Subsystems of and interlacing theorems for interface Sturm-Liouville systems
by Dennis Numan Winslow

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

Abstract:
This thesis deals with several important problems concerned with interface Sturm-Liouville systems. One of these problems is to determine how separation or interlacing theorems may be applied to the characteristic equations or auxiliary functions of such systems. Our attempt to solve this problem requires the introduction of the concept of a subsystem of a Sturm-Liouville interface system. Another problem is to discover if some results known to hold for ordinary Sturm— Liouville systems also hold for interface Sturm-Liouville systems. The first of these problems is solved by developing a theorem which asserts that the zeros of the auxiliary function of the entire system interlace the zeros of the products of the auxiliary functions of certain subsystems. This interlacing is used to solve the last problem by providing the basis for induction proofs which show that the set of characteristic values of any Sturm-Liouville interface system is countably infinite, contains a smallest element, is unbounded above, and has no finite limit point. Although information is also obtained about auxiliary functions of Sturm-Liouville interface systems and subsystems,, the main conclusions of this thesis are that the interlacing theorem is available for use in investigating characteristic values of Sturm-Liouville interface systems and that some important properties of characteristic values of ordinary Sturm-Liouville systems are also properties of characteristic values of interface Sturm-Liouville systems.

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Dennis Numan Winslow

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ABSTRACT

This thesis deals with several important problems concerned with interface Sturm-Liouville systems. One of these problems is to determine how separation or interlacing theorems may be applied to the characteristic equations or auxiliary functions of such systems. Our attempt to solve this problem requires the introduction of the concept of a subsystem of a Sturm-Liouville interface system. Another problem is to discover if some results known to hold for ordinary Sturm-Liouville systems also hold for interface Sturm-Liouville systems. The first of these problems is solved by developing a theorem which asserts that the zeros of the auxiliary function of the entire system interlace the zeros of the products of the auxiliary functions of certain subsystems. This interlacing is used to solve the last problem by providing the basis for induction proofs which show that the set of characteristic values of any Sturm-Liouville interface system is countably infinite, contains a smallest element, is unbounded above, and has no finite limit point. Although information is also obtained about auxiliary functions of Sturm-Liouville interface systems and subsystems, the main conclusions of this thesis are that the interlacing theorem is available for use in investigating characteristic values of Sturm-Liouville interface systems and that some important properties of characteristic values of ordinary Sturm-Liouville systems are also properties of characteristic values of interface Sturm-Liouville systems.

INTRODUCTION

Shortly after the advent of the differential calculus, nearly 320 years ago, various engineering, mathematical, and scientific problems gave rise to boundary value problems of one kind or another. Those involving differential equations in the real domain, with one or two point boundary conditions, were the first to receive serious attention. During the past century and a half, the mathematical literature concerning such systems has become very extensive and includes both practical and theoretical results of great importance.

It was not until 1897, however, that Nicoletti [18]/ introduced boundary conditions at more than two points into a differential system. He studied a single nth order equation together with boundary conditions at k points of an interval, under the restriction $k \leq n$. By 1908, Picone [19] had considered boundary conditions involving an infinite number of points along with integral boundary conditions. Wilder [32,33] subsequently introduced parameters into the differential equations. Independent of the earlier work of Nicoletti, Whyburn [29], after the manner of Carathéodory [10], developed a fundamental existence and uniqueness theorem for a quite general system involving n first-order ordinary differential equations with boundary conditions at $k \leq n$ points. With these points confined to an interval of definite length, and under a suitable Lipschitz condition, the solutions of the

/ Numbers within brackets refer to corresponding references of the bibliography.

system were shown to be continuous functions of the boundary points and values.

The studies of these individuals have been greatly expanded, and many new features of boundary value problems investigated, by other mathematicians. A variety of concepts and techniques such as Green's functions [15,29,30,34], Stieltje's integrals [6,25], matrix methods [1,2,7], and the calculus of variations [12,17,24,27] have been employed in the process. Much of the significant work done in this area, up until the year 1942, has already been reviewed in the literature [29], as have other major developments through the year 1955 [31]. Among the studies cited in these reviews, four are of particular interest to us because they have to do with interface characteristic value problems, as does this thesis.

Interface characteristic value problems involve differential equations on several adjoining open intervals together with a set of auxiliary conditions which apply at the ends of these intervals. At each point between neighboring intervals the relevant interface conditions relate left-hand and right-hand limiting values of a solution and its derivatives. The differential equations contain a parameter which may, or may not, appear in the boundary conditions.

