Numerical analysis for rectangular slabs under hydrostatic pressure
by Joseph Bozzay

A THESIS Submitted to the Graduate Committee in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering
Montana State University

Abstract:
The slab1 shown in Fig. la is divided into two series of 4-in. wide parallel orthogonal strips. Each strip can be considered a beam of an equivalent grid system. This group of rigidly connected continuous beams is mutually supported along the centerlines (dotted line). For example, the east-west and north-south grid strips are considered contin-uous over supports 3'-5', 4-6', and over supports 7-7', 8-8' The assumption is made that a beam segment such as 3'-4'-6'-5' is free to rotate and to twist at its supports. The resistances to bending and torsion along the edges 3'-4'and 5'-6' that are presumably neglected in the east-west beams are actually considered as vertical shear forces in the north-south beams® Because no openings exist between grid beams and because the cross section of any grid beam is identical to that of a comparable slab strip, the general appearance of the gridwork is no different from that of the slab.

Since the equivalent gridwork is under hydraulic pressure, the distribution of moments and torsions produced by this loading are based on relative stiffnesses and distribution factors.

The edge conditions in this case are assumed as follows! from Fig. la east edge - elastically built-in west edge north edge - free south edge - built-in or fixed The auxiliary forces P1, P3, P5, due to hydraulic pressure act-ing on the joints 1.3,5, are identical with the forces P2,P4,P6 act-ing on joints 2,4,6. That is P1= P2, P3= P4, P5= P6 consequently the deflections (D) produced by forces of tne same magnitude are also identical® In other words, D1=D2, D3= D4, D5= D6. The magnitude of these deflections are unknown, however, the fixed end moments due to these deflections can easily be computed in terms of the unknown displacements.

These fixed end moments will be converted into bending and torsion by a moment and torsion distribution process.

1 The slab shown in Fig. la is one of the two similar side walls of the rectangular water container the dimensions of which are given in Fig. 1c The algebraic sum of the reactions computed at each joint from the distributed moments has to be equal to the auxiliary force actually acting upon the joint. On the basis of this relation as many simultaneous equa-tions can be set up in terms of the unknown displacements as are needed for the determination of the arbitrary deflections.

After the values of the unknown deflections are computed by solving the simultaneous equations the bending and torque moments can be ob-tained at each joint by substituting the deflection values into express-ions resulting from the moment-torque distribution process.

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Chairman, Examining Committee

Dean, Graduate Division

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Joseph Bozzay

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ABSTRACT

The slab[1] shown in Fig. 1a is divided into two series of 4-in. wide, parallel orthogonal strips. Each strip can be considered a beam of an equivalent grid system. This group of rigidly connected continuous beams is mutually supported along the centerlines (dotted line). For example, the east-west and north-south grid strips are considered continuous over supports 3'-5', 4'-6', and over supports 7'-7', 8'-8'. The assumption is made that a beam segment such as 3'-4'-6'-5' is free to rotate and to twist at its supports. The resistances to bending and torsion along the edges 3'-4' and 5'-6' that are presumably neglected in the east-west beams are actually considered as vertical shear forces in the north-south beams. Because no openings exist between grid beams and because the cross section of any grid beam is identical to that of a comparable slab strip, the general appearance of the gridwork is no different from that of the slab.

Since the equivalent gridwork is under hydraulic pressure, the distribution of moments and torsions produced by this loading are based on relative stiffnesses and distribution factors.

The edge conditions in this case are assumed as follows; from Fig. 1a

east edge }
west edge } - elastically built-in

north edge - free

south edge - built-in or fixed

The auxiliary forces P_1, P_3, P_5 due to hydraulic pressure acting on the joints 1, 3, 5, are identical with the forces P_2, P_4, P_6 acting on joints 2, 4, 6. That is $P_1 = P_2$, $P_3 = P_4$, $P_5 = P_6$ consequently the deflections (D) produced by forces of the same magnitude are also identical. In other words, $D_1 = D_2$, $D_3 = D_4$, $D_5 = D_6$. The magnitude of these deflections are unknown, however, the fixed end moments due to these deflections can easily be computed in terms of the unknown displacements.

These fixed end moments will be converted into bending and torsion by a moment and torsion distribution process.

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[1]The slab shown in Fig. 1a is one of the two similar side walls of the rectangular water container the dimensions of which are given in Fig. 1a.

The algebraic sum of the reactions computed at each joint from the distributed moments has to be equal to the auxiliary force actually acting upon the joint. On the basis of this relation as many simultaneous equations can be set up in terms of the unknown displacements as are needed for the determination of the arbitrary deflections.

After the values of the unknown deflections are computed by solving the simultaneous equations the bending and torque moments can be obtained at each joint by substituting the deflection values into expressions resulting from the moment-torque distribution process.

INTRODUCTION

Object

The object of this thesis is to present an approximate numerical method for determining elastic deformations in the side walls of the rectangular water container shown in Fig. 1c. This analysis will be carried out without using the mathematical theory of elasticity which requires in most cases applications of partial differential equations of higher order and has proved to be impractical. In the following analysis only the loads acting perpendicular to the plane of the walls are considered.

The forces acting upon the side walls parallel to their planes have been neglected consequently the deflections obtained by this procedure are somewhat larger than the actual deflections.

The loads perpendicular to the walls, the wall thickness and the manner of support are identical for each side of the water tank, therefore, the calculations have been carried out for one side wall only.

Previous Investigation

A system of beams intersecting at right angles to one another can be made to form a gridwork that will yield a deflected surface similar to that of a slab when analyzed under normal loads. One method of determining the deflection in a gridwork of beams is presented by Miklos Hetenyi,[2] who assumes that the individual beams comprising the gridwork deflect without rotation at their intersections with other beams. The joint displacements are obtained by solutions of simultaneous differential equations.

[2]BEAMS ON AN ELASTIC FOUNDATION by Miklos Hetenyi, The University of Michigan Press, Ann Arbor, Michigan, 1946, pp 185-192.

With the assumption that no bending or torsion moment is transmitted at grid beam intersections, Stephen S. Timoshenko[3] employs a trigonometric series to express the elastic curves developed by the individual grid beams. However, a grid work, unable to transmit bending and torsional moments, is not analogous to a slab or plate in its action.

By the method presented here, moment and torsion transfer at the joints is taken into consideration. After the bending moments and torsional moments have been distributed over the grid, only one series of linear equations need be solved in order to define the deflection pattern. This is possible because the equations are written in terms of unknown deflections produced by auxiliary forces at the grid points. Bending moments and torsional moments can then be found without recourse to a second series of simultaneous equations.

