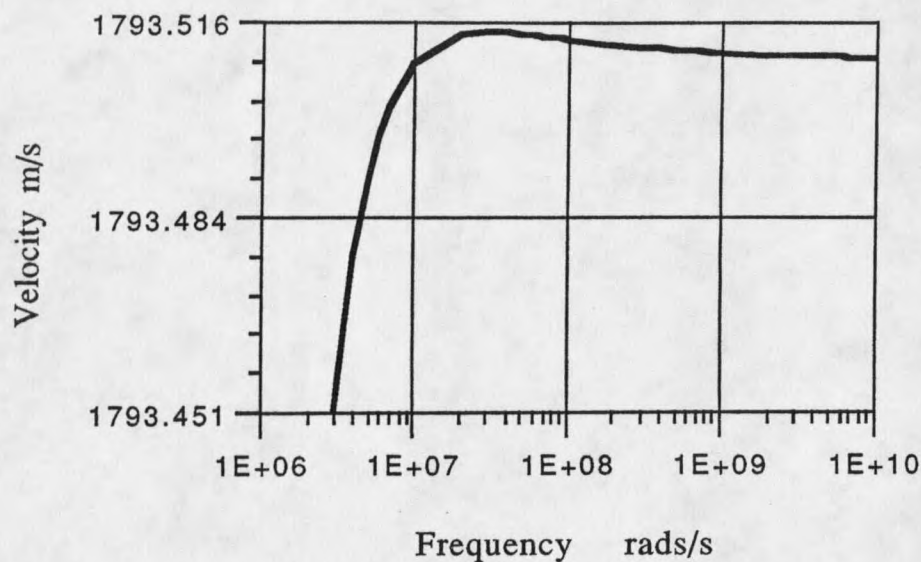


Figure 17: Propagation velocity for the second dilation mode in the high frequency regime with a density of 508 kg/m³..

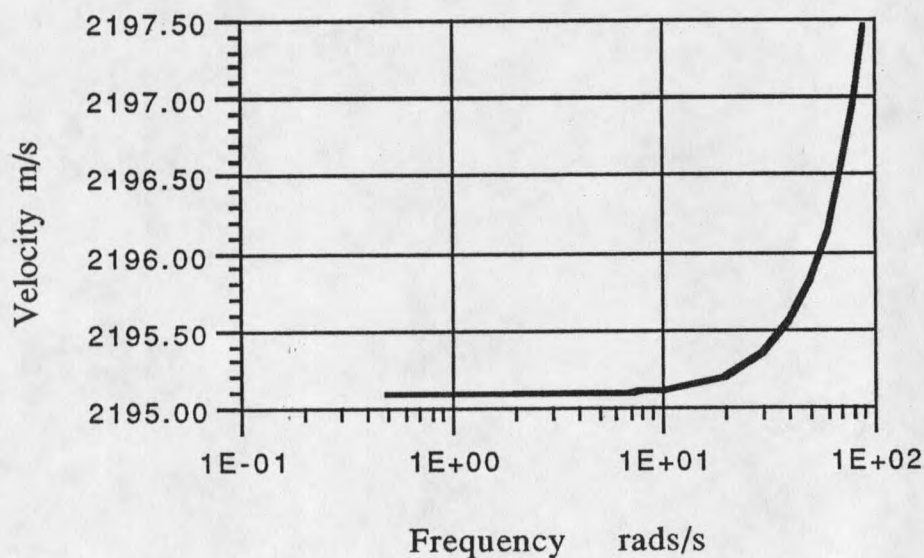


Figure 18: Propagation velocity for the second dilation mode in the low frequency regime with a density of 551 kg/m³..

