A constitutive equation for snow subjected to long-duration small strain-rates
by Bhushan Wasudeo Dandekar

A thesis submitted in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Civil Engineering
Montana State University

Abstract:
This paper develops a stress-strain equation for snow subjected to small strains and strain-rates. The stress is split into pure shear and volumetric effects. Pure shear is defined in terms of Newtonian fluid behavior.

Finite difference equations are developed to analyze the deformation of snow subjected to flexible surface loading and are numerically solved. The results are compared with the available experimental data. Effects of size, shape and density on the settlement of snow are discussed. Stress distribution in the snow-mass is plotted. The settlement and effect of load on settlement are found to agree with the experimental data. The stress distribution under the footing is similar to the stress distribution obtained by the Boussinesq solution for a point load on a homogeneous and isotropic elastic half space. For a good quantitative agreement between theory and experiment, the effects of equitemperature metamorphism should be included in such solutions.

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Date March 17, 1982

A CONSTITUTIVE EQUATION FOR SNOW SUBJECTED TO LONG-DURATION SMALL STRAIN-RATES

by

BHUSHAN WASUDEO DANDEKAR

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Approved:

Fred F. Videon
Chairperson, Graduate Committee

Theodore T. Williams
Head, Major Department

Michael P. Malone
Graduate Dean

MONTANA STATE UNIVERSITY
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ABSTRACT

This paper develops a stress-strain equation for snow subjected to small strains and strain-rates. The stress is split into pure shear and volumetric effects. Pure shear is defined in terms of Newtonian fluid behavior.

Finite difference equations are developed to analyze the deformation of snow subjected to flexible surface loading and are numerically solved. The results are compared with the available experimental data. Effects of size, shape and density on the settlement of snow are discussed. Stress distribution in the snow-mass is plotted. The settlement and effect of load on settlement are found to agree with the experimental data. The stress distribution under the footing is similar to the stress distribution obtained by the Boussinesq solution for a point load on a homogeneous and isotropic elastic half space. For a good quantitative agreement between theory and experiment, the effects of equitemperature metamorphism should be included in such solutions.

INTRODUCTION

Understanding the behaviour of snow-mass subjected to load distribution is important in the development of design criteria for foundations, snow roads and runways in the polar areas. This interest has also been in part due to increased recreational use of alpine areas and the increased construction of homes and roads in such areas. Any object placed at or near the surface of an ice cap moves downward with time as a combined result of natural densification of the snow and the penetration of the object. The penetration of an object is of primary interest to the design engineer since differential movements can impose severe stresses in structural elements.

Observations have shown that this penetration, or settlement, of footings into snow depends upon: snow density, temperature, time, load intensity, footing size, rigidity and shape (13). In particular the properties of snow are closely related to density and temperature. Cohesive snow subjected to non-destructive loading behaves as a viscoelastic material. Upon application of a fixed load, there is an instantaneous and a recoverable elastic deformation, followed by a decelerating primary creep, which is partly recoverable by relaxation. The final stage is a steady secondary creep which is not recoverable.

Under a constant load, gradual consolidation occurs which, depending on temperature and density, may lead to ultimate collapse. If the loading is sufficiently low, compaction alone may not lead to complete failure of the material, since moderate penetration in snow is usually limited by increasing resistance as the deforming snow compacts.

Snow is a granular or porous material comprising an ice-air mixture. Because of its high porosity it is capable of undergoing large deformations that are largely irreversible.

Specifically it is a nonlinear viscoelastic material exhibiting a high degree of compressibility. A considerable number of investigators have published papers on nonlinear constitutive equations for viscoelastic materials and snow. A brief overview of these existing models is presented here.

Kerr (6,7), treated the problem of foundations on snow by representing the snow mass in the form of elastic and viscous elements. Kerr's model is a viscoelastic modification of Pasternak foundation model, with viscous shear elements to simulate the shear interaction between snow under the loaded area and snow adjacent to the loaded area. Figure 1 shows the foundation model.

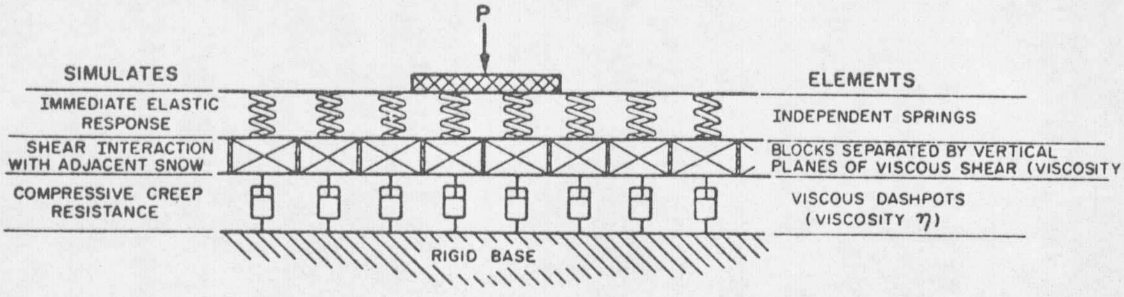


Figure 1. The Kerr viscoelastic foundation model.

The governing differential equation for vertical displacement of foundation is:

$$\mu\left(\frac{\partial^3 w}{\partial x^2\,\partial t} + \frac{\partial^3 w}{\partial y^2\,\partial t}\right) - \eta\,\frac{\partial w}{\partial t} + p\,(x,y,t) = 0 \qquad (1.1)$$

where μ = shear viscosity constant

η = compressible viscosity constant

Kerr assumed that a foundation under constant load settles at a constant rate. For long duration this is not quite so; as the loaded snow densifies, its deformation resistance increases. Therefore the validity of this model is restricted to small penetrations and low stresses.

Salm's constitutive equation (16) is similar to Hooke's law. The stress components T_{ij} are given by the equation

$$T_{ij} = \lambda V_{kk}\,\delta_{ij} + 2\mu\,V_{ij} \qquad (1.2)$$

where V_{ij} = rate of deformation

δ_{ij} = kronecker delta

Salm assumes the viscosity constants λ and μ to be functions of the basic invariants of deformation rates. This equation does not take into account that the change in volume is also partly due to the result of relative sliding between particles. This constitutive equation can be used for low strain-rates.

Behaviour of snow-mass under natural loading can be computed by the finite-element method; if stress-strain relationships for snow is available. Desrues et al. (4) have developed an incremental formulation of constitutive equation, which is particularly well suited for the finite element method. According to Desrues, the history of the stresses has an important influence on creep of snow. Therefore the Boltzman's superposition principle is used in developing the equation, which for uniaxial creep under constant stress is

$$\dot{E}(t) = \frac{B}{(At + B)^2} + C \tag{1.3}$$

E is the longitudinal strain, a superposed dot implies time differentiation. The first term on the right hand side represents the short term, i.e., elastic-plastic part of the deformation. A and B are functions of principal stress. C is the long-term element of the creep rate. The incremental formulation is

$$\dot{E}(t) = \sum_k \dot{E}_k (\Delta \sigma_k, \tau_k, t) \tag{1.4}$$

Where $\Delta \sigma_k$ is the stress increment, τ_k is the loading time of the stress increment and $[\tau_k - \tau_{k-1}]$ is the time increment.

The model gives good results for strains up to 10% and short term loading, but has not yet been extended to solve the long-term creep problem.

The constitutive equations discussed by Brown et al. (3) are

(a) Finite linear viscoelastic theory

(b) Second order finite viscoelastic theory

(c) The multiple integral representation

All the three equations represent nonlinear viscoelastic materials. In their general form these equations are too complicated to work with. The finite linear theory and second order theory appear to be representative equations to use when small and intermediate strain rates are involved. The multiple integral representation gives the best overall results when the full range of deformation rates are considered. A thermodynamic formulation developed by Brown (2) for nonlinear compressible viscoelastic materials, also gives very good results over a broad range of strain-rates, but this model is too complicated to use for practical purposes.

