



Stability, causality, and shock waves in the Israel-Stewart theory of relativistic dissipative fluids  
by Timothy Scott Olson

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in  
Physics

Montana State University

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Abstract:

The stability, causality, and hyperbolicity properties were analyzed for the Israel-Stewart theory of relativistic dissipative fluids formulated in the energy frame. The equilibria of the theory which are stable for small perturbations were found by constructing a Liapunov functional. The conditions which guarantee that small perturbations about equilibrium will propagate with velocities less than the speed of light and will obey a system of hyperbolic differential equations were determined by calculating the characteristic velocities. It was shown that the stability conditions are equivalent to the causality and hyperbolicity conditions. The behavior of the theory far from equilibrium was studied by considering the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas. The theory was shown to be hyperbolic for large deviations from equilibrium, and acausality implies instability in this example.

The plane steady shock wave solutions were also studied for the Israel-Stewart theory formulated in the Eckart frame. The theory was shown to fail to adequately describe the structure of strong shock waves. Physically acceptable solutions do not exist above a maximum upstream Mach number in any thermally nonconducting and viscous fluid described by the theory because the solutions become multiple-valued when the characteristic velocity is exceeded. It was also proven that physically acceptable solutions do not exist for thermally conducting and viscous fluids above either a maximum upstream Mach number, or else below a minimum downstream Mach number (or both). These limiting Mach numbers again correspond to the characteristic velocities of the fluid. Only extremely weak plane steady shock solutions can be single-valued in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas or for the degenerate free Fermi gas.

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THEORY OF RELATIVISTIC DISSIPATIVE FLUIDS

by

Timothy Scott Olson

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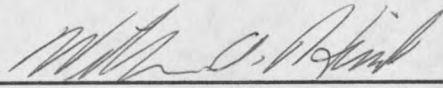
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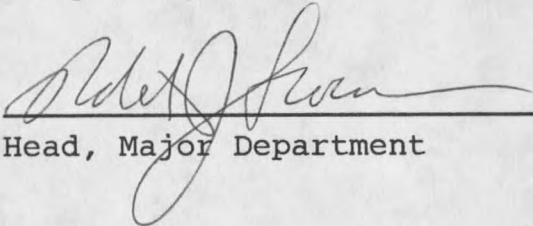
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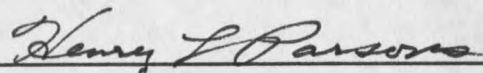
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## ABSTRACT

The stability, causality, and hyperbolicity properties were analyzed for the Israel-Stewart theory of relativistic dissipative fluids formulated in the energy frame. The equilibria of the theory which are stable for small perturbations were found by constructing a Liapunov functional. The conditions which guarantee that small perturbations about equilibrium will propagate with velocities less than the speed of light and will obey a system of hyperbolic differential equations were determined by calculating the characteristic velocities. It was shown that the stability conditions are equivalent to the causality and hyperbolicity conditions. The behavior of the theory far from equilibrium was studied by considering the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas. The theory was shown to be hyperbolic for large deviations from equilibrium, and acausality implies instability in this example.

The plane steady shock wave solutions were also studied for the Israel-Stewart theory formulated in the Eckart frame. The theory was shown to fail to adequately describe the structure of strong shock waves. Physically acceptable solutions do not exist above a maximum upstream Mach number in any thermally nonconducting and viscous fluid described by the theory because the solutions become multiple-valued when the characteristic velocity is exceeded. It was also proven that physically acceptable solutions do not exist for thermally conducting and viscous fluids above either a maximum upstream Mach number, or else below a minimum downstream Mach number (or both). These limiting Mach numbers again correspond to the characteristic velocities of the fluid. Only extremely weak plane steady shock solutions can be single-valued in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas or for the degenerate free Fermi gas.



## CHAPTER ONE

## INTRODUCTION

To make progress in the theoretical understanding of many physical systems, particularly in nuclear physics and relativistic astrophysics, a phenomenological theory of relativistic fluids that includes the effects of dissipation (thermal conductivity, bulk and shear viscosity) is often needed. The collision of two heavy nuclei moving at relativistic velocities (Clare and Strottman 1986), and the collapse of a stellar core which results in a supernova explosion and the formation of a neutron star are two physical systems of this type. These systems typically consist of very many particles, several hundred nucleons in the case of heavy nuclei collisions, of order a solar mass of nucleons for neutron star formation, and consequently it is very hard to follow the evolution of each individual particle of the system even numerically. Further, the matter in these problems can be in a highly excited state, the regime where the details of the fundamental interactions between the particles of the system are often poorly understood.

Because of these difficulties, use of the fluid approximation can be the best approach. Instead of a distribution of discrete particles, the matter is modeled as

a continuous fluid. The hope in this approach is that our inability to solve complicated many-body problems and our ignorance of the details of the interactions between the constituents of the system can be overcome by appealing to basic principles, such as conservation of energy and momentum, the generality of thermodynamics, and concepts of relativity theory.

Unfortunately, the classic theories of relativistic dissipative fluids proposed by Eckart (Eckart 1940; Misner, Thorne, and Wheeler 1973; Weinberg 1972, pp. 53-57) and by Landau and Lifshitz (1959, Sec. 127) are pathological. All the equilibrium states in these theories are unstable, regardless of the choice of equation of state of the fluid (Hiscock and Lindblom 1985). Further, linear perturbations around the equilibrium states can propagate acausally (faster than the speed of light) in these theories, and obey differential equations of a mixed hyperbolic-parabolic-elliptic type (Hiscock and Lindblom 1987). For example, water at room temperature and atmospheric pressure is predicted by the Eckart theory to be unstable and acausal: there exist plane wave perturbations which grow exponentially with a timescale of order  $10^{-35}$  seconds, and there are also plane wave perturbations which propagate at up to  $10^6$  times the speed of light. Clearly, these theories are not physically acceptable.

An alternative theory that is free of these pathologies was developed in the 1970's by Werner Israel and John Stewart (Israel 1976; Stewart 1977; Israel and Stewart 1979a, 1979b) building on earlier nonrelativistic work by Grad (1949) and Müller (1967, 1985). The Israel-Stewart theory has been shown to possess stable equilibrium states, perturbations around equilibrium propagate with velocities less than the speed of light, and obey a hyperbolic system of differential equations so that a well-posed initial value problem exists (Hiscock and Lindblom 1983, 1988b, 1989) if certain conditions on the equation of state and parameters in the theory are satisfied. Further, the conditions which guarantee that an equilibrium state of a fluid will be stable are equivalent to the conditions for perturbations to propagate causally and obey hyperbolic equations. Thus, at least near equilibrium, the Israel-Stewart theory is an acceptable theory of relativistic dissipative fluids.

The theory of Israel and Stewart is actually a collection of theories, each of which define the fluid four-velocity in terms of the fundamental tensors (the stress-energy tensor and particle number current vector) in a different way. One particular choice is to define the four-velocity to be parallel to the particle number current vector (hereafter referred to as the Israel-Stewart Eckart frame theory), as is done in the Eckart theory. It was for this particular choice of four-velocity definition that the conditions that guarantee

stability were first shown to be equivalent to the causality and hyperbolicity conditions near equilibrium (Hiscock and Lindblom 1983). Another natural choice is to define the four-velocity to be the unique timelike eigenvector of the stress-energy tensor (referred to as the Israel-Stewart energy frame theory), as is done in the Landau-Lifshitz theory.

One of the purposes of this dissertation is to study the stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory, both near equilibrium and far from equilibrium, in order to better understand how the predictions of the theory depend on the choice of four-velocity definition.

Chapter 2 is a review of the Israel-Stewart fluid theory. The stability, causality, and hyperbolicity properties of the energy frame theory near equilibrium are examined in Ch. 3 by linearizing the equations of motion of the theory about a fiducial background equilibrium state. The Liapunov functional techniques developed by Hiscock and Lindblom (1983) for the study of the linearized Israel-Stewart Eckart frame theory are employed to study the stability of the equilibria of the linearized energy frame theory. The causality and hyperbolicity properties of the linearized energy frame theory are studied by calculating the characteristic velocities. The stability, causality, and hyperbolicity properties of the energy frame theory far from equilibrium are investigated in

Ch. 4 by examining the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas.

Another purpose of this dissertation is to determine whether the Israel-Stewart theory is incapable of describing strong shock waves in all fluids.

The plane steady shock solutions of the Israel-Stewart Eckart frame theory for the ultrarelativistic Boltzmann gas have been studied by Majorana and Motta (1985). They concluded that physically acceptable plane steady shock solutions are guaranteed not to exist in the Israel-Stewart theory description of the ultrarelativistic Boltzmann gas above a certain upstream Mach number. However, it is unclear from their work how this breakdown depends on the choice of four-velocity definition or on the particular fluid.

The plane steady shock solutions of the Israel-Stewart theory are studied in Ch. 5. That analysis applies to any fluid described by the Israel-Stewart theory.

Chapter 6 is a summary of this dissertation. I also give my opinion of the present state of the theory of relativistic dissipative fluids. Units are chosen so that the speed of light  $c = 1$  throughout.

## CHAPTER TWO

## ISRAEL-STEWART FLUID THEORY

This chapter is a review of the Israel-Stewart theory of relativistic dissipative fluids. I begin with a brief history of the development of the theory.

History of the Israel-Stewart Theory

The Israel-Stewart theory arose out of developments from the kinetic theory of simple gases (Boltzmann gas, free Fermi and Bose gases). Grad (1949) found a method for the solution of the nonrelativistic Boltzmann equation different from the usual Chapman-Enskog approach. In the Grad method the molecular distribution function, which is the argument of the Boltzmann equation, is approximated by truncating its Hermite expansion after the first few terms. These terms are identified as the shear stress and heat flux, and a state of the fluid is defined by assigning values to two thermodynamic variables, the fluid velocity, the heat flux, and the shear stress. The Boltzmann equation then yields a closed system of hyperbolic differential equations for 13 independent fluid variables (there are 14 equations for 14 variables if the fluid has nonzero bulk viscosity). Grad argued that his method is possibly preferable to the Chapman-Enskog approach,

particularly for fluid flows which are not quasi-stationary. Linear perturbations in Grad's theory propagate with speeds of order the adiabatic sound speed.

Grad's approach was extended to relativistic simple gases by Stewart (1969, 1971, 1977), Anderson (1970), Marle (1969), Kranys (1970), and Israel and Stewart (1979a, 1979b). The Grad method predicts that linear perturbations propagate with three distinct characteristic velocities in the arbitrarily relativistic Boltzmann gas, each of which is less than the speed of light, and that these perturbations propagate via hyperbolic differential equations. Olson and Hiscock (1989) proved that the Grad method also predicts the degenerate and arbitrarily relativistic free Fermi gas to be stable for small perturbations, and these perturbations propagate causally and obey a hyperbolic system of differential equations.

By contrast, perturbations can propagate instantaneously in the Navier-Fourier-Stokes theory of nonrelativistic fluids because of the parabolic character of Fourier's law for heat flow (Landau and Lifshitz 1959, Sec. 51). This is disturbing even in a classical theory, since disturbances would be expected to propagate with some mean molecular speed. Müller (1967) found that the parabolic character of the Navier-Fourier-Stokes theory results from the neglect of terms quadratic in the heat flux, bulk stress, and shear stress in the expression for the entropy. By including these quadratic terms, Müller formulated a phenomenological theory of

nonrelativistic fluids which is consistent with the Grad nonrelativistic kinetic theory method. Israel (1976) extended Müller's theory to relativistic fluids. Israel and Stewart (Stewart 1977; Israel and Stewart 1979a, 1979b) showed that Israel's relativistic extension of Müller's phenomenological theory is consistent with the Grad relativistic kinetic theory method.

Hiscock and Lindblom (1983) showed that the Israel-Stewart Eckart frame theory (the Israel-Stewart theory that has the fluid four-velocity parallel to the particle number current) is causal and hyperbolic, not only in the kinetic theory of the simple gases where the equations of the theory reduce to the equations of the Grad method, but also for a far larger class of fluids. They proved that small perturbations about equilibrium will propagate causally and obey a hyperbolic system of differential equations if and only if the equilibrium state satisfies a certain set of inequalities. Further, an equilibrium state is guaranteed to be stable if this same set of inequalities is satisfied. These stability conditions are also useful for developing constraints on proposed models for the equation of state of nuclear matter (Olson and Hiscock 1989). In Ch. 3 (and also in Olson (1990)), I consider the stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory (the Israel-Stewart theory for which the fluid four-



velocity is an eigenvector of the stress-energy tensor) near equilibrium.

In the Israel-Stewart theory, the complete symmetric stress-energy tensor  $T^{ab}$  and the particle number current  $N^a$  are taken to be independent fields. Thus 14 independent fluid variables are necessary to describe the state of a single-component fluid away from equilibrium in the Israel-Stewart theory, rather than the usual 5 independent variables as in the Navier-Fourier-Stokes theory and the pathological Eckart and Landau-Lifshitz theories, and also for perfect fluids.

The Israel-Stewart theory reduces to the familiar dynamics of classical fluids for linear plane wave fluid perturbations which have wavelengths long compared to a typical mean-free-path (Hiscock and Lindblom 1987). In this limit, 9 of the 14 modes of an Israel-Stewart fluid are strongly damped, and the remaining 5 have propagation speeds which are relativistic generalizations of the Navier-Fourier-Stokes results. In other situations the dynamics of all 14 degrees of freedom can be important; the Israel-Stewart theory has been used to construct a one-fluid model of nonrelativistic superfluids that is mathematically equivalent to the Landau two-fluid model (Lindblom and Hiscock 1988).

There is some evidence that far from equilibrium the Israel-Stewart theory is unstable, acausal, and nonhyperbolic. Hiscock and Lindblom (1988a) considered an inviscid ultrarelativistic Boltzmann gas with planar symmetry in the

Israel-Stewart Eckart frame theory, and found that the theory predicts this gas to be unstable for very large deviations from equilibrium, in the sense that the fluid evolves away from equilibrium into the future, and these deviations propagate acausally by nonhyperbolic equations. However, this collapse of the theory only occurs very far from equilibrium, when the magnitude of the heat flux is of order the energy density of the fluid in the example considered, far beyond the regime of interest for realistic physical applications. In Ch. 4 (see also Hiscock and Olson (1989)), I examine the behavior of the inviscid ultrarelativistic Boltzmann gas in the Israel-Stewart energy frame theory far from equilibrium.

It has long been known that the Israel-Stewart theory is incapable of describing strong shock waves in the Boltzmann gas. Grad (1952) showed that for the nonrelativistic Boltzmann gas with zero bulk viscosity, the plane steady shock solutions of his kinetic theory method are multiple-valued, and hence physically unacceptable, above an upstream Mach number of approximately 1.65. Majorana and Motta (1985) found that the plane steady shock solutions of the Grad relativistic kinetic theory method are similarly physically unacceptable above a certain upstream Mach number for the ultrarelativistic Boltzmann gas with zero bulk viscosity. I consider in Ch. 5 (and also in Olson and Hiscock (1990b)) whether this breakdown is peculiar to the Boltzmann gas, or if the Israel-

Stewart theory is incapable of describing strong shock waves in any fluid.

### The Israel-Stewart Energy Frame Theory

The fundamental fields that describe a relativistic dissipative fluid in the Israel-Stewart theory are the particle number current  $N^a$  and the stress-energy tensor  $T^{ab}$ . The fluid four-velocity  $u^a$  is not considered to be a fundamental field; different choices of how the four-velocity is defined in terms of the particle number current and the stress-energy tensor lead to different theories. In the energy frame theory the fluid four-velocity is chosen to be the unique timelike eigenvector of the stress-energy tensor:

$$T^{ab}u_b = -\rho u^a, \quad (2.1)$$

where  $\rho$  is the energy density of the fluid, and the four-velocity is a timelike unit vector ( $u^a u_a = -1$ ). Thus the energy flux is always zero in the rest frame of the fluid. This section is a review of the Israel-Stewart energy frame theory based on the approach used by Olson (1990). The Israel-Stewart Eckart frame theory is reviewed in the next section.

The equations of motion of the fluid include the conservation laws

$$\nabla_a N^a = 0, \quad (2.2)$$

$$\nabla_a T^{ab} = 0 \quad , \quad (2.3)$$

where  $\nabla_a$  is the covariant derivative compatible with the metric tensor  $g^{ab}$ . I will only consider single-component fluids. The generalization to multiple, interacting components is straightforward (Israel 1976). The particle number current and the stress-energy tensor can then be written as

$$N^a = nu^a + v^a \quad , \quad (2.4)$$

$$T^{ab} = \rho u^a u^b + (p+\tau)q^{ab} + \tau^{ab} \quad , \quad (2.5)$$

where  $n$  is the number density of particles,  $p$  is the pressure, and  $q^{ab}$  is the projection tensor orthogonal to the fluid four-velocity ( $q^{ab}u_b = 0$ ):

$$q^{ab} = u^a u^b + g^{ab} \quad . \quad (2.6)$$

The decompositions in Eqs. (2.4) and (2.5) will be unique provided these fields are constrained as follows:

$$u_a v^a = u_a \tau^{ab} = \tau^a_a = \tau^{ab} - \tau^{ba} = 0 \quad . \quad (2.7)$$

Deviations from local equilibrium in the fluid are described by the three fields  $\tau$ ,  $v^a$ , and  $\tau^{ab}$ . The scalar field  $\tau$  can be interpreted as the bulk stress of the fluid,

$v^a$  is the particle diffusion current, and  $\tau^{ab}$  is the trace free shear stress tensor.

