Stability, causality, and shock waves in the Israel-Stewart theory of relativistic dissipative fluids
by Timothy Scott Olson

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

Abstract:
The stability, causality, and hyperbolicity properties were analyzed for the Israel-Stewart theory of relativistic dissipative fluids formulated in the energy frame. The equilibria of the theory which are stable for small perturbations were found by constructing a Liapunov functional. The conditions which guarantee that small perturbations about equilibrium will propagate with velocities less than the speed of light and will obey a system of hyperbolic differential equations were determined by calculating the characteristic velocities. It was shown that the stability conditions are equivalent to the causality and hyperbolicity conditions. The behavior of the theory far from equilibrium was studied by considering the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas. The theory was shown to be hyperbolic for large deviations from equilibrium, and acausality implies instability in this example.

The plane steady shock wave solutions were also studied for the Israel-Stewart theory formulated in the Eckart frame. The theory was shown to fail to adequately describe the structure of strong shock waves. Physically acceptable solutions do not exist above a maximum upstream Mach number in any thermally nonconducting and viscous fluid described by the theory because the solutions become multiple-valued when the characteristic velocity is exceeded. It was also proven that physically acceptable solutions do not exist for thermally conducting and viscous fluids above either a maximum upstream Mach number, or else below a minimum downstream Mach number (or both). These limiting Mach numbers again correspond to the characteristic velocities of the fluid. Only extremely weak plane steady shock solutions can be single-valued in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas or for the degenerate free Fermi gas.

STABILITY, CAUSALITY, AND SHOCK WAVES IN THE ISRAEL-STEWART THEORY OF RELATIVISTIC DISSIPATIVE FLUIDS

by

Timothy Scott Olson

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Physics

MONTANA STATE UNIVERSITY
Bozeman, Montana

April 1990

APPROVAL

of a thesis submitted by

Timothy Scott Olson

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

April 27, 1990 ______________________
Date

____
Chairperson, Graduate Committee

Approved for the Major Department

April 27, 1990 ______________________
Date

____
Head, Major Department

Approved for the College of Graduate Studies

May 3, 1990 ______________________
Date

Henry L Parsons ______________________
Graduate Dean

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a doctoral degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library. I further agree that copying of this thesis is allowable only for scholarly purposes, consistent with "fair use" as prescribed in the U.S. Copyright Law. Requests for extensive copying or reproduction of this thesis should be referred to University Microfilms International, 300 North Zeeb Road, Ann Arbor, Michigan 48106, to whom I have granted "the exclusive right to reproduce and distribute copies of the dissertation in and from microfilm and the right to reproduce and distribute by abstract in any format."

Signature Timothy S. Olson

Date April 27, 1990

ACKNOWLEDGMENTS

I wish to thank my advisor, Professor William A. Hiscock, for providing me with financial support from his National Science Foundation Grant, and for collaborating with me on several problems in relativistic fluid theory.

I also wish to express my gratitude to my wife Karen, and my daughter Erica, for their support during my years in graduate school.

TABLE OF CONTENTS

Page

1. INTRODUCTION 1

2. ISRAEL-STEWART FLUID THEORY 6

   History of the Israel-Stewart Theory 6

   The Israel-Stewart Energy Frame Theory 11

   The Israel-Stewart Eckart Frame Theory 19

3. THE ENERGY FRAME THEORY IN THE LINEAR REGIME 24

   Linearized Energy Frame Theory 24

   The Stability Conditions 26

   Causality and Hyperbolicity Conditions 35

   Stability is Equivalent to Causality and Hyperbolicity 41

   Conclusions 47

4. THE ENERGY FRAME THEORY IN THE NONLINEAR REGIME 49

   Introduction 49

   Acausality Implies Instability for the Inviscid Ultrarelativistic Boltzmann Gas with Planar Symmetry 52

   Conclusions 60

TABLE OF CONTENTS--Continued

Page

5. PLANE STEADY SHOCK WAVES IN ISRAEL-STEWART FLUIDS ... 63

Introduction 63

Plane Steady Shock Solutions 64

A Maximum Upstream Mach Number in Thermally Nonconducting, Viscous Fluids 69

Maximum Mach Numbers in Thermally Conducting, Viscous Fluids 76

Plane Steady Shocks in Simple Gases 84

Conclusions 88

6. SUMMARY 92

Dissertation Summary 92

The Theory of Relativistic Dissipative Fluids at Present 96

REFERENCES CITED 100

ABSTRACT

The stability, causality, and hyperbolicity properties were analyzed for the Israel-Stewart theory of relativistic dissipative fluids formulated in the energy frame. The equilibria of the theory which are stable for small perturbations were found by constructing a Liapunov functional. The conditions which guarantee that small perturbations about equilibrium will propagate with velocities less than the speed of light and will obey a system of hyperbolic differential equations were determined by calculating the characteristic velocities. It was shown that the stability conditions are equivalent to the causality and hyperbolicity conditions. The behavior of the theory far from equilibrium was studied by considering the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas. The theory was shown to be hyperbolic for large deviations from equilibrium, and acausality implies instability in this example.

The plane steady shock wave solutions were also studied for the Israel-Stewart theory formulated in the Eckart frame. The theory was shown to fail to adequately describe the structure of strong shock waves. Physically acceptable solutions do not exist above a maximum upstream Mach number in any thermally nonconducting and viscous fluid described by the theory because the solutions become multiple-valued when the characteristic velocity is exceeded. It was also proven that physically acceptable solutions do not exist for thermally conducting and viscous fluids above either a maximum upstream Mach number, or else below a minimum downstream Mach number (or both). These limiting Mach numbers again correspond to the characteristic velocities of the fluid. Only extremely weak plane steady shock solutions can be single-valued in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas or for the degenerate free Fermi gas.

CHAPTER ONE

INTRODUCTION

To make progress in the theoretical understanding of many physical systems, particularly in nuclear physics and relativistic astrophysics, a phenomenological theory of relativistic fluids that includes the effects of dissipation (thermal conductivity, bulk and shear viscosity) is often needed. The collision of two heavy nuclei moving at relativistic velocities (Clare and Strottman 1986), and the collapse of a stellar core which results in a supernova explosion and the formation of a neutron star are two physical systems of this type. These systems typically consist of very many particles, several hundred nucleons in the case of heavy nuclei collisions, of order a solar mass of nucleons for neutron star formation, and consequently it is very hard to follow the evolution of each individual particle of the system even numerically. Further, the matter in these problems can be in a highly excited state, the regime where the details of the fundamental interactions between the particles of the system are often poorly understood.

Because of these difficulties, use of the fluid approximation can be the best approach. Instead of a distribution of discrete particles, the matter is modeled as a continuous fluid. The hope in this approach is that our inability to solve complicated many-body problems and our ignorance of the details of the interactions between the constituents of the system can be overcome by appealing to basic principles, such as conservation of energy and momentum, the generality of thermodynamics, and concepts of relativity theory.

Unfortunately, the classic theories of relativistic dissipative fluids proposed by Eckart (Eckart 1940; Misner, Thorne, and Wheeler 1973; Weinberg 1972, pp. 53-57) and by Landau and Lifshitz (1959, Sec. 127) are pathological. All the equilibrium states in these theories are unstable, regardless of the choice of equation of state of the fluid (Hiscock and Lindblom 1985). Further, linear perturbations around the equilibrium states can propagate acausally (faster than the speed of light) in these theories, and obey differential equations of a mixed hyperbolic-parabolic-elliptic type (Hiscock and Lindblom 1987). For example, water at room temperature and atmospheric pressure is predicted by the Eckart theory to be unstable and acausal: there exist plane wave perturbations which grow exponentially with a timescale of order 10^{-35} seconds, and there are also plane wave perturbations which propagate at up to 10^{6} times the speed of light. Clearly, these theories are not physically acceptable.

An alternative theory that is free of these pathologies was developed in the 1970's by Werner Israel and John Stewart (Israel 1976; Stewart 1977; Israel and Stewart 1979a, 1979b) building on earlier nonrelativistic work by Grad (1949) and Müller (1967, 1985). The Israel-Stewart theory has been shown to possess stable equilibrium states, perturbations around equilibrium propagate with velocities less than the speed of light, and obey a hyperbolic system of differential equations so that a well-posed initial value problem exists (Hiscock and Lindblom 1983, 1988b, 1989) if certain conditions on the equation of state and parameters in the theory are satisfied. Further, the conditions which guarantee that an equilibrium state of a fluid will be stable are equivalent to the conditions for perturbations to propagate causally and obey hyperbolic equations. Thus, at least near equilibrium, the Israel-Stewart theory is an acceptable theory of relativistic dissipative fluids.

The theory of Israel and Stewart is actually a collection of theories, each of which define the fluid four-velocity in terms of the fundamental tensors (the stress-energy tensor and particle number current vector) in a different way. One particular choice is to define the four-velocity to be parallel to the particle number current vector (hereafter referred to as the Israel-Stewart Eckart frame theory), as is done in the Eckart theory. It was for this particular choice of four-velocity definition that the conditions that guarantee stability were first shown to be equivalent to the causality and hyperbolicity conditions near equilibrium (Hiscock and Lindblom 1983). Another natural choice is to define the four-velocity to be the unique timelike eigenvector of the stress-energy tensor (referred to as the Israel-Stewart energy frame theory), as is done in the Landau-Lifshitz theory.

One of the purposes of this dissertation is to study the stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory, both near equilibrium and far from equilibrium, in order to better understand how the predictions of the theory depend on the choice of four-velocity definition.

Chapter 2 is a review of the Israel-Stewart fluid theory. The stability, causality, and hyperbolicity properties of the energy frame theory near equilibrium are examined in Ch. 3 by linearizing the equations of motion of the theory about a fiducial background equilibrium state. The Liapunov functional techniques developed by Hiscock and Lindblom (1983) for the study of the linearized Israel-Stewart Eckart frame theory are employed to study the stability of the equilibria of the linearized energy frame theory. The causality and hyperbolicity properties of the linearized energy frame theory are studied by calculating the characteristic velocities. The stability, causality, and hyperbolicity properties of the energy frame theory far from equilibrium are investigated in

Ch. 4 by examining the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas.

