Existence and oscillation of solutions of certain functional differential equations
by Gary Wayne Grefsrud

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
Let g(t) and f(t,y) be known real valued functions.

In this paper we consider functional differential equations of the form x^n(t) + f(t,x(g(t))) = 0.

In Chapter II the problem of existence, uniqueness, and continuability of solutions is examined. If n = 2 and f(t,y) = a(t)y, a local solution of an initial value problem is constructed using the method of successive approximations. The solution, in fact, is unique and can be extended to the infinite interval (-∞,∞), With n = 2 and f(t,y) = a(t)y^γ, γ > 1, a local solution, which is unique, is again constructed.

Chapter III is devoted mainly to oscillatory 1 properties of solutions. With n even, ∫^∞ t^n-1a(t)dt < ∞ is found to be a necessary condition for x^(n) + a(t)f(x(g(t))) = 0 to have a bounded nonoscillatory solution. The condition is sufficient if n = 2. Conditions which insure that solutions of the equation x^(n)(t) + f(t,x(g(t))) = 0 are oscillatory or tend monitonically to zero are presented. The theorems given all cover the special case where f(t,y) = a(t)y^γ, γ > 1, with one result valid for 0 ≤ γ < 1. It is worth noting that if g(t) = t, all theorems given are well known.

EXISTENCE AND OSCILLATION OF SOLUTIONS OF CERTAIN FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

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CHAPTER I

INTRODUCTION

In general, a functional differential equation is defined as a differential equation in which the unknown function appears with various values of the argument [2]. In this paper we will restrict our attention to equations such as $x^{(n)}(t)+f(t,x(g(t))) = 0$. We will call the equation linear if it would be linear in the "normal" sense (disregarding the functional argument) and nonlinear otherwise. To be a solution of such an equation on an interval I, x(t) must be defined for $t \in I \cup g(I)$ and must satisfy the differential equation. If a solution, x(t), valid for large t, has arbitrarily large zero's x(t) is said to be oscillatory.

Prior to 1950, very little was known about solutions to functional differential equations although these types of equations were encountered in the eighteenth century by Euler while investigating the nature of curves similar to their own evolutes [5].

Since the 1950's numerous papers have been devoted to these equations in the special case where g(t) is a delay function of the form $t - \sigma(t)$, where $\sigma(t)$ is defined and nonnegative in some interval $[t_0,T]$. These equations are commonly referred to as "time delay differential equations", or "differential equations with retarded arguments". There a solution is sought which coincides with a given initial function $\theta(t)$ on an initial set $E_{t_0} = \{t \mid t - \sigma(t) < t_0$ for $t \geq t_0\}$ [2], [5]. If $g(t)$ is of the form $t - \sigma(t)$ with $\sigma(t) \leq 0$ on $[t_0, T]$, then the equation is referred to as a "differential equation with advanced argument".

Very little is known about solutions to functional differential equations if $g(t)$ is not a "delay function". Utz [11] posed the problem of existence and uniqueness of nontrivial solutions to $f'(t) = af(g(t))$ subject to $f(t_0) = f_0$.

Suppose $g(t)$ is a real valued continuous function defined on some interval D_g including the origin with corresponding range R_g. If $R_g \subset D_g$ and $g(t) \leq k$ in D_g, Ryder [8] has shown the existence of a solution for all $t \in D_g$ to $\bar{f}'(t) = A\bar{f}(g(t))$ satisfying $\bar{f}(t_0) = \bar{f}_0$ provided $\|Ak\| < 1$. Here A is a constant n x n real matrix with norm $\|A\|$, $\bar{f}$ an n-vector. In fact the solution is unique on any subinterval $N \subset D_g$ such that $g(N) \subset N$, where $0 \in N$.

Oberg [6] has shown the local existence of solutions of the equation $\dot{x}(t) = f(t, x(t), x(g(t, x(t))))$ satisfying $x(t_0) = x_0$ under the assumption that $f(t,x,y)$ and $g(t,x)$ satisfy uniform Lipschitz conditions with $g(t_0, x_0) = t_0$. Uniqueness then follows if $g(t,x)$ is contracting at (t_0, x_0), i.e. (t,x) close to (t_0,x_0) implies $|g(t,x) - t_0| \leq |t - t_0|$.

Chapter II of this paper is concerned with the existence and uniqueness of nontrivial solutions of the equation $\ddot{X}(t) + A(t)X^{\gamma}(G(t)) = 0$, satisfying $X(t_0) = c_1$, $\dot{X}(t_0) = c_2$. Suppose, however, we replace t by $t + t_0$. Then $x(t) = X(t + t_0)$ satisfies
$\ddot{x}(t) + a(t)\, x^{\gamma}(g(t)) = 0$, $x(0) = c_1$, $\dot{x}(0) = c_2$, where $g(t) = G(t + t_0) - t_0$, $a(t) = A(t + t_0)$. Thus the initial value problem can be translated to the origin. A problem equivalent to showing the existence of solutions to the initial value problem (at the origin), is showing the existence of a solution to the integral equation

$$x(t) = c_1 + c_2 t - \int_0^t (t - s)a(s)x^{\gamma}(g(s))ds.$$

This, of course, is exhibited easily by integrating the differential equation twice successively from 0 to t and interchanging the order of integration.

Several papers have been devoted to oscillation theorems associated with non-delay functional differential equations. Theorems of Bradley [1] and Waltman [12] closely related to this work appear in Chapter III of this paper.

CHAPTER II

EXISTENCE AND UNIQUENESS

We first consider the problem of existence and uniqueness of solutions to the linear second order functional differential equation

$$(1)\qquad \ddot{x}(t) + a(t)x(g(t)) = 0$$

subject to the initial conditions

$$(2)\qquad \begin{aligned} x(0) &= c_1 \\ \dot{x}(0) &= c_2. \end{aligned}$$

As we have shown in Chapter I, an equivalent problem is to solve the integral equation

$$(3)\qquad x(t) = c_1 + c_2 t - \int_0^t (t - s)a(s)x(g(s))ds.$$

To find a solution of (3) we employ the method of successive approximations and thus define

$$(4)\qquad \begin{aligned} x_0(t) &= c_1 + c_2 t \\ x_n(t) &= x_0(t) - \int_0^t (t - s)a(s)x_{n-1}(g(s))ds, \; n \geq 1. \end{aligned}$$

<u>Lemma II.1</u>. Let $g(t)$ and $a(t)$ be continuous on $[-\sigma,T]$, $\sigma \geq 0$, $T \geq 0$, such that $g([-\sigma,T]) \subset [-\sigma,T]$. Then $x_n(t)$, defined by (4), is continuous on $[-\sigma,T]$ and

$$(5)\qquad x_n(t) = x_0(t) + \sum_{k=1}^{n} (-1)^k g_k(t), \; n \geq 1,$$

where $g_k(t)$ is defined by

$$g_1(t) = \int_0^t (t - s)a(s)(c_1 + c_2 g(s))ds$$

(6)

$$g_n(t) = \int_0^t (t - s)a(s)g_{n-1}(g(s))ds, \quad n \geq 2.$$

PROOF: $g_1(t)$ is defined and continuous for all t in $[-\sigma,T]$ since a(t) and g(t) are continuous on $[-\sigma,T]$. We now assume $g_{k-1}(t)$ is defined and continuous for all t in $[-\sigma,T]$. Thus $g_{k-1}(g(t))$ is defined and continuous for all t in $[-\sigma,T]$ since $g([-\sigma,T]) \subset [-\sigma,T]$. Therefore, by (6) $g_k(t)$ is defined and continuous on $[-\sigma,T]$. Thus $g_n(t)$ is defined and continuous on $[-\sigma,T]$ for all n by induction.

To prove (5), we have, using (4) and (6),

$$x_1(t) = x_0(t) - \int_0^t (t - s)a(s)x_0(g(s))ds, \quad t \in [-\sigma,T],$$

$$= x_0(t) - \int_0^t (t - s)a(s)[c_1 + c_2 g(s)]ds,$$

or

$$x_1(t) = x_0(t) - g_1(t).$$

$$x_2(t) = x_0(t) - \int_0^t (t - s)a(s)x_1(g(s))ds, \quad t \in [-\sigma,T],$$

$$= x_0(t) - \int_0^t (t - s)a(s)[x_0(g(s)) - g_1(g(s))]ds.$$

$$x_2(t) = x_0(t) - \int_0^t (t - s)a(s)[c_1 + c_2 g(s)]ds$$

$$+ \int_0^t (t - s)a(s)g_1(g(s))ds$$

Thus

$$x_2(t) = x_0(t) - g_1(t) + g_2(t)$$

$$= x_0(t) + \sum_{k=1}^{2} (-1)^k g_k(t).$$

Now if

$$x_m(t) = x_0(t) + \sum_{k=1}^{m} (-1)^k g_k(t), \text{ then}$$

$$x_{m+1}(t) = x_0(t)$$

$$- \int_0^t (t - s)a(s)[x_0(g(s)) + \sum_{k=1}^{m} (-1)^k g_k(g(s))]ds$$

$$= x_0(t) - \int_0^t (t - s)a(s)[c_1 + c_2 g(s)]ds$$

$$+ \sum_{k=1}^{m} (-1)^{k+1} \int_0^t (t - s)a(s)g_k(g(s))ds$$

$$= x_0(t) - g_1(t) + \sum_{k=1}^{m} (-1)^{k+1} g_{k+1}(t)$$

or

$$x_{m+1}(t) = x_0(t) + \sum_{k=1}^{m+1} (-1)^k g_k(t) \qquad \text{Q.E.D.}$$

Note that if $a(t)$, $g(t)$ and $a(t)g(t)$ are integrable instead of continuous on $[-\sigma,T]$ the same result holds.

