Derivation of the equations of motion of a dissipative hydrodynamic system by a guage transformation of the lagrangian density
by George Hiroshi Saito

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Physics
Montana State University

Abstract:
This investigation is concerned with the derivation of the equations of motion of a real fluid from a Lagrangian density function. Dissipation of energy by viscous forces is introduced into the Lagrangian density function hy a hydrodynamic gauge transformation of the second kind analogous to the transformation used in electrodynamic systems. The equations of motion for the viscous fluid are analyzed, and are found consistent with the Navier-Stokes equation when the fluid flow is taken to he irrotational, the ratio of the fluid velocity to the velocity of sound is small, and the terms quadratic in the viscous coefficient are negligible.

The effects of the proposed transformation on the hydrodynamic field are analyzed through construction of the various densities- associated with the field. Employing the conservation of momentum, the Navier-Stokes equation for a viscous fluid is obtained. From this result, it is concluded that the validity of the hydrodynamic gauge transformation used to introduce viscosity into the equations of motion of a real fluid is established.

DERIVATION OF THE EQUATIONS OF MOTION OF A DISSIPATIVE HYDRODYNAMIC SYSTEM BY A GAUGE TRANSFORMATION OF THE LAGRANGIAN DENSITY

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A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

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ABSTRACT

This investigation is concerned with the derivation of the equations of motion of a real fluid from a Lagrangian density function. Dissipation of energy by viscous forces is introduced into the Lagrangian density function by a hydrodynamic gauge transformation of the second kind analogous to the transformation used in electrodynamic systems. The equations of motion for the viscous fluid are analyzed, and are found consistent with the Navier-Stokes equation when the fluid flow is taken to be irrotational, the ratio of the fluid velocity to the velocity of sound is small, and the terms quadratic in the viscous coefficient are negligible.

The effects of the proposed transformation on the hydrodynamic field are analyzed through construction of the various densities associated with the field. Employing the conservation of momentum, the Navier-Stokes equation for a viscous fluid is obtained. From this result, it is concluded that the validity of the hydrodynamic gauge transformation used to introduce viscosity into the equations of motion of a real fluid is established.

I

INTRODUCTION

The purpose of the study undertaken in this thesis is to construct a Lagrangian density function which provides the equations of motion of a dissipative, classical hydrodynamic system. Equations of motion obtained from the Lagrangian density function by means of the variational principle must be consistent with the Navier-Stokes equation of hydrodynamics. This is the criteria which will be used to determine the correct form of the Lagrangian density function for a real fluid.

For the treatment of a conservative hydrodynamic system, the variational principle has been used with success. However, for a nonconservative system in which there is a dissipation of energy, the treatment of the system is not as well defined. The difficulty arises in the formation of an appropriate Lagrangian density function for the system. Presently, there is no accepted, standard technique for the treatment of a general dissipative energy system by a variational principle. In a real fluid, it is known that the viscous stresses result in mechanical energy being dissipated from the system. This dissipative process takes place irreversibly as a continual degradation of system energy.

In this thesis, the attempt is made to introduce viscosity into the equations of motion of an irrotational fluid by considering the analogy between dissipative processes in hydrodynamics and electrodynamics and by employing a gauge transformation. This transformation, following electrodynamics, is called a gauge transformation of the second kind. It is shown that the transformation does not change the wave functions of a quantum hydrodynamic representation of the system.

To avoid confusion, a discussion of the established gauge transformation of hydrodynamics is included which is denoted as a gauge transformation of the first kind, in keeping with the electrodynamic nomenclature. In the following section, the discussion begins with a review of variational techniques.

II

FORMULATION OF THE VARIATIONAL PRINCIPLE

Introduction

Variational methods coupled with a canonical formalism are very versatile and have enjoyed widespread application. In addition, a classical canonical formalism for a system is a necessary prerequisite for the subsequent quantization of the system, Wentzel[1], Schiff[2], and Ito[3].

For this study, the techniques of the variational principle will be applied to attempt a classical canonical formalism of a dissipative hydrodynamic system. However, before developing the field equations from a generalized Lagrangian density, the methods used in classical mechanics will be described briefly, see for example Goldstein[4], Landau and Lifshitz[5], and Corben and Stehle[6].

Discrete Mechanical System

The equations of motion of classical particle mechanics are obtained from the Principle of Least Action or Hamilton's Principle which states that an action integral of the form

$$I = \int_{t_1}^{t_2} L \, dt \quad , \qquad (1)$$

has an extremum for the path of the motion in the time interval from t_1 to t_2, and the integrand function is defined to be the Lagrangian for the system.

The condition for the integral to be stationary is given by

$$\delta \int_{t_1}^{t_2} L\left(q, \frac{\partial q}{\partial t}, t\right) dt = 0 \quad ,$$

where it has been assumed that the Lagrangian is a function only of the generalized coordinate q and the first time derivative of the generalized coordinate. Carrying out the variation, the equation of motion for the system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad ,$$

with

$$\dot{q} = \frac{\partial q}{\partial t} \quad .$$

For a system with n degrees of freedom, as given by the number of generalized coordinates, the Lagrangian must be varied independently with respect to each of the generalized coordinates. For n generalized coordinates, the equations of motion for a discrete, conservative mechanical system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad ,$$

$$i = 1, 2, \cdots, n \quad .$$

The above set of second order equations represents the Euler-Lagrange differential equations for the system.

When the potential energy is a function of the generalized coordinates and the kinetic energy is quadratic in the generalized velocity coordinates, the Lagrangian can be expressed as

$$L = T - V \quad ,$$

where T is the kinetic energy,

and V is the potential energy.

When expressed in generalized coordinates, the Lagrangian formulation is invariant with respect to the coordinate system[4].

Hamilton's principle may be extended to nonconservative systems[4, 5] and is given in the form

$$\delta \int_{t_1}^{t_2} (T + W)\, dt = 0 \quad ,$$

where T is the kinetic energy,

and W is the virtual work.