In 1935, Reid [21] encountered such problems as a by-product while searching for functions which minimize the distance between several specified points when the functions are allowed to have discontinuities in their derivatives at the given points. Similar connections between the calculus of variations and certain interface problems were also explored by Mansfield [17] who, in 1941, found conditions for self-adjointness of interface characteristic value problems.

Perhaps, the first extensive study of Sturm-Liouville interface boundary value problems was undertaken by Sangren [22]. The left and right most subintervals of his system could extend to $-\infty$ and ∞, respectively. He showed that the related characteristic numbers were real and expansion theorems for the system were established. Applications of his results to problems in heat flow, potential and vibration theory, and nuclear reactor development were also pointed out.

Stallard [26], in 1955, studied matrix differential equations $Y' + PY = \lambda RY$ in an unknown matrix Y, with coefficient matrices P and R and a scalar parameter λ. The associated interface conditions were parameter free. He defined what is meant by a solution of such a system and gave existence and uniqueness theorems for solutions. His results also included conditions for self-adjointness, the development of Green's matrix, and expansion problems. He also gave conditions for self-adjointness of an interface system having $y'' + p_1y' + p_2y = \lambda ry$ as its differential equation, where p_1, p_2, and r are Lebesgue integrable functions.

During the past 25 years, boundary value problems have continued to provide a fruitful area of mathematical research.

By 1959, Barrett and Wylie [6] had used Stieltje's integrals to extend a number of results pertaining to classical Sturm-Liouville systems to systems whose boundary conditions contain the parameter. Applications of the theory to diffusion, heat flow, and vibration problems were also noted. That same year Barrett and Thorne [5] investigated Sturm-Liouville interface boundary value problems of the kind considered in this thesis. The following year, 1960, Sangren [23] solved some two dimensional membrane and heat flow problems which, upon separation of variables, resulted in interface characteristic value problems to which his earlier work was applied. During 1964 other boundary value problems involving interface systems were studied by Barrett [3].

A difficulty that is often encountered in connection with characteristic value problems is that of finding the related characteristic values. Frequently, it is either impractical or impossible to find their exact values. One must then resort to numerical methods for their computation. Separation and interlacing theorems which have proven to be useful in this regard were given by Barrett and Bendixen [4] in 1965.

An interface characteristic value problem of still another sort is that of Jerome [16], published in 1970. He defined linear operators L and L^* and then considered a differential equation of the form $L^*Ly = \lambda y$ together with a mixed set of parameter free boundary conditions of either ordinary or interface types. Under suitable restrictions on the operators involved, he proceeded to establish positiveness and multiplicity properties of the characteristic numbers, determined orthogonality conditions for the characteristic functions, and gave a formula for the solution of an equation of the form $L^*Ly = g(x)$.

A quite recent history updating some of the work done on interface systems is set forth in two 1975 publications of Brown [8,9]. These two articles have little to do with interface Sturm-Liouville systems, however.

Although a 1982 text by Wylie and Barrett [34] now discusses boundary value problems which entail interface Sturm-Liouville systems, the theory of such interface systems is still in an active state of development. The purpose of this thesis is to carry these investigations forward a little.

For the sake of clarity and easy reference, Chapter 1 begins with a characterization of just what is meant by an interface Sturm-Liouville system. After defining what is meant by characteristic values and characteristic functions of such systems, we state several well known theorems concerning these numbers and functions, and conditions under which they hold. Then characteristic equations and auxiliary functions of an interface system are derived. In Chapter 2 we introduce the notion of a subsystem and use a fundamental interlacing theorem to prove an interface interlacing theorem of considerable generality.

In Chapter 3, we apply our interface interlacing theorem to devise a subsystem method for locating characteristic numbers of an interface system within known intervals. In addition, we show that several well known properties of the characteristic numbers of ordinary Sturm-Liouville systems carry over to the characteristic numbers of interface Sturm-Liouville systems.

Ways of finding characteristic equations of both interface systems and their subsystems are discussed. This is done differently in

Chapters 2 and 3. A relation between auxiliary functions derived from different bases of the differential equations involved is also found.

Chapter 4 contains examples which serve to illustrate the theory of the first three chapters. Special attention is given to the interlacing phenomena of sets of characteristic numbers and the suborthogonality of sets of characteristic functions of the differential equations involved in various subsystems.

