EGO DEPLETION: AN ECONOMIC MODEL OF SELF-CONTROL

by

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I now think of economic theory as an art rather than a science. The virtues of explanatory power, simplicity, and elegance are shared by both econometrics and economic theory. However, econometrics, as a science, involves the consideration of a hypothesis along with empirics to lead to a statistical inference regarding the hypothesis. Theory, as I have come to know it, depends solely on the *form* of the hypothesis, whether formalized with mathematics or otherwise. While creativity, explanatory power, and elegance are important to good econometrics, these are the *sole crux* of economic theory. As such, I am grateful to all my math instructors who have taught me to appreciate the beauty in notation. Further, I am indebted to all those who have inspired creativity in me—professors, friends, family, philosophers, artists, and scientists.

Special thanks to my thesis committee members, Robert Fleck, Christiana Stoddard, and Timothy Fitzgerald. Much of the theory herein was inspired by numerous conversations with Francis Smart.

Boredom means you're off the Quality track, you're not seeing things freshly, you've lost your "beginner's mind" and your motorcycle is in great danger. Boredom means your gumption supply is low and must be replenished before anything else is done.

When you're bored, stop! Go to a show. Turn on the TV. Call it a day. Do anything but work on that machine. If you don't stop, the next thing that happens is the Big Mistake, and then all the boredom plus the Big Mistake combine together in one Sunday punch to knock all the gumption out of you and you are really stopped.

My favorite cure for boredom is sleep. It's very easy to get to sleep when bored and very hard to get bored after a long rest. My next favorite is coffee. I usually keep a pot plugged in while working on the machine. If these don't work it may mean deeper Quality problems are bothering you and distracting you from what's before you. The boredom is a signal that you should turn your attention to these problems—that's what you're doing anyway—and control them before continuing on the motorcycle...

Zen has something to say about boredom. Its main practice of "just sitting" has got to be the world's most boring activity—unless it's that Hindu practice of being buried alive. You don't do anything much; not move, not think, not care. What could be more boring? Yet in the center of all this boredom is the very thing Zen Buddhism seeks to teach. What is it? What is it at the very center of boredom that you're not seeing?

Robert M. Pirsig, *Zen and the Art of Motorcycle Maintenance: An Inquiry into Value*

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ABSTRACT

Philosophers, writers, and psychologists have studied and commented on the concept of willpower for thousands of years. Recently, behavioral economics has enjoyed a flurry of interest, and many economists have provided research—both theoretical and empirical—to bridge the gap between traditional microeconomics and contemporary evidence. Ego depletion is a relatively new view of self-control, demonstrated by psychologists in an experimental setting, that considers willpower to be a personal, renewable resource that is affected by an agent's actions. This paper proposes a fundamental framework that allows the phenomenon of ego depletion to coexist soundly with the traditional consumer microeconomic model. A formal generalized consumer model is proposed in which willpower is a depletable, renewable, unconstrained resource, and results are derived from specific cases. The conclusions are consistent with the theory of ego depletion, and many of the results illustrate the agent's optimal choices in a way that has not been previously presented.

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INTRODUCTION

An oddity that fascinated both Dostoyevski and Tolstoy, avoiding thinking about white bears is difficult. When an individual is asked to not think about white bears, the individual does just that—in fact, she cannot stop thinking about them (Wegner, Schneider, Carter, and White, 1987). While the implications of thought suppression are interesting for modern cognitive psychology, they are unexpected for economics. Not only does the individual struggle with avoiding thoughts of polar bears, but the individual will struggle with any other difficult task following the attempted thought suppression (Baumeister, Bratslavsky, Muraven, and Tice, 1998). This finding sheds light on the economic problem of self-control. While previous theories still explain many pieces of an economic agent's lack of self-control, the notion that current self-control hinders future self-control is a fundamental finding.

Economists have presented a wide variety of models to account for a perceived lack of self-control. The most basic explanation is that the agent in question changes her mind (e.g., she has time-inconsistent preferences) or that she gains information in the subsequent period. More useful theories include hyperbolic discounting and the rational addiction model. These theories all suggest different reasons why an individual may alter her predicted consumption in a future period. The reason could be that the individual is simply always more focused on the present than the future, as emphasized by hyperbolic discounting. Alternatively, the individual could be generating consumption capital, which affects her future consumption.

The primary motivation for this analysis of the self-control problem is to expand the explanatory power of the traditional consumer microeconomic model. To be sure, a model is useful if it explains one phenomenon particularly well while simplifying all others. A model that attempts to incorporate a multitude of phenomena is not as useful—either the mathematics become intractable or the concepts become muddled. In this vein, the present analysis emphasizes the phenomenon of ego depletion as it applies to the resource-allocating individual. Such a motivation is not an end in itself; rather, improved explanatory power allows economists to better understand how real-world individuals make decisions and execute them.

Indeed, this problem is especially important because it seems to hold true for everyone. Whereas some economic theories only apply to certain agents in particular circumstances, issues related to willpower and ego depletion affect everyone all the time. Therefore, if economists wish to understand the consumer, a complete account of ego depletion is necessary. Likewise, such a theory is useful to every non-economist as well—everyone must allocate their executive energy carefully, therefore a better understanding of how it is used will benefit everyone.

In this paper, the classic consumer model is expanded to explain the phenomenon of ego depletion. The model herein takes a basic, yet novel approach at modeling willpower, keeping the classic model nearly as simple, while capturing the essense of the ego depletion circumstance. The model attempts to show, for example, if an individual is considering working a full-time job and going back to school, then the choice to do them simultaneously may be unwise. Rather, she will likely be happier (or more successful) with a sequence where she quits her full-time job first and then starts taking classes a few weeks later (after a period of ego repletion). Alternatively, suppose that she normally goes to work and then goes to the gym on weekdays. If she knows that she will have to spend time with people whom she dislikes on Saturday, she would be wise to work a little less on Friday or forego the gym (she is conserving willpower for the dreaded event). These are only a couple examples of how this model applies to everyday life.

The present paper formulates a complete and cohesive general consumer microeconomic model that accounts for an agent's willpower (most commonly termed "self-control" in economics). The basic framework permits n goods in T time periods with personal discounting. It includes the use of a typical budget constraint. The model is then treated with a series of cases that demonstrate its suitability to laboratory evidence. Currently, microeconomic models of ego depletion only explain the optimal path of consumption for a single good over time (Ozdenoren, Salant, and Silverman, 2006). As such, the basic framework of the model herein is a fundamental and necessary contribution to the self-control literature in economics.

LITERATURE

This section is structured into two parts—the first covers the discussion in psychology, the other follows the discussion in economics. The psychology literature review will give some historical context and then quickly introduce the concepts to be discussed at length in the next section. The economic literature review will elaborate on the various models of self-control. This review is intended to explain concepts to be used in the following sections, as well as help illustrate the present paper's contribution.

Ego Depletion in Psychology

Like many contemporary psychology concepts, the concept of self-control was redefined as Freud founded the field of analytic psychology. Freud (1966) introduced two concepts of particular interest in this discussion—the pleasure principle and the reality principle. The pleasure principle states that individuals are entirely concerned with seeking pleasure and avoiding displeasure. This concept lies at the root of our psychological and biological beings. However, humans also have the capacity of foresight, and human capacities can operate outside the realm of instincts. As such, Freud coins a second concept called the reality principle—individuals take into account reality, in which, at times, momentary displeasure must be experienced to obtain greater pleasure. Furthermore, Freud writes that "the transition from the pleasure principle to the reality principle is one of the most important steps forward in the ego's development" (22:357). For example, the ability to delay gratification is a result of the transition from the pleasure principle to the reality principle (Mischel, 1996). Freud attributes the reality principle to the ego, while the id and superego serve two polar-opposite motivations.

Baumeister et al. (1998) explain that Freud "described the ego as the part of the psyche that must deal with the reality of the external world by mediating between conflicting inner and outer pressures. In his scheme, for example, a Victorian gentleman standing on the street might feel urged by his id to head for the brothel and by his superego to go to church, but it is ultimately left up to his ego to start his feet walking in one direction or the other. Freud also seems to have believed that the ego needed to use some energy in making such a decision" (page 1263). It is with this historical context that recent psychology literature has focused on self-control.

In his seminal 1975 paper, Ainslie argues that people know the costs of their impulsive actions, but they distort the costs in such a manner that contemporary benefits carry more weight. A similar theory was first proposed by Mowrer and Ullman (1945), and studies have found empirical support of this model of hyperbolic discounting (Blatt and Quinlan, 1967; Ainslie, 1975; McCown et al., 1987; Kirby and Herrnstein, 1995). Despite this newfound accord between theory and data, recent work in psychology has focused on Freud's idea that the ego uses energy to mediate between id and superego (or the present and future selves, respectively) according to the reality principle.

Muraven et al. (1998) propose a theory of regulatory depletion by writing that individuals "have a limited capacity for self-regulation, akin to having a limited supply of strength or energy" (page 774). They explain that this theory predicts that an individual's capacity to regulate herself is diminished if she regulated herself in the previous time period. The authors test this theory by taking two groups of individuals, assigning a regulatory task to only one group, and then assigning a subsequent regulatory task to both groups. If there is a statistical difference between the control and treatment groups in the final task, we can conclude that the previous regulatory task affected the proceeding one.

The results of such an experiment may seem self-evident, but on the contrary, a review of the literature on self-control reveals three pair-wise exclusive theories (Baumeister and Vohs, 2003). The first theory (which is akin to Freud's theory) regards self-control as reliant on a particular *personal strength* or *energy* and predicts that attempts at regulation will *impair* successive attempts due to *depletion*. The second theory views self-control as a *cognitive process* and predicts *short-term improvements* of the ability to self-regulate due to *priming*. The third theory considers self-control to be a *skill* and predicts *long-term improvements* of the ability to self-regulate due to *learning*. It is then clear that this experiment is testing a refutable hypothesis.[1]

[1] For further discussion of varying theories, see Muraven and Baumeister (2000); Baumeister (2002); Twenge and Baumeister (2002).