In this form, it is required that the first variation of the integral of the kinetic energy and the virtual work done by the forces on the system during the displacement must vanish. Other forms of the equations of motion for nonconservative and nonholonomic mechanical systems are well established in the literature[4, 5, 6].

Continuous Mechanical System

The treatment of the transition from a discrete mechanical system to a continuous mechanical system is given by Goldstein[4].

For this type of system, the action intergral to be varied has the form

$$I = \int \mathcal{L} \; dx_1 \, dx_2 \, dx_3 \, dx_4 \quad , \qquad (2)$$

where x_i represents x, y, z, and t.

The integrand function in equation (2) is often called the differential Lagrangian function or the Lagrangian density function since the Lagrangian can be expressed in the form

$$L = \int \mathcal{L} \; dx_1 \, dx_2 \, dx_3 \quad .$$

The significant difference between the Lagrangian density function and the integrand function of equation (1) is the role of the independent variables. For a discrete system, the independent variables are the generalized coordinates. In the transition to a continuous mechanical system, properties of the system become described by independent field variables with the coordinates of configuration space serving, to use an analogy, as a sort of book-keeping system to keep track of the independent variables. The coordinates no longer have the significance that they previously did in the description of a discrete, mechanical system.

The variational formalism may also be used to obtain the description of fields, Wentzel[1], and Landau and Lifshitz[7]. A classical field is specified by one or more independent functions of

space and time called field quantities. In hydrodynamics, the important field quantity is the velocity field which may be constructed from canonically conjugate field variables, where the canonical variables are obtained in the usual way from the Lagrangian density[1]. In this study, a classical continuum field will be considered in order to arrive at the equations of motion for the field.

Field Equations

For the present work, the Lagrangian density function is assumed to be a function only of the field variables, and the first space and time derivatives of the field variables. For simplification, only one field quantity, $m(x_j)$, will be expressed, i.e.,

$$\mathcal{L} = \mathcal{L}\left(m, \frac{\partial m}{\partial x_1}, \frac{\partial m}{\partial x_2}, \frac{\partial m}{\partial x_3}, \frac{\partial m}{\partial x_4}\right) ,$$

The action integral to be extremized is

$$I = \int \mathcal{L} \; dx_1 \, dx_2 \, dx_3 \, dx_4 \quad ,$$

where the integration is to be carried out over a definite region of configuration space. Suitable boundary conditions are applied such that the variables take on certain prescribed values at the boundary.

Generally following the techniques of Wentzel[1], Weinstock[8], and Fox[9], a one parameter family of comparison functions is introduced

$$\mathcal{M} = m(x_i) + \epsilon f(x_i) \quad ,$$

where it is assumed that the actual extremizing function is $m(x_1, x_2, x_3, x_4)$ and that the addition of an arbitrary function $f(x_1, x_2, x_3, x_4)$ generates a family of functions which reduce to the extremal function when the parameter epsilon assumes the value zero.

Abbreviating the details[1, 8, 9], the variation process leads to

$$\int f \left\{ \frac{\partial \mathcal{L}}{\partial m} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial m}{\partial x_i} \right)} \right\} dx^4 = 0 \quad ,$$

with dx^4 the volume element in configuration space.

Since the function $f(x_1, x_2, x_3, x_4)$ is arbitrary, the first variation is satisfied by

$$\frac{\partial \mathcal{L}}{\partial m} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial m}{\partial x_i} \right)} = 0 \quad .$$

In general for multiple field quantities, the equations of motion for the field take the form

$$\frac{\partial \mathcal{L}}{\partial m_i} - \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial m_i}{\partial x_j} \right)} = 0 , \quad (4)$$

where the implied double sum convention[7] has been employed and where m_i, $i = 1, 2, - - - , n$ represents the number n of independent field variables.

The result given by equation (4) is referred to as the Euler equations and are the equations of motion for the field. These field equations have been obtained assuming the Lagrangian density function to have as its arguments the field quantities and the first space and time derivatives only of the field variables. The formulation of a Lagrangian density function for a dissipative fluid system consistent with the field equations will be taken up in Section IV.

Navier-Stokes Equation

The Navier-Stokes equation is named in honor of Claud Louis Marie Henri Navier and George Gabriel Stokes, and is written as

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j} \tau_{ij} , \quad (5)$$

where u_i is the i th component of the velocity,

ρ is the fluid density,

P is the fluid pressure,

and τ_{ij} is the ij th component of the hydrodynamic stress tensor.

For the purposes of the present work, when the hydrodynamic stress tensor[10] is approximated to first degree in the viscous coefficient and when irrotational flow conditions are taken into account, the Navier-Stokes equation becomes

$$\rho \frac{D\bar{u}}{Dt} = -\nabla P + \frac{4}{3}\mu \nabla\left(\nabla \cdot \bar{u}\right) \quad ,$$

where μ is the coefficient of viscosity.

The velocity components will be used in the form which is designated as the Eulerian representation[11].

The substantive derivative operator, Aris[12] and Milne-Thompson[13], is expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\bar{v} \cdot \nabla\right) \quad .$$

A convenient identity of the velocities is

$$\left(\bar{v} \cdot \nabla\right)\bar{v} = \frac{1}{2}\nabla v^2 - \bar{v} \times \left(\nabla \times \bar{v}\right) \quad ,$$

where the quantity

$$\nabla \times \overline{v}$$

is the rotation or vorticity of the fluid.

In the following section, the development of variational techniques to various fluid systems will be briefly described.

III

APPLICATION OF THE VARIATIONAL PRINCIPLE TO HYDRODYNAMIC SYSTEMS

Introduction

The development and application of the variational principle to hydrodynamics has taken several forms and methods of approach. Basically, however, an appropriate Lagrangian density function is sought which will yield the Navier-Stokes equation subject to the boundary and flow conditions. Thus, a major obstacle to be overcome is the correct expression of the Lagrangian density function for a fluid system which will provide the proper equations of motion. Some of the applications of the variational principle to fluid systems will now be described briefly.