CHAPTER 1

INTERFACE STURM-LIOUVILLE SYSTEMS

1. Statement of the Problem and Related Theorems

Many problems in applied mathematics give rise to boundary value problems which, upon separation of variables, lead to differential systems of the type

(1) $\frac{d}{dx}[r(x)y_k'] + [\lambda p_k(x) + q_k(x)]y_k = 0, \quad x_{k-1} < x < x_k, \quad 1 \leq k \leq K$

(0;1) $U_0(y_1) \equiv a_1y_1'(a) + b_1y_1(a) = 0$

. . .

(k,1) $U_{k1}(y_k) + V_{k1}(y_{k+1}) \equiv [a_{k1}y_k'(x_k) + b_{k1}y_k(x_k)] + [c_{k1}y_{k+1}'(x_k) + d_{k1}y_{k+1}(x_k)] = 0$

$1 \leq k \leq K-1$

(k,2) $U_{k2}(y_k) + V_{k2}(y_{k+1}) \equiv [a_{k2}y_k'(x_k) + b_{k2}y_k(x_k)] + [c_{k2}y_{k+1}'(x_k) + d_{k2}y_{k+1}(x_k)] = 0$

. . .

(1;K) $V_K(y_K) \equiv c_Ky_K'(b) + d_Ky_K(b) = 0$

where $a = x_0$, $x_K = b$, and $2 \leq K$. In a variety of applications, it turns out that the coefficient function r admits of a single representation r(x) over the closed interval [a,b], where r is continuous and $r(x) > 0$, [5, p. 2]. On each subinterval $x_{k-1} < x < x_k$ of (a,b), p_k is continuous, $p_k(x) > 0$, q_k is continuous, and each p_k and q_k has finite limits at x_{k-1} and x_k. Expressions such as $y_k(x_k)$ and $y_k'(x_k)$ are to be interpreted as the left-hand limits

(2a) $y_k(x_k) = \lim_{x \to x_k^-} y_k(x)$ and $y'_k(x_k) = \lim_{x \to x_k^-} y'_k(x)$ $1 \leq k \leq K$.

Similarly, $y_{k+1}(x_k)$ and $y'_{k+1}(x_k)$ stand for the right-hand limits

(2b) $y_{k+1}(x_k) = \lim_{x \to x_k^+} y_{k+1}(x)$ and $y'_{k+1}(x_k) = \lim_{x \to x_k^+} y'_{k+1}(x)$ $0 \leq k \leq K-1$.

All of the constants a_1, b_1, ..., a_{k1}, ..., d_{k2}, ..., c_K, d_K are real.

Differential systems of this kind, involving boundary conditions of the type (0;1) and (1;K), and interface conditions of the sort (k,1) and (k,2) are referred to as interface Sturm-Liouville systems or Sturm-Liouville systems with interface conditions. Note that the semicolons in (0;1) and (1;K) serve to identify the boundary conditions, whereas commas are used in (k,1) and (k,2) to distinguish interface conditions. For brevity, we shall refer to the system just described as system (1 - 1;K).

A function y such that

(3) $$y(x) = y_k(x), \quad x_{k-1} < x < x_k, \quad 1 \leq k \leq K$$

is called a solution of (1) on the fundamental interval (a,b) if, for $1 \leq k \leq K$, both y_k and y'_k are continuous and y_k satisfies (1) on the interval (x_{k-1}, x_k). A solution y of (1) is said to be strictly nontrivial if there is no value of k for which $y_k(x) \equiv 0$. Interface systems having $y_k(x) \equiv 0$ for one or more values of k may be handled by considering two or more separate systems. Thus, from now on, only strictly nontrivial solutions for which each of the limits $y_k(x_k)$, $y'_k(x_k)$, $y_{k+1}(x_k)$, and $y'_{k+1}(x_k)$ exists, for the relevant values of k, will be called solutions of (1).

To make sure that the boundary conditions (0;1) and (1;K) are both present, the inequalities

(4) (a) $a_1^2 + b_1^2 \neq 0$ (b) $c_K^2 + d_K^2 \neq 0$

will be assumed to hold. With regard to the $2(K-1)$ interface conditions, we assume that

(5) (a) $|AB|_k |CD|_k \neq 0$ (b) $R_k = |CD|_k / |AB|_k > 0 \quad 1 \leq k \leq K-1$

where $|AB|_k$ and $|CD|_k$ are the determinants

(6) (a) $|AB|_k = \begin{vmatrix} a_{k1} & b_{k1} \\ a_{k2} & b_{k2} \end{vmatrix}$ (b) $|CD|_k = \begin{vmatrix} c_{k1} & d_{k1} \\ c_{k2} & d_{k2} \end{vmatrix} \quad 1 \leq k \leq K-1.$

In this thesis the only interface systems we shall consider will be those in which (4) and (5) are satisfied.

