Multi-functions and uniform spaces
by John Cecil James Harvey

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
Let (Y,V) be a uniform space, X be a topological space, and $F:X \to Y$ be a multi-function.

Chapter 1 introduces the concepts of F being quasi-continuous or uniformly continuous and examines some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions. A typical result is that if F is compact-valued then F is quasi-continuous if and only if F is continuous.

Chapter 2 is concerned with uniform convergence. A typical result if that if {Fd | d □ D} is a net of uniformly (quasi-) continuous multi-functions, which converges uniformly to F, then F is uniformly (quasi-) continuous .

Chapter 3 is concerned with the structure of P(Y), the power set of Y, with the Hausdorff uniformity and its induced topology. A one-one relationship is established between continuous single-valued functions into P(Y) with this topology and quasi-continuous multi-functions into Y.

Chapter 4 is about the semi-group of quasi-continuous real valued multi-functions. A typical result is that {x □ X | F(x) is unbounded) is open in X.

Chapter 5 is a listing of several properties of the set, CM(X,Y), of all continuous multi-functions from X into Y, with various topologies.

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## ABSTRACT

Let $(Y,\mathcal{V})$ be a uniform space, X be a topological space, and $F:X \to Y$ be a multi-function.

Chapter 1 introduces the concepts of F being quasi-continuous or uniformly continuous and examines some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions. A typical result is that if F is compact-valued then F is quasi-continuous if and only if F is continuous.

Chapter 2 is concerned with uniform convergence. A typical result if that if $\{F_d \mid d \in D\}$ is a net of uniformly (quasi-) continuous multi-functions, which converges uniformly to F, then F is uniformly (quasi-) continuous.

Chapter 3 is concerned with the structure of P(Y), the power set of Y, with the Hausdorff uniformity and its induced topology. A one-one relationship is established between continuous single-valued functions into P(Y) with this topology and quasi-continuous multi-functions into Y.

Chapter 4 is about the semi-group of quasi-continuous real valued multi-functions. A typical result is that $\{x \in X \mid F(x) \text{ is unbounded}\}$ is open in X.

Chapter 5 is a listing of several properties of the set, $C_M(X,Y)$, of all continuous multi-functions from X into Y, with various topologies.

## INTRODUCTION

Several authors have investigated properties of multi-functions satisfying various types of continuity. Our purpose in this paper will be to define a concept of a uniformly continuous multi-function. Then we will establish several properties of the uniformly continuous multi-functions.

In this work we shall assume that the definitions of the common topological terms are known. Among the symbols to be used are the following.

For a topological space X and a subset $A \subset X$, $\overline{A}$ is the closure of A, $A^{\circ}$ is the interior of A, $\partial(A)$ is the boundary of A, and X - A is the complement of A in X. $\emptyset$ will denote the empty set. R will denote the real numbers and if $a, b \in R$, $a \leq b$, then $[a,b] = \{x \mid a \leq x \leq b\}$.

By a uniform space we mean a pair $(X,\mathcal{U})$, where X is a topological space and $\mathcal{U}$ is a uniformity as defined in Kelley [9].

For a uniform space $(X,\mathcal{U})$, for any $x \in X$, $A \subset X$ and $U \in \mathcal{U}$; $U(x) = \{y \mid (x,y) \in U\}$; and $U[A] = \bigcup \{U(x) \mid x \in A\}$.

Note $\overline{A} = \bigcap \{U[A] \mid U \in \mathcal{U}\}$.

F is a <u>multi-function</u> <u>from</u> X <u>into</u> Y, $F:X \to Y$, if for each $x \in X$, F assigns a subset $F(x) \subset Y$. (Many authors require that for every $x \in X$, $F(x) \neq \emptyset$ and $F(x) = \overline{F(x)}$). The closedness of $F(x)$ is desireable if we are trying to establish separation properties, see Smithson [15].

For a multi-function $F:X \to Y$, X and Y topological spaces, we will say that F is compact-valued (closed-valued, open-valued) if $F(x)$ is compact (closed, open) for every $x \in X$.

F is upper semi-continuous (usc) at $x \in X$ if for every open set O in Y, if $F(x) \subset O$, then there is a neighborhood N of x such that $F(x') \subset O$ for every $x' \in N$.