We now show that the sequence of approximations, defined by (4), converges to a solution of (1) satisfying (2) for $t \in [-\sigma, T]$ under suitable restrictions on $a(t)$ and $g(t)$.

<u>Theorem II.1</u>. Let $|a(t)|g^2(t) \leq k$ for $t \in [-\sigma, T]$. Then under the assumptions of Lemma II.1, the sequence $\{x_n(t)\}$, defined by (4), converges uniformly on $[-\sigma, T]$ to a solution of (1) satisfying (2) provided $k < 2$.

PROOF: Since $g(t)$ and $a(t)$ are continuous on $[-\sigma, T]$, there exists k_1 such that $|a(t)||c_1 + c_2 g(t)| \leq k_1$ on $[-\sigma, T]$. Thus we have from (6),

$$|g_1(t)| \leq \left|\int_0^t |t - s||a(s)||c_1 + c_2 g(s)|ds\right|$$

$$\leq k_1 \left|\int_0^t |t - s| ds\right|$$

$$\leq \frac{k_1 t^2}{2}, \; t \in [-\sigma, T],$$

$$\leq \frac{k_1 I^2}{2} \text{ where } I = \max\{\sigma, T\}.$$

Again from (6) we have

$$|g_2(t)| \leq \left|\int_0^t |t - s||a(s)||g_1(g(s))|ds\right|$$

$$\leq \frac{k_1}{2}\left|\int_0^t |t - s| \, |a(s)|g^2(s)ds\right|, \; t \in [-\sigma, T].$$

$$|g_2(t)| \le \frac{k_1}{2} k \frac{t^2}{2}$$

$$\le \frac{k_1}{k} t^2 \left(\frac{k}{2}\right)^2$$

$$\le \frac{k_1}{k} I^2 \left(\frac{k}{2}\right)^2.$$

Now assume that

$$|g_m(t)| \le \frac{k_1}{k} t^2 \left(\frac{k}{2}\right)^m$$

$$\le \frac{k_1}{k} I^2 \left(\frac{k}{2}\right)^m.$$

Then we have

$$g_{m+1}(t) \le \left|\int_0^t |t - s|\, |a(s)|\, |g_m(g(s))|\, ds\right|$$

$$\le \left|\int_0^t |t - s|\, |a(s)| \frac{k_1}{k} g^2(s) \left(\frac{k}{2}\right)^m ds\right|, \quad t \in [-\sigma, T],$$

$$\le \frac{k_1}{k} \left(\frac{k}{2}\right)^m k \frac{t^2}{2}$$

$$\le \frac{k_1}{k} \left(\frac{k}{2}\right)^{m+1} t^2$$

$$\le \frac{k_1}{k} I^2 \left(\frac{k}{2}\right)^{m+1}.$$

From (5) we have

$$\lim_{n\to\infty} x_n(t) = x_0(t) + \sum_{j=1}^{\infty} (-1)^j g_j(t) \text{ and}$$

$$\sum_{j=1}^{\infty} |g_j(t)| \le \frac{k_1}{k} I^2 \sum_{j=1}^{\infty} \left(\frac{k}{2}\right)^j .$$

Thus $\{x_n(t)\}$ converges uniformly to some function $x(t)$ on $[-\sigma,T]$ provided $\frac{k}{2} < 1$ or $k < 2$, and $x(t)$ is continuous on $[-\sigma,T]$ since each $x_n(t)$ is.

To show that $x(t)$ satisfies (1) and (2) on $[-\sigma,T]$ we write

$$x_{n+1}(t) = x_0(t) - \int_0^t (t - s)a(s)x_n(g(s))ds, \quad -\sigma \leq t \leq T.$$

If we now take the limit as $n \to \infty$ and note that $\lim_{n\to\infty} x_n(g(t)) = x(g(t))$ uniformly on $[-\sigma,T]$ since $\lim_{n\to\infty} x_n(t) = x(t)$ on $[-\sigma,T]$ and $g([-\sigma,T]) \subset [-\sigma,T]$, we have

$$x(t) = x_0(t) - \int_0^t (t - s)a(s)x(g(s))ds.$$

i.e. $x(t)$ satisfies the integral equation (3) on $[-\sigma,T]$ and therefore is a solution of (1) satisfying (2) on $[-\sigma,T]$. Q.E.D.

Using bounds established in the proof of Theorem II.1, we can prove the following corollary.

Corollary II.1.1. Under the assumptions of Theorem II.1,

$$|x(t) - x_n(t)| \leq \frac{2k_1 t^2}{2-k}\left(\frac{k}{2}\right)^n, \quad t \in [-\sigma,T],$$ where the $x_n(t)$ are defined by (4) and $x(t) = \lim_{n\to\infty} x_n(t)$.

PROOF: From (5) we have that

$$x_n(t) = x_0(t) + \sum_{j=1}^{n} (-1)^j g_j(t) \text{ and thus}$$

$$x(t) = x_0(t) + \sum_{j=1}^{\infty} (-1)^j g_j(t).$$

Thus

$$x(t) - x_n(t) = \sum_{j=n+1}^{\infty} (-1)^j g_j(t) \quad \text{and,}$$

$$\begin{aligned} |x(t) - x_n(t)| &\le \sum_{j=n+1}^{\infty} |g_j(t)| \\ &\le \sum_{j=n+1}^{\infty} \frac{k_1}{k} t^2 \left(\frac{k}{2}\right)^j \\ &\le \frac{k_1}{k} t^2 \left(\frac{k}{2}\right)^{n+1} \sum_{j=0}^{\infty} \left(\frac{k}{2}\right)^j \end{aligned}$$

or

$$|x(t) - x_n(t)| \le \frac{2k_1 t^2}{2-k} \left(\frac{k}{2}\right)^n, \qquad \text{Q.E.D.}$$

Now suppose that $v(t)$ is any other solution of (1) on $[-\sigma, T]$ satisfying (2). Then we have

$$v(t) = c_1 + c_2 t - \int_0^t (t - s) a(s) v(g(s)) ds,$$

Thus if $g([-\sigma,T] \subset [-\sigma,T]$,

$$|v(t) - x_0(t)| \leq |\int_0^t |t - s||a(s)| |v(g(s))| ds|$$

$$\leq \frac{\alpha t^2}{2}, \ t \in [-\sigma,T], \text{ where } \alpha = \max |a(t)| |v(g(t))|$$

and

$$|v(t) - x_1(t)| \leq |\int_0^t |t - s| |a(s)| |x_0(g(s)) - v(g(s))| ds|$$

$$\leq \frac{\alpha}{2} |\int_0^t |t - s| |a(s)| g^2(s) ds|$$

$$\leq \frac{\alpha}{2} t^2 \frac{k}{2}.$$

Now suppose

$$|v(t) - x_m(t)| \leq \frac{\alpha}{2} t^2 (\frac{k}{2})^m.$$

Then

$$|v(t) - x_{m+1}(t)| \leq |\int_0^t |t - s| |a(s)| |x_m(g(s)) - v(g(s))| ds|$$

$$\leq \frac{\alpha}{2} (\frac{k}{2})^m |\int_0^t |t - s| |a(s)| g^2(s) ds|$$

$$\leq \frac{\alpha}{2} t^2 (\frac{k}{2})^{m+1}$$

i.e. $|v(t) - x_n(t)| \leq \frac{\alpha}{2} t^2 (\frac{k}{2})^n$ by induction.

Taking the limit as $n \to \infty$ we have

$$|v(t) - x(t)| \leq 0 \text{ if } k < 2$$

or $v(t) = x(t)$ on $[-\sigma, T]$.

We have thus proved:

Theorem II.2. Under the assumptions of Theorem II.1, the solution $x(t)$, of (1) on $[-\sigma, T]$ satisfying (2) is unique.

The results of the previous theorems can be illustrated by the following examples.

