On the geometry of affine tangent bundles with multiple and single vector fields as fibres
by Samuel Ralph Thompson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
ABSTRACT Recently (2) T. Okubo has shown that the frame (or principal, (1)p82) bundle F(Xn) of a differentiable manifold Xn of class Cr(r≥4) admits a tensor field Φ of type (1,1), rank nR, satisfying Φ3=-E , In this thesis I generalize the notion of a frame bundle and consider a fibre bundle Tm(Xn) having m (1< m< n) linearly independent tangent vectors as fibre. Following the method of T.Okubo it is shown that the bundle Tm (Xn) also admits such a tensor fieldΦ A symmetric affine connection is then assigned to Tm(Xn) and the differential geometry of Tm(Xn) is developed on the basis of this tensor structure and of the extended lift of a vector field in Xn. .

The latter portion of this thesis considers the case m=1; the so-called tangent bundle T(Xn) = T (Xn) , It is now well-known that the tangent bundle T(Xn), with Xn a Riemannian manifold, admits a tensor field F of type (1,1), rank 2n, satisfying F2= -E. It is shown here that T(Xn) also admits such a tensor field F when the base manifold is only an affinely connected space, A symmetric affine connection is introduced in T(Xn) on the same principle as before and the geometry of T(Xn) similarly discussed. In particular it is shown that the natural lift C (7),(3) of a path C in Xn is again a path in T(Xn).

Finally, an interesting aspect of this thesis from a technical point of view is the use made of the "adapted frames" as originated by K, Yarn and T. Okubo (6) as our reference frames instead of the usual "natural frames",, This choice of frames results in a number of simplifications in the calculations.

ON THE GEOMETRY OF AFFINE TANGENT BUNDLES WITH MULTIPLE AND SINGLE VECTOR FIELDS AS FIBRES

by

SAMUEL RALPH THOMPSON III

A thesis submitted to the Graduate Faculty in partial
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VITA

The author, Samuel Ralph Thompson III, was born on March 24, 1937 in St. Paul, Minnesota. His parents, Lt. Col. (retired) and Mrs. S. R. Thompson Jr., presently reside in San Antonio, Texas. The author attended secondary school in San Antonio where he was graduated from Thomas Jefferson High School in June, 1954 and subsequently attended St. Mary's University for one year. After which he attended the University of Dayton and received his Bachelor of Science degree in 1960 from this institution. In 1958 Mr. Thompson joined the Pi Mu Epsilon Mathematics Honor Fraternity and later in the year received from the Dayton Mathematics Club the Award of Excellence for Mathematics in the Junior Class.

Mr. Thompson received his Master of Arts degree from the University of Arizona in June, 1962 and was an NSF Cooperative Graduate Fellow there during the academic year 1962-1963. He was also an NSF fellow at the Summer Institute in Analysis held at Salt Lake City during the summer of 1961.

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ABSTRACT

Recently (2) T. Okubo has shown that the frame (or principal, (1)p82) bundle $F(X_n)$ of a differentiable manifold X_n of class C^r $(r \geq 4)$ admits a tensor field Φ of type (1,1), rank n^2, satisfying $\Phi^3 = -E$. In this thesis I generalize the notion of a frame bundle and consider a fibre bundle $T_m(X_n)$ having m $(1 < m < n)$ linearly independent tangent vectors as fibre. Following the method of T. Okubo it is shown that the bundle $T_m(X_n)$ also admits such a tensor field Φ. A symmetric affine connection is then assigned to $T_m(X_n)$ and the differential geometry of $T_m(X_n)$ is developed on the basis of this tensor structure and of the extended lift of a vector field in X_n.

The latter portion of this thesis considers the case $m = 1$; the so-called tangent bundle $T(X_n) \equiv T_1(X_n)$. It is now well-known that the tangent bundle $T(X_n)$, with X_n a Riemannian manifold, admits a tensor field F of type (1,1), rank 2n, satisfying $F^2 = -E$. It is shown here that $T(X_n)$ also admits such a tensor field F when the base manifold is only an affinely connected space. A symmetric affine connection is introduced in $T(X_n)$ on the same principle as before and the geometry of $T(X_n)$ similarly discussed. In particular it is shown that the natural lift $\bar{C}$ (7),(3) of a path C in X_n is again a path in $T(X_n)$.

Finally, an interesting aspect of this thesis from a technical point of view is the use made of the "adapted frames" as originated by K. Yano and T. Okubo (6) as our reference frames instead of the usual "natural frames". This choice of frames results in a number of simplifications in the calculations.

INTRODUCTION

Until recently (2) it had been an open problem as to the tensor structures admitted by the frame bundle $F(X_n)$ of a differentiable manifold X_n, that is, a kind of tangent bundle whose fibre is composed of n linearly independent vectors. This thesis has generalized the notion of a frame bundle and considers a fibre bundle $T_m(X_n)$ having m $(1 < m < n)$ linearly independent vectors as fibre and following the method of T. Okubo it is shown that $T_m(X_n)$ admits a tensor structure H of type (1,1), rank n^2, satisfying $H^3 - H = 0$, or equivalently a structure Φ satisfying $\Phi^3 = -E$. Moreover, H decomposes the unit tensor E of rank $n+mn$ into two complementary tensor fields P and Q of type (1,1) and of ranks n and mn, respectively; and H acts on P as an annihilator and on Q as an almost product structure.

In assignment of affine connections to $T_m(X_n)$ we have a pattern due to K. Yano and E. T. Davies (7) when the base space X_n is Riemannian. However, since the tensor structures considered in this investigation are coherent to the affine structure of the base manifold X_n; that is, independent of the metric structure, the geometry of $T_m(X_n)$ should be most profitably discussed in a framework free from metric considerations. The difficulty in the theory of connections of $T_m(X_n)$ lies partially in the fact that X_n is always reduced to a trivial affine space when the connection is given so that $T_m(X_n)$ is integrable, and mostly in the fact that a basic symmetric connection is required in our case to satisfy two criteria: that the parallel displacement of the horizontal lift (2), (7) of a vector field in X_n coincide with the parallelism of Levi-Civita, and that the natural lift of a path in X_n should be a path in $T_1(X_n)$.

This thesis gives a symmetric affine connection to the bundle $T_m(X_n)$ fulfilling the above requirements and discusses the geometry of $T_m(X_n)$ on the basis of the tensor structure H and of the extended lift of a vector field in X_n. The notion of the extended lift in $T_m(X_n)$ of a vector field in X_n being a generalization of what S. Sasaki (3) has called the extension of a vector field in the case of a bundle with a single tangent vector as fibre, that is, the case $m = 1$; and what K. Yano and E. T. Davies (7) have called the complete lift, again for the case $m = 1$.

As for a tangent bundle whose fibre is a single tangent vector it is now well-known that when the base manifold is Riemannian the bundle admits a tensor field F of type (1,1). rank 2n, satisfying $F^2 = -E$; (4) and F transforms the vertical distribution of $T_1(X_n)$ onto the horizontal distribution at each point of $\pi^{-1}(U)$, $U \subset X_n$. It is shown here that any tangent bundle $T_1(X_n)$ admits such a tensor field F when the base manifold is only an affinely connected space. A symmetric affine connection is introduced in $T_1(X_n)$ on the same principle as before and the geometry of $T_1(X_n)$ similarly discussed. In particular, it is shown that the natural lift $\bar{C}$ of a path C in X_n is again a path in $T_1(X_n)$ which generalizes results obtained by S. Sasaki (3) and by K. Yano and E. T. Davies (7).

Finally, it should be remarked that a systematic use of the "adapted frames" as originated by K. Yano and T. Okubo (6) is made throughout this thesis, and it is the author's opinion that the work required to do the same calculations with respect to the so-called "natural frames" would be prohibitive.

CHAPTER I

THE BUNDLE $T_m(X_n)$ WITH MULTIPLE TANGENT VECTORS $X^h_{(\alpha)}$ AS FIBRE

1. The existence of a tensor field H of type (1,1) in $T_m(X_n)$ satisfying $H^3 - H = 0$.

We adopt the following conventions for indices:

$$\begin{aligned} &a, b, c, d, h, i, j, k = 1, 2, \ldots, n \\ &\alpha, \beta, \gamma, \delta, \epsilon = 1, 2, \ldots, m \\ &\left.\begin{matrix} A, B, C, D, E \\ \lambda, \mu, \nu, \rho, \tau, \omega \end{matrix}\right\} = 1, 2, \ldots, n, n+1, \ldots, n+mn. \end{aligned}$$

and let X_n be an n-dimensional differentiable manifold of class C^r with $r \geq 4$. If we let $X^h_{(\alpha)}$ be the components of m $(1 < m < n)$ linearly independent vectors of X_n with respect to a local coordinate neighborhood $U(u^h)$, then we can consider the $(n + mn)$-dimensional differentiable manifold $T_m(X_n)$ with local coordinates $(u^h, X^h_{(\alpha)}) \equiv (u^h, X^h_{(1)}, \ldots, X^h_{(m)})$ in the neighborhood $U \times E^n \times \cdots \times E^n \equiv U \times E^{mn}$. Let $U(u^h)$ and $U'(u^{h'})$ be two overlapping coordinate neighborhoods of X_n. Then $U \times E^{mn}$ and $U' \times E^{mn}$ are overlapping coordinate neighborhoods in $T_m(X_n)$ and corresponding to the transformation $u^{h'} = u^{h'}(u^h)$ in $U \cap U'$ we have the transformation

$$(1.1)\quad \begin{cases} u^{h'} = u^{h'}(u^h) \\ X^{h'}_{(\alpha)} = \partial_h u^{h'} X^h_{(\alpha)} \quad \text{where } \partial_h \overset{\text{def}}{=} \dfrac{\partial}{\partial u^h} \end{cases}$$

In the future we shall write $h_\alpha = n\alpha + h$ so that we can describe the $n + mn$ coordinates in $T_m(X_n)$ as ξ^A where $\xi^h = u^h$ and $\xi^{h_\alpha} = X^h_{(\alpha)}$. (1.1) is now written

$$(1.2)\quad \left\{ \begin{array}{l} \xi^{h'} = \xi^{h'}(\xi^{h}) \\[1em] \xi^{h'_\alpha} = \partial_h \xi^{h'} \xi^{h_\alpha} \end{array} \right.$$

and for convenience we write (1.2) as $\xi^{A'} = \xi^{A'}(\xi^{A})$. Now

$$(1.3)\quad \left[\partial_A \xi^{A'}\right] \equiv \begin{pmatrix} \partial_h \xi^{h'} & \partial_{h_\alpha} \xi^{h'} \\ \partial_h \xi^{h'_\beta} & \partial_{h_\alpha} \xi^{h'_\beta} \end{pmatrix}$$

where $\partial_A = \frac{\partial}{\partial \xi^A}$; and in particular $\partial_{h_\alpha} = \frac{\partial}{\partial \xi^{h_\alpha}} = \frac{\partial}{\partial X^h_{(\alpha)}}$. Nows

$$\partial_h \xi^{h'} = \partial_h U^{h'} , \quad \partial_{h_\alpha} \xi^{h'} = 0$$

$$\partial_h \xi^{h'_\beta} = \partial_h (\partial_j U^{h'} X^j_{(\beta)}) = X^j_{(\beta)} \partial_h \partial_j U^{h'}$$

$$\partial_{h_\alpha} \xi^{h'_\beta} = \partial_{h_\alpha} (\partial_j U^{h'} X^j_{(\beta)}) = \partial_j U^{h'} \partial_{h_\alpha} X^j_{(\beta)} = \delta^\beta_\alpha \delta^j_h \partial_j U^{h} = \delta^\beta_\alpha \partial_h U^{h'}$$

and consequently (1.3.) becomes

$$(1.4)\quad \left[\partial_A \xi^{A'}\right] = \begin{array}{ccccccc}
 & \multicolumn{4}{c}{\overbrace{\hphantom{\partial_h U^{h'} \qquad \qquad \qquad \partial_h U^{h'}}}^{(m)}} & & \\
(n) & (n) & & & (n) & & \\
\partial_h U^{h'} & 0 & \cdots & \cdots & 0 & (n) & \\
X^j_{(1)} \partial_h \partial_j U^{h'} & \partial_h U^{h'} & & & & (n) & \\
X^j_{(2)} \partial_h \partial_j U^{h'} & & \ddots & & 0 & \vdots & \\
\vdots & 0 & & \ddots & & \vdots & \left.\right\}(m) \\
X^j_{(m)} \partial_h \partial_j U^{h'} & & & & \partial_h U^{h'} & (n) &
\end{array}$$

implying that

(1.5) $$\det\left[\partial_\Lambda \xi^{\Lambda'}\right] = \left(\det\left[\partial_h u^{h'}\right]\right)^{m+1}$$

From which we have the following (5,p. 214)

<u>Theorem 1.1</u>. If m is even, then the bundle $T_m(X_n)$ has the same orientation as X_n ; and if m is odd, $T_m(X_n)$ is always orientable.

T. Okubo (2) has told me that the frame bundle $F(X_n)$ of an n-dimensional differentiable manifold of class C^r, $r \geq 4$, admits a tensor field of type (1,1) whose cube is -E, the identity. Here we intend to show that $T_m(X_n)$ also admits such a tensor field. Following T. Okubo define a field of quantities $\mathbb{H}^A{}_B$ in $T_m(X_n)$ at a point $\xi^A(u^h, \mathbb{X}^h_{(\alpha)})$ by

(1.6) $$\mathbb{H}^A{}_B \equiv \begin{pmatrix} \mathbb{H}^h_j & \mathbb{H}^h_{j\alpha} \\ \mathbb{H}^{h_\beta}_j & \mathbb{H}^{h_\beta}_{j\alpha} \end{pmatrix} = \begin{array}{c|cccc|l} & \multicolumn{4}{c|}{\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}^{(m)}} & \\ (n) & (n) & \cdots & (n) & (n) & \\ \hline 0 & 0 & \cdots & 0 & 0 & (n) \\ \Gamma^h_{ji}\mathbb{X}^i_{(m)} & & & & \delta^h_j & (n) \\ \Gamma^h_{ji}\mathbb{X}^i_{(m-1)} & & 0 & \delta^h_j & & (n) \\ \vdots & & \iddots & & & \vdots \\ \Gamma^h_{ji}\mathbb{X}^i_{(1)} & \delta^h_j & & & 0 & (n) \end{array} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}(m)$$

where we now assume that X_n is an affine space with a symmetric linear connection whose components with respect to the local coordinates (u^h) are $\Gamma^h_{ji} = \Gamma^h_{ij}$. If $U(u^h)$ and $U'(u^{h'})$ are two overlapping coordinate neighborhoods, then Γ transforms according to the law (cf 1,p.79)

(1.7) $$\Gamma^{h'}_{j'i'} = \partial_h u^{h'}\left\{\partial_{j'} u^j \partial_{i'} u^i \Gamma^h_{ji} + \partial_{j'}\partial_{i'} u^h\right\}.$$

We denote by $\nabla_j v^h$ the covariant derivative of a vector field v:

$$\nabla_j v^h = \partial_j v^h + \Gamma^h_{ji} v^i$$

Theorem 1.2.

(1) $\mathbb{H}^A_B \mathbb{H}^B_C \mathbb{H}^C_D = \mathbb{H}^A_D$

(2) $\mathbb{H}^A_B$ are the components of a tensor field of type (1,1) in $T_m(X_n)$ with respect to the local coordinates (ξ^A).

Proof: For brevity we write Γ_α for the n × n matrix $\left[\Gamma^h_{ji} X^i_{(\alpha)}\right]$ and E for the n × n matrix $\left[\delta^h_j\right]$. Then

$$\mathbb{H}^A_B = \begin{array}{ccccc|l} & & (m) & & & \\ (n) & (n) & & (n) & (n) & \\ \hline 0 & 0 & \cdots & 0 & 0 & (n) \\ \Gamma_m & & & & E & (n) \\ \Gamma_{m-1} & & 0 & E & & (n) \\ \vdots & & \iddots & & 0 & \vdots \ \}(m) \\ \Gamma_1 & E & & & & (n) \end{array}$$

$$\mathbb{H}^A_B \mathbb{H}^B_C = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \iddots & & 0 \\ \Gamma_1 & E & & & \end{pmatrix}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \iddots & & 0 \\ \Gamma_1 & E & & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_1 & E & & & \\ \Gamma_2 & & E & 0 & \\ \vdots & & & \ddots & \\ \Gamma_m & 0 & & & E \end{pmatrix}$$

$$\underline{H}^A_B \underline{H}^B_C \underline{H}^C_D =$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_1 & E & & 0 & \\ \Gamma_2 & & E & & \\ \vdots & 0 & & \ddots & \\ \Gamma_m & & & & E \end{bmatrix}\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & & & E \\ \Gamma_{m-1} & 0 & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ \Gamma_1 & E & & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ \Gamma_1 & E & & & \end{bmatrix}.$$

Thus $\underline{H}^A_B \underline{H}^B_C \underline{H}^C_D = \underline{H}^A_D$ which proves (1). To prove (2) we must show:

$$\overline{\underline{H}}^{A'}_{B'} = \partial_A \xi^{A'} \partial_{B'} \xi^B \underline{H}^A_B$$

Now from (1.4) we see that

$$(1.8) \quad \left(\partial_{B'} \xi^B\right) = \begin{array}{c} \begin{array}{cc} (n) & \overbrace{(n) \quad (n) \quad \cdots \quad (n)}^{(m)} \end{array} \\ \begin{pmatrix} \partial_{h'} U^h & 0 & 0 & \cdots & 0 \\ X^{j'}_{(1)} \partial_{h'} \partial_{j'} U^h & \partial_{h'} U^h & & & 0 \\ X^{j'}_{(2)} \partial_{h'} \partial_{j'} U^h & & \partial_{h'} U^h & & \\ \vdots & & 0 & \ddots & \\ X^{j'}_{(m)} \partial_{h'} \partial_{j'} U^h & & & & \partial_{h'} U^h \end{pmatrix} \end{array}$$

For brevity we write T_α, Q_α in place of the n X n matrices $\left[X^j_{(\alpha)} \partial_h \partial_j U^{h'}\right]$ $\left[X^{j'}_{(\alpha)} \partial_{h'} \partial_{j'} U^h\right]$, respectively; and A, A' in place of the n X n matrices $\left(\partial_h U^{h'}\right)$ and $\left(\partial_{h'} U^h\right)$. Note that $AA' = A'A = E$. From (1.4) and (1.8) we now have

$$\left(\partial_A \xi^{A'}\right) = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ T_1 & A & & & \\ T_2 & & A & 0 & \\ \vdots & & 0 & \ddots & \\ T_m & & & & A \end{pmatrix}, \quad \left(\partial_{B'} \xi^{B}\right) = \begin{pmatrix} A' & 0 & 0 & \cdots & 0 \\ Q_1 & A' & & 0 & \\ Q_2 & & A' & & \\ \vdots & & 0 & \ddots & \\ Q_m & & & & A' \end{pmatrix}$$

$$\partial_A \xi^{A'} \Xi^{A}_{B} = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ T_1 & A & & 0 & \\ T_2 & & A & & \\ \vdots & 0 & & \ddots & \\ T_m & & & & A \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & 0 \\ \Gamma_1 & E & & & \end{pmatrix}$$

$$\doteq \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ A\Gamma_m & & 0 & & A \\ A\Gamma_{m-1} & & & A & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & 0 \\ A\Gamma_1 & A & & & \end{pmatrix}$$

$$\partial_A \xi^{A'} \Xi^{A}_{B} \partial_{B'} \xi^{B} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ A\Gamma_m & & 0 & & A \\ A\Gamma_{m-1} & & & A & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & 0 \\ A\Gamma_1 & A & & & \end{pmatrix} \begin{pmatrix} A' & 0 & 0 & \cdots & 0 \\ Q_1 & A' & & & 0 \\ Q_2 & & A' & & \\ \vdots & & 0 & \ddots & \\ Q_m & & & & A' \end{pmatrix}$$

$$\partial_A \xi^{A'} \Pi^A_B \partial_{B'} \xi^B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ A\Gamma_m A' + AQ_m & & 0 & & E \\ A\Gamma_{m-1} A' + AQ_{m-1} & & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ A\Gamma_1 A' + AQ_1 & E & & & 0 \end{pmatrix}$$

Now

$$\Pi^{A'}_{B'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma'_m & & 0 & & E \\ \Gamma'_{m-1} & & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ \Gamma'_1 & E & & & 0 \end{pmatrix}$$

where Γ'_α is the $n \times n$ matrix $\left(\Gamma^{h'}_{j'i'} \overline{X}^{i'}_{(\alpha)}\right)$. So we must show

$$\Gamma'_\alpha = A\Gamma_\alpha A' + AQ_\alpha$$

Now $A\Gamma_\alpha A' + AQ_\alpha$ is the $n \times n$ matrix

$$\left(A^{h'}_a \Gamma^a_{ib} \overline{X}^i_{(\alpha)} A^b_{j'} + A^{h'}_a A^a_{j'b'} \overline{X}^{b'}_{(\alpha)}\right)$$

where we have written $A^{h'}_a \equiv \partial_a u^{h'}$, $A^b_{j'} \equiv \partial_{j'} u^b$, and $A^a_{j'b'} \equiv \partial_{j'} \partial_{b'} u^a$. So it remains to show that

$$\Gamma^{h'}_{j'i'} \overline{X}^{i'}_{(\alpha)} = A^{h'}_a \Gamma^a_{ib} \overline{X}^i_{(\alpha)} A^b_{j'} + A^{h'}_a A^a_{j'b'} \overline{X}^{b'}_{(\alpha)} \tag{1.9}$$

Now from (1.7) we get

$$\Gamma^{h'}_{j'i'} = A^{h'}_{a} A^{b}_{j'} A^{c}_{i'} \Gamma^{a}_{bc} + A^{h'}_{a} A^{a}_{j'i'}$$

Also $X^{i'}_{(\alpha)} = A^{i'}_{d} X^{d}_{(\alpha)}$. So

$$\begin{aligned} \Gamma^{h'}_{j'i'} X^{i'}_{(\alpha)} &= A^{h'}_{a} A^{b}_{j'} A^{c}_{i'} A^{i'}_{d} \Gamma^{a}_{bc} X^{d}_{(\alpha)} + A^{h'}_{a} X^{i'}_{(\alpha)} A^{a}_{j'i'} \\ &= A^{h'}_{a} A^{b}_{j'} \Gamma^{a}_{bc} X^{c}_{(\alpha)} + A^{h'}_{a} X^{i'}_{(\alpha)} A^{a}_{j'i'} \\ &= A^{h'}_{a} \Gamma^{a}_{bi} X^{i}_{(\alpha)} A^{b}_{j'} + A^{h'}_{a} A^{a}_{j'b'} X^{b'}_{(\alpha)} \end{aligned}$$

which gives (1.9) since $\Gamma^{a}_{bi} = \Gamma^{a}_{ib}$. Thus $\mathbb{H}^{A'}_{B'} = \partial_A \xi^{A'} \partial_{B'} \xi^{B} \mathbb{H}^{A}_{B}$ and the proof is complete.