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[3] UBER DIE BIEGUNG VON TRAGERROSTEN by Stephen S. Timoshenko, Zeitschrift fur Angewandte Mathematik and Mechanik, Band 13, 1933, pp 153.

PROCEDURE

Step I

The arbitrary division of the slab into orthogonal strips that are considered the beams of the analogous gridwork (see Fig. 1a).

Step II

The determination of factors based on the individual beam's resistances to bending and to torsion for distributing the unbalanced moments at each grid beam joint.

A. Moment stiffness factors.

Definition of stiffness K: The moment necessary to produce a unit end rotation when the far end is fixed (Fig. 2d, 2e).

$\theta = \frac{LM}{4EI}$ for $\theta = 1$; $M = K = \frac{4EI}{L}$

The division of the unbalanced moment at a joint is to be made in direct proportion to the K values or in inverse ratio to the end rotations of the connecting members as caused by unit end moments. The end rotation caused by a unit end moment (end reaction in conjugate beam, Fig. 2e) is

$\theta = \frac{L}{4EI}$; $\frac{1}{\theta} = K = \frac{4EI}{L}$.

If the far end is pin-connected (Fig. 2a) the conjugate beam is another simple beam and the end rotation at the left is two-thirds of the area of the $\frac{M}{EI}$ diagram or $\frac{L}{3EI}$ (Fig. 2c). The ratio between the end rotations of two identical beams, one of which is pin-connected or simply supported at the far end and the other fixed, is as 4 to 3. The stiffness factor