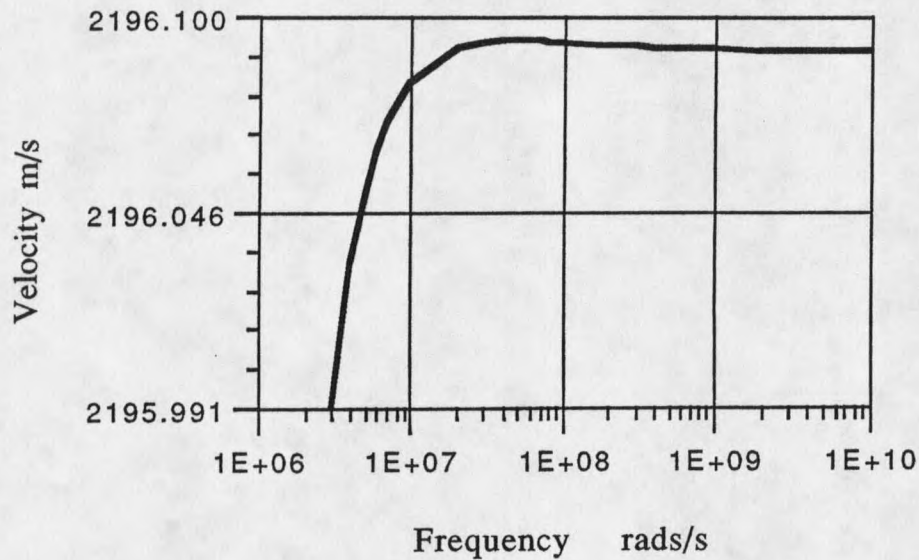


Figure 19: Propagation velocity for the second dilation mode in the high frequency regime with a density of 551 kg/m^3 .

Velocities for the second dilation mode are greater than those for the first dilation mode. This is in agreement with the "slow" and "fast" dilation waves found in both the porous media model of Biot [1956,I,II] and that of Atkins [1968]. Figure 20 and Figure 21 show attenuation coefficients for the second dilation mode. Results indicate that the attenuation coefficient increases with increasing frequency and decreases with increasing density.

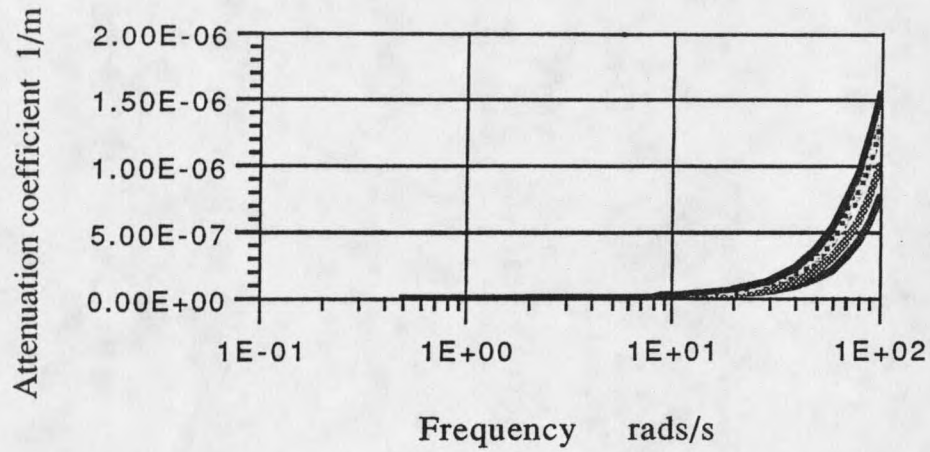


Figure 20: Attenuation coefficient for the second dilation mode in the low frequency regime.

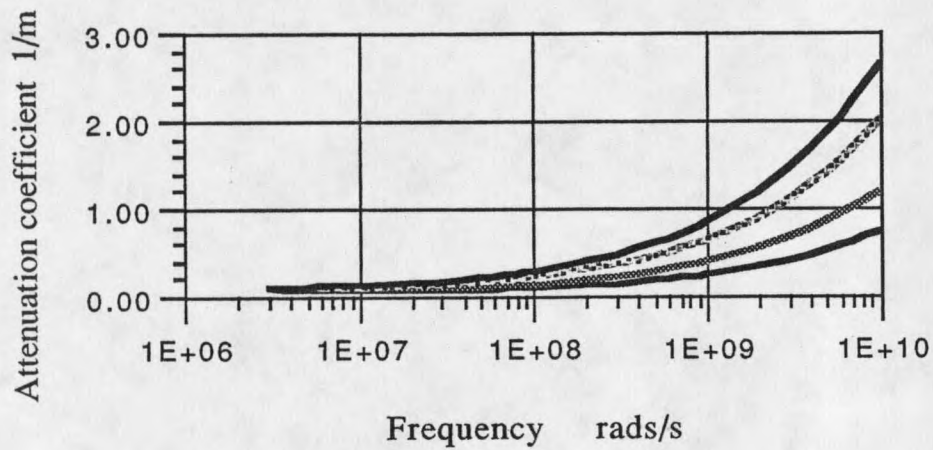


Figure 21: Attenuation coefficient for the second dilation mode in the high frequency regime.