Under hypotheses (4) and (5), it is known that there are values of λ, called <u>characteristic values</u> or <u>characteristic numbers</u>, for which the interface system has strictly nontrivial solutions. These solutions are called <u>characteristic functions</u>. The characteristic values and characteristic functions have the following important properties [5, pp. 3-13].

<u>Theorem 1</u> The characteristic values are real.

<u>Theorem 2</u> The characteristic values are simple roots of the characteristic equation.

<u>Theorem 3</u> The characteristic functions are real.

<u>Theorem 4</u> The characteristic functions are orthogonal over the fundamental interval (a,b) with respect to the weight function p defined by

$$p(x) = \left(\prod_{j=0}^{k-1} R_j \right) p_k(x) \qquad x_{k-1} < x < x_k \qquad 1 \leq k \leq K$$

where $R_0 = 1$.

2. A Fundamental Interlacing Theorem

If $F(\lambda) = 0$ is a characteristic equation of a Sturm-Liouville system, we shall call the function $F(\lambda)$ an <u>auxiliary function</u> of that system.

A convenient way of isolating one characteristic number from all the others of a Sturm-Liouville system containing just one pair of interface conditions, is to use the fact that the zeros of an auxiliary function of the system interlace the zeros of appropriately chosen products of simpler auxiliary functions of ordinary Sturm-Liouville systems which have no interface conditions [4, pp. 75-78]. In saying that the zeros of one function interlace the zeros of another, we mean the following:

<u>Definition 1</u> The zeros of a function $f(\lambda)$ <u>interlace</u> the zeros of a function $g(\lambda)$ on an interval I if, and only if, on this interval:

1. no simple zero of $g(\lambda)$ is a zero of $f(\lambda)$
2. each double zero of $g(\lambda)$ is a zero of $f(\lambda)$
3. all zeros of $f(\lambda)$ are simple whereas the zeros of $g(\lambda)$ are either simple or double
4. between any two consecutive zeros of $g(\lambda)$, whether simple or double, there lies exactly one zero of $f(\lambda)$.

Now let $F(\lambda)$ be any function which can be expressed in the form

$$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda) \tag{1}$$

where f_1, f_2, f_3, f_4 are continuously differentiable functions over an interval I, and define W_{ij} to be the wronskian

$$W_{ij}(\lambda) = \begin{vmatrix} f_i(\lambda) & f_j(\lambda) \\ f_i'(\lambda) & f_j'(\lambda) \end{vmatrix} \tag{2}$$

of f_i and f_j, $1 \leq i, j \leq 4$. Also suppose

(3) $$W_{12}(\lambda)W_{34}(\lambda) < 0 \quad \text{on } I.$$

Then, as a fundamental interlacing theorem we have [4, pp. 71-75]:

Theorem 1 The zeros of $F(\lambda)$ interlace the zeros of either of the product functions $f_1(\lambda)f_3(\lambda)$ or $f_2(\lambda)f_4(\lambda)$ on I.

In Chapter 2 we shall show how this theorem can be applied to the characteristic equation $F(\lambda) = 0$ of a general interface Sturm-Liouville system, or to $F(\lambda)$ itself, so as to isolate the characteristic numbers of the system from one another. Further properties of the characteristic numbers will also be established in Chapter 3.

3. Characteristic Equations and Auxiliary Functions of Interface Systems

Since, for $1 \leq k \leq K$, the same parameter λ occurs in each of the equations

(1) $$\frac{d}{dx}[r(x)y_k'] + [\lambda p_k(x) + q_k(x)]y_k = 0 \qquad x_{k-1} < x < x_k$$

solutions of these equations will not depend on x alone, as (3), Sec. 1, suggests, but on both λ and x. However, as in (1), a prime on y still denotes differentiation with respect to x. If $u_k(x,\lambda)$ and $v_k(x,\lambda)$ are linearly independent solutions of (1) on (x_{k-1},x_k), a complete solution of (1) on (a,b) is given by

(2) $$y(x,\lambda) = A_k u_k(x,\lambda) + B_k v_k(x,\lambda) \qquad x_{k-1} < x < x_k \quad 1 \leq k \leq K \quad .$$

where each A_k and B_k may depend on λ, but not on x.