Many experiments were conducted to test the first hypothesis that self-control depends on a personal strength or energy. The first study instructed the treatment group to control their emotions while watching a video; the control group was not told to regulate their emotions. Afterward, individuals' endurance at squeezing a hand-grip exerciser was measured. Those who had previously regulated themselves demonstrated a significantly shorter endurance (Muraven et al., 1998). Similarly, another experiment involved instructing the treatment group to avoid thinking about white bears; the control group received no such instructions. Individuals were subsequently given unsolvable anagrams—those who had not regulated their thoughts persisted on the anagrams statistically significantly longer. Other studies involved impulse control, dieting, and personal spending (Baumeister and Vohs, 2003; Vohs and Faber, 2007).

However, it seems that the third hypothesis—that self-control is a learned skill—carries some weight as well. Numerous studies involved instructing the treatment group to perform self-regulation exercises for two weeks, while the control had no such assignments. After two weeks, the treatment group demonstrated a significant improvement in self-control relative to the control (Muraven et al., 1999; Baumeister and Vohs, 2003). It then seems that the time horizon under consideration is crucial to the results; a time horizon of *two hours* will demonstrate ego depletion, but a time horizon of *two weeks* will demonstrate learning.

Interestingly, when an individual is offered a small amount of money to drink a foul beverage, they deplete more of their ego resource than when offered a large sum of money to drink the same beverage (Baumeister and Vohs, 2003). At first blush this may seem counter-intuitive, but perhaps ego depletion is simply dependent on the surplus of the transaction. For example, if a person is sustaining displeasure for a small gain, a large amount of willpower will be consumed in the process. However, if a person is sustaining the same displeasure for a large gain, a smaller amount of willpower will be consumed, presumably because the individual maintains energy due to the foresight of the large gain.

Decision-making was also linked to ego depletion through a series of experiments (Baumeister and Vohs, 2003). In one experiment, individuals were asked to choose between different consumption goods repeatedly, and they exhibited ego depletion relative to a control group. In another study, participants were asked to give a speech contrary to their personal beliefs. Those who were allowed more choices with regard to the speech demonstrated depletion relative to controls. As a result, ego depletion seems to encompass the decision-making process in addition to the executive process within the individual.

Self-Control in Economics

As microeconomic theory has advanced to incorporate more details of the actions of real-world consumers, a need to explain self-control within the microeconomic framework has emerged. Traditionally, when attempting to explain social phenomena, economists have avoided citing a change in consumer preferences as the basis. This is for good reason—preferences are the black box of consumer theory. Or, in the words of Stigler and Becker (1977), "The desires themselves are *data*" (page 76). Further, the utility function incorporates information that is unobservable in the real world; emphasizing the utility function as the source of any phenomena makes any hypothesis impossible to test in the field.[2]

To be sure, it is possible that the self-control problem is just an instance of changing preferences. For example, suppose that an individual has been avoiding purchasing Russian novels so that she may purchase a new computer. After a few months of diligently following her plan, perhaps she simply changes her mind, deciding that she likes Russian novels much more than she previously did, and that she will forgo the new computer so that she may spend her income on novels. This is an example of what is commonly known as time-inconsistent preferences. To simply explain this occurrence as a change in preferences is to say that we do not know the underlying reason why she changed her mind. If the reason is a change in relative prices, then the change can be reflected outside of the utility function, yielding useful predictions. Likewise, if the reason that she changed her intertemporal consumption bundle is that her income has changed, the goods have changed (perhaps the novels are now printed on acid-free paper or computers are less

[2]This statement assumes that use of discounting or expectation operators is done outside of the utility function.

cumbersome), or her information regarding the goods has changed, then we need not explain the phenomenon through the utility function.

Indeed, all of these alternative explanations are valid and are presumably testable in the field. Unfortunately, none of them explains the self-control phenomenon—neither changes in relative prices, income, nor a change in information about the goods. The problem of self-control goes beyond the traditional consumer model; with constant relative prices, income, information, and an exponential discount function, individuals often seem irrational (namely, inconsistent). As a result, economists have attempted to formulate models that are consistent with both the traditional framework and with the problem of self-control.[3]

Thaler (1990) presented a potpourri of anomalies, suggesting that assumptions such as fungibility do not hold in the real world. Fungibility is the idea that money has no labels or categorization—it is simply wealth with no intrinsic value. Indeed, based on evidence from the field, people treat various situations entirely differently; an individual may simultaneously demonstrate vastly different marginal propensities to save or personal discount rates. In the same vein, Akerlof (1991) gives motivation to the cause by writing that "although an analysis of behavioral pathology might initially appear to be outside the appropriate scope of economics, I shall argue that, in important instances, such pathology affects the performance of individuals and institutions in the economic and social domain" (page 1). There are certainly many reasons to seek economic models for self-control. Accurately modeling self-control can help explain other anomalous behavior. Also, incorporating real-world phenomena builds the validity of microeconomic theory. It is with these considerations that economists have sought to integrate numerous psychological theories into the field of economics.

Hyperbolic Discounting

Strotz's 1956 paper, titled "Myopia and Inconsistency in Dynamic Utility Maximization," was the first attempt at modeling self-control in economics (Ainslie, 1975; Laibson,

[3]For a general introduction to the assimilation of willpower into economics, see Loewenstein (2000); O'Donoghue and Rabin (2000); Loewenstein (1996).

1997; Camerer and Loewenstein, 2004). The theory in the paper shows how an individual can rationally change plans over time, even with constant tastes and preferences and unchanged information. Consider again our hypothetical individual who is deciding between spending her money on Russian novels today and saving her money to purchase a computer in a year. Today, she decides that she will buy a novel, but next week, she will forgo the novel and instead begin saving for the computer. However, when next week comes, she violates her previous plan and again buys a novel and forgoes saving money for another week. Strotz explains that this is not necessarily irrational, nor does it imply a change in tastes or preferences.

The essence of the proposed model is dynamic hyperbolic discounting. Strotz asserts that an individual's personal discount rate may not (and need not) be of the traditional exponential form. If we assume that the discount function is not exponential, then time-inconsistent actions are rational. As Strotz writes, "To-day it will be rational for a man to jettison his 'optimal' plan of yesterday, not because his tastes have changed in any unexpected way nor because his knowledge of the future is different, but because to-day he is a different person with a new discount function—the old one shifted forward in time" (page 173). A non-exponential discount function allows the individual to possess constant tastes and information, yet rationally change her plan.

Consumer Strategies for Expected Myopia

Strotz continues to explain how an individual will cope with her inability to follow a plan of action. It may be that she will take action to ensure that her future action is in accordance with her plan today. This is commonly known as utilizing a "commitment device." The traditional example of this is Homer's Odysseus ordering his crew to tie him to the mast of his ship as they pass the Sirens. Contemporary examples of a commitment device include a gym membership and a life insurance policy. In fact, if our hypothetical individual realizes that she'll never accomplish her long-term goal of purchasing a computer, she may decide to buy the computer on credit today, contractually forcing her future self to save for the computer. As a result, she will be getting the good she desires by utilizing a credit card as a commitment device. Obviously, the cost of commitment devices is not always trivial. Agents often pay a premium to force commitment to a plan of action.

Strotz argues that our individual may instead choose "consistent planning." Over time, she will learn that she keeps buying novels and pushing back her plan to save her income. At some point, if she doesn't choose to utilize a commitment device, she may eliminate the goal of saving altogether. At this point, she attempts to find the best plan that she will naturally follow in the future (she has learned that she will not follow her savings plan)—this plan may simply be to buy a Russian novel in each time period. Strotz goes a step further, attempting to demonstrate how an individual will calculate a plan for consumption that assumes consistent planning. Strotz's basic framework remains extremely useful, and there has been growing interest in hyperbolic discounting (and more generally, self-control) since his publication in 1955.[4]

Dual-Self Model

Schelling (1978) describes the self as a pair of individuals by writing, "Everybody behaves like two people, one who wants clean lungs and long life and another who adores tobacco, or one who wants a lean body and another who wants dessert. The two are in a continual contest for control; the 'straight' one often in command most of the time, but the wayward one needing only to get occasional control to spoil the other's best laid plan" (page 290). Thaler and Shefrin (1981) utilizes this idea by using two sets of preferences—often called a "dual-self model"—to help explain the self-control paradox. In this model, an individual agent is viewed as an organization of two people, a "planner" and a "doer." The planner attempts to maximize lifetime utility, while the doer attempts to maximize instantaneous utility. The planner can affect the doer's actions by creating incentives or altering preferences while permitting discretion by the doer, or the planner can impose constraints by creating rules for the doer. However, these controls are costly for the planner to impose—this assumption leads to many interesting results.

[4] See Pollak (1968); Blackorby et al. (1973); Peleg and Yaari (1973); Hammond (1976); Yaari (1977); Goldman (1979, 1980); Gul and Pesendorfer (2001, 2004, 2007); Dekel et al. (2009).

Thaler and Shefrin (1981) finds results that are contrary to predictions made from the traditional model. Consider two individuals with the same savings rate s. Ceteris paribus, if one must save $p < s$ because of a pension plan, the traditional model predicts that both individuals will continue saving the same amount s. However, in the proposed model, imposing constraints on the doer is costly, and this constraint is free (at least in the short-run). Because self-imposed constraints have increasing marginal cost, total savings increases. Indeed, this result is similar to those from empirical studies (page 399). Now consider two individuals who both make \$24,000 per year—one person earns \$2,000 per month, while the other earns \$1,500 monthly and a \$6,000 bonus in March. The proposed model predicts that the former person with the steady income stream will save less, while the latter person with the bonus will save more. Thaler and Shefrin believe that the withheld bonus is acting as a self-control device.[5] The authors also found empirical evidence in the literature to support another result, namely a discrepancy between an individual's marginal rate of time preference and her discount rate (page 403). Despite its simplicity, Thaler and Shefrin's model incorporates intrapersonal agency concepts into the traditional consumer model and arrives at results that help explain anomalous empirical findings.[6]

Rational Addiction

Simultaneous to this literature on self-control, another body of work was developing that deserves mention—namely, consumer models of habit and rational addiction. Beginning with Stigler and Becker (1977), titled "De Gustibus Non Est Disputandum," work on consumer habit and addiction blossomed.[7] This eventually led to Becker and Murphy's landmark paper, "A Theory of Rational Addiction" (1988). This entire development of the consumer model is notable for the concept of "consumption capital"—capital that is

[5]Recent evidence from Johnson et al. (2006) suggests that the Life Cycle-Permanent Income Hypothesis is indeed false. This study found that recipients of U.S. tax rebates in 2001 spent twenty to forty percent of the income gained on non-durable consumption goods in the following three months. Assuming diminishing marginal utility of consumption goods, if a third of the bonus is spent quickly on consumption goods, the bonus is larger than optimal.