Another purpose of this dissertation is to determine whether the Israel-Stewart theory is incapable of describing strong shock waves in all fluids.

The plane steady shock solutions of the Israel-Stewart Eckart frame theory for the ultrarelativistic Boltzmann gas have been studied by Majorana and Motta (1985). They concluded that physically acceptable plane steady shock solutions are guaranteed not to exist in the Israel-Stewart theory description of the ultrarelativistic Boltzmann gas above a certain upstream Mach number. However, it is unclear from their work how this breakdown depends on the choice of four-velocity definition or on the particular fluid.

The plane steady shock solutions of the Israel-Stewart theory are studied in Ch. 5. That analysis applies to any fluid described by the Israel-Stewart theory.

Chapter 6 is a summary of this dissertation. I also give my opinion of the present state of the theory of relativistic dissipative fluids. Units are chosen so that the speed of light c = 1 throughout.

CHAPTER TWO

ISRAEL-STEWART FLUID THEORY

This chapter is a review of the Israel-Stewart theory of relativistic dissipative fluids. I begin with a brief history of the development of the theory.

History of the Israel-Stewart Theory

The Israel-Stewart theory arose out of developments from the kinetic theory of simple gases (Boltzmann gas, free Fermi and Bose gases). Grad (1949) found a method for the solution of the nonrelativistic Boltzmann equation different from the usual Chapman-Enskog approach. In the Grad method the molecular distribution function, which is the argument of the Boltzmann equation, is approximated by truncating its Hermite expansion after the first few terms. These terms are identified as the shear stress and heat flux, and a state of the fluid is defined by assigning values to two thermodynamic variables, the fluid velocity, the heat flux, and the shear stress. The Boltzmann equation then yields a closed system of hyperbolic differential equations for 13 independent fluid variables (there are 14 equations for 14 variables if the fluid has nonzero bulk viscosity). Grad argued that his method is possibly preferable to the Chapman-Enskog approach, particularly for fluid flows which are not quasi-stationary. Linear perturbations in Grad's theory propagate with speeds of order the adiabatic sound speed.

Grad's approach was extended to relativistic simple gases by Stewart (1969, 1971, 1977), Anderson (1970), Marle (1969), Kranys (1970), and Israel and Stewart (1979a, 1979b). The Grad method predicts that linear perturbations propagate with three distinct characteristic velocities in the arbitrarily relativistic Boltzmann gas, each of which is less than the speed of light, and that these perturbations propagate via hyperbolic differential equations. Olson and Hiscock (1989) proved that the Grad method also predicts the degenerate and arbitrarily relativistic free Fermi gas to be stable for small perturbations, and these perturbations propagate causally and obey a hyperbolic system of differential equations.

By contrast, perturbations can propagate instantaneously in the Navier-Fourier-Stokes theory of nonrelativistic fluids because of the parabolic character of Fourier's law for heat flow (Landau and Lifshitz 1959, Sec. 51). This is disturbing even in a classical theory, since disturbances would be expected to propagate with some mean molecular speed. Müller (1967) found that the parabolic character of the Navier-Fourier-Stokes theory results from the neglect of terms quadratic in the heat flux, bulk stress, and shear stress in the expression for the entropy. By including these quadratic terms, Müller formulated a phenomenological theory of nonrelativistic fluids which is consistent with the Grad nonrelativistic kinetic theory method. Israel (1976) extended Müller's theory to relativistic fluids. Israel and Stewart (Stewart 1977; Israel and Stewart 1979a, 1979b) showed that Israel's relativistic extension of Müller's phenomenological theory is consistent with the Grad relativistic kinetic theory method.

Hiscock and Lindblom (1983) showed that the Israel-Stewart Eckart frame theory (the Israel-Stewart theory that has the fluid four-velocity parallel to the particle number current) is causal and hyperbolic, not only in the kinetic theory of the simple gases where the equations of the theory reduce to the equations of the Grad method, but also for a far larger class of fluids. They proved that small perturbations about equilibrium will propagate causally and obey a hyperbolic system of differential equations if and only if the equilibrium state satisfies a certain set of inequalities. Further, an equilibrium state is guaranteed to be stable if this same set of inequalities is satisfied. These stability conditions are also useful for developing constraints on proposed models for the equation of state of nuclear matter (Olson and Hiscock 1989). In Ch. 3 (and also in Olson (1990)), I consider the stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory (the Israel-Stewart theory for which the fluid four-velocity is an eigenvector of the stress-energy tensor) near equilibrium.

In the Israel-Stewart theory, the complete symmetric stress-energy tensor T^{ab} and the particle number current N^a are taken to be independent fields. Thus 14 independent fluid variables are necessary to describe the state of a single-component fluid away from equilibrium in the Israel-Stewart theory, rather than the usual 5 independent variables as in the Navier-Fourier-Stokes theory and the pathological Eckart and Landau-Lifshitz theories, and also for perfect fluids.

The Israel-Stewart theory reduces to the familiar dynamics of classical fluids for linear plane wave fluid perturbations which have wavelengths long compared to a typical mean-free-path (Hiscock and Lindblom 1987). In this limit, 9 of the 14 modes of an Israel-Stewart fluid are strongly damped, and the remaining 5 have propagation speeds which are relativistic generalizations of the Navier-Fourier-Stokes results. In other situations the dynamics of all 14 degrees of freedom can be important; the Israel-Stewart theory has been used to construct a one-fluid model of nonrelativistic superfluids that is mathematically equivalent to the Landau two-fluid model (Lindblom and Hiscock 1988).

There is some evidence that far from equilibrium the Israel-Stewart theory is unstable, acausal, and nonhyperbolic. Hiscock and Lindblom (1988a) considered an inviscid ultrarelativistic Boltzmann gas with planar symmetry in the

Israel-Stewart Eckart frame theory, and found that the theory predicts this gas to be unstable for very large deviations from equilibrium, in the sense that the fluid evolves away from equilibrium into the future, and these deviations propagate acausally by nonhyperbolic equations. However, this collapse of the theory only occurs very far from equilibrium, when the magnitude of the heat flux is of order the energy density of the fluid in the example considered, far beyond the regime of interest for realistic physical applications. In Ch. 4 (see also Hiscock and Olson (1989)), I examine the behavior of the inviscid ultrarelativistic Boltzmann gas in the Israel-Stewart energy frame theory far from equilibrium.

It has long been known that the Israel-Stewart theory is incapable of describing strong shock waves in the Boltzmann gas. Grad (1952) showed that for the nonrelativistic Boltzmann gas with zero bulk viscosity, the plane steady shock solutions of his kinetic theory method are multiple-valued, and hence physically unacceptable, above an upstream Mach number of approximately 1.65. Majorana and Motta (1985) found that the plane steady shock solutions of the Grad relativistic kinetic theory method are similarly physically unacceptable above a certain upstream Mach number for the ultrarelativistic Boltzmann gas with zero bulk viscosity. I consider in Ch. 5 (and also in Olson and Hiscock (1990b)) whether this breakdown is peculiar to the Boltzmann gas, or if the Israel-

Stewart theory is incapable of describing strong shock waves in any fluid.

The Israel-Stewart Energy Frame Theory

The fundamental fields that describe a relativistic dissipative fluid in the Israel-Stewart theory are the particle number current N^a and the stress-energy tensor T^{ab}. The fluid four-velocity u^a is not considered to be a fundamental field; different choices of how the four-velocity is defined in terms of the particle number current and the stress-energy tensor lead to different theories. In the energy frame theory the fluid four-velocity is chosen to be the unique timelike eigenvector of the stress-energy tensor:

$$T^{ab}u_b = -\rho u^a \quad , \tag{2.1}$$

where ρ is the energy density of the fluid, and the four-velocity is a timelike unit vector ($u^a u_a = -1$). Thus the energy flux is always zero in the rest frame of the fluid. This section is a review of the Israel-Stewart energy frame theory based on the approach used by Olson (1990). The Israel-Stewart Eckart frame theory is reviewed in the next section.

The equations of motion of the fluid include the conservation laws

$$\nabla_a N^a = 0 \quad , \tag{2.2}$$

$$\nabla_a T^{ab} = 0 \quad , \tag{2.3}$$

where ∇_a is the covariant derivative compatible with the metric tensor g^{ab}. I will only consider single-component fluids. The generalization to multiple, interacting components is straightforward (Israel 1976). The particle number current and the stress-energy tensor can then be written as

$$N^a = nu^a + \nu^a \quad , \tag{2.4}$$

$$T^{ab} = \rho u^a u^b + (p+\tau)q^{ab} + \tau^{ab} \quad , \tag{2.5}$$

where n is the number density of particles, p is the pressure, and q^{ab} is the projection tensor orthogonal to the fluid four-velocity ($q^{ab}u_b = 0$):

$$q^{ab} = u^a u^b + g^{ab} \quad . \tag{2.6}$$

The decompositions in Eqs. (2.4) and (2.5) will be unique provided these fields are constrained as follows:

$$u_a \nu^a = u_a \tau^{ab} = \tau^a{}_a = \tau^{ab} - \tau^{ba} = 0 \quad . \tag{2.7}$$

Deviations from local equilibrium in the fluid are described by the three fields τ, ν^a, and τ^{ab}. The scalar field τ can be interpreted as the bulk stress of the fluid,

ν^a is the particle diffusion current, and τ^{ab} is the trace free shear stress tensor.