Equations of Motion

An application of the variational method to the Lagrangian density function for a non-viscous or perfect fluid is given by Bateman[14]. The Lagrangian density is written as

$$\mathcal{L} = \rho \left\{ \frac{\partial \phi}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) \right\} + f(\rho) \quad ,$$

where ρ is the fluid density,

φ, β are components of the velocity potential,

α is an undetermined Lagrange multiplier,

u, v, w are components of the velocity,

and $f(\rho)$ is some function of the density,

with the velocity field defined as

$$v_i = \nabla_i \varphi + \alpha \nabla_i \beta \quad . \qquad (7)$$

Variation of this Lagrangian with respect to the variables ρ, α, and β gives the equations of motion for a perfect fluid, i.e., a fluid experiencing no dissipative forces

$$\rho \frac{D\bar{v}}{Dt} = -\nabla P \quad ,$$

where P is the fluid pressure and D/Dt is the substantive derivative operator.

Ito[3] employs a Clebsch transformation of the fluid velocity as in equation (7), with v_j replaced by $-v_j$, to develop the equations of motion. Considering a fluid system similar to that of the preceding paragraphs, the Lagrangian density for a perfect fluid is written as

$$\mathcal{L} = \rho \left\{ \frac{1}{2} v^2 - U(\rho) + \frac{D\varphi}{Dt} + \alpha \frac{D\beta}{Dt} \right\} ,$$

where $\overline{v}$ is the fluid velocity,

ρ is the fluid density,

u is the internal energy per unit mass,

α is an undetermined Lagrange multiplier,

and φ, β are components of the velocity.

The variation of the Lagrangian density[3] with respect to ρ, $\overline{v}$, φ, β, and α again gives Euler's equations of motion for the fluid.

In considering the case of an ideal fluid, a different formulation of Hamilton's principle is given by Herivel[15]. It is postulated that the treatment of a continuous system for which thermodynamic properties may not be disregarded should include both the potential energy and the intrinsic energy in the Lagrangian density. This generalization for a continuous system is expressed as

$$\delta \int_1^2 \{T - (u + V)\} \, dt = 0 \quad ,$$

where T is the kinetic energy,

U is the intrinsic energy,

and V is the potential energy.

This formulation leads directly to the equations of motion as before provided that there is no irreversible entropy production[15].

These treatments of an ideal fluid serve to demonstrate the power and elegance of the variational principle. However, a more

interesting subject is the consideration of the description of a fluid experiencing dissipative forces. These forces arise from the viscous stress or internal friction within the fluid which accounts for the mechanism of energy dissipation.

Dissipative Systems

Primarily, the variational principle has been confined to the description of reversible thermodynamic processes. In particular, these reversible processes are adiabatic processes whereby the entropy of the system is conserved. Investigations by others into the validity and extension of the variational principle to a dissipative system or an irreversible process will be described in the remainder of this section.

Historically, the first approach to a formulation of the equations of motion for a viscous fluid and dissipative energy system is described in Millikan[16]. A variational method is attempted for a viscous fluid under steady flow conditions. The Lagrangian density which is considered has the form

$$\mathcal{L} = \mathcal{L}\left(u, v, w, \nabla u, \nabla v, \nabla w\right) \quad ,$$

with the components of the velocity denoted by u, v, w, and the first order partial derivatives are with respect to the spatial coordinates. However, Millikan[16] concluded that this Lagrangian density would not produce the desired hydrodynamic relations.

A General Variation Principle

The formulation of a general variation principle which would be applicable to a wide variety of dissipative systems is attempted by Herivel[17]. This variation principle is stated as:

> "For a certain general class of irreversible processes there exists a dissipation function $\mathcal{F}$ such that the rate of change of the kinetic energy minus the rate of working of the mechanical forces (other than frictional force) acting on the system plus the dissipation function is stationary with respect to the velocities producing this change, the (generalized) accelerations of the system being kept constant during the course of the variation."

It is asserted by Herivel[17], that a general formulation of a variation principle for a dissipative process appears feasible. An important characteristic of a dissipative or irreversible process is that there does not exist a heat function analogous to a potential energy or internal energy function. Thus, to describe the most general thermodynamic process, rates of change of energy and not merely changes of energy must be considered[17]. For a verification of this principle, comparison is made with fluid systems whose equations of motion are already known.

For a perfect fluid[17], the variation expressed as

$$\delta \int \left\{ \frac{1}{2} \rho v^2 + \rho (U + V) \right\} dx^3 = 0 \quad ,$$

where U and V are the internal and potential energies, respectively, leads to the dynamic equations as before.

Extending this principle to the case of a viscous fluid, Herivel gives the variation as

$$\delta \int \left\{ \rho \bar{v} \cdot \frac{D\bar{v}}{Dt} - \rho \nabla \cdot \bar{v} + \rho \bar{v} \cdot \nabla V \right.$$

$$\left. + \frac{1}{2} \sum_i \sum_j \tau_{ij} \frac{\partial v_j}{\partial x_i} \right\} dx^4 = 0 \quad ,$$

where the rate of change of mechanical energy per unit mass is given by the term

$$\rho \nabla \cdot \bar{v} \quad ,$$

which represents energy expended against the hydrostatic pressure due to expansion of the system. The rate of dissipation of mechanical energy per unit volume as a result of the viscous forces is denoted by

$$\sum_i \sum_j \tau_{ij} \frac{\partial v_j}{\partial x_i} \quad .$$

Omitting the details of the variation[17], Herivel obtains a result consistent with the Navier-Stokes equation.

The applications of the variation principle enunciated by Herivel to cases where the equations of motion are known have proven correct. However, some aspects of the theory have been reviewed and it is reformulated[18] to be amenable for any thermodynamic system. The assumption used previously that the kinetic energy should be replaced by the sum of the kinetic and intrinsic energies is not correct when there is irreversible entropy production.