The value of the entropy per particle  $s$  is determined by the values of the energy density and the number density and the equation of state of the fluid:

$$s = s(\rho, n) \quad . \quad (2.8)$$

All other thermodynamic variables are then defined by the first law of thermodynamics,

$$ds = \frac{1}{Tn} d\rho - \frac{\rho+p}{Tn^2} dn \quad . \quad (2.9)$$

The pressure  $p$  is

$$p = -\rho - Tn^2 \left( \frac{\partial s}{\partial n} \right)_{\rho} \quad , \quad (2.10)$$

and the temperature  $T$  is

$$T = \frac{1}{n} \left( \frac{\partial s}{\partial \rho} \right)_n^{-1} \quad . \quad (2.11)$$

The theory is completed by implementing the second law of thermodynamics. The total entropy  $S(\Sigma)$  associated with a spacelike surface  $\Sigma$  is found by integrating the entropy current vector field  $s^a$  over this surface:

$$S(\Sigma) = \int_{\Sigma} s^a d\Sigma_a \quad . \quad (2.12)$$

The second law requires  $S(\Sigma)$  to be nondecreasing into the future, hence

$$S(\Sigma') - S(\Sigma) = \int \nabla_a s^a dV \geq 0 \quad , \quad (2.13)$$

for every spacelike surface  $\Sigma'$  to the future of  $\Sigma$ . In Eq. (2.13) the two surface integrals have been converted into a volume integral by Gauss's theorem. Equation (2.13) will always be satisfied provided

$$\nabla_a s^a \geq 0 \quad . \quad (2.14)$$

The theory of Landau and Lifshitz (1959, Sec. 127) now results if the entropy current is modeled by

$$s_{LL}^a = s n u^a - \theta v^a \quad . \quad (2.15)$$

The better behaved theory of Israel and Stewart models the entropy current by an expansion including all possible terms through quadratic order in the deviations from equilibrium:

$$s_{IS}^a = s n u^a - \theta v^a - \frac{1}{2} \left( \beta_0 \tau^2 + \beta_1 v^b v_b + \beta_2 \tau^{bc} \tau_{bc} \right) \frac{u^a}{T} \\ + \alpha_0 \tau \frac{v^a}{T} + \alpha_1 \tau^a_b \frac{v^b}{T} \quad . \quad (2.16)$$

The "second-order coefficients"  $\alpha_i$  and  $\beta_i$  are additional thermodynamic functions which are necessary to completely specify the evolution of the fluid. The  $\beta_i$ , which are

essentially relaxation times, describe the deviation of the physical entropy density from the thermodynamic entropy density, and the  $\alpha_i$  determine the magnitude of the particle diffusion - viscous couplings. The functional forms of these coefficients can in principle be extracted from experiments, or can be calculated from microscopic theory. Relativistic kinetic theory can be used to show that they are nonzero for simple gases (Israel and Stewart 1979b; Olson and Hiscock 1989).

The divergence of the entropy current in Eq. (2.16) can be computed and simplified with the aid of the conservation equations (Eqs. (2.2) and (2.3)). The result is

$$\begin{aligned}
T\nabla_a s^a = & T\left(\frac{\rho+p}{Tn} - s - \theta\right)\nabla_a v^a - \tau\left[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a v^a\right. \\
& + \frac{1}{2}T\tau\nabla_a\left(\beta_0 \frac{u^a}{T}\right) - \gamma_0 T v^a \nabla_a\left(\frac{\alpha_0}{T}\right)\left. - v^a\left[T\nabla_a \theta + \beta_1 u^b \nabla_b v_a\right.\right. \\
& - \alpha_0 \nabla_a \tau - \alpha_1 \nabla_b \tau_a^b + \frac{1}{2}T v_a \nabla_b\left(\beta_1 \frac{u^b}{T}\right) - (1-\gamma_0)T\tau\nabla_a\left(\frac{\alpha_0}{T}\right) \\
& \left. - (1-\gamma_1)T\tau_a^b \nabla_b\left(\frac{\alpha_1}{T}\right)\right] - \tau^{ab}\langle\nabla_a u_b + \beta_2 u^c \nabla_c \tau_{ab} - \alpha_1 \nabla_a v_b \\
& + \frac{1}{2}T\tau_{ab} \nabla_c\left(\beta_2 \frac{u^c}{T}\right) - \gamma_1 T v_a \nabla_b\left(\frac{\alpha_1}{T}\right)\rangle. \tag{2.17}
\end{aligned}$$

The brackets  $\langle \rangle$  which appear in Eq. (2.17) have the meaning:

$$\langle A_{ab} \rangle = \frac{1}{2} q_a^c q_b^d \left( A_{cd} + A_{dc} - \frac{2}{3} q_{cd} q^{ef} A_{ef} \right) , \quad (2.18)$$

for any second rank tensor  $A_{ab}$ . The new coefficients  $\gamma_0$  and  $\gamma_1$  appear because of the ambiguity in factoring the cross terms in Eq. (2.17) which involve  $\tau v^a \nabla_a (\alpha_0/T)$  and  $v^a \tau_a^b \nabla_b (\alpha_1/T)$ . The simplest way to guarantee that Eq. (2.17) is consistent with the second law of thermodynamics (Eq. (2.14)) is to require that

$$\theta = \frac{\rho+p}{Tn} - s , \quad (2.19)$$

$$\begin{aligned} \tau = -\zeta \left[ \nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a v^a \right. \\ \left. + \frac{1}{2} T \tau \nabla_a \left( \beta_0 \frac{u^a}{T} \right) - \gamma_0 T v^a \nabla_a \left( \frac{\alpha_0}{T} \right) \right] , \end{aligned} \quad (2.20)$$

$$\begin{aligned} v^a = -\sigma T q^{ab} \left[ T \nabla_b \theta + \beta_1 u^c \nabla_c v_b - \alpha_0 \nabla_b \tau - \alpha_1 \nabla_c \tau_b^c \right. \\ \left. + \frac{1}{2} T v_b \nabla_c \left( \beta_1 \frac{u^c}{T} \right) - (1-\gamma_0) T \tau \nabla_b \left( \frac{\alpha_0}{T} \right) \right. \\ \left. - (1-\gamma_1) T \tau_b^c \nabla_c \left( \frac{\alpha_1}{T} \right) + \gamma_2 v^c \nabla [b u_c] \right] , \end{aligned} \quad (2.21)$$

$$\begin{aligned} \tau^{ab} = -2\eta \langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a v^b + \frac{1}{2} T \tau^{ab} \nabla_c \left( \beta_2 \frac{u^c}{T} \right) \\ - \gamma_1 T v^a \nabla^b \left( \frac{\alpha_1}{T} \right) + \gamma_3 \tau_b^c \nabla [a u^c] \rangle . \end{aligned} \quad (2.22)$$



With these definitions the divergence of the entropy current takes the form

$$\nabla_a s^a = \frac{\tau}{\zeta T} + \frac{v^a v_a}{\sigma T^2} + \frac{\tau^{ab} \tau_{ab}}{2\eta T} \quad , \quad (2.23)$$

and is insured to be nonnegative provided that the three thermodynamic functions  $\zeta$ ,  $\sigma$ , and  $\eta$  are required to be positive. By taking the Newtonian limit of the theory, these functions may be identified as the bulk viscosity  $\zeta$ , the particle diffusion coefficient  $\sigma$  ( $\sigma(\rho+p)^2/n^2$  may be identified as the thermal conductivity), and the shear viscosity  $\eta$ . The terms multiplying  $\gamma_2$  and  $\gamma_3$  in Eqs. (2.21) and (2.22), which couple the particle diffusion vector and the shear stress tensor to the vorticity of the fluid, are the only additional terms which can be added to Eqs. (2.20)-(2.22) which agree with the results of kinetic theory and do not change the entropy generation equation, Eq. (2.17) (Hiscock and Lindblom 1983). The expressions for  $\tau$ ,  $v^a$ , and  $\tau^{ab}$  in the pathological Landau-Lifshitz theory (Landau and Lifshitz 1959, Sec. 127) result if all the coefficients  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$  are set equal to zero.

Equations (2.2), (2.3), and (2.20)-(2.22) form the complete set of equations of motion for the Israel-Stewart energy frame theory. The theory has 14 degrees of freedom, which can be chosen to be  $\tau$ , the three independent components of  $v^a$ , the five independent components of  $\tau^{ab}$ , the three

independent components of  $u^a$ , and two thermodynamic variables. There are 14 independent variables, rather than 5 as in the Navier-Fourier-Stokes or Landau-Lifshitz theories, because the dissipation fields  $\tau$ ,  $v^a$ , and  $\tau^{ab}$  obey differential equations instead of algebraic equations. The evolution of the fluid is completely determined provided that the equation of state, the dissipation coefficients  $\zeta$ ,  $\sigma$ ,  $\eta$ , and the second-order coefficients  $\alpha_i$ ,  $\beta_i$ ,  $\gamma_i$  are known thermodynamic functions. Gravitational interactions may be included by adding Einstein's equations,

$$G^{ab} = 8\pi T^{ab} \quad , \quad (2.24)$$

to this set of equations.

Equilibrium states of the theory are those for which the divergence of the entropy current vanishes. By Eq. (2.23), this implies that in equilibrium

$$\tau = v^a = \tau^{ab} = 0 \quad . \quad (2.25)$$

These conditions and the equations of motion (Eqs. (2.2), (2.3), and (2.20)-(2.22)) yield the following additional conditions for an equilibrium state:

$$\nabla_a u^a = 0 \quad , \quad (2.26)$$

$$q^{ab} \nabla_b \theta = 0 \quad , \quad (2.27)$$

$$\langle \nabla_a u_b \rangle = 0 \quad , \quad (2.28)$$

$$u^a \nabla_a n = 0 \quad , \quad (2.29)$$

$$u^a \nabla_a \rho = 0 \quad , \quad (2.30)$$

$$q^{ab} [\nabla_b p + (\rho+p) u^c \nabla_c u_b] = 0 \quad . \quad (2.31)$$

Equations (2.29) and (2.30) imply that all other thermodynamic variables must also be constant along the fluid flow lines because they depend on just two independent thermodynamic variables. These are the same equilibrium states as in the Landau-Lifshitz theory. The stability of these equilibrium states for small perturbations is studied in Ch. 3, and for large perturbations for the particular case of the ultrarelativistic Boltzmann gas in Ch. 4.

The notation used in this section differs slightly from that used originally by Israel (1976). The equations of motion for the Israel-Stewart energy frame theory in the notation of Israel (1976) result if the following substitutions are made in Eqs. (2.4) and (2.20)-(2.22):  $v^a \rightarrow -nq^a/(\rho+p)$ ,  $\sigma \rightarrow kn^2/(\rho+p)^2$ ,  $\beta_1 \rightarrow \beta_1(\rho+p)^2/n^2$ ,  $\alpha_i \rightarrow -\alpha_i(\rho+p)/n$ .

#### The Israel-Stewart Eckart Frame Theory

The only other commonly used Israel-Stewart theory is the Eckart frame theory. The plane steady shock solutions of the Eckart frame theory are studied in Ch. 5. The review of

the Eckart frame theory presented in this section is based on the approach used by Hiscock and Lindblom (1983, 1988b, 1989).

In the Israel-Stewart Eckart frame theory, just as in the energy frame theory, the fluid is described by a conserved particle number current and a conserved stress-energy tensor,

$$\nabla_a N^a = 0 \quad , \quad (2.32)$$

$$\nabla_a T^{ab} = 0 \quad , \quad (2.33)$$

but the fluid four-velocity is chosen so that the particle flux is always zero in the rest frame of the fluid,

$$N^a = n u^a \quad , \quad (2.34)$$

and the stress-energy tensor has the form

$$T^{ab} = \rho u^a u^b + (p+\tau)q^{ab} + q^a u^b + q^b u^a + \tau^{ab} \quad . \quad (2.35)$$

Deviations from local equilibrium are described by the bulk stress  $\tau$ , the heat flux  $q^a$ , and the shear stress  $\tau^{ab}$ , which are constrained to satisfy

$$q^a u_a = \tau^{ab} u_b = \tau^a_a = \tau_{ab} - \tau_{ba} = 0 \quad . \quad (2.36)$$

The equation of state of the fluid,  $s = s(\rho, n)$ , and the first law of thermodynamics (Eq. (2.9)), define the pressure  $p$  and the temperature  $T$  by Eqs. (2.10) and (2.11).

In the pathological Eckart (1940) theory, the entropy current is modeled as

$$s_E^a = s n u^a + \beta q^a \quad , \quad (2.37)$$

while in the better behaved Israel-Stewart Eckart frame theory, the entropy current is modeled by an expansion including all possible terms up to quadratic order in deviations from equilibrium,

$$\begin{aligned} s_{IS}^a = & s n u^a + \beta q^a - \frac{1}{2} \left( \beta_0 \tau^2 + \beta_1 q^b q_b + \beta_2 \tau^{bc} \tau_{bc} \right) \frac{u^a}{T} \\ & + \alpha_0 \tau \frac{q^a}{T} + \alpha_1 \tau^a_b \frac{q^b}{T} \quad . \end{aligned} \quad (2.38)$$

The divergence of the Eckart frame entropy current (Eq. (2.38)) is manifestly nonnegative,

$$\nabla_a s^a = \frac{\tau^2}{\zeta T} + \frac{q^a q_a}{\kappa T^2} + \frac{\tau^{ab} \tau_{ab}}{2\eta T} \quad , \quad (2.39)$$

provided the bulk viscosity  $\zeta$ , the thermal conductivity  $\kappa$ , and the shear viscosity  $\eta$  are positive thermodynamic functions, and if  $\beta$ ,  $\tau$ ,  $q^a$ , and  $\tau^{ab}$  are defined by

$$\beta = \frac{1}{T} \quad , \quad (2.40)$$

$$\begin{aligned} \tau = -\zeta \left[ \nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a \right. \\ \left. - \gamma_0 T q^a \nabla_a \left( \frac{\alpha_0}{T} \right) + \frac{1}{2} T \tau \nabla_a \left( \beta_0 \frac{u^a}{T} \right) \right] \end{aligned} \quad (2.41)$$

$$\begin{aligned} q^a = -\kappa T q^{ab} \left[ \frac{1}{T} \nabla_b T + u^c \nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau \right. \\ \left. - \alpha_1 \nabla_c \tau^c_b + \frac{1}{2} T q_b \nabla_c \left( \beta_1 \frac{u^c}{T} \right) - (1-\gamma_0) T \tau \nabla_b \left( \frac{\alpha_0}{T} \right) \right. \\ \left. - (1-\gamma_1) T \tau_b^c \nabla_c \left( \frac{\alpha_1}{T} \right) + \gamma_2 q^c \nabla_{[b} u_{c]} \right] \end{aligned} \quad (2.42)$$

$$\begin{aligned} \tau^{ab} = -2\eta \langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a q^b + \frac{1}{2} T \tau^{ab} \nabla_c \left( \beta_2 \frac{u^c}{T} \right) \\ \left. - \gamma_1 T q^a \nabla^b \left( \frac{\alpha_1}{T} \right) + \gamma_3 \tau_c^b \nabla^c [a u^c] \rangle \end{aligned} \quad (2.43)$$

Thus, the second law of thermodynamics will be satisfied if  $\beta$ ,  $\tau$ ,  $q^a$ , and  $\tau^{ab}$  are defined by Eqs. (2.40)-(2.43). The equations for  $\tau$ ,  $q^a$ , and  $\tau^{ab}$  in the Eckart theory result if each of the  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$  are set equal to zero.

Equations (2.32), (2.33), and (2.41)-(2.43) form the complete set of 14 equations for the 14 independent variables of the Israel-Stewart Eckart frame theory. The evolution of the fluid is completely determined provided that the equation of state, the dissipation coefficients  $\zeta$ ,  $\kappa$ ,  $\eta$ , and the second-order coefficients  $\alpha_i$ ,  $\beta_i$ ,  $\gamma_i$  are known thermodynamic

functions. Gravitational interactions may be included by adding Einstein's equations (Eq. (2.24)).

Equilibrium states of the theory are those for which the divergence of the entropy current vanishes. By Eq. (2.39), this implies that in equilibrium

$$\tau = q^a = \tau^{ab} = 0 \quad . \quad (2.44)$$

These conditions and the equations of motion (Eqs. (2.32), (2.33), and (2.41)-(2.43)) may be used to show that the equilibrium states of the Israel-Stewart Eckart frame theory are exactly the same as the equilibria of the Israel-Stewart energy frame theory (Eqs. (2.26)-(2.31)). So the definitions of the fluid four-velocity in the Israel-Stewart Eckart frame and energy frame theories differ only away from equilibrium. These are also the equilibrium states of the Eckart theory.

## CHAPTER THREE

## THE ENERGY FRAME THEORY IN THE LINEAR REGIME

The Israel-Stewart Eckart frame theory is an acceptable theory of relativistic dissipative fluids near equilibrium because it has stable equilibria, and small perturbations around equilibrium propagate causally and obey hyperbolic differential equations (Hiscock and Lindblom 1983). In this chapter, I investigate whether or not the Israel-Stewart energy frame theory is also a physically acceptable theory near equilibrium.

Linearized Energy Frame Theory

The equations of motion for small perturbations around the equilibrium states of the energy frame theory are derived in this section. These equations will be needed in the following sections to study the stability of the equilibria of the energy frame theory, and to determine under what conditions small perturbations propagate causally via a hyperbolic system of differential equations. The difference between the nonequilibrium value of a field  $Q$  at a given spacetime point and the value of this field at the same spacetime point in the fiducial equilibrium state will be denoted by  $\delta Q$ . Thus the fields  $\delta\rho$ ,  $\delta n$ ,  $\delta u^a$ ,  $\delta\tau$ ,  $\delta v^a$ ,  $\delta\tau^{ab}$ ,



and  $\delta g^{ab}$  describe the perturbations of the fluid about its equilibrium state. A variable without the prefix  $\delta$  (e.g.  $n$ ,  $\rho$ ,  $u^a$ ,  $g^{ab}$ , etc.) refers to the value of that field in the equilibrium state. The fields in the equilibrium state satisfy Eqs. (2.26)-(2.31).