As with the first mode, if poiseuille flow is assumed valid at high frequencies the attenuation coefficient approaches a constant value. Comparison with the first mode shows that the slow wave is highly attenuated relative to the second dilation mode. This is consistent with results from the porous media model of Biot. Results for the shear wave are presented in Figures 22 through Figure 31. These plots show that wave velocities increase with increasing frequency and increasing density and accurately approach the asymptotic behavior predicted by the limiting cases. Velocity values for the shear wave fall within the experimental values determined by Yamada, et al [1974]. Comparison with the dilation modes show that the shear wave travels faster than the first dilation wave mode but slower than the second dilation wave mode. This behavior is also seen in work done by Johnson [1978].

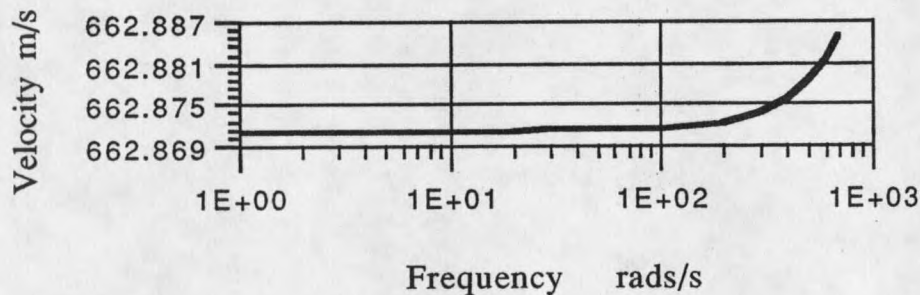


Figure 22: Propagation velocity for the shear wave in the low frequency regime with a density of 410 kg/m³.

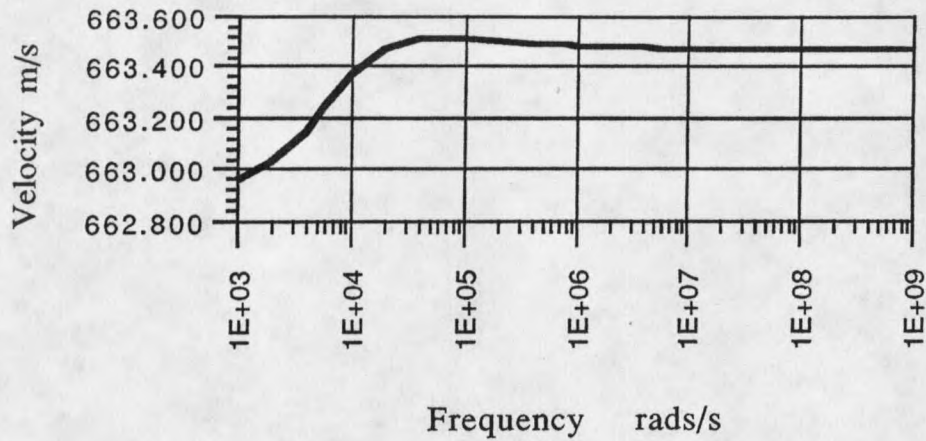


Figure 23: Propagation velocity for the shear wave in the high frequency regime with a density of 410 kg/m^3 .

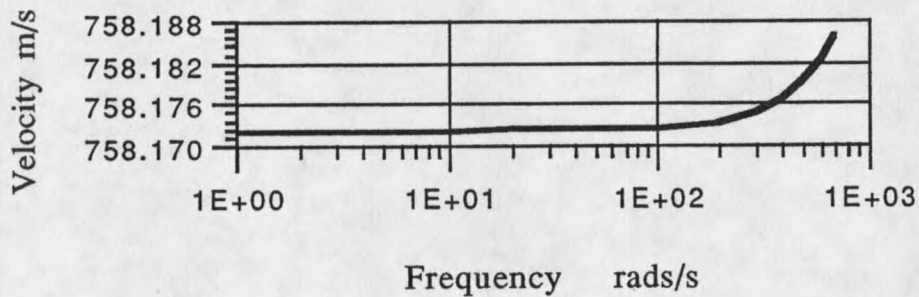


Figure 24: Propagation velocity for the shear wave in the low frequency regime with a density of 440 kg/m^3 .