The results discussed by Brown et al (3) shows that snow is an extremely nonlinear material, the stress response varying approximately with the cube of strain-rate. Additionally the results also indicate high-degree of compressibility of snow over the entire strain-rate range considered. This behaviour is too complicated to be represented by a simple constitutive equation. In this paper only small strain-rates are considered. The primary inent is to develop a constitutive equation for a snow-mass subjected to long-term loading, which can be used in developing design criteria for foundations on snow.

In developing the constitutive law, elastic strain and transient creep effects are not considered, since interest centers on the problems of long-duration loading. The stress tensor is separated into deviatoric and volumetric parts. The deviatoric stress is expressed in terms of strain-rates, and the volumetric stress or hydrostatic pressure is expressed by a constitutive equation. This constitutive equation is modified version of the original law developed by Brown (1). Finite difference numerical technique is used to solve the equilibrium equation for deformation rates. A flexible surface loading problem is solved. The discussion includes effect of load intensity on deformation, stress-distribution inside the snow-mass and comparison with the experimental data.

KINEMATICS AND FIELD EQUATIONS

This section covers a brief discussion of kinematics and field equations. Only the topics which are required for the development of constitutive equation are covered. A detailed discussion on this subject can be found in any standard textbook on continuum mechanics (5,9,10).

When a continuum is in motion, tensor quantities that are associated with specific particles change with time. Following two types of description of these changes are in common use:

(1) Referrential description, whose independent variables are the position X of the particle in an arbitrary chosen reference configuration, and the time t. When the reference configuration is chosen to be the actual initial configuration at time t = 0, the description is called the Lagrangian description.

(2) Spatial description, whose independent variables are the present position x occupied by the particle at the present time t. This description is often called the Eulerian description.

Figure 2 shows the displacement of a particle from its initial position X to the current configuration x, defined by the equations:

$$\underline{x} = \underline{X} + \underline{U}(X,t)$$

or

$$\underline{x} = \underline{x}(X,t) \qquad (2.1)$$

The deformation gradient referred to the undeformed configuration is denoted by $\underset{\sim}{F}$. The deformation gradient is defined as the tensor, whose rectangular cartesian components

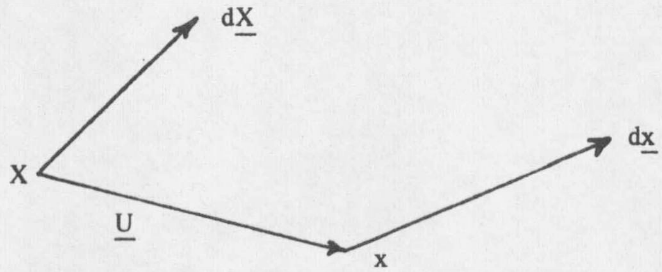


Figure 2. Displacement, stretch and rotation of a differential material vector $\underline{dX}$ to a new vector $\underline{dx}$.

are the partial derivatives $\partial x_k / \partial X_m$. It operates on a vector $\underline{dX}$ at X to associate with the vector $\underline{dx}$ at x. The vector $\underline{dx}$ is defined by the equation

$$\underline{dx} = \underset{\sim}{F}\, \underline{dX}$$

or

$$dx_k = \frac{\partial x_k}{\partial X_m} dX_m \tag{2.2}$$

In terms of displacements $\underset{\sim}{F}$, therefore has the the components

$$F_{IJ} = \begin{bmatrix} \left(1 + \frac{\partial U_1}{\partial X_1}\right) & \frac{\partial U_1}{\partial X_2} & \frac{\partial U_1}{\partial X_3} \\ \frac{\partial U_2}{\partial X_1} & \left(1 + \frac{\partial U_2}{\partial X_2}\right) & \frac{\partial U_2}{\partial X_3} \\ \frac{\partial U_3}{\partial X_1} & \frac{\partial U_3}{\partial X_2} & \left(1 + \frac{\partial U_3}{\partial X_3}\right) \end{bmatrix} \tag{2.3}$$

Rate of deformation tensor and spin tensor are important kinematic tensors. These are defined in terms of the velocity vector $\underline{v} = \underline{v}\,(x,t)$ expressed in terms of the spatial co-ordinates and the time. The rate of change of length and direction of the material element $\underline{dx}$ is given as

$$\frac{d}{dt}(d\underline{x}) = \underline{v}\,(x+dx,t) - \underline{v}(x,t) \tag{2.4}$$

or

$$d\underline{v} = \underset{\sim}{L}\, d\underline{x} \tag{2.5}$$

where

$$L_{IJ} = \frac{\partial v_I}{\partial x_J} \tag{2.6}$$

is the spatial gradient of the velocity. $\underset{\sim}{L}$ may be expressed as the sum of a symmetric tensor $\underset{\sim}{D}$ called the rate of deformation tensor and a skew-symmetric tensor $\underset{\sim}{W}$ called the spin tensor as follows. Let

$$\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T)$$

and

$$\underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{L} - \underset{\sim}{L}^T) \tag{2.7}$$

A superposed T implies the transpose of a tensor. Then

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W} \tag{2.8}$$

with rectangular cartesian components

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2.9}$$

$$W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{2.10}$$

The geometrical interpretation of tensor $\underset{\sim}{D}$ may be seen as follows. Let $d\underline{x}$ be a material vector, then

$$d\underline{x} \cdot d\underline{x} = (ds)^2 \tag{2.11}$$

hence

$$\frac{d}{dt}(ds)^2 = 2\, d\underline{x} \cdot \frac{d}{dt}(d\underline{x}) \tag{2.12}$$

But

$$\frac{d}{dt}(d\underline{x}) = \underset{\sim}{L}\, d\underline{x} \tag{2.13}$$

Therefore

$$\begin{aligned} \frac{d}{dt}(ds)^2 &= 2\, d\underline{x}\, \underset{\sim}{L}\, d\underline{x} \\ &= 2\, dx_i\, L_{ij}\, dx_j \\ &= 2\, dx_i\, D_{ij}\, dx_j + 2\, dx_i\, W_{ij}\, dx_j \end{aligned} \tag{2.14}$$

But as $\underset{\sim}{W}$ is skew-symmetric, the last term vanishes. Hence

$$\frac{d}{dt}(ds)^2 = 2\, dx_i\, D_{ij}\, dx_j = 2\, d\underline{x}\, \underset{\sim}{D}\, d\underline{x} \tag{2.15}$$

In other words, $\underset{\sim}{D}$ gives the rate of change of squared length $(ds)^2$ of the material vector $d\underline{x}$ at P. This is why it is called the rate of deformation tensor.

The strain tensors are defined so that they give change in squared length of the material vector $\underline{dX}$ as follows. The Lagrangian formulation is

$$(ds)^2 - (dS)^2 = 2\, \underline{dX}\, \underset{\sim}{E}\, \underline{dX}$$

$$(ds)^2 - (dS)^2 = 2\, dX_i\, E_{ij}\, dX_j \tag{2.16}$$

while the Eulerian formulation is

$$(ds)^2 - (dS)^2 = 2\, \underline{dx}\, \underset{\sim}{e}\, \underline{dx}$$

$$(ds)^2 - (dS)^2 = 2\, dx_i\, e_{ij}\, dx_j \tag{2.17}$$

The deformation tensor $\underset{\sim}{C}$ is closely related to the strain tensor. The Green deformation tensor $\underset{\sim}{C}$, referred to the undeformed configuration, gives the new squared length $(ds)^2$

$$(ds)^2 = \underline{dX}\, \underset{\sim}{C}\, \underline{dX} = dX_I\, C_{IJ}\, dX_J \tag{2.18}$$

Comparing equations (2.16) and (2.18), the following relationship can be deduced

$$2\, \underset{\sim}{E} = \underset{\sim}{C} - \underset{\sim}{1}$$

or

$$2\, E_{IJ} = C_{IJ} - \delta_{IJ} \tag{2.19}$$

$\underset{\sim}{1}$ is the identity tensor, with components equal to the kronecker delta, δ_{IJ}.