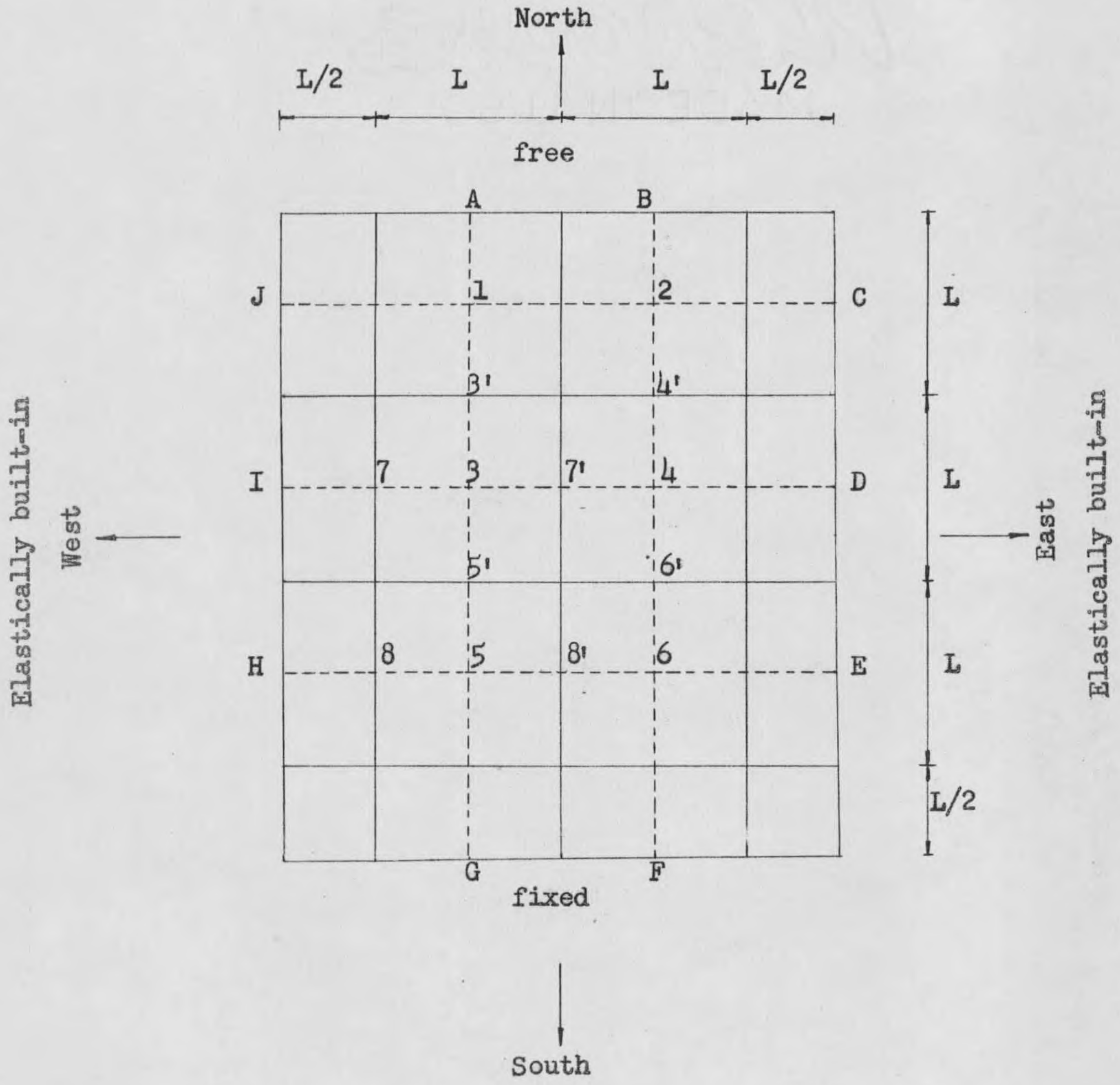

Fig. 1a Division of Slab

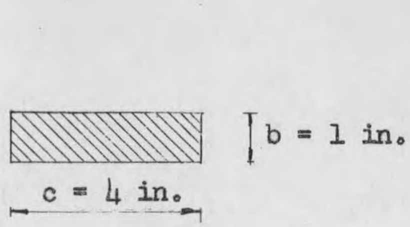

Fig. 1b Cross-section of a Strip

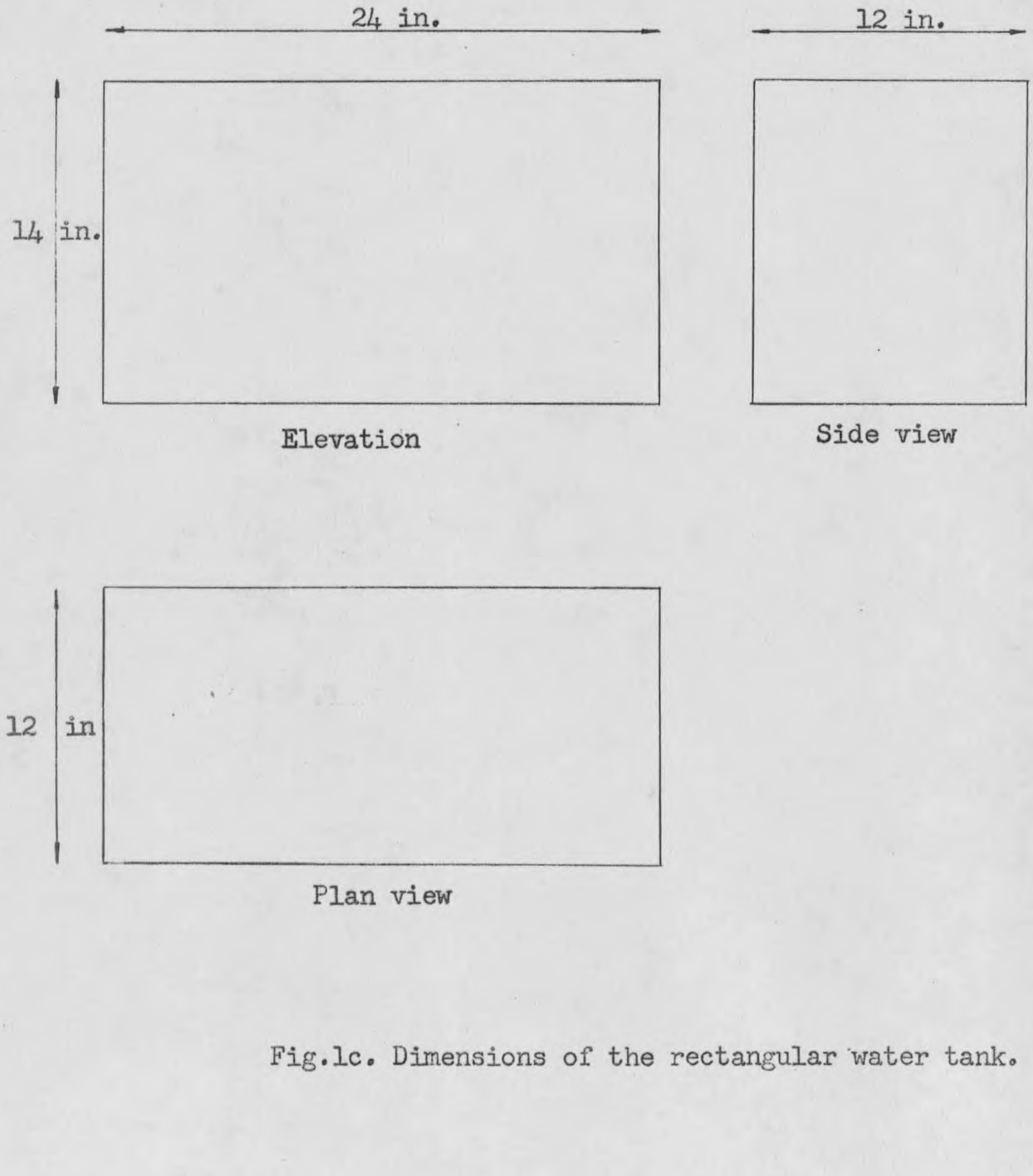

Fig.1c. Dimensions of the rectangular water tank.

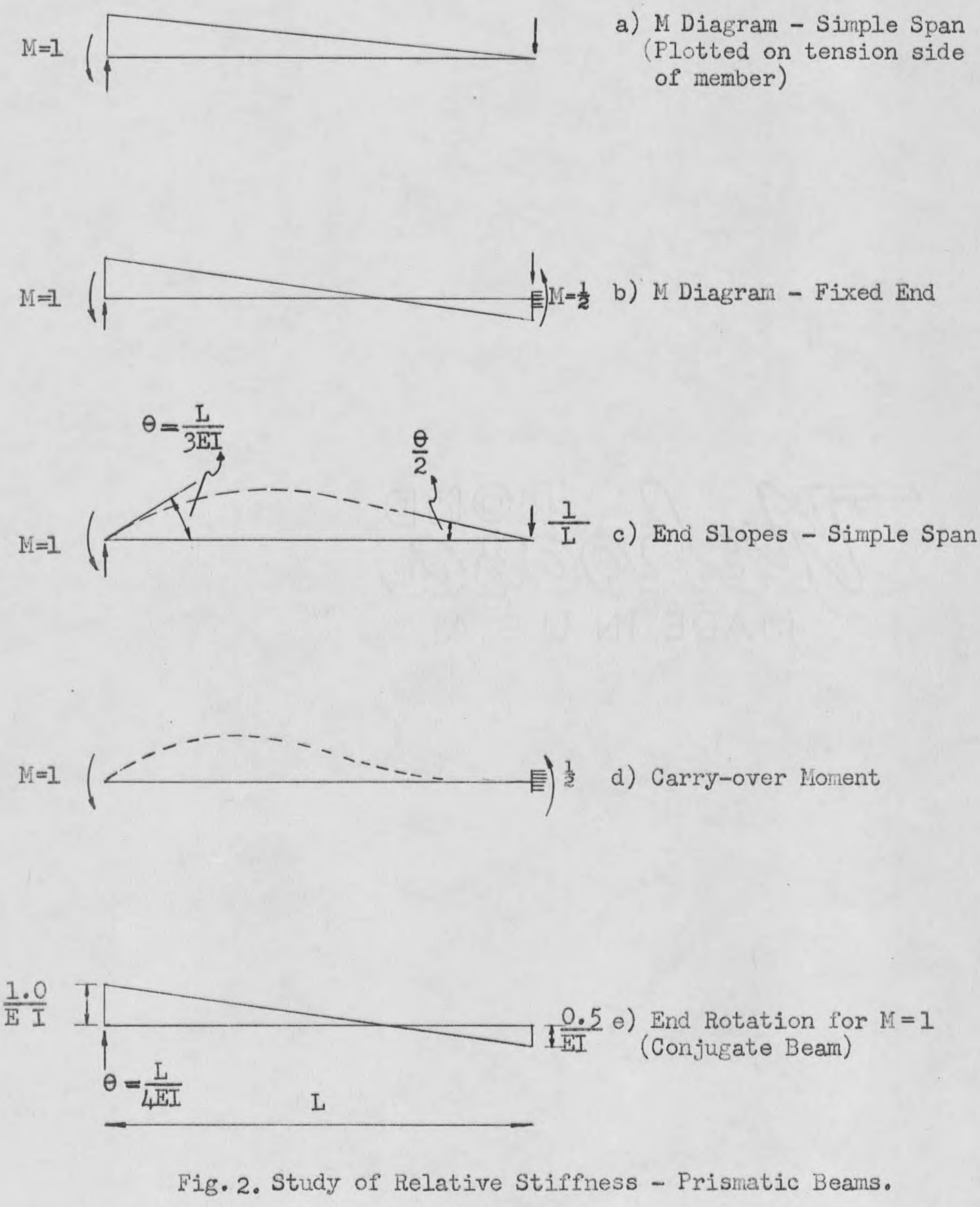

Fig. 2. Study of Relative Stiffness - Prismatic Beams.

when the far end is pin-connected accordingly becomes 75% of the stiffness factor when it is fixed. Then there is no fixed end moment or carry-over moment at the pin end.