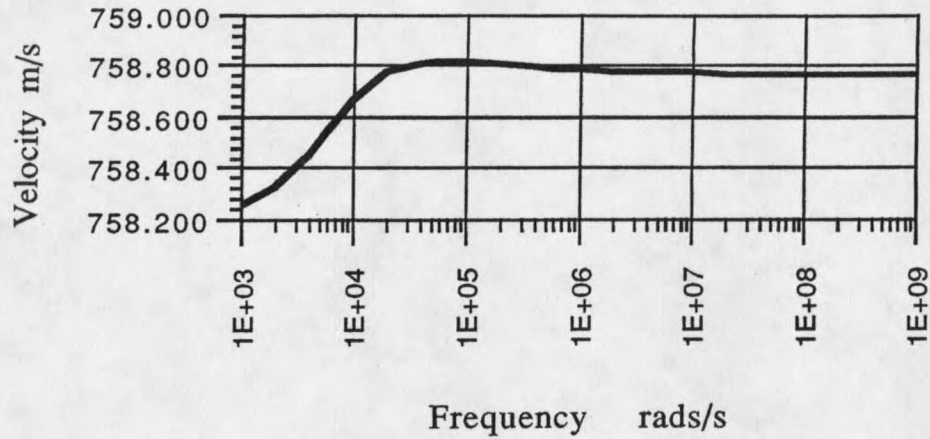


Figure 25: Propagation velocity for the shear wave in the high frequency regime with a density of 440 kg/m^3 .

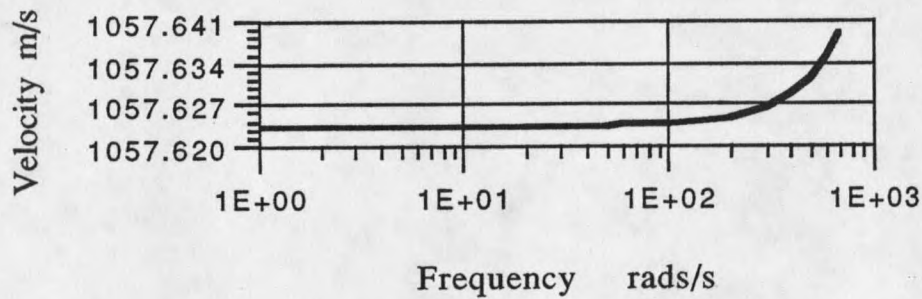


Figure 26: Propagation velocity for the shear wave in the low frequency regime with a density of 508 kg/m^3 .

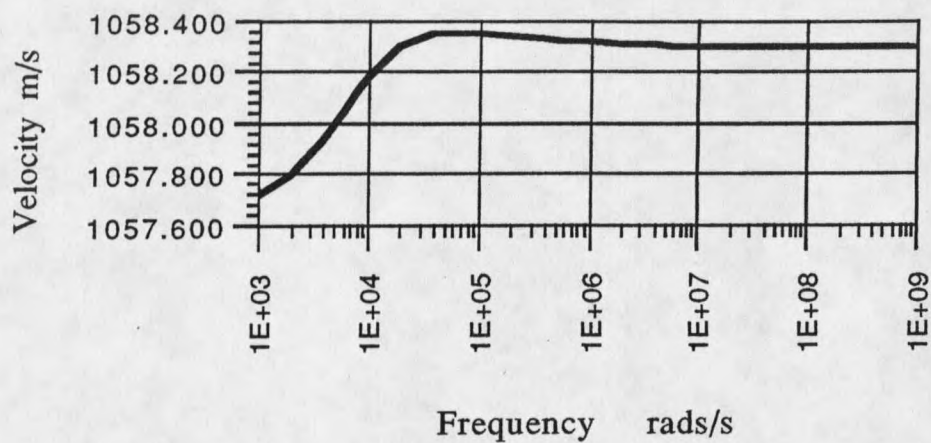


Figure 27: Propagation velocity for the shear wave in the high frequency regime with a density of 508 kg/m^3 .