The modified principle is reformulated as follows:

"For a general class of thermodynamic systems there exists a a dissipative function ϕ such that the rate of change of the kinetic energy plus intrinsic energy minus the rate at which the external forces are doing work on the system, minus ϕ, is stationary with respect to the velocities producing this change, the (generalized) rates of change of the velocities being regarded as constant for the purpose of the variation."

Using this formulation for a perfect fluid[18], the equations of motion are obtained as before. It is noted by Herivel that the results obtained for perfect and viscous fluids hold only for proper choice of the dissipation function. However, a problem arises in the selection of a proper dissipation function when the desired results are not known beforehand. This weakness in the above method has not been completely resolved.

Principle of Virtual Displacements

Additional work in the development of a variation principle for hydrodynamics is presented by Lieber, Wan, and Anderson[19]. This formulation of a variation principle differs from that of Herivel[17, 18] in that a complete account of the energy dissipation of the system is considered. The principle of virtual displacements for the flow of a viscous fluid expresses the condition that, "- - - the work done by the forces applied to a surface enclosing a real fluid is stationary for any real path satisfying boundary conditions assigned at the surfaces." To verify this principle, Lieber, Wan, and Anderson have applied it to a viscous fluid. Omitting the details[19], the equations of motion of a viscous fluid are produced as before.

Principle of Minimum Dissipation

The principle of minimum dissipation which is discussed by Lieber and Wan[20, 21] attempts to establish a formal connection between the principle of stationary dissipation and the Principle of Least Constraint of Gauss. For a fluid surface on which stresses are applied, "- - - the difference between the sum of the rate of work done by these stresses plus the body forces, and the dissipation rate plus the local temporal change of kinetic energy is stationary."

This principle is hypothesized upon the assumption that, subject to the constraints and boundary conditions, the time rate of dissipation of energy is a minimum. Lieber and Wan establish that sufficient

auxiliary conditions are given by the laws of conservation of mass and energy to prove the minimum dissipation principle.

To examine the results which would be obtained using this principle, several applications to viscous fluid flow are considered by Lieber and Wan. However, it is found[20, 21, 22] that an Euler equation is obtained which does not compare with the Navier-Stokes equation. This is interpreted by Lieber and Wan to imply that more solutions are obtained than are admitted by the Navier-Stokes equation.

The application of a minimum dissipation principle to a viscous fluid is considered by Dickerson[23]. To implement the minimum dissipation principle, Dickerson considers the total energy dissipation between two component systems to be a minimum. In addition, the velocity is separated into two parts

$$V_i = U_i + U_i' \quad ,$$

which is in effect performing a Clebsch transformation, and the difference of the time rates of dissipation of energy is represented by

$$I = \int (R - R') \, dx^3 \, dt \quad ,$$

where R is the time rate of dissipation of energy from the unprimed system,

R' is from the primed system,

and dx^3 is the differential volume element.

As given by Lamb[24], the time rate of dissipation of energy per unit volume is expressed by

$$R = -P \frac{\partial u_i}{\partial x_i} + \mu F \quad ,$$

where P is the fluid pressure,

μ is the coefficient of viscosity,

and F is a function of the spatial derivatives of the of the velocity components.

Carrying out the variation process[23], the Navier-Stokes equation is obtained for a viscous fluid.

The attempt has been made in this section to indicate some of the methods and techniques which have been devised for the study of dissipative hydrodynamic systems. In all of these treatments, the variational principle has been used exclusively. The following section considers the development of a Lagrangian density function for a real fluid by means of a hydrodynamic gauge transformation.

IV

DEVELOPMENT OF A HYDRODYNAMIC GAUGE TRANSFORMATION

Introduction

In the following sections, the attempt is made to introduce viscosity into the equations of motion of an irrotational fluid by considering a transformation of the wave functions representing a quantum hydrodynamic fluid. So as not to confuse the usual gauge transformation of hydrodynamics[3] with the transformation proposed in this study, the gauge transformation is reviewed below.

Gauge Transformation

It is convenient to represent the hydrodynamic velocity field in Clebsch's form as

$$v_j = -\nabla_j \varphi - \lambda \nabla_j \theta \qquad (8)$$

The scalar quantities φ, λ, and θ are not uniquely determined by the velocity field. Therefore, it is possible to find equivalent representations in which the velocity is invariant under transformation from one representation to another. Such a transformation is called a gauge transformation which in this study will be denoted as a gauge transformation of the second kind.

To review the established gauge transformation of hydrodynamics, the scalar product is formed from equation (8) and a virtual displacement at constant time is considered

$$\bar{v} \cdot d\bar{r} = -\nabla\varphi \cdot d\bar{r} - \lambda \nabla\theta \cdot d\bar{r} \quad ,$$

which becomes

$$\bar{v} \cdot d\bar{r} = -d\varphi - \lambda\, d\theta \quad .$$

For the left side of the above equation to be invariant under the transformation

$$\begin{aligned} \varphi &\rightarrow \varphi' \quad , \\ \lambda &\rightarrow \lambda' \quad , \\ \theta &\rightarrow \theta' \quad , \end{aligned}$$

the following is true

$$d(\varphi' - \varphi + \lambda'\theta') = \lambda\, d\theta + \theta' d\lambda' \quad .$$

Taking

$$\varphi' - \varphi + \lambda'\theta' = \theta\lambda' + w \quad ,$$

the above becomes

$$d(\theta\lambda' + w) = \theta\, d\lambda' + \lambda'\, d\theta + dw$$

$$= \lambda\, d\theta + \theta'\, d\lambda' \quad ,$$

or

$$dw = (\theta' - \theta)\, d\lambda' + (\lambda - \lambda')\, d\theta \quad .$$

Thus, the generating function of the transformation may be assumed to be a function of θ and λ', and it follows that

$$\theta' - \theta = \frac{\partial w}{\partial \lambda'} \quad ,$$

$$\lambda - \lambda' = \frac{\partial w}{\partial \theta} \quad .$$

The transformation is then constructed as

$$\rho\lambda' = \rho\lambda - \rho \frac{\partial w}{\partial \theta} \quad ,$$

$$\theta' = \theta + \rho \frac{\partial w}{\partial (\rho\lambda')} \quad ,$$

$$\phi' = \phi - \rho\lambda' \frac{\partial w}{\partial (\rho\lambda')} + w \quad ,$$

$$\rho \frac{\partial w}{\partial \rho} + \rho\lambda' \frac{\partial w}{\partial (\rho\lambda')} = 0 \quad .$$

These equations define the generating function which is found to be of the form

$$w = \frac{(\rho\lambda')}{\rho} f(\theta) \quad ,$$

where $f(\theta)$ is an arbitrary function of the parameter θ .