I will assume that all perturbation quantities  $\delta Q$  are small enough that their evolution is described well by the linearized equations of motion. I also assume for simplicity that  $\delta g^{ab} = 0$ . Hence the linearized equations of motion displayed below may only be applied to special-relativistic fluids (but the background geometry need not be flat spacetime), or to short wavelength perturbations of the fluid. The equations of motion (Eqs. (2.2), (2.3), and (2.20)-(2.22)) linearized about an arbitrary equilibrium state are then

$$\nabla_a \delta T^{ab} = 0 \quad , \quad (3.1)$$

$$\begin{aligned} \delta T^{ab} = & (\rho+p)(\delta u^a u^b + u^a \delta u^b) + \delta \rho u^a u^b \\ & + (\delta p + \delta \tau) q^{ab} + \delta \tau^{ab} \quad , \end{aligned} \quad (3.2)$$

$$u^a \nabla_a \delta n + \nabla_a (n \delta u^a) + \nabla_a \delta v^a = 0 \quad , \quad (3.3)$$

$$\begin{aligned} \delta \tau = & -\zeta \left[ \nabla_a \delta u^a + \beta_0 u^a \nabla_a \delta \tau - \alpha_0 \nabla_a \delta v^a \right. \\ & \left. + \frac{1}{2} T \delta \tau \nabla_a \left( \beta_0 \frac{u^a}{T} \right) - \gamma_0 T \delta v^a \nabla_a \left( \frac{\alpha_0}{T} \right) \right] \quad , \end{aligned} \quad (3.4)$$

$$\begin{aligned}
\delta v^a = & -\sigma T q^{ab} \left[ T \nabla_b \delta \theta + \beta_1 u^c \nabla_c \delta v_b - \alpha_0 \nabla_b \delta \tau - \alpha_1 \nabla_c \delta \tau_b^c \right. \\
& + \frac{1}{2} T \delta v_b \nabla_c \left( \beta_1 \frac{u^c}{T} \right) - (1-\gamma_0) T \delta \tau \nabla_b \left( \frac{\alpha_0}{T} \right) \\
& \left. - (1-\gamma_1) T \delta \tau_b^c \nabla_c \left( \frac{\alpha_1}{T} \right) + \gamma_2 \delta v^c \nabla_{[b} u_{c]} \right] , \quad (3.5)
\end{aligned}$$

$$\begin{aligned}
\delta \tau^{ab} = & -2\eta \langle \nabla^a \delta u^b + \delta u^a u^c \nabla_c u^b + \beta_2 u^c \nabla_c \delta \tau^{ab} - \alpha_1 \nabla^a \delta v^b \\
& + \frac{1}{2} T \delta \tau^{ab} \nabla_c \left( \beta_2 \frac{u^c}{T} \right) - \gamma_1 T \delta v^a \nabla^b \left( \frac{\alpha_1}{T} \right) + \gamma_3 \delta \tau_b^c \nabla_{[a} u_{c]} \rangle . \quad (3.6)
\end{aligned}$$

The derivative  $\nabla_a$  in Eqs. (3.1)-(3.6) is the covariant derivative compatible with the background spacetime metric  $g^{ab}$ . The projection tensor appearing in the definition of the brackets  $\langle \rangle$  (Eq. (2.18)) is the unperturbed projection tensor  $q^{ab}$ . The perturbation variables satisfy the constraints

$$u^a \delta v_a = u^a \delta \tau_{ab} = \delta \tau^a_a = \delta \tau^{ab} - \delta \tau^{ba} = u^a \delta u_a = 0 \quad (3.7)$$

because of the constraints in Eq. (2.7).

### The Stability Conditions

The stability of the equilibrium states of the Israel-Stewart energy frame theory for small perturbations are studied in this section by using the Liapunov functional techniques developed by Hiscock and Lindblom (1983) for the

analysis of the stability of the equilibria of the linearized Israel-Stewart Eckart frame theory. A Liapunov functional, a nonincreasing function of time which depends quadratically on the perturbation fields, is constructed. When this functional is nonnegative for all possible values of the perturbation fields the equilibrium state is stable, since in this case the functional is bounded below by zero. If the functional is negative for some values of the perturbation fields, then some of the perturbation fields will evolve towards infinity because the functional is unbounded below in this case, implying that the equilibrium state is unstable. This reasoning will be made rigorous in what follows.

A Liapunov functional can be constructed for the Israel-Stewart energy frame theory from an "energy" current  $E^a$  defined by

$$\begin{aligned}
 TE^a = & \delta T^a_b \delta u^b - \frac{1}{2}(\rho+p)u^a \delta u^b \delta u_b + T\delta\theta\delta v^a - \alpha_0 \delta\tau\delta v^a - \alpha_1 \delta\tau^a_b \delta v^b \\
 & + \frac{1}{2}(\rho+p)^{-1} \left[ \left( \frac{\partial\rho}{\partial p} \right)_s (\delta p)^2 + \left( \frac{\partial\rho}{\partial s} \right)_p \left( \frac{\partial p}{\partial s} \right)_\theta (\delta s)^2 \right] u^a \\
 & + \frac{1}{2} \left[ \beta_0 (\delta\tau)^2 + \beta_1 \delta v^b \delta v_b + \beta_2 \delta\tau^{bc} \delta\tau_{bc} \right] u^a \quad . \quad (3.8)
 \end{aligned}$$

The total energy  $E(\Sigma)$  associated with a spacelike surface  $\Sigma$  is

$$E(\Sigma) = \int_{\Sigma} E^a d\Sigma_a \quad (3.9)$$

The Liapunov functional  $E(\Sigma)$  will be nonincreasing into the future provided

$$E(\Sigma') - E(\Sigma) = \int \nabla_a E^a dV \leq 0 \quad , \quad (3.10)$$

for every spacelike surface  $\Sigma'$  to the future of  $\Sigma$ . Thus the Liapunov functional will be a nonincreasing function of time as long as the divergence of the energy current is not positive. The divergence of  $E^a$  in Eq. (3.8) may be calculated with the aid of the linearized equations (Eqs. (3.1)-(3.6)). The result is

$$\nabla_a E^a = - \left[ \frac{(\delta\tau)^2}{\zeta T} + \frac{\delta v^a \delta v_a}{\sigma T^2} + \frac{\delta\tau^{ab} \delta\tau_{ab}}{2\eta T} \right] \leq 0 \quad . \quad (3.11)$$

As required, this Liapunov functional is a nonincreasing function of time.

This energy current (Eq. (3.8)) was found by modifying the expression for the energy current of the Israel-Stewart Eckart frame theory found by Hiscock and Lindblom (1983). The structure of the two theories is similar enough that it is easy to see how to modify their expression to obtain an appropriate energy current for the Israel-Stewart energy frame theory.

It is useful to express the Liapunov functional in terms of an energy density:

$$E(\Sigma) = \int e \frac{u^a}{T} d^3x_a \quad . \quad (3.12)$$

The energy density  $e$  is defined by

$$e = TE^a t_a / u^b t_b \quad , \quad (3.13)$$

where the vector  $t^a$  is the future-directed unit normal to the spacelike surface  $\Sigma$ . The Liapunov functional  $E(\Sigma)$  will be nonnegative for all possible values of the perturbation fields if and only if the energy density  $e$  is a nonnegative functional of the perturbation fields at all points in the fluid.

It is possible to factor the energy density into the form

$$e = \frac{1}{2} \sum_A \Omega_A (\delta Z_A)^2 \quad . \quad (3.14)$$

The  $\Omega_A$  are functions of the thermodynamic variables of the equilibrium state given by

$$\Omega_1 = (\rho+p)^{-1} \left( \frac{\partial \rho}{\partial p} \right)_s \quad , \quad (3.15)$$

$$\Omega_2 = (\rho+p)^{-1} \left( \frac{\partial \rho}{\partial s} \right)_p \left( \frac{\partial p}{\partial s} \right)_e \quad , \quad (3.16)$$

$$\Omega_3 = (\rho+p) \left[ 1 - \lambda^2 \left( \frac{\partial p}{\partial \rho} \right)_s \right] - \left( \frac{1}{\beta_0} + \frac{2}{3\beta_2} + \frac{\kappa^2}{\Omega_6} \right) \lambda^2, \quad (3.17)$$

$$\Omega_4 = \rho+p - \frac{2(\rho+p)^2 \beta_2 + [n^2 \beta_1 - 2n(\rho+p)\alpha_1 - (\rho+p)] \lambda^2}{2(n^2 \beta_1 + \rho+p) \beta_2 - (n\alpha_1 + 1)^2 \lambda^2}, \quad (3.18)$$

$$\Omega_5 = \beta_0, \quad (3.19)$$

$$\Omega_6 = \frac{1}{(\rho+p)^2} \left[ \frac{n^2 \beta_1 + \rho+p}{\lambda^2} - \frac{(n\alpha_0 + 1)^2}{\beta_0} - \frac{2(n\alpha_1 + 1)^2}{3\beta_2} - \frac{1}{n} \left( \frac{\rho+p}{T} \right)^2 \left( \frac{\partial T}{\partial s} \right)_n \right], \quad (3.20)$$

$$\Omega_7 = \frac{1}{(\rho+p)^2} \left[ n^2 \beta_1 + \rho + p - \frac{(n\alpha_1 + 1)^2 \lambda^2}{2\beta_2} \right], \quad (3.21)$$

$$\Omega_8 = \beta_2. \quad (3.22)$$

The  $\delta Z_A$  are linearly independent combinations of the perturbation fields:

$$\delta Z_1 = \delta p + (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \left[ \lambda_a \delta u^a + \left( \frac{1}{n} - \frac{\rho+p}{Tn} \left( \frac{\partial T}{\partial p} \right)_s \right) \lambda_a \delta v^a \right], \quad (3.23)$$

$$\delta Z_2 = \delta s - \frac{(\rho+p)^2}{Tn} \left( \frac{\partial T}{\partial \rho} \right)_p \left( \frac{\partial s}{\partial p} \right)_\theta \lambda_a \delta v^a, \quad (3.24)$$

$$\delta Z_3 = \frac{1}{\lambda} \left( \lambda_a \delta u^a + \frac{1}{n} \lambda_a \delta v^a \right), \quad (3.25)$$

$$\delta Z_4^a = \gamma^a_b \delta u^b + \frac{1}{n} \gamma^a_b \delta v^b, \quad (3.26)$$

$$\delta Z_5 = \delta \tau + \frac{1}{\beta_0} \lambda_a \delta u^a - \frac{\alpha_0}{\beta_0} \lambda_a \delta v^a, \quad (3.27)$$

$$\delta Z_6 = - \frac{\rho+p}{n} \lambda_a \delta v^a + \frac{K}{\Omega_6} \left( \lambda_a \delta u^a + \frac{1}{n} \lambda_a \delta v^a \right), \quad (3.28)$$

$$\begin{aligned} \delta Z_7^a = & \frac{2(\rho+p)^2 \beta_2 - (\rho+p)(n\alpha_1+1)\lambda^2}{2(n^2 \beta_1 + \rho+p)\beta_2 - (n\alpha_1+1)^2 \lambda^2} \left( \delta u^b + \frac{1}{n} \delta v^b \right) \gamma^a_b \\ & - \frac{\rho+p}{n} \gamma^a_b \delta v^b, \end{aligned} \quad (3.29)$$

$$\delta Z_8^{ab} = \delta \tau^{ab} + \frac{1}{\beta_2} \langle \lambda^a \delta u^b \rangle - \frac{\alpha_1}{\beta_2} \langle \lambda^a \delta v^b \rangle. \quad (3.30)$$

In these equations  $\lambda^a$  is the velocity of observers moving along  $t^a$  relative to the rest frame of the background equilibrium state,

$$\lambda^a = q^{ab} t_b / u_c t^c, \quad (3.31)$$

$\lambda$  is the norm of  $\lambda^a$  (which can be shown to be bounded between zero and one), and  $K$  is defined as

$$K = \frac{1}{\lambda^2} - \frac{1}{\rho+p} \left[ \frac{n\alpha_0+1}{\beta_0} + \frac{2(n\alpha_1+1)}{3\beta_2} + \frac{n(\rho+p)}{T} \left( \frac{\partial T}{\partial n} \right)_s \right], \quad (3.32)$$

and the projection tensor  $\gamma^{ab}$  is given by

$$\gamma^{ab} = g^{ab} - \frac{1}{\lambda^2} \lambda^{a\lambda} \lambda^{\lambda b} \quad (3.33)$$

The requirement that the Liapunov functional be nonnegative is then equivalent to the condition that each of the  $\Omega_A$  be nonnegative,

$$\Omega_A \geq 0 \quad (3.34)$$

So Eq. (3.34) must be satisfied for all values of  $\lambda$  between zero and one (the speed of light) for the equilibrium state to be stable. The most restrictive conditions are obtained by setting  $\lambda = 1$ .

The stability conditions (Eq. (3.34)) have several interesting consequences. Because of the  $\Omega_3$ ,  $\Omega_5$ , and  $\Omega_8$  conditions,  $\beta_0$  and  $(3/2)\beta_2$  are nonnegative and bounded below by  $(\rho+p)^{-1}$ . The  $\Omega_4$  and  $\Omega_7$  conditions require  $\beta_1$  to be nonnegative. The square of the adiabatic sound speed  $(\partial p/\partial \rho)_s$  is bounded between zero and the speed of light because  $\Omega_1$  and  $\Omega_3$  are nonnegative. This does not imply that all perturbations of stable equilibrium states propagate subluminally; perturbations need not propagate with the adiabatic sound speed in a fluid theory having dissipation. The  $\Omega_2$  condition is the relativistic Schwarzschild criterion for stability against convection (Thorne 1967).

The connection between the sign of the energy functional and stability is established by the following theorems. Theorem A shows that a nonnegative energy functional is a



sufficient condition for stability.

Theorem A. If the stability conditions (Eq. (3.34)) are satisfied, then none of the perturbation variables will grow without bound (as measured by a square integral norm).

The proof of Theorem A is identical to the analogous theorem proven for the Israel-Stewart Eckart frame theory by Hiscock and Lindblom (1983, p. 494).

The following theorem falls somewhat short of showing that satisfaction of each of the conditions  $\Omega_A > 0$  is also a necessary condition for stability.

Theorem B. Suppose that the inequalities

$$0 < \left( \frac{\partial p}{\partial \rho} \right)_s < 1 \quad (3.35)$$

and

$$\left( \frac{\partial \rho}{\partial s} \right)_p \left( \frac{\partial p}{\partial s} \right)_\theta > 0 \quad (3.36)$$

are satisfied, and at least one of the stability conditions is violated. Suppose also that there do not exist solutions to the perturbation equations for which the square integral norms of the perturbation variables  $\delta\tau$ ,  $\delta v^a$ , and  $\delta\tau^{ab}$  evolve to zero, but these variables do not evolve to zero everywhere. Finally, assume in addition that the perturbation equations

only yield solutions for which there exists 14 linearly independent perturbation fields. Then there exist solutions to the perturbation equations which grow without bound.

Theorem B can be proven as follows. Assume that no unbounded solutions to the perturbation equations exist. This implies that the integral of the divergence of the energy current (Eq. (3.11)) must also evolve to zero since the Liapunov functional is bounded. It follows that the square integral norms of the fields  $\delta\tau$ ,  $\delta v^a$ , and  $\delta\tau^{ab}$  evolve to zero, and the energy density (Eq. (3.14)) evolves to

$$e = \frac{1}{2(\rho+p)} \left( \frac{\partial \rho}{\partial p} \right)_s \left[ \delta p + (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \lambda_a \delta u^a \right]^2 + \frac{\rho+p}{2} \gamma_{ab} \delta u^a \delta u^b \\ + \frac{\rho+p}{2\lambda^2} \left[ 1 - \lambda^2 \left( \frac{\partial p}{\partial \rho} \right)_s \right] \left( \lambda_a \delta u^a \right)^2 + \frac{1}{2(\rho+p)} \left( \frac{\partial \rho}{\partial s} \right)_p \left( \frac{\partial p}{\partial s} \right)_\theta (\delta s)^2, \quad (3.37)$$

because the perturbation fields  $\delta\tau$ ,  $\delta v^a$ ,  $\delta\tau^{ab}$  evolve to zero everywhere when their square integral norms do. When Eqs. (3.35) and (3.36) are satisfied this energy density is nonnegative. Therefore, the Liapunov functional is also nonnegative, implying that the energy of all possible perturbations is nonnegative because the Liapunov functional is a nonincreasing function of time. Thus all of the stability conditions  $\Omega_A > 0$  are satisfied because there exists 14 linearly independent perturbation fields, which proves Theorem B.

Theorem B fails to be a complete proof, even for the equilibria which have a real and causal adiabatic sound speed and are stable against convection (and so satisfy Eqs. (3.35) and (3.36)), because it has not been shown that the physically interesting solutions to the perturbation equations have 14 independent fluid variables with  $\delta\tau$ ,  $\delta v^a$ , and  $\delta\tau^{ab}$  evolving to zero everywhere when their square integral norms evolve to zero. The analogous theorem for the Eckart frame theory (Hiscock and Lindblom 1983, p. 494) has exactly the same deficiencies (Geroch and Lindblom 1990).

### Causality and Hyperbolicity Conditions

This section contains a derivation of the conditions which guarantee causal propagation of small perturbations around the equilibria of the Israel-Stewart energy frame theory. The conditions for the linearized equations of motion of the energy frame theory to form a symmetric hyperbolic system are also determined.

The system of equations for the linear perturbations (Eqs. (3.1)-(3.6)) has the form

$$A^A_B a^a_{\nu} \delta Y^B + B^A_B \delta Y^B = 0 \quad , \quad (3.38)$$

where the components of  $\delta Y^B$  are the fourteen perturbation fields  $\delta n$ ,  $\delta\rho$ ,  $\delta u^a$ ,  $\delta\tau$ ,  $\delta v^a$ ,  $\delta\tau^{ab}$ , and the index A labels one of the fourteen equations for the perturbation fields. The components of the matrices  $A^A_B a^a_{\nu}$  and  $B^A_B$  are functions of the

variables of the equilibrium state. The level surfaces of a scalar function  $\varphi$  satisfying

$$\det \left( A^A_B \nabla_a \varphi \right) = 0 \quad (3.39)$$

are the characteristic surfaces for these equations; the fields  $\delta Y^B$  cannot be freely specified on these surfaces (Courant and Hilbert 1962, p. 170). The characteristic velocities are the slopes of these surfaces; information propagates along the characteristic surfaces with these velocities.