[6]See also Fudenberg and Levine (2006); Loewenstein and O'Donoghue (2004, 2007).

[7]See also Pollak (1970); Ryder, Jr. and Heal (1973); Pollak (1976); Boyer (1978); Spinnewyn (1981); Boyer (1983); Iannaccone (1986); Gruber and Koszegi (2001); Laibson (2001); Bernheim and Rangel (2004); Benabou and Tirole (2004).

altered by current consumption and that affects future consumption. The use of consumption capital in the consumer model gives results that otherwise appear anomalous, while keeping the agent rational (in this case, consistent over time). These models also offer the benefit of being able to explain addictions to work, eating, and religion, as well as addictions to nicotine and heroin.

Saliency and Reference-Point Models

Despite the clarity and explanatory power of the rational addiction and consumer habit models, an effort to integrate more phenomena from psychology thrived. Akerlof (1991) emphasizes the importance of saliency to the consumer—indeed, this was also recognized as an important attribute of goods by Hoch and Loewenstein (1991). In other words, this is an issue of proximity (e.g., of time or distance) increasing impatience, which is akin to Strotz's model of impatience as hyperbolic discounting. Hoch and Loewenstein also emphasized the idea of a "reference point" playing a crucial role in valuation. These concepts are fundamentally related—they recognize the consumer's view of her environment as crucial. Around the same time, Loewenstein and Prelec (1992) presented four anomalies concerning the discounted-utility (DU) framework. These anomalies are "the common difference effect," "the absolute magnitude effect," "the gain-loss asymmetry," and "the delay-speedup asymmetry." The authors then present a more thorough mathematical treatment of a model that also uses the idea of a reference point to resolve these anomalies. Fischer (2001) puts forth a model that takes the concept of saliency a step further by modeling time as a discrete, exhaustible resource.

Naïveté

O'Donoghue and Rabin (1999) emphasized two distinctions—first, the difference between choices resulting in immediate costs versus immediate rewards, and second, whether individuals are naïve to their lack of self-control or if they recognize their future lack of self-control. Emphasis on these two factors leads to a matrix of results: if the individual is naïve and the action is costly in the present, she will procrastinate. Conversely, if the individual is non-naïve and the action is immediately costly, she will distrust her future self and perform the action now. However, the authors posit that being distrusting of one's future-self is not always beneficial—if the action involves a reward, naïveté results in over-estimation of future reward, and the naïve individual is more likely to wait for the reward. While the authors acknowledge that their model cannot accommodate many situations (page 118), the paper is certainly important for re-emphasizing naïveté and introducing a model that gives relevant results.[8]

Ego Depletion

The aforementioned psychological experiments on ego depletion were just conducted at the turn of the century. Accordingly, their introduction into the economic literature is even more recent. Ozdenoren et al. (2006) gives an optimal control model for consuming a good. The authors attempt to mathematically derive many results that are consistent with the ego depletion hypothesis. Unfortunately the model is only useful for the analysis of the consumption of a single good over time.

Houser et al. (2008) build on the previously mentioned paper by illustrating the concept with data. This study involved the collection of 2,800 observations of consumers waiting in grocery store lines. The primary finding is that as an individual spends more time waiting in line, she will be more likely to purchase an impulse item. The authors argue that these results suit the ego depletion theory particularly well.

The present paper attempts to add to the economic literature on ego depletion by offering a fundamental framework. The model, presented in the following chapter, explains these phenomena by using an elegant, yet powerful, approach.

[8] Naïveté is implicit in the analysis in Strotz (1956). See also O'Donoghue and Rabin (2000, 2003) for a more general discussion of self-control and naïveté. In addition, O'Donoghue and Rabin (2001) uses the concept of naïveté in a menu model. The main result of this paper is that choices of when to perform an action and which action to perform both add to the effect of procrastination. The authors also build on their aforementioned model of naïveté.

MODEL

Motivation

Modeling willpower as a depletable, renewable, unconstrained resource is desirable. Individuals utilitze their ego resource, depleting it so that they may execute difficult tasks. Over time, their ego resource is replenished naturally as they consume goods. There is no evidence that the resource is limited (Baumeister and Vohs, 2003). Instead, the laboratory experiments show that individuals use willpower for certain tasks while attempting to conserve it for other tasks—they do not exhaust the resource. For example, consider an individual running from a white bear. It could be that the individual has only enough willpower to run for ten miles, at which point she succumbs to dismemberment. However, it is more likely that at the tenth mile she is able to summon additional willpower at an extreme cost to her psychological and physiological well-being.

Ozdenoren et al. (2006) put forth a dynamic control problem in which the consumer divides a cake between time periods in order to maximize utility subject to a willpower constraint. However, neither experimental data nor introspection suggest that a willpower constraint is realistic. Instead, the model proposed in this paper assumes that individuals may always summon additional capacities for executing their desires, but this additional capacity comes at a significantly higher cost.

This paper proposes a model in which the individual faces the optimization problem of maximizing utility subject to a budget constraint and also subject to the cost of using willpower. This problem is similar to the firm's problem where capital is fixed in the short-run and labor is chosen. In the profit maximization model, results are driven by the second-order properties of q, the production function, and c, the cost function. Just as a firm may choose to avoid making an item at the margin because of a prohibitive production cost, consumers may choose to avoid consuming an item at the margin because of a prohibitive willpower cost.

Keeping this in mind, let us now return to the proposed consumer model. We begin with only one good, x. Suppose that the consumer's utility function, $u(x)$, gives us the benefit of consuming x units of the good. Under traditional circumstances, the consumer will exhaust her budget, M, buying as much x as possible ($x^* = \frac{M}{p}$, where p is the price of good x).

Suppose that, however, consuming x requires willpower. Let this cost be represented by $\omega(x)$, measured in utils. The problem is now as follows:

$$\max_x U \equiv u(x) - \omega(x) \tag{3.1}$$

$$\text{subject to } g \equiv M - px \geq 0 \tag{3.2}$$

Then, depending on the functional form of u and ω, the consumer may not spend her entire budget (or any of it). This would occur given first- and second-order conditions similar to those in the firm's optimization problem. An example of such a good may be running—it is beneficial, but after the first few miles, it creates disutility through the use of willpower.

This approach seemingly assumes that the *use* of willpower causes direct disutility. It is possible that willpower is only numéraire—meaning the use of willpower causes disutility only because some other good is foregone. The former case implies that the consumer mazimizes $U(x_0, x_1, \omega(x_0, x_1))$, which shows willpower directly affecting utility. Conversely, the latter case implies that the consumer mazimizes $U(x_0(\omega, k_0), x_1(\omega, k_1))$, where the consumer uses both willpower and capital to obtain goods, but willpower does not directly affect utility. Evidence in support of either view does not exist. These two forms are mathematically interchangeable, so we may continue using this model without any assumption on the channel through which willpower affects the consumer's utility.

General Model of Ego Depletion

With this in mind, let us begin formulating a consumer model of ego depletion beginning at the standard consumer optimization problem of utility maximization. This forumulation allows for n goods and T time periods using standard discounting:

$$\max_{x_{i,t}\forall i,t} U \equiv \sum_{t=0}^{T-1} \delta_t U_t(\mathbf{x}_t) \tag{3.3}$$

$$\text{subject to } g \equiv M - \sum_{t=0}^{T-1}\sum_{i=0}^{n-1} p_{i,t}x_{i,t} \geq 0 \tag{3.4}$$

$$\text{where } 0 < \delta_{T-1} \leq \delta_{T-2} \leq \cdots \leq \delta_1 \leq \delta_0 \equiv 1 \tag{3.5}$$

Using this notation, $p_{i,t}$ is the market price of good i at time t. $x_{i,t}$ is the choice variable for good i at time t. The sum in (3.4) is total expenditure—the entire statement requires that expenditure is equal to or less than income M. Let $\mathbf{x}_t$ be a vector of all n choice variables at time t. Up to this point, the model is commonplace.

However, we will make a modification to this standard model. Let $u_t(\mathbf{x}_t)$ be a standard contemporaneous utility function and $\omega_t(\mathbf{x}_t)$ be a contemporaneous disutility function. $\omega_t(\mathbf{x}_t)$ is disutility (or utility cost) for the use of willpower.

$$U_t \equiv u_t(\mathbf{x}_t) - \omega_t(\mathbf{x}_t)\theta(0,t) \tag{3.6}$$

$$\theta(\rho,t) \equiv \begin{cases} \gamma_{\rho+1}\omega_{t-1}(\mathbf{x}_{t-1})\theta(\rho+1,t-1) & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases} \tag{3.7}$$

The willpower function, θ, specified in (3.7) may seem overly complex, but γ must depend on how much willpower was expended in a given previous period and also how much time has passed since that expenditure. To facilitate these requirements, θ takes two arguments—ρ is the relative (elapsed) time, and t is the specified time. This specification will produce a system similar to the following:

$$U_0 \equiv u_0(\mathbf{x}_0) - \omega_0(\mathbf{x}_0) \tag{3.8}$$

$$U_1 \equiv u_1(\mathbf{x}_1) - \omega_1(\mathbf{x}_1)\gamma_1(\omega_0(\mathbf{x}_0)) \tag{3.9}$$

$$U_2 \equiv u_2(\mathbf{x}_2) - \omega_2(\mathbf{x}_2)\gamma_1(\omega_1(\mathbf{x}_1))\gamma_2(\omega_0(\mathbf{x}_0)) \tag{3.10}$$

$$U_3 \equiv u_3(\mathbf{x}_3) - \omega_3(\mathbf{x}_3)\gamma_1(\omega_2(\mathbf{x}_2))\gamma_2(\omega_1(\mathbf{x}_1))\gamma_3(\omega_0(\mathbf{x}_0)) \tag{3.11}$$

$$\vdots$$

A sequence of γ (i.e., $\gamma_1, \gamma_2, \gamma_3, \ldots$) represents a pattern of intertemporal ego effects. Experimental data suggest that upon exerting willpower, the agent is depleted for some number of time periods, then the agent is conditioned for some number of periods. If the agent is depleted, then $1 > \gamma$, making disutility amplified. If the agent is conditioned, then $0 < \gamma < 1$, making disutility deamplified. If there is no intertemporal ego effect, then $\gamma = 1$.