Far end fixed: $vK = \frac{4EI}{L}$; $v = 1$; Carry-over factor = $\frac{1}{2}$.

v is a factor which may be used to define the condition of end restraint, and it will vary from 0.75 to 1.00.

Far end pin-connected: $vK = 0.75 \frac{4EI}{L}$; $v = 0.75$; Carry-over factor = 0.

Far end elastically built-in: $vK = 0.875 \frac{4EI}{L}$; $v = 0.875$ (assumed) ; Carry-over factor = 0.25. In this case for v, the mean value between 0.75 and 1.00, 0.875 is assumed and the corresponding carry-over factor 0.25 is taken from the chart[4] referred to in the footnote.

B. Torsion stiffness factors.

The following expression for moment required to produce a unit angle twist in a rectangular beam has been taken from S. Timoshenko.[5]

$$T = \frac{Bb^3 cG}{L}$$

B is a factor the value of which depends upon the ratio $\frac{c}{b}$ (see Fig. 1b).

When $\frac{c}{b} = \frac{4}{1} = 4 \longrightarrow B = 0.281$

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[4] For an assumed value of v the corresponding value of carry-over factor can be taken from THEORY OF MODERN STEEL STRUCTURES by L. E. Grinter, Macmillan Co., New York, pp 160, Fig. 123.

[5] ELEMENTS OF STRENGTH OF MATERIALS by S. Timoshenko and G. H. Mac Cullough, D. Van Nostrand Co., Inc., New York, 1949, pp 265-266.

$G = 0.4E$ is taken

$$T = \frac{0.281\ b^3c\ 0.4E}{L} = \frac{0.1124\ b^3c\ E}{L}$$

$$I = \frac{b^3c}{12}$$ (Moment of inertia of the strip cross-section, Fig. 1b).

The expression for T can be written without changing its value as follows:

$$T = \frac{0.1124\ EI\ 12}{L} = \frac{1.3488\ EI}{L}$$

Carry-over factor is -1 for torsion for each member.

The factor $\frac{EI}{L}$ having identical values for moment and torsion stiffnesses cancels out in distribution factor computations.

TABLE I STIFFNESS AND CARRY-OVER FACTORS FOR MOMENT AND TORQUE

Joint No.	Member of Gridwork	Stiffness Factor Moment	Stiffness Factor Torque	Carry-over Factor Moment	Carry-over Factor Torque
	1-F	3.5	1.3488	0.25	- 1.00
	1-A	0.0	0.0000	0.00	- 1.00
1	1-2	4.0	1.3488	0.50	- 1.00
	1-3	4.0	1.3488	0.50	- 1.00
	2-1	4.0	1.3488	0.50	- 1.00
	2-B	0.0	0.0000	0.00	- 1.00
2	2-C	3.5	1.3488	0.25	- 1.00
	2-4	4.0	1.3488	0.50	- 1.00
	3-I	3.5	1.3488	0.25	- 1.00
	3-1	4.0	1.3488	0.50	- 1.00
3	3-4	4.0	1.3488	0.50	- 1.00
	3-5	4.0	1.3488	0.50	- 1.00
	4-3	4.0	1.3488	0.50	- 1.00
	4-2	4.0	1.3488	0.50	- 1.00
4	4-D	3.5	1.3488	0.25	- 1.00
	4-6	4.0	1.3488	0.50	- 1.00
	5-H	3.5	1.3488	0.25	- 1.00
	5-3	4.0	1.3488	0.50	- 1.00
5	5-6	4.0	1.3488	0.50	- 1.00
	5-G	4.0	1.3488	0.50	- 1.00
	6-5	4.0	1.3488	0.50	- 1.00
	6-4	4.0	1.3488	0.50	- 1.00
6	6-E	3.5	1.3488	0.25	- 1.00
	6-F	4.0	1.3488	0.50	- 1.00

TABLE II DISTRIBUTION FACTORS AT TYPICAL JOINTS 1 AND 3

Joint No.	Direction	Distribution Factor Moment	Distribution Factor Torque
	West	0.3955	0.2014
	North	0.0000	0.0000
1	East	0.4520	0.2014
	South	0.5972	0.1525
	West	0.3432	0.1261
	North	0.3739	0.1323
3	East	0.3922	0.1261
	South	0.3739	0.1323

Step III

The vertical displacement of each singular joint in the grid and consequent introduction of fixed-end moments on the beams. Joints where deflection produces a distinctive moment and torque pattern are known as singular joints.

Computation of fixed-end moments from displacements.

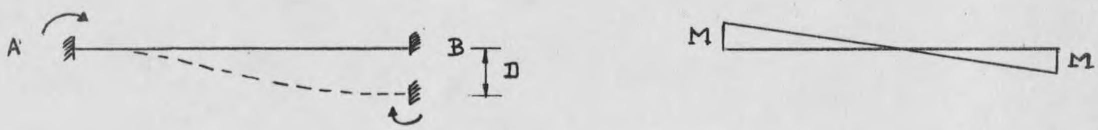

Fig. 3a Conjugate beam relations--far end fixed.

For a known (here assumed) settlement, the fixed-end moments are computed by the formula developed as follows:

From Fig. 3a the deflection of B from a tangent drawn at A is equal to the statical moment of the $\frac{M}{EI}$ diagram from A to B about B.

$D = \frac{M}{EI}\ \frac{L^2}{6}$; solving for M the fixed-end moments can be obtained:

$M = \frac{6EI}{L^2} D$

$D = \frac{ML^2}{3EI}$

$M = \frac{3EID}{L^2}$

Fig. 3b Conjugate beam relations--far end pin-connected.

When the far end is elastically built-in the mean value between $\frac{6EID}{L^2}$ and $\frac{3EID}{L^2}$ that is $\frac{4.5\ EID}{L^2}$ will be assumed.