Example II.1. This example shows that when $|a(t)|g^2(t) = 2$, there cannot, in general, exist a solution to (1) satisfying (2) for arbitrary initial values c_1 and c_2. Consider the initial value problem

$$(7) \qquad \ddot{x}(t) - ax\left(\sqrt{\tfrac{2}{a}}\right) = 0$$

$$(2) \qquad \begin{aligned} x(0) &= c_1 \\ \dot{x}(0) &= c_2 \end{aligned}$$

Here $a(t) = -a$, $a > 0$, constant, $g(t) = \sqrt{\frac{2}{a}}$. Note that $|a(t)|g^2(t) = 2$. Integrating (7) and using (2) we have

$$x(t) = c_1 + c_2 t + \frac{ax\left(\sqrt{\frac{2}{a}}\right)t^2}{2}$$

Now letting $t = \sqrt{\frac{2}{a}}$ we have

$$x(\sqrt{\tfrac{2}{a}}) = c_1 + c_2\sqrt{\tfrac{2}{a}} + x(\sqrt{\tfrac{2}{a}})$$

or

$$c_1 + c_2\sqrt{\tfrac{2}{a}} = 0.$$

Thus there exists a solution of (7) satisfying (2), valid on an interval including 0 and $\sqrt{\frac{2}{a}}$ only if c_1 and c_2 satisfy the condition

$$c_1 + c_2\sqrt{\tfrac{2}{a}} = 0.$$

<u>Example II.2.</u> This example shows that if there is a solution to (1) satisfying (2) with the condition $|a(t)|\,g^2(t) = 2$, the solution, in general, is not unique.

(8) $\qquad \ddot{x}(t) - 2e^{2t}x(e^{-t}) = 0$

(9) $\qquad x(0) = 0$, $\quad \dot{x}(0) = 0.$

Here $|a(t)|\,g^2(t) = 2.$

One can verify that two solutions to this initial value problem are:

$$x_1(t) = 0$$

and $\quad x_2(t) = t^2.$

Example II.3. An example of an initial value problem which possesses a locally unique solution is furnished by the following.

(10) $$\ddot{x}(t) - x(e^{-\frac{1}{(t+1)^2}}) = 0$$

(11) $$x(0) = 0$$ $$\dot{x}(0) = 1.$$

Here $a(t) = -1$, $g(t) = e^{-\frac{1}{(t+1)^2}}$. Note that on the interval $[a,b]$, where $-1 < a < 0$, $0 < b \leq 1$, $|a(t)|g^2(t) < 2$. Therefore one and only one solution exists on $[a,b]$, namely $x(t) = \log(t+1)$.

The natural question one now asks is that of continuability of solutions. The next theorem shows that, under suitable restrictions, a unique solution of (1) satisfying (2) can be extended to $(-\infty,\infty)$.

Theorem II.3. Let $g(t)$ and $a(t)$ be continuous for all t with $|a(t)|g^2(t) \leq k$. Suppose there exists positive monotone nondecreasing sequences $\{\sigma_i\}$ and $\{T_j\}$ with $\lim_{i\to\infty} \sigma_i = \lim_{j\to\infty} T_j = +\infty$ such that $g([-\sigma_i,T_j]) \subset [-\sigma_i,T_j]$ for all i and j. Then there exists a unique solution to (1) satisfying (2), valid on $(-\infty,\infty)$, provided $k < 2$.

PROOF: We first fix i and j and then note that Theorem II.1 and Theorem II.2 guarantee the existence of a unique solution, x(t), to (1) satisfying (2) on $[-\sigma_i, T_j]$. In fact $x(t) = \lim_{n\to\infty} x_n(t)$ where the $x_n(t)$ are defined by (4) on $[-\sigma_i, T_j]$.

However, now consider the sequence defined by (4) on the interval $[-\sigma_{i+1}, T_{j+1}]$.

i.e. $x_0(t) = c_1 + c_2 t$

$$x_n(t) = x_0(t) - \int_0^t (t - s)a(s)x_{n-1}(g(s))ds, \ t \in [-\sigma_{i+j} T_{j+1}].$$

We have $\lim_{n\to\infty} x_n(t) = X(t)$ on $[-\sigma_{i+j}, T_{j+1}]$ and X(t) is the unique solution of (1) satisfying (2) on $[-\sigma_{i+1}, T_{j+1}]$. But since $[-\sigma_i, T_j] \subset [-\sigma_{i+1}, T_{j+1}]$, X(t) = x(t) on $[-\sigma_i, T_j]$. Therefore, x(t) can be extended to $[-\sigma_{i+1}, T_{j+1}]$ and thus can be extended to $(-\infty, \infty)$. Q.E.D.

Remark. The condition on g(t), $(g([-\sigma_i, T_j]) \subset [-\sigma_i, T_j])$, states that g(t) could possibly oscillate about the straight line x = t, provided it does not oscillate "too rapidly".

We now consider the existence and uniqueness of solutions to the nonlinear functional differential equation,

(12) $\ddot{x}(t) + a(t)x^{\gamma}(g(t)) = 0$, $\gamma > 1$, the ratio of odd integers, satisfying the initial conditions (2).

As before we note that an equivalent problem is to find a solution to the integral equation

(13) $$x(t) = c_1 + c_2 t - \int_0^t (t - s)a(s)x^{\gamma}(g(s))ds.$$

Define

(14) $$x_0(t) = c_1 + c_2 t$$
$$x_n(t) = x_0(t) - \int_0^t (t - s)a(s)x^{\gamma}_{n-1}(g(s))\, ds,\ n \geq 1.$$

Lemma II.2. Suppose $a(t)$ and $g(t)$ are continuous on $|t| \leq \alpha$ with $|a(t)| \leq L$ on $|t| \leq \alpha$.

Let $h \leq \min\left\{\frac{\alpha}{2}, \sqrt{\frac{|c_2|\alpha}{L[|c_1| + |c_2|\alpha]^{\gamma}}}\right\}$ such that $g([-h,h]) \subset [-h,h]$. Then $x_n(t)$, defined by (14), is defined and continuous and $|x_n(t) - c_1| \leq b = \alpha|c_2|$ for $|t| \leq h$.

PROOF: $x_0(t) = c_1 + c_2 t$ is defined and continuous for all t. Suppose $x_m(t)$ is defined and continuous for

$|t| \leq h$, then $x_m(g(t))$ is defined and continuous on $|t| \leq h$ since $g([-h,h]) \subset [-h,h]$. Therefore, by (14) $x_{m+1}(t)$ is defined and continuous on $|t| \leq h$.

Furthermore,

$$|x_0(t) - c_1| = |c_2 t| \leq |c_2|\frac{\alpha}{2} = \frac{b}{2} < b \text{ if } |t| \leq h \leq \frac{\alpha}{2}$$

$$|x_1(t) - c_1| \leq |c_2 t| + L\left|\int_0^t |t - s|\,|x_0^\gamma(g(s))|ds\right|$$

$$\leq |c_2|\frac{\alpha}{2} + L[|c_1| + b]^\gamma \left|\int_0^t |t - s|ds\right|, \text{ for } |t| \leq h,$$

$$\leq |c_2|\frac{\alpha}{2} + L[|c_1| + b]^\gamma \frac{t^2}{2}$$

$$\leq |c_2|\frac{\alpha}{2} + L[|c_1| + b]^\gamma \frac{|c_2|\alpha}{2L[|c_1| + |c_2|\alpha]^\gamma}$$

$$\leq |c_2|\alpha = b, \text{ if } |t| \leq h.$$

Now suppose $|x_m(t) - c_1| \leq b$ for $|t| \leq h$.

Then, $|x_m(t)|^\gamma \leq [|c_1| + b]^\gamma$ if $|t| \leq h$

and therefore, $|x_m(g(t))|^\gamma \leq [|c_1| + b]^\gamma$ if $|t| \leq h$ since $g([-h,h]) \subset [-h,h]$.

Thus, $|x_{m+1}(t) - c_1| \leq |c_2 t| + L\left|\int_0^t |t - s|[|c_1| + b]^\gamma ds\right|$

$$\leq |c_2|\frac{\alpha}{2} + |c_2|\frac{\alpha}{2} = |c_2|\alpha = b, \text{ if } |t| \leq h. \quad \text{Q.E.D}$$

Lemma II.3. With the hypothesis of Lemma II.2,

$$|x_{n+1}(t) - x_n(t)| \le \frac{LM}{KB}t^2\left(\frac{KB}{2}\right)^{n+1} \text{ for } |t| \le h$$

where $M = [|c_1| + |c_2|\alpha]^\gamma$, $K = \gamma M^{\frac{\gamma-1}{\gamma}}$, $B = \max\limits_{[-h,h]} |a(t)|g^2(t)$ and $x_n(t)$ is defined by (14).