If we let $\Phi = -\frac{3}{2}\mathbb{H}^2 + \frac{\sqrt{3}}{2}\mathbb{H} - E$, then clearly $\Phi^3 = -E$; and thus the bundle $T_m(X_n)$ admits a tensor field of type (1,1) whose cube is -E.

If we define the tensor fields P and Q of type (1,1) by:

(a) $P = -\mathbb{H}\mathbb{H} + E$

(b) $Q = \mathbb{H}\mathbb{H}$

then we see that P and Q are of rank n and nm respectively and satisfy

(c) $P + Q = E$

(d) $PP = P,\ QQ = Q,\ PQ = QP = 0$

showing that P and Q are complementary. If we multiply (a) and (b) by H, we have

(e) $\mathbb{H}^2 P = -\mathbb{H}\mathbb{H}^3 + \mathbb{H}^2 E = -\mathbb{H}^2 + \mathbb{H}^2 = 0$

(f) $\mathbb{H}^2 Q = \mathbb{H}^2 \mathbb{H}^2 = \mathbb{H}\mathbb{H}^3 = \mathbb{H}\mathbb{H} = Q$

respectively.

Thus we have

Theorem 1.3. The Tensor field H decomposes the unit tensor field E of rank n+nm into the two complementary tensor fields P and Q of type (1,1) whose ranks are respectively n and nm. H acts on P as an annihilator and on Q as an almost product structure.

Later on we shall show that P and Q are the tensor fields coherent to the horizontal and vertical distributions of the multiple tangent bundle Tm(X_n). (6,p205) We also note that P and Q have the components

$$
\begin{array}{cl}
\begin{array}{cc} (n) & \overbrace{(n)\qquad (n)\quad\cdots\quad (n)}^{(m)} \end{array} & \\
\left(\begin{array}{ccccc}
\delta_j^h & 0 & 0 & \cdots & 0 \\
-\Gamma_{ji}^h X_{(1)}^i & & & & \\
-\Gamma_{ji}^h X_{(2)}^i & & & 0 & \\
\vdots & & & & \\
-\Gamma_{ji}^h X_{(m)}^i & & & &
\end{array}\right) &
\begin{array}{l} (n) \\ \left.\begin{array}{l} (n) \\ (n) \\ \vdots \\ (n) \end{array}\right\}(m) \end{array}
\end{array}
\qquad
\begin{array}{cl}
\begin{array}{cc} (n) & \overbrace{(n)\qquad (n)\quad\cdots\quad (n)}^{(m)} \end{array} & \\
\left(\begin{array}{ccccc}
0 & 0 & 0 & \cdots & 0 \\
\Gamma_{ji}^h X_{(1)}^i & \delta_j^h & & & \\
\Gamma_{ji}^h X_{(2)}^i & & \delta_j^h & 0 & \\
\vdots & & 0 & \ddots & \\
\Gamma_{ji}^h X_{(m)}^i & & & & \delta_j^h
\end{array}\right) &
\begin{array}{l} (n) \\ \left.\begin{array}{l} (n) \\ (n) \\ \vdots \\ (n) \end{array}\right\}(m) \end{array}
\end{array}
$$

2. The geometry of Tm(X_n). Adapted frames. (2), (6), (7).

For future reference we now make the following definition::

Definition 2.1. Let v be a vector field in X_n whose components with respect to the coordinate neighborhood (u^h) are v^h. Then we let

$$\bar{v}^A = (\bar{v}^h, \bar{v}^{h_\beta}) \equiv (v^h, X_{(\beta)}^a \partial_a v^h)$$

It is easily verified that $\bar{v}^A$ are the components of a contravariant vector field in $T_m(X_n)$ with respect to the coordinate neighborhood (u^h, $X_{(\alpha)}^h$). We shall call this vector field $\bar{v}^A$ the extended lift of v^h. In the case of a single tangent vector as fibre S. Sasaki (3) has called $\bar{v}^A$ an extension of v^h.

The fibres at a point (u_0^h) in X_n form a subspace of Tm(X_n) whose tangent space is spanned by the mn linearly independent vectors in $T_m(X_n)$ whose components with respect to (ξ^A) are

$$C_{j\alpha}^{\cdot\,A} = \left(C_{j\alpha}^{\cdot\,h}, C_{j\alpha}^{\cdot\,h_\beta}\right) = \left(0, \delta_j^h \delta_\alpha^\beta\right) \tag{2.1}$$

Moreover, if at each point P in $T_m(X_n)$ we consider the subspace of the tangent space to $T_m(X_n)$ at P consisting of those displacements $d\xi^A$ satisfying $d\xi^{h_\alpha} + \Gamma_{ji}^h X_{(\alpha)}^i d\xi^j = 0$, then it is clear that this plane is spanned by the n linearly independent vectors of $T_m(X_n)$ whose components with respect to $(u^h, X_{(\alpha)}^h)$ are

$$B_j^{\cdot\,A} = \left(B_j^{\cdot\,h}, B_j^{\cdot\,h_\beta}\right) = \left(\delta_j^h, -\Gamma_{ji}^h X_{(\beta)}^i\right) \tag{2.2}$$

We call these two distributions spanned by $B_j^{\ A}$ and $C_{j\alpha}^{\cdot\,A}$ respectively, the <u>horizontal</u> and <u>vertical distributions</u>. (2), (6), (7). Letting $A_\lambda^{\ A} \equiv \left(B_j^{\ A}, C_{j\alpha}^{\ A}\right)$, we see that $A_\lambda^{\ A}$ span the tangent space of $T_m(X_n)$. K. Yano and T. Okubo (6) have nominated $A_\lambda^{\ A}$ the <u>adapted frames</u> of $T_m(X_n)$ to distinguish them from the <u>natural frames</u> $\frac{\partial}{\partial \xi^A}$ of $T_m(X_n)$ corresponding to (ξ^A).

Consider the matrix $\left(A_\lambda^{\ A}\right) = \left(B_j^{\cdot A}\ C_{j\alpha}^{\cdot A}\right)$ of rank n + mn and let $\left(A^\lambda_{\ A}\right) = \left(B^j_{\cdot A}\ C^{j\alpha}_{\cdot A}\right)$ denote the inverse matrix. Then

$$\begin{cases} A^\lambda_{\ A} A_\mu^{\ B} = \delta_\mu^\lambda \\ A_\lambda^{\ A} A^\lambda_{\ B} = \delta_B^A \end{cases} \tag{2.3}$$

from which one has the following five relations:

$$(2.4)\quad \begin{cases} B^{h}_{\cdot A} B_{j}^{\cdot A} = \delta^{h}_{j} , \quad B^{h}_{\cdot A} C_{j\alpha}^{\cdot A} = \delta^{h}_{j\alpha} = 0 \\ C^{h_\beta}_{\cdot A} B_{j}^{\cdot A} = \delta^{h_\beta}_{j} = 0 , \quad C^{h_\beta}_{\cdot A} C_{j\alpha}^{\cdot A} = \delta^{h}_{j} \delta^{\beta}_{\alpha} \\ B_{j}^{\cdot A} B^{j}_{\cdot B} + C_{j\alpha}^{\cdot A} C^{\cdot j\alpha}_{\cdot B} = \delta^{A}_{B} \end{cases}$$

Using these relations the quantities $B^{h}_{\cdot j}$, $B^{h}_{\cdot j\alpha}$, $C^{h_\beta}_{\cdot j}$, $C^{h_\beta}_{\cdot j\alpha}$, can be obtained. Thus

$$B^{h}_{\cdot k} B_{j}^{\cdot k} + B^{h}_{\cdot k_\alpha} B_{j}^{\cdot k_\alpha} = \delta^{h}_{j}$$

$$\delta^{k}_{j} B^{h}_{\cdot k} - \Gamma^{k}_{ji} \overline{X}^{i}_{(\alpha)} B^{h}_{\cdot k_\alpha} = \delta^{h}_{j}$$

$$(2.5)\quad \underline{B^{h}_{\cdot j} - \Gamma^{k}_{ji} \overline{X}^{i}_{(\alpha)} B^{h}_{\cdot k_\alpha} = \delta^{h}_{j}}$$

$$B^{h}_{\cdot k} C_{j\alpha}^{\cdot k} + B^{h}_{\cdot k_\beta} C_{j\alpha}^{\cdot k_\beta} = 0$$

$$B^{h}_{\cdot k} \cdot 0 + B^{h}_{\cdot k_\beta} \delta^{\beta}_{\alpha} \delta^{k}_{j} = 0$$

$$(2.6)\quad \underline{B^{h}_{\cdot j\alpha} = 0}$$

$$C^{h_\beta}_{\cdot k} B_{j}^{\cdot k} + C^{h_\beta}_{\cdot k_\alpha} B_{j}^{\cdot k_\alpha} = 0$$

$$C^{h_\beta}_{\cdot k} \delta^{k}_{j} - C^{h_\beta}_{\cdot k_\alpha} \Gamma^{k}_{ji} \overline{X}^{i}_{(\alpha)} = 0$$

$$(2.7)\quad \underline{C^{h_\beta}_{\cdot j} - \Gamma^{k}_{ji} \overline{X}^{i}_{(\alpha)} C^{h_\beta}_{\cdot k_\alpha} = 0}$$

$$C^{h_\beta}_{\cdot k} C^{\cdot k}_{j\alpha} + C^{h_\beta}_{\cdot k_\gamma} C^{\cdot k_\gamma}_{j\alpha} = \delta^h_j \delta^\beta_\alpha$$

$$C^{h_\beta}_{\cdot k} \cdot 0 + C^{h_\beta}_{\cdot k_\gamma} \delta^k_j \delta^\gamma_\alpha = \delta^h_j \delta^\beta_\alpha$$

(2.8) $$\underline{C^{h_\beta}_{\cdot j\alpha} = \delta^h_j \delta^\beta_\alpha}$$

From (2.5) and (2.6) we have

$$B^h_{\cdot j} - \Gamma^k_{ji} \overline{X}^i_{(\alpha)} \cdot 0 = \delta^h_j$$

(2.9) $$\underline{B^h_{\cdot j} = \delta^h_j}$$

From (2.7) and (2.8) we have

$$C^{h_\beta}_{\cdot j} - \Gamma^k_{ji} \overline{X}^i_{(\alpha)} \delta^h_k \delta^\beta_\alpha = 0$$ (α is summed)

(2.10) $$\underline{C^{h_\beta}_{\cdot j} = \Gamma^h_{ji} \overline{X}^i_{(\beta)}}$$

From (2.6), (2.8), (2.9), and (2.10) we have

(2.11) $$\begin{cases} B^h_{\cdot A} \equiv (B^h_{\cdot j}, B^h_{\cdot j\alpha}) = (\delta^h_j, 0) \\ C^{h_\beta}_{\cdot A} \equiv (C^{h_\beta}_{\cdot j}, C^{h_\beta}_{\cdot j\alpha}) = (\Gamma^h_{ji} \overline{X}^i_{(\beta)}, \delta^h_j \delta^\beta_\alpha) \\ A^\lambda_{\cdot A} \equiv (A^h_{\cdot A}, A^{h_\beta}_{\cdot A}) = (B^h_{\cdot A}, C^{h_\beta}_{\cdot A}) \end{cases}$$

Now consider the differentials $d\xi^A$ dual to the natural frames $\frac{\partial}{\partial \xi^A}$, that is, $\left\langle \frac{\partial}{\partial \xi^A}, d\xi^B \right\rangle = \delta^B_A$ and let $\overline{X}_\lambda \overset{\text{def}}{=} A_\lambda{}^A \partial_A$ where $\partial_A = \frac{\partial}{\partial \xi^A}$;

Then the differentials $(d\xi)^\lambda \overset{\text{def}}{=} A^{\lambda}_{\cdot A}\, d\xi^A$ are dual to the adapted frames $\mathbb{X}_\lambda$. Indeed

$$\begin{aligned}\langle \mathbb{X}_\mu, (d\xi)^\lambda\rangle &= \langle A_\mu^{\;A}\partial_A, A^\lambda_{\;B}\, d\xi^B\rangle \\ &= A_\mu^{\;A}A^\lambda_{\;B}\langle \partial_A, d\xi^B\rangle = A_\mu^{\;A}A^\lambda_{\;B}\,\delta^B_A \\ &= A_\mu^{\;A}A^\lambda_{\;A} = \delta^\lambda_\mu\end{aligned}$$

In general the differentials $(d\xi)^\lambda$ are not exact, the round brackets indicating this fact. We let

(2.12) $$\Omega_{\mu\lambda}^{\;\cdot\cdot\,\nu} \overset{\text{def}}{=} A^\nu_{\;A}(\mathbb{X}_\mu A_\lambda^{\;A} - \mathbb{X}_\lambda A_\mu^{\;A})$$

Then the distributions are holonomic (involutive) if and only if $\Omega_{\mu\lambda}$ vanishes. (7) Recall that $\mathbb{X}_\lambda = A_\lambda^{\;A}\partial_A$ so that in particular from (2.11)

$$\mathbb{X}_j = B_j^{\;A}\partial_A = \partial_j - \Gamma^h_{ji}\,\mathbb{X}^i_{(\beta)}\,\partial_{h_\beta} \quad (\beta \text{ is summed})$$

$$\mathbb{X}_{j\alpha} = C_{j\alpha}^{\;\cdot A}\partial_A = \partial_{j\alpha}$$

Keeping these and (2.11) in mind let us calculate $\Omega_{\mu\lambda}^{\;\cdot\cdot\,\nu}$; first noting that $\Omega_{\mu\lambda}^{\;\cdot\cdot\,\nu}$ is skew-symmetric in the lower indices. That is,

$$\Omega_{\mu\lambda}^{\;\cdot\cdot\,\nu} = -\,\Omega_{\lambda\mu}^{\;\cdot\cdot\,\nu}$$

(2. 13) $$\begin{aligned}\Omega_{ji}^{\;\cdot\cdot\,h} &= B^h_{\;\cdot A}(\mathbb{X}_j B_i^{\;A} - \mathbb{X}_i B_j^{\;A}) \\ &= \delta^h_k(\mathbb{X}_j\,\delta^k_i - \mathbb{X}_i\,\delta^k_j) \\ &= 0\end{aligned}$$

$$(2.14)\quad \begin{aligned}\square\Omega\square_{j_\beta i}^{\cdot\cdot\;h} &= B^{h}_{\cdot A}\left(X_{j_\beta} B_i^{\cdot A} - X_i C_{j_\beta}^{\cdot A}\right)\\ &= \delta^h_k\left(X_{j_\beta}\delta^k_i - X_i 0\right)\\ &= 0\end{aligned}$$

$$(2.15)\quad \begin{aligned}\square\Omega\square_{j_\beta i_\alpha}^{\cdot\cdot\;h} &= B^{h}_{\cdot A}\left(X_{j_\beta} C_{i_\alpha}^{\cdot A} - X_{i_\alpha} C_{j_\beta}^{\cdot A}\right)\\ &= \delta^h_k\left(X_{j_\beta} 0 - X_{i_\alpha} 0\right)\\ &= 0\end{aligned}$$

$$(2.16)\quad \begin{aligned}\square\Omega\square_{ji}^{\cdot\cdot\;h\gamma} &= C^{h\gamma}_{\cdot A}\left(X_j B_i^{\cdot A} - X_i B_j^{\cdot A}\right)\\ &= C^{h\gamma}_{\cdot k}\left(X_j\delta^k_i - X_i\delta^k_j\right) + C^{h\gamma}_{\cdot k_\beta}\left(X_j B_i^{\cdot k_\beta} - X_i B_j^{\cdot k_\beta}\right)\\ &= \delta^h_k\delta^\gamma_\beta\left(X_j B_i^{\cdot k_\beta} - X_i B_j^{\cdot k_\beta}\right)\\ &= X_j B_i^{\cdot h\gamma} - X_i B_j^{\cdot h\gamma}\\ &= -\left\{X_j\left(\Gamma^h_{ia} X^a_{(\gamma)}\right) - X_i\left(\Gamma^h_{ja} X^a_{(\gamma)}\right)\right\}\end{aligned}$$

$$(2.17)\quad \begin{aligned}\square\Omega\square_{j_\beta i}^{\cdot\cdot\;h\gamma} &= C^{h\gamma}_{\cdot A}\left(X_{j_\beta} B_i^{\cdot A} - X_i C_{j_\beta}^{\cdot A}\right)\\ &= C^{h\gamma}_{\cdot k}\left(X_{j_\beta}\delta^k_i - X_i 0\right) + C^{h\gamma}_{\cdot k_\alpha}\left(X_{j_\beta} B_i^{\cdot k_\alpha} - X_i C_{j_\beta}^{\cdot k_\alpha}\right)\\ &= X_{j_\beta} B_i^{\cdot h\gamma} = X_{j_\beta}\left(-\Gamma^h_{ia} X^a_{(\gamma)}\right)\\ &= -\Gamma^h_{ia}\delta^a_j\delta^\gamma_\beta = -\Gamma^h_{ji}\delta^\gamma_\beta\end{aligned}$$

$$(2.18)\quad \Omega_{j_\beta i_\alpha}{}^{\cdot\cdot h_\gamma} = C^{h_\gamma}{}_{\cdot A}\left(X_{j_\beta} C_{i_\alpha}^{\cdot A} - X_{i_\alpha} C_{j_\beta}^{\cdot A}\right)$$

$$= C^{h_\gamma}{}_{\cdot k}\left(X_{j_\beta} 0 - X_{i_\alpha} 0\right) + C^{h_\gamma}{}_{\cdot k_\epsilon}\left(X_{j_\beta} \delta^k_i \delta^\epsilon_\alpha - X_{i_\alpha} \delta^k_j \delta^\epsilon_\beta\right)$$

$$= 0$$

Thus from (2.13) - (2.18) we get

$$(2.19)\quad \begin{cases} \Omega_{ji}{}^{\cdot\cdot h} = \Omega_{j_\beta i}{}^{\cdot\cdot h} = \Omega_{i j_\beta}{}^{\cdot\cdot h} = \Omega_{j_\beta i_\alpha}{}^{\cdot\cdot h} = \Omega_{j_\beta i_\alpha}{}^{\cdot\cdot h_\gamma} = 0 \\ \Omega_{ji}{}^{\cdot\cdot h_\gamma} = -\left\{ X_j\left(\Gamma^h_{ia} X^a_{(\gamma)}\right) - X_i\left(\Gamma^h_{ja} X^a_{(\gamma)}\right)\right\} \\ \Omega_{j_\beta i}{}^{\cdot\cdot h_\gamma} = -\Gamma^h_{ji}\,\delta^\gamma_\beta \end{cases}$$

Now

$$X_j\left(\Gamma^h_{ia} X^a_{(\gamma)}\right) = \partial_j\left(\Gamma^h_{ia} X^a_{(\gamma)}\right) - \Gamma^k_{jb} X^b_{(\beta)} \partial_{k_\beta}\left(\Gamma^h_{ia} X^a_{(\gamma)}\right)$$

$$= X^a_{(\gamma)} \partial_j \Gamma^h_{ia} - \Gamma^k_{jb} X^b_{(\beta)} \Gamma^h_{ia} \delta^a_k \delta^\gamma_\beta$$

$$= X^a_{(\gamma)} \partial_j \Gamma^h_{ia} - X^b_{(\gamma)} \Gamma^h_{ia} \Gamma^a_{jb}$$

$$= X^a_{(\gamma)}\left\{\partial_j \Gamma^h_{ia} - \Gamma^h_{ib} \Gamma^b_{ja}\right\}$$

Thus

$$X_j(\Gamma_{ia}^{h} X_{(\gamma)}^{a}) - X_i(\Gamma_{ja}^{h} X_{(\gamma)}^{a}) =$$

$$X_{(\gamma)}^{a}\{\partial_j \Gamma_{ia}^{h} - \partial_i \Gamma_{ja}^{h} + \Gamma_{ia}^{b}\Gamma_{jb}^{h} - \Gamma_{ja}^{b}\Gamma_{ib}^{h}\} =$$

$$X_{(\gamma)}^{a} R_{jia}{}^{h}$$

where $R_{jia}{}^{h}$ is the curvature affinor:

$$R_{jia}{}^{h} \stackrel{\text{def}}{=} \partial_j \Gamma_{ia}^{h} - \partial_i \Gamma_{ja}^{h} + \Gamma_{ia}^{b}\Gamma_{jb}^{h} - \Gamma_{ja}^{b}\Gamma_{ib}^{h}$$

So (2.19) becomes

$$(2.20)\quad \begin{cases} \Omega_{ji}^{\cdot\cdot h} = \Omega_{j\beta i}^{\cdot\cdot h} = \Omega_{ij\beta}^{\cdot\cdot h} = \Omega_{j\beta i\alpha}^{\cdot\cdot h} = \Omega_{j\beta i\alpha}^{\cdot\cdot h\gamma} = 0 \\ \Omega_{ji}^{\cdot\cdot h\gamma} = -R_{jia}{}^{h} X_{(\gamma)}^{a} \\ \Omega_{j\beta i}^{\cdot\cdot h\gamma} = -\Gamma_{ji}^{h}\delta_{\beta}^{\gamma} \end{cases}$$

From which we have

Theorem 1.2. The horizontal distribution $B_j^{\cdot A}$ is holonomic if and only if X_n is the trivial affine space A_n. (i.e., $\Gamma_{ji}^{h} = 0$.)