For example, consider again the individual running from the white bear. Suppose that she is running in period $t = 0$, so u_0 is the benefit of running, and ω_0 is the disutility from running (i.e., sore legs, cramps, cut up feet, et cetera). In period $t = 1$, as a result of prior disutility, the individual has her current disutility amplified by $\gamma_1 > 1$. Yet, suppose that both her feet and her will became calloused after a few nights' rest. Then in period $t = 2$, when she yet again must run from a bear, her disutility is deamplified by $0 < \gamma_2 < 1$. This illustrates how the willpower function, θ, amplifies or deamplifies disutility through the use of the parameter γ.

Because of the subscripts on γ, we may easily require that ego effects do not last for more than two periods. This would be accomplished by stating that $\gamma_j = 1\ \forall j > 2$. Alternatively, we could require that ego utilization amplifies disutility (the agent is depleted) for the following two periods, has no effect on the third, deamplifies disutility (the agent is conditioned) on the fourth, and has no effect thereafter. This is readily shown with the following:

$$1 < \gamma_1 \tag{3.12}$$

$$1 < \gamma_2 \tag{3.13}$$

$$\gamma_3 = 1 \tag{3.14}$$

$$0 < \gamma_4 < 1 \tag{3.15}$$

$$\gamma_j = 1 \qquad \forall j > 4 \tag{3.16}$$

Returning to the experiments discussed in section 2.1, the first study instructed the treatment group to control their emotions while watching a video. Later, these individuals' endurance at squeezing a hand-grip exerciser was measured. This can be represented mathematically as follows: $\omega_0 > 0$, as emotion control caused disutility. $\omega_1 > 0$, as squeezing the hand-grip exercizer caused disutility as well. However, the crucial point is that disutility from emotion control subsequently increased the disutility from the grip exercise. In this paper's formulation, this phenomenon is represented by $\gamma_1 > 1$. In words, emotion control disutility *amplified* the grip exercise disutility.

Continuing with the formulation of the general model, we will impose restrictions on the functions γ, u, and ω for completeness.

$$\gamma_j(\omega_t) > 0 \; \forall j, t \tag{3.17}$$

$$\frac{\partial \gamma_j(\omega_t)}{\partial \omega_t} \gtreqless 0 \; \forall j, t \tag{3.18}$$

$$\frac{\partial^2 \gamma_j(\omega_t)}{\partial \omega_t^2} \gtreqless 0 \; \forall j, t \tag{3.19}$$

The ego effect multiplier γ is always positive, as assumed in (3.17). This means that willpower utilization can never turn future disutility into utility nor completely eliminate it (we are not modeling Nietzsche's overman). We cannot assume a sign for γ_j' in the ω_j dimension—it could be increasing (i.e., if the agent is depleted) or decreasing (i.e., if the agent is conditioned). Therefore, we cannot assume any first-order sign in (3.18). Likewise, we cannot determine the sign of the second derivative in (3.19).

$$u_t(\mathbf{x}_t) \geq 0 \; \forall i, t \tag{3.20}$$

$$\frac{\partial u_t(\mathbf{x}_t)}{\partial x_{i,t}} > 0 \; \forall i, t \tag{3.21}$$

$$\frac{\partial^2 u_t(\mathbf{x}_t)}{\partial x_{i,t}^2} \gtreqless 0 \; \forall i, t \tag{3.22}$$

$$\omega_t(\mathbf{x}_t) \geq 0 \; \forall i, t \tag{3.23}$$

$$\frac{\partial \omega_t(\mathbf{x}_t)}{\partial x_{i,t}} > 0 \; \forall i, t \tag{3.24}$$

$$\frac{\partial^2 \omega_t(\mathbf{x}_t)}{\partial x_{i,t}^2} \gtreqless 0 \; \forall i, t \tag{3.25}$$

Each x is always an insatiable good in u (utility benefits), as shown in (3.20) and (3.21).

These goods may or may not exhibit diminishing marginal benefits, which is why (3.22) is ambiguous. Each x is always a bad in ω (utility costs), which is shown in (3.23) and (3.24). Like u, we cannot assume a sign for the second derivative of ω. Exercise, eating health food, and losing weight may be examples of $\partial^2\omega/\partial x_i^2 > 0$. This means that as we perform the difficult task, the task becomes more difficult. Foregoing smoking cigarettes, foregoing using cocaine, and avoiding junk food may be examples of $\partial^2\omega/\partial x_i^2 < 0$. This means that as we perform the difficult task, the task becomes easier. To keep the model general, we will not assume either sign, leaving (3.25) ambiguous.

This completes the generalized framework. We may now analyze the problem and attempt to derive results.

Simplest Case

Let us begin with the simplest analysis of our framework. In this case, we have two time periods and one good in each time period (e.g., $n = 1$ and $T = 2$). For simplicity, there are no time preferences; personal discounting has been ignored for this example ($\delta_1 = \delta_0 = 1$). We then have the following optimization problem:

$$\max_{x_0, x_1} U \equiv u_0(x_0) - \omega_0(x_0) + u_1(x_1) - \omega_1(x_1)\gamma_1(\omega_0(x_0)) \tag{3.26}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 \geq 0 \tag{3.27}$$

The resultant first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_0} \equiv \frac{\partial u_0(x_0)}{\partial x_0} - \frac{\partial \omega_0(x_0)}{\partial x_0}[1 + \omega_1(x_1)\gamma_1'(\omega_0(x_0))] - \lambda p_0 = 0 \tag{3.28}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} \equiv \frac{\partial u_1(x_1)}{\partial x_1} - \frac{\partial \omega_1(x_1)}{\partial x_1}\gamma_1(\omega_0(x_0)) - \lambda p_1 = 0 \tag{3.29}$$

As derived in the appendix (see section 4.1), we arrive at the following general comparative statics results, where μ is an arbitrary parameter that enters u_0, u_1, ω_0, or ω_1:

$$\frac{\partial x_0^*}{\partial \mu} \equiv \frac{\begin{vmatrix} -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{3.30}$$

$$\frac{\partial x_1^*}{\partial \mu} \equiv \frac{\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{3.31}$$

These are the generalized comparative static results for the $n = 1, T = 2$ case. While we can sign some of the elements in (3.30) and (3.31), we cannot sign them all. We know that $|H| > 0$ from our assumed second-order conditions. We also know that $\partial^2\mathcal{L}/\partial\lambda\partial x_t < 0\,\forall t$, $\partial^2\mathcal{L}/\partial\lambda^2 = 0$, and $\partial^2\mathcal{L}/\partial\lambda\partial\mu = 0$ by simple derivation. We do not know the sign of $\partial^2\mathcal{L}/\partial x_0^2$ or of the cross-effect, $\partial^2\mathcal{L}/\partial x_0\partial x_1$, from the border-preserving principal minors of $|H|$. We also do not know how the parameter μ enters the objective function, so we don't have information about $\partial^2\mathcal{L}/\partial x_0\partial\mu$ or $\partial^2\mathcal{L}/\partial x_1\partial\mu$.

These results are expected. Depending on the intertemporal cross-effect, changing a parameter will create a different response. Likewise, depending on the cross-effect between the parameter and each of the goods, the optimal consumption of the good might increase or decrease. Therefore, in order to sign the comparative statics in (3.30) and (3.31), we need to assume a functional form.

Let us specify u, w, and γ. This will potentially enable the aforementioned comparative statics to be derived. Consider the following:

$$U \equiv A_0 x_0 - B_0 x_0 + A_1 x_1 - B_1 x_1 \gamma_1(B_0 x_0) \tag{3.32}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{3.33}$$

$$\begin{aligned} \text{where } \; 0 &\leq x_i && \forall i \in \{0,1\} \\ 0 &< A_i && \forall i \in \{0,1\} \\ 0 &< B_i && \forall i \in \{0,1\} \\ 0 &< \gamma_1(z) && \forall z \in [0,\infty) \\ 0 &< \gamma_1'(z) && \forall z \in [0,\infty) \end{aligned} \tag{3.34}$$

Note that we are utilizing a simple linearly additive utility function. We are letting the residual ego multiplier (γ) be general, with simple restrictions which ensure that it illustrates ego depletion, as previously discussed. This chosen functional form has significant appeal. It is a standard separable utility function which has the ability to model a wide variety of goods. It is also the simplest form permitted by the second-order conditions to achieve a solution. We are now able to sign the comparative statics for the given specification. As derived in the appendix in section 4.1, we obtain:

$$\frac{\partial x_0^*}{\partial A_0} > 0 \qquad\qquad \frac{\partial x_1^*}{\partial A_0} < 0 \tag{3.35}$$

These results are telling of many effects. As A_0 increases, marginal utility of x_0 rises. As a result, the agent consumes more x_0. This is our commonplace comparative static result. However, because consuming more x_0 increases the cost of consuming x_1 in the following period, consumption of x_1 falls. This is interesting because A_0 raises the cost of x_1 through two channels—there is a substitution effect, but there is also a disutility effect. Because the individual has spent more income on x_0, she has less to spend on the other good, x_1. Further, because increased consumption of x_0 raises the marginal disutility from x_1, the individual consumes less x_1 for this reason as well. Let us now look at $\mu \equiv A_1$.

$$\frac{\partial x_1^*}{\partial A_1} \gtreqless 0 \qquad\qquad \frac{\partial x_0^*}{\partial A_1} < 0 \tag{3.36}$$

We cannot sign $\partial x_1^*/\partial A_1$, which means that we do not know if an increase in the benefit of x_1 will cause the agent to consume more, simply because that consumption is associated with disutility as well. However, we yet again get another interesting result in (3.36). This time, there is no ambiguity—increasing the benefit of x_1 leads the individual to consume less of x_0. This model is interesting and important precisely because there are three effects taking place. This model demonstrates an income effect, a substitution effect, and an ego depletion effect.