The combination $\rho\lambda'$ has been employed instead of λ' alone, as $\rho\lambda'$ and θ are conjugate variables, as are ρ and ϕ in hydrodynamics[3]. It is noticed that the so called phonon variables, ρ and ϕ , are not modified by the gauge transformation but the roton variables in the combination, $\rho\lambda'$ and θ , are changed by the transformation.

Proposed Hydrodynamic Gauge Transformation

In the following paragraphs, a different kind of gauge transformation will now be considered. It is known that a hydrodynamic analogy of sorts may be obtained from the Schrödinger equation by the transformation of the wave function

$$\psi = R\, e^{iS/\hbar} \quad ,$$

where R and S are real variables and

$$v_i = \frac{1}{m} \nabla_i S \quad .$$

If the analogy is restricted to the phonon variables only, i.e., ρ and ϕ, so that

$$\phi = -\frac{S}{m} \quad ,$$

then

$$\phi = \frac{\hbar}{2im} \ln \frac{\psi^*}{\psi} \quad , \qquad (9)$$

and in addition, it follows that

$$v_j = -\nabla_j \phi$$

$$= \frac{\hbar}{2im} \left\{ \frac{\psi^* \nabla_j \psi - \psi \nabla_j \psi^*}{\psi^* \psi} \right\} \quad ,$$

or

$$\rho \vec{v} = \frac{\hbar}{2i} \left\{ \psi^* \nabla \psi - \psi \nabla \psi^* \right\} \quad ,$$

with

$$\rho = m \psi^* \psi \quad . \qquad (10)$$

These equations of a quantum hydrodynamic representation suggest a scheme for introducing viscosity into the hydrodynamic equations. The following reasoning is purely heuristic and is not intended to be rigorous but merely suggests an analogy.

The internal friction or viscosity generates forces which disperse the energy in a given local disturbance of the hydrodynamic system in the same manner as it is recalled the coupling between the source and the field will do in electrodynamic systems. This coupling is introduced into electrodynamic systems[1] by a gauge transformation of the second kind as

$$\psi \rightarrow \psi \, e^{\frac{i\nu}{\hbar c} \xi(x)} .$$

The question posed is whether viscosity may be introduced into the hydrodynamic equations of motion by an analogous transformation. To this end, consider the gauge transformation

$$\psi \rightarrow \psi \, e^{\frac{i\mu}{\hbar \rho_0} \gamma \omega(\rho, \phi)} ,$$

where μ is the coefficient of viscosity,

ρ_0 is the mass density of the fluid at rest,

ω is, at the moment, an arbitrary function of the conjugate variables, ρ and ϕ ,

and γ is a constant.

From equations (9) and (10), the transformation gives

$$\varphi = \frac{\hbar}{2im} \ln \frac{\psi^*}{\psi}$$

$$= \frac{\hbar}{2im} \ln \frac{\psi'^*}{\psi'} - \frac{u'r'}{\rho_0} \omega(\rho', \varphi') \ ,$$

so that the consequence of the transformation is that

$$\left.\begin{array}{l} \varphi \to \varphi' - \dfrac{u'r'}{\rho_0} \omega(\rho', \varphi') \\ \rho \to \rho' \end{array}\right\} . \qquad (11)$$

This thesis is concerned with an analysis of the consequences of this hydrodynamic gauge transformation of the second kind upon the equations of motion of an irrotational fluid. The equations of motion of the hydrodynamic field to be analyzed are obtained from the Lagrangian density by the variational techniques of Section II.

Equations of Motion

From Appendix A, the Lagrangian density before transformation is written as

$$\mathcal{L} = \rho \left\{ \frac{1}{2} v^2 - \mathcal{E}(\rho) + \frac{D\varphi}{Dt} \right\} \ ,$$

where ρ is the fluid density,

$\overline{v}$ is the fluid velocity,

$\mathcal{E}$ is the internal energy,

and ϕ is the velocity potential function.

Under the hydrodynamic transformation of equation (11), the Lagrangian density becomes

$$\mathcal{L} \rightarrow \mathcal{L}' = \rho' \left\{ \frac{1}{2} v'^2 - \mathcal{E}(\rho') + \frac{D\phi'}{Dt} \right\} - \frac{\rho'}{\rho_0} \frac{D\pi'}{Dt} \quad , \qquad (12)$$

where

$$\pi' = \mu' r' w(\rho', \phi') \quad .$$

Dropping the primes and considering the variation of the transformed Lagrangian density with respect to the variables v_i, ρ, and ϕ, the variation of v_i yields

$$\rho v_i + \rho \nabla_i \phi - \frac{\rho}{\rho_0} \nabla_i \pi = 0 \quad , \qquad (13\ a)$$

or

$$v_i = - \nabla_i \phi + \frac{\mu r}{\rho_0} \nabla_i w \quad . \qquad (13\ b)$$

Again borrowing an interpretation from electrodynamics, it is suggested that equation (13 a) states that the momentum density of

the system is composed of a kinetic momentum plus a potential momentum, i.e., a momentum associated with the viscous forces in the fluid. Equation (13 b) represents the new form of the velocity with respect to the gauge transformation.