The value of the entropy per particle s is determined by the values of the energy density and the number density and the equation of state of the fluid:

$$s = s(\rho,n) \quad . \tag{2.8}$$

All other thermodynamic variables are then defined by the first law of thermodynamics,

$$ds = \frac{1}{Tn}\,d\rho - \frac{\rho+p}{Tn^2}\,dn \quad . \tag{2.9}$$

The pressure p is

$$p = -\rho - Tn^2\left(\frac{\partial s}{\partial n}\right)_\rho \quad , \tag{2.10}$$

and the temperature T is

$$T = \frac{1}{n}\left(\frac{\partial s}{\partial \rho}\right)_n^{-1} \quad . \tag{2.11}$$

The theory is completed by implementing the second law of thermodynamics. The total entropy $S(\Sigma)$ associated with a spacelike surface Σ is found by integrating the entropy current vector field s^a over this surface:

$$S(\Sigma) = \int_\Sigma s^a d\Sigma_a \quad . \tag{2.12}$$

The second law requires $S(\Sigma)$ to be nondecreasing into the future, hence

$$S(\Sigma') - S(\Sigma) = \int \nabla_a s^a dV \geq 0 \quad , \qquad (2.13)$$

for every spacelike surface Σ' to the future of Σ. In Eq. (2.13) the two surface integrals have been converted into a volume integral by Gauss's theorem. Equation (2.13) will always be satisfied provided

$$\nabla_a s^a \geq 0 \quad . \qquad (2.14)$$

The theory of Landau and Lifshitz (1959, Sec. 127) now results if the entropy current is modeled by

$$s^a_{LL} = snu^a - \theta\nu^a \quad . \qquad (2.15)$$

The better behaved theory of Israel and Stewart models the entropy current by an expansion including all possible terms through quadratic order in the deviations from equilibrium:

$$s^a_{IS} = snu^a - \theta\nu^a - \frac{1}{2}\left(\beta_0\tau^2 + \beta_1\nu^b\nu_b + \beta_2\tau^{bc}\tau_{bc}\right)\frac{u^a}{T}$$

$$+ \alpha_0\tau\frac{\nu^a}{T} + \alpha_1\tau^a{}_b\frac{\nu^b}{T} \quad . \qquad (2.16)$$

The "second-order coefficients" α_i and β_i are additional thermodynamic functions which are necessary to completely specify the evolution of the fluid. The β_i, which are essentially relaxation times, describe the deviation of the physical entropy density from the thermodynamic entropy density, and the α_i determine the magnitude of the particle diffusion - viscous couplings. The functional forms of these coefficients can in principle be extracted from experiments, or can be calculated from microscopic theory. Relativistic kinetic theory can be used to show that they are nonzero for simple gases (Israel and Stewart 1979b; Olson and Hiscock 1989).

The divergence of the entropy current in Eq. (2.16) can be computed and simplified with the aid of the conservation equations (Eqs. (2.2) and (2.3)). The result is

$$
\begin{aligned}
T\nabla_a s^a = {} & T\left(\frac{\rho+p}{Tn} - s - \Theta\right)\nabla_a \nu^a - \tau\Big[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a \nu^a \\
& + \tfrac{1}{2} T\tau \nabla_a\left(\beta_0 \frac{u^a}{T}\right) - \gamma_0 T \nu^a \nabla_a\left(\frac{\alpha_0}{T}\right)\Big] - \nu^a \Big[T\nabla_a \Theta + \beta_1 u^b \nabla_b \nu_a \\
& - \alpha_0 \nabla_a \tau - \alpha_1 \nabla_b \tau_a{}^b + \tfrac{1}{2} T\nu_a \nabla_b\left(\beta_1 \frac{u^b}{T}\right) - (1-\gamma_0) T\tau \nabla_a\left(\frac{\alpha_0}{T}\right) \\
& - (1-\gamma_1) T\tau_a{}^b \nabla_b\left(\frac{\alpha_1}{T}\right)\Big] - \tau^{ab}\langle \nabla_a u_b + \beta_2 u^c \nabla_c \tau_{ab} - \alpha_1 \nabla_a \nu_b \\
& + \tfrac{1}{2} T\tau_{ab} \nabla_c\left(\beta_2 \frac{u^c}{T}\right) - \gamma_1 T \nu_a \nabla_b\left(\frac{\alpha_1}{T}\right)\rangle \quad . \qquad (2.17)
\end{aligned}
$$

The brackets $\langle\ \rangle$ which appear in Eq. (2.17) have the meaning:

$$\langle A_{ab}\rangle = \tfrac{1}{2}q_a{}^c q_b{}^d\left(A_{cd} + A_{dc} - \tfrac{2}{3}q_{cd}q^{ef}A_{ef}\right) , \qquad (2.18)$$

for any second rank tensor A_{ab}. The new coefficients γ_0 and γ_1 appear because of the ambiguity in factoring the cross terms in Eq. (2.17) which involve $\tau\nu^a\nabla_a(\alpha_0/T)$ and $\nu^a\tau_a{}^b\nabla_b(\alpha_1/T)$. The simplest way to guarantee that Eq. (2.17) is consistent with the second law of thermodynamics (Eq. (2.14)) is to require that

$$\Theta = \frac{\rho+p}{Tn} - s , \qquad (2.19)$$

$$\tau = -\zeta\left[\nabla_a u^a + \beta_0 u^a\nabla_a\tau - \alpha_0\nabla_a\nu^a\right.$$
$$\left. + \tfrac{1}{2}T\tau\nabla_a\left(\beta_0\frac{u^a}{T}\right) - \gamma_0 T\nu^a\nabla_a\left(\frac{\alpha_0}{T}\right)\right] , \qquad (2.20)$$

$$\nu^a = -\sigma Tq^{ab}\left[T\nabla_b\Theta + \beta_1 u^c\nabla_c\nu_b - \alpha_0\nabla_b\tau - \alpha_1\nabla_c\tau_b{}^c\right.$$
$$+ \tfrac{1}{2}T\nu_b\nabla_c\left(\beta_1\frac{u^c}{T}\right) - (1-\gamma_0)T\tau\nabla_b\left(\frac{\alpha_0}{T}\right)$$
$$\left. - (1-\gamma_1)T\tau_b{}^c\nabla_c\left(\frac{\alpha_1}{T}\right) + \gamma_2\nu^c\nabla_{[b}u_{c]}\right] , \qquad (2.21)$$

$$\tau^{ab} = -2\eta\langle\nabla^a u^b + \beta_2 u^c\nabla_c\tau^{ab} - \alpha_1\nabla^a\nu^b + \tfrac{1}{2}T\tau^{ab}\nabla_c\left(\beta_2\frac{u^c}{T}\right)$$
$$- \gamma_1 T\nu^a\nabla^b\left(\frac{\alpha_1}{T}\right) + \gamma_3\tau^b{}_c\nabla^{[a}u^{c]}\rangle . \qquad (2.22)$$

With these definitions the divergence of the entropy current takes the form

$$\nabla_a s^a = \frac{\tau^2}{\zeta T} + \frac{\nu^a \nu_a}{\sigma T^2} + \frac{\tau^{ab}\tau_{ab}}{2\eta T} \quad , \qquad (2.23)$$

and is insured to be nonnegative provided that the three thermodynamic functions ζ, σ, and η are required to be positive. By taking the Newtonian limit of the theory, these functions may be identified as the bulk viscosity ζ, the particle diffusion coefficient σ ($\sigma(\rho+p)^2/n^2$ may be identified as the thermal conductivity), and the shear viscosity η. The terms multiplying γ_2 and γ_3 in Eqs. (2.21) and (2.22), which couple the particle diffusion vector and the shear stress tensor to the vorticity of the fluid, are the only additional terms which can be added to Eqs. (2.20)-(2.22) which agree with the results of kinetic theory and do not change the entropy generation equation, Eq. (2.17) (Hiscock and Lindblom 1983). The expressions for τ, ν^a, and τ^{ab} in the pathological Landau-Lifshitz theory (Landau and Lifshitz 1959, Sec. 127) result if all the coefficients α_i, β_i, and γ_i are set equal to zero.

Equations (2.2), (2.3), and (2.20)-(2.22) form the complete set of equations of motion for the Israel-Stewart energy frame theory. The theory has 14 degrees of freedom, which can be chosen to be τ, the three independent components of ν^a, the five independent components of τ^{ab}, the three independent components of u^a, and two thermodynamic variables. There are 14 independent variables, rather than 5 as in the Navier-Fourier-Stokes or Landau-Lifshitz theories, because the dissipation fields τ, ν^a, and τ^{ab} obey differential equations instead of algebraic equations. The evolution of the fluid is completely determined provided that the equation of state, the dissipation coefficients ζ, σ, η, and the second-order coefficients α_i, β_i, γ_i are known thermodynamic functions. Gravitational interactions may be included by adding Einstein's equations,

$$G^{ab} = 8\pi T^{ab} \quad , \qquad (2.24)$$

to this set of equations.

Equilibrium states of the theory are those for which the divergence of the entropy current vanishes. By Eq. (2.23), this implies that in equilibrium

$$\tau = \nu^a = \tau^{ab} = 0 \quad . \qquad (2.25)$$

These conditions and the equations of motion (Eqs. (2.2), (2.3), and (2.20)-(2.22)) yield the following additional conditions for an equilibrium state:

$$\nabla_a u^a = 0 \quad , \qquad (2.26)$$

$$q^{ab}\nabla_b \Theta = 0 \quad , \qquad (2.27)$$

$$\langle\nabla_a u_b\rangle = 0 \quad , \tag{2.28}$$

$$u^a\nabla_a n = 0 \quad , \tag{2.29}$$

$$u^a\nabla_a \rho = 0 \quad , \tag{2.30}$$

$$q^{ab}[\nabla_b p + (\rho+p)u^c\nabla_c u_b] = 0 \quad . \tag{2.31}$$

Equations (2.29) and (2.30) imply that all other thermodynamic variables must also be constant along the fluid flow lines because they depend on just two independent thermodynamic variables. These are the same equilibrium states as in the Landau-Lifshitz theory. The stability of these equilibrium states for small perturbations is studied in Ch. 3, and for large perturbations for the particular case of the ultrarelativistic Boltzmann gas in Ch. 4.