F is lower semi-continuous (lsc) at $x \in X$ provided that for every open set O in Y, if $F(x) \cap O \neq \emptyset$, there is a neighborhood N of x such that $F(x') \cap O \neq \emptyset$ for every $x' \in N$.

F is continuous at $x \in X$, provided that F is both usc and lsc at x. F is continuous (usc, lsc) if for every $x \in X$, F is continuous (usc, lsc) at x.

We shall need the following two theorems:

0.1 Theorem.

If i) $(x, \mathcal{U})$ is a uniform space,

ii) A is a compact subset of X,

iii) O is an open set in X, and

iv) $A \subset O$,

then there is a $V \in \mathcal{U}$ such that $V[A] \subset O$.

Proof: Kelley [9] Theorem 33, Chapter 36.

0.2 Definition.

A uniform space $(X, \mathcal{U})$ is a <u>uniformly locally connected</u> space if for each $U \in \mathcal{U}$, there is a $W \in \mathcal{U}$ such that $W \subset U$ and $W(x)$ is connected for every $x \in X$.

0.3 Theorem.

If i) $(X, \mathcal{U})$ is a uniformly locally connected space,

ii) $A \subset X$ such that $\partial(A)$ is compact,

iii) $O$ is an open set in $X$, and

iv) $\overline{A} \subset O$,

then there is a $U \in \mathcal{U}$ such that $U[\overline{A}] \subset O$.

Proof: By Theorem 0.1 since $\partial(A)$ is compact, there is a $V \in \mathcal{U}$ such that $V[\partial(A)] \subset O$. There is a $W \in \mathcal{U}$, such that $W$ is symmetric and $W^2 \subset V$. Also there is a $U \in \mathcal{U}$, such that $U \subset W$ and $U(x)$ is connected for each $x \in X$. Then $U[\overline{A}] \subset O$. For; let $x \in \overline{A}$. If $x \in \partial(A)$, $U(x) \subset V(x) \subset O$. If $U(x) \cap \partial(A) \neq \emptyset$, there is a $y \in \partial(A)$ such that $y \in U(x) \subset W(x)$. Since $W$ is symmetric, $x \in W(y)$. Thus $U(x) \subset W[W(y)] = W^2(y) \subset V(y) \subset O$.

Suppose $U(x) \cap \partial(A) = \emptyset$. Let $M_1 = U(x) \cap (X-\overline{A})$ and

$M_2 = U(x) \cap \overline{A} = U(x) \cap (A^\circ \cup \partial(A)) =$

$(U(x)) \cap A^\circ) \cup (U(x) \cap \partial(A)) = U(x) \cap A^\circ.$

But $U(x) = M_1 \cup M_2$ and $x \in M_2$; therefore, since $U(x)$ is connected, $M_1 = \emptyset$. Thus $U(x) = M_2 \subset A \subset O$.

# CHAPTER 1: QUASI-CONTINUOUS AND UNIFORMLY CONTINUOUS MULTI-FUNCTIONS.

In studying continuous multi-functions into a uniform space it is apparent that a different definition of continuity would be useful since, e.g., the log function, log: $C \to C$, is not a continuous multi-function in the usual sense. Ratner, in his dissertation, defined concepts which he called metrically continuous and uniformly continuous, which for metric spaces are the same as the concepts of quási-continuous and uniformly continuous.

1.1 Definition.

Let i) $X$ be a topological space,

ii) $(Y, \mathcal{V})$ be a uniform space, and

iii) $F:X \to Y$ be a multi-function,

then $F$ is <u>quasi-continuous</u> (<u>q.c.</u>) if both of the following are satisfied:

1) For each $V \in \mathcal{V}$, and each $x \in X$, there is a neighborhood $N$ of $x$ such that $F(x') \subset V[F(x)]$ for each $x' \in N$.

2) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood $N$ of $x$ such that $F(x') \cap V(y) \neq \emptyset$ for each $x' \in N$ and each $y' \in F(x)$.

If a multi-function satisfies condition 1, we will say that it is <u>quasi-upper semi-continuous</u> (qusc), if it satisfies 2, we will say that it is quasi-lower semi-continuous (qlsc).