Expressions for the strain and Green deformation tensors in terms of the deformation gradient are obtained as follows

$$(ds)^2 = \underline{dx} \cdot \underline{dx} = (\underline{dX}\, \underset{\sim}{F}^T)(\underset{\sim}{F}\, \underline{dX}) = \underline{dX}\, (\underset{\sim}{F}^T \underset{\sim}{F})\, \underline{dX} \tag{2.20}$$

Hence

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}$$

or

$$C_{IJ} = \frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J} \tag{2.21}$$

An expression for the strain tensor can be obtained from equations (2.19) and (2.21).

$$\underset{\sim}{E} = \frac{1}{2} [\underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{1}]$$

or

$$E_{IJ} = \frac{1}{2} [\frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_j} - \delta_{IJ}] \tag{2.22}$$

From the above equations it follows that strain and Green deformation tensors are symmetric. The Green deformation tensor reduces to unit tensor $\underset{\sim}{1}$, when there is no strain, while the strain tensor reduces to zero. Using the relation $\underline{x} = \underline{X} + \underline{U}(X,t)$, the strain tensor can be expressed in terms of displacements

$$E_{IJ} = \frac{1}{2} [\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_k}{\partial X_I} \frac{\partial U_k}{\partial X_J}] \tag{2.23}$$

For small strain the higher order term is neglected.

The strain rate and rate of deformation tensors are related as follows. Material derivative of equation (2.16) gives

$$\frac{d}{dt} (ds)^2 = 2\, d\underline{X} \frac{d\underset{\sim}{E}}{dt} d\underline{X} \tag{2.24}$$

Since (ds) and $d\underline{x}$ are constants. But the equation (2.15) shows that

$$\frac{d}{dt} (ds)^2 = 2\, d\underline{x}\, \underset{\sim}{D}\, d\underline{x}$$

hence

$$\frac{d}{dt} (ds)^2 = 2\, (d\underline{X}\, \underset{\sim}{F}^T)\, \underset{\sim}{D}\, (\underset{\sim}{F}\, d\underline{X})$$

$$= 2\, d\underline{X}\, (\underset{\sim}{F}^T \underset{\sim}{D}\, \underset{\sim}{F})\, d\underline{X} \tag{2.25}$$

Comparing this with equation (2.24), for any arbitrary $d\underline{X}$

$$\frac{d\underset{\sim}{E}}{dt} = \underset{\sim}{F}^T \underset{\sim}{D}\, \underset{\sim}{F}$$

or

$$\frac{dE_{IJ}}{dt} = F_{mI}\, D_{mn}\, F_{nJ} \tag{2.26}$$

The equation of conservation of mass or the continuity equation in spatial coordinates is given by the equation

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{v} = 0 \tag{2.27}$$

where ρ is the density and v is the velocity expressed in terms of the spatial coordinates. This completes the kinematical description of the motion of a continuum.

The external forces acting on a body are classified in continuum mechanics in two kinds: body forces and surface forces. Body forces are usually reckoned per unit mass. Surface forces act on a surface and are reckoned per unit area of the surface across which they act.

The surface traction at a point of a surface is defined through the definition of a stress vector. Figure 3 shows the surface S, which has a unit normal vector $\underline{n}$. The stress vector $\underline{t}_n$ is defined as the limit of the ratio $\Delta f/\Delta S$ as $\Delta S \rightarrow 0$. That is

$$\underline{t}_n = \lim_{\Delta S \rightarrow 0} = \frac{df}{dS} \tag{2.28}$$

The subscript n is used here, since the stress vector acting across a surface at a material point depends on the orientation of the surface, which is given by the unit normal vector $\underline{n}$.

Let σ be a transformation such that if $\underline{n}$ is a unit normal vector to a surface, then the stress vector on the surface is given by

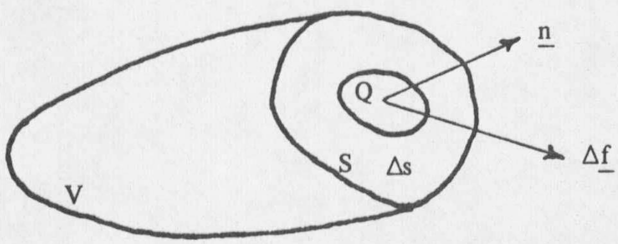


Figure 3. Surface traction $\underline{\Delta f}$ on area ΔS.

$$\underline{t}_n = \underline{n}\,\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \underline{n} \tag{2.29}$$

$\underset{\sim}{\sigma}$ is called the stress tensor or the Cauchy stress tensor. In component form the relation is

$$t_{ni} = \sigma_{ji}\, n_j \tag{2.30}$$

It can be shown that the Cauchy stress tensor is a second order symmetric tensor.

The momentum principle for a continuum states that the time rate of change of the total momentum equals the vector sum of all the external forces acting on the continuum, provided Newton's third law of action and reaction governs the internal forces. The equations in spatial coordinates are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho\, \frac{dv_i}{dt} \tag{2.31}$$

In the special case of static equilibrium of the medium, important in solid mechanics, the acceleration is zero and the equations of motions reduce to the partial differential equations of equilibrium.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho bi = 0 \qquad (2.32)$$

where ρ is the density and b is the body force. These equations do not contain any kinematic variables. But they are not in general sufficient to determine the stress distribution, since there are only three first order partial differential equations for six independent stress components. Additional equations must be found, and these in general include the constitutive equations defining the nature of the particular material under consideration.

Cauchy stress tensor considers a continuum in a strained state or it is defined in Eulerian formulation. A stress tensor referred to the strained state is a natural physical concept. However, in development of constitutive laws stresses are related to strains. Hence, if strains were referred to the initial configuration, it would be convenient to define stresses similarly with respect to the initial configuration. The two Piola-Kirchoff stress tensors discussed below are two alternative definitions of stresses in the reference state.

As shown in Figure 4, the vector $d\underline{P}$ denotes the force acting on a surface element dS with a unit normal $\underline{n}$. $d\underline{P}_o$ denotes the corresponding force assigned to the corresponding original area dS_o with a unit normal $\underline{n}_o$. If σ_{ij} is the stress tensor referred to the deformed configuration, then Cauchy's relation is

$$dP_i = \sigma_{ji} n_j \qquad (2.33)$$

Now the first Piola-Kirchoff stress tensor $\underset{\sim}{T}^\circ$ gives the actual force $d\underline{P}$, but it is reckoned per unit area of the undeformed dS_o. Thus the force is expressed as

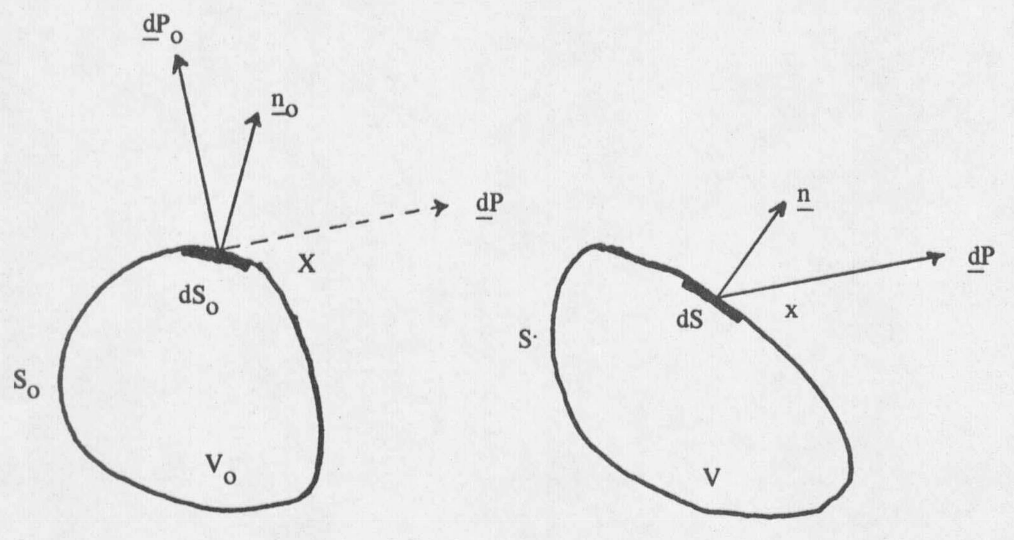


Figure 4. The corresponding force vectors in the original and deformed state of a body.