The Nijenhuis tensor plays an important role in the theory of spaces of complex and almost complex structure. Indeed, it is known (5) that if a tensor F of type (1,1) satisfies $F^2 = -E$, then F induces a complex structure if and only if its Nijenhuis tensor vanishes. We shall now show that

Theorem 1.2. can be restated in terms of the Nijenhuis tensor N_{CB}^{A} of $\mathbb{H}_B^A$. N_{CB}^{A} is defined by (5,p.227), (6)

$$(2.21)\quad N_{CB}^{A} = \mathbb{H}_C^M(\partial_M \mathbb{H}_B^A - \partial_B \mathbb{H}_M^A) - \mathbb{H}_B^M(\partial_M \mathbb{H}_C^A - \partial_C \mathbb{H}_M^A)$$

We now compute the components of N_{CB}^{A} first noting that $N_{BC}^{A} = -N_{CB}^{A}$. Since $\mathbb{H}_B^h = 0$ we see from (2.21) that

$$N_{ji}^{h} = N_{ji\alpha}^{h} = N_{j\beta i}^{h} = N_{j\beta i\alpha}^{h} = 0$$

Now

$$\begin{aligned}
N_{ji}^{h\gamma} &= \mathbb{H}_j^M(\partial_M \mathbb{H}_i^{h\gamma} - \partial_i \mathbb{H}_M^{h\gamma}) - \mathbb{H}_i^M(\partial_M \mathbb{H}_j^{h\gamma} - \partial_j \mathbb{H}_M^{h\gamma}) \\
&= \mathbb{H}_j^{k_\beta}(\partial_{k_\beta} \mathbb{H}_i^{h\gamma} - \cancelto{0}{\partial_i \mathbb{H}_{k_\beta}^{h\gamma}}) - \mathbb{H}_i^{k_\beta}(\partial_{k_\beta} \mathbb{H}_j^{h\gamma} - \cancelto{0}{\partial_j \mathbb{H}_{k_\beta}^{h\gamma}}) \\
&= \mathbb{H}_j^{k_\beta}\partial_{k_\beta} \mathbb{H}_i^{h\gamma} - \mathbb{H}_i^{k_\beta}\partial_{k_\beta} \mathbb{H}_j^{h\gamma} \\
&= \Gamma_{jl}^{k} \mathbb{X}_{(m+1-\beta)}^{l}\, \partial_{k_\beta}(\Gamma_{ir}^{h} \mathbb{X}_{(m+1-\gamma)}^{r}) - \Gamma_{il}^{k} \mathbb{X}_{(m+1-\beta)}^{l}\, \partial_{k_\beta}(\Gamma_{jr}^{h} \mathbb{X}_{(m+1-\gamma)}^{r}) \\
&= \Gamma_{jl}^{k}\Gamma_{ir}^{h} \mathbb{X}_{(m+1-\beta)}^{l}\, \delta_k^r \delta_{(m+1-\gamma)}^{\beta} - \Gamma_{il}^{k}\Gamma_{jr}^{h} \mathbb{X}_{(m+1-\beta)}^{l}\, \delta_k^r \delta_{(m+1-\gamma)}^{\beta} \\
&= \Gamma_{jl}^{k}\Gamma_{ik}^{h} \mathbb{X}_{(\gamma)}^{l} - \mathbb{X}_{(\gamma)}^{l}\Gamma_{il}^{k}\Gamma_{jk}^{h} \\
&= (\Gamma_{jl}^{k}\Gamma_{ik}^{h} - \Gamma_{il}^{k}\Gamma_{jk}^{h})\mathbb{X}_{(\gamma)}^{l}
\end{aligned}$$

$$\begin{aligned}
N_{j\beta i\alpha}^{h\gamma} &= \mathbb{H}_{j\beta}^M(\cancelto{0}{\partial_M \mathbb{H}_{i\alpha}^{h\gamma}} - \partial_{i\alpha} \mathbb{H}_M^{h\gamma}) - \mathbb{H}_{i\alpha}^M(\cancelto{0}{\partial_M \mathbb{H}_{j\beta}^{h\gamma}} - \partial_{j\beta} \mathbb{H}_M^{h\gamma}) \\
&= -\mathbb{H}_{j\beta}^{k\delta}\cancelto{0}{\partial_{i\alpha} \mathbb{H}_{k\delta}^{h\gamma}} + \mathbb{H}_{i\alpha}^{k\delta}\cancelto{0}{\partial_{j\beta} \mathbb{H}_{k\delta}^{h\gamma}} = 0
\end{aligned}$$

$$N_{j\beta i}{}^{h\gamma} = \mathbb{H}^{M}_{j_\beta}(\partial_M \mathbb{H}_i^{h\gamma} - \partial_i \mathbb{H}_M^{h\gamma}) - \mathbb{H}_i^M(\cancelto{0}{\partial_M} \mathbb{H}_{j_\beta}^{h\gamma} - \partial_{j_\beta}\mathbb{H}_M^{h\gamma})$$

$$= \mathbb{H}_{j_\beta}^{k_\alpha}(\partial_{k_\alpha}\mathbb{H}_i^{h\gamma}) = \delta_j^k \delta^\alpha_{(m+1-\beta)} \partial_{k_\alpha}(\Gamma_{il}^h \mathbb{X}^l_{(m+1-\gamma)})$$

$$= \delta_j^k \delta^\alpha_{(m+1-\beta)} \Gamma_{il}^h \delta_k^l \delta^{(m+1-\gamma)}_\alpha = \Gamma_{ij}^h \delta^\alpha_{(m+1-\beta)} \delta^{(m+1-\gamma)}_\alpha$$

$$= \Gamma_{ji}^h \delta_\beta^\gamma = - \Omega_{j\beta i}{}^{h\gamma}$$

So we have

$$(2.22)\quad \begin{cases} N_{ji}{}^h = N_{j\beta i}{}^h = N_{ji\alpha}{}^h = N_{j\beta i\alpha}{}^h = N_{j\beta i\alpha}{}^{h\gamma} = 0 \\ N_{j\beta i}{}^{h\gamma} = -N_{ij_\beta}{}^{h\gamma} = +\Gamma_{ji}^h \delta_\beta^\gamma = -\Omega_{j\beta i}{}^{h\gamma} \\ N_{ji}{}^{h\gamma} = (\Gamma_{jl}^k \Gamma_{ik}^h - \Gamma_{il}^k \Gamma_{jk}^h)\mathbb{X}^l_{(\gamma)} \end{cases}$$

From which we see that $N_{CB}^A = 0$ if and only if $\Gamma_{ji}^h = 0$ and so we have the following restatement of Theorem 1.2:

Theorem 1.3. In order that $T_m(X_n)$ be integrable, or that the Nijenhuis tensor of H vanish, it is necessary and sufficient that X_n be the trivial affine space. (i.e., that $\Gamma_{ji}^h = 0$)

3. Derivation of $(\underset{(v^A)}{\mathcal{L}} \mathbb{H})_\mu^\lambda$ with respect to an arbitrary vector field v^A in $T_m(X_n)$.

First we compute $(\mathbb{H})_\mu^\lambda \overset{\text{def}}{=} A_\Lambda^\lambda A_\mu^B \mathbb{H}_B^\Lambda$, the components of our tensor field $\mathbb{H}_B^A$ with respect to the adapted frames. Now

$$\mathbb{H}^{A}_{B} = \begin{pmatrix} \mathbb{H}^{h}_{j} & \mathbb{H}^{h}_{j\alpha} \\ \mathbb{H}^{h_\beta}_{j} & \mathbb{H}^{h_\beta}_{j\alpha} \end{pmatrix} = \begin{array}{c} \qquad (n) \qquad \overbrace{(n)\ \cdots\cdots\ (n)\quad (n)}^{(m)} \\ \left(\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 0 \\ \Gamma^{h}_{ji}X^{i}_{(m)} & & 0 & & \delta^{h}_{j} \\ \Gamma^{h}_{ji}X^{i}_{(m-1)} & & & \delta^{h}_{j} & \\ \vdots & & \cdot & & \\ \Gamma^{h}_{ji}X^{i}_{(1)} & \delta^{h}_{j} & & & 0 \end{array}\right) \begin{array}{l} (n) \\ \left.\begin{array}{l} (n) \\ (n) \\ \vdots \\ (n) \end{array}\right\} (m) \end{array} \end{array}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_{m} & & & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \cdot & & \\ \Gamma_{1} & E & & & \end{pmatrix}.$$

Also

$$A^{\lambda}_{B} = \begin{pmatrix} A^{h}_{\cdot j} & A^{h}_{\cdot j\alpha} \\ A^{h_\beta}_{\cdot j} & A^{h_\beta}_{\cdot j\alpha} \end{pmatrix} = \begin{pmatrix} B^{h}_{\cdot j} & B^{h}_{\cdot j\alpha} \\ C^{h_\beta}_{\cdot j} & C^{h_\beta}_{\cdot j\alpha} \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ \Gamma_{1} & E & & & 0 \\ \Gamma_{2} & & E & & \\ \vdots & & & \ddots & \\ \Gamma_{m} & & 0 & & E \end{pmatrix}$$

$$A_{\mu}^{A} = \begin{pmatrix} A_{j}^{\cdot h} & A_{j\alpha}^{\cdot h} \\ A_{j}^{\cdot h_\beta} & A_{j\alpha}^{\cdot h_\beta} \end{pmatrix} = \begin{pmatrix} B_{j}^{\cdot h} & B_{j\alpha}^{\cdot h} \\ C_{j}^{\cdot h_\beta} & C_{j\alpha}^{\cdot h_\beta} \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ -\Gamma_{1} & E & & & 0 \\ -\Gamma_{2} & & E & & \\ \vdots & & & \ddots & \\ -\Gamma_{m} & 0 & & & E \end{pmatrix}$$

Now

$$A^{\lambda}{}_{A}\mathbb{H}^{A}{}_{B} = \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ \Gamma_1 & E & & & \\ \Gamma_2 & & E & 0 & \\ \vdots & 0 & & \ddots & \\ \Gamma_m & & & & E \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & & & E \\ \Gamma_{m-1} & 0 & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ \Gamma_1 & E & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & & & E \\ \Gamma_{m-1} & 0 & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ \Gamma_1 & E & & & \end{pmatrix}$$

$$A^{\lambda}{}_{A}\mathbb{H}^{A}{}_{B}A_{\mu}{}^{B} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & & & E \\ \Gamma_{m-1} & 0 & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ \Gamma_1 & E & & & \end{pmatrix} \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ -\Gamma_1 & E & & & \\ -\Gamma_2 & & E & 0 & \\ \vdots & 0 & & \ddots & \\ -\Gamma_m & & & & E \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & & & & E \\ 0 & 0 & & E & \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & \\ 0 & E & & & \end{pmatrix}$$

So

$$(3.1) \qquad (\mathbb{H})^{\lambda}{}_{\mu} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & & & & \delta^h_j \\ 0 & 0 & & \delta^h_j & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ 0 & \delta^h_j & & 0 & \end{pmatrix}$$

Now let $V^A(\xi)$ be a vector field in $T_m(X_n)$ and consider the infinitesimal transformation $\bar{\xi}^A = \xi^A + V^A(\xi)\,\delta t$. With respect to V^A the Lie derivative $\mathcal{L}\mathbb{H}^A_B$ of our tensor field $\mathbb{H}^A_B$ is given by (2), (5), (7)

$$(3.2) \qquad \underset{(V^D)}{\mathcal{L}}\mathbb{H}^A_B = V^D\partial_D\mathbb{H}^A_B - \mathbb{H}^C_B\partial_C V^A + \mathbb{H}^A_D\partial_B V^D$$

Now let

$$(3.3) \qquad (\underset{(V^D)}{\mathcal{L}}\mathbb{H})^{\lambda}_{\mu} \overset{\text{def}}{=} A^{\lambda}{}_{A}A_{\mu}{}^{B}(\underset{(V^D)}{\mathcal{L}}\mathbb{H}^A_B)$$

be the components of $\pounds\mathbb{H}^A_B$ with respect to the adapted frames, and since $(\mathbb{H})^\lambda_\mu = {A^\lambda}_A {A_\mu}^B \mathbb{H}^A_B$ we have

$$(3.4) \qquad \mathbb{H}^A_B = {A_\lambda}^A {A^\mu}_B (\mathbb{H})^\lambda_\mu$$

Recalling that $\partial_D = {A^\nu}_D \mathbb{X}_\nu$, (3.2) can be written

$$(3.5) \qquad \underset{(V^D)}{\pounds}\mathbb{H}^A_B = {A^\nu}_D V^D \mathbb{X}_\nu \mathbb{H}^A_B - {A^\nu}_C \mathbb{H}^C_B \mathbb{X}_\nu V^A + {A^\nu}_B \mathbb{H}^A_D \mathbb{X}_\nu V^D$$
$$= {A^\nu}_D V^D \mathbb{X}_\nu ({A_\lambda}^A {A^\mu}_B (\mathbb{H})^\lambda_\mu) - \boxed{{A^\nu}_C {A_\lambda}^C} {A^\mu}_B (\mathbb{H})^\lambda_\mu \mathbb{X}_\nu V^A$$
$$+ {A^\nu}_B {A_\lambda}^A {A^\mu}_D (\mathbb{H})^\lambda_\mu \mathbb{X}_\nu V^D$$

Now let $(V)^\omega = {A^\omega}_D V^D$ be the components of the vector field v with respect to the adapted frames. Then we can write $V^D = {A_\omega}^D (V)^\omega$ so that by (2.3), (3.5) becomes

$$(3.6) \qquad \underset{(V^D)}{\pounds}\mathbb{H}^A_B = \boxed{{A^\nu}_D {A_\omega}^D} (V)^\omega \mathbb{X}_\nu ({A_\lambda}^A {A^\mu}_B (\mathbb{H})^\lambda_\mu)$$
$$- \boxed{{A^\nu}_C {A_\lambda}^C} {A^\mu}_B (\mathbb{H})^\lambda_\mu \mathbb{X}_\nu ({A_\omega}^A (V)^\omega)$$
$$+ {A^\nu}_B {A_\lambda}^A {A^\mu}_D (\mathbb{H})^\lambda_\mu \mathbb{X}_\nu ({A_\omega}^D (V)^\omega)$$
$$= (V)^\omega \mathbb{X}_\omega ({A_\lambda}^A {A^\mu}_B (\mathbb{H})^\lambda_\mu) - {A^\mu}_B (\mathbb{H})^\lambda_\mu \mathbb{X}_\lambda ({A_\omega}^A (V)^\omega)$$
$$+ {A^\nu}_B {A_\lambda}^A {A^\mu}_D (\mathbb{H})^\lambda_\mu \mathbb{X}_\nu ({A_\omega}^D (V)^\omega)$$

From (3.3), (3.6), and (2.3) we have

$$(3.7)\quad \left(\underset{(V)}{\mathcal{L}}\mathbb{H}\right)^{\lambda}_{\mu} = A^{\lambda}{}_{A} A_{\mu}{}^{B} (V)^{\omega} \mathbb{X}_{\omega} \left(A_{\tau}{}^{A} A^{\rho}{}_{B} (\mathbb{H})^{\tau}_{\rho}\right)$$

$$- A^{\lambda}{}_{A} \boxed{A_{\mu}{}^{B} A^{\rho}{}_{B}} (\mathbb{H})^{\tau}_{\rho} \mathbb{X}_{\tau} \left(A_{\omega}{}^{A} (V)^{\omega}\right)$$

$$+ \boxed{A^{\lambda}{}_{A} A_{\tau}{}^{A}} \boxed{A_{\mu}{}^{B} A^{\nu}{}_{B}} A^{\rho}{}_{D} (\mathbb{H})^{\tau}_{\rho} \mathbb{X}_{\nu} \left(A_{\omega}{}^{D} (V)^{\omega}\right)$$

$$= A^{\lambda}{}_{A} A_{\mu}{}^{B} (V)^{\omega} \mathbb{X}_{\omega} \left(A_{\tau}{}^{A} A^{\rho}{}_{B} (\mathbb{H})^{\tau}_{\rho}\right)$$

$$- A^{\lambda}{}_{A} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} \left(A_{\omega}{}^{A} (V)^{\omega}\right)$$

$$+ A^{\rho}{}_{D} (\mathbb{H})^{\lambda}_{\rho} \mathbb{X}_{\mu} \left(A_{\omega}{}^{D} (V)^{\omega}\right)$$

We now examine the individual terms in (3.7).

$$(3.8)\quad A^{\lambda}{}_{A} A_{\mu}{}^{B} (V)^{\omega} \mathbb{X}_{\omega} \left(A_{\tau}{}^{A} A^{\rho}{}_{B} (\mathbb{H})^{\tau}_{\rho}\right) =$$

$$\boxed{A^{\lambda}{}_{A} A_{\tau}{}^{A}} \boxed{A_{\mu}{}^{B} A^{\rho}{}_{B}} (V)^{\omega} \mathbb{X}_{\omega} (\mathbb{H})^{\tau}_{\rho} +$$

$$\boxed{A^{\lambda}{}_{A} A_{\tau}{}^{A}} A_{\mu}{}^{B} (V)^{\omega} (\mathbb{H})^{\tau}_{\rho} \mathbb{X}_{\omega} A^{\rho}{}_{B} +$$

$$A^{\lambda}{}_{A} \boxed{A_{\mu}{}^{B} A^{\rho}{}_{B}} (V)^{\omega} (\mathbb{H})^{\tau}_{\rho} \mathbb{X}_{\omega} A_{\tau}{}^{A} =$$

$$(V)^{\omega} \mathbb{X}_{\omega} (\mathbb{H})^{\lambda}_{\mu} + A_{\mu}{}^{B} (V)^{\omega} (\mathbb{H})^{\lambda}_{\rho} \mathbb{X}_{\omega} A^{\rho}{}_{B} + A^{\lambda}{}_{A} (V)^{\omega} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\omega} A_{\tau}{}^{A}$$

$$(3.9)\quad A^{\lambda}{}_{A} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} \left(A_{\omega}{}^{A} (V)^{\omega}\right) =$$

$$\boxed{A^{\lambda}{}_{A} A_{\omega}{}^{A}} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} (V)^{\omega} + A^{\lambda}{}_{A} (V)^{\omega} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} A_{\omega}{}^{A} =$$

$$(\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} (V)^{\lambda} + A^{\lambda}{}_{A} (V)^{\omega} (\mathbb{H})^{\tau}_{\mu} \mathbb{X}_{\tau} A_{\omega}{}^{A}$$

(3.10) $A^{\rho}{}_{D}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}(A_{\omega}{}^{D}(V)^{\omega}) =$

$$\boxed{A^{\rho}_{D}A_{\omega}{}^{D}}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}(V)^{\omega} + A^{\rho}{}_{D}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}A_{\omega}{}^{D} =$$

$$(\mathbb{H})^{\lambda}_{\omega}\,\mathbb{X}_{\mu}(V)^{\omega} + A^{\rho}{}_{D}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}A_{\omega}{}^{D}$$