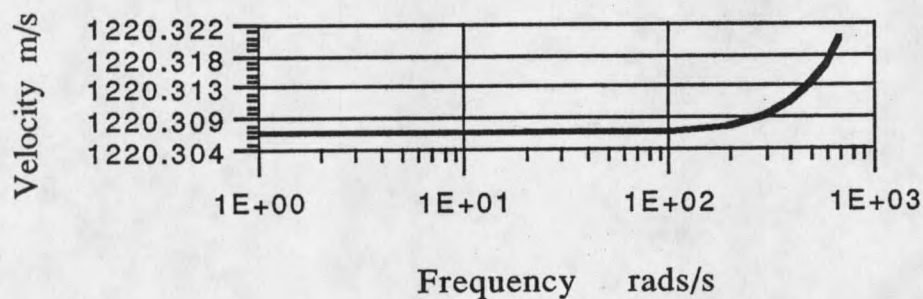


Figure 28: Propagation velocity for the shear wave in the low frequency regime with a density of 551 kg/m^3 .

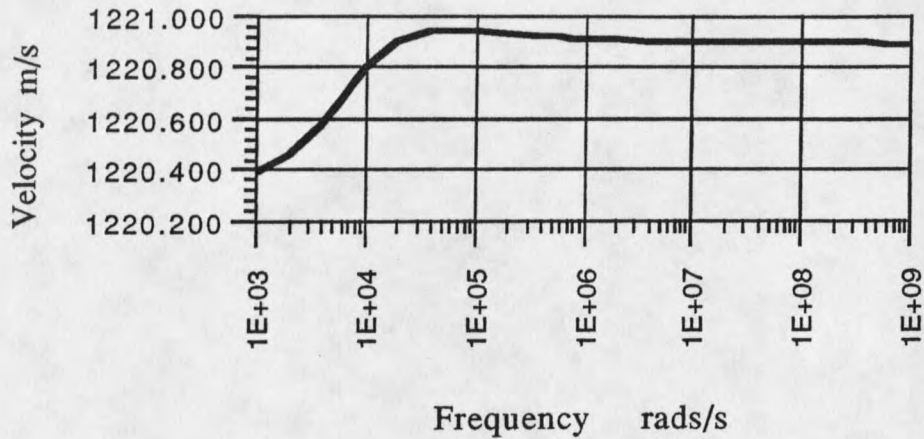


Figure 29: Propagation velocity for the shear wave in the high frequency regime with a density of 551 kg/m^3 .

Attenuation coefficients for the shear wave are shown in Figure 29 and Figure 30. These plots indicate that the attenuation coefficient increases with increasing frequency and decreases with increasing density. Comparison with the dilation waves shows that it is the least attenuated of the three waves although the attenuation of the second dilation wave mode is very similar. As with the other waves the attenuation coefficient tends toward a constant value when poiseuille flow is assumed valid at high frequencies.

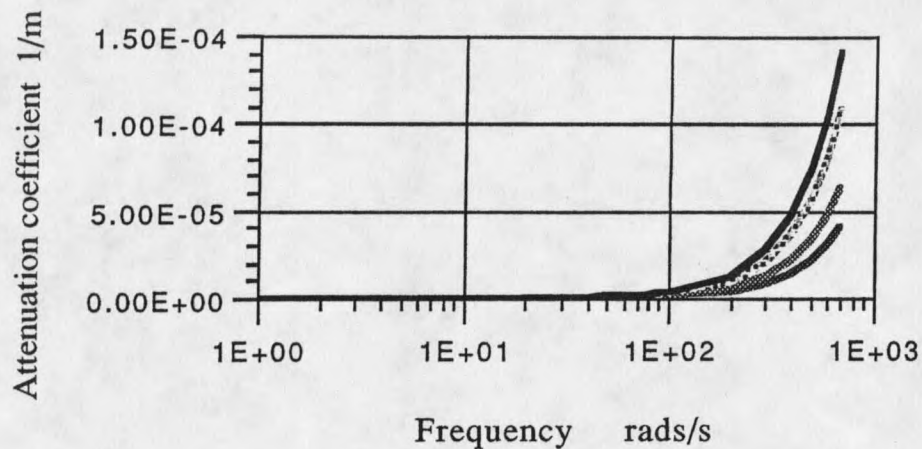


Figure 30: Attenuation coefficient for the shear wave in the low frequency regime.