TABLE III FIXED END MOMENTS IN TERMS OF $\frac{EI}{L^2}$ D DUE TO ARBITRARY DEFLECTIONS INTRODUCED AT SINGULAR JOINTS

Joint No.	1	3	5
Deflection	D_1	D_3	D_5
	$M_{F1} = +4.5 = M_{1F}$	$M_{I3} = +4.5 = M_{3I}$	$M_{H5} = +4.5 = M_{5H}$
Fixed end moment	$M_{12} = -6.0 = M_{21}$	$M_{34} = -6.0 = M_{43}$	$M_{56} = -6.0 = M_{65}$
	$M_{31} = +6.0 = M_{13}$	$M_{53} = +6.0 = M_{35}$	$M_{G5} = +6.0 = M_{5G}$
	$M_{1A} = 0 = M_{A1}$	$M_{31} = -6.0 = M_{13}$	$M_{53} = -6.0 = M_{35}$

Sign Convention: The signs of the fixed end moments are in accordance with the moment distribution convention when viewed from the south and the east respectively.

Step IV

The conversion of the fixed-end moments into bending and twisting moments by a moment and torque distribution process.

For convenience, moments are represented by the constant coefficients 4.5 and 6 multiplied by 1000.

TABLE IV MOMENT DISTRIBUTION EAST-WEST, TORSION DISTRIBUTION NORTH-SOUTH

	E	W	N	S	E	W	N	S	E	W
	0	0.3955	0	0.1525	0.4520	0.4520	0	0.1525	0.3955	0
J ——— 1 ——— 2 ——— C										
	4500	4500			-6000	-6000				
	148	593	0	229	678	339				
				- 30	1279	2559	0	863	2239	560
	- 123	- 494	0	- 191	- 565	- 282	0	- 108		
				46	88	176		59	154	38
	- 13	- 53	0	- 20	- 61	- 30		- 19		
				7	11	22	0	7	19	5
	- 2	- 7	0	- 3	- 8	- 4		- 3		
				1	1	3	0	1	3	1
	4510	4539	0	38	-4578	-3217	0	800	2415	604

	E	W	N	S	E	W	N	S	E	W
	0	0.3432	0.1323	0.1323	0.3922	0.3922	0.1323	0.1323	0.3432	0
I ——— 3 ——— 4 ——— D										
			- 229				- 863			
	20	79	30	30	90	45				
			191	- 4	160	321	108	108	281	70
	- 30	- 119	- 46	- 46	- 136	- 68	- 59	- 14		
			20	9	27	55	19	19	48	12
	- 5	- 19	- 7	- 7	- 22	- 11	- 7	- 3		
			3	1	4	8	3	3	7	2
	- 1	- 3	- 1	- 1	- 3	- 1	- 1	- 1		
	- 16	- 62	- 39	- 18	120	350	- 799	113	336	84

	E	W	N	S	E	W	N	S	E	W
	0	0.3432	0.1323	0.1323	0.3922	0.3922	0.1323	0.1323	0.3432	0
H ——— 5 ——— 6 ——— E										
			- 30				- 108			
	3	10	4	4	12	6				
			46		20	40	14	14	35	9
	- 6	- 23	- 9	- 9	- 26	- 13	- 19			
			7		4	9	3	3	8	2
	- 1	- 4	- 1	- 1	- 4	- 2	- 3			
			1		1	2	1	1	1	0
	- 4	- 17	17	- 6	6	42	- 112	18	44	11
			- 4				- 14			
			9				- 3			
			1				- 1			
			6				- 18			
			G				F			

TABLE V MOMENT DISTRIBUTION NORTH-SOUTH, TORSION DISTRIBUTION EAST-WEST

J ——— 1 ——— 2 ——— C

E	W	N	S	E	W	N	S	E	W
0	0.2014	0	0.5972	0.2014	0.2014	0	0.5972	0.2014	0
			6000						
1208	-1208	0	-3583	-1208	1208				
			- 787	243	- 243	0	- 721	- 243	243
- 110	110	0	325	110	- 110		- 32		
			- 62	- 29	29	0	85	29	- 29
- 18	18	0	54	18	- 18		- 20		
			- 10	- 8	8	0	23	8	- 8
- 4	4		10	4	- 4		- 4		
			- 2	- 1	1		5	1	- 1
1076	-1076	0	1947	- 870	870	0	- 664	- 205	205

I ——— 3 ——— 4 ——— D

E	W	N	S	E	W	N	S	E	W
0	0.1261	0.3739	0.3739	0.1261	0.1261	0.3739	0.3739	0.1261	0
		6000				- 360			
		-1791							
531	- 531	-1574	-1574	- 531	531				
		162	147	22	- 22	- 64	- 64	- 22	22
42	- 42	- 124	- 124	- 42	42	42	24	- 14	14
7	- 7	- 21	- 21	- 7	7	11	5		
		5	2	3	- 3	- 9	- 9	- 3	3
1	- 1	- 4	- 4	- 1	1	2	1		
581	- 581	2680	-1559	- 542	542	- 419	- 84	- 40	40

H ——— 5 ——— 6 ——— E

E	W	N	S	E	W	N	S	E	W
0	0.1261	0.3739	0.3739	0.1261	0.1261	0.3739	0.3739	0.1261	0
		- 787				- 32			
- 99	99	294	294	99	- 99				
		- 62		- 17	17	49	49	17	- 17
- 10	10	30	30	10	- 10	- 20			
		- 10		- 4	4	11	11	4	- 4
- 2	2	5	5	2	- 2	- 4			
		- 2		- 1	1	2	2	1	- 1
- 111	111	- 531	330	90	- 90	6	62	22	- 22

G	F
147	24
15	5
2	1
164	30

TABLE VI MOMENT DISTRIBUTION EAST-WEST, TORSION DISTRIBUTION NORTH-SOUTH

Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.3955	0	0.1525	0.4520	0.4520	0	0.1525	0.3955	0
J – 1 – 2 – C			1				2			C
Moments	-15	-62	0	-56	+120	+351	0	-689	+338	+84
Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.3432	0.1323	0.1323	0.3922	0.3922	0.1323	0.1323	0.3432	0
I – 3 – 4 – D			3				4			D
Moments	+4522	+4584	+56	+50	-4690	-3522	+689	+708	+2124	+531
Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.3432	0.1323	0.1323	0.3922	0.3922	0.1323	0.1323	0.3432	0
H – 5 – 6 – E			5				6			E
Moments	-10	-45	-50	-18	+113	+307	-708	+111	+289	+72
			+18				-111			
			G				F			

TABLE VII MOMENT DISTRIBUTION NORTH-SOUTH, TORSION DISTRIBUTION EAST-WEST

Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.2014	0	0.5972	0.2014	0.2014	0	0.5972	0.2014	0
J – 1 – 2 – C			1				2			C
Moments	-1297	+1297	0	-2314	+1021	-1021	0	+744	+276	-276
Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.1261	0.3739	0.3739	0.1261	0.1261	0.3739	0.3739	0.1261	0
I – 3 – 4 – D			3				4			D
Moments	+109	-109	-4402	+4570	-60	+60	+262	-271	-49	+49
Member	E	W	N	S	E	W	N	S	E	W
Distribution factors	0	0.1261	0.3739	0.3739	0.1261	0.1261	0.3739	0.3739	0.1261	0
H – 5 – 6 – E			5				6			E
Moments	+747	-747	+3625	-2214	-663	+663	-325	-252	-84	+84
			-1107				-125			
			G				F			