Putting (3.8), (3.9), (3.10) into (3.7) we obtain

(3.11)
$$\begin{aligned}
\underset{(V^{\rho})}{(\mathcal{L}\mathbb{H})^{\lambda}_{\mu}} &= (V)^{\omega}\,\mathbb{X}_{\omega}(\mathbb{H})^{\lambda}_{\mu} + A_{\mu}{}^{B}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\omega}A^{\rho}{}_{B} \\
&+ A^{\lambda}{}_{A}(V)^{\omega}(\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\omega}A_{\tau}{}^{A} - (\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\tau}(V)^{\lambda} \\
&- A^{\lambda}{}_{A}(V)^{\omega}(\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\tau}A_{\omega}{}^{A} + (\mathbb{H})^{\lambda}_{\omega}\,\mathbb{X}_{\mu}(V)^{\omega} \\
&+ A^{\rho}{}_{D}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}A_{\omega}{}^{D} \\
&= (V)^{\omega}\,\mathbb{X}_{\omega}(\mathbb{H})^{\lambda}_{\mu} - (\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\tau}(V)^{\lambda} + (\mathbb{H})^{\lambda}_{\omega}\,\mathbb{X}_{\mu}(V)^{\omega} \\
&+ \left[A^{\lambda}{}_{A}(V)^{\omega}(\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\omega}A_{\tau}{}^{A} - A^{\lambda}{}_{A}(V)^{\omega}(\mathbb{H})^{\tau}_{\mu}\,\mathbb{X}_{\tau}A_{\omega}{}^{A}\right] \\
&+ \left[A^{\rho}{}_{D}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\mu}A_{\omega}{}^{D} + A_{\mu}{}^{B}(V)^{\omega}(\mathbb{H})^{\lambda}_{\rho}\,\mathbb{X}_{\omega}A^{\rho}{}_{B}\right]
\end{aligned}$$

Recall that $\Omega_{\mu\lambda}{}^{\cdot\cdot\nu} = A^{\nu}{}_{A}(\mathbb{X}_{\mu}A_{\lambda}{}^{A} - \mathbb{X}_{\lambda}A_{\mu}{}^{A})$**. With this in mind we examine the last four terms of (3.11).**

(3.12) $$A^\lambda{}_A(\mathbb{V})^\omega(\mathbb{H})^\tau_\mu\,\mathbb{X}_\omega A_\tau{}^A - A^\lambda{}_A(\mathbb{V})^\omega(\mathbb{H})^\tau_\mu\,\mathbb{X}_\tau A_\omega{}^A =$$
$$(\mathbb{V})^\omega(\mathbb{H})^\tau_\mu A^\lambda{}_A\left(\mathbb{X}_\omega A_\tau{}^A - \mathbb{X}_\tau A_\omega{}^A\right) = (\mathbb{V})^\omega(\mathbb{H})^\tau_\mu\,\overset{\frown}{\square\square}{}_{\omega\tau}{}^{\cdot\cdot\lambda}$$

Now from (2.3), $A^\rho{}_B A_\mu{}^B = \delta^\rho_\mu$ so that

$$A^\rho{}_B\,\mathbb{X}_\omega A_\mu{}^B + A_\mu{}^B\,\mathbb{X}_\omega A^\rho{}_B = 0$$ implying that

$$A_\mu{}^B\,\mathbb{X}_\omega A^\rho{}_B = -A^\rho{}_B\,\mathbb{X}_\omega A_\mu{}^B.$$

Thus

(3.13) $$A^\rho{}_D(\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\mathbb{X}_\mu A_\omega{}^D + (\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho A_\mu{}^B\,\mathbb{X}_\omega A^\rho{}_B =$$
$$A^\rho{}_B(\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\mathbb{X}_\mu A_\omega{}^D - A^\rho{}_B(\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\mathbb{X}_\omega A_\mu{}^B =$$
$$(\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho A^\rho{}_B\left(\mathbb{X}_\mu A_\omega{}^B - \mathbb{X}_\omega A_\mu{}^B\right) = (\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\overset{\frown}{\square\square}{}_{\mu\omega}{}^{\cdot\cdot\rho} =$$
$$-(\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\overset{\frown}{\square\square}{}_{\omega\mu}{}^{\cdot\cdot\rho}$$

Substituting (3.12) and (3.13) into (3.11) we get

(3.14) $$\boxed{\begin{aligned}\Big(\underset{(VD)}{\pounds}\mathbb{H}\Big)^\lambda_\mu &= (\mathbb{V})^\omega\,\mathbb{X}_\omega(\mathbb{H})^\lambda_\mu - (\mathbb{H})^\tau_\mu\,\mathbb{X}_\tau(\mathbb{V})^\lambda + (\mathbb{H})^\lambda_\omega\,\mathbb{X}_\mu(\mathbb{V})^\omega\\ &\quad + (\mathbb{V})^\omega(\mathbb{H})^\tau_\mu\,\overset{\frown}{\square\square}{}_{\omega\tau}{}^{\cdot\cdot\lambda} - (\mathbb{V})^\omega(\mathbb{H})^\lambda_\rho\,\overset{\frown}{\square\square}{}_{\omega\mu}{}^{\cdot\cdot\rho}.\end{aligned}}$$

4. Necessary and sufficient conditions that $(\underset{(\bar{v}^A)}{\pounds}\mathbb{H})^{\lambda}_{\mu} = 0$ where $\bar{v}^A$ is the extended lift of a vector field v^h in X_n.

Let v be a vector field in X_n with components v^h with respect to the coordinate neighborhood (u^h). Taking the extended lift $\bar{v}^A$ (Definition 2.1) of v^h as the generator of an infinitesimal transformation we now wish to compute

$$(\underset{(\bar{v}^A)}{\pounds}\mathbb{H})^{\lambda}_{\mu} = \left((\pounds\mathbb{H})^{h}_{j},\ (\pounds\mathbb{H})^{h}_{j\alpha},\ (\pounds\mathbb{H})^{h_\beta}_{j},\ (\pounds\mathbb{H})^{h_\beta}_{j\alpha}\right)$$

Let us first compute $(\bar{v})^{\lambda} = A^{\lambda}{}_{A}\bar{v}^{A}$, the components of the extended lift $\bar{v}^A$ with respect to the adapted frames.

$$\begin{aligned}
(\bar{v})^{h} &= B^{h}{}_{A}\bar{v}^{A} \\
&= B^{h}{}_{\cdot j}\bar{v}^{j} + B^{h}{}_{\cdot j_\alpha}\bar{v}^{j_\alpha} \\
&= \delta^{h}_{j}v^{j} + C\cdot\bar{v}^{j_\alpha} \\
&= v^{h}
\end{aligned}$$

$$\begin{aligned}
(\bar{v})^{h_\beta} &= C^{h_\beta}{}_{\cdot A}\bar{v}^{A} \\
&= C^{h_\beta}{}_{\cdot j}\bar{v}^{j} + C^{h_\beta}{}_{\cdot j_\alpha}\bar{v}^{j_\alpha} \\
&= \Gamma^{h}_{ji}\mathbb{X}^{i}_{(\beta)}v^{i} + \delta^{h}_{j}\delta^{\beta}_{\alpha}\mathbb{X}^{a}_{(\alpha)}\partial_{a}v^{j} \\
&= \Gamma^{h}_{ji}\mathbb{X}^{i}_{(\beta)}v^{j} + \mathbb{X}^{a}_{(\beta)}\partial_{a}v^{h} \\
&= \mathbb{X}^{a}_{(\beta)}\left\{\partial_{a}v^{h} + \Gamma^{h}_{ja}v^{j}\right\} \\
&= \mathbb{X}^{a}_{(\beta)}\nabla_{a}v^{h}
\end{aligned}$$

So

$$(4.1)\quad (\bar{v})^{\lambda} = \left((\bar{v})^{h}, (\bar{v})^{h_\beta}\right) = \left(v^{h}, \bar{X}^{a}_{(\beta)} \nabla_a v^{h}\right).$$

Now from (3.14) we have

$$(4.2)\quad \left(\underset{(\bar{v}^{\Lambda})}{\pounds}\mathbb{H}\right)^{\lambda}_{\mu} = (\bar{v})^{\omega} \cancelto{0}{X_{\omega}} (\mathbb{H})^{\lambda}_{\mu} - (\mathbb{H})^{\omega}_{\mu} X_{\omega} (\bar{v})^{\lambda} + (\mathbb{H})^{\lambda}_{\omega} X_{\mu} (\bar{v})^{\omega}$$
$$+ (\bar{v})^{\omega} (\mathbb{H})^{\nu}_{\mu} \Omega_{\omega\nu}{}^{\cdot\cdot\lambda} - (\bar{v})^{\omega} (\mathbb{H})^{\lambda}_{\nu} \Omega_{\omega\mu}{}^{\cdot\cdot\nu}$$

where, of course $(\mathbb{H})^{\lambda}_{\mu}$ is given by (3.1).

$$(4.3)\quad \left(\underset{(\bar{v}^{\Lambda})}{\pounds}\mathbb{H}\right)^{h}_{j} = -(\mathbb{H})^{\omega}_{j} X_{\omega} (\bar{v})^{h} + (\mathbb{H})^{h}_{\omega} X_{j} (\bar{v})^{\omega}$$
$$+ (\bar{v})^{\omega} (\mathbb{H})^{\nu}_{j} \Omega_{\omega\nu}{}^{\cdot\cdot h} - (\bar{v})^{\omega} (\mathbb{H})^{h}_{\nu} \Omega_{\omega j}{}^{\cdot\cdot\nu}$$
$$= -0 \cdot X_{\omega} (\bar{v})^{h} + 0 \cdot X_{j} (\bar{v})^{\omega}$$
$$+ (\bar{v})^{\omega} \cdot 0 \cdot \Omega_{\omega\nu}{}^{\cdot\cdot h} - (\bar{v})^{\omega} \cdot 0 \cdot \Omega_{\omega j}{}^{\nu}$$
$$= \underline{0}$$

$$(4.4)\quad \left(\underset{(\bar{v}^{\Lambda})}{\pounds}\mathbb{H}\right)^{h}_{j_\alpha} = -(\mathbb{H})^{\omega}_{j_\alpha} X_{\omega} (\bar{v})^{h} + (\mathbb{H})^{h}_{\omega} X_{j_\alpha} (\bar{v})^{\omega}$$
$$+ (\bar{v})^{\omega} (\mathbb{H})^{\nu}_{j_\alpha} \Omega_{\omega\nu}{}^{\cdot\cdot h} - (\bar{v})^{\omega} (\mathbb{H})^{h}_{\nu} \Omega_{\omega j_\alpha}{}^{\cdot\cdot\nu}$$
$$= -(\mathbb{H})^{k_\beta}_{j_\alpha} \bar{X}_{k_\beta} (\bar{v})^{h} + 0 \cdot \bar{X}_{j_\alpha} (\bar{v})^{\omega}$$
$$+ (\bar{v})^{\omega} \cdot (\mathbb{H})^{\nu}_{j_\alpha} \cdot 0 - (\bar{v})^{\omega}\, 0 \cdot \Omega_{\omega j_\alpha}{}^{\nu}$$
$$= -(\mathbb{H})^{k_\beta}_{j_\alpha} \partial_{k_\beta} v^{h}$$

$= \underline{0}$ since v^{h} is independent of ξ^{k_β}.

$$
\begin{aligned}
(4.5)\quad \big(\underset{(\bar{v}^A)}{\pounds}\mathbb{H}\big)_j^{h_\beta} &= -(\mathbb{H})_j^{\omega}\,\mathbb{X}_\omega(\bar{v})^{h_\beta} + (\mathbb{H})_\omega^{h_\beta}\,\mathbb{X}_j(\bar{v})^{\omega} \\
&\quad + (\bar{v})^{\omega}(\mathbb{H})_j^{\nu}\,\overset{\frown}{\square\square}{}_{\omega\nu}^{\cdot\cdot h_\beta} - (\bar{v})^{\omega}(\mathbb{H})_\nu^{h_\beta}\,\overset{\frown}{\square\square}{}_{\omega j}^{\cdot\cdot \nu} \\
&= -0\cdot\mathbb{X}_\omega(\bar{v})^{h_\beta} + (\mathbb{H})_{k_\alpha}^{h_\beta}\,\mathbb{X}_j(\bar{v})^{k_\alpha} \\
&\quad + (\bar{v})^{\omega}\,0\cdot\overset{\frown}{\square\square}{}_{\omega\nu}^{\cdot\cdot h_\beta} - (\bar{v})^{\omega}(\mathbb{H})_{k_\alpha}^{h_\beta}\,\overset{\frown}{\square\square}{}_{\omega j}^{\cdot\cdot k_\alpha} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\,\mathbb{X}_j\big(\mathbb{X}^a_{(\alpha)}\nabla_a v^k\big) - v^i(\mathbb{H})_{k_\alpha}^{h_\beta}\,\overset{\frown}{\square\square}{}_{ij}^{\cdot k_\alpha} \\
&\quad - (\bar{v})^{i_\gamma}(\mathbb{H})_{k_\alpha}^{h_\beta}\,\overset{\frown}{\square\square}{}_{i_\gamma j}^{k_\alpha} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\Big\{\mathbb{X}^a_{(\alpha)}\partial_j(\nabla_a v^k) - \nabla_a v^h\,\mathbb{X}^i_{(\gamma)}\Gamma^b_{ji}\,\partial_{b_\gamma}(\mathbb{X}^a_{(\alpha)}) \\
&\qquad\qquad - v^i R_{jia}{}^{k}\,\mathbb{X}^a_{(\alpha)} + (\bar{v})^{i_\gamma}\delta^\alpha_\gamma\Gamma^k_{ji}\Big\} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\Big\{\mathbb{X}^a_{(\alpha)}\partial_j(\nabla_a v^k) - \nabla_a v^k\,\mathbb{X}^i_{(\gamma)}\Gamma^b_{ji}\,\delta^a_b\delta^\gamma_\alpha \\
&\qquad\qquad - v^i R_{jia}{}^{k}\,\mathbb{X}^a_{(\alpha)} + \mathbb{X}^a_{(\alpha)}\nabla_a v^i\,\Gamma^k_{ji}\Big\} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\,\mathbb{X}^a_{(\alpha)}\Big\{\partial_j(\nabla_a v^k) + \nabla_a v^i\,\Gamma^k_{ji} - \nabla_b v^k\,\Gamma^b_{ja} \\
&\qquad\qquad - R_{jia}{}^{k}\,v^i\Big\} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\,\mathbb{X}^a_{-(\alpha)}\Big\{\nabla_j\nabla_a v^k + R_{ija}{}^{k}\,v^i\Big\} \\
&= (\mathbb{H})_{k_\alpha}^{h_\beta}\,\mathbb{X}^a_{(\alpha)}\big(\underset{(v^i)}{\pounds}\Gamma^k_{ja}\big) \quad \text{(see (7) p. 9)} \\
&= \big(\pounds\Gamma^h_{ja}\big)\,\mathbb{X}^a_{(m+1-\beta)}
\end{aligned}
$$

$$
\begin{aligned}
(4.6)\quad \Big(\underset{(\bar{v}^{\wedge})}{\pounds}\,\mathbb{H}\Big)_{j\alpha}^{h\beta} &= -(\mathbb{H})_{j\alpha}^{\omega}\,\mathbb{X}_{\omega}(\bar{v})^{h\beta} + (\mathbb{H})_{\omega}^{h\beta}\,\mathbb{X}_{j\alpha}(\bar{v})^{\omega} \\
&\quad + (\bar{v})^{\omega}(\mathbb{H})_{j\alpha}^{\nu}\,\overset{\frown}{\square\square}_{\omega\nu}{}^{h\beta} - (\bar{v})^{\omega}(\mathbb{H})_{\nu}^{h\beta}\,\overset{\frown}{\square\square}_{\omega j\alpha}{}^{\cdot\cdot\,\nu} \\
&= -(\mathbb{H})_{j\alpha}^{k\gamma}\,\mathbb{X}_{k\gamma}(\mathbb{X}^{a}_{(\beta)}\nabla_a v^h) + (\mathbb{H})_{k\gamma}^{h\beta}\,\mathbb{X}_{j\alpha}(\mathbb{X}^{a}_{(\gamma)}\nabla_a v^k) \\
&\quad + (\bar{v})^{\omega}(\mathbb{H})_{j\alpha}^{k\gamma}\,\overset{\frown}{\square\square}_{\omega k\gamma}{}^{\cdot\, h\beta} - (\bar{v})^{\omega}(\mathbb{H})_{k\gamma}^{h\beta}\,\overset{\frown}{\square\square}_{\omega j\alpha}{}^{\cdot\cdot\, k\gamma} \\
&= -(\mathbb{H})_{j\alpha}^{k\gamma}\,\nabla_a v^h\,\mathbb{X}_{k\gamma}(\mathbb{X}^{a}_{(\beta)}) + (\mathbb{H})_{k\gamma}^{h\beta}\,\nabla_a v^k\,\mathbb{X}_{j\alpha}(\mathbb{X}^{a}_{(\gamma)}) \\
&\quad + (\bar{v})^{i}(\mathbb{H})_{j\alpha}^{k\gamma}\,\overset{\frown}{\square\square}_{ik\gamma}{}^{h\beta} - (\bar{v})^{i}(\mathbb{H})_{k\gamma}^{h\beta}\,\overset{\frown}{\square\square}_{ij\alpha}{}^{k\gamma} \\
&= -(\mathbb{H})_{j\alpha}^{k\gamma}\,\nabla_a v^h\,\delta^a_k\,\delta^\beta_\gamma + (\mathbb{H})_{k\gamma}^{h\beta}\,\nabla_a v^k\,\delta^a_j\,\delta^\gamma_\alpha \\
&\quad + v^i(\mathbb{H})_{j\alpha}^{k\gamma}\,\delta^\beta_\gamma\,\Gamma^h_{ki} - v^i(\mathbb{H})_{k\gamma}^{h\beta}\,\delta^\gamma_\alpha\,\Gamma^k_{ji} \\
&= -(\mathbb{H})_{j\alpha}^{k\beta}\,\nabla_k v^h + (\mathbb{H})_{k\alpha}^{h\beta}\,\nabla_j v^k + (\mathbb{H})_{j\alpha}^{k\beta}\,v^i\,\Gamma^h_{ki} \\
&\quad - (\mathbb{H})_{k\alpha}^{h\beta}\,v^i\,\Gamma^k_{ji} \\
&= -(\mathbb{H})_{j\alpha}^{k\beta}\{\nabla_k v^h - \Gamma^h_{ki}v^i\} + (\mathbb{H})_{k\alpha}^{h\beta}\{\nabla_j v^k - \Gamma^k_{ji}v^i\} \\
&= -(\mathbb{H})_{j\alpha}^{k\beta}\,\partial_k v^h + (\mathbb{H})_{k\alpha}^{h\beta}\,\partial_j v^k \\
&= \begin{cases} 0 & \text{if } \alpha+\beta \neq m+1 \\ -\delta^k_j\,\partial_k v^h + \delta^h_k\,\partial_j v^k = 0 & \text{if } \alpha+\beta = m+1 \end{cases} \\
&= 0
\end{aligned}
$$

So from (4.3) through (4.6) we have

$$\boxed{\begin{aligned} &(\underset{(\bar{v}^{A})}{\mathcal{L}}\mathbb{H})^{h}_{j} = (\underset{(\bar{v}^{A})}{\mathcal{L}}\mathbb{H})^{h}_{j\alpha} = (\underset{(\bar{v}^{A})}{\mathcal{L}}\mathbb{H})^{h_\beta}_{j\alpha} = 0 \\ &(\underset{(\bar{v}^{A})}{\mathcal{L}}\mathbb{H})^{h_\beta}_{j} = (\underset{(v^{i})}{\mathcal{L}}\Gamma^{h}_{ja})\,\bar{X}^{a}_{(m+1-\beta)} \end{aligned}} \tag{4.7}$$

Thus $\underset{(\bar{v}^{C})}{\mathcal{L}}\mathbb{H}^{A}_{B} = A_{\lambda}{}^{A}A^{\mu}{}_{B}(\underset{(\bar{v}^{C})}{\mathcal{L}}\mathbb{H})^{\lambda}_{\mu} = 0$ if and only if $\underset{(v^{k})}{\mathcal{L}}\Gamma^{h}_{ji} = 0$ and hence we have:

Theorem 4.1. A necessary and sufficient condition that $\mathbb{H}^{A}_{B}$ be invariant under an infinitesimal transformation generated by the extended lift of a vector field v in X_n is that v define an affine collineation (see (7) p.9) in X_n.

Again if we define a tensor field $\overline{\Phi}$ by $\overline{\Phi} = -\frac{3}{2}\mathbb{H}^{2} + \frac{\sqrt{3}}{2}\mathbb{H} - E$, then $\overline{\Phi}$ is of rank n + mn, satisfies $\overline{\Phi}^{3} = -E$, and the above theorem also holds good for $\overline{\Phi}$.