$$(T^{\circ}{}_{ji} n_{oj})\, dS_o = dP_i = (\sigma_{ji} n_j)\, dS \qquad (2.34)$$

The second Piola-Kirchoff stress tensor $\underset{\sim}{T}$ is defined as

$$\underline{dP}_o = (\underline{n}_o \underset{\sim}{T})\, dS_o = \underset{\sim}{F}^{-1}\, \underline{dP}$$

or

$$dP_{oi} = (T_{ji} n_{oj})\, dS_o = \frac{\partial X_i}{\partial x_j}\, dP_j \qquad (2.35)$$

Using the law of conservation of mass it can be shown that (10) the deformed and the undeformed areas are related by

$$\underline{n}\, dS = \frac{\rho_o}{\rho}\, \underline{n}_o \underset{\sim}{F}^{-1}\, dS_o \qquad (2.36)$$

Substituting the above relation in the right hand side of equation (2.34) and dividing by dS_o, the following relation is obtained

$$\underline{n}_o [\underset{\sim}{T}^\circ - \frac{\rho_o}{\rho} \underset{\sim}{F}^{-1} \underset{\sim}{\sigma}] = 0 \tag{2.37}$$

Hence for any arbitrary unit vector $\underline{n}_o$

$$\underset{\sim}{T}^\circ = \frac{\rho_o}{\rho} \underset{\sim}{F}^{-1} \underset{\sim}{\sigma}$$

or

$$T^\circ_{IJ} = \frac{\rho_o}{\rho} \frac{\partial X_J}{\partial x_m} \sigma_{mI} \tag{2.38}$$

Equation (2.35) gives

$$(\underline{n}_o \underset{\sim}{T})\, dS_o = \underset{\sim}{F}^{-1}\, \underline{dP} = \underset{\sim}{F}^{-1}\, (\underline{n}\, \underset{\sim}{\sigma})\, dS \tag{2.39}$$

Therefore

$$(\underline{n}_o \underset{\sim}{T})\, dS_o = \underset{\sim}{F}^{-1} (\frac{\rho_o}{\rho} \underline{n}_o\, dS_o \underset{\sim}{F}^{-1} \underset{\sim}{\sigma}) \tag{2.39}$$

$$= [\frac{\rho_o}{\rho} \underline{n}_o\, dS_o \underset{\sim}{F}^{-1} \underset{\sim}{\sigma}]\, (\underset{\sim}{F}^{-1})^T \tag{2.40}$$

Hence for any arbitrary unit vector $\underline{n}_o$

$$\underset{\sim}{T} = \frac{\rho_o}{\rho} \underset{\sim}{F}^{-1} \underset{\sim}{\sigma} (\underset{\sim}{F}^{-1})^T$$

$$T_{ij} = \frac{\rho_o}{\rho} \frac{\partial X_j}{\partial x_k} \sigma_{kl} \frac{\partial X_i}{\partial x_l} \tag{2.41}$$

The relation between two Piola-Kirchoff tensors is given by the equation

$$\underset{\sim}{T} = \underset{\sim}{T}^{\circ} (F^{-1})^{T}$$

or

$$T_{ij} = T^{\circ}_{ik} \frac{\partial X_k}{\partial x_j} \tag{2.42}$$

The equations of motion in the reference state, in terms of Piola-Kirchoff stress tensors are (10):

$$\frac{\partial}{\partial X_J}(T^{\circ}_{Ji}) + \rho_o b_{oi} = \frac{d^2 x_i}{dt^2} \tag{2.43}$$

and

$$\frac{\partial}{\partial X_J}(T_{JI} F_{iI}) + \rho_o b_{oi} = \frac{d^2 x_i}{dt^2} \tag{2.44}$$

CONSTITUTIVE LAW

Snow is a highly rate sensitive material, and usually for any problem some range of strain-rates can be designated. The low strain rates are defined to be less than 10^{-6} S^{-1} to 10^{-5} S^{-1}, and high strain-rates are those which exceed 10^{-5} S^{-1} (15). In this work only low strain-rates will be considered.

The material behavior is assumed to be a linear visco-elastic material as long as the stress state is below some critical level. Once this critical state is reached the material behavior becomes non-linear.

The problem to be discussed later in more detail is one which can involve finite-strain and large deformation. Lagrangian formulation is adapted to define the strains. As discussed earlier, the stresses should also be referred with respect to the reference configuration. Therefore one of the Piola-Kirchoff stress tensors will be used. The first Piola-Kirchoff stress tensor is unsymmetric and it will be inconvenient to use in a stress-strain law in which the strain tensor is symmetric. The second Piola-Kirchoff stress tensor is symmetric and so it is more suitable for this purpose. This stress tensor will be denoted by $\underset{\sim}{T}$ with components T_{IJ} (I = 1,2,3, J = 1,2,3). It relates with the Cauchy stress tensor by the relation

$$\underset{\sim}{T} = \frac{\rho_o}{\rho} \underset{\sim}{F}^{-1} \underset{\sim}{\sigma} (\underset{\sim}{F}^{-1})^T \tag{3.1}$$

where ρ_o is the initial density

ρ is the instantaneous density

$\underset{\sim}{F}$ is the deformation gradient tensor

$\underset{\sim}{\sigma}$ is the Cauchy stress tensor

A superscript T implies the transpose of the tensor.

Mechanical behaviour is facilitated by separating volumetric effects from pure shear, or deviatoric effects. The Cauchy stress tensor can be split into deviatoric components $\underset{\sim}{S}$, which tend to change the shape of an element without change of volume, and hydrostatic pressure which only brings change in specific volume

$$\underset{\sim}{\sigma} = \underset{\sim}{S} - p\underset{\sim}{1} \tag{3.2}$$

The hydrostatic pressure p is defined in terms of density ratio, α,

$$\alpha = \frac{\rho_m}{\rho} \tag{3.3}$$

Where ρ_m and ρ are respectively mass densities of ice and snow. Brown (1) has developed a volumetric constitutive law for snow subjected to large strain-rates. The material is treated as a suspension of air voids in a matrix material of polycrystalline ice, and then a pore collapse model is considered.

$$p(t) = \frac{J}{3\alpha} e^{(-\phi\alpha/\alpha_o)} \ln\left(\frac{\alpha}{\alpha - 1}\right) \quad 2(S_o - C) + C \ln\left[\frac{(-\dot{\alpha}A)^2}{\alpha(\alpha - 1)}\right] \tag{3.4}$$

J, C, ϕ and A are all material constants. $\dot{\alpha}$ is the rate of change of α. The above equation describes behaviour of snow for deformations which are predominantly volumetric. Snow, when subjected to shearing stresses distorts. This distortion is associated with slipping between the grains and a rearrangement of the particles into a denser packing; consequently the volume decreases. The overall strain of a snow mass will be the result of deformation of individual particles and partly the result of relative sliding between particles. The above equation does not take into account the deviatoric deformations, and it would not be expected to represent volumetric behaviour when significant shear stresses are involved.