Descriptive Case

This section presents a case of the ego depletion model with one good in each of three time periods (e.g., $n = 1$ and $T = 3$). This case will permit many phenomena to be analyzed. Again, for simplicity, there are no time preferences; personal discounting has been ignored for this example ($\delta_2 = \delta_1 = \delta_0 = 1$). Consider this optimization problem:

$$\begin{aligned} \max_{x_t \forall t} U \equiv \quad & u_0(x_0) - \omega_0(x_0) \\ & + u_1(x_1) - \omega_1(x_1)\gamma_1(\omega_0(x_0)) \\ & + u_2(x_2) - \omega_2(x_2)\gamma_1(\omega_1(x_1))\gamma_2(\omega_0(x_0)) \end{aligned} \tag{3.37}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{3.38}$$

We must define a Lagrangian for this problem to utilize the constraint.

$$\mathcal{L} \equiv U + \lambda g \tag{3.39}$$

We arrive at the following first-order conditions (FOCs).

$$\frac{\partial \mathcal{L}}{\partial x_0} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{3.40}$$

Following the steps outlined in section 4.1 of the appendix, using Cramer's Rule, we may find comparative statics with respect to an arbitrary parameter μ.

$$\frac{\partial x_0^*}{\partial \mu} \equiv \frac{\begin{vmatrix} -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{3.41}$$

$$\frac{\partial x_1^*}{\partial \mu} \equiv \frac{\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{3.42}$$

$$\frac{\partial x_2^*}{\partial \mu} \equiv \frac{\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{3.43}$$

At this level of generality, we cannot sign many of the elements in the numerators in (3.41), (3.42), or (3.43). In order to sign these numerators and therefore sign these comparative statics, let us specify u and ω. In this case, we will use additively separable utility and disutility functions.

$$
\begin{aligned}
U \equiv \quad & A_0 x_0^{\alpha_0} - B_0 x_0^{\beta_0} \\
& + A_1 x_1^{\alpha_1} - B_1 x_1^{\beta_1} \gamma_1(B_0 x_0^{\beta_0}) \\
& + A_2 x_2^{\alpha_2} - B_2 x_2^{\beta_2} \gamma_1(B_1 x_1^{\beta_1}) \gamma_2(B_0 x_0^{\beta_0})
\end{aligned} \tag{3.44}
$$

$$
\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{3.45}
$$

where $\forall t \in \{0, 1, 2\}$,

$$
\begin{aligned}
& 0 \leq x_t \\
& 0 < A_t \\
& 0 < B_t \\
& 0 < \alpha_t < 1 \\
& 1 < \beta_t
\end{aligned} \tag{3.46}
$$

First, we simply assume that the gamma function demonstrates ego depletion in both future time periods. From our intuition of the problem in this case, we know these:

$$
\gamma_1 > 1 \qquad \gamma_1' > 0 \qquad \gamma_1'' \geq 0 \tag{3.47}
$$

$$
\gamma_2 > 1 \qquad \gamma_2' > 0 \qquad \gamma_2'' \geq 0 \tag{3.48}
$$

Now let us assume that the gamma function demonstrates ego depletion in the second time period and ego conditioning (learning) in the third. Again, using our intuition of the problem, we specify the following:

$$
\gamma_1 > 1 \qquad \gamma_1' > 0 \qquad \gamma_1'' \geq 0 \tag{3.49}
$$

$$
0 < \gamma_2 < 1 \qquad \gamma_2' < 0 \qquad \gamma_2'' > 0 \tag{3.50}
$$

We now have two distinct cases which we may use to derive results. Unfortunately, using such a descriptive specification has shortfalls. We are only able to readily sign three comparative statics, as outlined in section 4.1 of the appendix.

$$\frac{\partial x_0^*}{\partial A_0} > 0 \qquad \frac{\partial x_1^*}{\partial A_1} > 0 \qquad \frac{\partial x_2^*}{\partial A_2} > 0 \tag{3.51}$$

These are commonplace comparative statics, as they follow directly from the Conjugate-Pairs theorem. While this descriptive model is not as accessible as the simple model, it provides a crucial example of the full potential of this framework. With small, additional assumptions, or a more exact specification of U, useful results may be generated.

DISCUSSION

The present paper begins by studying experimental phenomena involving willpower. Numerous experiments have demonstrated that if an individual completes a difficult task, then the individual will find a subsequent task even more difficult (Muraven et al., 1998). The first study instructed the treatment group to control their emotions while watching a video; the control group was not told to regulate their emotions. Afterward, individuals' endurance at squeezing a hand-grip exerciser was measured. Those who had previously regulated themselves demonstrated a significantly shorter endurance (ibid.).

Another experiment involved instructing the treatment group to avoid thinking about white bears; the control group received no such instructions. Individuals were subsequently given unsolvable anagrams—those who had not previously regulated their thoughts persisted on the anagrams statistically significantly longer. Other studies involved impulse control, dieting, and personal spending (Baumeister and Vohs, 2003; Vohs and Faber, 2007).

The model proposed in this paper is well-suited to explain these results in the realm of economics. We begin by expanding the utility-maximization model to include both utility and disutility. In these experiments, utility may be a small reward of chocolate, monetary compensation for participating in the study, or good feelings from doing as the laboratory worker instructs. Disutility is the result of the difficult tasks. The first study involved emotion regulation and exercising one's hand. The disutility in these cases may be psychological distress or fatigue, for example.

Accordingly, the model set out in this paper permits each good to carry both utility and disutility components. The principal notion is that disutility from the first period *amplifies* disutility in the second period. To this end, in this model, each instance of disutility is affected by prior disutility. In the simplest case, generating disutility in the first time period amplifies the disutility in the second time period.

The model demonstrates its suitability to the discussed experiments in equations (3.35) and (3.36). Let us begin by looking at the comparative statics with respect to A_0.

$$\frac{\partial x_0^*}{\partial A_0} > 0 \qquad\qquad \frac{\partial x_1^*}{\partial A_0} < 0$$

In this case, let us suppose that x_0 represents thought suppression of white bears, and x_1 represents exercise. As thought suppression has a higher payoff (as A_0 increases), then the individual optimally chooses to persist at the thought suppression. As a result of the increased thought suppression, she exercises less. There are two reasons for these findings. The first is the common substitution effect. The individual has a limited budget, so as the marginal rate of substitution of payoffs changes, then she optimally suppresses more thoughts and exercises less. However, there is a novel effect here, as well. As the individual suppresses more thoughts, then she experiences an increasing disutility, so she exercises even less.

Now we will study the comparative statics with respect to A_1.

$$\frac{\partial x_1^*}{\partial A_1} \gtreqless 0 \qquad\qquad \frac{\partial x_0^*}{\partial A_1} < 0$$

Now we are looking at the optimal consumption bundle with respect to the payoff from exercising. As the payoff from exercising increases, then the individual would like to exercise more. However, there is a disutility associated with exercising. As a result, we may know that she chooses to optimally exercise more or less depending on the circumstances. We do know that she will suppress fewer thoughts. These findings are in accord with those from the experiments.

Conclusion

This paper has posited willpower as a depletable, renewable, unconstrained resource. Individuals utilize their ego resource, depleting it so that they may execute difficult tasks. Over time, their ego resource is replenished naturally as they consume goods. This is a new general microeconomic theory of self-control—by using the concept of ego depletion, a variety of goods can be put into a framework that demonstrates how an agent will reallocate willpower when a variety of parameters (i.e., information, payoffs, discomfort, prices, or income) change.

This theory is not presented as a comprehensive alternative to the various other economic models of self-control. To be sure, the present model does not account for myopia or naïveté. It parallels the theory of rational addiction, as the agent possesses a stock of ego, similar to the idea of consumption capital. This model *does*, however, account for a potpourri of experiments conducted that demonstrate that *current self-control hinders future self-control*.

Thus, this model makes an important contribution to the field of economics. While the concept of ego depletion has been discussed, it has not yet been incorporated into the consumer model. This paper has shown a usable, elegant model which permits a high degree of flexibility, while still realistically accounting for ego depletion.

REFERENCES

Ainslie, George. "Specious reward: A behavioral theory of impulsiveness and impulse control." *Psychological Bulletin* 82, 4: (1975) 463–496.

Akerlof, George A. "Procrastination and obedience." *The American Economic Review* 81, 2: (1991) 1–19.

Baumeister, Roy F. "Yielding to temptation: Self-control failure, impulsive purchasing, and consumer behavior." *Journal of Consumer Research* 28, 4: (2002) 670–676.

Baumeister, Roy F., Ellen Bratslavsky, Mark Muraven, and Dianne M. Tice. "Ego depletion: Is the active self a limited resource?" *Journal of Personality and Social Psychology* 74, 5: (1998) 1252–1265.

Baumeister, Roy F., and Kathleen D. Vohs. *Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice*, Russell Sage Foundation, 2003, chapter 6, "Willpower, choice, and self-control", 13–86.

Becker, Gary S., and Kevin M. Murphy. "A theory of rational addiction." *The Journal of Political Economy* 96, 4: (1988) 675–700.

Benabou, Roland, and Jean Tirole. "Willpower and personal rules." *Journal of Political Economy* 112, 4: (2004) 848–886.

Bernheim, B. Douglas, and Antonio Rangel. "Addiction and cue-triggered decision processes." *American Economic Review* 94, 5: (2004) 1558–1590.

Blackorby, Charles, David Nissen, Daniel Primont, and R. Robert Russell. "Consistent intertemporal decision making." *The Review of Economic Studies* 40, 2: (1973) 239–248.

Blatt, Sidney J., and Paul Quinlan. "Punctual and procrastinating students: a study of temporal parameters." *Journal of Consulting Psychology* 31, 2: (1967) 169–174.

Boyer, Marcel. "A habit forming optimal growth model." *International Economic Review* 19, 3: (1978) 585–609.

———. "Rational demand and expenditures patterns under habit formation." *Journal of Economic Theory* 31, 1: (1983) 27–53.

Camerer, Colin F., and George Loewenstein. *Advances in behavioral economics*, Russell Sage Foundation, 2004, chapter 1, "Behavorial Economics: Past, Present, Future", 3–51.

Dekel, Eddie, Barton L. Lipman, and Aldo Rustichini. "Temptation-driven preferences." *Review of Economic Studies* 76, 3: (2009) 937–971.

Fischer, Carolyn. "Read this paper later: Procrastination with time-consistent preferences." *Journal of Economic Behavior and Organization* 46, 3: (2001) 249–269.