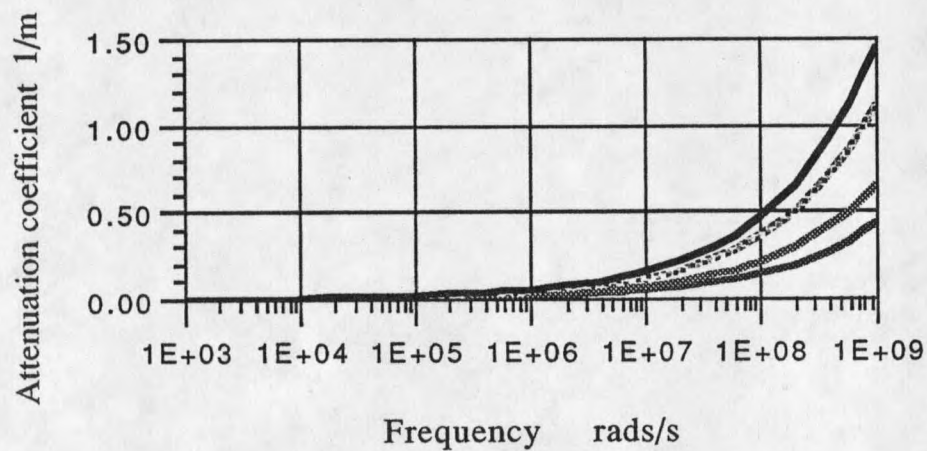
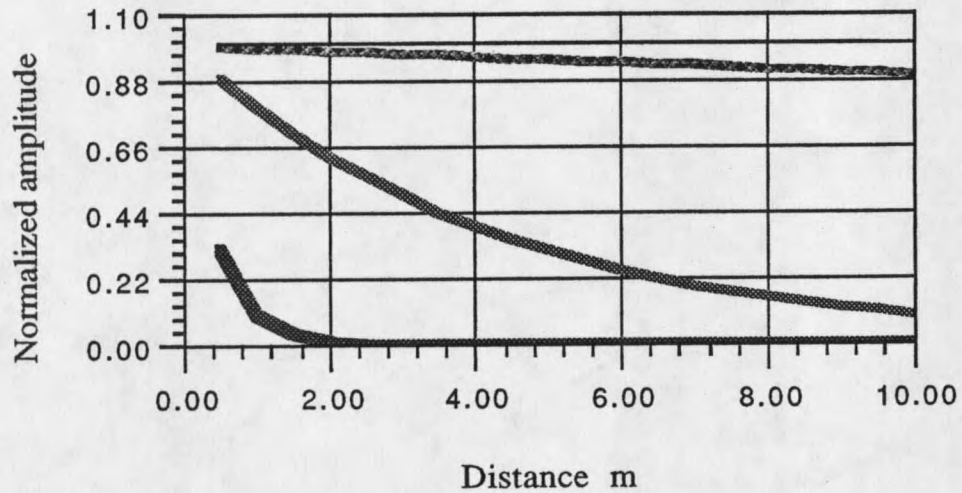


Figure 31: Attenuation coefficients for the shear wave in the high frequency regime.

Figure 32 and Figure 33 shows the normalized amplitude attenuation of all three waves for three different frequencies. In Figure 32 the dilation wave modes are plotted. Note that the first dilation mode is highly attenuated relative to the second mode. For $\omega = 600$ rads/s the amplitude of the first dilation mode is very close to zero and thus does not show up on the plot.