Equation (3.4) is valid for strain-rates greater than 10^{-5} S^{-1}. The problem to be considered in this study falls in the category of low strain-rates ($< 10^{-6}$ S^{-1}). The above equation can be reduced to the following simplified form

$$p(t) = k\dot{\alpha} \tag{3.5}$$

$$k = \frac{Je^{(-\phi\alpha/\alpha_0)}}{3\alpha} C \eta_0 \ln\left(\frac{\alpha}{\alpha - 1}\right) \tag{3.6}$$

where J, C, ϕ are properties of ice and η_0 is the initial shear viscosity. Equations (3.5)-(3.6) can be used to calculate hydrostatic pressure for small strain-rates.

For low strain-rates deformation takes place by a slow flow of the snow structure. At the stresses that produce strain-rates in this range little rearrangement of the ice grains takes place. The behaviour can be represented as a fluid. Cauchy deviatoric stress $\underset{\sim}{S}$ is represented as Newtonian fluid behavior.

$$\underset{\sim}{S} = 2\mu\left[\underset{\sim}{D} - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{D})\underset{\sim}{1}\right] \tag{3.7}$$

where $\mathrm{tr}(\underset{\sim}{D})$ is the trace of $\underset{\sim}{D}$, equal to the first invariant of $\underset{\sim}{D}$. μ is the density dependent viscosity term. If $\underline{v} = v_i \underline{e}_i$ is the velocity vector, then the rate of deformation tensor $\underset{\sim}{D}$ is defined as

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{3.8}$$

The above equation is in Eulerian coordinates x_i. To express $\underset{\sim}{D}$ in terms of Lagrangian coordinates X_i, equation (2.26) is used.

$$\frac{d\mathbf{E}}{dt} = \mathbf{F}^T \mathbf{D}\,\mathbf{F}$$

where, $\mathbf{E} = 1/2(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ is the Lagrangian strain tensor. Then $\mathbf{D}$ is given by the equation

$$\mathbf{D} = (\mathbf{F}^T)^{-1}\,\dot{\mathbf{E}}\,\mathbf{F}^{-1} \qquad (3.9)$$

Note that, since $\mathbf{x} = \mathbf{X} + \mathbf{U}$, $F_{IJ} = \delta_{IJ} + \partial U_I/\partial X_J$, where $\mathbf{U}$ is the displacement vector. Hence for small strain, wwen displacement gradient components are small compared to unity the equation (3.8) reduces to

$$\frac{d\mathbf{E}}{dt} \simeq \mathbf{1}\,\mathbf{D}\,\mathbf{1} = \mathbf{D} \qquad (3.10)$$

Under high stresses displacement gradient components are significant and introduce non-linearities. Using equation (3.9) $\mathbf{S}$ in terms of strain-rate is given as

$$\mathbf{S} = 2\mu[(\mathbf{F}^{-1})^T\,\dot{\mathbf{E}}\,\mathbf{F}^{-1} - \frac{1}{3}\mathrm{tr}(\mathbf{C}^{-1}\,\dot{\mathbf{E}})\,\mathbf{1}] \qquad (3.11)$$

where, Green deformation tensor $\mathbf{C}$ is defined as $\mathbf{C} = \mathbf{F}^T\,\mathbf{F}$. The above equation can be expressed in the following form:

$$\mathbf{S} = (\mathbf{F}^{-1})^T\,[2\mu\,\dot{\mathbf{E}} - \frac{2}{3}\mu\,\mathrm{tr}(\mathbf{C}^{-1}\,\dot{\mathbf{E}})\,\mathbf{C}]\,\mathbf{F}^{-1} \qquad (3.12)$$

Using equations (3.1), (3.2) and (3.12), second Piola-Kirchoff stress tensor $\mathbf{T}$ is expressed in terms of Lagrangian strain-rate tensor and hydrostatic pressure; to complete the constitutive equation. This is done as follows:

$$\mathbf{T} = \frac{\rho_o}{\rho}\,\mathbf{F}^{-1}\,\mathbf{S}\,(\mathbf{F}^{-1})^T - \frac{\rho_o}{\rho}\,\mathbf{F}^{-1}\,p\,\mathbf{1}\,(\mathbf{F}^{-1})^T \qquad (3.13)$$

$$\mathbf{T} = \frac{\rho_o}{\rho}\,\mathbf{F}^{-1}\,(\mathbf{F}^{-1})^T\,[2\mu\,\dot{\mathbf{E}} - \frac{2}{3}\,\mu\,\mathrm{tr}(\mathbf{C}^{-1}\,\dot{\mathbf{E}})\,\mathbf{C}]\,\mathbf{F}^{-1}\,(\mathbf{F}^{-1})^T - \frac{\rho_o}{\rho}\,p\,\mathbf{F}^{-1}\,(\mathbf{F}^{-1})^T$$

$$(3.14)$$

$$\underset{\sim}{T} = 2(\mu \frac{\rho_o}{\rho}) \underset{\sim}{C}^{-1} \underset{\sim}{\dot{E}} \underset{\sim}{C}^{-1} - \frac{2}{3} (\mu \frac{\rho_o}{\rho}) \underset{\sim}{C}^{-1} \, tr(\underset{\sim}{C}^{-1} \underset{\sim}{\dot{E}}) - \frac{\rho_o}{\rho} \, p \, \underset{\sim}{C}^{-1} \tag{3.15}$$

The viscosity term μ is a function of density,

$$\eta = \mu \frac{\rho_o}{\rho} = \eta \, (\rho, \rho_o) \tag{3.16}$$

Then the constitutive law is

$$\underset{\sim}{T} = 2 \eta \underset{\sim}{C}^{-1} \underset{\sim}{\dot{E}} \underset{\sim}{C}^{-1} - \frac{2}{3} \eta \underset{\sim}{C}^{-1} \, tr(\underset{\sim}{C}^{-1} \underset{\sim}{\dot{E}}) - \frac{\rho_o}{\rho} \, p \, \underset{\sim}{C}^{-1}$$

$$T_{IJ} = 2 \eta \, C^{-1}_{IK} \, \dot{E}_{kL} \, C^{-1}_{LJ} - \frac{2}{3} \eta \, C^{-1}_{IJ} \, (C^{-1}_{KL} \, \dot{E}_{LK}) - \frac{\rho_o}{\rho} \, p \, C^{-1}_{IJ} \tag{3.17}$$

For most quasistatic problems in snowmechanics gravity body forces are significant, and they should be incorporated into the equilibrium equations. These equations for negligible acceleration and in Lagrangian description are

$$\frac{\partial}{\partial X_J} (T_{JI} \, F_{iI}) + \rho_o bo_i = \frac{d^2 x_i}{dt^2} \tag{3.18}$$

The equation of conservation of mass or continuity equation in Eulerian description has been stated in the previous section. Material form of continuity equation accounts for change in density and volume. In material coordinates this equation can be put in the form (10)

$$\frac{\alpha}{\alpha_o} = |J| \tag{3.19}$$

where $|J|$ is the absolute value of Jacobian determinant J.

$$J = \det |\underset{\sim}{F}| \tag{3.20}$$

For a given problem, equations (3.17–3.20) can be solved numerically, when appropriate boundary and initial conditions are specified. A detailed discussion is presented in the next section.