Freud, Sigmund. *The Complete Introductory Lectures on Psychoanalysis*, W. W. Norton and Company, 1966, chapter 22, "Some Thoughts on Development and Regression—Aetiology", 339–357.

Fudenberg, Drew, and David K. Levine. "A dual-self model of impulse control." *American Economic Review* 96, 5: (2006) 1449–1476.

Goldman, Steven M. “Intertemporally inconsistent preferences and the rate of consumption.” *Econometrica* 47, 3: (1979) 621–626.

———. “Consistent plans.” *The Review of Economic Studies* 47: (1980) 533–537.

Gruber, Jonathan, and Botond Koszegi. “Is Addiction ‘Rational’? Theory and Evidence.” *Quarterly Journal of Economics* 116, 4: (2001) 1261–1303.

Gul, Faruk, and Wolfgang Pesendorfer. “Temptation and self-control.” *Econometrica* 69, 6: (2001) 1403–1435.

———. “Self-control and the theory of consumption.” *Econometrica* 72, 1: (2004) 119–158.

———. “Harmful addiction.” *Review of Economic Studies* 74, 1: (2007) 147–172.

Hammond, Peter J. “Changing tastes and coherent dynamic choice.” *The Review of Economic Studies* 43, 1: (1976) 159–173.

Hoch, Steven J., and George F. Loewenstein. “Time-inconsistent preferences and consumer self-control.” *Journal of Consumer Research* 17: (1991) 492–507.

Houser, Daniel, David H. Reiley, and Michael B. Urbancic. “Checking Out Temptation: A Natural Experiment with Purchases at the Grocery Register.”, 2008. `http://repec.ices-gmu.org/RePEc/pdf/1008.pdf`. Unpublished.

Iannaccone, Laurence R. “Addiction and satiation.” *Economics Letters* 21, 1: (1986) 95–99.

Johnson, David S., Jonathan A. Parker, and Nicholas S. Souleles. “Household expenditure and the income tax rebates of 2001.” *The American Economic Review* 96, 5: (2006) 1589–1610.

Kirby, Kris N., and RJ. Herrnstein. “Preference reversals due to myopic discounting of delayed reward.” *Psychological Science* 6, 2: (1995) 83–89.

Laibson, David. “Golden Eggs and Hyperbolic Discounting.” *Quarterly Journal of Economics* 112, 2: (1997) 443–477.

———. “A Cue-Theory of Consumption.” *Quarterly Journal of Economics* 116, 1: (2001) 81–119.

Loewenstein, George. “Out of control: Visceral influences on behavior.” *Organizational Behavior and Human Decision Processes* 65, 3: (1996) 272–292.

———. “Willpower: A decision-theorist’s perspective.” *Law and Philosophy* 19, 1: (2000) 51–76.

Loewenstein, George, and Ted O’Donoghue. “Animal spirits: Affective and deliberative processes in economic behavior.”, 2004. `http://ssrn.com/abstract=539843`. Unpublished.

———. “The heat of the moment: Modeling interactions between affect and deliberation.”, 2007. `http://www.arts.cornell.edu/econ/edo1/heat.pdf`. Unpublished.

Loewenstein, George, and Drazen Prelec. “Anomalies in intertemporal choice: Evidence and an interpretation.” *The Quarterly Journal of Economics* 107, 2: (1992) 573–597.

McCown, Willian, Thomas Petzel, and Patricia Rupert. "An experimental study of some hypothesized behaviors and personality variables of college student procrastinators." *Personality and Individual Differences* 8, 6: (1987) 781–786.

Mischel, Walter. *The Psychology of Action: Linking Cognition and Motivation to Behavior*, Guilford Press, 1996, chapter 9, "From good intentions to willpower", 197–218.

Mowrer, O. H., and A. D. Ullman. "Time as a determinant in integrative learning." *Psychological Review* 52, 2: (1945) 61–90.

Muraven, Mark, and Roy F. Baumeister. "Self-regulation and depletion of limited resources: Does self-control resemble a muscle?" *Psychological Bulletin* 126, 2: (2000) 247–259.

Muraven, Mark, Roy F. Baumeister, and Dianne M. Tice. "Longitudinal improvement of self-regulation through practice: Building self-control strength through repeated exercise." *Journal of Social Psychology* 139, 4: (1999) 446–457.

Muraven, Mark, Dianne M. Tice, and Roy F. Baumeister. "Self-control as limited resource: Regulatory depletion patterns." *Journal of Personality and Social Psychology* 74: (1998) 774–789.

O'Donoghue, Ted, and Matthew Rabin. "Doing it now or later." *American Economic Review* 89, 1: (1999) 103–124.

———. "The economics of immediate gratification." *Journal of Behavioral Decision Making* 13, 2: (2000) 233–250.

———. "Choice and Procrastination." *Quarterly Journal of Economics* 116, 1: (2001) 121–160.

———. *Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice*, Russell Sage Foundation, 2003, chapter 7, "Self awareness and self control", 217–243.

Ozdenoren, Emre, Stephan Salant, and Dan Silverman. "Willpower and the Optimal Control of Visceral Urges.", 2006. `http://ssrn.com/abstract=906756`. NBER Working Paper No. W12278.

Peleg, Bezalel, and Menahem E. Yaari. "On the existence of a consistent course of action when tastes are changing." *The Review of Economic Studies* 40, 3: (1973) 391–401.

Pollak, R. A. "Consistent planning." *The Review of Economic Studies* 35, 2: (1968) 201–208.

———. "Habit formation and dynamic demand functions." *The Journal of Political Economy* 78, 4: (1970) 745–763.

———. "Habit formation and long-run utility functions." *Journal of Economic Theory* 13, 2: (1976) 272–297.

Ryder, Harl E., Jr., and Geoffrey M. Heal. "Optimal growth with intertemporally dependent preferences." *The Review of Economic Studies* 40, 1: (1973) 1–31.

Schelling, T. C. "Egonomics, or the art of self-management." *The American Economic Review* 68, 2: (1978) 290–294.

Spinnewyn, Frans. "Rational habit formation." *European Economic Review* 15, 1: (1981) 91–109.

Stigler, George J., and Gary S. Becker. "De gustibus non est disputandum." *The American Economic Review* 67, 2: (1977) 76–90.

Strotz, R. H. "Myopia and inconsistency in dynamic utility maximization." *The Review of Economic Studies* 23, 3: (1956) 165–180.

Thaler, Richard H. "Anomalies: Saving, fungibility, and mental accounts." *The Journal of Economic Perspectives* 4, 1: (1990) 193–205.

Thaler, Richard H., and H. M. Shefrin. "An economic theory of self-control." *The Journal of Political Economy* 89, 2: (1981) 392–406.

Twenge, Jean M., and Roy F. Baumeister. *Self and Identity: Personal, Social, and Symbolic*, Lawrence Erlbaum, 2002, chapter 3, "Self-control: A limited yet renewable resource", 57–70.

Vohs, Kathleen D., and Ronald J. Faber. "Spent resources: Self-regulatory resource availability affects impulse buying." *Journal of Consumer Research* 33, 4: (2007) 537–547.

Wegner, Daniel M., David J. Schneider, Samuel R. Carter, and Teri L. White. "Paradoxical effects of thought suppression." *Journal of Personality and Social Psychology* 53, 1: (1987) 5–13.

Yaari, Menahem E. "Consistent Utilization of an Exhaustible Resource, Or, How to Eat an Appetite-Arousing Cake." *Center for Research in Mathematical Economics and Game Theory Research Memorandum* 26.

APPENDIX A

DERIVATIONS

Simplest Case

Suppose that the problem involves two time periods and one good in each time period (e.g., $T = 2$ and $n = 1$). Personal discounting has been ignored for this example ($\delta_1 = \delta_0 = 1$). We assume that the constraint is binding. We then have the following optimization problem:

$$\max_{x_0, x_1} U \equiv u_0(x_0) - \omega_0(x_0) + u_1(x_1) - \omega_1(x_1)\gamma_1(\omega_0(x_0)) \tag{5.1}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 = 0 \tag{5.2}$$

Consider the Lagrangian:

$$\mathcal{L} \equiv U + \lambda g \tag{5.3}$$

We then have these first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_0} \equiv \frac{\partial u_0(x_0)}{\partial x_0} - \frac{\partial \omega_0(x_0)}{\partial x_0}[1 + \omega_1(x_1)\gamma_1'(\omega_0(x_0))] - \lambda p_0 = 0 \tag{5.4}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} \equiv \frac{\partial u_1(x_1)}{\partial x_1} - \frac{\partial \omega_1(x_1)}{\partial x_1}\gamma_1(\omega_0(x_0)) - \lambda p_1 = 0 \tag{5.5}$$

Let us construct the bordered Hessian matrix.

$$H \equiv \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_0^2} & \frac{\partial \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial \mathcal{L}}{\partial x_1^2} & \frac{\partial \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial \mathcal{L}}{\partial \lambda^2} \end{bmatrix} \tag{5.6}$$

$$\frac{\partial \mathcal{L}^*}{\partial x_0} \equiv 0 \quad \frac{\partial \mathcal{L}^*}{\partial x_1} \equiv 0 \quad \frac{\partial \mathcal{L}^*}{\partial \lambda} \equiv 0 \tag{5.7}$$

We can then attempt to find comparative statics for the arbitrary parameter μ. We begin by differentiating the system with respect to μ.

$$\frac{\partial^2 \mathcal{L}}{\partial x_0^2}\frac{\partial x_0^*}{\partial \mu} + \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1}\frac{\partial x_1^*}{\partial \mu} + \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda}\frac{\partial \lambda^*}{\partial \mu} + \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} \equiv 0 \tag{5.8}$$

$$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1^2}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu}\equiv 0 \tag{5.9}$$

$$\frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda^2}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu}\equiv 0 \tag{5.10}$$

Then we arrange this system into matrix notation.

$$\begin{bmatrix}\frac{\partial^2\mathcal{L}}{\partial x_0^2} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0} & \frac{\partial^2\mathcal{L}}{\partial x_1^2} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{bmatrix}\begin{pmatrix}\frac{\partial x_0^*}{\partial\mu}\\ \frac{\partial x_1^*}{\partial\mu}\\ \frac{\partial\lambda^*}{\partial\mu}\end{pmatrix}\equiv\begin{pmatrix}-\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu}\\ -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu}\\ -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu}\end{pmatrix} \tag{5.11}$$