Varying φ gives

$$-\frac{\partial \rho}{\partial t} - \nabla_\alpha (\rho v_\alpha) + \frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\frac{\partial \eta}{\partial \varphi}\right)$$

$$+ \nabla_\alpha \left(\frac{\rho}{\rho_0} v_\alpha \frac{\partial \eta}{\partial \varphi}\right) - \frac{\rho}{\rho_0}\frac{D\varphi}{Dt}\frac{\partial^2 \eta}{\partial \varphi^2} = 0 \quad .$$

Expanding and collecting terms, the above equation becomes after simplification

$$-\left\{1 - \frac{1}{\rho_0}\frac{\partial \eta}{\partial \varphi}\right\}\left\{\frac{\partial \rho}{\partial t} + \nabla_\alpha (\rho v_\alpha)\right\}$$

$$+ \frac{\rho}{\rho_0}\frac{\partial^2 \eta}{\partial \rho \partial \varphi}\left\{\frac{\partial \rho}{\partial t} + v_\alpha \nabla_\alpha \rho\right\} = 0 \quad ,$$

which implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \bar{v}) = 0 \quad , \qquad (14)$$

if the function η does not depend on φ but is a function of ρ alone.

Varying ρ gives

$$\frac{1}{2} v^2 - \varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho} + \frac{D\varphi}{Dt} - \frac{1}{\rho_0} \frac{D\eta}{Dt}$$

$$- \frac{\rho}{\rho_0} \frac{\partial^2 \eta}{\partial \rho^2} \frac{D\rho}{Dt} + \frac{1}{\rho_0} \frac{\partial \eta}{\partial \rho} \left\{ \frac{\partial \rho}{\partial t} + \nabla_2 (\rho v_2) \right\}$$

$$+ \frac{\rho}{\rho_0} \frac{D}{Dt} \left(\frac{\partial \eta}{\partial \rho} \right) = 0 \quad ,$$

which becomes after employing the continuity equation and simplification

$$\frac{1}{2} v^2 - \varepsilon(\rho) - \rho \frac{\partial \varepsilon(\rho)}{\partial \rho} + \frac{D\varphi}{Dt} = 0 \quad . \quad (15)$$

For irrotational flow, the substantive derivative of the velocity becomes

$$\frac{D v_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{1}{2} \nabla_i v^2 \quad ,$$

or with equation (13 b)

$$\frac{D v_i}{Dt} = - \nabla_i \left\{ \frac{\partial \varphi}{\partial t} - \frac{1}{2} v^2 \right\}$$

$$+ \frac{\mu r}{\rho_0} \frac{\partial}{\partial t} \nabla_i w \quad .$$

Equation (15) is written as

$$\frac{\partial \phi}{\partial t} = \varepsilon(\rho) + \rho \frac{\partial \varepsilon(\rho)}{\partial \rho} - \frac{1}{2} v^2 - v_\alpha \nabla_\alpha \phi ,$$

so that the equations of motion become

$$\frac{D v_i}{D t} = - \nabla_i \left\{ \varepsilon(\rho) + \rho \frac{\partial \varepsilon(\rho)}{\partial \rho} - v^2 - v_\alpha \nabla_\alpha \phi - \frac{\mu n}{\rho_0} \frac{\partial \omega}{\partial t} \right\} ,$$

or

$$\frac{D v_i}{D t} = - \nabla_i \left\{ \varepsilon(\rho) + \rho \frac{\partial \varepsilon(\rho)}{\partial \rho} \right\} + \frac{\mu n}{\rho_0} \nabla_i \left\{ v_\alpha \nabla_\alpha \omega + \frac{\partial \omega}{\partial t} \right\} ,$$

which becomes

$$\frac{D v_i}{D t} = - \frac{1}{\rho} \nabla_i P + \frac{\mu n}{\rho_0} \nabla_i \left\{ v_\alpha \nabla_\alpha \omega + \frac{\partial \omega}{\partial t} \right\} , \qquad (16)$$

when the internal energy is assumed a function only of the density and it is assumed that

$$\frac{\partial \mathcal{E}(\rho)}{\partial \rho} = \frac{P}{\rho^2} \quad ,$$

where P is the fluid pressure.

From the condition used to obtain the continuity equation, if it is assumed that

$$w = \ln \rho \quad , \qquad (17)$$

then equation (16) becomes

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho} \nabla P - \frac{\mu \eta}{\rho_0} \nabla \left(\nabla \cdot \bar{v} \right) . \qquad (18)$$

This equation becomes the Navier-Stokes equation for irrotational flow of a viscous fluid

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho} \nabla P + \frac{4}{3} \frac{\mu}{\rho} \nabla \left(\nabla \cdot \bar{v} \right) , \qquad (19)$$

when the constant is evaluated as

$$\gamma = -\frac{4}{3} \quad ,$$

and the flow velocities are assumed small compared to the velocity

of sound, i.e.,

$$\frac{v^2}{v_s^2} < 1 \quad ,$$

so that

$$\frac{\mu}{\rho_0} \nabla(\nabla \cdot \bar{v}) \sim \frac{\mu}{\rho} \nabla(\nabla \cdot \bar{v}) + O\left(\frac{v^2}{v_s^2}\right) \quad ,$$

where the last term is of order $\frac{v^2}{v_s^2}$ and may be neglected.

Discussion

The simplest way to analyze the effect of the hydrodynamic gauge transformation upon the hydrodynamic field is to construct the various densities associated with the field. These quantities are found from the Lagrangian density function following Wentzel[1], and Landau and Lifshitz[7], as reviewed in Appendix B.

The Lagrangian density

$$\mathcal{L} = \rho \left\{ \frac{1}{2} v^2 - \varepsilon + \frac{D\varphi}{Dt} \right\} - \frac{\rho}{\rho_0} \frac{D\eta}{Dt} \quad ,$$

written in terms of the conjugate variables ρ and ϕ is

$$\mathcal{L} = \rho \left\{ - \frac{(\nabla \phi)^2}{2} - \varepsilon + \frac{\partial \phi}{\partial t} \right\}$$

$$+ \frac{\mu n \rho}{\rho_0} \nabla_\alpha v_\alpha \quad ,$$

where use has been made of equation (13 b), which becomes after simplification

$$\mathcal{L} = \rho \left\{ - \frac{(\nabla \phi)^2}{2} - \varepsilon(\rho) + \frac{\partial \phi}{\partial t} \right\}$$

$$- \frac{\mu n}{\rho_0} \nabla^2 \phi + O(\mu^2) \quad ,$$

where now

$$\eta = \mu n \ln \rho \quad ,$$

and the terms that are quadratic in the viscous coefficient are assumed to be negligible.