The characteristic velocities are most easily found by solving Eq. (3.39) with the following choice of coordinates. At some point in the fluid choose a Cartesian coordinate system which is comoving with the fluid and which has the  $x^1$  direction along the direction of spatial variation of  $\varphi$ . Then at this point

$$g^{ab} \partial_a \partial_b = -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 \quad , \quad (3.40)$$

$$u^a \partial_a = \partial_0 \quad , \quad (3.41)$$

and

$$\varphi = \varphi(x^0, x^1) \quad . \quad (3.42)$$

In this coordinate system the equation for the characteristic velocities (Eq. (3.39)) is

$$\det \left( v A_B^{A 0} - A_B^{A 1} \right) = 0 \quad , \quad (3.43)$$

where  $v$  is the characteristic velocity defined by

$$v = - \frac{\partial_0 \varphi}{\partial_1 \varphi} \quad . \quad (3.44)$$

When the set of fourteen perturbation fields is chosen as

$$\delta Y^B = \left( T \delta \theta, \frac{\delta T}{T}, \delta \tau, \delta u^1, \delta v^1, \delta \tau^{11}, \delta u^2, \delta v^2, \delta \tau^{21}, \right. \\ \left. \delta u^3, \delta v^3, \delta \tau^{31}, \delta \tau^{22} - \delta \tau^{33}, \delta \tau^{32} \right) \quad , \quad (3.45)$$

the characteristic matrix block diagonalizes:

$$v A_B^{A 0} - A_B^{A 1} = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & S \end{bmatrix} \quad . \quad (3.46)$$

The matrices  $Q$ ,  $R$ , and  $S$  are defined as

$$Q = \begin{bmatrix} \frac{v}{T} \left( \frac{\partial n}{\partial \theta} \right)_T & \frac{v}{T} \left( \frac{\partial \rho}{\partial \theta} \right)_T & 0 & -n & -1 & 0 \\ \frac{v}{T} \left( \frac{\partial \rho}{\partial \theta} \right)_T & v T \left( \frac{\partial \rho}{\partial T} \right)_\theta & 0 & -(\rho+p) & 0 & 0 \\ 0 & 0 & v \beta_0 & -1 & \alpha_0 & 0 \\ -n & -(\rho+p) & -1 & v(\rho+p) & 0 & -1 \\ -1 & 0 & \alpha_0 & 0 & v \beta_1 & \alpha_1 \\ 0 & 0 & 0 & -1 & \alpha_1 & \frac{3}{2} v \beta_2 \end{bmatrix} \quad , \quad (3.47)$$

$$\mathbf{R} = \begin{bmatrix} v(\rho+p) & 0 & -1 \\ 0 & v\beta_1 & \alpha_1 \\ -1 & \alpha_1 & 2v\beta_2 \end{bmatrix}, \quad (3.48)$$

$$\mathbf{S} = \begin{bmatrix} 2v\beta_2 & 0 \\ 0 & v\beta_2 \end{bmatrix}. \quad (3.49)$$

Thus the equation for the characteristic velocities (Eq. (3.39)) takes the form

$$\det \mathbf{Q} (\det \mathbf{R})^2 \det \mathbf{S} = 0, \quad (3.50)$$

where the determinants are

$$\det \mathbf{Q} = \frac{3}{2}v^2 (Av^4 + Bv^2 + C) \left[ \left( \frac{\partial n}{\partial \theta} \right)_{\mathbf{T}} \left( \frac{\partial \rho}{\partial \mathbf{T}} \right)_{\theta} - \left( \frac{\partial n}{\partial \mathbf{T}} \right)_{\theta} \left( \frac{\partial \rho}{\partial \theta} \right)_{\mathbf{T}} \right], \quad (3.51)$$

$$\det \mathbf{R} = v \left[ 2(\rho+p)\beta_1\beta_2v^2 - (\rho+p)\alpha_1^2 - \beta_1 \right], \quad (3.52)$$

$$\det \mathbf{S} = 2(v\beta_2)^2. \quad (3.53)$$

The coefficients of the quartic in Eq. (3.51) are

$$A = (\rho+p)\beta_0\beta_1\beta_2, \quad (3.54)$$

$$B = -(\rho+p)D - \left[ \left( \frac{n}{\rho+p} \right)^2 \beta_1 + \frac{1}{\rho+p} \right] E - 2F, \quad (3.55)$$

$$c = \left( \frac{n}{\rho+p} \right)^2 \frac{DE - F^2}{\beta_0 \beta_2} \quad (3.56)$$

The functions D, E, and F are defined as

$$D = \frac{\beta_0 \beta_2}{n^2} \left[ \frac{(n\alpha_0 + 1)^2}{\beta_0} + \frac{2(n\alpha_1 + 1)^2}{3\beta_2} + \frac{1}{n} \left( \frac{\rho+p}{T} \right)^2 \left( \frac{\partial T}{\partial s} \right)_n \right] \quad (3.57)$$

$$E = \left( \frac{\rho+p}{n} \right)^2 \beta_0 \beta_2 \left[ (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s + \frac{1}{\beta_0} + \frac{2}{3\beta_2} \right] \quad (3.58)$$

$$F = - \frac{\rho+p}{n^2} \beta_0 \beta_2 \left[ \frac{n\alpha_0 + 1}{\beta_0} + \frac{2(n\alpha_1 + 1)}{3\beta_2} + \frac{n(\rho+p)}{T} \left( \frac{\partial T}{\partial n} \right)_s \right] \quad (3.59)$$

The set of characteristic velocities is found by setting each of  $\det Q$ ,  $\det R$ , and  $\det S$  to zero. The two characteristic velocities found by setting  $\det S$  to zero both vanish. The matrix  $R$  has one vanishing characteristic velocity, and two nonvanishing characteristic velocities given by

$$v_T^2 = \frac{(\rho+p)\alpha_1^2 + \beta_1}{2(\rho+p)\beta_1\beta_2} \quad (3.60)$$

These are the transverse characteristic velocities since the matrix  $R$  describes perturbations tangent to the characteristic surfaces, and so transverse to the direction of propagation. The matrix  $Q$  has two vanishing characteristic velocities, and four nonvanishing characteristic velocities which satisfy

$$Av_L^4 + Bv_L^2 + C = 0 \quad (3.61)$$

These are the longitudinal characteristic velocities.

The conditions for the system of linear perturbation equations (Eq. (3.38)) to form a hyperbolic system will now be determined. The most common definition of hyperbolicity, which requires the characteristic velocities to satisfy conditions such as reality, cannot be used because the characteristic velocities of the energy frame theory are not all distinct. Since the matrices  $A_B^{A a}$  are symmetric, the system of perturbation equations forms a symmetric hyperbolic system if some linear combination of the  $A_B^{A a}$  were positive definite. The system of perturbation equations would then have a well-posed initial value problem (Courant and Hilbert 1962, p. 593).

A necessary and sufficient condition for a matrix to be positive definite is that the determinant of each leading principal submatrix be positive. The matrix  $A_B^{A 0}$  is itself positive definite if and only if the following inequalities are satisfied:

$$\left(\frac{\partial n}{\partial \theta}\right)_T > 0 \quad , \quad (3.62)$$

$$\left(\frac{\partial \rho}{\partial T}\right)_n > 0 \quad , \quad (3.63)$$

$$\beta_i > 0 \quad , \quad (3.64)$$



for each of  $i = 0, 1, 2$ . Thus a sufficient condition for a positive definite linear combination of the  $A_B^A a$  to exist is for each of these inequalities to be satisfied. Equations (3.62) and (3.63) also imply that  $(\partial\rho/\partial T)_0 > 0$ .

Satisfaction of the inequalities in Eqs. (3.62)-(3.64) is also a necessary condition for a positive definite linear combination of the  $A_B^A a$  to exist. The matrix  $A_B^A 0$  must be positive definite because only  $A_B^A 0$  has nonzero diagonal elements and none of the other  $A_B^A a$  have nonzero off-diagonal elements in the locations where the off-diagonal elements of  $A_B^A 0$  are nonzero. Therefore, the perturbation equations (Eq. (3.38)) form a symmetric hyperbolic system if and only if the inequalities in Eqs. (3.62)-(3.64) are satisfied. However, it is unknown whether or not these are the weakest conditions necessary for the theory to be hyperbolic because it has not been proven that satisfaction of Eqs. (3.62)-(3.64) are also necessary conditions for the theory to be hyperbolic.

#### Stability is Equivalent to Causality and Hyperbolicity

The connection between stability, causality, and hyperbolicity in the Israel-Stewart energy frame theory is established in this section. The proofs of the results of this section are very similar to the corresponding proofs for the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983). In what follows it will be assumed that the stability conditions are strictly satisfied,  $\Omega_A(\lambda^2) > 0$ . One

relationship is that stability implies causality and hyperbolicity: the linear perturbations of a fluid described by the Israel-Stewart energy frame theory propagate causally (subluminally) and obey a symmetric hyperbolic system of equations when the stability conditions are strictly satisfied.

The proof is as follows. Equations (3.18), (3.21), and (3.60) can be used to show

$$1 - v_T^2 = \frac{\Omega_4(1) \cdot \Omega_7(1)}{\Omega_4(0) \cdot \Omega_7(0)} > 0 \quad (3.65)$$

when  $\Omega_4$  and  $\Omega_7$  are positive. Also  $v_T^2 > 0$ , since  $\beta_1 > 0$  and  $\beta_2 > 0$  when the stability conditions are strictly satisfied. Thus the transverse velocities are real and bounded between zero and the speed of light:

$$0 < v_T^2 < 1 \quad (3.66)$$

Recall that the squares of the longitudinal characteristic velocities  $x = v_L^2$  are the roots of the quadratic

$$P(x) = Ax^2 + Bx + C \quad (3.67)$$

The squares of the longitudinal velocities will be real if and only if the discriminant is nonnegative. Equations (3.54)-(3.59) can be used to show

$$\begin{aligned}
B^2 - 4AC = & \frac{\rho+p}{n^2 \beta_1 + \rho+p} \left[ \left[ (\rho+p)D + \left[ \left( \frac{n}{\rho+p} \right)^2 \beta_1 + \frac{1}{\rho+p} \right] E + 2 \left( \frac{n^2 \beta_1}{\rho+p} + 1 \right) F \right]^2 \right. \\
& \left. + \frac{n^2 \beta_1}{\rho+p} \left[ (\rho+p)D - \left[ \left( \frac{n}{\rho+p} \right)^2 \beta_1 + \frac{1}{\rho+p} \right] E \right]^2 \right] \geq 0 \quad (3.68)
\end{aligned}$$

when the stability conditions are satisfied. So, if the stability conditions are satisfied, then the squares of the longitudinal characteristic velocities are real.

The zeros of the quadratic  $P(x)$  can be seen to lie between  $x = 0$  and  $x = 1$  when the stability conditions are strictly satisfied from the following quantities:

$$\begin{aligned}
P(0) = C = & \frac{2}{3}(\alpha_1 - \alpha_0)^2 + \frac{2}{3}n \left( \frac{\partial p}{\partial n} \right)_s \left[ \alpha_1 + \frac{1}{n} - \frac{\rho+p}{Tn} \left( \frac{\partial T}{\partial p} \right)_s \right]^2 \beta_0 \\
& + \left( \frac{\rho+p}{Tn} \right)^2 \left( \frac{\partial T}{\partial s} \right)_p \left( \frac{\partial p}{\partial n} \right)_s \beta_0 \beta_2 + n \left( \frac{\partial p}{\partial n} \right)_s \left[ \alpha_0 + \frac{1}{n} - \frac{\rho+p}{Tn} \left( \frac{\partial T}{\partial p} \right)_s \right]^2 \beta_2 \\
& + \frac{1}{n} \left( \frac{\rho+p}{Tn} \right)^2 \left( \frac{\partial T}{\partial s} \right)_p \left( \beta_2 + \frac{2}{3}\beta_0 \right) > 0 \quad , \quad (3.69)
\end{aligned}$$

$$\frac{dP(0)}{dx} = B = \left( \frac{\rho+p}{n} \right)^2 \left( \frac{\partial \theta}{\partial s} \right)_\rho \beta_0 \beta_2 - (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \beta_0 \beta_1 \beta_2$$

$$-(\rho+p) \left( \alpha_0^2 \beta_2 + \frac{2}{3} \alpha_1^2 \beta_0 \right) - \beta_1 \beta_2 - \frac{2}{3} \beta_0 \beta_1 < 0 \quad , \quad (3.70)$$

$$P(1) = A + B + C = \left(\frac{\rho+p}{n}\right)^2 \Omega_3(1)\Omega_6(1)\beta_0\beta_2 > 0 \quad , \quad (3.71)$$

$$\begin{aligned} \frac{dP(1)}{dx} = 2A + B &= \beta_0\beta_1\beta_2 \left[ \Omega_3(1) + \frac{K^2(1)}{\Omega_6(1)} \right] + \frac{\rho+p}{n^2} \beta_0\beta_2\Omega_3(1) \\ &+ \frac{(\rho+p)\beta_0\beta_2}{n^2\Omega_6(1)} \left[ (\rho+p)\Omega_6(1) - K(1) \right]^2 > 0 \quad . \quad (3.72) \end{aligned}$$

Thus, the two zeros of  $P(x)$  must be real and lie between  $x = 0$  and  $x = 1$  because  $P(0) > 0$ ,  $dP(0)/dx < 0$ ,  $P(1) > 0$ , and  $dP(1)/dx > 0$ :

$$0 < v_L^2 < 1 \quad . \quad (3.73)$$

Equations (3.69)-(3.72) may be verified from the definitions of  $A$ ,  $B$ , and  $C$ . The thermodynamic derivatives  $(\partial p/\partial n)_s$  and  $(\partial T/\partial s)_p$  are each guaranteed to be positive, and  $(\partial \theta/\partial s)_p$  is negative, as a consequence of the first law of thermodynamics and the positivity of  $\Omega_1$  and  $\Omega_2$  (see Hiscock and Lindblom 1983, p. 482).

The hyperbolicity conditions (Eqs. (3.62)-(3.64)) are easily seen to be satisfied when the stability conditions are strictly satisfied. Therefore, linear perturbations propagate causally via a symmetric hyperbolic system of equations when the stability criteria are strictly satisfied.

The converse theorem also holds: the Israel-Stewart energy frame theory is stable for linear perturbations when

these perturbations propagate causally and obey a symmetric hyperbolic system of equations (in the sense of Eqs. (3.62)-(3.64)). This will be proven by showing that the causality conditions (in particular Eqs. (3.65), (3.71), and (3.72)) and the hyperbolicity criteria (Eqs. (3.62)-(3.64)) imply the positivity of each of the  $\Omega_A$ .

The stability conditions  $\Omega_5$  and  $\Omega_8$  are positive when Eq. (3.64) is satisfied. The stability conditions  $\Omega_1$  and  $\Omega_2$  may be expressed as

$$\Omega_1 = \left[ Tn^2 \left( \frac{\partial \theta}{\partial n} \right)_T + \frac{1}{Tn^2} \left( \frac{\partial p}{\partial s} \right)_n^2 \left( \frac{\partial \rho}{\partial T} \right)_n \right]^{-1}, \quad (3.74)$$

$$\Omega_2 = \frac{T^2 n^4}{\rho + p} \left( \frac{\partial T}{\partial \rho} \right)_\theta \left( \frac{\partial \rho}{\partial p} \right)_s \left[ \left( \frac{\partial \theta}{\partial n} \right)_T + \frac{1}{T^2} \left( \frac{\partial \rho}{\partial n} \right)_T^2 \left( \frac{\partial T}{\partial \rho} \right)_n \right]. \quad (3.75)$$

Thus  $\Omega_1$  and  $\Omega_2$  are also positive when the hyperbolicity conditions are satisfied.

Equation (3.71) and the positivity of  $\Omega_5$  and  $\Omega_8$  require  $\Omega_3(1)$  and  $\Omega_6(1)$  to be nonzero and to have the same sign. This and Eq. (3.72) require  $\Omega_3(1)$  and  $\Omega_6(1)$  to be positive. This guarantees the positivity of  $\Omega_6(\lambda^2)$  since  $\Omega_6(\lambda^2) \geq \Omega_6(1)$  for all  $\lambda^2 < 1$  (see Eq. (3.20)). Further,  $\Omega_7$  is also positive because

$$\begin{aligned} \Omega_7(\lambda^2) \geq \Omega_7(1) = \Omega_6(1) + \frac{1}{\beta_0} \left( \frac{n\alpha_0 + 1}{\rho + p} \right)^2 \\ + \frac{1}{6\beta_2} \left( \frac{n\alpha_1 + 1}{\rho + p} \right)^2 + \frac{1}{T^2 n} \left( \frac{\partial T}{\partial S} \right)_n > 0 \end{aligned} \quad (3.76)$$

for  $\lambda^2 < 1$ .

The minima of  $\Omega_3(\lambda^2)$  and  $\Omega_4(\lambda^2)$  can be determined from the following derivatives:

$$\begin{aligned} \frac{d\Omega_3}{d(\lambda^2)} = -\frac{1}{c_2} \left[ \left( \frac{c_3}{c_3 - \lambda^2 c_2} \right)^2 - 1 \right] \left( c_1 + \frac{c_2}{c_3} \right)^2 - \frac{1}{c_3^2} \left[ \frac{1}{\beta_0} \left[ \left( \frac{n}{\rho + p} \right)^2 \beta_1 \right. \right. \\ \left. \left. - \frac{n\alpha_0}{\rho + p} \right]^2 + \frac{2}{3\beta_2} \left[ \left( \frac{n}{\rho + p} \right)^2 \beta_1 - \frac{n\alpha_1}{\rho + p} \right]^2 + \frac{1}{T^2 n} \left( \frac{\partial T}{\partial S} \right)_p \right. \\ \left. + (\rho + p) \left( \frac{\partial p}{\partial \rho} \right)_s \left[ c_3 - \frac{1}{n} \left( \frac{\partial p}{\partial \rho} \right)_n \left( \frac{\partial n}{\partial p} \right)_s \right]^2 \right] \quad , \end{aligned} \quad (3.77)$$

$$\frac{d\Omega_4}{d(\lambda^2)} = -2\beta_2 \left[ \frac{n^2 \beta_1 - n(\rho + p)\alpha_1}{2(n^2 \beta_1 + \rho + p)\beta_2 - (n\alpha_1 + 1)^2 \lambda^2} \right]^2 \quad . \quad (3.78)$$

The quantities  $c_1$ ,  $c_2$ , and  $c_3$  are defined as

$$c_1 = K(\lambda^2) - \frac{1}{\lambda^2} \quad , \quad (3.79)$$

$$c_2 = \frac{c_3}{\lambda^2} - \Omega_6 \quad , \quad (3.80)$$

$$c_3 = \left( \frac{n}{\rho+p} \right)^2 \beta_1 + \frac{1}{\rho+p} \quad (3.81)$$

When the hyperbolicity conditions are satisfied,  $d\Omega_3/d(\lambda^2) \leq 0$  and  $d\Omega_4/d(\lambda^2) \leq 0$  since  $c_2$  and  $c_3$  are nonnegative in this case. It follows that the minima of  $\Omega_3(\lambda^2)$  and  $\Omega_4(\lambda^2)$  in the relevant range of  $\lambda^2$  occur at  $\lambda^2 = 1$ . So  $\Omega_3(\lambda^2)$  is positive because  $\Omega_3(1)$  is positive. The positivity of  $\Omega_4(1)$  is seen from Eqs. (3.65) and (3.76) and the positivity of  $\Omega_4(0)$ :

$$\Omega_4(0) = \frac{n^2(\rho+p)\beta_1}{n^2\beta_1 + \rho + p} > 0 \quad (3.82)$$

Therefore  $\Omega_4(\lambda^2)$  is positive also. Thus each of the  $\Omega_A(\lambda^2)$  are positive when the causality and hyperbolicity conditions are satisfied.