The notation used in this section differs slightly from that used originally by Israel (1976). The equations of motion for the Israel-Stewart energy frame theory in the notation of Israel (1976) result if the following substitutions are made in Eqs. (2.4) and (2.20)-(2.22): $\nu^a \to -nq^a/(\rho+p)$, $\sigma \to \kappa n^2/(\rho+p)^2$, $\beta_1 \to \beta_1(\rho+p)^2/n^2$, $\alpha_i \to -\alpha_i(\rho+p)/n$.

The Israel-Stewart Eckart Frame Theory

The only other commonly used Israel-Stewart theory is the Eckart frame theory. The plane steady shock solutions of the Eckart frame theory are studied in Ch. 5. The review of the Eckart frame theory presented in this section is based on the approach used by Hiscock and Lindblom (1983, 1988b, 1989).

In the Israel-Stewart Eckart frame theory, just as in the energy frame theory, the fluid is described by a conserved particle number current and a conserved stress-energy tensor,

$$\nabla_a N^a = 0 \quad , \tag{2.32}$$

$$\nabla_a T^{ab} = 0 \quad , \tag{2.33}$$

but the fluid four-velocity is chosen so that the particle flux is always zero in the rest frame of the fluid,

$$N^a = nu^a \quad , \tag{2.34}$$

and the stress-energy tensor has the form

$$T^{ab} = \rho u^a u^b + (p+\tau)q^{ab} + q^a u^b + q^b u^a + \tau^{ab} \quad . \tag{2.35}$$

Deviations from local equilibrium are described by the bulk stress τ, the heat flux q^a, and the shear stress τ^{ab}, which are constrained to satisfy

$$q^a u_a = \tau^{ab} u_b = \tau^a{}_a = \tau_{ab} - \tau_{ba} = 0 \quad . \tag{2.36}$$

The equation of state of the fluid, $s = s(\rho,n)$, and the first law of thermodynamics (Eq. (2.9)), define the pressure p and the temperature T by Eqs. (2.10) and (2.11).

In the pathological Eckart (1940) theory, the entropy current is modeled as

$$s_E^a = snu^a + \beta q^a \quad , \tag{2.37}$$

while in the better behaved Israel-Stewart Eckart frame theory, the entropy current is modeled by an expansion including all possible terms up to quadratic order in deviations from equilibrium,

$$s_{IS}^a = snu^a + \beta q^a - \frac{1}{2}\left(\beta_0\tau^2 + \beta_1 q^b q_b + \beta_2 \tau^{bc}\tau_{bc}\right)\frac{u^a}{T}$$

$$+ \alpha_0 \tau \frac{q^a}{T} + \alpha_1 \tau^a{}_b \frac{q^b}{T} \quad . \tag{2.38}$$

The divergence of the Eckart frame entropy current (Eq. (2.38)) is manifestly nonnegative,

$$\nabla_a s^a = \frac{\tau^2}{\zeta T} + \frac{q^a q_a}{\kappa T^2} + \frac{\tau^{ab}\tau_{ab}}{2\eta T} \quad , \tag{2.39}$$

provided the bulk viscosity ζ, the thermal conductivity κ, and the shear viscosity η are positive thermodynamic functions, and if β, τ, q^a, and τ^{ab} are defined by

$$\beta = \frac{1}{T} \quad , \tag{2.40}$$

$$\tau = -\zeta\Big[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a$$

$$- \gamma_0 T q^a \nabla_a\left(\frac{\alpha_0}{T}\right) + \frac{1}{2} T\tau \nabla_a\left(\beta_0 \frac{u^a}{T}\right)\Big] \quad , \tag{2.41}$$

$$q^a = -\kappa T q^{ab}\Big[\frac{1}{T}\nabla_b T + u^c \nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau$$

$$- \alpha_1 \nabla_c \tau^c{}_b + \frac{1}{2} T q_b \nabla_c\left(\beta_1 \frac{u^c}{T}\right) - (1-\gamma_0) T\tau \nabla_b\left(\frac{\alpha_0}{T}\right)$$

$$- (1-\gamma_1) T\tau_b{}^c \nabla_c\left(\frac{\alpha_1}{T}\right) + \gamma_2 q^c \nabla_{[b} u_{c]}\Big] \quad , \tag{2.42}$$

$$\tau^{ab} = -2\eta\langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a q^b + \frac{1}{2} T\tau^{ab} \nabla_c\left(\beta_2 \frac{u^c}{T}\right)$$

$$- \gamma_1 T q^a \nabla^b\left(\frac{\alpha_1}{T}\right) + \gamma_3 \tau_c{}^b \nabla^{[a} u^{c]}\rangle \quad . \tag{2.43}$$

Thus, the second law of thermodynamics will be satisfied if β, τ, q^a, and τ^{ab} are defined by Eqs. (2.40)-(2.43). The equations for τ, q^a, and τ^{ab} in the Eckart theory result if each of the α_i, β_i, and γ_i are set equal to zero.

Equations (2.32), (2.33), and (2.41)-(2.43) form the complete set of 14 equations for the 14 independent variables of the Israel-Stewart Eckart frame theory. The evolution of the fluid is completely determined provided that the equation of state, the dissipation coefficients ζ, κ, η, and the second-order coefficients α_i, β_i, γ_i are known thermodynamic functions. Gravitational interactions may be included by adding Einstein's equations (Eq. (2.24)).

Equilibrium states of the theory are those for which the divergence of the entropy current vanishes. By Eq. (2.39), this implies that in equilibrium

$$\tau = q^a = \tau^{ab} = 0 \quad . \tag{2.44}$$

These conditions and the equations of motion (Eqs. (2.32), (2.33), and (2.41)-(2.43)) may be used to show that the equilibrium states of the Israel-Stewart Eckart frame theory are exactly the same as the equilibria of the Israel-Stewart energy frame theory (Eqs. (2.26)-(2.31)). So the definitions of the fluid four-velocity in the Israel-Stewart Eckart frame and energy frame theories differ only away from equilibrium. These are also the equilibrium states of the Eckart theory.

CHAPTER THREE

THE ENERGY FRAME THEORY IN THE LINEAR REGIME

The Israel-Stewart Eckart frame theory is an acceptable theory of relativistic dissipative fluids near equilibrium because it has stable equilibria, and small perturbations around equilibrium propagate causally and obey hyperbolic differential equations (Hiscock and Lindblom 1983). In this chapter, I investigate whether or not the Israel-Stewart energy frame theory is also a physically acceptable theory near equilibrium.

Linearized Energy Frame Theory

The equations of motion for small perturbations around the equilibrium states of the energy frame theory are derived in this section. These equations will be needed in the following sections to study the stability of the equilibria of the energy frame theory, and to determine under what conditions small perturbations propagate causally via a hyperbolic system of differential equations. The difference between the nonequilibrium value of a field Q at a given spacetime point and the value of this field at the same spacetime point in the fiducial equilibrium state will be denoted by δQ. Thus the fields $\delta\rho$, δn, δu^a, $\delta\tau$, $\delta\nu^a$, $\delta\tau^{ab}$, and δg^{ab} describe the perturbations of the fluid about its equilibrium state. A variable without the prefix δ (e.g. n, ρ, u^a, g^{ab}, etc.) refers to the value of that field in the equilibrium state. The fields in the equilibrium state satisfy Eqs. (2.26)-(2.31).

I will assume that all perturbation quantities δQ are small enough that their evolution is described well by the linearized equations of motion. I also assume for simplicity that $\delta g^{ab} = 0$. Hence the linearized equations of motion displayed below may only be applied to special-relativistic fluids (but the background geometry need not be flat spacetime), or to short wavelength perturbations of the fluid. The equations of motion (Eqs. (2.2), (2.3), and (2.20)-(2.22)) linearized about an arbitrary equilibrium state are then

$$\nabla_a \delta T^{ab} = 0 \quad , \tag{3.1}$$

$$\delta T^{ab} = (\rho+p)(\delta u^a u^b + u^a \delta u^b) + \delta\rho u^a u^b$$

$$+ (\delta p + \delta\tau) q^{ab} + \delta\tau^{ab} \quad , \tag{3.2}$$

$$u^a \nabla_a \delta n + \nabla_a (n \delta u^a) + \nabla_a \delta\nu^a = 0 \quad , \tag{3.3}$$

$$\delta\tau = -\zeta \Big[\nabla_a \delta u^a + \beta_0 u^a \nabla_a \delta\tau - \alpha_0 \nabla_a \delta\nu^a$$

$$+ \tfrac{1}{2} T \delta\tau \nabla_a \left(\beta_0 \frac{u^a}{T} \right) - \gamma_0 T \delta\nu^a \nabla_a \left(\frac{\alpha_0}{T} \right) \Big] \quad , \tag{3.4}$$

$$\delta\nu^a = -\sigma T q^{ab}\Big[T\nabla_b\delta\Theta + \beta_1 u^c\nabla_c\delta\nu_b - \alpha_0\nabla_b\delta\tau - \alpha_1\nabla_c\delta\tau_b{}^c$$

$$+ \tfrac{1}{2}T\delta\nu_b\nabla_c\left(\beta_1 \frac{u^c}{T}\right) - (1-\gamma_0)T\delta\tau\nabla_b\left(\frac{\alpha_0}{T}\right)$$

$$- (1-\gamma_1)T\delta\tau_b{}^c\nabla_c\left(\frac{\alpha_1}{T}\right) + \gamma_2\delta\nu^c\nabla_{[b}u_{c]}\Big] \quad , \qquad (3.5)$$

$$\delta\tau^{ab} = -2\eta\langle\nabla^a\delta u^b + \delta u^a u^c\nabla_c u^b + \beta_2 u^c\nabla_c\delta\tau^{ab} - \alpha_1\nabla^a\delta\nu^b$$

$$+ \tfrac{1}{2}T\delta\tau^{ab}\nabla_c\left(\beta_2 \frac{u^c}{T}\right) - \gamma_1 T\delta\nu^a\nabla^b\left(\frac{\alpha_1}{T}\right) + \gamma_3\delta\tau^b{}_c\nabla^{[a}u^{c]}\rangle. \qquad (3.6)$$

The derivative ∇_a in Eqs. (3.1)-(3.6) is the covariant derivative compatible with the background spacetime metric g^{ab}. The projection tensor appearing in the definition of the brackets < > (Eq. (2.18)) is the unperturbed projection tensor q^{ab}. The perturbation variables satisfy the constraints

$$u^a\delta\nu_a = u^a\delta\tau_{ab} = \delta\tau^a{}_a = \delta\tau^{ab} - \delta\tau^{ba} = u^a\delta u_a = 0 \qquad (3.7)$$

because of the constraints in Eq. (2.7).