PROOF:

$$|x_1(t) - x_0(t)| \le \left|\int_0^t |t - s|\,|a(s)|\,|c_1 + c_2 g(s)|^\gamma ds\right|$$

$$\le LM\frac{t^2}{2}, \quad \text{if } |t| \le h.$$

$$\le \frac{LM}{KB}t^2\left(\frac{KB}{2}\right).$$

Now suppose

$$|x_m(t) - x_{m-1}(t)| \le \frac{LM}{KB}t^2\left(\frac{KB}{2}\right)^m \quad \text{for } |t| \le h.$$

Then $|x_{m+1}(t) - x_m(t)|$

$$\le \left|\int_0^t |t - s|\,|a(s)|\,|x_m^\gamma(g(s)) - x_{m-1}^\gamma(g(s))|\,ds\right|$$

$$\le \left|\int_0^t |t - s|\,|a(s)|\,\gamma|x_s^{\gamma-1}|\,|x_m(g(s)) - x_{m-1}(g(s))|\,ds\right|$$

by the law of the mean. Thus

$|x_{m+1}(t) - x_m(t)|$

$$\le \gamma[|c_1| + |c_2|\alpha]^{\gamma-1}\,\frac{LM}{KB}\left(\frac{KB}{2}\right)^m\left|\int_0^t |t - s|\,|a(s)|g^2(s)ds\right|$$

$$\le \frac{LM}{KB}t^2\left(\frac{KB}{2}\right)^{m+1} \quad \text{if } |t| \le h. \quad \text{Q.E.D.}$$

Lemma II.4. Under the assumptions of Lemma II.2, the sequence $\{x_n(t)\}$, defined by (14), converges uniformly on $|t| \leq h$ to a solution of (12) satisfying (2) provided $KB < 2$, i.e. $|a(t)|g^2(t) < \dfrac{2}{\gamma[|c_1| + |c_2|\alpha]^{\gamma-1}}$.

PROOF: Write

$$|x_n(t)| \leq |x_0(t)| + \sum_{j=1}^{\infty} |x_j(t) - x_{j-1}(t)|$$

and $$\sum_{j=1}^{\infty} |x_j(t) - x_{j-1}(t)| \leq \frac{LM}{KB} t^2 \sum_{j=1}^{\infty} \left(\frac{KB}{2}\right)^j .$$

Thus by the Weierstrass M test $\{x_n(t)\}$ converges uniformly to say $x(t)$ on $|t| \leq h$ if $KB < 2$.

By writing $x_n(t) = c_1 + c_2 t - \int_0^t (t - s)a(s)x^{\gamma}_{n-1}(g(s))ds$ and using the uniform convergence of the $x_n(t)$ (and the $x_n(g(t))$) we see that $x(t)$ is a solution of (12) satisfying (2) on $|t| \leq h$. Q.E.D.

Lemma II.5. Suppose the conditions of Lemma II.2 and Lemma II.4 are satisfied. Let $x(t) = \lim_{n\to\infty} x_n(t)$ where the $x_n(t)$ are defined by (14) and let $v(t)$ be any other solution of (12) satisfying (2) valid on $|t| \leq h$. Then $v(t) = x(t)$ for $|t| \leq h_1 \leq h$, where h_1 is such that $|v(t)| \leq |c_1| + |c_2|\alpha$ on $|t| \leq h_1$ and $g([-h_1, h_1]) \subset [-h_1, h_1]$.

PROOF: Since $v(t)$ is a solution of (12) satisfying (2) we have $v(t) = c_1 + c_2 t - \int_0^t (t - s)a(s)v^{\gamma}(g(s))ds$

Therefore from (14) we have

$$|v(t) - x_0(t)| \leq \left|\int_0^t |t - s|\,|a(s)|\,|v^{\gamma}(g(s))|ds\right|$$

$$\leq L\left|\int_0^t |t - s|\,|v^{\gamma}(g(s))|ds\right|$$

$$\leq L[\,|c_1| + |c_2|\alpha]^{\gamma}\frac{t^2}{2}$$

$$\leq LM\frac{t^2}{2}, \qquad |t| \leq h_1.$$

$$|v(t) - x_1(t)| \leq \left|\int_0^t |t - s|\,|a(s)|\,|v^{\gamma}(g(s)) - x_0^{\gamma}(g(s))|ds\right|$$

$$\leq K\left|\int_0^t |t - s|\,|a(s)|\,|v(g(s)) - x_0(g(s))|ds\right|$$

$$\leq K\frac{LM}{2}\left|\int_0^t |t - s|\,|a(s)|g^2(s)ds\right|$$

$$\leq \frac{ML}{KB}\left(\frac{KB}{2}\right)^2 t^2$$

Thus by induction,

$$|v(t) - x_n(t)| \leq \frac{ML}{KB}\left(\frac{KB}{2}\right)^{n+1} t^2, \text{ for all } n,\ |t| \leq h_1$$

thus $\lim_{n\to\infty} |v(t) - x_n(t)| = |v(t) - x(t)| = 0$ if $KB < 2$

i.e. $v(t) = x(t)$ for $|t| \leq h_1$. Q.E.D.

Summarizing the results of Lemma's II.2 through II.5 we have

<u>Theorem II.4</u>. Suppose $a(t)$ and $g(t)$ are continuous on $|t| \le \alpha$ with $|a(t)| \le L$ on $|t| \le \alpha$. Let

$$h \le \min\left\{\frac{\alpha}{2}, \sqrt{\frac{|c_2|\alpha}{L[|c_1| + |c_2|\alpha]^{\gamma}}}\right\} \text{ such that } g([-h,h]) \subset [-h,h].$$

Then there exists a solution, $x(t)$, of (12) satisfying (2) on $t \le h$ provided $|a(t)|g^2(t) < \dfrac{2}{\gamma[|c_1| + |c_2|\alpha]^{\gamma-1}}$.

The solution is unique on $|t| \le h_1 \le h$ if h_1 is such that $|x(t)| \le |c_1| + |c_2|\alpha$ on $|t| \le h_1$ and $g([-h_1,h_1]) \subset [-h_1,h_1]$.

CHAPTER III

PROPERTIES OF SOLUTIONS

While the majority of this chapter will be devoted to oscillation theory, we first prove a boundedness theorem associated with solutions of (1) satisfying (2) in the special case where $a(t) = a$, constant, $a > 0$, the proof given being completely dependent upon the constructions used in the existence proofs in Chapter II.

Theorem III.1. Suppose the conditions of Theorem II.3 are satisfied. Let $a(t) = a > 0$, a constant. Let $x(t)$ be the unique solution to (1) satisfying (2), valid on $(-\infty, \infty)$. Then $\frac{x(t)}{t^2}$ is bounded as $t \to \infty$.

PROOF: Recall that $x(t) = \lim_{n\to\infty} x_n(t)$ where $x_n(t) = c_1 + c_2 t + \sum_{j=1}^{n} (-1)^j g_j(t)$. Using the fact that $ag^2(t) \leq k < 2$, one can easily verify (as in the proof of Theorem II.1) that

$$|g_j(t)| \leq \frac{a|c_1| + |c_2|\sqrt{2a}}{k} t^2 \left(\frac{k}{2}\right)^j .$$

Thus we have

$$|x_n(t)| \leq |c_1| + |c_2|\,|t| + \frac{a|c_1| + |c_2|\sqrt{2a}}{k} t^2 \sum_{j=1}^{n} \left(\frac{k}{2}\right)^j$$

$$\leq |c_1| + |c_2|\,|t| + \frac{a|c_1| + |c_2|\sqrt{2a}}{k} t^2 \left(\frac{2}{2-k}\right).$$

Thus $\frac{|x_n(t)|}{t^2} \leq \frac{|c_1| + |c_2||t|}{t^2} + \frac{a|c_1| + |c_2|\sqrt{2a}}{k}(\frac{2}{2-k})$.

Thus $\frac{|x(t)|}{t^2} \leq \frac{|c_1| + |c_2||t|}{t^2} + \frac{a|c_1| + |c_2|\sqrt{2a}}{k}(\frac{2}{2-k})$.

and $\frac{|x(t)|}{t^2} \leq \frac{a|c_1| + |c_2|\sqrt{2a}}{k}(\frac{2}{2-k})$.

i.e. $\frac{x(t)}{t^2}$ is bounded as $t \to \infty$. Q.E.D.

We now turn our attention to oscillatory properties of solutions of differential equations with functional arguments. That solutions of

$$\ddot{x}(t) + a(t)x(t) = 0$$

and $\ddot{x}(t) + a(t)x(g(t) = 0$

do not behave the same for large t even if $g(t) \to \infty$ as $t \to \infty$ is shown by the following example due to Waltman [12]. Consider the equation

$$\ddot{x}(t) + \frac{1}{2t^2}x(\frac{t}{4}) = 0.$$

This equation has the nonoscillatory solution $x(t) = t^{\frac{1}{2}}$ although all solutions of

$\ddot{x}(t) + \frac{1}{2t^2}x(t) = 0$ are oscillatory.

The following theorem is also due to Waltman.

Theorem III.2. Let $x(t)$ be a solution of $\ddot{x}(t) + a(t)f(x(t),x(g(t))) = 0$ valid for large t. Suppose

(a) $a(t) \geq 0$,

(b) $\lim_{t\to\infty} g(t) = +\infty$,

(c) $f(x,y)$ is nondecreasing in x and y,

(d) when x and y are of one sign, f has that sign.