### Conclusions

I have shown in this chapter that satisfaction of each of the stability conditions (Eq. (3.34)) is a sufficient condition for the stability of the equilibria of the Israel-Stewart energy frame theory for small perturbations. I also argued that satisfaction of each of the stability conditions may also be a necessary condition for the stability of all equilibrium states which have a real adiabatic sound speed and are stable against convection. Thus, just as for the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983), the

equilibria of the energy frame theory are stable for small perturbations if the equilibrium state satisfies a certain set of inequalities.

I have also proven in this chapter that small perturbations in the Israel-Stewart energy frame theory propagate causally and obey hyperbolic differential equations if and only if the equilibrium state satisfies the stability conditions (Eq. (3.34)). Thus, just as in the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1983), the stability conditions are equivalent to the causality and hyperbolicity conditions.



## CHAPTER FOUR

## THE ENERGY FRAME THEORY IN THE NONLINEAR REGIME

Introduction

The stability, causality, and hyperbolicity properties of the Israel-Stewart energy frame theory far from equilibrium are studied in this chapter. Unfortunately, it does not seem feasible at present to perform a stability analysis that would apply generically to any fluid described by the full nonlinear theory, as was done for the linearized theory in Ch. 3, because of the complexity of the theory outside the linear regime. However information about the nonlinear properties of the theory can still be obtained by considering a specific example; in this chapter I examine the plane symmetric motions of an inviscid Boltzmann gas at ultrarelativistic temperatures ( $kT \gg m$ , where  $m$  is the rest mass of a fluid particle).

The Israel-Stewart theories are constructed by assuming the entropy current is given by an expansion in the deviations from equilibrium which is truncated after the quadratic terms. Thus it is not reasonable to expect any of these theories to correctly describe the dynamics of a fluid which is extremely far from equilibrium. Also, if the deviations from equilibrium are too large, the fundamental variables of the theory will not have the properties associated with classical

fluids. Specifically, an observer with four-velocity  $\xi^a$  would measure the energy density of the fluid to be

$$\rho_\xi = T^{ab} \xi_a \xi_b \quad . \quad (4.1)$$

The stress-energy tensor  $T^{ab}$  would violate the weak energy condition (Hawking and Ellis 1973) if this observer could measure a negative fluid energy density ( $\rho_\xi < 0$ ). Equation (2.35) can be used to show that an inviscid fluid described by the Eckart frame theory will violate the weak energy condition if

$$\rho + pv^2 - 2|q|v_q < 0 \quad . \quad (4.2)$$

In Eq. (4.2),  $\rho$  is the fluid energy density and  $p$  is the pressure, both measured in the rest frame of the fluid,  $|q|$  is the magnitude of the heat flux  $q^a$  ( $q^2 = q^a q_a$ ), and  $v_q$  is the component of the velocity  $v$  of the observer relative to the fluid rest frame in the direction of the heat flux. The tightest constraint is obtained by setting  $v_q \rightarrow 1$  (and so also  $v \rightarrow 1$ ):

$$\frac{|q|}{\rho+p} > \frac{1}{2} \quad . \quad (4.3)$$

The weak energy condition will be violated in the Israel-Stewart Eckart frame theory for inviscid fluids having heat fluxes which are large enough to satisfy Eq. (4.3). For the

inviscid ultrarelativistic Boltzmann gas, which has the equation of state

$$p = \frac{1}{3}\rho = nkT \quad , \quad (4.4)$$

the weak energy condition is violated in the Eckart frame theory when  $|q|/\rho > 2/3$ .

Hiscock and Lindblom (1988a) have also studied the plane symmetric motions of an inviscid ultrarelativistic Boltzmann gas, but in the context of the Israel-Stewart Eckart frame theory. They found that the Eckart frame theory is causal and hyperbolic only when  $|q|/\rho > 0.1303$ . The theory is also unstable when  $|q|/\rho > 0.5597$  in that if the heat flux is this large, the fluid will evolve away from equilibrium rather than towards it. (These values are corrections to those found by Hiscock and Lindblom (1988a). The values they published were obtained using an incorrect value for the second-order coefficient  $\beta_1$ .) Thus the Eckart frame theory breaks down well before the weak energy condition is violated, at least for this particular fluid.

For an inviscid fluid described by the Israel-Stewart energy frame theory, Eq. (2.5) can be used to show that an observer with four-velocity  $\xi^a$  would measure the fluid energy density to be

$$\rho_\xi = \frac{\rho + pv^2}{1 - v^2} \quad . \quad (4.5)$$

So the weak energy condition is always satisfied in the Israel-Stewart energy frame theory because the velocity  $v$  of the observer relative to the fluid is always less than the speed of light ( $v < 1$ ). But, by Eq. (2.4), the particle number current will be spacelike,

$$N_a N^a = -n^2 + v^2 > 0 \quad , \quad (4.6)$$

if the magnitude  $|v|$  of the particle diffusion current  $v^a$  ( $v^2 = v^a v_a$ ) is greater than the number density of particles as measured in the fluid rest frame ( $|v| > n$ ). Some observers could then measure a negative particle number density. Thus, the energy frame theory cannot be expected to correctly describe the dynamics of any inviscid fluid that is so far from equilibrium that  $|v| > n$ .

Acausality Implies Instability for the Inviscid  
Ultrarelativistic Boltzmann Gas with Planar Symmetry

The equations of motion of the Israel-Stewart energy frame theory for inviscid fluids are Eqs. (2.2), (2.3), and (2.21) with  $\tau$  and  $\tau^{ab}$  set to zero. For simplicity I will assume that the background spacetime is Minkowski space, and only the plane symmetric motions of the fluid will be considered. Choosing Cartesian coordinates with the x-axis aligned with the direction of spatial variation of the fluid, the equations of motion then take on a particularly simple form if the four-velocity is written in terms of the rapidity

$\psi$ ,

$$u^a = (\cosh\psi, \sinh\psi, 0, 0) \quad . \quad (4.7)$$

The particle diffusion current  $v^a$ , which must be orthogonal to the four-velocity, becomes

$$v^a = v(\sinh\psi, \cosh\psi, 0, 0) \quad . \quad (4.8)$$

The equations of motion are then

$$\begin{aligned} & \cosh\psi \partial_t n + \sinh\psi \partial_x n + (n \sinh\psi + v \cosh\psi) \partial_t \psi \\ & + (n \cosh\psi + v \sinh\psi) \partial_x \psi + \sinh\psi \partial_t v + \cosh\psi \partial_x v = 0 \quad , \quad (4.9) \end{aligned}$$

$$\begin{aligned} & \cosh\psi \partial_t \rho + \sinh\psi \partial_x \rho + (\rho + p) \left( \sinh\psi \partial_t \psi \right. \\ & \left. + \cosh\psi \partial_x \psi \right) = 0 \quad , \quad (4.10) \end{aligned}$$

$$\begin{aligned} & \sinh\psi \partial_t p + \cosh\psi \partial_x p + (\rho + p) \left( \cosh\psi \partial_t \psi \right. \\ & \left. + \sinh\psi \partial_x \psi \right) = 0 \quad , \quad (4.11) \end{aligned}$$

$$\begin{aligned} & \frac{v}{\sigma T} + T \left( \sinh\psi \partial_t \theta + \cosh\psi \partial_x \theta \right) + \beta_1 \left( \cosh\psi \partial_t v + \sinh\psi \partial_x v \right) \\ & + \frac{1}{2} v \left( \cosh\psi \partial_t \beta_1 + \sinh\psi \partial_x \beta_1 \right) + \frac{1}{2} \beta_1 v \left( \sinh\psi \partial_t \psi \right. \\ & \left. + \cosh\psi \partial_x \psi \right) - \frac{\beta_1 v}{2T} \left( \cosh\psi \partial_t T + \sinh\psi \partial_x T \right) = 0 \quad , \quad (4.12) \end{aligned}$$

where  $\theta$  is the thermodynamic potential defined in Eq. (2.19).

Here I examine the dynamics contained in Eqs. (4.9)-(4.12) for a Boltzmann gas in the ultrarelativistic limit, for which the equation of state is Eq. (4.4), and

$$\theta = k \ln \left[ \frac{n}{8\pi g} \left( \frac{h}{kT} \right)^2 \right] , \quad (4.13)$$

where  $k$  is Boltzmann's constant,  $h$  is Planck's constant, and  $g$  is the spin weight of the fluid particles. I also take the second-order coefficient  $\beta_1$  to have the value

$$\beta_1 = \frac{20\lambda p}{n^2} \quad (4.14)$$

with  $\lambda > 0$ . Setting  $\lambda = 0$  would yield the pathological Landau-Lifshitz theory, while  $\lambda = 1$  corresponds to the value obtained for  $\beta_1$  from relativistic kinetic theory (Israel and Stewart 1979b).

The equation of state and the values for  $\beta_1$  and  $\theta$  (Eqs. (4.4), (4.13), and (4.14)) may be used to write the equations of motion (Eqs. (4.9)-(4.12)) as a set of four coupled differential equations for four independent fluid variables by eliminating  $\beta_1$ ,  $p$ ,  $T$ , and  $\theta$  in favor of  $\rho$  and  $n$ . The resulting four equations may be written symbolically in the form

$$A^A_B a_a \partial_a Y^B + B^A = 0 , \quad (4.15)$$

where the index A labels one of the four equations of motion, and the index B runs over the fluid variables:  $Y^B = (\rho, n, \psi, v)$ . The components of the matrices  $A^A{}_B{}^a$  and  $B^A$  depend on the fluid variables  $Y^B$ , but not their derivatives. The matrices  $A^A{}_B{}^a$  are nonzero only when the spacetime index a takes on the value t or x, due to the assumption of planar symmetry.

The set of equations of motion will be hyperbolic if there exist four real and distinct characteristic velocities for the system (Courant and Hilbert 1962, Ch. VI). The characteristic velocities v are the roots of the equation

$$\det\left(vA^A{}_B{}^t - A^A{}_B{}^x\right) = 0 \quad . \quad (4.16)$$

This equation is most easily solved in the rest frame of the fluid, where the rapidity  $\psi$  is zero. Equation (4.16) may then be written as

$$v^4 - \frac{v}{2n} v^3 - \left(\frac{1}{3} + \frac{1}{5\lambda}\right) v^2 + \frac{v}{6n} v + \frac{1}{15\lambda} = 0 \quad . \quad (4.17)$$

Equation (4.17) may be factored:

$$\left(v^2 - \frac{1}{3}\right)\left(v^2 - \frac{v}{2n} v - \frac{1}{5\lambda}\right) = 0 \quad . \quad (4.18)$$

The characteristic velocities are then given by  $v = \pm(1/3)^{1/2}$ , and

$$v = \frac{v}{4n} \pm \left[\frac{v^2}{16n^2} + \frac{1}{5\lambda}\right]^{1/2} \quad . \quad (4.19)$$

So for all values of  $\lambda > 0$ , and for arbitrarily large deviations from equilibrium  $v/n$ , there are always four real and distinct characteristic velocities (ignoring the exceptional cases where  $v/n$  has the precise value needed for one of the roots of Eq. (4.19) to be equal to  $\pm(1/3)^{1/2}$ ; there are then only three distinct characteristic velocities, and a different treatment is needed to establish hyperbolicity). The Israel-Stewart energy frame theory, applied to an inviscid ultrarelativistic Boltzmann gas with planar symmetry, is then always hyperbolic, no matter how large the deviation from equilibrium.

Although the energy frame theory is always hyperbolic for the special case of the inviscid ultrarelativistic Boltzmann gas, it is not causal for arbitrarily large deviations from equilibrium. The characteristic velocities will all be less than the speed of light for  $\lambda > 1/5$  only if

$$|v|/n < \frac{2(5\lambda - 1)}{5\lambda} \quad (4.20)$$

For the kinetic theory value of  $\beta_1$  ( $\lambda = 1$ ), the energy frame theory will predict acausal propagation when  $|v|/n > 8/5$ . This deviation from equilibrium is considerably larger than the critical deviation at which the particle number current vector becomes spacelike,  $|v|/n > 1$ . Accordingly, this acausal behavior does not indicate a significant flaw in the theory. By contrast, in the Israel-Stewart Eckart frame theory applied to the ultrarelativistic Boltzmann gas, the



equations of motion are nonhyperbolic for  $|q|/\rho > 0.1303$ , while the critical deviation at which the weak energy condition is violated (where one might reasonably expect the theory to break down) is significantly larger,  $|q|/\rho > 2/3$ .

Also of interest is the stability of the energy frame theory for large deviations from equilibrium. The energy frame theory is unstable for inviscid fluids if the deviation from equilibrium (measured in a dimensionless fashion as the ratio  $|v|/n$ ) grows into the future, rather than decreasing towards an equilibrium state. It is known that the original Eckart and Landau-Lifshitz theories are generically unstable even at the level of linear perturbation theory (Hiscock and Lindblom 1985), while both the Eckart frame and energy frame Israel-Stewart theories are linearly stable if the second-order coefficients have values which result in linear perturbation equations which are causal and hyperbolic (Hiscock and Lindblom 1983; Olson 1990). However, a spatially homogeneous mode in the Israel-Stewart Eckart frame theory has been shown to be unstable when applied to the ultrarelativistic Boltzmann gas if the initial deviation from equilibrium is not small, but exceeds  $|q|/\rho > 0.5597$  (Hiscock and Lindblom 1988a).

I will now analyze the dynamics of the spatially homogeneous modes of the ultrarelativistic Boltzmann gas in the Israel-Stewart energy frame theory. The equations of motion are given by Eqs. (4.9)-(4.12) with all x-derivative

terms set equal to zero. These equations then can be written as

$$\frac{dN}{dt} = \frac{d}{dt}(n \cosh \psi + v \sinh \psi) = 0 \quad , \quad (4.21)$$

$$\frac{dE}{dt} = \frac{d}{dt}(\rho \cosh^2 \psi + p \sinh^2 \psi) = 0 \quad , \quad (4.22)$$

$$\frac{dP}{dt} = \frac{d}{dt}[(\rho+p) \cosh \psi \sinh \psi] = 0 \quad , \quad (4.23)$$

$$\frac{v}{\sigma T} + T \sinh \psi \frac{d\theta}{dt} + \beta_1 \cosh \psi \frac{dv}{dt} + \frac{1}{2} T v \frac{d}{dt} \left( \frac{\beta_1}{T} \cosh \psi \right) = 0 \quad . \quad (4.24)$$

Equations (4.21)-(4.23) may be integrated immediately. The resulting integration constants are the total number density of the fluid  $N$ , the total energy density of the fluid  $E$ , and the total momentum density of the fluid  $P$ . Both  $N$  and  $E$  must be positive constants. For the ultrarelativistic Boltzmann gas, the integrals of Eqs. (4.22) and (4.23) imply that the energy density, pressure, and rapidity are constant for the spatially homogeneous modes,  $\rho = \rho_0$ ,  $p = p_0$ , and  $\psi = \psi_0$ . The rapidity  $\psi_0$  and the momentum density  $P$  are positive (negative) if the fluid flow is in the positive (negative)  $x$  direction. The number density may then be written as a simple function of  $\psi_0$  and the particle diffusion magnitude:

$$n = \frac{N - v \sinh \psi_0}{\cosh \psi_0} = \frac{N}{\cosh \psi_0 + \frac{v}{n} \sinh \psi_0} \quad . \quad (4.25)$$

Note that this number density changes sign for large values of  $v$ . If  $n$  is to be interpreted as the particle number density, and hence must be a nonnegative quantity, then the strongest constraint on  $v$  is obtained by setting  $\psi_0 \rightarrow \pm\infty$ . In that case,  $n$  will be nonnegative as long as  $|v|/n < 1$ , exactly the restriction needed to guarantee that the particle number current is timelike. Equations (4.9)-(4.12) may be used to produce the following expression for  $dv/dt$ :

$$\frac{1}{v} \frac{dv}{dt} = - \frac{n \cosh \psi_0}{4\sigma k T^2 \left( 5\lambda \cosh^2 \psi_0 - \sinh^2 \psi_0 + \frac{5\lambda v}{2n} \cosh \psi_0 \sinh \psi_0 \right)}. \quad (4.26)$$

The fluid will be stable for the spatially homogeneous modes provided the right-hand side of Eq. (4.26) is negative. The spatially homogeneous zero momentum mode, the mode with  $P = 0$  (and so also  $\psi_0 = 0$ ), is always stable; no matter how large the initial particle diffusion  $v$ , the fluid will decay exponentially towards equilibrium. For all values of  $\lambda > 1/5$ , the right hand side will be negative for all values of  $\psi_0$  if

$$|v|/n < \frac{2(5\lambda - 1)}{5\lambda}. \quad (4.27)$$

This condition is exactly the same constraint that must be satisfied for the characteristic velocities to be less than the speed of light (Eq. (4.20)).