The Stability Conditions

The stability of the equilibrium states of the Israel-Stewart energy frame theory for small perturbations are studied in this section by using the Liapunov functional techniques developed by Hiscock and Lindblom (1983) for the analysis of the stability of the equilibria of the linearized Israel-Stewart Eckart frame theory. A Liapunov functional, a nonincreasing function of time which depends quadratically on the perturbation fields, is constructed. When this functional is nonnegative for all possible values of the perturbation fields the equilibrium state is stable, since in this case the functional is bounded below by zero. If the functional is negative for some values of the perturbation fields, then some of the perturbation fields will evolve towards infinity because the functional is unbounded below in this case, implying that the equilibrium state is unstable. This reasoning will be made rigorous in what follows.

A Liapunov functional can be constructed for the Israel-Stewart energy frame theory from an "energy" current E^a defined by

$$TE^a = \delta T^a{}_b \delta u^b - \tfrac{1}{2}(\rho+p)u^a \delta u^b \delta u_b + T\delta\Theta\delta\nu^a - \alpha_0 \delta\tau\delta\nu^a - \alpha_1 \delta\tau^a{}_b \delta\nu^b$$
$$+ \tfrac{1}{2}(\rho+p)^{-1}\left[\left(\frac{\partial\rho}{\partial p}\right)_s (\delta p)^2 + \left(\frac{\partial\rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta (\delta s)^2\right]u^a$$
$$+ \tfrac{1}{2}\left[\beta_0(\delta\tau)^2 + \beta_1 \delta\nu^b \delta\nu_b + \beta_2 \delta\tau^{bc}\delta\tau_{bc}\right]u^a \quad . \qquad (3.8)$$

The total energy $E(\Sigma)$ associated with a spacelike surface Σ is

$$E(\Sigma) = \int_\Sigma E^a d\Sigma_a \quad . \tag{3.9}$$

The Liapunov functional $E(\Sigma)$ will be nonincreasing into the future provided

$$E(\Sigma') - E(\Sigma) = \int \nabla_a E^a \, dV \leq 0 \quad , \tag{3.10}$$

for every spacelike surface Σ' to the future of Σ. Thus the Liapunov functional will be a nonincreasing function of time as long as the divergence of the energy current is not positive. The divergence of E^a in Eq. (3.8) may be calculated with the aid of the linearized equations (Eqs. (3.1)-(3.6)). The result is

$$\nabla_a E^a = - \left[\frac{(\delta\tau)^2}{\zeta T} + \frac{\delta\nu^a \delta\nu_a}{\sigma T^2} + \frac{\delta\tau^{ab}\delta\tau_{ab}}{2\eta T} \right] \leq 0 \quad . \tag{3.11}$$

As required, this Liapunov functional is a nonincreasing function of time.

This energy current (Eq. (3.8)) was found by modifying the expression for the energy current of the Israel-Stewart Eckart frame theory found by Hiscock and Lindblom (1983). The structure of the two theories is similar enough that it is easy to see how to modify their expression to obtain an appropriate energy current for the Israel-Stewart energy frame theory.

It is useful to express the Liapunov functional in terms of an energy density:

$$E(\Sigma) = \int e \frac{u^a}{T} d^3x_a \quad . \tag{3.12}$$

The energy density e is defined by

$$e = TE^a t_a / u^b t_b \quad , \tag{3.13}$$

where the vector t^a is the future-directed unit normal to the spacelike surface Σ. The Liapunov functional E(Σ) will be nonnegative for all possible values of the perturbation fields if and only if the energy density e is a nonnegative functional of the perturbation fields at all points in the fluid.

It is possible to factor the energy density into the form

$$e = \frac{1}{2} \sum_A \Omega_A (\delta Z_A)^2 \quad . \tag{3.14}$$

The Ω_A are functions of the thermodynamic variables of the equilibrium state given by

$$\Omega_1 = (\rho+p)^{-1}\left(\frac{\partial\rho}{\partial p}\right)_s \quad , \tag{3.15}$$

$$\Omega_2 = (\rho+p)^{-1}\left(\frac{\partial\rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta \quad , \tag{3.16}$$

$$\Omega_3 = (\rho+p)\left[1-\lambda^2\left(\frac{\partial p}{\partial \rho}\right)_s\right] - \left(\frac{1}{\beta_0} + \frac{2}{3\beta_2} + \frac{K^2}{\Omega_6}\right)\lambda^2 \quad , \qquad (3.17)$$

$$\Omega_4 = \rho+p - \frac{2(\rho+p)^2\beta_2 + [n^2\beta_1 - 2n(\rho+p)\alpha_1 - (\rho+p)]\lambda^2}{2\left(n^2\beta_1 + \rho+p\right)\beta_2 - \left(n\alpha_1 + 1\right)^2\lambda^2} \quad , \qquad (3.18)$$

$$\Omega_5 = \beta_0 \quad , \qquad (3.19)$$

$$\Omega_6 = \frac{1}{(\rho+p)^2}\left[\frac{n^2\beta_1 + \rho+p}{\lambda^2} - \frac{(n\alpha_0 + 1)^2}{\beta_0} - \frac{2(n\alpha_1 + 1)^2}{3\beta_2} - \frac{1}{n}\left(\frac{\rho+p}{T}\right)^2\left(\frac{\partial T}{\partial s}\right)_n\right] \quad , \qquad (3.20)$$

$$\Omega_7 = \frac{1}{(\rho+p)^2}\left[n^2\beta_1 + \rho + p - \frac{(n\alpha_1 + 1)^2\lambda^2}{2\beta_2}\right] \quad , \qquad (3.21)$$

$$\Omega_8 = \beta_2 \quad . \qquad (3.22)$$

The δZ_A are linearly independent combinations of the perturbation fields:

$$\delta Z_1 = \delta p + (\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s\left[\lambda_a\delta u^a + \left(\frac{1}{n} - \frac{\rho+p}{Tn}\left(\frac{\partial T}{\partial p}\right)_s\right)\lambda_a\delta\nu^a\right] \quad , \qquad (3.23)$$

$$\delta Z_2 = \delta s - \frac{(\rho+p)^2}{Tn}\left(\frac{\partial T}{\partial \rho}\right)_p\left(\frac{\partial s}{\partial p}\right)_\Theta \lambda_a\delta\nu^a \quad , \qquad (3.24)$$

$$\delta Z_3 = \frac{1}{\lambda}\left(\lambda_a\delta u^a + \frac{1}{n}\lambda_a\delta\nu^a\right) \quad , \qquad (3.25)$$

$$\delta Z_4^a = \gamma^a{}_b \delta u^b + \frac{1}{n}\gamma^a{}_b \delta \nu^b \quad , \tag{3.26}$$

$$\delta Z_5 = \delta\tau + \frac{1}{\beta_0}\lambda_a \delta u^a - \frac{\alpha_0}{\beta_0}\lambda_a \delta\nu^a \quad , \tag{3.27}$$

$$\delta Z_6 = -\frac{\rho+p}{n}\lambda_a \delta\nu^a + \frac{K}{\Omega_6}\left(\lambda_a \delta u^a + \frac{1}{n}\lambda_a \delta\nu^a\right) \quad , \tag{3.28}$$

$$\delta Z_7^a = \frac{2(\rho+p)^2\beta_2 - (\rho+p)(n\alpha_1+1)\lambda^2}{2(n^2\beta_1+\rho+p)\beta_2 - (n\alpha_1+1)^2\lambda^2}\left(\delta u^b + \frac{1}{n}\delta\nu^b\right)\gamma^a{}_b$$

$$- \frac{\rho+p}{n}\gamma^a{}_b \delta\nu^b \quad , \tag{3.29}$$

$$\delta Z_8^{ab} = \delta\tau^{ab} + \frac{1}{\beta_2}\langle\lambda^a \delta u^b\rangle - \frac{\alpha_1}{\beta_2}\langle\lambda^a \delta\nu^b\rangle \quad . \tag{3.30}$$

In these equations λ^a is the velocity of observers moving along t^a relative to the rest frame of the background equilibrium state,

$$\lambda^a = q^{ab} t_b / u_c t^c \quad , \tag{3.31}$$

λ is the norm of λ^a (which can be shown to be bounded between zero and one), and K is defined as

$$K = \frac{1}{\lambda^2} - \frac{1}{\rho+p}\left[\frac{n\alpha_0+1}{\beta_0} + \frac{2(n\alpha_1+1)}{3\beta_2} + \frac{n(\rho+p)}{T}\left(\frac{\partial T}{\partial n}\right)_s\right] \quad , \tag{3.32}$$

and the projection tensor γ^{ab} is given by

$$\gamma^{ab} = g^{ab} - \frac{1}{\lambda^2}\lambda^a\lambda^b \quad . \tag{3.33}$$

The requirement that the Liapunov functional be nonnegative is then equivalent to the condition that each of the Ω_A be nonnegative,

$$\Omega_A \geq 0 \quad . \tag{3.34}$$

So Eq. (3.34) must be satisfied for all values of λ between zero and one (the speed of light) for the equilibrium state to be stable. The most restrictive conditions are obtained by setting $\lambda = 1$.