Then if $\int^{\infty} a(t)dt = +\infty$, $x(t)$ is oscillatory.

We also list for reference the following theorems due to Bradley [1] associated with the equation

(15) $(r(t)y'(t))' + p(t)f(y(t),y(g(t))) = 0$ where the following assumptions are made on f and g:

(i) $g(t)\to\infty$ as $t\to\infty$,

(ii) if y and v are of one sign, then $f(y,v)$ has that sign,

(iii) $f(y,v)$ is bounded away from zero when y and v are.

Lemma III.1. If $p(t) \geq 0$, $(p(t) \not\equiv 0)$, $r(t) > 0$, $\int^{\infty}\frac{1}{r(t)}dt = \infty$, conditions (i) - (iii) hold and y is a solution of (15) that is positive, (negative), for large t, then $y'(t) \geq 0$ $(y'(t) \leq 0)$, for large t.

Theorem III.3. If $p(t) \geq 0$, $r(t) > 0$, $\int^{\infty}p(t)dt = \infty$, $\int^{\infty}\frac{1}{r(t)}dt = \infty$, and conditions (i) - (iii) hold, then any solution of (15) that exists on a ray (a,∞) has arbitrarily large zeros.

Bradley also proved the following boundedness and nonoscillation theorem.

Theorem III.4. If (i) - (iii) hold, $p(t) \geq 0$ and y is a nonoscillatory solution of (15) on an interval (a,∞), then there are nonnegative constants k_1, k_2 such that $|y(t)| \leq k_1 + k_2\int_{t_0}^{t} \frac{1}{r(s)}ds$. In particular, if $\int^{\infty}\frac{1}{r(s)}ds < \infty$, then all solutions existing on (a,∞) are oscillatory or bounded.

We note that both Theorems III.2 and III.3 imply that all solutions of (12) $(\ddot{x}(t) + a(t)x^{\gamma}(g(t)) = 0)$, valid for large t, are oscillatory if $a(t) \geq 0$, $\lim_{t\to\infty} g(t) = \infty$, $\gamma > 0$ is the ratio of odd integers.

Comment. Suppose $\gamma(> 0) = \frac{2m}{2n+1}$, m and n integers, if $a(t)$ does not change sign for t sufficiently large then solutions of (12), valid for large t, cannot oscillate.

PROOF: Suppose $a(t) \geq 0$ for $t > t_1$. Then from (12) $x(t)$ is concave down for $t > t_1$ and $\dot{x}(t) \leq 0$ or $\dot{x}(t) > 0$ for $t > t_2 \geq t_1$. Thus $x(t)$ cannot oscillate. If $a(t) \leq 0$ for $t > t_1$ a similar argument produces the desired conclusion. Q.E.D.

Recently, a number of papers have appeared on oscillatory properties of equations of the form $x^{(n)} + a(t)f(x,x',x'',\dots,x^{(n-1)}) = 0$. (See [4], [7], [9]). We now generalize some of the results obtained in the above papers to the equations

(16) $\qquad x^{(n)}(t) + a(t)f(x(g(t))) = 0$, and

(17) $\qquad x^{(n)}(t) + f(t,x(g(t))) = 0$

Before stating the theorems, we give the following lemmas, which can be found in [9].

<u>Lemma III.2</u>. Suppose $f(t) \in C^k[a,\infty)$, $f(t) \geq 0$ and $f^{(k)}(t)$ is monotone. Then exactly one of the following is true:

(i) $\lim_{t\to\infty} f^{(k)}(t) = 0$,

(ii) $\lim_{t\to\infty} f^{(k)}(t) > 0$ and $f(t),\dots,f^{(k-1)}(t)$ tend to ∞ as $t\to\infty$.

<u>Lemma III.3</u>. If $y(t) \in C^n[a,\infty)$, $y(t) \geq 0$ and $y^{(n)}(t) \leq 0$ on $[a,\infty)$, then exactly one of the following is true:

(I) $y'(t),\dots,y^{(n-1)}(t)$ tend monotonically to zero as $t\to\infty$.

(II) There is an odd integer k, $1 \leq k \leq n-1$, such that $\lim_{t\to\infty} y^{(n-j)}(t) = 0$ for $1 \leq j \leq k-1$, $\lim_{t\to\infty} y^{(n-k)}(t) \geq 0$,

$\lim_{t\to\infty} y^{(n-k-1)}(t) > 0$ and $y(t), y'(t), \ldots, y^{(n-k-2)}(t)$ tend to ∞ as $t \to \infty$.

Analogous statements can be made if $y(t) \leq 0$ and $y^{(n)}(t) \geq 0$ on $[a,\infty)$.

Lemma III.4. If $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are absolutely continuous and of constant sign on the interval $[t_0,\infty)$, and $y^{(n)}(t)y(t) \leq 0$, then there exists an integer $q, 0 \leq q \leq n-1$, which is even if n is odd and odd if n is even, so that

$$|y(t)| \geq \frac{(t - t_0)^{n-1}}{(n-1)\ldots(n-q)}\left|y^{(n-1)}(2^{n-q-1}t)\right|, \quad t \geq t_0$$

The results of Lemma's III.2 and III.3 will be used throughout the remainder of this paper, while Lemma III.4 is needed to prove Theorem III.9.

Theorem III.5. Suppose that n is even and

(i) $a(t) \geq 0$ for t sufficiently large,

(ii) $\lim_{t\to\infty} g(t) = +\infty$,

(iii) $yf(y) > 0$ $(y \neq 0)$, $f(y)$ continuous on $(-\infty,\infty)$.

Then a necessary condition for equation (16) to have a bounded nonoscillatory solution is $\int^{\infty} t^{n-1}a(t)dt < \infty$.

PROOF: Let $x(t)$ be a bounded nonoscillatory solution of (16). Suppose $x(t) > 0$ for t sufficiently large. Thus, since $\lim_{t\to\infty} g(t) = +\infty$, we have that $x(g(t)) > 0$ for t sufficiently large. Hence, pick T large enough so that $a(t) \geq 0$, $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T$. We have (for $t \geq T$), using Lemma III.3, $x^{(n-1)}(t) \geq 0$, $x^{(n-2)}(t) \leq 0, \ldots, \dot{x}(t) \geq 0$; $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 1,\ldots,n-1$. Thus, $x(t)$ is a nondecreasing function and since $x(t) > 0$ and is bounded we have, $\lim_{t\to\infty} x(t) = L > 0$ and $\lim_{t\to\infty} x(g(t)) = L$.

Also from (16),

$$x^{(n-1)}(s) \geq \int_s^\infty a(u)f(x(g(u)))du. \tag{18}$$

An integration of (18) $(n-2)$ times from t to ∞ yields

$$(-1)^n\dot{x}(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)f(x(g(u)))du \tag{19}$$

and integrating (19) from s to t where $T \leq s \leq t$ we have

$$x(t) - x(s) \geq \int_s^t \frac{(u-s)^{n-1}}{(n-1)!} a(u)f(x(g(u)))du.$$

Now using the continuity of f we may choose $T_1 \geq T$ such that for $t \geq T_1$, $f(x(g(t))) \geq \frac{1}{2}f(L) = M$. Hence for $T \leq T_1 \leq s \leq t$ we have

$$x(t) - x(s) \geq \frac{M}{(n-1)!}\int_s^t (u - s)^{n-1} a(u)du. \tag{20}$$

Now letting $t \to \infty$ in (20) we have

$$\int_s^\infty (u - s)^{n-1} a(u)du < \infty.$$

Then for $t \geq 2s$ we have

$$\int_t^\infty \left(\frac{u}{2}\right)^{n-1} a(u)du < \int_t^\infty (u - s)^{n-1} a(u)du < \infty.$$

i.e. $\int_t^\infty u^{n-1} a(u)du < \infty.$

If $x(t) < 0$ for $t \geq T$ a similar proof yields the desired result. Q.E.D.

In the following theorem we establish necessary conditions for solutions of equation (16), when $n = 2$, to be oscillatory.

<u>Theorem III.6</u>. Under the hypothesis

(i) there exists $t_1 > 0$ such that $g(t) \geq t_1$ for all $t \geq t_1$,

(ii) $g(t)$ is continuous on $[0,\infty)$,

(iii) $f(y)$ is continuous on $(-\infty,\infty)$ with $yf(y) > 0$ for $y \neq 0$,

(iv) $|f(y_1)| \leq |f(y_2)|$ if $|y_1| \leq |y_2|$,

(v) there exists a positive solution to the inequality $f(t) \leq \beta t$, $\beta > 0$,

(vi) $a(t) \geq 0$ and locally integrable on $[0,\infty)$ with $a(t)$ not identically zero on any subinterval of $[0,\infty)$, if

(21) $\int^{\infty} ta(t)dt < \infty$,

there exists a bounded nonoscillatory solution of (16) when $n = 2$.