Based on this analysis, it can be concluded that acausality implies instability in the Israel-Stewart energy

frame theory applied to the spatially homogeneous modes of the inviscid ultrarelativistic Boltzmann gas. This result is an extension of the known connection between stability, causality, and hyperbolicity in the linearized Israel-Stewart energy frame and Eckart frame theories (Olson 1990; Hiscock and Lindblom 1983). While not as general, since this analysis is restricted to a particular equation of state and to spatially homogeneous planar motions for the analysis of stability, it is nonetheless interesting since it is not possible to extend these results to the nonlinear regime for the Israel-Stewart Eckart frame theory (Hiscock and Lindblom 1988a).

### Conclusions

In this section I compare what is now known about the dynamics of the Israel-Stewart Eckart frame and energy frame theories far from equilibrium. Both theories have been studied in the inviscid limit for an ultrarelativistic Boltzmann gas with planar symmetry. Specializing to the kinetic theory value of the second-order coefficient  $\beta_1$  (by choosing  $\lambda = 1$ ), the following comparisons can be made. The Eckart frame theory is causal (each characteristic velocity is less than speed of light) and hyperbolic (the characteristic velocities are real and distinct) only if  $|q|/\rho < 0.1303$ , while the energy frame theory is hyperbolic for arbitrarily large deviations from equilibrium, and is causal

for  $|v|/n < 8/5$ . Further, the value of  $|v|/n$  for which the energy frame theory becomes acausal is considerably larger than the value where the particle number current becomes spacelike, but the Eckart theory breaks down well before the weak energy condition is violated.

The spatially homogeneous zero-momentum mode in the Eckart frame theory is stable only if  $|q|/\rho < 0.5597$ , while the zero-momentum mode in the energy frame theory is stable for arbitrarily large deviations from equilibrium. In the nonlinearized Eckart frame theory no connection has been found between stability, causality and hyperbolicity. In the energy frame theory acausality implies instability; there is a spatially homogeneous mode of the fluid which does become unstable precisely when one characteristic velocity exceeds the speed of light (at  $|v|/n = 8/5$ ). Thus the connection between causality, hyperbolicity, and stability which was proven for the linearized theories (Hiscock and Lindblom 1983; Olson 1990) may apply also to the full nonlinearized Israel-Stewart energy frame theory. On the basis of the present analysis it can at least be concluded that acausality implies instability for the inviscid ultrarelativistic Boltzmann gas with planar symmetry in the energy frame theory.

These comparisons are evidence that the Israel-Stewart energy frame theory may be less pathological than the Israel-Stewart Eckart frame theory far from equilibrium. The energy frame theory would thus seem to be the best candidate for use

in possible applications, such as modeling high-energy astrophysical situations such as supernovae, or relativistic heavy-ion collisions. However, this conclusion is based on the behavior of these theories for just one physical system, and for different initial states of the gas because the initial numerical values of the components of the particle number current and stress-energy tensor in the two theories are different.

## CHAPTER FIVE

## PLANE STEADY SHOCK WAVES IN ISRAEL-STEWART FLUIDS

Introduction

An important application of the theory of relativistic fluids is to shock waves in dissipative fluids. Study of the structure of relativistic shock waves may prove helpful for the understanding of supernovae, the early universe, and heavy-ion collisions. For example, the possible transition from the nucleon phase to a quark-gluon plasma in relativistic heavy nuclei collisions is believed to be sensitive to the width of the shock front (Clare and Strottman 1986).

A remaining fundamental question about the Israel-Stewart theory is whether it is capable of adequately describing the structure of strong shock waves. Other authors have suggested that the Israel-Stewart theory fails to describe strong shock waves adequately, but these analyses are restricted to particular kinetic theory limits of the Israel-Stewart theory. Grad (1952) (see also Anile and Majorana 1981) showed that the plane steady shock solutions of his kinetic theory method, which are the same as the Israel-Stewart theory solutions when kinetic theory values are used for the second-order coefficients, are multiple-valued above an upstream Mach number of 1.65 for the nonrelativistic Boltzmann gas with zero

bulk viscosity. Majorana and Motta (1985) found that physically acceptable solutions which describe plane steady shock waves in an ultrarelativistic Boltzmann gas with zero bulk viscosity, with the second-order coefficients taken to have their kinetic theory values, do not exist for strong shocks.

The purpose of this chapter is to investigate whether the Israel-Stewart theory is incapable of describing strong shock waves in all fluids, not only in the particular kinetic theory limits of the theory considered to date. Some preliminary work on this question has also been done by Hiscock and Lindblom (unpublished), in particular on the connection between the characteristic velocities and the shock strength limits.

### Plane Steady Shock Solutions

In this section the system of differential equations which yield the plane steady shock solutions of the Israel-Stewart Eckart frame theory of relativistic dissipative fluids will be derived. I choose to use the Israel-Stewart Eckart frame theory in order to most easily compare with the conclusions reached by Grad and by Majorana and Motta. The equations of motion for a single-component fluid described by the Eckart frame theory are Eqs. (2.32), (2.33), (2.41), (2.42), and (2.43). Of interest here are the solutions of these equations which describe a plane steady shock wave. For



simplicity I shall assume that the background spacetime is flat Minkowski space, and the  $x$ -axis will be chosen to point in the direction of the plane symmetric fluid flow. Thus, for the plane steady solutions, all fluid variables depend on just the  $x$  coordinate. In these coordinates the four-velocity may be written as  $u^a = (\gamma, \gamma v, 0, 0)$ , where  $v$  is the velocity of the fluid at  $x$  and  $\gamma = (1 - v^2)^{-1/2}$ . The constraints in Eq. (2.36) require the heat flow vector to take the form  $q^a = q(\gamma v, \gamma, 0, 0)$ , and imply that there is only one independent component of the shear stress tensor which will be taken to be  $\tau^{xx} = \sigma \gamma^2$ .

The conservation laws (Eqs. (2.32) and (2.33)) may be integrated immediately:

$$n\gamma v = N \quad , \quad (5.1)$$

$$(\rho + p + \tau + \sigma)v + q(1 + v^2) = E/\gamma^2 \quad , \quad (5.2)$$

$$\rho v^2 + p + \tau + \sigma + 2qv = P/\gamma^2 \quad . \quad (5.3)$$

If the plane steady solutions are to describe a shock wave, we must impose the boundary condition that the fluid tends to an equilibrium state ( $\tau=q=\sigma=0$ ) for  $x \rightarrow \pm\infty$ . Further, the fluid velocity must be supersonic in front of the shock (velocity greater than the adiabatic sound speed  $v_s$ ,

$$v_s = \left( \frac{\partial p}{\partial \rho} \right)_s^{1/2} \quad , \quad (5.4)$$

in the equilibrium state at  $x = -\infty$ ) and subsonic behind the shock (velocity less than  $v_s$  in the equilibrium state at  $x = +\infty$ ). The positive constants  $N$ ,  $E$ , and  $P$  in Eqs. (5.1)-(5.3) are then determined by picking the values of any two thermodynamic variables and a supersonic fluid velocity for the equilibrium state at  $x = -\infty$ . The shock strength is defined by either the upstream Mach number  $M^-$ ,

$$M^- = \frac{v(-\infty)}{v_s(-\infty)} \quad , \quad (5.5)$$

or by the downstream Mach number  $M^+$ ,

$$M^+ = \frac{v(+\infty)}{v_s(+\infty)} \quad . \quad (5.6)$$

The limit of a weak shock is  $M^-, M^+ \rightarrow 1$ . Increasing the value of  $v$  at  $x = -\infty$  for a given equilibrium state increases the shock strength.

Equation (5.1) implies that all thermodynamic functions may be written as functions of  $v$  and  $\rho$ . It follows from Eqs. (5.2) and (5.3) that  $q$  and  $\sigma$  may be written as functions of  $v$ ,  $\rho$ , and  $\tau$ :

$$q = E - (P + \rho)v \quad , \quad (5.7)$$

$$\sigma = P(1 + v^2) + v^2\rho - p(\rho, v) - \tau - 2vE \quad . \quad (5.8)$$

Equations (5.7) and (5.8) may be used to show that the plane steady solutions of Eqs. (2.41)-(2.43) form a three-

dimensional autonomous system:

$$M \begin{bmatrix} \frac{dv}{dx} \\ \frac{d\rho}{dx} \\ \frac{d\tau}{dx} \end{bmatrix} = -\frac{1}{\gamma} \begin{bmatrix} \tau \\ \frac{q}{\kappa T} \\ \frac{3\sigma}{4\eta} \end{bmatrix}, \quad (5.9)$$

where the components of the matrix  $M$  are

$$M = \begin{bmatrix} M_{11}\gamma^2 & M_{12} & \beta_0 v \\ M_{21}\gamma^2 & M_{22} & \alpha_1 - \alpha_0 \\ M_{31}\gamma^2 & M_{32} & -\frac{3}{2}\beta_2 v \end{bmatrix}, \quad (5.10)$$

$$M_{11} = 1 + \alpha_0(\rho + p + \tau + \sigma + qv) + \frac{1}{2}\beta_0\tau - \frac{1}{2}ng(\beta_0)\tau + n\gamma_0g(\alpha_0)\frac{q}{v}, \quad (5.11)$$

$$M_{12} = \alpha_0 v + \frac{1}{2}f(\beta_0)\tau v - \gamma_0 f(\alpha_0)q, \quad (5.12)$$

$$M_{21} = v - \frac{n}{vT}\left(\frac{\partial T}{\partial n}\right)_\rho - \frac{n}{v}\left(\frac{\partial p}{\partial n}\right)_\rho \alpha_1 - \beta_1(\rho + p + \tau)v - (\alpha_1 + \beta_1)\sigma v + \left(\frac{1}{2} - 2v^2\right)\beta_1 q + 2\alpha_1 q \gamma^{-2} - \frac{1}{2}ng(\beta_1)q + (1-\gamma_0)ng(\alpha_0)\frac{\tau}{v} + (1-\gamma_1)ng(\alpha_1)\frac{\sigma}{v}, \quad (5.13)$$

$$M_{22} = \frac{1}{T} \left( \frac{\partial T}{\partial \rho} \right)_n + \left( \frac{\partial p}{\partial \rho} \right)_n \alpha_1 - \left( \alpha_1 + \beta_1 \right) v^2 + \frac{1}{2} f(\beta_1) qv$$

$$- (1-\gamma_0) f(\alpha_0) \tau - (1-\gamma_1) f(\alpha_1) \sigma \quad , \quad (5.14)$$

$$M_{31} = 1 + \frac{3}{2} n \left( \frac{\partial p}{\partial n} \right)_\rho \beta_2 + \alpha_1 (\rho + p + \tau + \sigma + 2qv) - 3\beta_2 q \gamma^{-2} v$$

$$+ \frac{3}{4} \beta_2 \sigma - \frac{3}{4} n g(\beta_2) \sigma + n \gamma_1 g(\alpha_1) \frac{q}{v} \quad , \quad (5.15)$$

$$M_{32} = \frac{3}{2} \beta_2 v^3 - \frac{3}{2} \left( \frac{\partial p}{\partial \rho} \right)_n \beta_2 v + \alpha_1 v$$

$$- \gamma_1 f(\alpha_1) q + \frac{3}{4} f(\beta_2) \sigma v \quad . \quad (5.16)$$

In Eqs. (5.11)-(5.16),  $f(F)$  and  $g(F)$  are defined as

$$f(F) = \left( \frac{\partial F}{\partial \rho} \right)_n - \frac{F}{T} \left( \frac{\partial T}{\partial \rho} \right)_n \quad , \quad (5.17)$$

$$g(F) = \left( \frac{\partial F}{\partial n} \right)_\rho - \frac{F}{T} \left( \frac{\partial T}{\partial n} \right)_\rho \quad , \quad (5.18)$$

and  $F$  is any one of the thermodynamic functions  $\alpha_i$ ,  $\beta_i$ .

Not all the plane steady shock solutions of Eq. (5.9) are physically acceptable. In particular, we require the solutions to be finite, continuous, single-valued, and stable. (Additional conditions for physically acceptable solutions could be imposed, such as uniqueness and structural stability (Majorana and Motta 1985), but will not be needed for the present work.) When the determinant of  $M$  in Eq. (5.9)

vanishes, the slope of the solution for each fluid variable diverges. Thus the solution will be multiple-valued whenever  $\det \mathbf{M} = 0$ .

A Maximum Upstream Mach Number in Thermally  
Nonconducting, Viscous Fluids

I show in this section that physically acceptable plane steady shock solutions do not exist above a maximum upstream Mach number in the Israel-Stewart Eckart frame theory for any thermally nonconducting ( $\kappa = 0$ ) fluid. Physically acceptable solutions do not exist when the characteristic velocity is exceeded. In the next section I consider the extension of this result to thermally conducting fluids.

Setting the thermal conductivity  $\kappa$  to zero implies, by Eq. (2.42), that  $q = 0$ . All thermodynamic variables then can be written as functions of  $v$  because  $n$  and  $\rho$  can by using Eqs. (5.1) and (5.7). Thus, for the thermally nonconducting case, Eq. (5.9) reduces to a two-dimensional autonomous system:

$$\mathbf{N} \begin{bmatrix} \frac{dv}{dx} \\ \frac{d\tau}{dx} \end{bmatrix} = -\frac{1}{\gamma} \begin{bmatrix} \frac{\tau}{\zeta} \\ \frac{3\sigma}{4\eta} \end{bmatrix} \quad (5.19)$$

The components of the 2x2 matrix  $\mathbf{N}$  are

$$\mathbf{N} = \begin{bmatrix} N_{11}\gamma^2 & \beta_0 v \\ N_{21}\gamma^2 & -\frac{3}{2}\beta_2 v \end{bmatrix}, \quad (5.20)$$

$$N_{11} = 1 + \frac{1}{2} \left[ \beta_0 - (\rho+p+\tau+\sigma)f(\beta_0) - ng(\beta_0) \right] \tau, \quad (5.21)$$

$$N_{21} = 1 + \frac{3}{2} \left[ (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s + \left( \frac{\partial p}{\partial \rho} \right)_n (\tau+\sigma) - (\rho+p+\tau+\sigma)v^2 \right] \beta_2 \\ + \frac{3}{4} \left[ \beta_2 - (\rho+p+\tau+\sigma)f(\beta_2) - ng(\beta_2) \right] \sigma, \quad (5.22)$$

and its determinant is

$$\det N = \frac{3}{2} \gamma^2 v (Av^2 + B), \quad (5.23)$$

where

$$A = \beta_0 \beta_2 (\rho+p+\tau+\sigma), \quad (5.24)$$

$$B = -\beta_2 - \frac{2}{3} \beta_0 - (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \beta_0 \beta_2 \\ + \frac{1}{2} \beta_2 \left[ (\rho+p+\tau+\sigma)f(\beta_0) + ng(\beta_0) - \beta_0 \left[ 1 + 2 \left( \frac{\partial p}{\partial \rho} \right)_n \right] \right] \tau \\ + \frac{1}{2} \beta_0 \left[ (\rho+p+\tau+\sigma)f(\beta_2) + ng(\beta_2) \right. \\ \left. - \beta_2 \left[ 1 + 2 \left( \frac{\partial p}{\partial \rho} \right)_n \right] \right] \sigma. \quad (5.25)$$

Solutions of Eq. (5.19) will be multiple-valued whenever the determinant in Eq. (5.23) vanishes. Thus, a physically acceptable solution will not have  $v^2 = -B/A$  for any value of  $x$ . (The determinant in Eq. (5.23) also vanishes for  $v = 0$ , but this is not a physically acceptable solution because  $n$  and  $\rho$  diverge for  $v \rightarrow 0$  by Eqs. (5.1) and (5.7)). In the equilibrium states at  $x = \pm\infty$ ,

$$-\frac{B}{A} = \left(\frac{\partial p}{\partial \rho}\right)_s + (\rho+p)^{-1} \left(\frac{1}{\beta_0} + \frac{2}{3\beta_2}\right) . \quad (5.26)$$

The equilibria of the Israel-Stewart theory are stable, in the sense that small perturbations away from equilibrium do not evolve in time to become large departures from equilibrium, if the thermodynamic variables and second-order coefficients satisfy a certain set of inequalities. The equilibria of the nonconducting Israel-Stewart Eckart frame theory for the plane steady shock solutions on a flat Minkowski space background satisfy Eqs. (2.26)-(2.31). The Liapunov functional techniques of Ch. 3 may be used to show that these equilibria will be stable if each of the following inequalities is satisfied:

$$(\rho+p)^{-1} \left(\frac{\partial \rho}{\partial p}\right)_s \geq 0 , \quad (5.27)$$

$$(\rho+p)^{-1} \left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_\theta \geq 0 , \quad (5.28)$$

$$(\rho+p) \left[ 1 - \left( \frac{\partial p}{\partial \rho} \right)_s \right] - \left( \frac{1}{\beta_0} + \frac{2}{3\beta_2} \right) \geq 0 \quad , \quad (5.29)$$

$$\rho+p - \frac{1}{2\beta_2} \geq 0 \quad , \quad (5.30)$$

$$\beta_0 \geq 0 \quad , \quad (5.31)$$

$$\beta_2 \geq 0 \quad . \quad (5.32)$$

Equations (5.27) and (5.29) require the square of the adiabatic sound speed to be bounded between zero and the speed of light. Equation (5.28) is the relativistic Schwarzschild criterion for stability against convection (Thorne 1967). Equations (5.29), (5.31), and (5.32) require the value of  $-B/A$  in equilibrium (Eq. (5.26)) to be bounded between  $(\partial p/\partial \rho)_s$  and the speed of light.

The existence of a maximum Mach number is established by the following theorem.

**THEOREM A.** Physically acceptable plane steady shock solutions (i.e. finite, continuous, and single-valued solutions for the fluid variables and stable equilibrium states at  $x = \pm\infty$ ) do not exist in the Israel-Stewart theory of thermally nonconducting, viscous fluids above a maximum upstream Mach number.