Once a complete solution $y(x,\lambda)$ of (1) on (a,b) is known, a characteristic equation for the interface system (1 - 1;K), Sec. 1, can be found by requiring $y(x,\lambda)$ to satisfy the auxiliary conditions

(0;1) through (1;K). When all of these conditions, except (1;K), are imposed on $y(x,\lambda)$, a system of $2K - 1$ homogeneous linear algebraic equations is obtained in the $2K$ parameters A_k and B_k. Since such a system always has nontrivial solutions, there are solutions of (1) on (a,b) which satisfy all of the auxiliary conditions, except (1;K). In fact, if $u(x,\lambda)$ is any one of these, they all have the form [5, pp. 9–11].

$$y(x,\lambda) = Au(x,\lambda) \qquad A \neq 0 \tag{3}$$

where A may depend on λ, but not on x or k.

Imposing (1;K) on (3), we get

$$V_K(y) \equiv c_K y'(b,\lambda) + d_K y(b,\lambda) = 0 \tag{4}$$

as a <u>characteristic equation</u>, and

$$V_K(y) \equiv c_K y'(b,\lambda) + d_K y(b,\lambda) \tag{5}$$

as an <u>auxiliary function</u> of our interface Sturm–Liouville system.

Since $V_K(y) = AV_K(u)$, (4) is simply the equation

$$V_K(u) \equiv c_K u'(b,\lambda) + d_K u(b,\lambda) = 0 \tag{6}$$

multiplied through by $A \neq 0$. Hence (4) and (6) are equivalent, and so the function

$$F(\lambda) \equiv V_K(u) \equiv c_K u'(b,\lambda) + d_K u(b,\lambda) \tag{7}$$

has the same zeros as (5). The solutions of (6), i.e. the zeros of (7), are the <u>characteristic numbers</u> of system (1 – 1;K). Because these numbers remain unchanged, no matter what particular function of the family (3) is identified as $u(x,\lambda)$, it is customary to refer to (6) as <u>the</u> characteristic equation, and to (7) as <u>the</u> auxiliary function of the interface system, even though (6) and (7) may change by a nonzero multiple with each different choice of u.

CHAPTER 2

AN INTERFACE INTERLACING THEOREM

1. Special Representations of the Auxiliary Function

In this chapter we shall show that the fundamental interlacing theorem, Sec. 2, Chap. 1, can be applied to the auxiliary function $F(\lambda)$ of an interface Sturm-Liouville system for the purpose of isolating characteristic numbers of the system within intervals. We shall derive our result by proving that $F(\lambda)$ can be written in the form

(1) $$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

required by the fundamental interlacing theorem, where for all real values of λ, each $f_i(\lambda)$, $1 \le i \le 4$, is continuously differentiable and $W_{12}(\lambda)W_{34}(\lambda) < 0$.

A number of distinct representations of F, each having the structure (1) may be found through the use of determinants (as is done in Sec. 3, Chap. 3) or as follows. With k fixed, and $1 \le k < K$, let $u(x,\lambda)$ now be any specific solution of (1), Sec. 1, Chap. 1, on (a,x_k) which satisfies all auxiliary conditions of the interface system involving values of x less than x_k, and let

(2) $$u(x,\lambda) = u_j(x,\lambda) \qquad x_{j-1} < x < x_j \qquad 1 \le j \le k$$

be a representation of u. By analogy with (3), Sec. 3, Chap. 1, all solutions of this kind have the form $Au(x,\lambda)$. Similarly, suppose $v(x,\lambda)$ is any particular solution of (1), Sec. 1, Chap. 1, on (x_k,b) which satisfies all auxiliary conditions involving values of x greater than x_k, and let

(3) $$v(x,\lambda) = v_j(x,\lambda) \qquad x_{j-1} < x < x_j \qquad k+1 \leq j \leq K$$

be a representation of v. All such solutions have the form $Bv(x,\lambda)$. Thus, every solution of the differential equations of system (1 - 1; K) on (a,b) which satisfies all of the auxiliary conditions, except the two interface conditions at x_k, has a representation in terms of $u(x,\lambda)$ and $v(x,\lambda)$ of the form

(4) $$y(x,\lambda) = \begin{cases} Au(x,\lambda) & \text{on } (a,x_k) \quad A \neq 0 \\ Bv(x,\lambda) & \text{on } (x_k,b) \quad B \neq 0 \end{cases}$$

where A and B may depend on λ, but not on x or k. Imposing the interface conditions (k,1) and (k,2) on (4) we have