PROOF: Assuming that $\int^{\infty} ta(t)dt < \infty$, we consider the integral equation

(22) $$x(t) = \frac{M}{2} + t\int_t^{\infty} a(s)f(x(g(s)))ds + \int_{t_1}^{t} sa(s)f(x(g(s)))ds$$

where t_1 is chosen to satisfy (i). We now note that (v) implies the existence of some number $M > 0$ such that

(23) $$\int_{t_1}^{\infty} sa(s)ds \le \frac{M}{2f(M)}.$$

Now we define a sequence $\{x_k(t)\}$ by

$$x_0(t) = \frac{M}{2}$$

(24)

$$x_k(t) = \frac{M}{2} + t\int_t^{\infty} a(s)f(x_{k-1}(g(s)))ds + \int_{t_1}^{t} sa(s)f(x_{k-1}(g(s)))ds.$$

One concludes that $x_k(t)$, $k = 0, 1, 2, \ldots$, is defined and continuous and, in fact, is positive on $[t_1, \infty)$.

$x_1(t) = \frac{M}{2} + t\int_t^{\infty} a(s)f(\frac{M}{2})ds + \int_{t_1}^{t} sa(s)f(\frac{M}{2})ds$ and thus

$$\frac{M}{2} \le x_1(t) \le \frac{M}{2} + f(M)\int_{t_1}^{\infty} sa(s)ds \le M.$$

i.e. $\frac{M}{2} \le x_1(t) \le M$ and $\frac{M}{2} \le x_1(g(t)) \le M$.

Assuming that $\frac{M}{2} \leq x_k(t) \leq M$ (and therefore $\frac{M}{2} \leq x_k(g(t)) \leq M$), we obtain

$$\frac{M}{2} \leq x_{k+1}(t) \leq \frac{M}{2} + \int_{t_1}^{\infty} sa(s)f(M)ds \leq M.$$

We have thus established

$$(25) \quad \begin{aligned} &\frac{M}{2} \leq x_k(t) \leq M \quad \text{and} \\ &\frac{M}{2} \leq x_k(g(t)) \leq M, \ k = 0,1,2,\ldots \end{aligned}$$

It now remains to be shown that

$$(26) \quad x_k(t) \geq x_{k-1}(t) \text{ and } x_k(g(t)) \geq x_{k-1}(g(t)), \ k = 1,2,\ldots$$

From (25), $x_1(t) \geq x_0(t)$ and hence $x_1(g(t)) \geq x_0(g(t))$. Now suppose that $x_k(t) \geq x_{k-1}(t)$ and $x_k(g(t)) \geq x_{k-1}(g(t))$. Then

$$\begin{aligned} x_k(t) &= \frac{M}{2} + t\int_t^{\infty} a(s)f(x_{k-1}(g(s)))ds + \int_{t_1}^{\infty} sa(s)f(x_{k-1}(g(s)))ds \\ &\leq \frac{M}{2} + t\int_t^{\infty} a(s)f(x_k(g(s)))ds + \int_{t_1}^{\infty} sa(s)f(x_k(g(s)))ds \\ &\leq x_{k+1}(t). \end{aligned}$$

Hence (26) is established and we have that the sequence $\{x_k(t)\}$ converges to some function $x(t)$ for $t \geq t_1$ and indeed $\frac{M}{2} \leq x(t) \leq M$; $\frac{M}{2} \leq x(g(t)) \leq M$ for $t \geq t_1$.

We now must establish that $x(t)$ is a solution of the integral equation (22) and thus a solution (nonoscillatory) of (16). For any $\epsilon > 0$, choose T large enough so that $\int_T^\infty sa(s)ds < \frac{\epsilon}{2f(M)}$. Then we have

$$|x_k(t) - \frac{M}{2} - t\int_t^\infty a(s)f(x(g(s)))ds - \int_{t_1}^t sa(s)f(x(g(s)))ds|$$

$$\leq t\int_t^\infty a(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds$$

$$+\int_{t_1}^t sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds$$

$$\leq \int_t^T sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds$$

$$+\int_{t_1}^t sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds$$

$$+\int_T^\infty sa(s)f(x_{k-1}(g(s)))ds + \int_T^\infty sa(s)f(x(g(s)))ds$$

$$\leq \int_{t_1}^T sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds + \epsilon.$$

Letting $k \to \infty$ we obtain

$$|x(t) - \frac{M}{2} - t\int_t^\infty a(s)f(x(g(s)))ds - \int_{t_1}^t sa(s)f(x(g(s)))ds| \leq \epsilon$$

using the continuity of f and the Monotone Convergence Theorem. Thus x(t) is a nonoscillatory solution of (16). Q.E.D.

Restricting our attention now to equation (17), we make the following assumptions:

(27)

(i) $g(t) \geq t - c$ for t sufficiently large, $c > 0$, constant,

(ii) $f(t,y)$ is continuous in $S = [0,\infty)\times(-\infty,\infty)$,

(iii) $a(t)\Phi(y) \leq f(t,y)$ if $y > 0$ and $f(t,y) \leq b(t)\Psi(y)$ if $y < 0$, $(t,x)\in S$, where

(iv) $a(t)$ and $b(t)$ are nonnegative and locally integrable on $[0,\infty)$ and neither $a(t)$ nor $b(t)$ is identically zero on any subinterval of $[0,\infty)$,

(v) $\Phi(y)$ and $\Psi(y)$ are nondecreasing with $y\Phi(y) > 0$ and $y\Psi(y) > 0$ on $(-\infty,\infty)$ for $y \neq 0$,

(vi) there exists positive constants β and δ such that $\Phi(\lambda y) = \lambda^{\beta}\Phi(y)$, $\Psi(\lambda y) = \lambda^{\delta}\Psi(y)$, λ constant,

(vii) for some $\alpha > 0$

$$\int_{\alpha}^{\infty}\frac{du}{\Phi(u)} < \infty \text{ and } \int_{-\alpha}^{-\infty}\frac{du}{\Psi(u)} < \infty.$$

<u>Theorem III.7</u>. Let $x(t)$ be a solution of (17), valid for large t, which is nonoscillatory. If n is odd, assume $\lim_{t\to\infty} x(t) \neq 0$. Suppose conditions (i) - (vi) of (27) are satisfied. Then there exists a positive number k such

that $\frac{\emptyset(x(g(t)))}{\emptyset(x(t))} \geq k$ if x(t) is eventually positive and $\frac{\psi(x(g(t)))}{\psi(x(t))} \geq k$ if x(t) is eventually negative for t sufficiently large.

PROOF: Let x(t) be a nonoscillatory solution of (17). Suppose x(t) is positive for $t \geq T$, and pick T large enough so that $x(t - c) > 0$ for $t \geq T$. From (17) we have

(28)

$$x^{(n)}(t) = -f(t,x(g(t))) \leq -a(t)\emptyset(x(g(t))) \leq 0 \text{ if } t \geq T.$$

Thus from Lemmas III.2 and III.3, x(t) satisfies one of the following:

(1) $\ddot{x}(t) \geq 0$, $\dot{x}(t) \leq 0$ for t sufficiently large, $\lim_{t\to\infty} \dot{x}(t) = 0$, $\lim_{t\to\infty} x(t) = L > 0$.

(2) $\ddot{x}(t) \leq 0$, $\dot{x}(t) \geq 0$ for t sufficiently large.

(3) $\ddot{x}(t) \geq 0$, $\dot{x}(t) \leq 0$ for t sufficiently large, with $x(t), \dot{x}(t), \ldots, x^{(n-k-2)}(t)$ tending to ∞ as $t \to \infty$, $x^{(n-k-1)}(t)$ increasing to $L(0 < L \leq \infty)$, $x^{(n-k)}(t)$ decreasing to $M(M \geq 0)$, and $x^{(n-k+1)}(t), \ldots, x^{(n-1)}(t)$, tending to zero as $t \to \infty$.

Suppose case (1) applies. Since x(t) is decreasing to $L > 0$, and $g(t) \geq t - c$ we have that $\lim_{t\to\infty} x(g(t)) = L$ and thus $\lim_{t\to\infty} \frac{x(g(t))}{x(t)} = \frac{L}{L} = 1$. Therefore for t large enough, say $t \geq T$, $\frac{x(g(t))}{x(t)} \geq \frac{1}{2}$.