TABLE VIII MOMENT DISTRIBUTION EAST-WEST, TORSION DISTRIBUTION NORTH-SOUTH

	E	W	N		S	E	W	N		S	E	W	
	0	0.3955	0		0.1525	0.4520	0.4520	0		0.1525	0.3955	0	
J				1					2				C
	-4	-18	0		+12	+6	+45	0		-94	+49	+12	
	E	W	N		S	E	W	N		S	E	W	
	0	0.3432	0.1323		0.1323	0.3922	0.3922	0.1323		0.1323	0.3432	0	
I				3					4				D
	-10	-45	-12		-57	+113	+306	+94		-689	+290	+72	
	E	W	N		S	E	W	N		S	E	W	
	0	0.3432	0.1323		0.1323	0.3922	0.3922	0.1323		0.1323	0.3432	0	
H				5					6				E
	+4526	+4602	+57		+39	-4699	-3567	+689		+801	+2076	+518	
			-39					-801					
				G					F				

TABLE IX MOMENT DISTRIBUTION NORTH-SOUTH, TORSION DISTRIBUTION EAST-WEST

	E	W	N		S	E	W	N		S	E	W	
	0	0.2014	0		0.5972	0.2014	0.2014	0		0.5972	0.2014	0	
J				1					2				C
	+274	-274	0		+454	-181	+181	0		-87	-94	+94	
	E	W	N		S	E	W	N		S	E	W	
	0	0.1261	0.3739		0.3739	0.1261	0.1261	0.3739		0.3739	0.1261	0	
I				3					4				D
	-855	+855	+2131		-3708	+721	-721	+258		+331	+134	-134	
	E	W	N		S	E	W	N		S	E	W	
	0	0.1261	0.3739		0.3739	0.1261	0.1261	0.3739		0.3739	0.1261	0	
H				5					6				E
	+166	-166	-5224		+5510	-120	+120	+59		-133	-46	+46	
			+5757					-66					
				G					F				

Step V

The computation of reactions at all joints in terms of the unknown displacements

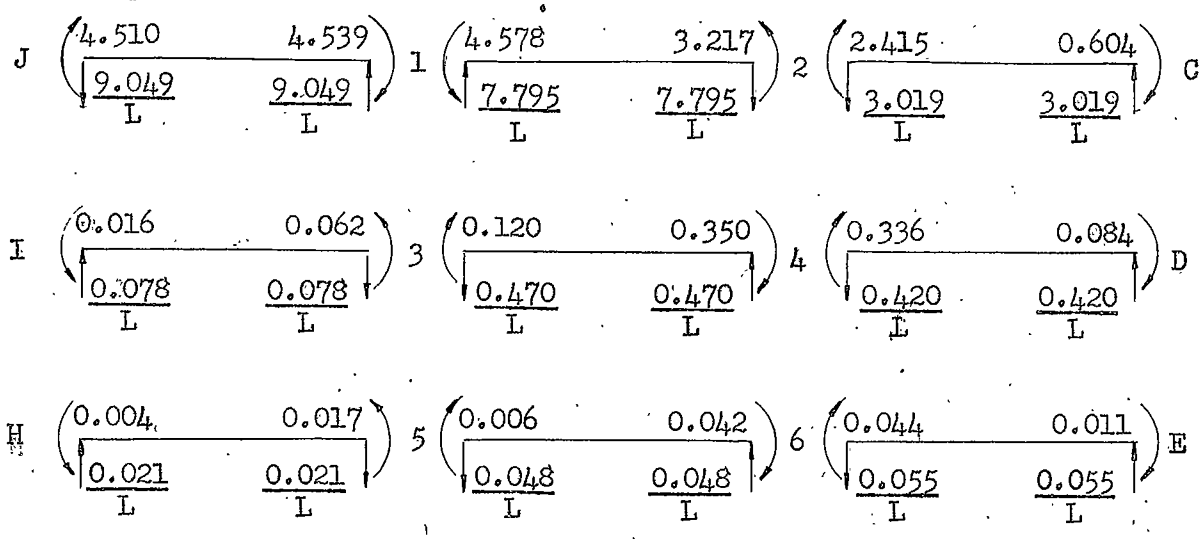

Fig. 4a. Moments and reactions on grid beams (from Table IV)

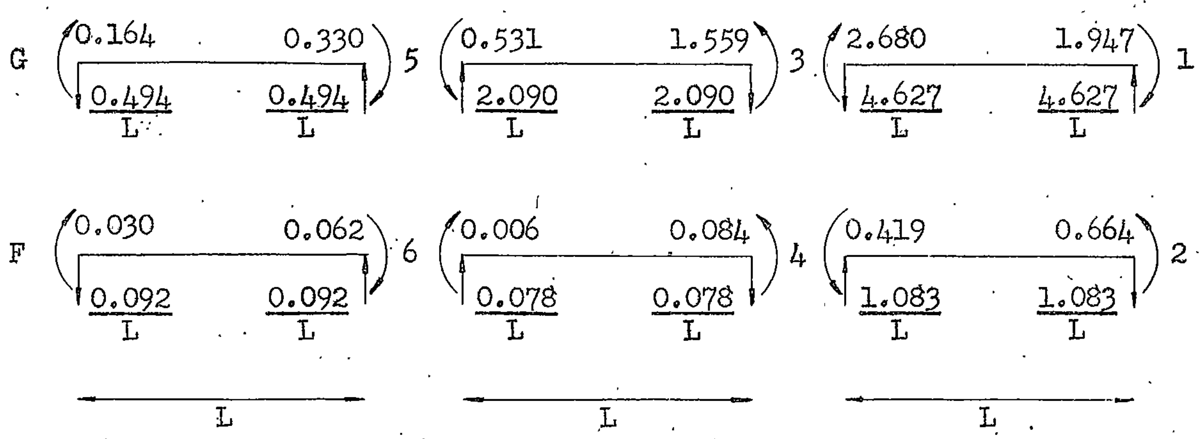

Fig. 4b. Moments and reactions on grid beams (from Table V)

Sign convention for reactions :