(5) $$\begin{bmatrix} U_{k1}(u) & V_{k1}(v) \\ U_{k2}(u) & V_{k2}(v) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which, for each k such that $1 \leq k < K$, gives

(6) $$F(\lambda) \equiv U_{k1}(u)V_{k2}(v) - U_{k2}(u)V_{k1}(v)$$

as a representation of the auxiliary function F. That these K - 1 representations all have the structure (1) is obvious, for we may set

$$f_1(\lambda) = U_{k1}(u), \quad f_2(\lambda) = U_{k2}(u), \quad f_3(\lambda) = V_{k1}(v), \quad f_4(\lambda) = V_{k2}(v).$$

For each fixed k, the function u as presently defined plays much the same role as did the function u appearing in (3), Sec. 3, Chap. 1. Indeed, if $U_{k1}(y_k) = 0$, instead of $V_K(y_K) = 0$, is taken as a right-hand boundary condition, then V_K gets replaced by U_{k1} in (6) and (7) of Sec. 3, Chap. 1. Thus, we can immediately interpret $U_{k1}(u)$ as the auxiliary function of the Sturm-Liouville system

(1:k) $\frac{d}{dx}[r(x)y_j'] + [\lambda p_j(x) + q_j(x)]y_j = 0, \quad x_{j-1} < x < x_j, \quad 1 \le j \le k$

(0;1) $U_0(y_1) = 0$

. . .

(j,1) $U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0$

$1 \le j \le k-1$

(j,2) $U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0$

. . .

(1;k) $U_{k1}(y_k) = 0$

where, if $k = 1$, there are no interface conditions. If, in this system, the right-hand boundary condition (1;k) is replaced by $U_{k2}(y_k) = 0$, the resulting system has $U_{k2}(u)$ as its auxiliary function.

Recalling the meaning (3) of v, and noting the effect of taking $V_{k1}(y_{k+1}) = 0$ as the relevant left-hand boundary condition, we similarly find that $V_{k1}(v)$ represents the auxiliary function of the Sturm-Liouville system

(k+1:K) $\frac{d}{dx}[r(x)y_j']+[\lambda p_j(x) + q_j(x)]y_j = 0, \quad x_{j-1} < x < x_j, \quad k+1 \le j \le K$

(k;1) $V_{k1}(y_{k+1}) = 0$

. . .

(j,1) $U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0$

$k+1 \le j \le K-1$

(j,2) $U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0$

. . .

(1;K) $V_K(y_K) = 0$

where there are no interface conditions in the system if $k = K-1$. If, in this system, the left-hand boundary condition (k;1) is replaced by $V_{k2}(y_{k+1}) = 0$, the resulting system has $V_{k2}(v)$ as its auxiliary function.

2. Subsystems of Interface Sturm-Liouville Systems

Clearly, the differential systems of Sec. 1 having $U_{k1}(u)$, $U_{k2}(u)$, $V_{k1}(v)$, and $V_{k2}(v)$ as auxiliary functions, all involve differential equations and auxiliary conditions which are closely related to, if not included among, those of system (1 - 1;K), Sec. 1, Chap. 1. It is therefore natural to refer to the first four of these five systems as <u>subsystems</u> of the last one. To describe this connection precisely, let us introduce the <u>Kronecker</u> <u>delta</u>

$$(1) \qquad \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \qquad j,k = 0, 1, 2, 3, \ldots$$

and for each fixed set of integers {i, h, m, n}, define a subsystem of system (1 - 1;K) to be a differential system of the type:

$$(i+1:h) \quad \frac{d}{dx}[r(x)y_j'] + [\lambda p_j(x) + q_j(x)]y_j = 0, \; x_{j-1} < x < x_j, \; i+1 \leq j \leq h$$

$$(i;m) \quad \delta_{0i}U_0(y_1) + (1 - \delta_{0i})[\delta_{1m}V_{i1}(y_{i+1}) + \delta_{2m}V_{i2}(y_{i+1})] = 0$$

$$\cdots$$

$$(j,1) \quad U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0$$

$$(j,2) \quad U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0 \qquad i+1 \leq j \leq h-1$$

$$\cdots$$

$$(n;h) \quad \delta_{hK}V_K(y_K) + (1 - \delta_{hK})[\delta_{n1}U_{h1}(y_h) + \delta_{n2}U_{h2}(y_h)] = 0$$

where $0 \leq i \leq h-1$, $1 \leq h \leq K$, and $m,n = 1,2$. We shall identify these subsystems by the notation (i;m - n;h), rather than by the notation (i+1:h - n;h) which fails to specify the relevant value of m when $0 < i$.