The proof of Theorem A is as follows. Consider all the finite and continuous shock solutions of Eq. (5.19) which connect stable equilibrium states at  $x = -\infty$  and  $x = +\infty$ . Suppose also that  $v^2(-\infty) > -B/A|_{x = -\infty}$ . For a shock solution  $v^2(+\infty) < (\partial p / \partial \rho)_s|_{x = +\infty}$ , so  $v^2(+\infty) < -B/A|_{x = +\infty}$  because  $-B/A > (\partial p / \partial \rho)_s$  for stable equilibria. Therefore, for these solutions there must be at least one value of  $x$  such that  $v^2 = -B/A$  because  $v$  and  $-B/A$  are continuous functions of  $x$ . These solutions will not be physically acceptable because they are multiple-valued. Therefore, physically acceptable plane steady shock solutions do not exist above the upstream Mach number  $M_{\max}^-$ , where

$$M_{\max}^- = 1 + (\rho + p)^{-1} \left( \frac{\partial \rho}{\partial p} \right)_s \left( \frac{1}{\beta_0} + \frac{2}{3\beta_2} \right) \quad (5.33)$$

Note that Eq. (5.33) is not a least upper bound for the existence of a physically acceptable solution; the solution for a particular fluid could become multiple-valued for  $M^- < M_{\max}^-$  if there is one value of  $x$  for which  $v^2 = -B/A$ .

Also of interest is to determine what property of the Israel-Stewart theory is responsible for the existence of a maximum Mach number. I now show that the plane steady shock solutions become multiple-valued when the fluid velocity exceeds the nonzero characteristic velocity of the nonconducting Israel-Stewart theory.

The system of equations for the plane symmetric motions of a thermally nonconducting fluid in the Israel-Stewart theory (Eqs. (2.32), (2.33), (2.41), and (2.43) with  $q$  and all  $y$  and  $z$  derivatives set to zero) has the form

$$A_B^A \mu_{\partial \mu} Y^B + B^A = 0 \quad , \quad (5.34)$$

where the components of  $Y^B$  are the 5 fields  $\rho, n, \psi, \tau, \sigma$  (the rapidity  $\psi$  is defined by  $u^\mu = (\cosh \psi, \sinh \psi, 0, 0)$ ), and the index  $A$  labels one of the 5 equations for these fields. The components of the matrices  $A_B^A \mu_{\partial \mu}$  and  $B^A$  do not contain time or space derivatives of the  $Y^B$ .

The level surfaces of a scalar function  $\phi(t, x)$  satisfying

$$\det \left( A_B^A \mu_{\partial \mu} \phi \right) = 0 \quad (5.35)$$

are the characteristic surfaces for these equations; the fields  $Y^B$  cannot be freely specified on these surfaces (Courant and Hilbert 1962, p. 170). The characteristic velocities, defined by

$$v = \frac{\partial_t \phi}{\partial_x \phi} \quad , \quad (5.36)$$

are the slopes of these surfaces. Discontinuities in the initial data propagate along the characteristic surfaces with these velocities.

The characteristic velocities are calculated by setting the determinant of the matrix  $vA_B^A t + A_B^A x$  to zero, where

$$vA_B^A t + A_B^A x = \begin{bmatrix} 0 & v & n & 0 & 0 \\ v & 0 & \rho+p+\tau+\sigma & 0 & 0 \\ \left(\frac{\partial p}{\partial \rho}\right)_n & \left(\frac{\partial p}{\partial n}\right)_\rho & (\rho+p+\tau+\sigma)v & 1 & 1 \\ \frac{1}{2}f(\beta_0)\tau v & \frac{1}{2}g(\beta_0)\tau v & 1 + \frac{1}{2}\beta_0\tau & \beta_0 v & 0 \\ \frac{3}{4}f(\beta_2)\sigma v & \frac{3}{4}g(\beta_2)\sigma v & 1 + \frac{3}{4}\beta_2\sigma & 0 & \frac{3}{2}\beta_2 v \end{bmatrix} \quad (5.37)$$

The characteristic velocities are then found to be

$$\frac{3}{2}v^3 \left( Av^2 + B \right) = 0 \quad , \quad (5.38)$$

where A and B are defined in Eqs. (5.24) and (5.25). The only nonzero characteristic velocity is  $v^2 = -B/A$ , which is the same as the condition for the plane steady solutions to be multiple-valued.

The nonconducting Israel-Stewart theory fails to adequately describe strong shock waves because it is a hyperbolic theory with causal characteristic velocities. When the shock solution is multiple-valued (a value of  $x$  where  $v^2$

=  $-B/A$ ), information cannot propagate fast enough ahead of this point to round off the abrupt shock front because  $v^2 = -B/A$ , the characteristic velocity, is also the velocity with which information propagates away from this shock front. So the fluid in front of the shock has not responded to the impending shock since no information about the shock has yet to reach this region. Thus, there is a maximum upstream Mach number for plane steady shocks in the Israel-Stewart theory as a direct consequence of the causality and hyperbolicity properties of the theory.

#### Maximum Mach Numbers in Thermally Conducting, Viscous Fluids

In this section I show that physically acceptable plane steady shock solutions do not exist in the Israel-Stewart theory in any thermally conducting, viscous fluid either above a maximum upstream Mach number, or below a minimum downstream Mach number. When either (or both) of these Mach numbers is crossed the solution is assured to be multiple-valued.

The solutions of Eq. (5.9) will be multiple-valued when the determinant of the matrix  $\mathbb{M}$  vanishes. This determinant has the form

$$\det \mathbb{M} = - \frac{3\gamma^2}{2v} \left( \tilde{A}v^6 + \tilde{B}v^5 + \tilde{C}v^4 + \tilde{D}v^3 + \tilde{E}v^2 + \tilde{F}v + \tilde{G} \right) \quad (5.39)$$

I choose not to display the coefficients of the powers of  $v$  in Eq. (5.39) because they are very long expressions when the

dissipation fields  $\tau$ ,  $q^a$ ,  $\tau^{ab}$  are nonzero. The solution will be multiple-valued when  $v(x)$  is one of the roots of the polynomial in Eq. (5.39). For the equilibrium states at  $x = \pm\infty$ , Eq. (5.39) becomes

$$\det \mathbf{M} = -\frac{3}{2}\gamma^2 v \left( \tilde{A}v^4 + \tilde{C}v^2 + \tilde{E} \right) , \quad (5.40)$$

where the equilibrium values ( $\tau=q^a=\tau^{ab}=0$ ) of the coefficients of the quartic are

$$\tilde{A} = \beta_0 \beta_2 [(\rho+p)\beta_1 - 1] , \quad (5.41)$$

$$\begin{aligned} \tilde{C} = & -(\rho+p) \left( \alpha_0^2 \beta_2 + \frac{2}{3} \alpha_1^2 \beta_0 \right) - (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \beta_0 \beta_1 \beta_2 - \left( \beta_2 + \frac{2}{3} \beta_0 \right) \beta_1 \\ & - 2 \left( \alpha_0 \beta_2 + \frac{2}{3} \alpha_1 \beta_0 \right) + \left[ \left( \frac{\partial p}{\partial \rho} \right)_s + \left( \frac{\partial \theta}{\partial s} \right)_\rho \right] \beta_0 \beta_2 , \end{aligned} \quad (5.42)$$

$$\begin{aligned} \tilde{E} = & \frac{2}{3} (\alpha_1 - \alpha_0)^2 + (\rho+p) \left( \frac{\partial p}{\partial \rho} \right)_s \left( \alpha_0^2 \beta_2 + \frac{2}{3} \alpha_1^2 \beta_0 \right) \\ & + \frac{1}{T^2 n} \left( \frac{\partial T}{\partial s} \right)_n \left( \beta_2 + \frac{2}{3} \beta_0 \right) + \frac{2n}{T} \left( \frac{\partial T}{\partial n} \right)_s \left( \alpha_0 \beta_2 + \frac{2}{3} \alpha_1 \beta_0 \right) \\ & - n^2 \left( \frac{\partial T}{\partial n} \right)_\theta \left( \frac{\partial \theta}{\partial \rho} \right)_n \beta_0 \beta_2 . \end{aligned} \quad (5.43)$$

The characteristic velocities for thermally conducting, viscous fluids can be calculated using the same procedure as was used for nonconducting fluids in the previous section.

The system of equations for the plane symmetric motions of a thermally conducting, viscous fluid has the form of Eq. (5.34), where in this case the components of  $Y^B$  are the 6 fields  $\rho, n, \psi, \tau, q, \sigma$ . The characteristic velocities  $v$  are the roots of

$$\det \left( v A_B^A t + A_B^A x \right) = 0 \quad (5.44)$$

For thermally conducting and viscous fluids,

$$v A_B^A t + A_B^A x =$$

$$\begin{bmatrix} 0 & v & n & 0 & 0 & 0 \\ v & 0 & \rho+p+\tau+\sigma+2qv & 0 & 1 & 0 \\ \left(\frac{\partial p}{\partial \rho}\right)_n & \left(\frac{\partial p}{\partial n}\right)_\rho & (\rho+p+\tau+\sigma)v+2q & 1 & v & 1 \\ a_1 & a_2 & 1+\frac{1}{2}\beta_0\tau-\alpha_0qv & \beta_0v & -\alpha_0 & 0 \\ b_1 & b_2 & (1-\alpha_1\sigma)v+\frac{1}{2}\beta_1q & -\alpha_0 & \beta_1v & -\alpha_1 \\ c_1 & c_2 & 1+\frac{3}{4}\beta_2\sigma & 0 & -\alpha_1 & \frac{3}{2}\beta_2v \end{bmatrix} \quad (5.45)$$

where

$$a_1 = \frac{1}{2}f(\beta_0)\tau v - \gamma_0 f(\alpha_0)q \quad (5.46)$$

$$a_2 = \frac{1}{2}g(\beta_0)\tau v - \gamma_0 g(\alpha_0)q \quad , \quad (5.47)$$

$$b_1 = \frac{1}{T}\left(\frac{\partial T}{\partial \rho}\right)_n + (\gamma_0 - 1)f(\alpha_0)\tau + \frac{1}{2}f(\beta_1)qv + (\gamma_1 - 1)f(\alpha_1)\sigma \quad , \quad (5.48)$$

$$b_2 = \frac{1}{T}\left(\frac{\partial T}{\partial n}\right)_\rho + (\gamma_0 - 1)g(\alpha_0)\tau + \frac{1}{2}g(\beta_1)qv + (\gamma_1 - 1)g(\alpha_1)\sigma \quad , \quad (5.49)$$

$$c_1 = \frac{3}{4}f(\beta_2)\sigma v - \gamma_1 f(\alpha_1)q \quad , \quad (5.50)$$

$$c_2 = \frac{3}{4}g(\beta_2)\sigma v - \gamma_1 g(\alpha_1)q \quad . \quad (5.51)$$

The characteristic equation (Eq. (5.44)) is then

$$- \frac{3}{2}\left(\tilde{A}v^6 + \tilde{B}v^5 + \tilde{C}v^4 + \tilde{D}v^3 + \tilde{E}v^2 + \tilde{F}v + \tilde{G}\right) = 0 \quad , \quad (5.52)$$

where each of the coefficients of the polynomial in Eq. (5.52) are exactly the same as the coefficients of the polynomial in Eq. (5.39). Just as in the thermally nonconducting case, the solution becomes multiple-valued at all values of  $x$  where the fluid velocity equals one of the characteristic velocities. For the equilibrium states at  $x = \pm\infty$ , Eq. (5.52) has the form

$$- \frac{3}{2}v^2 \left( \tilde{A}v^4 + \tilde{C}v^2 + \tilde{E} \right) = 0 \quad . \quad (5.53)$$

The two roots of the quartic in Eq. (5.53) are the two nonzero characteristic velocities for plane symmetric motions about an equilibrium state in a thermally conducting, viscous fluid in the Israel-Stewart Eckart frame theory. The Liapunov

functional techniques of Ch. 3 may be used to show (Hiscock and Lindblom 1983) that these characteristic velocities are real and less than the speed of light if and only if the corresponding equilibria are stable. Label the characteristic velocities for the equilibrium state at  $x = -\infty$  by  $v_1^-$  and  $v_2^-$  (ordered so that  $v_1^- \geq v_2^-$ ), and the characteristic velocities for the equilibrium state at  $x = +\infty$  by  $v_1^+$  and  $v_2^+$  (with  $v_1^+ \geq v_2^+$ ). These characteristic velocities may be ordered relative to the adiabatic sound speed  $v_s$  at  $x = \pm\infty$  in 9 distinct ways:

CLASS 1:

a.  $v_1^-, v_2^- < v_s(-\infty); v_1^+, v_2^+ \geq v_s(+\infty)$ .

b.  $v_1^-, v_2^- < v_s(-\infty); v_1^+ \geq v_s(+\infty) \geq v_2^+$ .

c.  $v_1^- \geq v_s(-\infty) \geq v_2^-; v_1^+, v_2^+ > v_s(+\infty)$ .

CLASS 2:

a.  $v_1^- \geq v_s(-\infty) \geq v_2^-; v_1^+ \geq v_s(+\infty) \geq v_2^+$ .

b.  $v_1^-, v_2^- \leq v_s(-\infty); v_1^+, v_2^+ < v_s(+\infty)$ .

c.  $v_1^-, v_2^- > v_s(-\infty); v_1^+, v_2^+ \geq v_s(+\infty)$ .



CLASS 3:

$$a. v_1^- > v_s(-\infty) \geq v_2^-; v_1^+, v_2^+ < v_s(+\infty).$$

$$b. v_1^-, v_2^- > v_s(-\infty); v_1^+ \geq v_s(+\infty) > v_2^+.$$

$$c. v_1^-, v_2^- > v_s(-\infty); v_1^+, v_2^+ < v_s(+\infty).$$

We then have the following theorem.

**THEOREM B.** Physically acceptable plane steady shock solutions (i.e. finite, continuous, and single-valued solutions for the fluid variables and stable equilibrium states at  $x = \pm\infty$ ) do not exist in the Israel-Stewart theory for thermally conducting, viscous fluids either above a maximum upstream Mach number, or below a minimum downstream Mach number.

Theorem B is proven as follows. Consider all the finite and continuous shock solutions of Eq. (5.9) which connect stable equilibrium states at  $x = \pm\infty$ . The 6 roots of the characteristic equation (Eq. (5.52)) are continuous functions of  $x$ , so there must be at least one continuous curve  $u_1(x)$  and at least one continuous curve  $u_2(x)$  such that  $u_1(x)$  and  $u_2(x)$  are roots of the characteristic equation for all  $x$ ,  $u_1(-\infty) = v_1^-$ ,  $u_1(+\infty) = v_1^+$  or  $u_1(+\infty) = v_2^+$ ,  $u_2(-\infty) = v_2^-$ , and  $u_2(+\infty) = v_1^+$  or  $u_2(+\infty) = v_2^+$ . A solution that has the fluid velocity

$v(x) = u_1(x)$  or  $v(x) = u_2(x)$  is multiple-valued at that value of  $x$ .

For solutions having equilibrium states with characteristic velocities in Class 1a, there is at least one value of  $x$  where  $v = u_1$  because  $v$  and  $u_1$  are continuous functions of  $x$ , and because  $v(-\infty) > v_s(-\infty)$  and  $v(+\infty) < v_s(+\infty)$  for a shock solution. For solutions having equilibrium states in Class 1c, there is at least one value of  $x$  where  $v = u_2$ . Solutions in Class 1b have at least one value of  $x$  where  $v = u_1$ , or at least one value of  $x$  where  $v = u_2$ . In each of these cases the solution is multiple-valued for all upstream and downstream Mach numbers. Thus, solutions having equilibrium states in Class 1 are not physically acceptable.

When the equilibrium states have characteristic velocities in Class 2, there can be physically acceptable solutions. Consider the solutions of Class 2a which have  $u_1(+\infty) = v_1^+$  and  $u_2(+\infty) = v_2^+$ . Any of these solutions which also have  $v(-\infty) > v_1^-$  or  $v(+\infty) < v_2^+$  will be multiple-valued because there is at least one value of  $x$  where  $v = u_1$  or at least one value of  $x$  where  $v = u_2$ . So physically acceptable solutions are assured not to exist above the upstream Mach number  $M_{\max}^- = v_1^-/v_s(-\infty)$ , and no physically acceptable solutions exist below the minimum downstream Mach number  $M_{\min}^+ = v_2^+/v_s(+\infty)$ . The solutions of Class 2a which have  $u_2(+\infty) = v_1^+$  are multiple-valued. Solutions of Class 2b have a minimum downstream Mach number  $M_{\min}^+ = v_1^+/v_s(+\infty)$ , and solutions of

Class 2c have a maximum upstream Mach number  $M_{\max}^- = v_2^-/v_s(-\infty)$ . Thus, for solutions having equilibrium states in Class 2, there are no physically acceptable solutions either above a maximum upstream Mach number, or below a minimum downstream Mach number. Note that I have not established the existence of physically acceptable solutions for weaker shocks; the solution for a particular fluid could become physically unacceptable for weaker shocks. I have shown that stronger shocks, those which have  $M^- > M_{\max}^-$  or  $M^+ < M_{\min}^+$ , are guaranteed to be physically unacceptable.

Physically acceptable weak shock solutions do not exist for solutions having equilibrium states with characteristic velocities in Class 3. The weak shock solutions of Class 3a, those which have  $v(-\infty) < v_1^-$  and  $v(+\infty) > v_1^+$ , are not physically acceptable because they are multiple-valued. The solutions of Class 3a which have  $v(-\infty) > v_1^-$  are assured to be multiple-valued for  $M^+ < v_1^+/v_s(+\infty)$ . When  $u_2(+\infty) = v_2^+$ , the solutions of Class 3a which have  $v(-\infty) < v_1^-$  can be physically acceptable provided that  $v_2^+ < v(+\infty) < v_1^+$ , but become multiple-valued when  $M^- > v_1^-/v_s(-\infty)$  or when  $M^+ < v_2^+/v_s(+\infty)$ . When  $u_2(+\infty) = v_1^+$  and  $v(-\infty) < v_1^-$  the solution is multiple-valued. So physically acceptable solutions of Class 3a do not exist for weak shocks, and do not exist above either a maximum upstream Mach number or below a minimum downstream Mach number. Similarly, it may be shown that no physically acceptable solutions of Class 3b and 3c exist for weak shocks,

and do not exist either above a maximum upstream Mach number or below a minimum downstream Mach number. This completes the proof of Theorem B.