Using Cramer's rule, we may solve for $\partial x_0^*/\partial\mu$, $\partial x_1^*/\partial\mu$, and $\partial\lambda^*/\partial\mu$.

$$\frac{\partial x_0^*}{\partial\mu}\equiv\frac{\begin{vmatrix}-\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\\ -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_1^2} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\\ -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{vmatrix}}{|H|} \tag{5.12}$$

$$\frac{\partial x_1^*}{\partial\mu}\equiv\frac{\begin{vmatrix}\frac{\partial^2\mathcal{L}}{\partial x_0^2} & -\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0} & -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0} & -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{vmatrix}}{|H|} \tag{5.13}$$

$$\frac{\partial\lambda^*}{\partial\mu}\equiv\frac{\begin{vmatrix}\frac{\partial^2\mathcal{L}}{\partial x_0^2} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_1} & -\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu}\\ \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0} & \frac{\partial^2\mathcal{L}}{\partial x_1^2} & -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu}\\ \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1} & -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu}\end{vmatrix}}{|H|} \tag{5.14}$$

Let us specify u, w, and γ. This will enable the aforementioned comparative statics to be derived. Consider the following:

$$U \equiv A_0 x_0 - B_0 x_0 + A_1 x_1 - B_1 x_1 \gamma_1(B_0 x_0) \tag{5.15}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{5.16}$$

$$\begin{aligned} \text{where } \; & 0 \leq x_i && \forall i \in \{0,1\} \\ & 0 < A_i && \forall i \in \{0,1\} \\ & 0 < B_i && \forall i \in \{0,1\} \\ & 0 < \gamma_1(z) && \forall z \in [0,\infty) \\ & 0 < \gamma_1'(z) && \forall z \in [0,\infty) \end{aligned} \tag{5.17}$$

Because $\partial\gamma/\partial x_0 > 0$ is assumed, we can sign the following:

$$\frac{\partial^2 \mathcal{L}}{\partial x_0^2} = 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1^2} = 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} < 0 \tag{5.18}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = 0 \tag{5.19}$$

Again, because $\partial\gamma/\partial x_0 > 0$, we can sign the following:

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial A_0} > 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial A_1} = 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial B_1} < 0 \tag{5.20}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial A_0} = 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial A_1} > 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial B_1} = 0 \tag{5.21}$$

We can consequently sign the following comparative statics:

$$\frac{\partial x_0^*}{\partial A_0} > 0 \qquad \frac{\partial x_1^*}{\partial A_0} < 0 \tag{5.22}$$

$$\frac{\partial x_1^*}{\partial A_1} \gtreqless 0 \qquad \frac{\partial x_0^*}{\partial A_1} < 0 \tag{5.23}$$

Descriptive Case

Suppose that this problem involves three time periods and one good in each time period (e.g., $T = 3$ and $n = 1$). Personal discounting has again been ignored ($\delta_2 = \delta_1 = \delta_0 = 1$). We then have the following optimization problem:

$$\begin{aligned} \max_{x_t \forall t} U \equiv \quad & u_0(x_0) - \omega_0(x_0) \\ & + u_1(x_1) - \omega_1(x_1)\gamma_1(\omega_0(x_0)) \\ & + u_2(x_2) - \omega_2(x_2)\gamma_1(\omega_1(x_1))\gamma_2(\omega_0(x_0)) \end{aligned} \tag{5.24}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{5.25}$$

We will assume that the constraint is binding.

$$\mathcal{L} \equiv U + \lambda g \tag{5.26}$$

We arrive at the following first-order conditions (FOCs).

$$\frac{\partial \mathcal{L}}{\partial x_0} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{5.27}$$

The Hessian matrix for this problem is defined as follows.

$$H \equiv \begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{bmatrix} \tag{5.28}$$

To ensure a solution to the system of FOCs, we must assume that this Hessian matrix has full rank.

$$|H| \neq 0 \tag{5.29}$$

Substituting the solution (e.g., optimal arguments) into the system of FOCs, we obtain $\mathcal{L}^*$.

$$\frac{\mathcal{L}^*}{\partial x_0} \equiv 0 \quad \frac{\mathcal{L}^*}{\partial x_1} \equiv 0 \quad \frac{\mathcal{L}^*}{\partial x_2} \equiv 0 \quad \frac{\mathcal{L}^*}{\partial \lambda} \equiv 0 \tag{5.30}$$

Assuming the following sufficient second-order conditions (SOCs), we know that $\mathcal{L}^*$ specifies a maximum.

$$|H_{(0,1)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} < 0 \tag{5.31}$$

$$|H_{(0,2)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} < 0 \tag{5.32}$$

$$|H_{(1,2)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} < 0 \tag{5.33}$$

$$|H_{(0)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} > 0 \tag{5.34}$$

$$|H_{(1)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} > 0 \tag{5.35}$$

$$|H_{(2)}| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} > 0 \tag{5.36}$$

$$|H| \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix} < 0 \tag{5.37}$$

Fully differentiating the system in (5.30), we arrive at the following statements.

$$\frac{\partial^2\mathcal{L}}{\partial x_0^2}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_0\partial x_1}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_0\partial x_2}\frac{\partial x_2^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu}\equiv 0 \tag{5.38}$$

$$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1^2}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2}\frac{\partial x_2^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu}\equiv 0 \tag{5.39}$$

$$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_0}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_2^2}\frac{\partial x_2^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_2\partial\lambda}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial x_2\partial\mu}\equiv 0 \tag{5.40}$$

$$\frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0}\frac{\partial x_0^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1}\frac{\partial x_1^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_2}\frac{\partial x_2^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda^2}\frac{\partial\lambda^*}{\partial\mu}+\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu}\equiv 0 \tag{5.41}$$

Let us rewrite this system using matrix notation.

$$H\begin{pmatrix}\frac{\partial x_0^*}{\partial\mu}\\ \frac{\partial x_1^*}{\partial\mu}\\ \frac{\partial x_2^*}{\partial\mu}\\ \frac{\partial\lambda^*}{\partial\mu}\end{pmatrix}\equiv\begin{pmatrix}-\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu}\\ -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu}\\ -\frac{\partial^2\mathcal{L}}{\partial x_2\partial\mu}\\ -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu}\end{pmatrix} \tag{5.42}$$

Using Cramer's Rule, we may find comparative statics with respect to an arbitrary parameter μ.

$$\frac{\partial x_0^*}{\partial\mu}\equiv\frac{\begin{vmatrix}-\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\\ -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_1^2} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\\ -\frac{\partial^2\mathcal{L}}{\partial x_2\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial x_2^2} & \frac{\partial^2\mathcal{L}}{\partial x_2\partial\lambda}\\ -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_1} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{vmatrix}}{|H|} \tag{5.43}$$

$$\frac{\partial x_1^*}{\partial\mu}\equiv\frac{\begin{vmatrix}\frac{\partial^2\mathcal{L}}{\partial x_0^2} & -\frac{\partial^2\mathcal{L}}{\partial x_0\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial x_0\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_0} & -\frac{\partial^2\mathcal{L}}{\partial x_1\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial x_1\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial x_2\partial x_0} & -\frac{\partial^2\mathcal{L}}{\partial x_2\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial x_2^2} & \frac{\partial^2\mathcal{L}}{\partial x_2\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_0} & -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\mu} & \frac{\partial^2\mathcal{L}}{\partial\lambda\partial x_2} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{vmatrix}}{|H|} \tag{5.44}$$

$$\frac{\partial x_2^*}{\partial \mu} \equiv \frac{\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_0^2} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_0} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{vmatrix}}{|H|} \tag{5.45}$$

At this level of generality, we cannot sign many of the elements in the numerators in (5.43), (5.44), or (5.45). To pursue these comparative statics, let us define u and ω. Consider the following:

$$\begin{aligned} U \equiv \quad & A_0 x_0^{\alpha_0} - B_0 x_0^{\beta_0} \\ & + A_1 x_1^{\alpha_1} - B_1 x_1^{\beta_1} \gamma_1(B_0 x_0^{\beta_0}) \\ & + A_2 x_2^{\alpha_2} - B_2 x_2^{\beta_2} \gamma_1(B_1 x_1^{\beta_1}) \gamma_2(B_0 x_0^{\beta_0}) \end{aligned} \tag{5.46}$$

$$\text{subject to } g \equiv M - p_0 x_0 - p_1 x_1 - p_2 x_2 \geq 0 \tag{5.47}$$

where $\forall t \in \{0, 1, 2\}$,

$$\begin{aligned} & 0 \leq x_t \\ & 0 < A_t \\ & 0 < B_t \\ & 0 < \alpha_t < 1 \\ & 1 < \beta_t \end{aligned} \tag{5.48}$$