5. Introduction of a symmetric affine connection $\textcircled{H}^{A}_{BC}$ into $T_m(X_n)$.

We now introduce into $T_m(X_n)$ (which can always be done, see (1)p.78) a symmetric affine connection whose components with respect to the natural frames are $\textcircled{H}^{A}_{BC}$. To find its components $\hat{\textcircled{H}}^{\lambda}_{\mu\nu}$ with respect to the adapted frames $A_{\lambda}{}^{A} = (B_{j}{}^{A}, C_{j\alpha}{}^{A})$ we write $\omega^{A}_{C} = \textcircled{H}^{A}_{BC}\,d\xi^{B}$ and $\omega^{\lambda}_{\nu} = \hat{\textcircled{H}}^{\lambda}_{\mu\nu}(d\xi)^{\mu}$. Then ω^{A}_{C} and ω^{λ}_{ν} must satisfy (see (1)p.77)

$$dp_{\lambda}^{A} + \omega^{A}_{C}\,p^{C}_{\lambda} == p^{A}_{\mu}\,\omega^{\mu}_{\lambda}$$

where in this case

$p_\lambda^A = A_\lambda^A$ So $dp_\lambda^A = \mathbb{X}_\mu A_\lambda^A (d\xi)^\mu$, substituting

$$\mathbb{X}_\mu A_\lambda^A (d\xi)^\mu + \textcircled{H}{}^A_{BC}\, d\xi^B A_\lambda^C = A_\mu^A \hat{\textcircled{H}}{}^\mu_{\nu\lambda} (d\xi)^\nu$$

But $d\xi^B = A_\nu^B (d\xi)^\nu$ so that

$$(\mathbb{X}_\nu A_\lambda^A + \textcircled{H}{}^A_{BC} A_\nu^B A_\lambda^C)(d\xi)^\nu = A_\mu^A \hat{\textcircled{H}}{}^\mu_{\nu\lambda} (d\xi)^\nu$$

So $A_\mu^A \hat{\textcircled{H}}{}^\mu_{\nu\lambda} = \mathbb{X}_\nu A_\lambda^A + \textcircled{H}{}^A_{BC} A_\nu^B A_\lambda^C$ and hence

$$\hat{\textcircled{H}}{}^\lambda_{\mu\nu} = A^\lambda_A (\mathbb{X}_\mu A_\lambda^A + \textcircled{H}{}^A_{BC} A_\mu^B A_\nu^C) \tag{5.1}$$

from which we observe that

$$\hat{\textcircled{H}}{}^\lambda_{\mu\nu} - \hat{\textcircled{H}}{}^\lambda_{\nu\mu} = A^\lambda_A (\mathbb{X}_\mu A_\nu^A - \mathbb{X}_\nu A_\mu^A) = \Omega_{\mu\nu}{}^\lambda \tag{5.2}$$

which has been previously derived (2.20).

There would be various ways of introducing $\textcircled{H}{}^A_{BC}$ into $\xi^A(x^h, \mathbb{X}_{(\beta)}^h)$ however, we now assume these components in the natural frames should satisfy: $\textcircled{H}{}^A_{BC} = \textcircled{H}{}^A_{CB}$ $(\Gamma^h_{ji} = \Gamma^h_{ij})$ and

$$\left\{\begin{aligned}
&\textcircled{H}{}^h_{ji} = \textcircled{H}{}^h_{ij} = \Gamma^h_{ji}\\
&\textcircled{H}{}^h_{j_\beta i} = \textcircled{H}{}^h_{i j_\beta} = \textcircled{H}{}^h_{j_\beta i_\alpha} = 0\\
&\textcircled{H}{}^{h_\gamma}_{j_\beta i} = \textcircled{H}{}^{h_\gamma}_{i j_\beta} = \delta^\gamma_\beta \Gamma^h_{ji}\\
&\textcircled{H}{}^{h_\gamma}_{j_\beta i_\alpha} = 0\\
&\textcircled{H}{}^{h_\gamma}_{ji} = -\mathbb{X}^b_{(\gamma)} \Gamma^h_{ba} \Gamma^a_{ji} + \tfrac{1}{2}\{\mathbb{X}_j(\mathbb{X}^b_{(\gamma)} \Gamma^h_{bi}) + \mathbb{X}_i(\mathbb{X}^b_{(\gamma)} \Gamma^h_{bj})\}\\
&\qquad + \mathbb{X}^a_{(\gamma)} \Gamma^h_{bi} \Gamma^b_{ja} + \mathbb{X}^a_{(\gamma)} \Gamma^h_{bj} \Gamma^b_{ia}
\end{aligned}\right. \tag{5.3}$$

Using (5.1) and (5.3) we now compute the components $\hat{\textcircled{H}}{}^{\lambda}_{\mu\nu}$.

$$(5.4)\quad \hat{\textcircled{H}}{}^{h}_{ji} = B^{h}_{\cdot A}\left(X_j B_i^{\cdot A} + \textcircled{H}{}^{A}_{BC} B_j^{\cdot B} B_i^{\cdot C}\right)$$

$$= \delta^{h}_{k}\left(\cancelto{0}{X_j \delta_i^{k}} + \textcircled{H}{}^{k}_{BC} B_j^{\cdot B} B_i^{\cdot C}\right)$$

$$= \textcircled{H}{}^{h}_{BC} B_j^{\cdot B} B_i^{\cdot C} = \textcircled{H}{}^{h}_{ab} B_j^{\cdot a} B_i^{\cdot b}$$

$$= \Gamma^{h}_{ab}\delta^{a}_{j}\delta^{b}_{i} = \Gamma^{h}_{ji}$$

$$(5.5)\quad \hat{\textcircled{H}}{}^{h}_{j_\beta i} = B^{h}_{\cdot A}\left(X_{j_\beta} B_i^{\cdot A} + \textcircled{H}{}^{A}_{BC} C_{j_\beta}^{\cdot B} B_i^{\cdot C}\right)$$

$$= \delta^{h}_{k}\left(\cancelto{0}{X_{j_\beta} \delta_i^{k}} + \textcircled{H}{}^{k}_{BC} C_{j_\beta}^{\cdot B} B_i^{\cdot C}\right)$$

$$= \textcircled{H}{}^{h}_{BC} C_{j_\beta}^{\cdot B} C_i^{\cdot C} = \textcircled{H}{}^{h}_{ab} C_{j_\beta}^{\cdot a} B_i^{\cdot b}$$

$$= \Gamma^{h}_{ab}\cdot 0 \cdot \delta^{b}_{i} = 0$$

$$(5.6)\quad \hat{\textcircled{H}}{}^{h}_{j i_\alpha} = B^{h}_{\cdot A}\left(X_j C_{i_\alpha}^{A} + \textcircled{H}{}^{A}_{BC} B_j^{\cdot B} C_{i_\alpha}^{\cdot C}\right)$$

$$= X_j C_{i_\alpha}^{h} + \textcircled{H}{}^{h}_{BC} B_j^{\cdot B} C_{i_\alpha}^{C}$$

$$= \cancelto{0}{X_j 0} + \Gamma^{h}_{ab} B_j^{\cdot a} C_{i_\alpha}^{\cdot b}$$

$$= \Gamma^{h}_{ab}\delta^{a}_{j}\, 0 = 0$$

$$(5.7)\quad \hat{\textcircled{H}}{}^{h}_{j_\beta i_\alpha} = B^{h}_{\cdot A}\left(X_{j_\beta} C_{i_\alpha}^{A} + \textcircled{H}{}^{A}_{BC} C_{j_\beta}^{\cdot B} C_{i_\alpha}^{C}\right)$$

$$= X_{j_\beta} C_{i_\alpha}^{\cdot h} + \textcircled{H}{}^{h}_{BC} C_{j_\beta}^{\cdot B} C_{i_\alpha}^{\cdot C}$$

$$= \cancelto{0}{\mathbb{X}_{j_\beta}}\, 0 + \textcircled{H}^{h}_{ab} C^{\cdot a}_{j_\beta} C^{\cdot b}_{1_\alpha}$$

$$= \Gamma^{h}_{ab}\, 0 \cdot 0 = 0$$

$$
\begin{aligned}
(5.8)\quad \hat{\textcircled{H}}^{h\gamma}_{ji} &= C^{h\gamma}_{\cdot A}\left(\mathbb{X}_j B^{\cdot A}_{i} + \textcircled{H}^{A}_{BC} B^{\cdot B}_{j} B^{\cdot C}_{i}\right) \\
&= C^{h\gamma}_{\cdot k}\left(\mathbb{X}_j B^{\cdot k}_{i} + \textcircled{H}^{k}_{BC} B^{\cdot B}_{j} B^{\cdot C}_{i}\right) \\
&\quad + C^{h\gamma}_{\cdot k_\beta}\left(\mathbb{X}_j B^{\cdot k_\beta}_{i} + \textcircled{H}^{k_\beta}_{BC} B^{\cdot B}_{j} B^{\cdot C}_{i}\right) \\
&= \Gamma^{h}_{ka} \mathbb{X}^{a}_{(\gamma)}\left(\cancelto{0}{\mathbb{X}_j \delta^{k}_{i}} + \Gamma^{k}_{bc} \delta^{b}_{j} \delta^{c}_{i}\right) \\
&\quad + \mathbb{X}_j B^{h\gamma}_{i} + \textcircled{H}^{h\gamma}_{BC} B^{\cdot B}_{j} B^{\cdot C}_{i} \\
&= \mathbb{X}^{a}_{(\gamma)} \Gamma^{k}_{ji} \Gamma^{h}_{ka} - \mathbb{X}_j\left(\Gamma^{h}_{ia} \mathbb{X}^{a}_{(\gamma)}\right) + \textcircled{H}^{h\gamma}_{ji} \\
&\quad + \textcircled{H}^{h\gamma}_{a_\alpha b} B^{\cdot a_\alpha}_{j} B^{\cdot b}_{i} + \textcircled{H}^{h\gamma}_{a b_\beta} B^{\cdot a}_{j} B^{\cdot b_\beta}_{i} \\
&= \mathbb{X}^{a}_{(\gamma)} \Gamma^{h}_{ka} \Gamma^{k}_{ji} - \mathbb{X}_j\left(\mathbb{X}^{a}_{(\gamma)} \Gamma^{h}_{ia}\right) + \textcircled{H}^{h\gamma}_{ji} \\
&\quad + \delta^{\gamma}_{\alpha} \Gamma^{h}_{ab}\left(-\mathbb{X}^{k}_{(\alpha)} \Gamma^{a}_{jk}\right)\delta^{b}_{i} + \delta^{\gamma}_{\beta} \Gamma^{h}_{ab} \delta^{a}_{j}\left(-\mathbb{X}^{k}_{(\beta)} \Gamma^{b}_{ik}\right) \\
&= \mathbb{X}^{a}_{(\gamma)} \Gamma^{h}_{ka} \Gamma^{k}_{ji} - \mathbb{X}_j\left(\mathbb{X}^{a}_{(\gamma)} \Gamma^{h}_{ia}\right) + \textcircled{H}^{h\gamma}_{ji} \\
&\quad - \mathbb{X}^{k}_{(\gamma)} \Gamma^{h}_{ia} \Gamma^{a}_{jk} - \mathbb{X}^{k}_{(\gamma)} \Gamma^{h}_{jb} \Gamma^{b}_{ik}
\end{aligned}
$$

$$= \underset{\sim\sim\sim}{X^a_{(\gamma)}\Gamma^h_{ka}\Gamma^k_{ji}} - \cancel{X^k_{(\gamma)}\Gamma^h_{ia}\Gamma^a_{jk}} - \cancel{X^k_{(\gamma)}\Gamma^h_{jb}\Gamma^b_{ik}}$$

$$- X_j(X^a_{(\gamma)}\Gamma^h_{ia}) - \underset{\sim\sim\sim}{X^b_{(\gamma)}\Gamma^h_{ba}\Gamma^a_{ji}} + \cancel{X^a_{(\gamma)}\Gamma^h_{bi}\Gamma^b_{ja}} + \cancel{X^a_{(\gamma)}\Gamma^h_{bj}\Gamma^b_{ia}}$$

$$+ \frac{1}{2}\left\{X_j(X^b_{(\gamma)}\Gamma^h_{bi}) + X_i(X^b_{(\gamma)}\Gamma^h_{bj})\right\}$$

$$= \frac{1}{2}\left\{X_i(X^b_{(\gamma)}\Gamma^h_{bj}) - X_j(X^b_{(\gamma)}\Gamma^h_{bi})\right\}$$

$$= \frac{1}{2}\,\Omega_{ji}^{\cdot\cdot h\gamma}$$ from (3.19).

(5.9) $$\hat{\textcircled{H}}{}^{h\gamma}_{j_\beta i} = C^{h\gamma}_{\cdot A}\left(X_{j_\beta}B_i^{\cdot A} + \textcircled{H}^A_{BC}C_{j_\beta}^{\cdot B}B_i^{\cdot C}\right)$$

$$= C^{h\gamma}_{\cdot k}\left(X_{j_\beta}B_i^{\cdot k} + \textcircled{H}^k_{BC}C_{j_\beta}^{\cdot B}B_i^{\cdot C}\right)$$

$$+ C^{h\gamma}_{\cdot k_\alpha}\left(X_{j_\beta}B_i^{\cdot k_\alpha} + \textcircled{H}^{k_\alpha}_{BC}C_{j_\beta}^{\cdot B}B_i^{\cdot C}\right)$$

$$= \Gamma^h_{ka}X^a_{(\gamma)}\left(\cancelto{0}{X_{j_\beta}}\delta^k_i + \textcircled{H}^k_{bd}\cancelto{0}{C_{j_\beta}^{\cdot b}}B_i^{\cdot d}\right)$$

$$+ X_{j_\beta}B_i^{\cdot h\gamma} + \textcircled{H}^{h\gamma}_{BC}C_{j_\beta}^{B}B_i^{\cdot C}$$

$$= X_{j_\beta}(-X^a_{(\gamma)}\Gamma^h_{ia}) + \textcircled{H}^{h\gamma}_{ab}\cancelto{0}{C_{j_\beta}^{\cdot a}}B_i^{\cdot b}$$

$$+ \textcircled{H}^{h\gamma}_{ab_\alpha}\cancelto{0}{C_{j_\beta}^{\cdot a}}B_i^{\cdot b_\alpha} + \textcircled{H}^{h\gamma}_{a_\alpha b}C_{j_\beta}^{\cdot a_\alpha}B_i^{\cdot b}$$

$$= -\Gamma^h_{ia}X_{j_\beta}(X^a_{(\gamma)}) + \textcircled{H}^h_{a_\alpha b}\delta^a_j\delta^\alpha_\beta\delta^b_i$$

$$= -\Gamma^h_{ia}\delta^a_j\delta^\gamma_\beta + \textcircled{H}^{h\gamma}_{j_\beta i} = 0$$

$$
\begin{aligned}
\text{(5.10)}\quad \hat{\textcircled{H}}{}^{h\gamma}_{ji_\alpha} &= C^{h\gamma}_{A}\left(\overline{\underline{X}}_j C_{i_\alpha}^{A} + \textcircled{H}{}^{A}_{BC} B_j^{B} C_{i_\alpha}^{C}\right)\\
&= C^{h\gamma}_{\cdot k}\left(\overline{\underline{X}}_j C_{i_\alpha}^{\cdot k} + \textcircled{H}{}^{k}_{BC} B_j^{\cdot B} C_{i_\alpha}^{\cdot C}\right)\\
&\quad + C^{h\gamma}_{\cdot k_\beta}\left(\overline{\underline{X}}_j C_{i_\alpha}^{\cdot k_\beta} + \textcircled{H}{}^{k_\beta}_{BC} B_j^{B} C_{i_\alpha}^{C}\right)\\
&= \overline{\underline{X}}_j\, 0 + \Gamma^{k}_{ab}\delta^{a}_{j}\, 0 + \overline{\underline{X}}_j C_{i_\alpha}^{h\gamma} + \textcircled{H}{}^{h\gamma}_{BC} B_j^{\cdot B} C_{i_\alpha}^{\cdot C}\\
&= \textcircled{H}{}^{h\gamma}_{ab} B_j^{\cdot a} C_{i_\alpha}^{\cdot b} + \textcircled{H}{}^{h\gamma}_{a_\beta b} B_j^{\cdot a_\beta} C_{i_\alpha}^{\cdot b} + \textcircled{H}{}^{h\gamma}_{ab_\beta} B_j^{\cdot a} C_{i_\alpha}^{\cdot b_\beta}\\
&= \textcircled{H}{}^{h\gamma}_{ab_\beta}\delta^{a}_{j}\delta^{b}_{i}\delta^{\beta}_{\alpha} = \textcircled{H}{}^{h\gamma}_{ji_\alpha} = \delta^{\gamma}_{\alpha}\Gamma^{h}_{ji}
\end{aligned}
$$

$$
\begin{aligned}
\text{(5.11)}\quad \hat{\textcircled{H}}{}^{h\gamma}_{j_\beta i_\alpha} &= C^{h\gamma}_{\cdot A}\left(\overline{\underline{X}}_{j_\beta} C_{i_\alpha}^{A} + \textcircled{H}{}^{A}_{BC} C_{j_\beta}^{\cdot B} C_{i_\alpha}^{\cdot C}\right)\\
&= C^{h\gamma}_{\cdot k}\left(\cancelto{0}{\overline{\underline{X}}_{j_\beta} C_{i_\alpha}^{\cdot k}} + \textcircled{H}{}^{k}_{BC} \cancelto{0}{C_{j_\beta}^{\cdot B}} C_{i_\alpha}^{\cdot C}\right)\\
&\quad + C^{h\gamma}_{\cdot k_\delta}\left(\overline{\underline{X}}_{j_\beta} C_{i_\alpha}^{\cdot k_\delta} + \textcircled{H}{}^{k_\delta}_{BC} C_{j_\beta}^{\cdot B} C_{i_\alpha}^{\cdot C}\right)\\
&= \overline{\underline{X}}_{j_\beta} C_{i_\alpha}^{\cdot h\gamma} + \textcircled{H}{}^{h\gamma}_{BC} C_{j_\beta}^{\cdot B} C_{i_\alpha}^{\cdot C} = 0
\end{aligned}
$$

Putting (5.4)-(5.11) together we get:

$$
\text{(5.12)}\quad
\boxed{
\begin{aligned}
&\hat{\textcircled{H}}{}^{h}_{ji} = \hat{\textcircled{H}}{}^{h}_{ij} = \Gamma^{h}_{ji}\,, && \hat{\textcircled{H}}{}^{h}_{j_\beta i_\alpha} = 0\\
&\hat{\textcircled{H}}{}^{h}_{j_\beta i} = 0 && \hat{\textcircled{H}}{}^{h\gamma}_{ji} = \frac{1}{2}\Omega^{\cdot\cdot h\gamma}_{ji} = -\frac{1}{2}R_{jia}{}^{h}\,\overline{\underline{X}}{}^{a}_{(\gamma)}\\
&\hat{\textcircled{H}}{}^{h}_{ji_\alpha} = 0 && \hat{\textcircled{H}}{}^{h\gamma}_{j_\beta i} = 0\\
&\hat{\textcircled{H}}{}^{h\gamma}_{j_\beta i_\alpha} = 0 && \hat{\textcircled{H}}{}^{h\gamma}_{ji_\alpha} = \delta^{\gamma}_{\alpha}\Gamma^{h}_{ji}
\end{aligned}
}
$$

It is easily seen that (5.12) satisfies the relation (5.2).