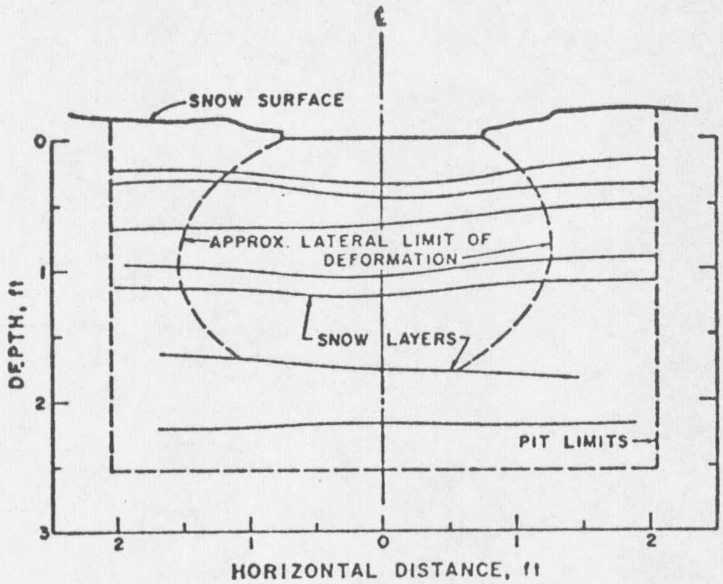
SNOW MASS SUBJECTED TO UNIFORMLY DISTRIBUTED FLEXIBLE SURFACE LOADING

The settlement of foundations, platforms, or stationary tracked vehicles into snow can be accurately predicted under exceptional conditions; because of the complexity of mechanical properties of snow and disturbing influences of temperature. Nevertheless a theoretical analysis of settlement phenomena is important because the results at least permit the engineer to recognize the factor that determines the magnitude and the distribution of the settlement.

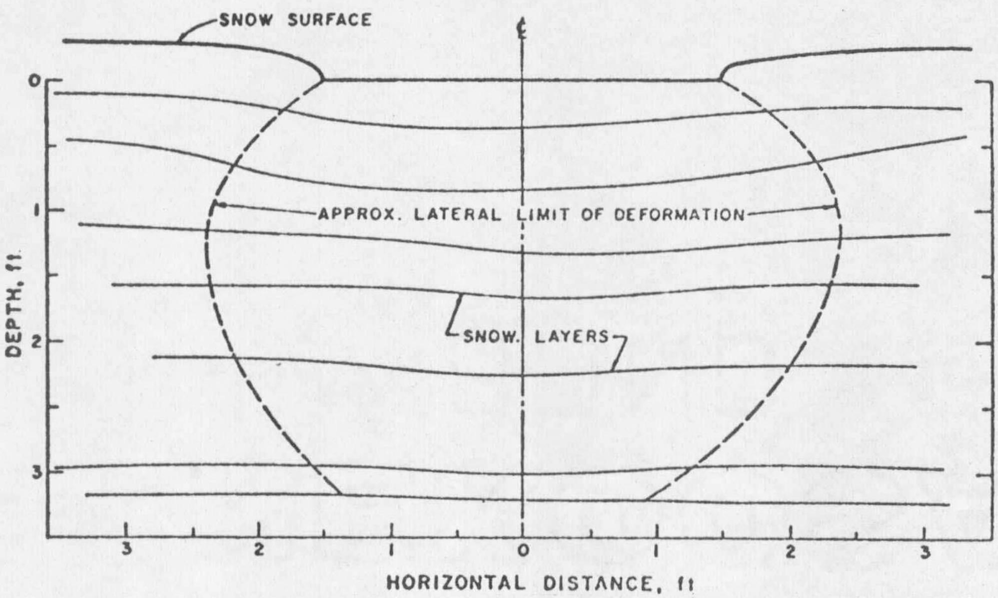
The problem of flexible surface loading is similar to the problem of foundations on snow. The only experimental data available is from Camp Century, Greenland, where a series of nine spread test footings were installed on undisturbed snow (13). These tests were continuously observed for a period of two years. The influence of footing load, size and shape on settlement were investigated. The effects of uncontrolled parameters such as temperature and density were recorded. The results from these experiments are compared to the theoretical results obtained in this paper.

Figure 5 shows the experimentally observed deformation zone or the pressure bulb (13). If the loading is symmetrical, then the pressure bulb can be assumed symmetrical about the center of the footing. The width of the deformation zone is approximately between 1½ and 2 times the footing width. The vertical extent of deformation would be between 1 and 1½ times the footing width. For simplicity, consider a rectangular area underneath the footing as shown in Figure 6. The dimensions of this rectangle are larger than the observed size of the pressure bulb. The average temperature is assumed constant.

The constitutive equation (3.17) in component form can be written as



(a) Snow deformation beneath 18 × 18 in footing, 1000 psf. (13)



(b) Snow deformation beneath 36 × 36 in footing, 2000 psf. (13)

Figure 5. Experimentally observed influence zone.

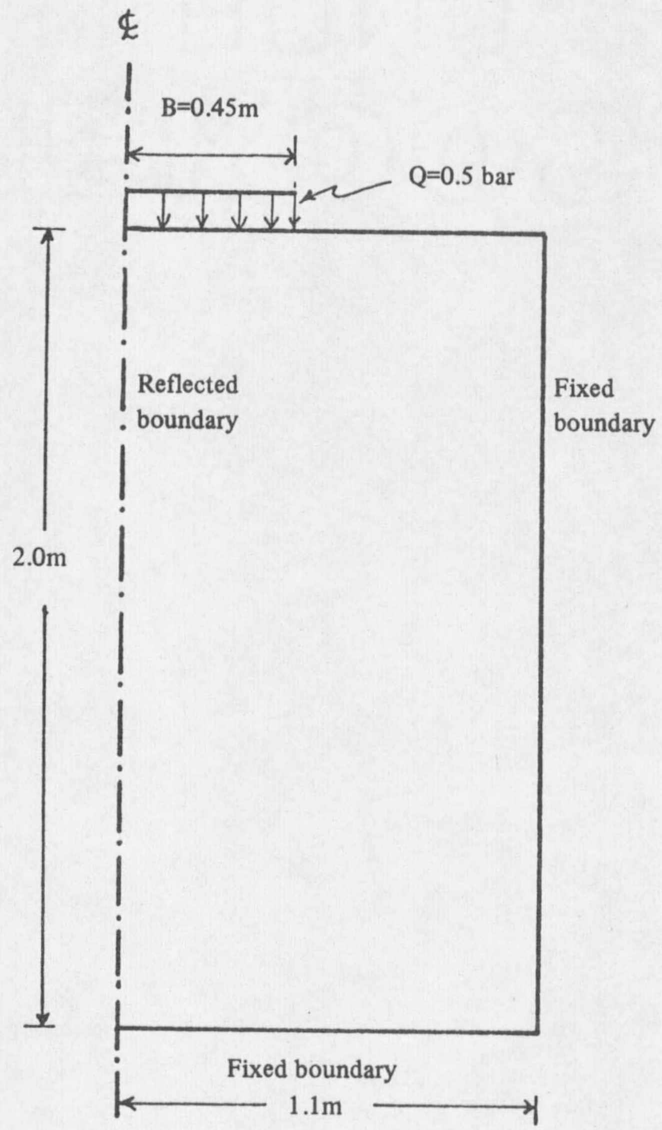


Figure 6. Standard problem.

$$T_{IJ} = 2\,B_{IJKL}\dot{E}_{LK} - \frac{\rho_o}{\rho}\,p\,C^{-1}_{IJ} \tag{4.1}$$

where

$$B_{IJKL} = \eta C^{-1}_{IK}\;C^{-1}_{LJ} \;-\; \frac{1}{3}\,\eta C^{-1}_{IJ}\;C^{-1}_{KL} \tag{4.2}$$

The repeated index denotes the summation. Using the above expression for $\underset{\sim}{T}$, the equilibrium equations (3.18) take form

$$2\,\frac{\partial}{\partial X_I}(F_{iJ}\,B_{IJKL})\;\dot{E}_{LK} + 2\,F_{iJ}\,B_{IJKL}\,\frac{\partial \dot{E}_{LK}}{\partial X_I} - [F_{iJ}\,\frac{\rho_o}{\rho}\;C^{-1}_{IJ}\;]\frac{\partial p}{\partial X_I} \tag{4.3}$$

$$-\;\frac{\partial}{\partial X_I}(F_{iJ}\,\frac{\rho_o}{\rho}\,C^{-1}_{IJ})\;p + \rho_o b_{oi} = 0$$