We then know that:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_0} = A_0 \alpha_0 x_0^{\alpha_0 - 1} - B_0 \beta_0 x_0^{\beta_0 - 1} [1 + B_1 x_1^{\beta_1} \gamma_1'(B_0 x_0^{\beta_0}) \\ + B_2 x_2^{\beta_2} \gamma_1(B_1 x_1^{\beta_1}) \gamma_2'(B_0 x_0^{\beta_0})] - \lambda p_0 \end{aligned} \tag{5.49}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} = & A_1 \alpha_1 x_1^{\alpha_1 - 1} \\ & - B_1 \beta_1 x_1^{\beta_1 - 1} [\gamma_1(B_0 x_0^{\beta_0}) + B_2 x_2^{\beta_2} \gamma_1'(B_1 x_1^{\beta_1}) \gamma_2(B_0 x_0^{\beta_0})] - \lambda p_1 \end{aligned} \tag{5.50}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = A_2 \alpha_2 x_2^{\alpha_2 - 1} - B_2 \beta_2 x_2^{\beta_2 - 1} \gamma_1(B_1 x_1^{\beta_1}) \gamma_2(B_0 x_0^{\beta_0}) - \lambda p_2 \tag{5.51}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_0 x_0 - p_1 x_1 - p_2 x_2 \tag{5.52}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0^2} = A_0\alpha_0(\alpha_0-1)x_0^{\alpha_0-2} - B_0\beta_0(\beta_0-1)x_0^{\beta_0-2}[1+B_1x_1^{\beta_1}\gamma_1'(B_0x_0^{\beta_0})+B_2x_2^{\beta_2}\gamma_1(B_1x_1^{\beta_1})\gamma_2'(B_0x_0^{\beta_0})] - (B_0\beta_0x_0^{\beta_0-1})^2[B_1x_1^{\beta_1}\gamma_1''(B_0x_0^{\beta_0})+B_2x_2^{\beta_2}\gamma_1(B_1x_1^{\beta_1})\gamma_2''(B_0x_0^{\beta_0})] \tag{5.53}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} = A_1\alpha_1(\alpha_1-1)x_1^{\alpha_1-2} - B_1\beta_1(\beta_1-1)x_1^{\beta_1-2}[\gamma_1(B_0x_0^{\beta_0})+B_2x_2^{\beta_2}\gamma_1'(B_1x_1^{\beta_1})\gamma_2(B_0x_0^{\beta_0})] - (B_1\beta_1x_1^{\beta_1-1})^2B_2x_2^{\beta_2}\gamma_1''(B_1x_1^{\beta_1})\gamma_2(B_0x_0^{\beta_0}) \tag{5.54}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2^2} = A_2\alpha_2(\alpha_2-1)x_2^{\alpha_2-2} - B_2\beta_2(\beta_2-1)x_2^{\beta_2-2}\gamma_1(B_1x_1^{\beta_1})\gamma_2(B_0x_0^{\beta_0}) \tag{5.55}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0\partial x_1} = -B_0\beta_0x_0^{\beta_0-1}B_1\beta_1x_1^{\beta_1-1}[\gamma_1'(B_0x_0^{\beta_0}) + B_2x_2^{\beta_2}\gamma_1'(B_1x_1^{\beta_1})\gamma_2'(B_0x_0^{\beta_0})] \tag{5.56}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0\partial x_2} = -B_0\beta_0x_0^{\beta_0-1}B_2\beta_2x_2^{\beta_2-1}\gamma_1(B_1x_1^{\beta_1})\gamma_2'(B_0x_0^{\beta_0}) \tag{5.57}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1\partial x_2} = -B_1\beta_1x_1^{\beta_1-1}B_2\beta_2x_2^{\beta_2-1}\gamma_1'(B_1x_1^{\beta_1})\gamma_2(B_0x_0^{\beta_0}) \tag{5.58}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda\partial x_t} = -p_t \ \forall t \in \{0,1,2\} \tag{5.59}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = 0 \tag{5.60}$$

We can also differentiate with respect to parameters so that we may sign the comparative statics.

$$\frac{\partial^2 \mathcal{L}}{\partial x_0\partial A_0} = \alpha_0x_0^{\alpha_0-1} \tag{5.61}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1\partial A_0} = 0 \tag{5.62}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2\partial A_0} = 0 \tag{5.63}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda\partial A_0} = 0 \tag{5.64}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0\partial A_1} = 0 \tag{5.65}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1\partial A_1} = \alpha_1x_1^{\alpha_1-1} \tag{5.66}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2\partial A_1} = 0 \tag{5.67}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial A_1} = 0 \tag{5.68}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial A_2} = 0 \tag{5.69}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial A_2} = 0 \tag{5.70}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial A_2} = \alpha_2 x_2^{\alpha_2 - 1} \tag{5.71}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial A_2} = 0 \tag{5.72}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial B_0} =& -\beta_0 x_0^{\beta_0 - 1}[1 + B_1 x_1^{\beta_1}\gamma_1'(B_0 x_0^{\beta_0}) + B_2 x_2^{\beta_2}\gamma_1(B_1 x_1^{\beta_1})\gamma_2'(B_0 x_0^{\beta_0})] \\ & - B_0\beta_0 x_0^{\beta_0 - 1} x_0^{\beta_0}[B_1 x_1^{\beta_1}\gamma_1''(B_0 x_0^{\beta_0}) \\ & \qquad + B_2 x_2^{\beta_2}\gamma_1(B_1 x_1^{\beta_1})\gamma_2''(B_0 x_0^{\beta_0})]\end{aligned} \tag{5.73}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial B_0} = -B_1\beta_1 x_1^{\beta_1 - 1} x_0^{\beta_0}[\gamma_1'(B_0 x_0^{\beta_0}) + B_2 x_2^{\beta_2}\gamma_1'(B_1 x_1^{\beta_1})\gamma_2'(B_0 x_0^{\beta_0})] \tag{5.74}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial B_0} = -B_2\beta_2 x_2^{\beta_2 - 1} x_0^{\beta_0}\gamma_1(B_1 x_1^{\beta_1})\gamma_2'(B_0 x_0^{\beta_0}) \tag{5.75}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial B_0} = 0 \tag{5.76}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial B_1} = -B_0\beta_0 x_0^{\beta_0 - 1} x_1^{\beta_1}[\gamma_1'(B_0 x_0^{\beta_0}) + B_2 x_2^{\beta_2}\gamma_1'(B_1 x_1^{\beta_1})\gamma_2'(B_0 x_0^{\beta_0})] \tag{5.77}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial B_1} =& -\beta_1 x_1^{\beta_1 - 1}[\gamma_1(B_0 x_0^{\beta_0}) + B_2 x_2^{\beta_2}\gamma_1'(B_1 x_1^{\beta_1})\gamma_2(B_0 x_0^{\beta_0})] \\ & - B_1\beta_1 x_1^{\beta_1 - 1} x_1^{\beta_1} B_2 x_2^{\beta_2}\gamma_1''(B_1 x_1^{\beta_1})\gamma_2(B_0 x_0^{\beta_0})\end{aligned} \tag{5.78}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial B_1} = -B_2\beta_2 x_2^{\beta_2 - 1} x_1^{\beta_1}\gamma_1'(B_1 x_1^{\beta_1})\gamma_2(B_0 x_0^{\beta_0}) \tag{5.79}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial B_1} = 0 \tag{5.80}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial B_2} = -B_0\beta_0 x_0^{\beta_0 - 1} x_2^{\beta_2}\gamma_1(B_1 x_1^{\beta_1})\gamma_2'(B_0 x_0^{\beta_0}) \tag{5.81}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial B_2} = -B_1\beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2}\gamma_1'(B_1 x_1^{\beta_1})\gamma_2(B_0 x_0^{\beta_0}) \tag{5.82}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial B_2} = -\beta_2 x_2^{\beta_2 - 1}\gamma_1(B_1 x_1^{\beta_1})\gamma_2(B_0 x_0^{\beta_0}) \tag{5.83}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial B_2} = 0 \tag{5.84}$$

Case One: Depletion in period $t = 1$, depletion in period $t = 2$. In this case, we simply assume that the gamma function demonstrates ego depletion in both future time periods. From our intuition of the problem in this case, we know these:

$$\gamma_1 > 1 \qquad \gamma_1' > 0 \qquad \gamma_1'' \geq 0 \tag{5.85}$$

$$\gamma_2 > 1 \qquad \gamma_2' > 0 \qquad \gamma_2'' \geq 0 \tag{5.86}$$

We then arrive at these signs:

$$\frac{\partial^2 \mathcal{L}}{\partial x_0^2} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1^2} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_2^2} < 0 \tag{5.87}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} < 0 \tag{5.88}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} < 0 \tag{5.89}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = 0 \tag{5.90}$$

Case Two: Depletion in period $t = 1$, learning in period $t = 2$. In this case, we assume that the gamma function demonstrates ego repletion (e.g., learning). Again, from our intuition of the problem in this case, we know these:

$$\gamma_1 > 1 \qquad \gamma_1' > 0 \qquad \gamma_1'' \geq 0 \tag{5.91}$$

$$0 < \gamma_2 < 1 \qquad \gamma_2' < 0 \qquad \gamma_2'' > 0 \tag{5.92}$$

However, we must make the following two additional assumptions in order to sign $\frac{\partial^2 \mathcal{L}}{\partial x_0^2}$ and $\frac{\partial^2 \mathcal{L}}{\partial x_0 x_1}$, respectively:

$$1 + B_1 x_1^{\beta_1} \gamma_1'(B_0 x_0^{\beta_0}) > B_2 x_2^{\beta_2} \gamma_1(B_1 x_1^{\beta_1}) \gamma_2'(B_0 x_0^{\beta_0}) \tag{5.93}$$

$$\gamma_1'(B_0 x_0^{\beta_0}) > B_2 x_2^{\beta_2} \gamma_1'(B_1 x_1^{\beta_1}) \gamma_2'(B_0 x_0^{\beta_0}) \tag{5.94}$$

We then arrive at these signs:

$$\frac{\partial^2 \mathcal{L}}{\partial x_0^2} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1^2} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_2^2} < 0 \tag{5.95}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_1} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_0 \partial x_2} > 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} < 0 \tag{5.96}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_0 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} < 0 \qquad \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} < 0 \tag{5.97}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = 0 \tag{5.98}$$

Both cases. However, due to the generality of this specification, we can only sign three comparative statics.

$$\frac{\partial x_0^*}{\partial A_0} > 0 \qquad \frac{\partial x_1^*}{\partial A_1} > 0 \qquad \frac{\partial x_2^*}{\partial A_2} > 0 \tag{5.99}$$

The following comparative statics cannot be readily signed.

$$\frac{\partial x_0^*}{\partial A_1} \qquad \frac{\partial x_1^*}{\partial A_2} \qquad \frac{\partial x_2^*}{\partial A_0} \tag{5.100}$$

$$\frac{\partial x_0^*}{\partial A_2} \qquad \frac{\partial x_1^*}{\partial A_0} \qquad \frac{\partial x_2^*}{\partial A_1} \tag{5.101}$$

$$\frac{\partial x_0^*}{\partial B_0} \qquad \frac{\partial x_1^*}{\partial B_0} \qquad \frac{\partial x_2^*}{\partial B_0} \tag{5.102}$$

$$\frac{\partial x_0^*}{\partial B_1} \qquad \frac{\partial x_1^*}{\partial B_1} \qquad \frac{\partial x_2^*}{\partial B_1} \tag{5.103}$$

$$\frac{\partial x_0^*}{\partial B_2} \qquad \frac{\partial x_1^*}{\partial B_2} \qquad \frac{\partial x_2^*}{\partial B_2} \tag{5.104}$$