We now find a necessary and sufficient condition that $\nabla_\lambda(\mathbb{H})^\mu_\nu = 0$.

where

(5.13) $$\nabla_\lambda(\mathbb{H})^\mu_\nu = \cancelto{0}{X_\lambda(\mathbb{H})^\mu_\nu} + \hat{\textcircled{H}}^\mu_{\lambda\omega}(\mathbb{H})^\omega_\nu - \hat{\textcircled{H}}^\omega_{\lambda\nu}(\mathbb{H})^\mu_\omega .$$

Now

$$\begin{aligned}\nabla_j(\mathbb{H})^h_i &= \hat{\textcircled{H}}^h_{j\omega}(\mathbb{H})^\omega_i - \hat{\textcircled{H}}^\omega_{ji}(\mathbb{H})^h_\omega \\ &= \hat{\textcircled{H}}^h_{j\omega}\cdot 0 - \hat{\textcircled{H}}^\omega_{ji}\cdot 0 = 0\end{aligned}$$

$$\begin{aligned}\nabla_j(\mathbb{H})^h_{i_\alpha} &= \hat{\textcircled{H}}^h_{j\omega}(\mathbb{H})^\omega_{i_\alpha} - \hat{\textcircled{H}}^\omega_{ji_\alpha}(\mathbb{H})^h_\omega \\ &= \hat{\textcircled{H}}^h_{jk_\beta}(\mathbb{H})^{k_\beta}_{i_\alpha} - \hat{\textcircled{H}}^\omega_{ji_\alpha}\cdot 0 \\ &= 0\cdot(\mathbb{H})^{k_\beta}_{i_\alpha} = 0\end{aligned}$$

$$\begin{aligned}\nabla_{j_\beta}(\mathbb{H})^h_i &= \hat{\textcircled{H}}^h_{j_\beta\omega}(\mathbb{H})^\omega_i - \hat{\textcircled{H}}^\omega_{j_\beta i}(\mathbb{H})^h_\omega \\ &= \hat{\textcircled{H}}^h_{j_\beta\omega}\, 0 - \hat{\textcircled{H}}^\omega_{j_\beta i}\, 0 = 0\end{aligned}$$

$$\begin{aligned}\nabla_{j_\beta}(\mathbb{H})^h_{i_\alpha} &= \hat{\textcircled{H}}^h_{j_\beta\omega}(\mathbb{H})^\omega_{i_\alpha} - \hat{\textcircled{H}}^\omega_{j_\beta i_\alpha}(\mathbb{H})^h_\omega \\ &= 0\cdot(\mathbb{H})^\omega_{i_\alpha} - \hat{\textcircled{H}}^\omega_{j_\beta i_\alpha}\cdot 0 = 0\end{aligned}$$

$$\begin{aligned}\nabla_j (\mathbb{H})^{h\gamma}_{i} &= \hat{\textcircled{H}}^{h\gamma}_{j\omega} (\mathbb{H})^{\omega}_{i} - \hat{\textcircled{H}}^{\omega}_{ji} (\mathbb{H})^{h\gamma}_{\omega} \\ &= \hat{\textcircled{H}}^{h\gamma}_{j\omega}\, 0 - \hat{\textcircled{H}}^{k_\beta}_{ji} (\mathbb{H})^{h\gamma}_{k_\beta} \\ &= \tfrac{1}{2} (\mathbb{H})^{h\gamma}_{k_\beta} R_{jia}{}^{k} X^{a}_{(\beta)} = \tfrac{1}{2} R_{jia}{}^{h} X^{a}_{(m+1-\gamma)}\end{aligned}$$

$$\begin{aligned}\nabla_{j_\beta} (\mathbb{H})^{h\gamma}_{i} &= \hat{\textcircled{H}}^{h\gamma}_{j_\beta\omega} (\mathbb{H})^{\omega}_{i} - \hat{\textcircled{H}}^{\omega}_{j_\beta i} (\mathbb{H})^{h\gamma}_{\omega} \\ &= \hat{\textcircled{H}}^{h\gamma}_{j_\beta\omega}\, 0 - \hat{\textcircled{H}}^{k_\alpha}_{j_\beta i} (\mathbb{H})^{h\gamma}_{k_\alpha} \\ &= -0\, (\mathbb{H})^{h\gamma}_{k_\alpha} = 0\end{aligned}$$

$$\begin{aligned}\nabla_j (\mathbb{H})^{h\gamma}_{i_\alpha} &= \hat{\textcircled{H}}^{h\gamma}_{j\omega} (\mathbb{H})^{\omega}_{i_\alpha} - \hat{\textcircled{H}}^{\omega}_{ji_\alpha} (\mathbb{H})^{h\gamma}_{\omega} \\ &= \hat{\textcircled{H}}^{h\gamma}_{jk_\beta} (\mathbb{H})^{k_\beta}_{i_\alpha} - \hat{\textcircled{H}}^{k_\beta}_{ji_\alpha} (\mathbb{H})^{h\gamma}_{k_\beta} \\ &= (\mathbb{H})^{k_\beta}_{i_\alpha} \delta^{\gamma}_{\beta} \Gamma^{h}_{jk} - (\mathbb{H})^{h\gamma}_{k_\beta} \delta^{\beta}_{\alpha} \Gamma^{k}_{ji} \\ &= (\mathbb{H})^{k_\gamma}_{i_\alpha} \Gamma^{h}_{jk} - (\mathbb{H})^{h\gamma}_{k_\alpha} \Gamma^{k}_{ji} \\ &= \begin{cases} 0 \cdot \Gamma^{h}_{jk} - 0\, \Gamma^{k}_{ji} = 0 & \text{if } \gamma+\alpha \neq m+1 \\ \delta^{k}_{i} \Gamma^{h}_{jk} - \delta^{h}_{k} \Gamma^{k}_{ji} = 0 & \text{if } \gamma+\alpha = m+1 \end{cases} \\ &= 0\end{aligned}$$

$$\begin{aligned}\nabla_{j\beta}(\mathbb{H})^{h\gamma}_{i\alpha} &= \hat{\textcircled{H}}{}^{h\gamma}_{j\beta\,\omega}(\mathbb{H})^{\omega}_{i\alpha} - \hat{\textcircled{H}}{}^{\omega}_{j\beta\, i\alpha}(\mathbb{H})^{h\gamma}_{\omega} \\ &= \hat{\textcircled{H}}{}^{h\gamma}_{j\beta\, k\delta}(\mathbb{H})^{k\delta}_{i\alpha} - \hat{\textcircled{H}}{}^{k\delta}_{j\beta\, i\alpha}(\mathbb{H})^{h\gamma}_{k\delta} \\ &= 0\cdot(\mathbb{H})^{k\delta}_{i\alpha} - 0\cdot(\mathbb{H})^{h\gamma}_{k\delta} = 0\end{aligned}$$

So we have:

$$\boxed{\begin{array}{ll} \nabla_j(\mathbb{H})^h_i = 0 & \nabla_j(\mathbb{H})^{h\gamma}_i = \frac{1}{2}R_{jia}{}^h\mathbb{X}^a_{(m+1-\gamma)} \\ \nabla_{j\beta}(\mathbb{H})^h_i = 0 & \nabla_{j\beta}(\mathbb{H})^{h\gamma}_i = 0 \\ \nabla_j(\mathbb{H})^h_{i\alpha} = 0 & \nabla_j(\mathbb{H})^{h\gamma}_{i\alpha} = 0 \\ \nabla_{j\beta}(\mathbb{H})^h_{i\alpha} = 0 & \nabla_{j\beta}(\mathbb{H})^{h\gamma}_{i\alpha} = 0 \end{array}} \tag{5.14}$$

from which we see that $\nabla_\lambda(\mathbb{H})^\mu_\nu = 0$ if and only if $R_{jia}{}^h = 0$ and thus we have the following:

Theorem 5.1: A necessary and sufficient condition that the covariant derivative of H vanish is that X_n be flat.

Next we determine the necessary and sufficient condition that the complete lift $\bar{v}^A$ of a vector field v^i in X_n be a parallel vector field in $T_m(X_n)$, that is, that $\nabla_\lambda(\bar{v})^\mu = 0$ where

$$\nabla_\lambda(\bar{v})^\mu = \mathbb{X}_\lambda(\bar{v})^\mu + \hat{\textcircled{H}}{}^{\mu}_{\lambda\nu}(\bar{v})^\nu \tag{5.15}$$

$$
\begin{aligned}
\nabla_j (\bar{v})^h &= X_j (\bar{v})^h + \hat{\textcircled{H}}{}^{h}_{j\nu} (\bar{v})^\nu \\
&= X_j v^h + \hat{\textcircled{H}}{}^{h}_{ji} v^i \\
&= \partial_j v^h - \Gamma^k_{ja} X^a_{(\alpha)} \cancelto{0}{\partial_{k_\alpha}} v^h + \Gamma^h_{ji} v^i \\
&= \partial_j v^h + \Gamma^h_{ji} v^i = \nabla_j v^h
\end{aligned}
$$

$$
\begin{aligned}
\nabla_j (\bar{v})^{h_\beta} &= X_j (\bar{v})^{h_\beta} + \hat{\textcircled{H}}{}^{h_\beta}_{j\nu} (\bar{v})^\nu \\
&= X_j (X^a_{(\beta)} \nabla_a v^h) + \hat{\textcircled{H}}{}^{h_\beta}_{ji} v^i + \hat{\textcircled{H}}{}^{h_\beta}_{ji_\alpha} (X^a_{(\alpha)} \nabla_a v^i) \\
&= X^a_{(\beta)} \partial_j (\nabla_a v^h) - \Gamma^k_{jb} X^b_{(\alpha)} \nabla_a v^h \partial_{k_\alpha} (X^a_{(\beta)}) \\
&\quad - \frac{1}{2} X^a_{(\beta)} R_{jia}{}^h v^i + \delta^\beta_\alpha \Gamma^h_{ji} X^a_{(\alpha)} \nabla_a v^i \\
&= X^a_{(\beta)} \partial_j (\nabla_a v^h) - \Gamma^k_{jb} X^b_{(\alpha)} \nabla_a v^h \delta^a_k \delta^\alpha_\beta \\
&\quad - \frac{1}{2} X^a_{(\beta)} R_{jia}{}^h v^i + X^a_{(\beta)} \Gamma^h_{ji} \nabla_a v^i \\
&= X^a_{(\beta)} \left\{ \partial_j (\nabla_a v^h) + \Gamma^h_{ji} \nabla_a v^i - \Gamma^k_{ja} \nabla_k v^h - \frac{1}{2} R_{jia}{}^h v^i \right\} \\
&= X^a_{(\beta)} \left\{ \nabla_j \nabla_a v^h - \frac{1}{2} R_{jia}{}^h v^i \right\} \\
&= \frac{1}{2} X^a_{(\beta)} \left\{ \underset{(v^i)}{\pounds} \Gamma^h_{ja} + \nabla_j \nabla_a v^h \right\}
\end{aligned}
$$

$$\begin{aligned}\nabla_{j\alpha}(\bar{v})^h &= X_{j\alpha}(\bar{v})^h + \hat{\textcircled{H}}{}^h_{j\alpha\nu}(\bar{v})^\nu \\ &= \cancelto{0}{X_{j\alpha}} v^h + 0\cdot(\bar{v})^\nu = 0\end{aligned}$$

$$\begin{aligned}\nabla_{j\alpha}(\bar{v})^{h_\beta} &= X_{j\alpha}(\bar{v})^{h_\beta} + \hat{\textcircled{H}}{}^{h_\beta}_{j\alpha\nu}(\bar{v})^\nu \\ &= X_{j\alpha}(X^a_{(\beta)}\nabla_a v^h) + 0\cdot(\bar{v})^\nu \\ &= (\nabla_a v^h)\delta^a_j\delta^\beta_\alpha = \delta^\beta_\alpha \nabla_j v^h\end{aligned}$$

Putting these together we get

$$\boxed{\begin{aligned}&\nabla_j(\bar{v})^h = \nabla_j v^h = \partial_j v^h + \Gamma^h_{ji} v^i \\ &\nabla_{j\alpha}(\bar{v})^h = 0 \\ &\nabla_j(\bar{v})^{h_\beta} = \frac{1}{2}\left\{\nabla_j\nabla_a v^h + \underset{(v^i)}{\pounds}\Gamma^h_{ja}\right\} X^a_{(\beta)} \\ &\nabla_{j\alpha}(\bar{v})^{h_\beta} = \delta^\beta_\alpha \nabla_j v^h\end{aligned}} \tag{5.16}$$

from which we see that $\nabla_\mu(\bar{v})^\lambda = 0$ if and only if $\nabla_j v^h = 0$ and $\underset{(v^a)}{\pounds}\Gamma^h_{ji} = 0$ and so we have:

Theorem 5.2. A necessary and sufficient condition that the extended lift $\bar{v}^A$ of v^i be a parallel vector field in $T_m(X_n)$ is that v^i be a parallel vector field in X_n and define there an affine collineation. (5)

CHAPTER II

THE TANGENT BUNDLE $T(X) \equiv T_1(X_n)$

1. The existence of an almost complex structure F, $F^2 = -E$ for $T(X_n)$.

In this case the letters $X^h_{(\alpha)}$, $\alpha=1,2,\dots,m$ of the preceding chapter are reduced to a single letter X^h, and the Greek letters $\lambda, \mu, \nu, \rho, \tau, \omega$ now run from 1 to 2n. On writing $u^h = \xi^h$, $X^h = \xi^{h*}$ we have the point transformation of $T(X_n)$ as

$$\left\{ \begin{array}{l} \xi^{h'} = \xi^{h'}(\xi^h) \\ \xi^{h*'} = A^{h'}_h \xi^{h*} \quad \text{where } A^{h'}_h = \partial_h \xi^{h'} \end{array} \right. \tag{1.1}$$

which we write more simply as $\xi^{A'} = f^{A'}(\xi^A)$ where the capital Roman letters A,B,C,... take the values $1,\dots,n,n+1,\dots,2n$. Also

$$\frac{\partial \xi^{h*'}}{\partial \xi^k} = \frac{\partial}{\partial \xi^k}\left(A^{h'}_h \xi^{h*}\right) = A^{h'}_{kh}\xi^{h*} \text{ where } A^{h'}_{kh} = \frac{\partial A^{h'}_h}{\partial \xi^k} = \partial_k A^{h'}_h.$$

And $\dfrac{\partial \xi^{h*'}}{\partial \xi^{k*}} = A^{h'}_k$

so that

$$\left(\frac{\partial \xi^{A'}}{\partial \xi^A}\right) = \begin{pmatrix} \dfrac{\partial \xi^{h'}}{\partial \xi^h} & \dfrac{\partial \xi^{h'}}{\partial \xi^{h*}} \\ \dfrac{\partial \xi^{h*'}}{\partial \xi^h} & \dfrac{\partial \xi^{h*'}}{\partial \xi^{h*}} \end{pmatrix} = \begin{pmatrix} A^{h'}_h & 0 \\ A^{h'}_{jh}\xi^{j*} & A^{h'}_h \end{pmatrix} \tag{1.2}$$

from which

$$\det\left[\frac{\partial \xi^{A'}}{\partial \xi^A}\right] = \left(\det\left[A^{h'}_h\right]\right)^2 > 0$$

and thus the tangent bundle $T(X_n)$ is always orientable (5, p214).

Definition 1.1. (4),(5) A 2n-dimensional differentiable manifold M_{2n} of class C^r, $r \geq 2$, admitting a tensor field F^A_B of type (1,1) satisfying $F^A_B F^B_C = -\delta^A_C$ is said to be an almost complex space and the tensor field F^A_B is said to assign to the manifold an almost complex structure. A vector field V^A in M_{2n} is said to be almost analytic if $\underset{(V^c)}{\mathcal{L}} F^A_B = 0$ where $\mathcal{L}$ is the operator of Lie derivation.

When the base differentiable manifold of class C^r, $(r \geq 4)$, is Riemannian, S. Tachibana (4) has shown that the tangent bundle $T(X_n)$ admits an almost complex structure F, $F^2 = -E$; this structure was also derived by K, Yano and E. T. Davies (7) by the use of the one form $\omega = g_{ij} \overline{X}^j d\xi^i$. We shall seek now an almost complex structure for an affine tangent bundle.

In this case the adapted frames $A_\lambda^{\cdot A}$ are composed of n vectors $B_j^{\cdot A}$ and n vectors $C_{j*}^{\cdot A}$ having components

$$(1.3) \qquad B_j^{\cdot A} = (\delta^h_j, -\Gamma^h_j), \quad C_{j*}^{\cdot A} = (0, \delta^h_j)$$

where we have written $\Gamma^h_j \overset{def}{=} X^i \Gamma^h_{ij} = \xi^{i*} \Gamma^h_{ij}$. The dual $A^\lambda_{\cdot A}$ will therefore be composed of vectors $B^h_{\cdot A}$ and $C^{h*}_{\cdot A}$ having components

$$(1.4) \qquad B^h_{\cdot A} = (\delta^h_j, 0), \quad C^{h*}_{\cdot A} = (\Gamma^h_j, \delta^h_j).$$

Theorem 1.1. At each point (ξ^A) of $T(X_n)$ there exists a field of quantities F^A_B satisfying

(i) $\quad F^A_B C_{j*}^{\cdot B} = B_j^{\cdot A}$

(ii) $\quad F^A_B F^B_C = -\delta^A_C$

Proof:

Now

$$F^A_B = \begin{pmatrix} F^h_i & F^h_{i*} \\ F^{h*}_i & F^{h*}_{i*} \end{pmatrix} \qquad C_{j*}^{\ A} = \begin{pmatrix} C_{j*}^{\ i} \\ C_{j*}^{\ i*} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta^i_j \end{pmatrix}$$

Thus

$$F^A_B C_{j*}^{\ B} = \begin{pmatrix} F^h_i & F^h_{i*} \\ F^{h*}_i & F^{h*}_{i*} \end{pmatrix} \begin{pmatrix} 0 \\ \delta^i_j \end{pmatrix} = \begin{pmatrix} F^h_{j*} \\ F^{h*}_{j*} \end{pmatrix}$$

From condition (i) we must have

$$\begin{pmatrix} F^h_{j*} \\ F^{h*}_{j*} \end{pmatrix} = \begin{pmatrix} B_j^{\cdot h} \\ B_j^{\cdot h*} \end{pmatrix} = \begin{pmatrix} \delta^h_j \\ -\Gamma^h_j \end{pmatrix} \quad \text{implying that}$$

$$(1.5) \begin{cases} F^h_{j*} = \delta^h_j \\ F^{h*}_{j*} = -\Gamma^h_j \end{cases} \quad \text{Thus} \quad F^A_B = \begin{pmatrix} F^h_i & \delta^h_i \\ F^{h*}_i & -\Gamma^h_i \end{pmatrix}.$$

So condition (ii) takes the form

$$\begin{pmatrix} F^h_i & \delta^h_i \\ F^{h*}_i & -\Gamma^h_i \end{pmatrix}\begin{pmatrix} F^i_j & \delta^i_j \\ F^{i*}_j & -\Gamma^i_j \end{pmatrix} = \begin{pmatrix} -\delta^h_j & 0 \\ 0 & -\delta^h_j \end{pmatrix}$$

$$\left(\begin{array}{c:c} F^h_i F^i_j + F^{h*}_j & F^h_j - \Gamma^h_j \\ \hdashline F^{h*}_i F^i_j - \Gamma^h_i F^{i*}_j & F^{h*}_j + \Gamma^h_i \Gamma^i_j \end{array}\right) = \begin{pmatrix} -\delta^h_j & 0 \\ 0 & -\delta^h_j \end{pmatrix}$$

Thus we obtain

(a) $F^h_i F^i_j + F^{h*}_j = -\delta^h_j$

(b) $F^{h*}_i F^i_j - \Gamma^h_i F^{i*}_j = 0$

(c) $F^h_j - \Gamma^h_j = 0$

(d) $F^{h*}_j + \Gamma^h_i \Gamma^i_j = -\delta^h_j$

From (c) and (d) we get

(1.6) $F^h_j = \Gamma^h_j \quad , \quad F^{h*}_j = -\delta^h_j - \Gamma^h_i \Gamma^i_j$

and clearly these satisfy (a) and (b). Combining (1.5) and (1.6) we get

$$(1.7)\quad \boxed{F^A_B = \begin{pmatrix} \Gamma^h_j & \delta^h_j \\ -\delta^h_j - \Gamma^h_a \Gamma^a_j & -\Gamma^h_j \end{pmatrix}}$$

Theorem 1.2. The field of quantities (1.7) described in Theorem 1.1 is a tensor field globally defined on $T(X_n)$.