Thus, the quantities ρ and ϕ are readily seen to be conjugate variables for

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \pi_\phi = \rho \quad .$$

The momentum density defined as

$$\mathcal{G}_\alpha = -\frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \quad ,$$

becomes

$$\mathcal{G}_\alpha = -\frac{\partial \varphi}{\partial x_\alpha} \rho$$

$$= \rho v_\alpha - \frac{\rho}{\rho_0} \mu r \nabla_\alpha \omega \quad ,$$

using equation (13 b) and it is seen, as stated previously, that there is a momentum density associated with the viscous field.

The companion Hamiltonian density of the field is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} = \rho \dot{\varphi} - \mathcal{L}$$

$$= -\rho \left\{ -\frac{1}{2} v^2 - \varepsilon + \frac{\mu r}{\rho_0 \rho} v_\alpha \nabla_\alpha \rho \right\}$$

$$+ \frac{\mu r}{\rho_0} \left\{ \frac{\partial \rho}{\partial t} + v_\alpha \nabla_\alpha \rho \right\}$$

$$= \rho \left\{ \frac{1}{2} v^2 + \varepsilon \right\} - \frac{\mu r}{\rho_0} \nabla_\alpha (\rho v_\alpha) \quad ,$$

by substituting the expression for the Lagrangian density and eliminating terms involving φ . Hence, the energy density of the fluid has decreased from that of a perfect fluid by the presence of

a term proportional to the divergence of the kinetic momentum.

The energy density current is seen to be

$$\mathcal{S}_\alpha = \dot{\phi}\,\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x_\alpha}\right)} = -\rho\,\dot{\phi}\,\nabla_\alpha \phi$$

$$= -(\mathcal{H} + \mathcal{L})\,\nabla_\alpha \phi$$

$$= \mathcal{H} v_\alpha - \frac{\mu n}{\rho_0 \rho}\,\nabla_\alpha \rho\,(\mathcal{H} + \mathcal{L}) + \mathcal{L} v_\alpha$$

$$= \mathcal{H} v_\alpha + v_\alpha \left\{ P + \frac{\mu n}{\rho_0}\,\rho\,\nabla \cdot \bar{v} \right\}$$

$$- \frac{\mu n}{\rho_0}\,\nabla_\alpha \rho\,(\mathcal{H} + \mathcal{L}) \quad ,$$

when the Lagrangian density is expressed in the form

$$\mathcal{L} = P + \frac{\mu n}{\rho_0}\,\rho\,\nabla_\alpha v_\alpha \quad .$$

Actually, the energy density current

$$\mathcal{S}_\alpha = v_\alpha \rho \left\{ \mathcal{E}(\rho) + \frac{1}{2} v^2 \right\}$$

$$+ v_\alpha \left\{ P - \frac{\mu n}{\rho_0}\, v_\alpha \nabla_\alpha \rho \right\} \quad ,$$

has decreased from that of an ideal fluid because of the decrease in the Hamiltonian density with the pressure modified as

$$P \to P' - \frac{\mu' n'}{\rho_0}\,(\bar{v}' \cdot \nabla \rho') \quad .$$

Finally, the stress tensor or energy momentum tensor

$$\tau_{\alpha i} = -\frac{\partial \varphi}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_\alpha}\right)} + \mathcal{L}\,\delta_{\alpha i} \quad ,$$

becomes

$$\tau_{\alpha i} = \frac{\partial \varphi}{\partial x_i} \rho \frac{\partial \varphi}{\partial x_\alpha} + \mathcal{L}\,\delta_{\alpha i}$$

$$= \rho\, v_i\, v_\alpha - \frac{\mu\nu}{\rho_0}\left\{ v_\alpha \nabla_i \rho + v_i \nabla_\alpha \rho \right\} + \mathcal{L}\,\delta_{\alpha i} \quad ,$$

or

$$\tau_{\alpha i} = \rho\, v_i\, v_\alpha - \frac{\mu\nu}{\rho_0}\left\{ v_\alpha \nabla_i \rho + v_i \nabla_\alpha \rho \right\} + \left\{ p + \frac{\mu\nu\rho}{\rho_0} \nabla \cdot \bar{v} \right\} \delta_{\alpha i} \quad .$$

The above stress tensor employed with the momentum density, $\mathcal{G}_i$, in the equation expressing the conservation of momentum

$$\dot{\mathcal{G}}_i + \nabla_\alpha \tau_{\alpha i} = 0 \quad ,$$

gives the Navier-Stokes equation.

For the above expression becomes

$$\mathcal{G}_{ij} = \frac{\partial}{\partial t}(\rho v_j) - \frac{\partial}{\partial t}\left\{\frac{\mu_r}{\rho_0}\nabla_j \rho\right\}$$

$$= -\nabla_k\left\{\rho v_j v_k - \frac{\mu_r}{\rho_0}\left(v_k \nabla_j \rho + v_j \nabla_k \rho\right)\right\}$$

$$-\nabla_j\left\{p + \frac{\mu_r \rho}{\rho_0}\nabla_k v_k\right\} \quad ,$$

or upon rearranging terms

$$\rho \frac{Dv_j}{Dt} = -\nabla_j p + \frac{\mu_r}{\rho_0}\nabla_k\left\{v_k \nabla_j \rho + v_j \nabla_k \rho\right\}$$

$$-\frac{\mu_r}{\rho_0}\nabla_j\left\{\rho \nabla_k v_k - \frac{\partial \rho}{\partial t}\right\}$$

$$= -\nabla_j p - \frac{\mu_r \rho}{\rho_0}\nabla_j\left(\nabla_k v_k\right)$$

$$-\frac{\mu_r}{\rho_0}\nabla_k \rho \nabla_k v_k \quad .$$

since the last term is of order $\frac{v^2}{v_s^2}$, it may be neglected.