In either case (2) or case (3) we have, since $\dot{x}(t) \geq 0$,

$$x(g(t)) \geq x(t - c) \text{ and } \frac{x(g(t))}{x(t)} \geq \frac{x(t - c)}{x(t)} .$$

If case (2) applies, let f be the function whose graph is the line tangent to the graph of x at $(t - c, x(t - c))$ for some fixed $t \geq T$. That is

$$f(s) = \dot{x}(t - c)(s - t + c) + x(t - c).$$

Since x is concave down,

$$\frac{x(g(t))}{x(t)} \geq \frac{x(t - c)}{x(t)} > \frac{x(t - c)}{f(t)} = \frac{f(t - c)}{f(t)} .$$

Let $z = \dfrac{-x(t - c)}{\dot{x}(t - c)} + t - c$. Then $f(z) = 0$ and because of similar triangles we have

$$(29) \quad \frac{f(t - c)}{f(t)} = \frac{z - t + c}{z - t} = \frac{x(t - c)}{x(t - c) + c\dot{x}(t - c)} .$$

But $\dot{x}(t - c)$ is decreasing so the last member of (29) increases to a positive limit k_1 as t tends to ∞. Hence $\dfrac{x(g(t))}{x(t)} \geq k_1$ for $t \geq T$.

Now suppose case (3) applies. Consider

$$\lim_{t \to \infty} \frac{x(t - c)}{x(t)} \text{ which is of the form } \frac{\infty}{\infty}.$$

Using L'Hopital's rule a sufficient number of times we obtain

$$\lim_{t \to \infty} \frac{x(t - c)}{x(t)} = \ldots = \lim_{t \to \infty} \frac{x^{(n-k-1)}(t - c)}{x^{(n-k-1)}(t)} .$$

If L (in case 3) is finite we are done since then $\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \frac{L}{L} = 1$. When $L = \infty$, then again using L'Hopital's rule we have

$$\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \ldots = \lim_{t\to\infty} \frac{x^{(n-k)}(t-c)}{x^{(n-k)}(t)}.$$

If M (in case 3) is positive again we are done since $\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \frac{M}{M} = 1$. However, if M is zero we then claim that $\lim_{t\to\infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} = 1$ since

$$\lim_{t\to\infty} [x^{(n-k-1)}(t) - x^{(n-k-1)}(t-c)]$$

$$= \lim_{t\to\infty} x^{(n-k)}(\xi)c = 0, \quad t - c < \xi < t.$$

Thus

$$\left|\frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} - 1\right| = \left|\frac{x^{(n-k-1)}(t-c) - x^{(n-k-1)}(t)}{x^{(n-k-1)}(t)}\right|$$

$$< \frac{\epsilon\, x^{(n-k-1)}(t_1)}{x^{(n-k-1)}(t_1)} < \epsilon$$ where $t_1 \geq T$, is such that $x^{(n-k-1)}(t_1) > 0$. Summarizing we have $\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = 1$.

Thus for t large enough, $\frac{x(g(t))}{x(t)} \geq \frac{x(t-c)}{x(t)} > \frac{1}{2}$.

Now letting $k_2 = \min\left\{\frac{1}{2}, k_1\right\}$, we have $\frac{x(g(t))}{x(t)} \geq k_2$ for $t \geq T_1 \geq T$ and

$$\frac{\phi(x(g(t)))}{\phi(x(t))} \geq \frac{\phi(k_2 x(t))}{\phi(x(t))} = k_2^\beta \frac{\phi(x(t))}{\phi(x(t))} = k_2^\beta = k.$$

Now suppose $x(t)$ is a nonoscillatory solution of (17) which is negative for $t \geq T$. Again, pick T large enough so that $x(t - c) < 0$ for $t \geq T$. Then (28) becomes

(30)

$$x^{(n)}(t) = -f(t,x(g(t))) \geq -b(t)\psi(x(g(t))) \geq 0 \text{ if } t \geq T,$$

and we find that $x(t)$ must satisfy one of the following:

(1) $\ddot{x}(t) \leq 0$, $\dot{x}(t) \geq 0$ for t sufficiently large, $\lim_{t\to\infty} \dot{x}(t) = 0$, $\lim_{t\to\infty} x(t) = L < 0$,

(2) $\ddot{x}(t) \geq 0$, $\dot{x}(t) \leq 0$ for t sufficiently large,

(3) $\ddot{x}(t) \leq 0$, $\dot{x}(t) \leq 0$ for t sufficiently large, with $x(t)$, $\dot{x}(t),\ldots,x^{(n-k-2)}(t)$ tending to $-\infty$ as $t\to\infty$, $x^{(n-k-1)}(t)$ decreasing to $L(-\infty \leq L < 0)$, $x^{(n-k)}(t)$ increasing to M $(M \leq 0)$, and $x^{(n-k+1)}(t),\ldots,x^{(n-1)}(t)$ tending to zero as $t\to\infty$.

If case (1) applies, we have that $\lim_{t\to\infty} x(g(t)) = L$ since $g(t) \geq t - c$ and $x(t)$ is decreasing to $L < 0$. Thus

$$\lim_{t\to\infty} \frac{x(g(t))}{x(t)} = \frac{L}{L} = 1.$$

In either case (2) or (3), $g(t) \geq t - c$ implies $x(g(t)) \leq x(t - c)$ and $|x(g(t))| \geq |x(t - c)|$ with

$$\frac{x(g(t))}{x(t)} = \left|\frac{x(g(t))}{x(t)}\right| \geq \left|\frac{x(t - c)}{x(t))}\right| = \frac{x(t - c)}{x(t)} .$$

If we now use arguments similar to those used when $x(t) > 0$, we obtain the desired conclusion. Q.E.D.

We are now ready to state our main results.

Theorem III.8. If $g(t)$ satisfies (i) and $f(t,y)$ satisfies (ii) - (vii) of (27) and in addition

$$(31) \qquad \int_0^\infty t^{n-1} a(t)dt = \int_0^\infty t^{n-1} b(t)dt = \infty,$$

then if n is even, each solution of (17), valid for large t, is oscillatory, while if n is odd, each solution of (17) valid for large t is either oscillatory or it tends monotonically to zero together with its first n-1 derivatives.

PROOF: Suppose $x(t)$ is a nonoscillatory solution of (17), valid for large t. Assume $x(t)$ is eventually positive. Thus since $\lim_{t\to\infty} g(t) = \infty$, $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T$. From (17)

$$(32) \qquad x^{(n)}(t) = -f(t, x(g(t))) \leq -a(t)\phi(x(g(t))) \leq 0.$$

Thus by Lemma III.2 $x^{(n-1)}(t)$ decreases to a nonnegative limit, so from (32) we obtain

(33) $$x^{(n-1)}(s) \geq \int_s^\infty a(u)\phi(x(g(u)))du.$$

Suppose case I of Lemma III.3 holds. Then an integration of (33) n-2 times from t to ∞ yields

(34) $$(-1)^{(n-2)}\dot{x}(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(u-2)!} a(u)\phi(x(g(u)))du.$$

If n is even, integrating (34) from T to $t \geq T$,

$$x(t) \geq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(x(g(u)))du.$$

Since ϕ is nondecreasing

(35) $$\phi(x(t))/\phi\left[\int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(x(g(u)))du\right] \geq 1.$$

If we now multiply (35) by $\frac{(t-T)^{n-1}}{(n-1)!} a(t)\frac{\phi(x(g(t)))}{\phi(x(t))}$ and integrate from r to s we get, after a change of variable on the left

(36)
$$\int_R^S \frac{du}{\phi(u)} \geq \int_r^s \frac{(t-T)^{n-1}}{(n-1)!} a(t)\frac{\phi(x(g(t)))}{\phi(x(t))}dt \geq k\int_r^s \frac{(t-T)^{n-1}}{(n-1)!} a(t)dt$$

where

$$R = \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(x(g(u)))du$$

and

$$S = \int_T^s \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(x(g(u)))du.$$

Now if by an appropriate choice of r, we can make $R \geq \alpha$, then the left hand side of (36) is bounded above for all $s > r$, hence $\int_0^\infty t^{n-1}a(t)dt < \infty$. If this is not possible then for all $r \geq T$

$$\alpha > \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du \geq \Phi(x(g(T)))\int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u)du$$

and the result again follows.

If n is odd, then (34) becomes

$$(37) \qquad -\dot{x}(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du \geq 0.$$

So $x(t)$ decreases to a limit $L \geq 0$.

Suppose $L > 0$. Then integrating (37) from T to ∞,

$$x(T) > x(T) - L \geq \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du$$

$$\geq \Phi(L)\int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!} a(u)du,$$

using the monotonicity of Φ. But this implies

$\int_0^\infty t^{n-1}a(t)dt < \infty$.