Upward positive ↑ , Downward negative ↓

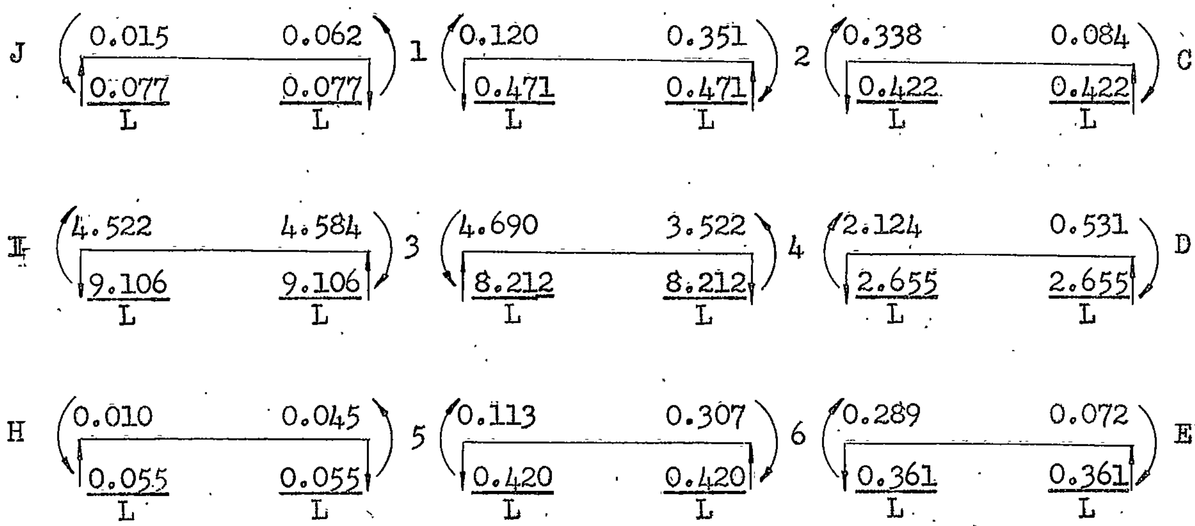

Fig. 5a. Moments and reactions on grid beams (from Table VI)

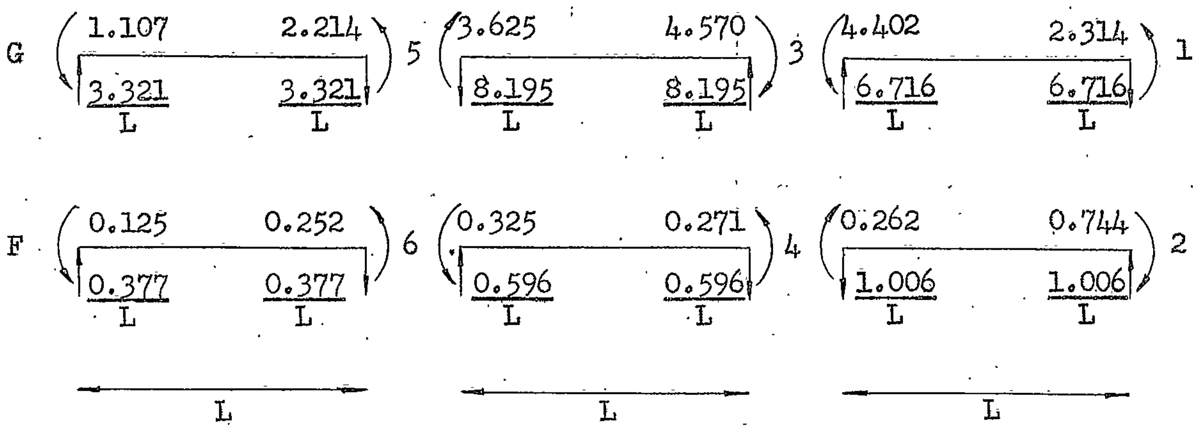

Fig. 5b. Moments and reactions on grid beams (from Table VII)

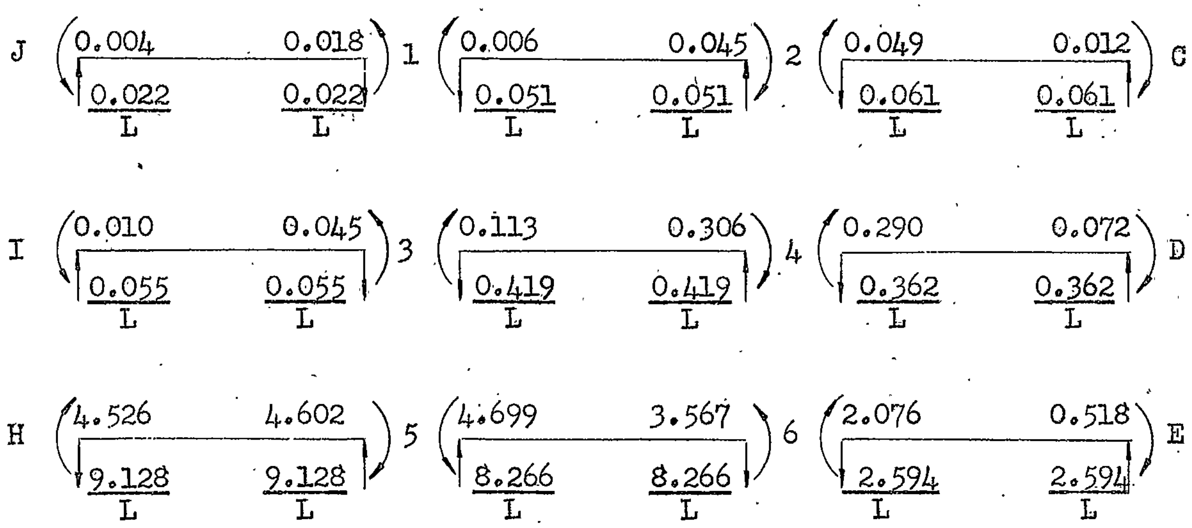

Fig. 6a. Moments and reactions on grid beams (from Table VIII)

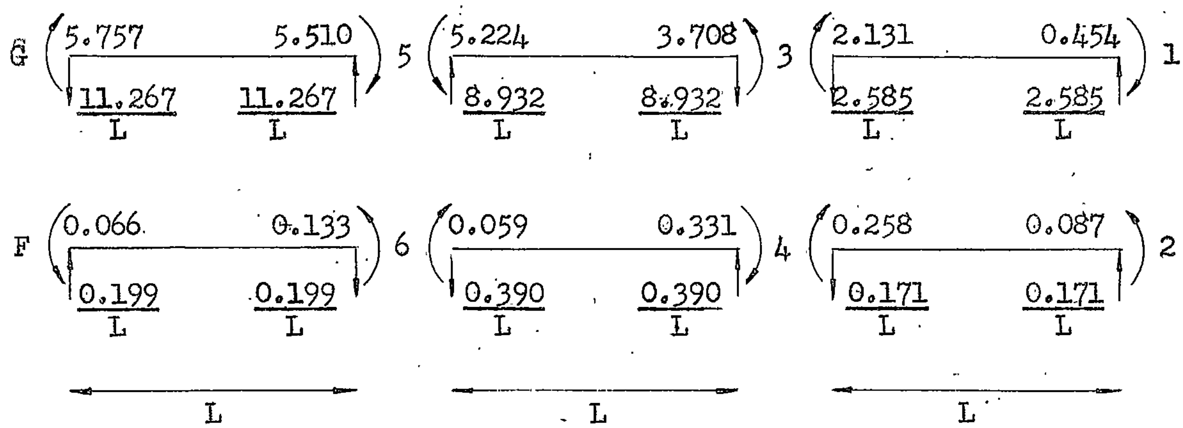

Fig. 6b. Moments and reactions on grid beams (from Table IX)

Step VI

Computation of Deflections.