1.2 Lemma.

Let i) $X$ be a topological space,

ii) $(Y, \mathcal{V})$ be a uniform space, and

iii) $F: X \to Y$ be a multi-function.

Then the following are equivalent:

1) $F$ is quasi-lower semi-continuous.

2) For each $x \in X$ and each $V \in \mathcal{V}$, there is a neighborhood $N$ of $x$ such that $F(x) \subset V[F(x')]$ for each $x' \in N$.

Proof: Suppose that $F$ is qlsc and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$, such that $W$ is symmetric and $W \subset V$. Since $F$ is qlsc there is a neighborhood $N$ of $x$ such that for every $x' \in N$, $F(x') \cap W(y) \neq \emptyset$ for every $y \in F(x)$. Since $W$ is symmetric, if $x' \in N$, for each $y \in F(x)$, $y \in W[F(x')] \subset V[F(x')]$ thus $F(x) \subset V[F(x')]$.

Now suppose condition 2 holds. If $x \in X$ and $V \in \mathcal{V}$, let $W \in \mathcal{V}$ such that $W$ is symmetric and $W \subset V$. There is a neighborhood $N$ of $x$ such that $F(x) \subset W[F(x')]$ for each $x' \in N$. So for any $y \in F(x)$ and any $x' \in N$, $y \in W[F(x')]$. Since $W$ is symmetric there is a $y' \in F(x')$ such that $y' \in W(y)$. Thus $\emptyset \neq F(x') \cap W(y) \subset F(x') \cap V(y)$.

Since y was arbitrary, F is qlsc.

1.3 Theorem.

Let i) X be a topological space,

ii) $(Y,\mathcal{V})$ be a uniform space, and

iii) $F:X \rightarrow Y$ be a multi-function,

then the following are equivalent:

1) F is quasi-continuous

2) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood N of x such that for any $x' \in N$, $F(x') \subset V[F(x)]$ and $F(x) \subset V[F(x')]$.

3) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood N of x such that for any two points $x_1, x_2 \in N$, $F(x_1) \subset V[F(x_2)]$.

Proof: We will show that 1 implies 2, 2 implies 3 and 3 implies 1.

Suppose 1 holds and that $x \in X$ and $V \in \mathcal{V}$ are given. Since F is qusc there is a neighborhood $N_1$ of x such that $F(x') \subset V[F(x)]$ for each $x' \in N$. Since F is qlsc and by lemma 1.2 there is a neightborhood $N_2$ of x such that $F(x) \subset V[F(x')]$ for each $x' \in N_2$. Let $N = N_1 \cap N_2$.

Suppose 2 holds, and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$, such that W is symmetric and $W^2 \subset V$.

There is a neighborhood N of x such that for any $x' \in N$, $F(x') \subset W[F(x)]$ and $F(x) \subset W[F(x')]$, let $x_1, x_2 \in N$. Then $F(x_1) \subset W[F(x)]$ and $F(x) \subset W[F(x_2)]$ so $F(x_1) \subset W[F(x)] \subset W \circ W[F(x_2)] = W^2[F(x_2)] \subset V[F(x_2)]$.

Suppose 3 holds and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a neighborhood N of x such that for any pair $x_1, x_2 \in N$, $F(x_1) \subset V[F(x_2)]$. So for any $x' \in N$, $F(x') \subset V[F(x)]$. Thus F is qusc. Also $F(x) \subset V[F(x')]$ for each $x' \in N$, so by lemma 1.3 F is qlsc.

1.4 Definition.

Let i) $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces, and

ii) $F: X \to Y$ be a multi-function.

Then F is uniformly continuous (uc) if F satisfies either of the following conditions:

1) For every $V \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that for each pair $(x,x') \in U$, $F(x') \subset V[F(x)]$.

2) For every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \cap V(y) \neq \emptyset$ for every $y \in F(x)$.

1.5 Lemma.

If i) $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces, and

ii) $F: X \to Y$ is a multi-function,

then conditions 1 and 2 of definition 1.4 are equivalent.