The stability conditions (Eq. (3.34)) have several interesting consequences. Because of the Ω_3, Ω_5, and Ω_8 conditions, β_0 and $(3/2)\beta_2$ are nonnegative and bounded below by $(\rho+p)^{-1}$. The Ω_4 and Ω_7 conditions require β_1 to be nonnegative. The square of the adiabatic sound speed $(\partial p/\partial\rho)_s$ is bounded between zero and the speed of light because Ω_1 and Ω_3 are nonnegative. This does not imply that all perturbations of stable equilibrium states propagate subluminally; perturbations need not propagate with the adiabatic sound speed in a fluid theory having dissipation. The Ω_2 condition is the relativistic Schwarzschild criterion for stability against convection (Thorne 1967).

The connection between the sign of the energy functional and stability is established by the following theorems. Theorem A shows that a nonnegative energy functional is a sufficient condition for stability.

Theorem A. If the stability conditions (Eq. (3.34)) are satisfied, then none of the perturbation variables will grow without bound (as measured by a square integral norm).

The proof of Theorem A is identical to the analogous theorem proven for the Israel-Stewart Eckart frame theory by Hiscock and Lindblom (1983, p. 494).

The following theorem falls somewhat short of showing that satisfaction of each of the conditions $\Omega_A > 0$ is also a necessary condition for stability.

Theorem B. Suppose that the inequalities

$$0 < \left(\frac{\partial p}{\partial \rho}\right)_s < 1 \qquad (3.35)$$

and

$$\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\Theta > 0 \qquad (3.36)$$

are satisfied, and at least one of the stability conditions is violated. Suppose also that there do not exist solutions to the perturbation equations for which the square integral norms of the perturbation variables $\delta\tau$, $\delta\nu^a$, and $\delta\tau^{ab}$ evolve to zero, but these variables do not evolve to zero everywhere. Finally, assume in addition that the perturbation equations only yield solutions for which there exists 14 linearly independent perturbation fields. Then there exist solutions to the perturbation equations which grow without bound.

Theorem B can be proven as follows. Assume that no unbounded solutions to the perturbation equations exist. This implies that the integral of the divergence of the energy current (Eq. (3.11)) must also evolve to zero since the Liapunov functional is bounded. It follows that the square integral norms of the fields $\delta\tau$, $\delta\nu^a$, and $\delta\tau^{ab}$ evolve to zero, and the energy density (Eq. (3.14)) evolves to

$$e = \frac{1}{2(\rho+p)}\left(\frac{\partial\rho}{\partial p}\right)_s\left[\delta p + (\rho+p)\left(\frac{\partial p}{\partial\rho}\right)_s\lambda_a\delta u^a\right]^2 + \frac{\rho+p}{2}\gamma_{ab}\delta u^a\delta u^b$$

$$+ \frac{\rho+p}{2\lambda^2}\left[1-\lambda^2\left(\frac{\partial p}{\partial\rho}\right)_s\right]\left(\lambda_a\delta u^a\right)^2 + \frac{1}{2(\rho+p)}\left(\frac{\partial\rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta(\delta s)^2, \quad (3.37)$$

because the perturbation fields $\delta\tau$, $\delta\nu^a$, $\delta\tau^{ab}$ evolve to zero everywhere when their square integral norms do. When Eqs. (3.35) and (3.36) are satisfied this energy density is nonnegative. Therefore, the Liapunov functional is also nonnegative, implying that the energy of all possible perturbations is nonnegative because the Liapunov functional is a nonincreasing function of time. Thus all of the stability conditions $\Omega_A > 0$ are satisfied because there exists 14 linearly independent perturbation fields, which proves Theorem B.

Theorem B fails to be a complete proof, even for the equilibria which have a real and causal adiabatic sound speed and are stable against convection (and so satisfy Eqs. (3.35) and (3.36)), because it has not been shown that the physically interesting solutions to the perturbation equations have 14 independent fluid variables with $\delta\tau$, $\delta\nu^a$, and $\delta\tau^{ab}$ evolving to zero everywhere when their square integral norms evolve to zero. The analogous theorem for the Eckart frame theory (Hiscock and Lindblom 1983, p. 494) has exactly the same deficiencies (Geroch and Lindblom 1990).

Causality and Hyperbolicity Conditions

This section contains a derivation of the conditions which guarantee causal propagation of small perturbations around the equilibria of the Israel-Stewart energy frame theory. The conditions for the linearized equations of motion of the energy frame theory to form a symmetric hyperbolic system are also determined.

The system of equations for the linear perturbations (Eqs. (3.1)-(3.6)) has the form

$$\mathbf{A}^A{}_B{}^a \nabla_a \delta Y^B + \mathbf{B}^A{}_B \delta Y^B = 0 \quad , \tag{3.38}$$

where the components of δY^B are the fourteen perturbation fields δn, $\delta\rho$, δu^a, $\delta\tau$, $\delta\nu^a$, $\delta\tau^{ab}$, and the index A labels one of the fourteen equations for the perturbation fields. The components of the matrices $\mathbf{A}^A{}_B{}^a$ and $\mathbf{B}^A{}_B$ are functions of the variables of the equilibrium state. The level surfaces of a scalar function φ satisfying

$$\det\left(A^{A}{}_{B}{}^{a}\nabla_{a}\varphi \right) = 0 \tag{3.39}$$

are the characteristic surfaces for these equations; the fields δY^{B} cannot be freely specified on these surfaces (Courant and Hilbert 1962, p. 170). The characteristic velocities are the slopes of these surfaces; information propagates along the characteristic surfaces with these velocities.

The characteristic velocities are most easily found by solving Eq. (3.39) with the following choice of coordinates. At some point in the fluid choose a Cartesian coordinate system which is comoving with the fluid and which has the x^1 direction along the direction of spatial variation of φ. Then at this point

$$g^{ab}\partial_a\partial_b = -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 \quad , \tag{3.40}$$

$$u^a\partial_a = \partial_0 \quad , \tag{3.41}$$

and

$$\varphi = \varphi(x^0, x^1) \quad . \tag{3.42}$$

In this coordinate system the equation for the characteristic velocities (Eq. (3.39)) is

$$\det\left(vA^A{}_B{}^0 - A^A{}_B{}^1 \right) = 0 \quad , \tag{3.43}$$

where v is the characteristic velocity defined by

$$v = -\frac{\partial_0\varphi}{\partial_1\varphi} \quad . \tag{3.44}$$

When the set of fourteen perturbation fields is chosen as

$$\delta Y^B = \Big(T\delta\Theta,\ \frac{\delta T}{T},\ \delta\tau,\ \delta u^1,\ \delta\nu^1,\ \delta\tau^{11},\ \delta u^2,\ \delta\nu^2,\ \delta\tau^{21},$$

$$\delta u^3,\ \delta\nu^3,\ \delta\tau^{31},\ \delta\tau^{22} - \delta\tau^{33},\ \delta\tau^{32}\Big) \quad , \tag{3.45}$$

the characteristic matrix block diagonalizes:

$$vA^A{}_B{}^0 - A^A{}_B{}^1 = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & \mathbf{S} \end{bmatrix} \quad . \tag{3.46}$$

The matrices **Q**, **R**, and **S** are defined as

$$\mathbf{Q} = \begin{bmatrix} \frac{v}{T}\left(\frac{\partial n}{\partial\Theta}\right)_T & \frac{v}{T}\left(\frac{\partial\rho}{\partial\Theta}\right)_T & 0 & -n & -1 & 0 \\ \frac{v}{T}\left(\frac{\partial\rho}{\partial\Theta}\right)_T & vT\left(\frac{\partial\rho}{\partial T}\right)_\Theta & 0 & -(\rho+p) & 0 & 0 \\ 0 & 0 & v\beta_0 & -1 & \alpha_0 & 0 \\ -n & -(\rho+p) & -1 & v(\rho+p) & 0 & -1 \\ -1 & 0 & \alpha_0 & 0 & v\beta_1 & \alpha_1 \\ 0 & 0 & 0 & -1 & \alpha_1 & \frac{3}{2}v\beta_2 \end{bmatrix} , \tag{3.47}$$

$$\mathbf{R} = \begin{bmatrix} v(\rho+p) & 0 & -1 \\ 0 & v\beta_1 & \alpha_1 \\ -1 & \alpha_1 & 2v\beta_2 \end{bmatrix} \quad , \qquad (3.48)$$

$$\mathbf{S} = \begin{bmatrix} 2v\beta_2 & 0 \\ 0 & v\beta_2 \end{bmatrix} \quad . \qquad (3.49)$$

Thus the equation for the characteristic velocities (Eq. (3.39)) takes the form

$$\det \mathbf{Q}\ (\det \mathbf{R})^2 \det \mathbf{S} = 0 \quad , \qquad (3.50)$$

where the determinants are

$$\det \mathbf{Q} = \tfrac{3}{2}v^2\left(Av^4 + Bv^2 + C\right)\left[\left(\frac{\partial n}{\partial \Theta}\right)_T\left(\frac{\partial \rho}{\partial T}\right)_\Theta - \left(\frac{\partial n}{\partial T}\right)_\Theta\left(\frac{\partial \rho}{\partial \Theta}\right)_T\right] \quad , \qquad (3.51)$$

$$\det \mathbf{R} = v\left[\ 2(\rho+p)\beta_1\beta_2 v^2 - (\rho+p)\alpha_1{}^2 - \beta_1\ \right] \quad , \qquad (3.52)$$

$$\det \mathbf{S} = 2(v\beta_2)^2 \quad . \qquad (3.53)$$

The coefficients of the quartic in Eq. (3.51) are

$$A = (\rho+p)\beta_0\beta_1\beta_2 \quad , \qquad (3.54)$$

$$B = -(\rho+p)D - \left[\left(\frac{n}{\rho+p}\right)^2\beta_1 + \frac{1}{\rho+p}\ \right]E - 2F \quad , \qquad (3.55)$$

$$C = \left(\frac{n}{\rho+p}\right)^2 \frac{DE - F^2}{\beta_0\beta_2} \quad . \qquad (3.56)$$