So in the thermally conducting case the solutions become multiple-valued whenever the fluid velocity crosses any one of the characteristic velocities of the fluid. Just as in the nonconducting case, this breakdown of the theory is due to the fact that the Israel-Stewart theory is hyperbolic with causal characteristic velocities. Each characteristic velocity is associated with the propagation of information in a particular mode of the fluid. When the fluid velocity at  $x$  is below  $N$  of the characteristic velocities, then the solution will in general have nonzero support in each of the  $N$  corresponding modes of the fluid. If the solution for the fluid velocity becomes larger than one or more of these  $N$  characteristic velocities at some other value of  $x$ , then these corresponding modes can no longer propagate fast enough to keep up with the flow. The result is the solution becomes multiple-valued at that point.

#### Plane Steady Shocks in Simple Gases

The second-order coefficients, the thermodynamic functions  $\alpha_i$  and  $\beta_i$ , have been calculated for the Boltzmann gas (Israel and Stewart 1979b) and the degenerate free Fermi gas (Olson and Hiscock 1989) using relativistic kinetic theory. In this section I use these expressions to show that

the only physically acceptable plane steady shock solutions that can exist in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas and the degenerate free Fermi gas are extremely weak.

The two nonzero characteristic velocities for plane symmetric perturbations about equilibrium in the Israel-Stewart theory are the two roots  $v_1^2$ ,  $v_2^2$  of

$$\tilde{A}v^4 + \tilde{C}v^2 + \tilde{E} = 0 \quad . \quad (5.54)$$

The coefficients of this quartic are defined in Eqs. (5.41)-(5.43). For the Boltzmann gas in the nonrelativistic limit ( $\beta = m/kT \rightarrow \infty$ ),  $v_1^2 \approx 5.18\beta^{-1}$  and  $v_2^2 \approx 1.35\beta^{-1}$ . These characteristic velocities are in Class 2a (using the classification scheme of the previous section) because the square of the adiabatic sound speed is  $v_s^2 = 5/3\beta \approx 1.67\beta^{-1}$  in the nonrelativistic limit. Thus, by Theorem B of the last section, there exists a maximum upstream Mach number  $M_{\max}^- \approx 1.76$ , and a minimum downstream Mach number  $M_{\min}^+ \approx 0.90$ . Stronger shocks are assured to be multiple-valued, and so physically unacceptable.

Grad (1952) considered the plane steady shock solutions of his kinetic theory method for the nonrelativistic Boltzmann gas, which are the same as the Israel-Stewart theory solutions when kinetic theory values are used for the second-order coefficients. Grad found these solutions to be multiple-valued above an upstream Mach number of 1.65, rather than 1.76

as I have found, because he assumed the bulk viscosity to be identically zero for this gas. However, the bulk viscosity should not be set identically to zero for this gas because it is small, but nonzero, at nonzero temperatures (Weinberg 1972, p. 57), and a nonzero bulk viscosity has a significant effect on the characteristic velocities, and hence a significant effect on the maximum shock strength.

In the ultrarelativistic limit ( $\beta \rightarrow 0$ ), the two nonzero characteristic velocities for the Boltzmann gas are

$$v_1^2 = \frac{3}{5} \left[ 1 - \left| o(\beta^2 \ln \beta) \right| \right] , \quad (5.55)$$

$$v_2^2 = \frac{1}{3} \left[ 1 - \left| o(\beta^2 \ln \beta) \right| \right] . \quad (5.56)$$

The square of the adiabatic sound speed is

$$v_s^2 = \frac{1}{3} \left[ 1 - \left| o(\beta^2 \ln \beta) \right| \right] , \quad (5.57)$$

so these characteristic velocities are in Class 2a because  $v_1^2 > v_s^2 > v_2^2$ . Thus, there is a maximum upstream Mach number  $M_{\max}^- = (9/5)^{1/2} [1 - |o(\beta^2 \ln \beta)|]$ , and a minimum downstream Mach number  $M_{\min}^+ = 1 - |o(\beta^2 \ln \beta)|$ . So physically acceptable solutions can exist only for extremely weak shocks in the ultrarelativistic limit. For example,  $M_{\min}^+ = 1 - |o(10^{-9})|$  for  $\beta = 10^{-5}$ .

Majorana and Motta (1985) also considered the plane steady shock solutions of the ultrarelativistic Boltzmann gas. They found that physically acceptable solutions exist provided a maximum upstream Mach number is not exceeded. (Majorana and Motta used wrong expressions for the second-order coefficients  $\alpha_1$  and  $\beta_1$ . When the correct expressions (Israel and Stewart 1979b) are used,  $\alpha_1 = -1/2p$  and  $\beta_1 = 3/2p$ , the maximum upstream Mach number is found to be  $M_{\max}^- \approx 1.32$ ). However, they assumed that the fluid has zero bulk viscosity. Although it is true that the bulk viscosity vanishes in the ultrarelativistic limit ( $\beta \rightarrow 0$ ), it is nonzero, although small, for finite temperatures (Israel 1963; Anderson 1970; Stewart 1971). When the bulk viscosity of the ultrarelativistic Boltzmann gas is nonzero (which it is for finite temperatures), only extremely weak plane steady shock solutions are physically acceptable because the solutions are multiple-valued when  $M^+ < M_{\min}^+$ .

The two nonzero characteristic velocities for plane symmetric motions about the equilibrium states of the degenerate free Fermi gas (the two roots of Eq. (5.54)) may be determined using the expressions for the second-order coefficients calculated by Olson and Hiscock (1989). The result is

$$v_1^2 = \frac{9}{5}v_s^2 + O(\beta v)^{-2}, \quad (5.58)$$

$$v_2^2 = v_s^2 + O(\beta v)^{-2} \quad , \quad (5.59)$$

where the adiabatic sound speed  $v_s$  is

$$v_s^2 = \frac{v^2 + 2v}{3(v + 1)^2} + O(\beta v)^{-2} \quad . \quad (5.60)$$

The thermodynamic function  $v$  is the non-rest mass part of the chemical potential per unit mass,

$$mv = \frac{\rho + p}{n} - Ts - m \quad , \quad (5.61)$$

and the degenerate limit is  $\beta v \gg 1$ . Depending on the sign of the  $O(\beta v)^{-2}$  term in Eq. (5.59), these characteristic velocities are either in Class 2a or 2c. In either case, physically acceptable solutions can only exist for extremely weak shocks in the degenerate free Fermi gas. I have not explicitly calculated this  $O(\beta v)^{-2}$  correction because this calculation is extremely tedious due to very many cancellations in the Sommerfeld expansion.

### Conclusions

I have shown that the Israel-Stewart Eckart frame theory fails to adequately describe the structure of strong plane steady shock waves in any single-component fluid, in that the solutions become multiple-valued for strong shocks. In this section I consider whether this is a serious pathology of the theory.



A basic requirement of an acceptable theory of relativistic dissipative fluids is that it be a hyperbolic theory with causal characteristic velocities. In any such theory, which includes both the Israel-Stewart Eckart frame and energy frame theories, the plane steady shock solutions will become multiple-valued when the fluid velocity crosses one of the characteristic velocities because the characteristic velocities are the speeds with which information propagates ahead of a discontinuity. The existence of a maximum shock strength, beyond which physically acceptable solutions do not exist, is a generic property of these theories. Thus, the failure of the Israel-Stewart theories to adequately describe strong shock waves should be considered a failure of the fluid approximation, rather than a pathology peculiar to the Israel-Stewart theories.

Theories of the Israel-Stewart type are derived by truncating an expansion in deviations from equilibrium after second order, analogous to the Grad method for the solution of the Boltzmann equation (Grad 1949). So the Israel-Stewart theory should not be expected to adequately describe very strong shock waves because the higher-order terms will be important when the fluid is far from equilibrium. At least in the case of a nonrelativistic Boltzmann gas with zero bulk viscosity, the plane steady shock solutions become multiple-valued precisely when these higher-order terms become important (Grad 1952). However, it seems unlikely that the

neglect of higher-order terms could explain the failure of the Israel-Stewart Eckart frame theory to adequately describe plane steady shock waves for all but extremely weak shocks in the ultrarelativistic Boltzmann gas and the degenerate free Fermi gas.

Although a candidate theory of relativistic dissipative fluids should perhaps be expected to be capable of at least describing weak shocks in these simple gases, the failure of the Israel-Stewart Eckart frame theory to do so is not necessarily serious. The ultrarelativistic Boltzmann gas is not applicable to realistic physical systems because the existence of antiparticles, which will be present in large numbers at ultrarelativistic temperatures, is ignored in this model. Further, there also will be massless particles (photons) present which, when appropriate interactions with the degenerate free Fermi gas are taken into account, will allow for the transmission of information at light speed.

It might also be hoped that physically acceptable plane steady shock solutions exist for strong shocks in the Israel-Stewart energy frame theory for the ultrarelativistic Boltzmann gas and the degenerate free Fermi gas, since there is evidence that the energy frame theory is stable, causal, and hyperbolic much farther from equilibrium than the Israel-Stewart Eckart frame theory (Hiscock and Olson 1989). Unfortunately, this is not the case because the Eckart frame theory characteristic velocities for these simple gases are

identical to the energy frame theory characteristic velocities when the kinetic theory values are used for the second-order coefficients. Thus, physically acceptable plane steady shock solutions only exist for extremely weak shocks in both the Eckart frame and energy frame theories for these simple gases.

A separate question is whether or not the Israel-Stewart theory is capable of describing discontinuous shock waves. For many applications shock waves are most simply treated by piecing together continuous solutions at judiciously chosen points. Shock waves are thus modeled as discontinuities in the fluid variables at discrete points, rather than regions of nonzero extent. This procedure is justified when the detailed structure of the shock wave is not important to the evolution of the fluid. Discontinuous solutions can be given mathematical meaning in theories of divergence type, such as the theories developed by Liu, Müller, Ruggeri (1986), and by Geroch and Lindblom (1990). These theories may be more appropriate for these applications than the Israel-Stewart theories, because the Israel-Stewart theories are not known to be of divergence type.

## CHAPTER SIX

## SUMMARY

This chapter begins with a summary of this dissertation. I conclude with my opinion of the present state of the theory of relativistic dissipative fluids.

Dissertation Summary

The first part of this dissertation was a study of the Israel-Stewart energy frame theory. The energy frame theory was reviewed in Ch. 2. In Ch. 3, the behavior of the energy frame theory near equilibrium was examined. The stability of the equilibrium states of the theory for small perturbations was determined by first linearizing the equations of motion. A set of stability conditions was derived by constructing a Liapunov functional, a nonincreasing function of time which is quadratic in the perturbation fields. Satisfaction of these stability conditions is a sufficient condition for the stability of the energy frame equilibria. A somewhat incomplete proof that satisfaction of these stability conditions is also a necessary condition for the stability of those equilibria that have a real adiabatic sound speed and are stable against convection was also given.

The causality and hyperbolicity properties of the Israel-Stewart energy frame theory near equilibrium were also studied in Ch. 3. It was found that perturbations can propagate with three distinct characteristic velocities in the energy frame theory. The conditions which must be satisfied for the perturbation equations to form a symmetric hyperbolic system, so that a well-posed initial value problem exists, were also obtained.

The main conclusion from the analysis of the properties of the Israel-Stewart energy frame theory near equilibrium performed in Ch. 3 is that the stability conditions are equivalent to the causality and hyperbolicity conditions. If an equilibrium state of a fluid described by the energy frame theory satisfies each of the stability conditions, then small perturbations about this equilibrium state will propagate subluminally and obey a symmetric hyperbolic system of equations. The converse is also true: if the perturbations of an equilibrium state propagate causally via a symmetric hyperbolic system of equations, then this equilibrium state is stable. Thus, just as for the linearized Israel-Stewart Eckart frame theory, stability is equivalent to causality and hyperbolicity in the linearized Israel-Stewart energy frame theory.

The behavior of the Israel-Stewart energy frame theory far from equilibrium was examined in Ch. 4 by considering the plane symmetric motions of an inviscid ultrarelativistic

Boltzmann gas. The main conclusion of this chapter was that the behavior of the energy frame theory far from equilibrium is significantly better than the Israel-Stewart Eckart frame theory for this particular fluid. The Eckart frame theory becomes acausal and nonhyperbolic, and the spatially homogeneous zero momentum mode becomes unstable, well before the weak energy condition is violated. Thus the Eckart frame theory breaks down well before it might reasonably be expected to do so. There is no connection between stability, causality, and hyperbolicity in the Eckart frame theory: the theory does not become unstable, acausal, and nonhyperbolic at the same value of the heat flux.

By contrast, the energy frame theory was shown to be hyperbolic for the inviscid ultrarelativistic Boltzmann gas for arbitrarily large values of the particle diffusion. In addition, the energy frame theory becomes acausal for this particular gas only very far from equilibrium, well into the regime where the particle number current is spacelike. The spatially homogeneous zero momentum mode for the inviscid ultrarelativistic Boltzmann gas in the energy frame theory is stable for arbitrarily large deviations from equilibrium. There is a connection between stability and causality far from equilibrium in the energy frame theory. It was shown that acausality implies instability in the energy frame theory, at least for the case of the inviscid ultrarelativistic Boltzmann gas, because there is a spatially homogeneous mode which

becomes unstable precisely at a value of the particle diffusion for which one of the characteristic velocities exceeds the speed of light. Thus the energy frame theory does not break down before it would be expected to, not until after the particle number current becomes spacelike.

The second part of this dissertation was a study of the plane steady shock wave solutions of the Israel-Stewart Eckart frame theory. The Eckart frame theory was reviewed in Ch. 2. The system of differential equations which has solutions that describe plane steady shock waves was derived in Ch. 5. Physically acceptable solutions were required to be finite, continuous, single-valued, and stable. The condition for a solution to be single-valued was found.

The Israel-Stewart theory was shown to fail to adequately describe the structure of strong shock waves. Physically acceptable solutions do not exist above a maximum upstream Mach number in any thermally nonconducting, viscous fluid described by the Israel-Stewart theory. Solutions describing shock waves with upstream Mach numbers larger than this maximum are guaranteed to be multiple-valued because the characteristic velocity is exceeded; information then cannot propagate fast enough ahead of the shock wave to round off the abrupt shock front. It was also proven that physically acceptable solutions do not exist for thermally conducting, viscous fluids above either a maximum upstream Mach number, or else below a minimum downstream Mach number. When one of

these Mach numbers is crossed the solution is assured to be multiple-valued, and so not physically acceptable. Again, these limiting Mach numbers correspond to the characteristic velocities of the fluid.

It was also found in Ch. 5 that the only physically acceptable plane steady shock solutions that can exist in the Israel-Stewart theory for the ultrarelativistic Boltzmann gas are extremely weak shocks. Also, only extremely weak physically acceptable plane steady shock solutions can exist in the Israel-Stewart theory for the free Fermi gas at any temperature, including relativistic temperatures, for which the gas is degenerate.

#### The Theory of Relativistic Dissipative Fluids at Present

I believe the Israel-Stewart energy frame theory should be the theory of choice for use in applications to problems requiring a phenomenological theory of relativistic fluids that includes the effects of dissipation. The older theories due to Eckart (1940) and Landau and Lifshitz (1959, Sec. 127) are pathological in that the equilibria of these theories are generically unstable (Hiscock and Lindblom 1983), and perturbations can propagate faster than the speed of light in these theories (Hiscock and Lindblom 1987). Although the Israel-Stewart Eckart frame and energy frame theories are equally well-behaved near equilibrium, since both theories have stable equilibria and perturbations about an equilibrium



state will propagate causally via hyperbolic equations if and only if the equilibrium state satisfies the stability conditions, the energy frame theory is well-behaved much farther from equilibrium than the Eckart frame theory for the inviscid ultrarelativistic Boltzmann gas with planar symmetry. More work is necessary to determine whether the energy frame theory is generically better than the Eckart frame theory far from equilibrium.

The failure of the Israel-Stewart theory to adequately describe the structure of strong shock waves is a drawback of the theory. However, I believe this will be a property of any causal and hyperbolic, and hence physically acceptable, theory of relativistic dissipative fluids. Thus the breakdown of the theory in this case should be regarded as a failure of the fluid approximation, not a failure peculiar to the Israel-Stewart theory.

There are a few other theories of relativistic dissipative fluids that have been developed recently. A phenomenological theory of inviscid fluids developed by Carter (1989), the "regular" theory, has been studied for small perturbations about equilibrium by Olson and Hiscock (1990a). It was found that Carter's theory when linearized is equivalent to the inviscid limit of the linearized Israel-Stewart Eckart frame theory for a particular choice of the second-order coefficients  $\beta_1$  and  $\gamma_2$ . The functional forms of these second-order coefficients are derivable from the

equation of state of the fluid in Carter's theory, unlike in the Israel-Stewart theory where these functions are not derivable from the equation of state, but are separately chosen thermodynamic functions. Thus Carter's theory would seem to have a distinct practical advantage over the Israel-Stewart theory. Unfortunately, the particular functional forms chosen for these second-order coefficients in Carter's theory leads to unacceptable behavior for the simple gases. Carter's theory predicts that both the nonrelativistic Boltzmann gas and the degenerate free Fermi gas (at all temperatures for which the gas can be considered degenerate) are unstable and acausal (Olson and Hiscock 1990a). For example, argon at atmospheric pressure and room temperature, treated as a nonrelativistic Boltzmann gas, is predicted by Carter's theory to have perturbations propagating at around 3.6 times the speed of light. By contrast, the Israel-Stewart theory predicts these gases to be stable and causal. Consequently, I do not believe Carter's theory to be a generally acceptable alternative to the inviscid Israel-Stewart theory.

Another class of relativistic dissipative fluid theories are the theories of divergence type (Liu, Müller, Ruggeri 1986; Geroch and Lindblom 1990). As I noted in Ch. 5, these theories may prove to be better-suited to the study of discontinuous shock waves. Some members of this class of theories are stable and causal near equilibrium. The main

difficulty I see with the application of these theories is that the dynamical variables used to describe a fluid in these theories are not easily interpreted in terms of the usual variables of classical fluid mechanics (a fluid energy density, heat flux, etc.). Perhaps after further study the theories of divergence type will be found to be better for applications than the Israel-Stewart theories. But, at present I believe the Israel-Stewart energy frame theory is the best choice.

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