Proof: For convenience of notation in this rather tedious proof we write $\dot{\xi}^h$ in place of ξ^{h*}. We must show that

$$(1.8)\quad F^{A'}_{B'} = \frac{\partial \xi^{A'}}{\partial \xi^A} \frac{\partial \xi^B}{\partial \xi^{B'}} F^A_B$$

where $\xi^{A'} = \xi^{A'}(\xi)$ is a coordinate transformation of the form (1.1) Now the inverse transormation $\xi^A = \xi^A(\xi^{A'})$ can be written

$$(1.9)\quad \begin{cases} \xi^h = \xi^h(\xi^{h'}) \\ \dot{\xi}^h = A^h{}_{h'} \dot{\xi}^{h'} \text{ where } A^h_{h'} = \partial_{h'} \xi^h \end{cases}$$

So from (1.2) we have

$$(1.10)\quad \left(\frac{\partial \xi^A}{\partial \xi^{A'}}\right) = \begin{pmatrix} A^h{}_{h'} & 0 \\ A^h_{j'h'} \dot{\xi}^{j'} & A^h{}_{h'} \end{pmatrix}$$

First consider $F^A_B \dfrac{\partial \xi^B}{\partial \xi^{B'}}$ From (1.7) and (1.10) we have

$$F^A_B \frac{\partial \xi^B}{\partial \xi^{B'}} = \left(\begin{array}{c|c} \Gamma^h_a & \delta^h_a \\ \hline -\delta^h_a - \Gamma^h_b \Gamma^b_a & -\Gamma^h_a \end{array}\right) \begin{pmatrix} A^a_{h'} & 0 \\ A^a_{b'h'} \dot{\xi}^{b'} & A^a_{h'} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \Gamma^h_a A^a_{h'} + A^h_{b'h'} \dot{\xi}^{b'} & A^h_{h'} \\ \hline -A^h_{h'} - \Gamma^h_b \Gamma^b_a A^a_{h'} - \Gamma^h_a A^a_{b'h'} \dot{\xi}^{b'} & -\Gamma^h_a A^a_{h'} \end{array}\right)$$

Recall that $A^{h'}_a A^a_{j'} = \delta^{h'}_{j'}$ and consider

$$\frac{\partial \xi^{A'}}{\partial \xi^A} F^A_B \frac{\partial \xi^B}{\partial \xi^{B'}} =$$

$$\left(\begin{array}{c|c} A^{h'}_j & 0 \\ \hline A^{h'}_{cj} \dot{\xi}^c & A^{h'}_j \end{array}\right) \left(\begin{array}{c|c} \Gamma^j_a A^a_{j'} + A^j_{b'j'} \dot{\xi}^{b'} & A^j_{j'} \\ \hline -A^j_{j'} - \Gamma^j_b \Gamma^b_a A^a_{j'} - \Gamma^j_a A^a_{b'j'} \dot{\xi}^{b'} & -\Gamma^j_a A^a_{j'} \end{array}\right)$$

$$\frac{\partial \bar{\xi}^{A'}}{\partial \xi^{A}} F^{A}_{B} \frac{\partial \xi^{B}}{\partial \xi^{B'}} =$$

$$\left(\begin{array}{c|c} A^{h'}_{j}\Gamma^{j}_{a}A^{a}_{j'} + A^{h'}_{j}A^{j}_{b'j'}\dot{\xi}^{b'} & \delta^{h'}_{j'} \\ \hline A^{h'}_{cj}\dot{\xi}^{c}\Gamma^{j}_{a}A^{a}_{j'} + A^{h'}_{cj}\dot{\xi}^{c}A^{j}_{b'j'}\dot{\xi}^{b'} - \delta^{h'}_{j'} - A^{h'}_{j}\Gamma^{j}_{b}\Gamma^{b}_{a}A^{a}_{j'} - A^{h'}_{j}\Gamma^{j}_{a}A^{a}_{b'j'}\dot{\xi}^{b'} & A^{h'}_{cj}\dot{\xi}^{c}A^{j}_{j'} - A^{h'}_{j}\Gamma^{j}_{a}A^{a}_{j'} \end{array}\right)$$

Now by (1.7) we must show that the above is equal to

$$F^{A'}_{B'} = \left(\begin{array}{c|c} \Gamma^{h'}_{j'} & \delta^{h'}_{j'} \\ \hline -\delta^{h'}_{j'} - \Gamma^{h'}_{a'}\Gamma^{a'}_{j} & -\Gamma^{h'}_{j'} \end{array}\right)$$

So we must show

(1.11) $$\Gamma^{h'}_{j'} = A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{j'} + A^{h'}_{a}A^{a}_{b'j'}\dot{\xi}^{b'}$$

(1.12) $$-\Gamma^{h'}_{j'} = A^{h'}_{ab}\dot{\xi}^{a}A^{b}_{j'} - A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{j'}$$

(1.13) $$-\delta^{h'}_{j'} - \Gamma^{h'}_{a'}\Gamma^{a'}_{j'} = A^{h'}_{ab}\dot{\xi}^{a}\Gamma^{b}_{c}A^{c}_{j'} + A^{h'}_{ac}\dot{\xi}^{a}A^{c}_{b'j'}\dot{\xi}^{b'} - \delta^{h'}_{j'}$$
$$- A^{h'}_{c}\Gamma^{c}_{a}\Gamma^{a}_{b}A^{b}_{j'} - A^{h'}_{c}\Gamma^{c}_{a}A^{a}_{b'j'}\dot{\xi}^{b'}$$

and (1.13) is equivalent to

(1.14) $$\Gamma^{h'}_{a'}\Gamma^{a'}_{j'} = A^{h'}_{c}\Gamma^{c}_{a}\Gamma^{a}_{b}A^{b}_{j'} + A^{h'}_{c}\Gamma^{c}_{a}A^{a}_{b'j'}\dot{\xi}^{b'}$$
$$- A^{h'}_{ab}\dot{\xi}^{a}\Gamma^{b}_{c}A^{c}_{j'} - A^{h'}_{ac}\dot{\xi}^{a}A^{c}_{b'j'}\dot{\xi}^{b'}$$

We first verify (1.11). Since Γ^{h}_{ji} is a connection we have

$$\Gamma^{h'}_{i'j'} = \frac{\partial\xi^{h'}}{\partial\xi^{a}}\left\{\frac{\partial\xi^{b}}{\partial\xi^{i'}}\frac{\partial\xi^{c}}{\partial\xi^{j'}}\Gamma^{a}_{bc} + \frac{\partial^{2}\xi^{a}}{\partial\xi^{i'}\partial\xi^{j'}}\right\}$$

and since $\dot{\xi}^{i'} = A^{i'}_{d}\dot{\xi}^{d}$ we have

$$\Gamma^{h'}_{j'} = \dot{\xi}^{i'}\Gamma^{h'}_{i'j'} = A^{i'}_{d}\dot{\xi}^{d}A^{h'}_{a}\left\{A^{b}_{i'}A^{c}_{j'}\Gamma^{a}_{bc} + A^{a}_{i'j'}\right\}$$
$$= A^{h'}_{a}A^{c}_{j'}\dot{\xi}^{b}\Gamma^{a}_{bc} + A^{h'}_{a}A^{a}_{b'j'}\dot{\xi}^{b'}$$

So

$$\Gamma^{h'}_{j'} = A^{h'}_{a}\Gamma^{a}_{c}A^{c}_{j'} + A^{h'}_{a}A^{a}_{b'j'}\dot{\xi}^{b'} \qquad \text{since} \qquad \Gamma^{a}_{c} = \dot{\xi}^{b}\Gamma^{a}_{bc}$$

which proves (1.11). Now

$$\frac{\partial \xi^{A'}}{\partial \xi^{B'}} = \frac{\partial \xi^{A'}}{\partial \xi^{A}}\frac{\partial \xi^{A}}{\partial \xi^{B'}} = \delta^{A'}_{B'}$$

In particular

$$\frac{\partial \dot{\xi}^{h'}}{\partial \xi^{a}}\frac{\partial \xi^{a}}{\partial \xi^{j'}} + \frac{\partial \dot{\xi}^{h'}}{\partial \dot{\xi}^{b}}\frac{\partial \dot{\xi}^{b}}{\partial \xi^{j'}} = \delta^{n+h}_{j} = 0$$

or

$$A^{h'}_{ba}\dot{\xi}^{b}A^{a}_{j'} + A^{h'}_{b}A^{b}_{a'j'}\dot{\xi}^{a'} = 0$$

which yields

(1.15) $$A^{h'}_{a}A^{a}_{b'j'}\dot{\xi}^{b'} = -A^{h'}_{ab}\dot{\xi}^{a}A^{b}_{j'}$$

Substituting (1.15) into (1.11) we obtain

$$\Gamma^{h'}_{j'} = A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{j'} - A^{h'}_{ab}\dot{\xi}^{a}A^{b}_{j'}$$

$$-\Gamma^{h'}_{j'} = A^{h'}_{ab}\dot{\xi}^{a}A^{b}_{j'} - A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{j'}$$

which proves (1.12). It remains to verify (1.14). From (1.11) we obtain

$$
\begin{aligned}
(1.16)\quad \Gamma^{h'}_{k'}\Gamma^{k'}_{j'} &= (A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{k'} + A^{h'}_{a}A^{a}_{b'k'}\dot{\xi}^{b})(A^{k'}_{d}\Gamma^{d}_{c}A^{c}_{j'} + A^{k'}_{d}A^{d}_{e'j'}\dot{\xi}^{e'}) \\
&= A^{h'}_{b}\boxed{A^{a}_{k'}A^{k'}_{d}}A^{c}_{j'}\Gamma^{b}_{a}\Gamma^{d}_{c} + A^{h'}_{a}A^{c}_{j'}A^{k'}_{d}\Gamma^{d}_{c}A^{a}_{b'k'}\dot{\xi}^{b'} \\
&\quad + A^{h'}_{b}\boxed{A^{a}_{k'}A^{k'}_{d}}\Gamma^{b}_{a}A^{d}_{e'j'}\dot{\xi}^{e'} + A^{h'}_{a}A^{k'}_{d}A^{a}_{b'k'}\dot{\xi}^{b'}A^{d}_{e'j'}\dot{\xi}^{e'} \\
&= A^{h'}_{b}A^{c}_{j'}\Gamma^{b}_{a}\Gamma^{a}_{c} + A^{h'}_{b}\Gamma^{b}_{a}A^{a}_{e'j'}\dot{\xi}^{e'} \\
&\quad + A^{h'}_{a}A^{a}_{b'k'}\dot{\xi}^{b'}A^{k'}_{d}A^{c}_{j'}\Gamma^{d}_{c} + A^{h'}_{a}A^{a}_{b'k'}\dot{\xi}^{b'}A^{k'}_{d}A^{d}_{e'j'}\dot{\xi}^{e'}
\end{aligned}
$$

We see that the first two terms agree with the first two terms of (1.14).

Now from (1.15) we get

$$
(1.17)\quad \left\{
\begin{aligned}
A^{h'}_{a}A^{a}_{b'k'}\dot{\xi}^{b'}A^{k'}_{d}A^{c}_{j}\Gamma^{d}_{c} &= -A^{h'}_{ab}\dot{\xi}^{a}\boxed{A^{b}_{k'}A^{k'}_{d}}A^{c}_{j'}\Gamma^{d}_{c} \\
&= -A^{h'}_{ab}\dot{\xi}^{a}A^{c}_{j'}\Gamma^{b}_{c} \\
A^{h'}_{a}A^{a}_{b'k'}\dot{\xi}^{b'}A^{k'}_{d}A^{d}_{e'j'}\dot{\xi}^{e'} &= -A^{h'}_{ab}\dot{\xi}^{a}\boxed{A^{b}_{k'}A^{k'}_{d}}A^{d}_{e'j'}\dot{\xi}^{e'}. \\
&= -A^{h'}_{ab}\dot{\xi}^{a}A^{b}_{e'j'}\dot{\xi}^{e'}.
\end{aligned}
\right.
$$

Substituting (1.17) into (1.16) yields (1.14) and completes the proof of the tensor character of F^{A}_{B}.

Let (F) denote the components of F with respect to the adapted frames, that is, $(F)_\mu^{\ \lambda} = A_\mu^{\ B} A^\lambda_{\ A} F^A_B$. Then

$$\begin{aligned}(F)_j^{\ h} &= B_j^{\ B} B^h_{\cdot A} F^A_B = B_j^{\ B} \delta^h_k F^k_B = B_j^{\ B} F^h_B \\ &= \delta^k_j F^h_k - \Gamma^k_j F^h_{k*} = F^h_j - \Gamma^k_j \delta^h_k \\ &= \Gamma^h_j - \Gamma^h_j = 0\end{aligned}$$

$$\begin{aligned}(F)^h_{j*} &= C_{j*}^{\ B} B^h_A F^A_B = C_{j*}^{\ B} \delta^h_k F^k_B = C_{j*}^{\ B} F^h_B \\ &= \delta^{k*}_{j*} F^h_{k*} = F^h_{j*} = \delta^h_j\end{aligned}$$

$$\begin{aligned}(F)_j^{\ h*} &= B_j^{\ B} C^{h*}_A F^A_B = B_j^{\ B} \Gamma^h_k F^k_B + B_j^{\ B} \delta^{h*}_{k*} F^{k*}_B \\ &= B_j^{\ B} \Gamma^h_k F^k_B + B_j^{\ B} F^{h*}_B = B_j^{\ B}(\Gamma^h_k F^k_B + F^{h*}_B) \\ &= \delta^i_j(\Gamma^h_k F^k_i + F^{h*}_i) - \Gamma^i_j(\Gamma^h_k F^k_{i*} + F^{h*}_{i*}) \\ &= \Gamma^h_k F^k_j + F^{h*}_j - \Gamma^i_j \Gamma^h_k \delta^k_i + \Gamma^i_j \Gamma^h_i \\ &= \cancel{\Gamma^h_k \Gamma^k_j} - \delta^h_j - \cancel{\Gamma^h_i \Gamma^i_j} - \cancel{\Gamma^h_i \Gamma^i_j} + \cancel{\Gamma^h_i \Gamma^i_j} \\ &= -\delta^h_j\end{aligned}$$

$$\begin{aligned}(F)^{h*}_{j*} &= C_{j*}^{\ B} C^{h*}_A F^A_B = \delta^{k*}_{j*} C^{h*}_A F^A_{k*} = C^{h*}_A F^A_{j*} \\ &= \Gamma^h_k F^k_{j*} + \delta^{h*}_{k*} F^{k*}_{j*} = \Gamma^h_k \delta^k_j + F^{h*}_{j*} \\ &= \Gamma^h_j - \Gamma^h_j = 0\end{aligned}$$

Thus

$$(F)^\lambda_\mu = \begin{pmatrix} (F)^h_j & (F)^h_{j^*} \\ (F)^{h^*}_j & (F)^{h^*}_{j^*} \end{pmatrix} = \begin{pmatrix} 0 & \delta^h_j \\ -\delta^h_j & 0 \end{pmatrix} \tag{1.18}$$

As in Chapter I we let $X_\lambda \overset{\text{def}}{=} A_\lambda^{\ A}\partial_A$ where $\partial_A = \frac{\partial}{\partial \xi^A}$ and the differentials $(d\xi)^\lambda \overset{\text{def}}{=} A^\lambda_{\ A}\, d\xi^A$ are dual to the X_λ. Again, the differentials $(d\xi)^\lambda$ are not in general exact, the condition of holonomity (involutiveness) being the vanishing of the quantity:

$$\Omega_{\mu\nu}^{\cdot\cdot\lambda} = A^\lambda_{\ A}(X_\mu A_\nu^{\ A} - X_\nu A_\mu^{\ A}) \tag{1.19}$$

From (2.20) in Chapter I we have that the components of $\Omega_{\mu\nu}^{\cdot\cdot\lambda}$ in this case are:

$$\begin{cases} \Omega_{ji}^{\ \ h} = \Omega_{j^*i}^{\ \ h} = \Omega_{ij^*}^{\ \ h} = \Omega_{j^*i^*}^{\ \ h} = \Omega_{j^*i^*}^{\ \ h^*} = 0 \\ \Omega_{ji}^{\ \ h^*} = X_i \Gamma^h_j - X_j \Gamma^h_i = -R_{jia}^{\ \ \ h}\xi^{a^*} \\ \Omega_{j^*i}^{\ \ h^*} = -\Gamma^h_{ji} \end{cases} \tag{1.20}$$

Let V^A be an arbitrary vector field in $T(X_n)$ and consider the infinitesimal deformation $\xi^{A'} = \xi^A + V^A(\xi)\delta t$. With respect to V^A the Lie derivative of our tensor field F^A_B is given by (6)

$$\underset{(V^D)}{\pounds} F^A_B = V^D\partial_D F^A_B - F^C_B\partial_C V^A + F^A_D\partial_B V^D \tag{1.21}$$

As before we let $(\pounds F)^\lambda_\mu = A^\lambda_{\ A} A_\mu^{\ B}(\pounds F^A_B)$ be the components of $\pounds F^A_B$

with respect to the adapted frames. Then the same calculation performed in Chapter I goes over and we have

(1.22) $$\boxed{\begin{aligned}\underset{(V^0)}{(\pounds F)}{}^{\lambda}_{\mu} = {} & (V)^{\omega} X_{\omega}(F)^{\lambda}_{\mu} - (F)^{\tau}_{\mu} X_{\tau}(V)^{\lambda} + (F)^{\lambda}_{\omega} X_{\mu}(V)^{\omega} \\ & + (V)^{\omega}(F)^{\tau}_{\mu}\,\Omega_{\omega\tau}{}^{\lambda} - (V)^{\omega}(F)^{\lambda}_{\rho}\,\Omega_{\omega\mu}{}^{\rho}\end{aligned}}$$

2. Necessary and sufficient conditions that the extended lift of a vector field in X_n be almost analytic.

For the case of a single tangent vector $\xi^{h*} = X^h$ as fibre, the entended lift $\bar{v}^A$ of a vector field v^h in X_n becomes (Chapter I, Definition 2.1)

(2.1) $$\bar{v}^A = (v^h,\ \xi^{a*}\partial_a v^h)$$

and from Chapter I (4.1) the components $(\bar{v})^{\lambda} = A^{\lambda}{}_{A}\,\bar{v}^A$ of $\bar{v}^A$ with respect to the adapted frames are

(2.2) $$(\bar{v})^{\lambda} = (v^h,\ \xi^{a*}\nabla_a v^h)$$

Taking now the extended lift $\bar{v}^A$ of v^h as the generator of an infinitesimal transformation we compute

$$\underset{(\bar{v}^A)}{(\pounds F)}{}^{\lambda}_{\mu} = \left\{ (\pounds F)^{h}_{j},\ (\pounds F)^{h}_{j*},\ (\pounds F)^{h*}_{j},\ (\pounds F)^{h*}_{j*} \right\}$$

Now by (1.22)

$$\begin{aligned}\underset{(\bar{v}^A)}{(\pounds F)}{}^{\lambda}_{\mu} = {} & (\bar{v})^{\omega} \cancelto{0}{X_{\omega}(F)^{\lambda}_{\mu}} - (F)^{\tau}_{\mu} X_{\tau}(\bar{v})^{\lambda} + (F)^{\lambda}_{\omega} X_{\mu}(\bar{v})^{\omega} \\ & + (\bar{v})^{\omega}(F)^{\tau}_{\mu}\,\Omega_{\omega\tau}{}^{\lambda} - (\bar{v})^{\omega}(F)^{\lambda}_{\rho}\,\Omega_{\omega\mu}{}^{\rho}\end{aligned}$$

where $(F)^{\lambda}_{\mu}$ is given by (1.18). So

$$
\begin{aligned}
(\pounds F)^h_j &= -(F)^\tau_j X_\lambda(\bar v)^h + (F)^h_\omega X_j(\bar v)^\omega \\
&\quad + (\bar v)^\omega \cancelto{0}{(F)^\tau_j \sqsupset\!\Omega\!\sqsubset_{\omega\tau}{}^h} - (\bar v)^\omega (F)^h_\rho \sqsupset\!\Omega\!\sqsubset_{\omega j}{}^\rho \\
&= \delta^{k*}_j X_{k*} v^h + \delta^h_k X_j (\bar v)^{k*} - (\bar v)^\omega \delta^h_k \sqsupset\!\Omega\!\sqsubset_{\omega j}{}^{k*} \\
&= \cancelto{0}{X_{j*}} v^h + X_j \xi^{a*} \nabla_a v^h - (\bar v)^\omega \sqsupset\!\Omega\!\sqsubset_{\omega j}{}^{h*} \\
&= \partial_j(\xi^{a*}\nabla_a v^h) - \Gamma^k_j \partial_{k*}(\xi^{a*}\nabla_a v^h) - v^i \sqsupset\!\Omega\!\sqsubset_{ij}{}^{h*} - (\bar v)^{i*} \sqsupset\!\Omega\!\sqsubset_{i*j}{}^{h*} \\
&= \xi^{a*}\partial_j(\nabla_a v^h) - \xi^{a*}\Gamma^k_{ja}\nabla_k v^h + v^i R_{ija}{}^h \xi^{a*} - \xi^{a*}\nabla_a v^i \Gamma^h_{ji} \\
&= \xi^{a*}\left\{\partial_j(\nabla_a v^h) + \nabla_a v^i \Gamma^h_{ji} - \nabla_k v^h \Gamma^k_{ja} + R_{ija}{}^h v^i\right\} \\
&= \xi^{a*}\left\{\nabla_j \nabla_a v^h + R_{ija}{}^h v^i\right\}
\end{aligned}
$$

Now (6,p9)

$$\underset{(v^i)}{\pounds}\Gamma^h_{ja} = \nabla_j \nabla_a v^h + R_{ija}{}^h v^i$$

So that

$$(2.3) \qquad \underset{(\bar v^D)}{(\pounds F)^h_j} = \left(\underset{(v^i)}{\pounds}\Gamma^h_{ja}\right)\xi^{a*}$$