Proof: Suppose that $F$ satisfies condition 1. Let $V \in \mathcal{V}$ be given. There is a $W \in \mathcal{V}$ such that $W$ is symmetric and $W \subset V$. There is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \subset W[F(x)]$. Without loss of generality we may suppose that $U$ is symmetric. Then $(x,x') \in U$ implies that $F(x) \subset W[F(x')]$. For any $y \in F(x)$, $y \in W[F(x')]$ so there is a $y' \in F(x')$ such that $(y,y') \in W$. But then $F(x') \cap W(y) \neq \emptyset$. So $F$ satisfies condition 2.

Suppose $F$ satisfies condition 2. Let $V \in \mathcal{V}$. There is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \cap V(y) \neq \emptyset$ for each $y \in F(x)$. Then for each $y \in F(x)$ there is a $y' \in F(x')$ such that $(y,y') \in V$. Without loss of generality we may assume that $V$ and $U$ are symmetric. Then for $(x,x') \in U$ and for each $y \in F(x)$ there is a $y' \in F(x')$ such that $(y,y') \in V$. Then also $(y',y) \in V$ so $y \in V(y') \subset V[F(x')]$. Thus $F(x) \subset V[F(x')]$. Since $U$ was symmetric we also have $(x',x) \in U$ so $F(x') \subset V[F(x)]$.

Using our definition of continuity, if $F:Y \rightarrow X$ and $G:Z \rightarrow Y$ are continuous multi-functions then $F \circ G:Z \rightarrow X$ is a continuous multi-function. However the composition of two closed valued continuous multi-functions need not be closed valued.

1.6 Example.

Let $X = Z = [0,1]$, $Y = [1,\infty)$.

Define $G:Z \to Y$ by $G(z) = [1,\frac{1}{z}]$; $z \in (0,1]$,

$$G(0) = [1,\infty);$$

$$F:Y \to X \text{ by } F(y) = [\tfrac{1}{x},1];$$

then $F\circ G(0) = (0,1]$ is not closed.

However the following is true.

1.7 Theorem.

Let i) Z and Y are topological spaces,

ii) X is a regular space,

iii) $G:Z \to Y$ is a compact-valued continuous multi-function, and

iv) $F:Y \to X$ is a continuous closed-valued multi-function,

then $F\circ G$ is a closed valued function.

Proof: Let $z \in Z$, we will shown that if $x \notin F\circ G(z)$ then $x \notin \overline{F\circ G(z)}$. Suppose $x \in F(y)$. Since Z is regular, there exists open sets $O_1$, $O_2$ in X such that $F(y) \subset O_1$, $x \in O_2$ and $O_1 \cap O_2 = \emptyset$. Since F is continuous, there exists a neighborhood N of y, such that if $y' \in N$, then $F(y') \subset O_1$.

Suppose $x \notin F \circ G(z) = F[G(z)]$. Since $G(z)$ is compact, there exists finite sets $\{N_1, N_2, \ldots, N_n\}$ of open sets in Y and $\{O_{11}, O_{12}, \ldots, O_{1n}\}$ and $\{O_{21}, O_{22}, \ldots, O_{2n}\}$ of open sets in X such that

i) $G(z) \subset \bigcup_{i=1}^{n} N_i$,

ii) $F[N_i] \subset O_{1i}$, $i=1, 2, \ldots, n$,

iii) $x \in O_{2i}$, $i=1, 2, \ldots, n$; and

iv) $O_{1i} \cap O_{2i} = \emptyset$, $i=1, 2, \ldots, n$.

Then

$F \circ G(z) = F[G(z)] \subset F[\bigcup_{i=1}^{n} N_i] \subset \bigcup_{i=1}^{n} F[N_i] \subset \bigcup_{i=1}^{n} O_{1i}$. Now

$x \in \bigcap_{i=1}^{n} O_{2i}$, and $(\bigcup_{i=1}^{n} O_{1i}) \cap (\bigcap_{i=1}^{n} O_{2i}) = \emptyset$.

So $x \notin \overline{F \circ G(z)}$.

The above shows that if F is a closed-valued continuous multi-function into a regular space and K is a compact subset of the domain of F then F[K] is closed.