Computation of total reactions R_T at Joints 1, 3, and 5.

Joint 1: $R_{T1} = R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16}$

Joint 3: $R_{T3} = R_{31} + R_{32} + R_{33} + R_{34} + R_{35} + R_{36}$

Joint 5: $R_{T5} = R_{51} + R_{52} + R_{53} + R_{54} + R_{55} + R_{56}$

R_{13} = reaction at Joint 1 due to deflection D_3.

R_{T1} = total reaction at Joint 1 that is the sum of reactions at Joint 1 produced by displacements D_1 to D_6.

By the theorem of reciprocity the reaction $R_{21} = 11.897 \frac{EI}{L^3} D_1$ at junction 2 caused by the deflection D_1 at Joint 1 is equal to the reaction at Joint 1 caused by an equal deflection D_2 at Joint 2. That is $R_{21} = R_{12}$.

From the theorem of reciprocity it follows that

$R_{12} = R_{21}$, $R_{32} = R_{41}$, $R_{52} = R_{51}$ when $D_1 = D_2$

$R_{23} = R_{14}$, $R_{34} = R_{43}$, $R_{54} = R_{63}$ when $D_3 = D_4$

$R_{16} = R_{25}$, $R_{36} = R_{45}$, $R_{56} = R_{65}$ when $D_5 = D_6$

Using the above theory the total reactions can be written

$R_{T1} = R_{11} + R_{21} + R_{13} + R_{23} + R_{15} + R_{25}$

$R_{T3} = R_{31} + R_{41} + R_{33} + R_{43} + R_{35} + R_{45}$

$R_{T5} = R_{51} + R_{61} + R_{53} + R_{63} + R_{55} + R_{65}$

Equating the total reactions R_{T1}, R_{T3}, R_{T5} to the auxiliary forces P_1, P_3, P_5 due to the hydraulic pressure acting on Joints 1, 3, and 5 three simultaneous equations can be obtained for the determination of the unknown displacements.

$R_{T1} = P_1$; $R_{T3} = P_3$; $R_{T5} = P_5$

Substituting the values from Table X for the reactions and multiplying through by $\frac{L^3}{EI}$ the simultaneous equations can be written as follows:

$$21.471D_1 - 11.897D_1 - 7.264D_3 + 1.055D_3 + 2.512D_5 + 0.161D_5 = P_1 \frac{L^3}{EI}$$

$$1.055D_1 - 7.265D_1 + 32.229D_3 - 12.469D_3 - 11.991D_5 + 0.276D_5 = P_3 \frac{L^3}{EI}$$

$$2.515D_1 + 0.163D_1 - 11.991D_3 + 0.278D_3 + 37.593D_5 - 11.449D_5 = P_5 \frac{L^3}{EI}$$

$$9.574D_1 - 6.209D_3 + 2.673D_5 = P_1 \frac{L^3}{EI}$$

$$-6.210D_1 + 19.760D_3 - 11.715D_5 = P_3 \frac{L^3}{EI}$$

$$2.678D_1 - 11.713D_3 + 26.144D_5 = P_5 \frac{L^3}{EI}$$

It is evident from the above equations that the deflections D_1, D_3, and D_5 determine the deflected surface of the slab.

Solving the three simultaneous equations by determinants

$$D_1 = \frac{L^3}{EI} \; \frac{379.387P_1 + 131.019P_3 + 19.920P_5}{2871.976}$$

$$D_3 = \frac{L^3}{EI} \; \frac{130.981P_1 + 243.145P_3 + 95.560P_5}{2871.976}$$

$$D_5 = \frac{L^3}{EI} \; \frac{19.821P_1 + 95.512P_3 + 150.624P_5}{2871.976}$$

Computation of auxiliary forces $P_1 = P_2$; $P_3 = P_4$; $P_5 = P_6$.

When the slab is subjected to hydraulic pressure which is increasing with the depth the equivalent gridwork is loaded at the Joints as follows:

Joint 1 and 2: $P_1 = P_2 = 1.156$ lb.

Joint 3 and 4: $P_3 = P_4 = 3.468$ lb.

Joint 5 and 6: $P_5 = P_6 = 5.780$ lb.

Evaluation of the factor $\frac{EI}{L^3}$

$L = 4$ in. ; $I = 1/3$ in.^{4} (see Fig. 1b)

E the modulus of elasticity for concrete will be assumed as

$1000\, f'_c$. If $f'_c = 3000$ p.s.i. $E = 3 \quad 10^6$ p.s.i.

$EI = 10^6$ p.s.i. $\quad \frac{EI}{L^3} = \frac{10^6}{64}$ lb./in.

Table X Reactions and Deflections at Joints Caused by D Displacements

Joint No.	Displacements D_1	D_2	D_3	D_4	D_5	D_6	P auxiliary forces in lbs.
	a) Summation of Reactions in Terms of $\frac{EI}{L^3}$ D						
1	21.471	-11.897	- 7.264	1.055	2.512	0.161	1.156
2	-11.897	21.471	1.055	- 7.264	0.161	2.512	1.156
3	- 7.265	1.055	32.229	-12.469	-11.991	0.276	3.468
4	1.055	- 7.265	-12.469	32.229	0.276	-11.991	3.468
5	2.515	0.163	-11.991	0.278	37.593	-11.449	5.780
6	0.163	2.515	0.278	-11.991	-11.449	37.593	5.780
	b) Values of Deflections in Terms of $\frac{L^3}{EI}$						
	0.351	0.351	0.539	0.539	0.426	0.426	
	c) Final Values of Deflections in inches						
	$22.464 \; 10^{-6}$	$22.464 \; 10^{-6}$	$34.496 \; 10^{-6}$	$34.496 \; 10^{-6}$	$27.264 \; 10^{-6}$	$27.264 \; 10^{-6}$	