For a plane-strain condition ($x_3 = X_3$), the above equations $i = 1,2$) can be solved for strain-rates. For the three unknowns $\dot{E}_{11}$, $\dot{E}_{12} = \dot{E}_{21}$ and $\dot{E}_{22}$, one more equation is needed. The third equation can be obtained by using the compatibility equation. To avoid computational difficulty, equilibrium equations are first developed in terms of $\dot{x}_i$. The Lagrangian strain-rate tensor $\underset{\sim}{\dot{E}}$ is given by the expression

$$\dot{E}_{IJ} = \frac{1}{2}\,(\dot{F}_{KI}\,F_{KJ} + F_{KI}\,\dot{F}_{KJ}) \tag{4.4}$$

Using the above expression and equation (4.3), the equilibrium equations are expressed in terms $\dot{x}_i$.

$$\alpha_{ilIL}\,\frac{\partial^2 \dot{x}_l}{\partial X_I \partial X_L} + \beta_{ilIK}\,\frac{\partial^2 \dot{x}_l}{\partial X_I \partial X_K} + \gamma_{ilL}\,\frac{\partial \dot{x}_l}{\partial X_L} + \delta_{ilK}\,\frac{\partial \dot{x}_l}{\partial X_K}$$

$$+\;\epsilon_i + \rho_o b_{oi} = 0 \tag{4.5}$$

where

$$B_{IJKL} = \eta\, C^{-1}_{IK}\, C^{-1}_{LJ} - \frac{\eta}{3} C^{-1}_{IJ}\, C^{-1}_{KL}$$

$$\alpha_{ilIL} = F_{iJ}\, B_{IJKL}\, F_{lK}$$

$$\beta_{ilLK} = F_{iJ}\, B_{IJKL}\, F_{lL} \tag{4.6}$$

$$\gamma_{ilL} = \frac{\partial}{\partial X_I}(F_{iJ}\, B_{IJKL}\, F_{lK})$$

$$\sigma_{ilK} = \frac{\partial}{\partial X_I}(F_{iJ}\, B_{IJKL}\, F_{lL})$$

$$\epsilon_i = \frac{\partial}{\partial X_I}(\frac{\rho_o}{\rho} F_{iJ}\, p\, C^{-1}_{IJ})$$

The above second order partial differential equations are linear in $\dot{x}_i$. Note that the coefficients are functions of current deformations, and not the functions of the rate of change of deformations $\dot{x}_i$. These two differential equations (i = 1,2) yield a system of simultaneous linear equations in $\dot{x}_i$. If t^k is the current time, then

$$t^{k+1} = t^k + \Delta t \tag{4.7}$$

$$x_1^{k+1} = x_1^k + \dot{x}_1 \;\Delta t \tag{4.8}$$

To solve the differential equations (4.5), a central difference scheme is adapted. The expanded form of the equilibrium equation for i = 1 is:

$$(\alpha_{1111}+\beta_{1111})\frac{\partial^2\dot{x}_1}{\partial X_1^2}+(\alpha_{1112}+\alpha_{1121}+\beta_{1112}+\beta_{1121})\frac{\partial^2\dot{x}_1}{\partial X_1\partial X_2}$$

$$+(\alpha_{1122}+\beta_{1122})\frac{\partial^2\dot{x}_1}{\partial X_2^2}+(\alpha_{1211}+\beta_{1211})\frac{\partial^2\dot{x}_2}{\partial X_1^2}+(\alpha_{1212}+\alpha_{1221}+$$

$$\beta_{1212}+\beta_{1221})\frac{\partial^2\dot{x}_2}{\partial X_1\partial X_2}+(\alpha_{1222}+\beta_{1222})\frac{\partial^2\dot{x}_2}{\partial X_2^2}+(\gamma_{111}+\delta_{111})\frac{\partial\dot{x}_1}{\partial X_1}$$

$$+(\gamma_{112}+\delta_{112})\frac{\partial\dot{x}_1}{\partial X_2}+(\gamma_{121}+\delta_{121})\frac{\partial\dot{x}_2}{\partial X_1}+(\gamma_{122}+\delta_{122})\frac{\partial\dot{x}_2}{\partial X_2}+\epsilon_1=0 \tag{4.9}$$

For i = 2 the equation has the body force term ($\rho_O g$) in it, where g is the gravitational acceleration.

Figure 7 shows a typical finite difference mesh. The dotted lines denote fictitious mesh points to be used in the calculations for the mesh. Central differencing with respect to X_1 and X_2 gives the following types of terms.

$$\frac{\partial\dot{x}}{\partial X_1}=(\dot{x}_{i,j+1}-\dot{x}_{i,j-1})/2\Delta X_1 \tag{4.10}$$

$$\frac{\partial^2\dot{x}}{\partial X_1^2}=(\dot{x}_{i,j+1}-2\dot{x}_{1,j}+\dot{x}_{i,j-1})/\Delta X_1^2 \tag{4.11}$$

$$\frac{\partial^2\dot{x}}{\partial X_1\partial X_2}=(\dot{x}_{i+1,j+1}-\dot{x}_{i+1,j+1}-\dot{x}_{i-1,j+1}-\dot{x}_{i+1,j-1})/4\Delta X_1\Delta X_2 \tag{4.12}$$

Substitution of the above relations in the equilibrium equations (4.9), yields the following simultaneous linear equations at an interior node I,J

(Let $\dot{x}=\dot{x}_1$, $\dot{y}=\dot{x}_2$, $X=X_1$ and $Y=X_2$)

$$\sum_{i=I-1}^{I+1}\sum_{j=J-1}^{J+1}(A_{11ij}\dot{x}_{i,j}+A_{12ij}\dot{y}_{i,j})=-\epsilon_1 \tag{4.13}$$

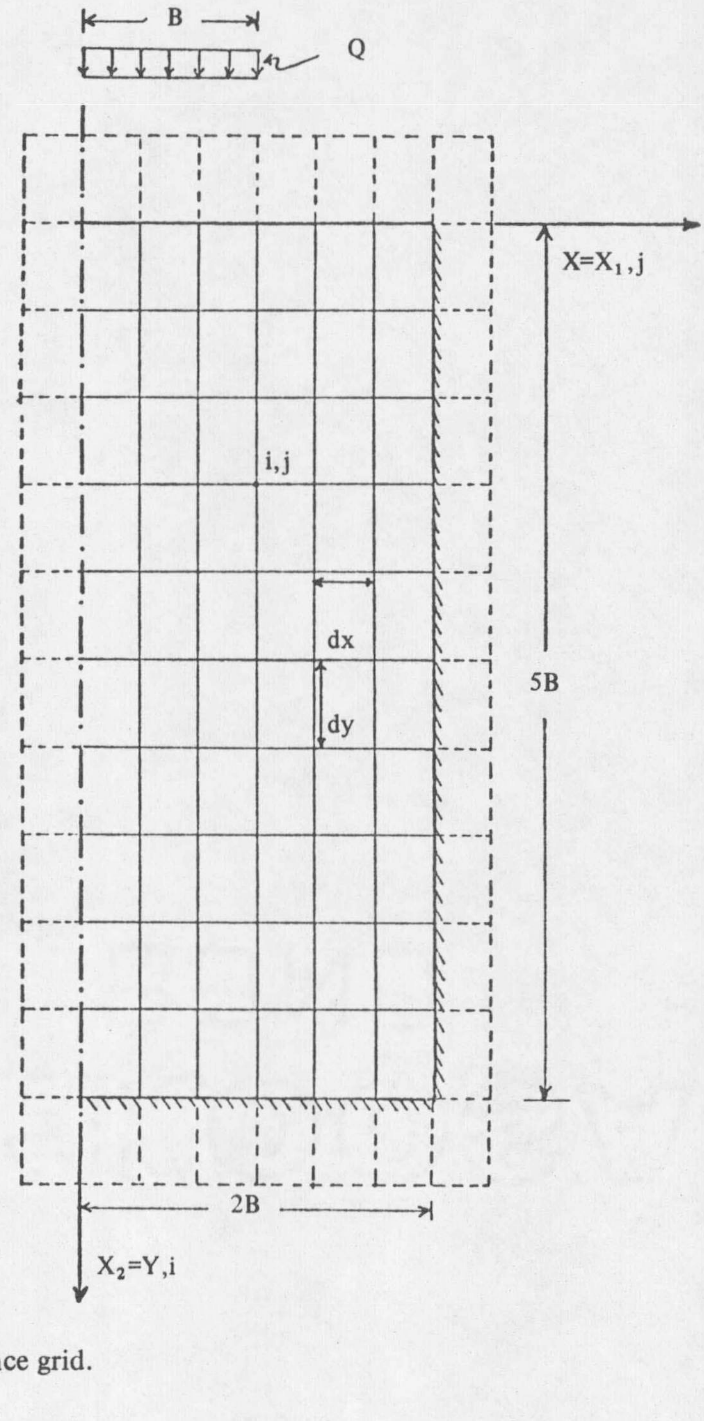


Figure 7. Finite difference grid.