1.8 Lemma.

If i) Z, Y are topological spaces,

ii) $(X,\mathcal{U})$ is a uniform space,

iii) $G:Z \to Y$ is continuous compact-valued multi-function, and

iv) $F:Y \to X$ is quasi-continuous,

then $F \circ G$ is quasi-continuous

Proof: Let $W \in \mathcal{U}$, $z \in Z$. For each $y \in G(z)$ there is, by Theorem 1.3, an open neighborhood $M$ of $y$ such that if $y', y'' \in M$, then $F(y') \subset W[F(y'')]$. Since $G(z)$ is compact there is a finite set of open sets $\{M_1, \ldots, M_n\}$ and a set of points $\{y_1, y_2, \ldots, y_n\}$ such that

i) $y_i \in M_i \cap G(z)$, $i=1, 2, \ldots, n$,

ii) $G(z) \subset \bigcup_{i=1}^{n} M_i$, and

iii) if $y', y'' \in M_i$, $F(y') \subset W[F(y'')]$ for $i=1, 2, \ldots, n$.

There is a neighborhood $N$ of $z$ such that if $z' \in N$ then $G(z') \subset \bigcup_{i=1}^{n} M_i$ and $G(z') \cap M_i \neq \emptyset$ $i=1, \ldots, n$. Let $z' \in N$. Then $F \circ G(z') = F[G(z')] \subset F[\bigcup_{i=1}^{n} M_i] \subset \bigcup_{i=1}^{n} F[M_i]$ $\subset \bigcup_{i=1}^{n} W[F(y_i)] \subset W[F \circ G(z)]$.

For each i=1, ..., n, there is a point $y'_i \in G(z') \cap M_i$.

Then $F \circ G(z) \subset F[\bigcup_{i=1}^{n} M_i] \subset \bigcup_{i=1}^{n} W[F(y'_i)] \subset W[F \circ G(z')]$.

Thus F∘G is quasi-continuous by Theorem 1.3.

To show that the requirement of G(z) being compact for each $z \in Z$ is essential consider the following:

1.9 Example

Let $X = Y = Z = [0,\infty)$.

Define $G:Z \to Y$ by $G(z) = [0,\frac{1}{z}]$ $z \in (0,\infty)$,

$= [0,\infty)$ $z = 0$,

$F:Y \to X$ by $F(y) = \{y\}$

Then G is continuous and F is quasi-continuous, but F∘G is not quasi-continuous.

1.10 Example.

Let $X = Y = Z = [0,\infty)$.

Define $G:Z \to Y$ by $G(z) = \{z + k \mid k = 0, 1, 2, 3, . . .\}$

$F:Y \to X$ by $F(y) = \{y^2\}$

Then G and F are both quasi-continuous but F∘G is not quasi-continuous.

For uniformly continuous multi-functions we achieve a better result.

1.11 Theorem.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, and ii) $F:Y \to X$ and $G:Z \to Y$ are uniformly continuous, then $F \circ G$: $Z \to X$ is uniformly continuous.

Proof: Let $U \in \mathcal{U}$. Then there is a symmetric $V \in \mathcal{V}$ such that for any pair $(y,y') \in V$, $F(y') \subset U[F(y)]$, and $W \in \mathcal{W}$ such that if $(z,z') \in W$, $G(z') \subset V[G(z)]$. Let $(z,z') \in W$ and $x' \in F \circ G(z')$. There is a $y' \in G(z')$ such that $x' \in F(y')$. Since $G(z') \subset V[G(z)]$ and $V$ is symmetric there is a $y \in G(z)$ such that $(y,y') \in V$. Then $x' \in F(y') \subset V[F(y)]$. Therefore $F \circ G(z') \subset V[F \circ G(z)]$.

The next few theorems will establish some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions.

In Ratner's dissertation [13] he established theorems of the following sort to establish the connection between the definitions of upper semi-continuity and lower semi-continuity for multi-functions and the traditional definitions for single valued functions.

"If $F:X \to R$ is a lower semi-continuous multi-function, such that for every $x \in X$, $F(x)$ is

> bounded above, and $f:X \to R$ is defined by $f(x) = \text{lub}\{F(x)\}$ then $f$ is a lower semi-continuous single-valued function."

The following theorem achieves a similar result for uniformly continuous multi-functions.

1.12 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $f, g:X \to R$ are functions such that $f(x) \leq g(x)$, for every $x \in X$, and

iii) $F:X \to R$ is a multi-function such that for every $x \in X$, $F(x) = [f(x), g(x)]$,

then $F$ is uniformly continuous iff $f$ and $g$ are uniformly continuous.