Now suppose that case II of Lemma III.3 holds. Integrating (32) a sufficient number of times we have

$$(38) \qquad x^{(n-k)}(t) \geq \int_t^\infty \frac{(u-t)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))du.$$

Since $x^{(j)}(t)$ increases to ∞, $j < n-k-1$, there exists $t_1 \geq T$ such that $x^{(j)}(t) > 0$ for $t \geq t_1$, $j = 0,\ldots,n-k-1$. Integrating (38) from t_1 to $t > t_1$,

$$x^{(n-j-1)}(t) \geq \int_{t_1}^{t} \int_{s}^{\infty} \frac{(u-s)^{k-1}}{(k-1)!} a(u)\tilde{\phi}(x(g(u)))du$$

$$\geq \int_{t}^{\infty} \frac{(u-t_1)^k - (u-t)^k}{k!} a(u)\tilde{\phi}(x(g(u)))du.$$

So

$$(39) \qquad x^{(n-j-1)}(t) > \int_{t}^{\infty} \frac{(t-t_1)^k}{k!} a(u)\tilde{\phi}(x(g(u)))du.$$

Integrating (39) successively $n-j-2$ times from t_1 to t we obtain

$$(40) \qquad \dot{x}(t) > \int_{t}^{\infty} \frac{(t-t_1)^{n-2}}{(n-2)!} a(u)\tilde{\phi}(x(g(u)))du$$

and integrating (40) from t_1 to t gives

$$x(t) > \int_{t_1}^{t} \frac{(u-t_1)^{n-1}}{(n-1)!} a(u)\tilde{\phi}(x(g(u)))du.$$

Now the proof proceeds as in case I.

If $x(t)$ is a solution of (17), valid for large t, such that $x(t) < 0$ for $t \geq T$, the proof is the same except $a(t)$ and $\tilde{\phi}(u)$ are replaced respectively by $b(t)$ and $\psi(u)$, and the sense of appropriate inequalities are changed. Q.E.D

In the next theorem, condition (vi) of (27) is changed so that equation (17) includes the special case

$$x^{(n)} + a(t)\, x^{\alpha}(g(t)) = 0, \quad 0 \leq \alpha < 1,$$

α the ratio of odd integers.

<u>Theorem III.9</u>. Let $g(t)$ satisfy (i) and $f(t,y)$ satisfy (ii) - (v) of (27). In addition suppose $f(t,y)$ satisfies

(vii) there exists positive constants λ_0, M ,N and constants β,γ, where $0 \leq \beta < 1$, $0 \leq \gamma < 1$, such that

$$\bar{\phi}(\lambda y) \geq M\lambda^{\beta}\bar{\phi}(y), \quad y > 0,$$

$$\psi(\lambda y) \leq N\lambda^{\gamma}\psi(y), \quad y < 0, \quad \lambda \geq \lambda_0 > 0.$$

Then if

(41) $$\int^{\infty} t^{(n-1)\beta} a(t)dt = \int^{\infty} t^{(n-1)\gamma} b(t)dt = +\infty,$$ each solution of (17), valid for large t, is oscillatory when n is even and is either oscillatory or tends to zero with its first n-1 derivative if n is odd.

PROOF: We first consider the case when n is even. Let $x(t)$ be a nonoscillatory solution of (17) such that $x(t)$ and $x(t - c)$ are positive for $t \geq T$. By Lemma III.3, $\dot{x}(t) \geq 0$, so $x(t)$ is nondecreasing. Also $x^{(n)}(t) \leq 0$ so $x^{(n-1)}(t)$ is nonincreasing and positive on $[T,\infty)$. Thus by Lemma III.4 we have

(42) $$x(t) \geq x(2^{1-n}t) \geq At^{n-1}x^{(n-1)}(t)$$

for $t \geq 2^nT = t_1$, where $A = \frac{2^{-n^2}}{(n-1)!}$.

Because of condition (iii), $x(t)$ satisfies

(43) $$x^{(n)}(t) + a(t)\phi(x(g(t))) \leq 0.$$

Since $x(t)$ is nondecreasing, $kx(t - c) \geq \lambda_0$ for

$k \geq \frac{\lambda_0}{x(t_1 - c)}$, $t \geq t_1$ and

$\phi(x(g(t))) \geq \phi(x(t - c)) \geq k^\beta x^\beta(t - c)\phi(\frac{1}{k})M$ by the monotonicity of ϕ and x, and (viii). Now letting

$B = k^\beta\phi(\frac{1}{k})M > 0$, we have

$$x^{(n)}(t) + Ba(t)x^\beta(t - c) \leq 0 \quad \text{for } t \geq t_1$$

and from (42)

$$x^{(n)}(t) + A^\beta Ba(t)(t - c)^{(n-1)\beta}[x^{(n-1)}(t - c)]^\beta \leq 0, \ t \geq t_1+c.$$

Dividing by $[x^{(n-1)}(t)]^\beta$ and integrating from $t_1 + c$ to t we obtain

$$\int_{x^{(n-1)}(t_1+c)}^{x^{(n-1)}(t)} \frac{dx}{x^\beta} + A^\beta B\int_{t_1+c}^{t} (s-c)^{(n-1)\beta}a(s)\left[\frac{x^{(n-1)}(s-c)}{x^{(n-1)}(s)}\right]^\beta ds \leq 0.$$

But since $x^{(n-1)}(t)$ is nonincreasing, $\frac{x^{(n-1)}(s - c)}{x^{(n-1)}(s)} \geq 1$.

Thus we have

(44)

$$\int_{x^{(n-1)}(t_1+c)}^{x^{(n-1)}(t)} \frac{dx}{x^{\beta}} + A^{\beta}B\int_{t_1+c}^{t} (s - c)^{(n-1)\beta} a(s)ds \leq 0.$$

However $s - c \geq \frac{t_1 s}{(t_1 + c)}$ for $s \geq t_1 + c$, and we obtain

(45)

$$\int_{x^{(n-1)}(t_1+c)}^{x^{(n-1)}(t)} \frac{dx}{x^{\beta}} + A^{\beta}B\left(\frac{t_1}{t_1 + c}\right)^{(n-1)\beta}\int_{t_1+c}^{t} s^{(n-1)\beta} a(s)ds \leq 0.$$

But since $0 > \int_{x^{(n-1)}(t_1+c)}^{x^{(n-1)}(t)} \frac{dx}{x^{\beta}} \geq \int_{b}^{0} \frac{dx}{x^{\beta}}$, $0 < b < \infty$ and the latter integral is finite for $\beta < 1$, we have a contradiction of (45) as $t \to \infty$ if $\int^{\infty} s^{(n-1)\beta} a(s)ds = +\infty$. Thus $x(t)$ is oscillatory.

The case where $x(t)$ and $x(t - c)$ are negative for $t \geq T$ can be handled in the same way and yields a contradiction to the fact that $\int^{\infty} t^{(n-1)\gamma} b(t)dt = +\infty$. The inequalities (42) and (43) are reversed with $a(t)\phi(x(g(t)))$ replaced by $b(t)\psi(x(g(t)))$, and inequalities (44) and (45) being in the same direction with x being replaced by $-x$.

Now suppose n is odd and $x(t)$ does not approach zero. Then $|x^{(n-1)}(t)|$ is still nonincreasing and

$$|x(t)| = \left|\frac{x(t)}{x(2^{1-n}t)}\right| \left|x(2^{1-n}t)\right|$$

$$\geq \inf_{t \geq T}\left|\frac{x(t)}{x(2^{1-n}t)}\right| A \left|x^{(n-1)}(t)\right| t^{n-1}, \quad t \geq t_1.$$

So that $|x(t)| \geq B_1 t^{n-1} |x^{(n-1)}(t)|$ for constant B_1 and the preceeding proof again yields a contradiction to the existence of a nonoscillatory solution of (17). Q.E.D.

REFERENCES

[1] J. S. Bradley, Oscillation Theorems for a Second-Order Delay Equation, to appear.

[2] L. E. El'sgol'ts, Introduction to the Theory of Differential Equations with Deviating Arguments, Holden-Day, Inc., San Francisco, London, Amsterdam, 1966.

[3] E. L. Ince, Ordinary Differential Equations, Dover Publications, New York, 1944. (Reprint).

[4] J. W. Macki and J. S. W. Wong, Oscillation of Solutions to Second-Order Non-linear Differential Equations, Pacific J. Math. 24 (1968), pp. 111-117.

[5] S. B. Norkin, Differential Equations of the Second-Order with Deviating Argument, to appear.

[6] R. J. Oberg, On the Local Existence of Solutions of Certain Functional Differential Equations, Proc. Amer. Math. Soc. Vol. 20, No. 2, Feb. 1969, pp.295.

[7] Hiroshi Onose, Oscillatory Property of Ordinary Differential Equations of Arbitrary Order, J. Diff. Eqs. Vol. 7, 1970, pp. 454-458.

[8] Gerald H. Ryder, Solutions of a Functional Differential Equation, Amer. Math. Monthly Vol. 76, No. 9, Nov. 1969, pp. 1031-1033.

[9] Gerald H. Ryder and David V. V. Wend, Oscillation of Solutions of Certain Ordinary Differential Equations of nth Order, Proc. Amer. Math. Soc. Vol. 25, No. 3, July 1970, pp. 463-369.

[10] R. A. Struble, Nonlinear Differential Equations, Mc-Graw - Hill Book Company, New York, Toronto, London, 1962.

[11] W. R. Utz, The Equation $f'(x) = af(g(x))$, Bull. Amer. Math. Soc., 71 (1963) pp. 138.

[12] Paul Waltman, A Note on an Oscillation Criterion for an Equation with a Functional Argument, to appear.