Continuing

$$\underset{(\bar{v}^{D})}{(\pounds F)}{}^{h}_{j*} = -(F)^{\tau}_{j*}X_{\tau}(\bar{v})^{h} + (F)^{h}_{\omega}X_{j*}(\bar{v})^{\omega} + (\bar{v})^{\omega}(F)^{\tau}_{j*}\cancelto{0}{\sqsupset\!\Omega\!\sqsubset_{\omega\tau}{}^{h}}$$

$$-(\bar{v})^{\omega}(F)^{h}_{\rho}\sqsupset\!\Omega\!\sqsubset_{\omega j*}{}^{\rho}$$

$$= -\delta^{k}_{j}X_{k}v^{h} + \delta^{h}_{k}X_{j*}(\bar{v})^{k*} - (\bar{v})^{\omega}\delta^{h}_{k}\sqsupset\!\Omega\!\sqsubset_{\omega j*}{}^{k*}$$

$$= -X_{j}v^{h} + X_{j*}(\xi^{a*}\nabla_{a}v^{h}) - v^{i}\sqsupset\!\Omega\!\sqsubset_{ij*}{}^{h*} - (\bar{v})^{i*}\cancelto{0}{\sqsupset\!\Omega\!\sqsubset_{i*j*}{}^{h*}}$$

$$= -\partial_{j}v^{h} + \nabla_{j}v^{h} - v^{i}\Gamma^{h}_{ji}$$

$$= \nabla_{j}v^{h} - (\partial_{j}v^{h} + v^{i}\Gamma^{h}_{ji}) = 0$$ Thus

$$(2.4)\qquad \underset{(\bar{v}^{D})}{(\pounds F)}{}^{h}_{j*} = 0$$

Also

$$\underset{(\bar{v}^{D})}{(\pounds F)}{}^{h*}_{j} = -(F)^{\tau}_{j}X_{\tau}(\bar{v})^{h*} + (F)^{h*}_{\omega}X_{j}(\bar{v})^{\omega} + (\bar{v})^{\omega}(F)^{\tau}_{j}\sqsupset\!\Omega\!\sqsubset_{\omega\tau}{}^{h*}$$

$$-(\bar{v})^{\omega}(F)^{h*}_{\rho}\sqsupset\!\Omega\!\sqsubset_{\omega j}{}^{\rho}$$

$$= \delta^{k}_{j}X_{k*}(\xi^{a*}\nabla_{a}v^{h}) - \delta^{h}_{k}X_{j}v^{k} - (\bar{v})^{\omega}\delta^{k}_{j}\sqsupset\!\Omega\!\sqsubset_{\omega k*}{}^{h*}$$

$$+ (\bar{v})^{\omega}\delta^{h}_{k}\cancelto{0}{\sqsupset\!\Omega\!\sqsubset_{\omega j}{}^{k}}$$

$$= X_{j*}(\xi^{a*}\nabla_{a}v^{h}) - X_{j}v^{h} - (\bar{v})^{\omega}\sqsupset\!\Omega\!\sqsubset_{\omega j*}{}^{h*}$$

$$= \nabla_j v^h - \partial_j v^h - (\bar{v})^i \sqsubset\!\Omega\!\sqsupset_{ij*}{}^{h*} - (\bar{v})^{i*} \cancelto{0}{\sqsubset\!\Omega\!\sqsupset_{i*j*}{}^{h*}}$$

$$= v^i \Gamma^h_{ji} - v^i \Gamma^h_{ji} == 0$$ Thus

(2.5) $$\underset{(\bar{v}^D)}{(\pounds F)}{}_j{}^{h*} = 0$$

Finally

$$\underset{(\bar{v}^D)}{(\pounds F)}{}_{j*}{}^{h*} = -(F)_{j*}{}^{\tau} \overline{\underline{X}}_\tau (\bar{v})^{h*} + (F)_\omega{}^{h*} \overline{\underline{X}}_{j*} (\bar{v})^\omega + (\bar{v})^\omega (F)_{j*}{}^{\tau} \sqsubset\!\Omega\!\sqsupset_{\omega\tau}{}^{h*}$$

$$- (\bar{v})^\omega (F)_\rho{}^{h*} \sqsubset\!\Omega\!\sqsupset_{\omega j*}{}^{\rho}$$

$$= -\delta_j^k \overline{\underline{X}}_k (\xi^{a*} \nabla_a v^h) - \delta_k^h \overline{\underline{X}}_{j*} v^k + (\bar{v})^\omega \delta_j^k \sqsubset\!\Omega\!\sqsupset_{\omega k}{}^{h*}$$

$$+ (\bar{v})^\omega \delta_k^h \cancelto{0}{\sqsubset\!\Omega\!\sqsupset_{\omega j*}{}^{k}}$$

$$= -\overline{\underline{X}}_j (\xi^{a*} \nabla_a v^h) - \cancelto{0}{\overline{\underline{X}}_{j*}} v^h + (\bar{v})^\omega \sqsubset\!\Omega\!\sqsupset_{\omega j}{}^{h*}$$

$$= -\partial_j (\xi^{a*} \nabla_a v^h) + \Gamma_j^k \partial_{k*} (\xi^{a*} \nabla_a v^h) + v^i \sqsubset\!\Omega\!\sqsupset_{ij}{}^{h*}$$

$$+ \xi^{a*} \nabla_a v^i \sqsubset\!\Omega\!\sqsupset_{i*j}{}^{h*}$$

$$= -\xi^{a*} \partial_j (\nabla_a v^h) + \xi^{a*} \Gamma^k_{ja} \nabla_k v^h - v^i R_{ija}{}^h \xi^{a*}$$

$$- \xi^{a*} \nabla_a v^i \Gamma^h_{ji}$$

$$= -\xi^{a*}\{\partial_j(\nabla_a v^h) + \nabla_a v^i\,\Gamma_{ji}^h - \nabla_k v^h\,\Gamma_{ja}^k + v^i R_{ija}{}^h\}$$

$$= -\xi^{a*}\{\nabla_j\nabla_a v^h + v^i R_{ija}{}^h\} = -\xi^{a*}(\underset{(v^i)}{\pounds}\Gamma_{ja}^h) \quad \text{Thus}$$

$$(2.6) \quad (\underset{(\bar{v}^D)}{\pounds} F)_{j*}^{h*} = -(\underset{(v^i)}{\pounds}\Gamma_{ja}^h)\xi^{a*}$$

So

$$(2.7) \quad \boxed{\begin{array}{ll} (\underset{(\bar{v}^D)}{\pounds} F)_j^h = (\underset{(v^i)}{\pounds}\Gamma_{ja}^h)\xi^{a*}, & (\underset{(\bar{v}^D)}{\pounds} F)_j^{h*} = 0 \\ (\underset{(\bar{v}^D)}{\pounds} F)_{j*}^h = 0 & (\underset{(\bar{v}^D)}{\pounds} F)_{j*}^{h*} = -(\underset{(v^i)}{\pounds}\Gamma_{ja}^h)\xi^{a*} \end{array}}$$

Now from (2.7) we see that $(\underset{(\bar{v}^D)}{\pounds} F)_\mu^\lambda = 0$ if and only if $\underset{(v^h)}{\pounds}\Gamma_{ji}^h = 0$ Moreover from the relation $(\underset{(\bar{v}^D)}{\pounds} F)_\mu^\lambda = A^\lambda{}_A A_\mu{}^B \pounds F_B^A$ we see that $\underset{(\bar{v}^D)}{\pounds} F_B^A = 0$ if and only if $(\underset{(\bar{v}^D)}{\pounds} F)_\mu^\lambda = 0$. So we have the following theorem:

Theorem 2.1. A necessary and sufficient condition that the extended lift $\bar{v}^A$ of v^i be an almost analytic vector field in T(X_n) is that v^i define an affine collineation in X_n. (i.e., that $\underset{(v^h)}{\pounds}\Gamma_{ji}^h = 0$)

3. Introduction of a symmetric affine connection $\textcircled{\scriptsize H}{}^A_{BC}$ into T(X_n).

We now introduce an affine connection $\textcircled{\scriptsize H}{}^A_{BC}$ on the same principle as that used in Chapter I, obtaining from (5.12) the following components with respect to the adapted frames:

$$
(3.1)\quad \begin{cases}
\hat{\textcircled{H}}{}^{h}_{ji} = \Gamma^{h}_{ji} \\
\hat{\textcircled{H}}{}^{h}_{j^*i} = \hat{\textcircled{H}}{}^{h}_{ji^*} = \hat{\textcircled{H}}{}^{h}_{j^*i^*} = 0 \\
\hat{\textcircled{H}}{}^{h^*}_{ji} = -\frac{1}{2} R_{jia}{}^{h} \xi^{a^*} \\
\hat{\textcircled{H}}{}^{h^*}_{j^*i} = \hat{\textcircled{H}}{}^{h^*}_{j^*i^*} = 0 \\
\hat{\textcircled{H}}{}^{h^*}_{ji^*} = \Gamma^{h}_{ji}
\end{cases}
$$

We now find the necessary and sufficient conditions that $\textcircled{H}{}^{A}_{BC}$ be an F-connection, that is, that $\nabla_\mu (F)^\lambda_\nu = 0$ where

$$
(3.2)\quad \nabla_\mu (F)^\lambda_\nu = \cancelto{0}{\partial_\mu (F)^\lambda_\nu} + \hat{\textcircled{H}}{}^{\lambda}_{\mu\omega} (F)^\omega_\nu - \hat{\textcircled{H}}{}^{\omega}_{\mu\nu} (F)^\lambda_\omega
$$

$$
\begin{aligned}
\nabla_j (F)^h_i &= \hat{\textcircled{H}}{}^{h}_{j\omega} (F)^\omega_i - \hat{\textcircled{H}}{}^{\omega}_{ji} (F)^h_\omega = -\hat{\textcircled{H}}{}^{h}_{jk^*} \delta^k_i - \hat{\textcircled{H}}{}^{k^*}_{ji} \delta^h_k \\
&= -\hat{\textcircled{H}}{}^{h}_{ji^*} - \hat{\textcircled{H}}{}^{h^*}_{ji} = -0 + \frac{1}{2} R_{jia}{}^{h} \xi^{a^*} \\
&= \underline{\frac{1}{2} R_{jia}{}^{h} \xi^{a^*}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_j (F)^h_{i^*} &= \hat{\textcircled{H}}{}^{h}_{j\omega} (F)^\omega_{i^*} - \hat{\textcircled{H}}{}^{\omega}_{ji^*} (F)^h_\omega = -\hat{\textcircled{H}}{}^{h}_{jk} \delta^k_i - \hat{\textcircled{H}}{}^{k^*}_{ji^*} \delta^h_k \\
&= \hat{\textcircled{H}}{}^{h}_{ji} - \hat{\textcircled{H}}{}^{h^*}_{ji^*} = \Gamma^h_{ji} - \Gamma^h_{ji} \\
&= \underline{0}
\end{aligned}
$$

$$\nabla_j (F)_i^{h*} = \hat{\textcircled{H}}{}^{h*}_{j\omega}(F)^{\omega}_{i} - \hat{\textcircled{H}}{}^{\omega}_{ji}(F)^{h*}_{\omega} = -\hat{\textcircled{H}}{}^{h*}_{jk*}\delta^k_i + \hat{\textcircled{H}}{}^{k}_{ji}\delta^h_k$$

$$= -\hat{\textcircled{H}}{}^{h*}_{ji*} + \hat{\textcircled{H}}{}^{h}_{ji} = -\Gamma^h_{ji} + \Gamma^h_{ji} = 0$$

$$\nabla_j (F)_{i*}^{h*} = \hat{\textcircled{H}}{}^{h*}_{j\omega}(F)^{\omega}_{i*} - \hat{\textcircled{H}}{}^{\omega}_{ji*}(F)^{h*}_{\omega} = \hat{\textcircled{H}}{}^{h*}_{jk}\delta^k_i + \hat{\textcircled{H}}{}^{k}_{ji*}\delta^h_k$$

$$= \hat{\textcircled{H}}{}^{h*}_{ji} + \hat{\textcircled{H}}{}^{h}_{ji*} = -\frac{1}{2} R_{jia}{}^{h}\xi^{a*} + 0 = -\frac{1}{2} R_{jia}{}^{h}\xi^{a*}$$

$$\nabla_{j*} (F)_i^{h} = \hat{\textcircled{H}}{}^{h}_{j*\omega}(F)^{\omega}_{i} - \hat{\textcircled{H}}{}^{\omega}_{j*i}(F)^{h}_{\omega} = -\hat{\textcircled{H}}{}^{h}_{j*k*}\delta^k_i - \hat{\textcircled{H}}{}^{k*}_{j*i}\delta^h_k$$

$$= -\hat{\textcircled{H}}{}^{h}_{j*i*} - \hat{\textcircled{H}}{}^{h*}_{j*i} = -0 - 0 = 0$$

$$\nabla_{j*} (F)_{i*}^{h} = \hat{\textcircled{H}}{}^{h}_{j*\omega}(F)^{\omega}_{i*} - \hat{\textcircled{H}}{}^{\omega}_{j*i*}(F)^{h}_{\omega} = \hat{\textcircled{H}}{}^{h}_{j*k}\delta^k_i - \hat{\textcircled{H}}{}^{k*}_{j*i*}\delta^h_k$$

$$= \hat{\textcircled{H}}{}^{h}_{j*i*} - \hat{\textcircled{H}}{}^{h*}_{j*i*} = 0 - 0 = 0$$

$$\nabla_{j*} (F)_i^{h*} = \hat{\textcircled{H}}{}^{h*}_{j*\omega}(F)^{\omega}_{i} - \hat{\textcircled{H}}{}^{\omega}_{j*i}(F)^{h*}_{\omega} = -\hat{\textcircled{H}}{}^{h*}_{j*k*}\delta^k_i + \hat{\textcircled{H}}{}^{k}_{j*i}\delta^h_k$$

$$= -\hat{\textcircled{H}}{}^{h*}_{j*i*} + \hat{\textcircled{H}}{}^{h}_{j*i} = -0 + 0 = 0$$

$$\nabla_{j*} (F)_{i*}^{h*} = \hat{\textcircled{H}}{}^{h*}_{j*\omega}(F)^{\omega}_{i*} - \hat{\textcircled{H}}{}^{\omega}_{j*i*}(F)^{h*}_{\omega} = \hat{\textcircled{H}}{}^{h*}_{j*k}\delta^k_i + \hat{\textcircled{H}}{}^{k}_{j*i*}\delta^h_k$$

$$= \hat{\textcircled{H}}{}^{h*}_{j*i} + \hat{\textcircled{H}}{}^{h}_{j*i*} = 0 + 0 = 0$$

Putting these together (3.2) has the components:

$$
(3.3)\begin{cases}
\nabla_j(F)^h_i = \frac{1}{2} R_{jia}{}^h \xi^{a*} & \nabla_j(F)^{h*}_i = 0 \\
\nabla_j(F)^h_{i*} = 0 & \nabla_j(F)^{h*}_{i*} = -\frac{1}{2} R_{jia}{}^h \xi^{a*} \\
\nabla_{j*}(F)^h_i = 0 & \nabla_{j*}(F)^{h*}_i = 0 \\
\nabla_{j*}(F)^h_{i*} = 0 & \nabla_{j*}(F)^{h*}_{i*} = 0
\end{cases}
$$

Thus $\nabla_\mu(F)^\lambda_\nu = 0$ if and only if $R_{jia}{}^h = 0$ and hence we have the following:

<u>Theorem 3.1</u>. A necessary and sufficient condition that $\textcircled{H}{}^A_{BC}$ be an F-connection is that the base space X_n be flat.

Now for the covariant derivative $\nabla_\mu(\bar{v})^\lambda$ of the extended lift $\bar{v}^A$ in $T(X_n)$ of a vector field v^h in X_n we have from (5.16) that

$$
(3.4)\begin{cases}
\nabla_j(\bar{v})^h = \nabla_j v^h \\
\nabla_{j*}(\bar{v})^h = 0 \\
\nabla_j(\bar{v})^{h*} = \frac{1}{2}\left\{\nabla_j \nabla_a v^h + \underset{(v)}{\pounds} \Gamma^h_{ja}\right\} \overset{\xi^{a*}}{\bar{X}^a} \\
\nabla_{j*}(\bar{v})^{h*} = \nabla_j v^h
\end{cases}
$$

so that we have the same conclusion here as Theorem 5.2; namely

<u>Theorem 3.2</u>. A necessary and sufficient condition that the extended lift $\bar{v}^A$ of v^h be a parallel vector field in $T(X_n)$ is that v^h be a parallel vector field in X_n and define there an affine collineation.

4. <u>Paths in T(Xn)</u>.

We begin this section with the following definition; (3), (7)

Definition 4.1. Suppose that C: $x^h = u^h(t)$ is a curve in X_n and $v^h(t)$ a vector field defined along C. Then in the tangent bundle $T(X_n)$ we can define a curve $\bar{C}$: $\xi^A = \xi^A(t)$ called the lift of C by $\xi^h(t) = u^h(t)$, $\xi^{h*}(t) = v^h(t)$. When $v^h(t)$ is taken as the tangent vector field to C, that is, $v^h(t) = \frac{du^h}{dt}$, then $\bar{C}$ is called the natural lift of C.

Let us now consider the differential equations of a path $\xi^A = \xi^A(t)$ in $T(X_n)$; which have the usual form

$$(4.1) \qquad \frac{\delta^2 \xi^A}{dt^2} \equiv \frac{d^2 \xi^A}{dt^2} + \text{\textcircled{H}}^A_{CB} \frac{d\xi^C}{dt} \frac{d\xi^B}{dt} = 0$$

Referring (4.1) to the adapted frames they are expressed in the form

$$(4.2) \qquad \frac{d}{dt} \frac{\omega^\lambda}{dt} + \hat{\text{\textcircled{H}}}^\lambda_{\mu\nu} \frac{\omega^\mu}{dt} \frac{\omega^\nu}{dt} = 0$$

where we have put (7)

$$(4.3) \qquad \begin{cases} \omega^h = B^h{}_A d\xi^A = d\xi^h = du^h \\ \omega^{h*} = C^{h*}_A d\xi^A = \delta\xi^{h*} = \delta \bar{X}^h \end{cases}$$

For $\lambda = h$ (4.2) becomes

$$\frac{d}{dt} \frac{\omega^h}{dt} + \hat{\text{\textcircled{H}}}^h_{ji} \frac{\omega^j}{dt} \frac{\omega^i}{dt} + \hat{\text{\textcircled{H}}}^h_{ji*} \frac{\omega^j}{dt} \frac{\omega^{i*}}{dt} + \hat{\text{\textcircled{H}}}^h_{j*i} \frac{\omega^{j*}}{dt} \frac{\omega^i}{dt}$$

$$+ \hat{\text{\textcircled{H}}}^h_{j*i*} \frac{\omega^{j*}}{dt} \frac{\omega^{i*}}{dt} = 0$$

which in virtue of (3.1) and (4.3) becomes

$$\frac{d}{dt}\left(\frac{du^h}{dt}\right) + \Gamma^h_{ji}\frac{du^j}{dt}\frac{du^i}{dt} = 0 .$$

That is,

$$\frac{\delta^2 u^h}{dt^2} \equiv \frac{d^2 u^h}{dt^2} + \Gamma^h_{ji}\frac{du^j}{dt}\frac{du^i}{dt} = 0 \qquad (4.4)$$

Taking $\lambda = h^*$ in (4.2) we have

$$\frac{d}{dt}\frac{\omega^{h*}}{dt} + \hat{\textcircled{H}}{}^{h*}_{ji}\frac{\omega^j}{dt}\frac{\omega^i}{dt} + \hat{\textcircled{H}}{}^{h*}_{ji*}\frac{\omega^j}{dt}\frac{\omega^{i*}}{dt} + \hat{\textcircled{H}}{}^{h*}_{j*i}\frac{\omega^{j*}}{dt}\frac{\omega^i}{dt}$$
$$+ \hat{\textcircled{H}}{}^{h*}_{j*i*}\frac{\omega^{j*}}{dt}\frac{\omega^{i*}}{dt} = 0$$

Substituting (3.1) and (4.3) we get

$$\frac{d}{dt}\left(\frac{\delta X^h}{dt}\right) - \frac{1}{2}R_{jia}{}^h X^a\frac{du^j}{dt}\frac{du^i}{dt} + \Gamma^h_{ji}\frac{du^j}{dt}\frac{du^i}{dt} = 0$$

But $R_{jia}{}^h$ is skew-symmetric in the first two lower indices, that is, $R_{jia}{}^h = -R_{ija}{}^h$ so that

$$R_{jia}{}^h\frac{du^j}{dt}\frac{du^i}{dt} = 0$$ (In particular, $R_{iia}{}^h = 0$, i not summed)

So

$$\frac{\delta^2 X^h}{dt^2} \equiv \frac{d}{dt}\left(\frac{\delta X^h}{dt}\right) + \Gamma^h_{ji}\frac{\delta X^j}{dt}\frac{du^i}{dt} = 0$$

Thus the differential equations (4.2) become

$$(4.5)\quad \begin{cases} \text{(a)} \quad \dfrac{\delta^2 u^h}{dt^2} = \dfrac{\delta}{dt}\left(\dfrac{\delta u^h}{dt}\right) = 0 \\[2ex] \text{(b)} \quad \dfrac{\delta^2 X^h}{dt^2} = \dfrac{\delta}{dt}\left(\dfrac{\delta X^h}{dt}\right) = 0 \end{cases}$$

Since a curve C: $u^h = u^h(t)$ is a path in X_n if and only if $\dfrac{\delta^2 u^h}{dt^2} = \dfrac{d^2 u^h}{dt^2} + \Gamma^h_{ji}\dfrac{du^j}{dt}\dfrac{du^i}{dt} = 0$ if we take $\dfrac{du^h}{dt}$ for X^h, that is, if we consider the natural lift $\bar{C}$ of a path C we have for (b) the equations $\dfrac{\delta}{dt}\left(\dfrac{\delta^2 u^h}{dt^2}\right) = 0$ and consequently the following theorem holds.

Theorem 4.1. The natural lift of a path in the base space X_n is always a path in $T(X_n)$.

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