The functions D, E, and F are defined as

$$D = \frac{\beta_0\beta_2}{n^2}\left[\frac{(n\alpha_0+1)^2}{\beta_0} + \frac{2(n\alpha_1+1)^2}{3\beta_2} + \frac{1}{n}\left(\frac{\rho+p}{T}\right)^2\left(\frac{\partial T}{\partial s}\right)_n\right] , \qquad (3.57)$$

$$E = \left(\frac{\rho+p}{n}\right)^2 \beta_0\beta_2 \left[(\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s + \frac{1}{\beta_0} + \frac{2}{3\beta_2}\right] , \qquad (3.58)$$

$$F = -\frac{\rho+p}{n^2}\beta_0\beta_2\left[\frac{n\alpha_0+1}{\beta_0} + \frac{2(n\alpha_1+1)}{3\beta_2} + \frac{n(\rho+p)}{T}\left(\frac{\partial T}{\partial n}\right)_s\right] . \qquad (3.59)$$

The set of characteristic velocities is found by setting each of det **Q**, det **R**, and det **S** to zero. The two characteristic velocities found by setting det **S** to zero both vanish. The matrix **R** has one vanishing characteristic velocity, and two nonvanishing characteristic velocities given by

$$v_T^{\,2} = \frac{(\rho+p)\alpha_1^{\,2} + \beta_1}{2(\rho+p)\beta_1\beta_2} \quad . \qquad (3.60)$$

These are the transverse characteristic velocities since the matrix **R** describes perturbations tangent to the characteristic surfaces, and so transverse to the direction of propagation. The matrix **Q** has two vanishing characteristic velocities, and four nonvanishing characteristic velocities which satisfy

$$Av_L^4 + Bv_L^2 + C = 0 \quad . \tag{3.61}$$

These are the longitudinal characteristic velocities.

The conditions for the system of linear perturbation equations (Eq. (3.38)) to form a hyperbolic system will now be determined. The most common definition of hyperbolicity, which requires the characteristic velocities to satisfy conditions such as reality, cannot be used because the characteristic velocities of the energy frame theory are not all distinct. Since the matrices $\mathbf{A}^A{}_B{}^a$ are symmetric, the system of perturbation equations forms a symmetric hyperbolic system if some linear combination of the $\mathbf{A}^A{}_B{}^a$ were positive definite. The system of perturbation equations would then have a well-posed initial value problem (Courant and Hilbert 1962, p. 593).

A necessary and sufficient condition for a matrix to be positive definite is that the determinant of each leading principal submatrix be positive. The matrix $A^A{}_B{}^0$ is itself positive definite if and only if the following inequalities are satisfied:

$$\left(\frac{\partial n}{\partial \Theta}\right)_T > 0 \quad , \tag{3.62}$$

$$\left(\frac{\partial \rho}{\partial T}\right)_n > 0 \quad , \tag{3.63}$$

$$\beta_i > 0 \quad , \tag{3.64}$$

for each of $i = 0,1,2$. Thus a sufficient condition for a positive definite linear combination of the $A^{A}{}_{B}{}^{a}$ to exist is for each of these inequalities to be satisfied. Equations (3.62) and (3.63) also imply that $(\partial\rho/\partial T)_{\Theta} > 0$.

Satisfaction of the inequalities in Eqs. (3.62)-(3.64) is also a necessary condition for a positive definite linear combination of the $A^{A}{}_{B}{}^{a}$ to exist. The matrix $A^{A}{}_{B}{}^{0}$ must be positive definite because only $A^{A}{}_{B}{}^{0}$ has nonzero diagonal elements and none of the other $A^{A}{}_{B}{}^{a}$ have nonzero off-diagonal elements in the locations where the off-diagonal elements of $A^{A}{}_{B}{}^{0}$ are nonzero. Therefore, the perturbation equations (Eq. (3.38)) form a symmetric hyperbolic system if and only if the inequalities in Eqs. (3.62)-(3.64) are satisfied. However, it is unknown whether or not these are the weakest conditions necessary for the theory to be hyperbolic because it has not been proven that satisfaction of Eqs. (3.62)-(3.64) are also necessary conditions for the theory to be hyperbolic.

Stability is Equivalent to Causality and Hyperbolicity

The connection between stability, causality, and hyperbolicity in the Israel-Stewart energy frame theory is established in this section. The proofs of the results of this section are very similar to the corresponding proofs for the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983). In what follows it will be assumed that the stability conditions are strictly satisfied, $\Omega_A(\lambda^2) > 0$. One relationship is that stability implies causality and hyperbolicity: the linear perturbations of a fluid described by the Israel-Stewart energy frame theory propagate causally (subluminally) and obey a symmetric hyperbolic system of equations when the stability conditions are strictly satisfied.

The proof is as follows. Equations (3.18), (3.21), and (3.60) can be used to show

$$1 - v_T^{\,2} = \frac{\Omega_4(1)\ \Omega_7(1)}{\Omega_4(0)\ \Omega_7(0)} > 0 \tag{3.65}$$

when Ω_4 and Ω_7 are positive. Also $v_T^{\,2} > 0$, since $\beta_1 > 0$ and $\beta_2 > 0$ when the stability conditions are strictly satisfied. Thus the transverse velocities are real and bounded between zero and the speed of light:

$$0 < v_T^{\,2} < 1 \quad . \tag{3.66}$$

Recall that the squares of the longitudinal characteristic velocities $x = v_L^{\,2}$ are the roots of the quadratic

$$P(x) = Ax^2 + Bx + C \quad . \tag{3.67}$$

The squares of the longitudinal velocities will be real if and only if the discriminant is nonnegative. Equations (3.54)-(3.59) can be used to show

$$B^2-4AC = \frac{\rho+p}{n^2\beta_1+\rho+p}\left[\left[(\rho+p)D + \left[\left(\frac{n}{\rho+p}\right)^2\beta_1 + \frac{1}{\rho+p}\right]E + 2\left(\frac{n^2\beta_1}{\rho+p}+1\right)F\right]^2\right.$$

$$\left. + \frac{n^2\beta_1}{\rho+p}\left[(\rho+p)D - \left[\left(\frac{n}{\rho+p}\right)^2\beta_1 + \frac{1}{\rho+p}\right]E\right]^2\right] \geq 0 \qquad (3.68)$$

when the stability conditions are satisfied. So, if the stability conditions are satisfied, then the squares of the longitudinal characteristic velocities are real.

The zeros of the quadratic $P(x)$ can be seen to lie between $x = 0$ and $x = 1$ when the stability conditions are strictly satisfied from the following quantities:

$$P(0) = C = \tfrac{2}{3}(\alpha_1 - \alpha_0)^2 + \tfrac{2}{3}n\left(\frac{\partial p}{\partial n}\right)_s\left[\alpha_1 + \frac{1}{n} - \frac{\rho+p}{Tn}\left(\frac{\partial T}{\partial p}\right)_s\right]^2\beta_0$$

$$+ \left(\frac{\rho+p}{Tn}\right)^2\left(\frac{\partial T}{\partial s}\right)_p\left(\frac{\partial p}{\partial n}\right)_s\beta_0\beta_2 + n\left(\frac{\partial p}{\partial n}\right)_s\left[\alpha_0 + \frac{1}{n} - \frac{\rho+p}{Tn}\left(\frac{\partial T}{\partial p}\right)_s\right]^2\beta_2$$

$$+ \frac{1}{n}\left(\frac{\rho+p}{Tn}\right)^2\left(\frac{\partial T}{\partial s}\right)_p\left(\beta_2 + \tfrac{2}{3}\beta_0\right) > 0 \quad , \qquad (3.69)$$

$$\frac{dP(0)}{dx} = B = \left(\frac{\rho+p}{n}\right)^2\left(\frac{\partial \theta}{\partial s}\right)_\rho\beta_0\beta_2 - (\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s\beta_0\beta_1\beta_2$$

$$-(\rho+p)\left(\alpha_0^2\beta_2 + \tfrac{2}{3}\alpha_1^2\beta_0\right) - \beta_1\beta_2 - \tfrac{2}{3}\beta_0\beta_1 < 0 \quad , \qquad (3.70)$$

$$P(1) = A + B + C = \left(\frac{\rho+p}{n}\right)^2 \Omega_3(1)\Omega_6(1)\beta_0\beta_2 > 0 \quad , \qquad (3.71)$$

$$\frac{dP(1)}{dx} = 2A + B = \beta_0\beta_1\beta_2 \left[\Omega_3(1) + \frac{K^2(1)}{\Omega_6(1)}\right] + \frac{\rho+p}{n^2}\beta_0\beta_2\Omega_3(1)$$

$$+ \frac{(\rho+p)\beta_0\beta_2}{n^2\Omega_6(1)} \left[(\rho+p)\Omega_6(1) - K(1) \right]^2 > 0 \quad . \qquad (3.72)$$

Thus, the two zeros of $P(x)$ must be real and lie between $x = 0$ and $x = 1$ because $P(0) > 0$, $dP(0)/dx < 0$, $P(1) > 0$, and $dP(1)/dx > 0$:

$$0 < v_L^2 < 1 \quad . \qquad (3.73)$$

Equations (3.69)-(3.72) may be verified from the definitions of A, B, and C. The thermodynamic derivatives $(\partial p/\partial n)_s$ and $(\partial T/\partial s)_p$ are each guaranteed to be positive, and $(\partial\theta/\partial s)_\rho$ is negative, as a consequence of the first law of thermodynamics and the positivity of Ω_1 and Ω_2 (see Hiscock and Lindblom 1983, p. 482).