As is evident from (i+1:h), if $i < h-1$, the differential equations belonging to a subsystem of an interface system whose fundamental interval is (a,b) must be solved on two or more of the successive subintervals originally identified with (a,b). For i = 0, the left-hand boundary condition (i;m) is $U_0(y_1) = 0$, as in system (0;1 - 1;k), Sec. 1, and the value of m is irrelevant. If $0 < i$, condition (i;m) becomes $V_{i1}(y_{i+1}) = 0$, or $V_{i2}(y_{i+1}) = 0$, according as m = 1, or m = 2, and, for either value of m, it follows from (5) and (6b), Sec. 1, Chap. 1, that $c_{im}^2 + d_{im}^2 \neq 0$. For h = K, the right-hand boundary condition (n;h) is $V_K(y_K) = 0$, as in system (k;1 - 1;K), Sec. 1, and the value of n is irrelevant. If $h < K$, condition (n;h) becomes $U_{h1}(y_h) = 0$, or $U_{h2}(y_h) = 0$, according as n = 1, or n = 2, and, in either case, (5) and (6a), Sec. 1, Chap. 1, imply $a_{hn}^2 + b_{hn}^2 \neq 0$. Finally, the interface conditions (j,1) and (j,2) are contained among those of the original interface system. Thus, we see that every subsystem (i;m - n;h) of (1 - 1;K), Sec. 1, Chap. 1, for which $i < h - 1$, is itself an interface Sturm-Liouville system which has (x_i, x_h) as its fundamental interval. To all such subsystems, Theorems 1 through 4, Sec. 1, Chap. 1, apply. In particular, Theorem 4 yields an important <u>suborthogonality property</u> of the characteristic functions of each subsystem whose fundamental interval is (x_i, x_h): it is that these characteristic functions are orthogonal over (x_i, x_h) with respect to the weight function p given by

$$(2) \qquad p(x) = \left(\prod_{s=i}^{j-1} R_s\right) p_j(x) \qquad x_{j-1} < x < x_j \qquad i+1 \leq j \leq h$$

where $R_i = 1$ and, for $i+1 \leq s \leq h-1$, R_s is defined by (5b) and (6), Sec. 1, Chap. 1.

A subsystem, for which $i = h-1$, is an ordinary Sturm-Liouville system with no interface conditions. On the other hand, if $i = 0$ and $h = K$, the subsystem $(0;m - n;K)$ and the original interface system $(1 - 1;K)$ are the same. In other words, every interface system is a subsystem of itself. A proper subsystem of an interface system is a subsystem which is distinct from the interface system.

3. Interlacing Properties of the Characteristic Numbers of Certain Subsystems

Having now defined what is meant by a subsystem of an interface system, let us henceforth denote system $(1 - 1;K)$, Sec. 1, Chap. 1, by $(1 - 1;K)$, and its subsystem $(i;m - n;h)$, Sec. 2, by $(i;m - n;h)$, and simply remember the sections where these concepts are defined.

In Sec. 1, we found that, for each fixed k such that $1 \leq k < K$, the functions

(1) $$f_1(\lambda) = U_{k1}(u) \equiv a_{k1}u'(x_k,\lambda) + b_{k1}u(x_k,\lambda)$$

(2) $$f_2(\lambda) = U_{k2}(u) \equiv a_{k2}u'(x_k,\lambda) + b_{k2}u(x_k,\lambda)$$

(3) $$f_3(\lambda) = V_{k1}(v) \equiv c_{k1}v'(x_k,\lambda) + d_{k1}v(x_k,\lambda)$$

(4) $$f_4(\lambda) = V_{k2}(v) \equiv c_{k2}v'(x_k,\lambda) + d_{k2}v(x_k,\lambda)$$

could be interpreted as auxiliary functions of the respective subsystems $(0;m - 1;k)$, $(0;m - 2;k)$, $(k;1 - n;K)$, and $(k;2 - n;K)$. We also saw that the auxiliary function of $(1 - 1;K)$ could be written as

(5) $$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

which has the structure required by the fundamental interlacing theorem, Sec. 2, Chap. 1. As has been pointed out, each of the functions f_i, $1 \leq i \leq 4$, is an auxiliary function of either an ordinary, or else an interface, Sturm-Liouville subsystem of (1 - 1;K). Any one of the f_i which is itself the auxiliary function of an interface subsystem can, in the same way that (5) was derived, be represented by a difference of products of auxiliary functions of subsystems of that interface subsystem.