$$\sum_{i=I-1}^{I+1} \sum_{j=J-1}^{J+1} (A_{21ij}\, \dot{x}_{i,j} + A_{22ij}\, \dot{y}_{i,j}) = -\epsilon_2 - \rho_o g \tag{4.14}$$

where

$$\begin{aligned}
A_{ij11} &= (\alpha_{ij12} + \alpha_{ij21} + \beta_{ij12} + \beta_{ij21})/4\Delta X\Delta Y \\
A_{ij12} &= (\alpha_{ij11} + \beta_{ij11})/\Delta X^2 - (\gamma_{ij1} + \delta_{ij1})/2\Delta X \\
A_{ij21} &= (\alpha_{ij22} + \beta_{ij22})/\Delta Y^2 - (\gamma_{ij2} + \delta_{ij2})/2\Delta Y \\
A_{ij22} &= -2[(\alpha_{ij11} + \beta_{ij11})/\Delta X^2 + (\alpha_{ij22} + \beta_{ij22})/\Delta Y^2] \\
A_{ij}23 &= (\alpha_{ij22} + \beta_{ij22})/\Delta Y^2 + (\gamma_{ij2} + \delta_{ij2})/2\Delta Y \\
A_{ij23} &= (\alpha_{ij22} + \beta_{ij22})/\Delta Y^2 + (\gamma_{ij2} + \delta_{ij2})/2\Delta Y \\
A_{ij32} &= (\alpha_{ij11} + \beta_{ij11})/\Delta X^2 + (\gamma_{ij1} + \delta_{ij1})/2\Delta X \\
A_{ij33} &= -A_{ij31} = -A_{ij13} = A_{ij11}
\end{aligned} \tag{4.15}$$

Boundary Conditions

To solve equations (4.13) and (4.14) the following boundary conditions are assumed. The size of the pressure bulb is chosen larger than the size of the pressure bulb observed experimentally by Reed (13); therefore the vertical wall and the bottom of the bulb can be assumed fixed. Due to symmetry it follows that the lateral displacement along the center line is zero. The boundary conditions assumed are

(1) Fixed boundary

$$x = X \text{ and } y = Y \tag{4.16}$$

(2) Reflected boundary

$$\dot{x} = 0\ ;\ x = X$$

$$\frac{d\dot{y}}{dX} = \frac{dy}{dX} = 0 \tag{4.17}$$

To derive the boundary conditions for the upper surface, completely vertical surface loading is assumed. Q is the pressure measured per unit undeformed surface.

The force $\underline{dP}$ on the deformed area as shown in Figure 8 is

$$\underline{dP} = -Q\, dS_o\, \underline{e}_2 \tag{4.18}$$

where dA is the undeformed area, and $\underline{e}_2$ is the unit vector in Y direction. The first Piola-Kirchoff stress tensor $\underset{\sim}{T}^\circ$, and the Cauchy stress tensor $\underset{\sim}{\sigma}$ are related to the $\underline{dP}$ by the equation (2.34)

$$\underline{dP} = -QdS_o\,\underline{e}_2 = (\underline{n}\,\underset{\sim}{\sigma})\, ds = (\underline{n}_o\,\underset{\sim}{T}^\circ)\, dS_o \tag{4.19}$$

where $\underline{n}_o = -\underline{e}_2$ is unit normal to the undeformed surface. Equation (2.42) shows that

$$\underset{\sim}{T} = \underset{\sim}{T}^\circ\,(\underset{\sim}{F}^{-1})^T \tag{4.20}$$

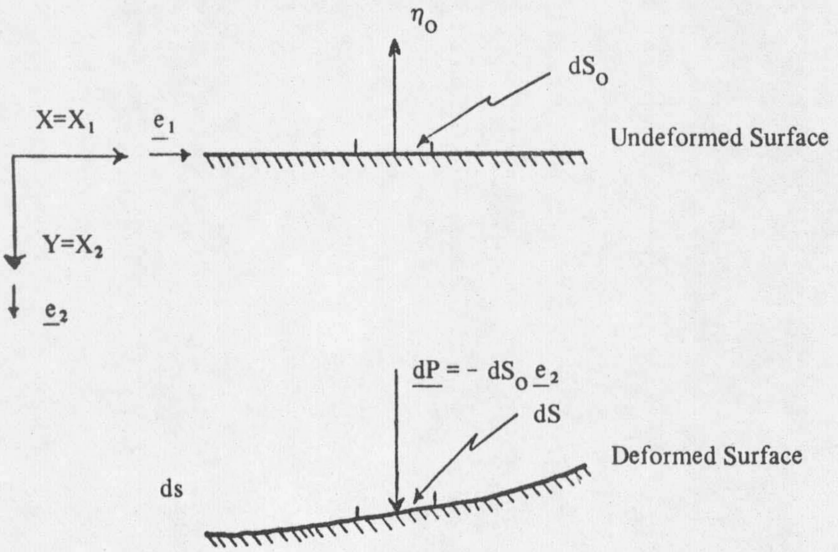


Figure 8. The normal stress vector on the deformed surface.

or

$$\underset{\sim}{T}^{\circ} = \underset{\sim}{T}\,\underset{\sim}{F}^{T}$$

Hence

$$Q\,\underline{e}_2 = -\,\underline{e}_2\,(\underset{\sim}{T}\,\underset{\sim}{F}^{T}) \tag{4.21}$$

The rectangular cartesian components of this relation are

$$T_{2J}\,\frac{\partial x_1}{\partial X_J} = 0 \tag{4.22}$$

and

$$T_{2J}\,\frac{\partial x_2}{\partial X_J} = -\,Q \tag{4.23}$$

Using the constitutive equation (4.1), (4.4) and the above equations the following two differential equations for the upper surface are obtained

$$(B_{2JKL}F_{1J}F_{mK})\,\frac{\partial \dot{x}_m}{\partial X_L} + (B_{2JKL}F_{1J}F_{mL})\,\frac{\partial \dot{x}_m}{\partial X_K} = \frac{\rho_o}{\rho}\,p\,C^{-1}_{2J}\,F_{1J} \tag{4.24}$$

$$(B_{2JKL}F_{2J}F_{mK})\,\frac{\partial \dot{x}_m}{\partial X_L} + (B_{2JKL}F_{2J}F_{mL})\,\frac{\partial \dot{x}_m}{\partial X_K} = \frac{\rho_o}{\rho}\,p\,C^{-1}_{2J}\,F_{2J} - Q \tag{4.25}$$

Central differencing these equations, a relation between fictitious grid points outside the upper surface and grid points inside the domain is obtained. This relation is substituted back into equations (4.13) and (4.14). Let (o,j), (1,j) and (2,j) denote mesh points on a layer outside the surface, on the surface and a layer below the surface respectively. Then the final expressions are