Therefore, as found previously, the equations of motion for the hydrodynamic field are

$$\rho \frac{D\bar{v}}{Dt} = -\nabla P - \frac{\mu\eta\rho}{\rho_0} \nabla(\nabla \cdot \bar{v}) .$$

Thus, the hydrodynamic gauge transformation has yielded equations of motion of the hydrodynamic field which are equivalent to the Navier-Stokes equation for irrotational viscous fluid motion. This agreement of the equations of motion establishes the validity of the hydrodynamic gauge transformation of this study.

V

SUMMARY

In this study, it is proposed that an investigation of a dissipative, classical hydrodynamic system can be undertaken in a manner similar to that employed for dissipative processes in an electrodynamic system. An examination of the gauge transformation used to introduce coupling between source and field in electrodynamic systems suggested a means whereby viscosity may be introduced into the hydrodynamic system under consideration.

For the present work, a proposed hydrodynamic gauge transformation of the second kind is examined to determine the validity of the transformation for the introduction of viscous effects into the equations of motion. Using variational techniques to obtain the dynamic equations of the fluid, the validity of the hydrodynamic gauge transformation proposed in this study is established by the equivalence of the result with the Navier-Stokes equation. This agreement of the equations of motion of the fluid with the Navier-Stokes equation is limited by the consideration of an irrotational fluid flow, and in the mathematical analysis by the assumptions that the fluid flow velocity is small compared to the velocity of sound and that the terms quadratic in the viscous coefficient are negligible.

An additional outcome of the application of the hydrodynamic gauge transformation is the observation that the momentum density of

the fluid field can be considered to be composed of both a kinetic momentum and a potential momentum. Using an electrodynamic interpretation, this potential momentum is associated with the internal friction or viscous forces of the hydrodynamic system.

The consequences of the hydrodynamic gauge transformation are analyzed through the construction of the various densities associated with the hydrodynamic field. Forming the field densities in the usual manner, the equations of motion of the hydrodynamic field are found from the conservation of momentum. The equivalence of the resulting equations of motion of the hydrodynamic field with the Navier-Stokes equation substantiates the hydrodynamic gauge transformation of the second kind as proposed in this study for a classical, dissipative hydrodynamic system.

APPENDIX

Part A

Lagrangian Density Function

For this study, the Lagrangian density function of a real fluid is formed in a direct fashion by manipulating the velocity potential function. Although the fluid system under consideration is a dissipative one, the Lagrangian density of an ideal fluid is formed[3, 14] and the viscous effect of the dissipative fluid is introduced through the proposed hydrodynamic gauge transformation.

The Lagrangian density of an ideal fluid is constructed by considering fluid flow within an enclosed volume with the kinetic energy of the fluid given by

$$T = \int \frac{1}{2} \rho \, v^2 \, dx^3 \quad ,$$

where ρ is the fluid density

$\overline{v}$ is the fluid velocity

and dx^3 is the differential volume element.

Correspondingly, the total internal energy of the fluid is

$$U = \int \rho \, \varepsilon(\rho) \, dx^3 \quad ,$$

where it is assumed that the internal energy per unit mass, $\varepsilon(\rho)$, is a function solely of the density.

Lastly, the term

$$\int \rho \frac{D\phi}{Dt} \, dx^3$$

is incorporated where ϕ is called the velocity potential function. The expression used for the velocity in this study is a particular form of the general Clebsch velocity[15]. In this study, it is through this term that the viscous effects in a real fluid are introduced into the Lagrangian density.

Collecting the above quantities, the action integral for the system is formed

$$I = \int \mathcal{L} \, dx^3 \, dt \quad ,$$

where the Lagrangian density function is

$$\mathcal{L} = \rho \left\{ \frac{1}{2} v^2 - \varepsilon(\rho) + \frac{D\phi}{Dt} \right\} \quad ,$$

and where ρ is the fluid density,

$\vec{v}$ is the fluid velocity,

$\varepsilon(\rho)$ is the internal energy,

and ϕ is the velocity potential function.

Part B

Hydrodynamic Field Densities

The consequences of the proposed hydrodynamic gauge transformation are examined through construction of the various field densities in a manner following Wentzel[1], and Landau and Lifshitz[7]. When the Lagrangian density is expressed as

$$\mathcal{L} = \mathcal{L}(\rho, \varphi) \quad ,$$

the canonically conjugate variables are determined from the relations

$$\frac{\partial \mathcal{L}}{\partial \dot{\Lambda}_k} = \pi_k \quad .$$

In the usual nomenclature of field theory[1], the quantity π_k is denoted as the momentum canonically conjugate to the field variables.

In order to have conservation of momentum[1, 7], the momentum density of the field is taken to be

$$\mathcal{G}_\nu = - \frac{\partial \Lambda}{\partial x_\nu} \pi_k \quad ,$$

where Λ is the field variable.

Application of the conservation of energy to the total energy

of a classical, continuum field[1, 7] leads to the continuity equation

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \mathcal{S} = 0 \quad ,$$

where $\mathcal{H}$ is the Hamiltonian density or the energy density of the field defined by

$$\mathcal{H} = \pi_\alpha \dot{q}_\alpha - \mathcal{L} \quad ,$$

and $\mathcal{S}$ is the energy density current

$$\mathcal{S}_\alpha = \dot{\Lambda}_k \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Lambda_k}{\partial x_\alpha}\right)} \quad .$$

A stress tensor which is in agreement with the definitions of the energy density and the energy density current of the field is given as

$$\tau_{\alpha j} = - \frac{\partial \Lambda_k}{\partial x_j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Lambda_k}{\partial x_\alpha}\right)} + \mathcal{L}\, \delta_{\alpha j} \quad ,$$

and is called the canonical energy momentum tensor[1].

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