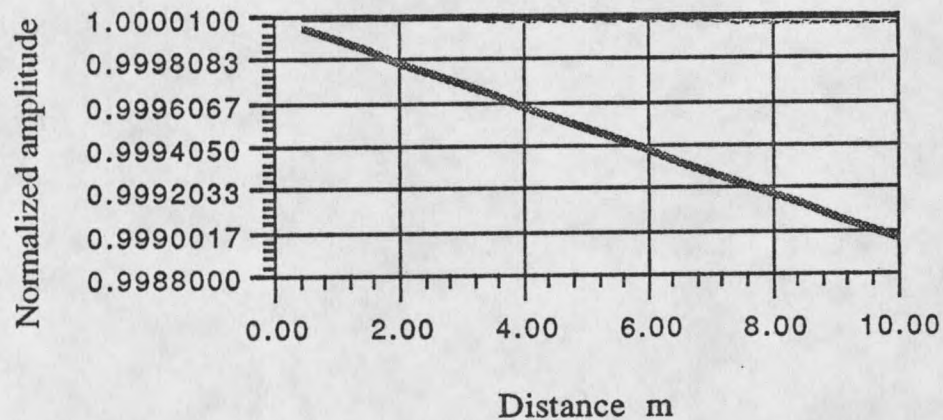


- 1st dilation mode at 10 rads/s
- 1st dilation mode at 100 rads/s
- 2nd dilation mode at 10 rads/s
- · - · - 2nd dilation mode at 100 rads/s
- × × × × 2nd dilation mode at 600 rads/s

Figure 32: Normalized amplitude attenuation for the two dilation waves with a density of 410 kg/m^3 .

Also note that the amplitude for the second dilation mode for the frequencies 10 rads/s and 100 rads/s are so similar that they appear as one line on the plot.

Figure 33 shows the amplitude attenuation for the shear wave. All three waves exhibit increasing amplitude decay with increasing frequency, the first dilation mode being the most attenuated and the shear wave being the least attenuated.



— 10 rads/s - - - 100 rads/s - · - 600 rads/s

Figure 33: Normalized amplitude attenuation for the shear wave with a density of 410 kg/m^3 .

Summary

In a porous media consisting of a linear elastic solid and an elastic fluid three waves, two dilational modes and a shear wave

can propagate. The first dilational wave mode is associated with the fluid in the pore space while the second dilational wave mode and the shear wave are supported by the solid elastic frame. A diffusive nature at low frequency, nondispersive character at high frequency, slow wave speed and high attenuation characterize the first dilation wave mode. The second dilation wave mode exhibits little attenuation with nondispersive character in both the low frequency and high frequency limits. Propagation velocity for the second mode is the fastest of all three waves. The shear wave travels faster than the slow dilation wave but at speeds less than that of the second dilation wave. In the low and high frequency limits the shear wave propagates without dispersion. It is the least attenuated of all the waves. Wave velocities of the three wave types increase with increasing frequency and increasing density, while the attenuation coefficients increase with increasing frequency and decrease with increasing density for all three waves. At high frequencies the first dilation mode is so attenuated that the amplitude approaches zero very quickly and thus is hard to detect with experimental techniques. High frequency theoretical velocities show good agreement with sonic wave velocities and attenuation coefficients exhibit behavior that is known to exist in snow.

An obvious problem with the approach used in this investigation is the inability to address the frequency transition range adequately. Nondimensionalizing the equations could

possibly expand the range of frequencies over which the binomial expansions are valid thereby reducing or eliminating the problem. As this is a simple model of a complex physical phenomena it could be expanded in several directions. More experimental data is required such as values for the coupling coefficients which would allow the stress of one constituent to depend on the strain of the other constituent. Existing data for snow is primarily confined to the higher densities. While this may be adequate for maritime and polar climates it is inappropriate for continental climates where the snowpack exists at much lower densities. Formulating the governing equations in spherical coordinates would allow geometric spreading to influence the behavior of the model and would be a more realistic approximation of the wave behavior which occurs in avalanche control. This investigation modeled the void spaces as parallel cylindrical tubes. A more complex representation of the void spaces would be in order such as allowing void spaces to be sinuous or take on different crosssectional shapes or diameters. Scattering effects due to the porous geometry would then play a greater role in the attenuation characteristics. Finally as snow and other geologic materials usually exist as stratified material, boundary effects should be investigated. Due to the intractable nature of the analytical approach it is felt that a numerical analysis of the problem may be more appropriate especially with regard to boundary effects and different loading conditions.

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