By applying the process just mentioned to the auxiliary function of each new interface subsystem evolved, it is possible to express $F(\lambda)$ solely in terms of a finite number of sums and differences of products of auxiliary functions of ordinary Sturm-Liouville systems. But, as is well know, the auxiliary functions of all these Sturm-Liouville systems are analytic for all real values of their argument, [14, pp. 388-389]. Hence, $F(\lambda)$ is analytic on $(-\infty,\infty)$. Moreover, the functions $f_i(\lambda)$ of (5) are also analytic and, therefore, continuously differentiable, as required by Theorem 1, Sec. 2, Chap. 1, on $(-\infty,\infty)$. Summarizing these comments, we have:

Lemma 1 For all real values of its argument, the auxiliary function of an interface Sturm-Liouville system is analytic.

Lemma 2 For all real values of λ, and for each fixed value of k such that $1 \leq k < K$, the functions $f_i(\lambda)$, defined by Eqs. (1) through (4) have continuous derivatives of all orders.

Having shown that $F(\lambda)$ has the form (5) and that each $f_i(\lambda)$, as given by (1) through (4), is continuously differentiable, we must still prove that the remaining hypothesis of the fundamental interlacing theorem holds before that theorem can be applied to $F(\lambda)$ and $f_1(\lambda)f_3(\lambda)$, or to $F(\lambda)$ and $f_2(\lambda)f_4(\lambda)$. That is, we must show that, for all real values of λ

$$(6) \qquad W_{12}(\lambda)W_{34}(\lambda) = \begin{vmatrix} f_1(\lambda) & f_2(\lambda) \\ f_1'(\lambda) & f_2'(\lambda) \end{vmatrix} \begin{vmatrix} f_3(\lambda) & f_4(\lambda) \\ f_3'(\lambda) & f_4'(\lambda) \end{vmatrix} < 0.$$

Of course, the primes here signify differentiation with respect to λ.

Once (6) has been verified, the following interface interlacing theorem will be established.

Theorem 1 For each k such that $1 \leq k < K$, let the corresponding functions $f_i(\lambda)$, $1 \leq i \leq 4$, be those defined by Eqs. (1) through (4). For each set $\{f_1, f_2, f_3, f_4\}$ thus defined, the zeros of the auxiliary function $F(\lambda)$ of system $(1 - 1;K)$ interlace the zeros of either of the product functions $f_1(\lambda)f_3(\lambda)$ or $f_2(\lambda)f_4(\lambda)$ on $-\infty < \lambda < \infty$.

Proof that $W_{12}(\lambda)W_{34}(\lambda) < 0$ First, the structure of $W_{12}(\lambda)W_{34}(\lambda)$ will be investigated. Making use of Eqs. (1) and (2), and familiar properties of determinants, we obtain

$$(7) \quad W_{12}(\lambda) = \begin{vmatrix} f_1(\lambda) & f_2(\lambda) \\ f_1'(\lambda) & f_2'(\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} a_{k1}u'(x_k,\lambda)+b_{k1}u(x_k,\lambda) & a_{k2}u'(x_k,\lambda)+b_{k2}u(x_k,\lambda) \\ a_{k1}\frac{d}{d\lambda}u'(x_k,\lambda)+b_{k1}\frac{d}{d\lambda}u(x_k,\lambda) & a_{k2}\frac{d}{d\lambda}u'(x_k,\lambda)+b_{k2}\frac{d}{d\lambda}u(x_k,\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} u'(x_k,\lambda) & u(x_k,\lambda) \\ \frac{d}{d\lambda}u'(x_k,\lambda) & \frac{d}{d\lambda}u(x_k,\lambda) \end{vmatrix} \begin{vmatrix} a_{k1} & a_{k2} \\ b_{k1} & b_{k2} \end{vmatrix}$$

$$\equiv W[u'(x_k,\lambda), u(x_k,\lambda)] \; |AB|_k$$

where $W[u'(x_k,\lambda),u(x_k,\lambda)]$ is the wronskian of $u'(x_k,\lambda)$ and $u(x_k,\lambda)$ with respect to the independent variable λ.

Similarly, through the use of Eqs. (3) and (4), we find

$$(8) \qquad W_{34}(\lambda) = W[v'(x_k,\lambda), \; v(x_k,\lambda)] \, |CD|_k.$$