The hyperbolicity conditions (Eqs. (3.62)-(3.64)) are easily seen to be satisfied when the stability conditions are strictly satisfied. Therefore, linear perturbations propagate causally via a symmetric hyperbolic system of equations when the stability criteria are strictly satisfied.

The converse theorem also holds: the Israel-Stewart energy frame theory is stable for linear perturbations when these perturbations propagate causally and obey a symmetric hyperbolic system of equations (in the sense of Eqs. (3.62)-(3.64)). This will be proven by showing that the causality conditions (in particular Eqs. (3.65), (3.71), and (3.72)) and the hyperbolicity criteria (Eqs. (3.62)-(3.64)) imply the positivity of each of the Ω_A.

The stability conditions Ω_5 and Ω_8 are positive when Eq. (3.64) is satisfied. The stability conditions Ω_1 and Ω_2 may be expressed as

$$\Omega_1 = \left[Tn^2\left(\frac{\partial\Theta}{\partial n}\right)_T + \frac{1}{Tn^2}\left(\frac{\partial p}{\partial s}\right)_n^2\left(\frac{\partial \rho}{\partial T}\right)_n \right]^{-1} , \tag{3.74}$$

$$\Omega_2 = \frac{T^2n^4}{\rho+p}\left(\frac{\partial T}{\partial \rho}\right)_\Theta\left(\frac{\partial \rho}{\partial p}\right)_s\left[\left(\frac{\partial\Theta}{\partial n}\right)_T + \frac{1}{T^2}\left(\frac{\partial \rho}{\partial n}\right)_T^2\left(\frac{\partial T}{\partial \rho}\right)_n \right] . \tag{3.75}$$

Thus Ω_1 and Ω_2 are also positive when the hyperbolicity conditions are satisfied.

Equation (3.71) and the positivity of Ω_5 and Ω_8 require $\Omega_3(1)$ and $\Omega_6(1)$ to be nonzero and to have the same sign. This and Eq. (3.72) require $\Omega_3(1)$ and $\Omega_6(1)$ to be positive. This guarantees the positivity of $\Omega_6(\lambda^2)$ since $\Omega_6(\lambda^2) \geq \Omega_6(1)$ for all $\lambda^2 < 1$ (see Eq. (3.20)). Further, Ω_7 is also positive because

$$\Omega_7(\lambda^2) \geq \Omega_7(1) = \Omega_6(1) + \frac{1}{\beta_0}\left(\frac{n\alpha_0 + 1}{\rho+p}\right)^2$$

$$+ \frac{1}{6\beta_2}\left(\frac{n\alpha_1 + 1}{\rho+p}\right)^2 + \frac{1}{T^2 n}\left(\frac{\partial T}{\partial s}\right)_n > 0 \qquad (3.76)$$

for $\lambda^2 < 1$.

The minima of $\Omega_3(\lambda^2)$ and $\Omega_4(\lambda^2)$ can be determined from the following derivatives:

$$\frac{d\Omega_3}{d(\lambda^2)} = -\frac{1}{c_2}\left[\left(\frac{c_3}{c_3 - \lambda^2 c_2}\right)^2 - 1\right]\left(c_1 + \frac{c_2}{c_3}\right)^2 - \frac{1}{c_3{}^2}\left[\frac{1}{\beta_0}\left[\left(\frac{n}{\rho+p}\right)^2 \beta_1\right.\right.$$

$$\left. - \frac{n\alpha_0}{\rho+p}\right]^2 + \frac{2}{3\beta_2}\left[\left(\frac{n}{\rho+p}\right)^2 \beta_1 - \frac{n\alpha_1}{\rho+p}\right]^2 + \frac{1}{T^2 n}\left(\frac{\partial T}{\partial s}\right)_p$$

$$\left. + (\rho+p)\left(\frac{\partial p}{\partial \rho}\right)_s\left[c_3 - \frac{1}{n}\left(\frac{\partial p}{\partial \rho}\right)_n\left(\frac{\partial n}{\partial p}\right)_s\right]^2\right] , \qquad (3.77)$$

$$\frac{d\Omega_4}{d(\lambda^2)} = -2\beta_2\left[\frac{n^2\beta_1 - n(\rho+p)\alpha_1}{2(n^2\beta_1 + \rho+p)\beta_2 - (n\alpha_1 + 1)^2\lambda^2}\right]^2 . \qquad (3.78)$$

The quantities c_1, c_2, and c_3 are defined as

$$c_1 = K(\lambda^2) - \frac{1}{\lambda^2} , \qquad (3.79)$$

$$c_2 = \frac{c_3}{\lambda^2} - \Omega_6 , \qquad (3.80)$$

$$c_3 = \left(\frac{n}{\rho+p}\right)^2 \beta_1 + \frac{1}{\rho+p} \quad . \tag{3.81}$$

When the hyperbolicity conditions are satisfied, $d\Omega_3/d(\lambda^2) \leq 0$ and $d\Omega_4/d(\lambda^2) \leq 0$ since c_2 and c_3 are nonnegative in this case. It follows that the minima of $\Omega_3(\lambda^2)$ and $\Omega_4(\lambda^2)$ in the relevant range of λ^2 occur at $\lambda^2 = 1$. So $\Omega_3(\lambda^2)$ is positive because $\Omega_3(1)$ is positive. The positivity of $\Omega_4(1)$ is seen from Eqs. (3.65) and (3.76) and the positivity of $\Omega_4(0)$:

$$\Omega_4(0) = \frac{n^2(\rho+p)\beta_1}{n^2\beta_1 + \rho + p} > 0 \quad . \tag{3.82}$$

Therefore $\Omega_4(\lambda^2)$ is positive also. Thus each of the $\Omega_A(\lambda^2)$ are positive when the causality and hyperbolicity conditions are satisfied.

Conclusions

I have shown in this chapter that satisfaction of each of the stability conditions (Eq. (3.34)) is a sufficient condition for the stability of the equilibria of the Israel-Stewart energy frame theory for small perturbations. I also argued that satisfaction of each of the stability conditions may also be a necessary condition for the stability of all equilibrium states which have a real adiabatic sound speed and are stable against convection. Thus, just as for the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983), the equilibria of the energy frame theory are stable for small perturbations if the equilibrium state satisfies a certain set of inequalities.

I have also proven in this chapter that small perturbations in the Israel-Stewart energy frame theory propagate causally and obey hyperbolic differential equations if and only if the equilibrium state satisfies the stability conditions (Eq. (3.34)). Thus, just as in the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983), the stability conditions are equivalent to the causality and hyperbolicity conditions.

CHAPTER FOUR

THE ENERGY FRAME THEORY IN THE NONLINEAR REGIME

Introduction

The stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory far from equilibrium are studied in this chapter. Unfortunately, it does not seem feasible at present to perform a stability analysis that would apply generically to any fluid described by the full nonlinear theory, as was done for the linearized theory in Ch. 3, because of the complexity of the theory outside the linear regime. However information about the nonlinear properties of the theory can still be obtained by considering a specific example; in this chapter I examine the plane symmetric motions of an inviscid Boltzmann gas at ultrarelativistic temperatures ($kT \gg m$, where m is the rest mass of a fluid particle).

The Israel-Stewart theories are constructed by assuming the entropy current is given by an expansion in the deviations from equilibrium which is truncated after the quadratic terms. Thus it is not reasonable to expect any of the these theories to correctly describe the dynamics of a fluid which is extremely far from equilibrium. Also, if the deviations from equilibrium are too large, the fundamental variables of the theory will not have the properties associated with classical fluids. Specifically, an observer with four-velocity ξ^a would measure the energy density of the fluid to be

$$\rho_\xi = T^{ab}\xi_a\xi_b \quad . \tag{4.1}$$

The stress-energy tensor T^{ab} would violate the weak energy condition (Hawking and Ellis 1973) if this observer could measure a negative fluid energy density ($\rho_\xi < 0$). Equation (2.35) can be used to show that an inviscid fluid described by the Eckart frame theory will violate the weak energy condition if

$$\rho + pv^2 - 2|q|v_q < 0 \quad . \tag{4.2}$$

In Eq. (4.2), ρ is the fluid energy density and p is the pressure, both measured in the rest frame of the fluid, $|q|$ is the magnitude of the heat flux q^a ($q^2 = q^a q_a$), and v_q is the component of the velocity v of the observer relative to the fluid rest frame in the direction of the heat flux. The tightest constraint is obtained by setting $v_q \to 1$ (and so also $v \to 1$):

$$\frac{|q|}{\rho+p} > \frac{1}{2} \quad . \tag{4.3}$$

The weak energy condition will be violated in the Israel-Stewart Eckart frame theory for inviscid fluids having heat fluxes which are large enough to satisfy Eq. (4.3). For the inviscid ultrarelativistic Boltzmann gas, which has the equation of state

$$p = \frac{1}{3}\rho = nkT \quad , \tag{4.4}$$

the weak energy condition is violated in the Eckart frame theory when $|q|/\rho > 2/3$.

Hiscock and Lindblom (1988a) have also studied the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas, but in the context of the Israel-Stewart Eckart frame theory. They found that the Eckart frame theory is causal and hyperbolic only when $|q|/\rho > 0.1303$. The theory is also unstable when $|q|/\rho > 0.5597$ in that if the heat flux is this large, the fluid will evolve away from equilibrium rather than towards it. (These values are corrections to those found by Hiscock and Lindblom (1988a). The values they published were obtained using an incorrect value for the second-order coefficient β_1.) Thus the Eckart frame theory breaks down well before the weak energy condition is violated, at least for this particular fluid.

For an inviscid fluid described by the Israel-Stewart energy frame theory, Eq. (2.5) can be used to show that an observer with four-velocity ξ^a would measure the fluid energy density to be

$$\rho_\xi = \frac{\rho + pv^2}{1 - v^2} \quad . \tag{4.5}$$