Proof: Suppose $f$ and $g$ are both uniformly continuous. Let $r > 0$ be given. Let

$V_r = \{(r_1, r_2) \in R \times R \mid |r_1 - r_2| < r\}$. Then there exists $U \in \mathcal{U}$, such that if $(x_1, x_2) \in U$ then

$f(x_2) \in V_r(f(x_1))$ and $g(x_2) \in V_r(g(x_1))$.

Then $F(x_2) = [f(x_2), g(x_2)]$

$$\subset (f(x_1) - r, g(x_1) + r)$$

$$= V_r[F(x_1)].$$

Suppose $F$ is a uniformly continuous multi-function.

Let $r > 0$ be given. There exists a symmetric $U \in \mathcal{U}$, such that if $(x_1, x_2) \in U$, then $F(x_2) \subset V_r[F(x_1)]$. Then $[f(x_2), g(x_2)] \subset V_r[F(x_1)] = (f(x_1) - r, g(x_1) + r)$, so $f(x_2) > f(x_1) - r$. But $(x_2, x_1) \in U$ if $(x_1, x_2) \in U$ so that $F(x_1) \subset V_r[F(x_2)]$. Thus $f(x_1) > f(x_2) - r$, then $f(x_1) - r < f(x_2) < f(x_1) + r$. So $f(x_2) \in V_r(f(x_1))$. Similarly $g(x_2) \in V_r(g(x_1))$.

1.13 Lemma.

If i) $X$ is a topological space,

ii) $(Y, \mathcal{V})$ is a uniform space,

iii) $x \in X$, and

iv) $F: X \to Y$ is a multi-function

such that $F(x)$ is compact;

the the following are equivalent:

1) $F$ is lower semi-continuous at $x$.

2) $F$ is quasi-lower semi-continuous at $x$.

Proof: Obviously 2 implies 1. For the converse suppose 1 is true, so that given any $O$ open in $Y$ such that $F(x) \cap O \neq \emptyset$, there is a neighborhood $N_1$ of $x$ such that $F(x') \cap O \neq \emptyset$ for every $x' \in N_1$. Let $V \in \mathcal{V}$ be given. Then there is a $W \in \mathcal{V}$, such that $W$ is symmetric and $W^2 \in V$. Since $F(x)$ is compact there is a finite set

$\{y_1, y_2, \ldots, y_n\}$, $F(x) \subset \bigcup_{i=1}^{n} W(y_i)$. Since F is lsc at x there is a neighborhood $N_i$ of x such that $F(x') \cap W(y_i) \neq \emptyset$ for every i=1, ..., n and for every $x' \in N_i$. Now if $y \in F(x)$ and $x' \in \bigcap_{i=1}^{n} N_i$ there is a $y_i$ such that $y \in W(y_i)$, and then

$$\emptyset \neq F(x') \cap W(y_i) \subset F(x') \cap W^2(y) \subset F(x') \cap V(y).$$

So F is qlsc.

1.14 Lemma.

i) X is a topological space,

ii) $(Y, \mathcal{U})$ is uniform space,

iii) $x \in X$, and

iv) $F: X \to Y$ is a multi-function such that F(x) is compact,

then the following are equivalent:

1) F is upper semi-continuous at x.

2) F is quasi upper semi-continuous at x.

Proof: 1 obviously implies 2. For the converse use Theorem 0.1.

1.15 Theorem

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ is a uniform space,

iii) $x \in X$, and

iv) $F:X \to Y$ is a multi-function such that $F(x)$ is compact;

then the following are equivalent;

1) F is continuous at x.

2) F is quasi-continuous at x.

Proof: Lemmas 1.13 and 1.14.

1.16 Corollary.

If i) X is a topological space,

ii) $(Y, \mathcal{U})$ is a uniform space, and

iii) $F:X \to Y$ is a compact-valued multi-function;

then the following are equivalent:

1) F is continuous.

2) F is quasi-continuous.

For a finite set S let #S denote the number of elements in S.

1.17 Theorem.

If i) X is a topological space,

ii) $(Y, \mathcal{U})$ is a $T_2$ uniform space, and

iii) $F:X \to Y$ is a multi-function such that

$\#F(x') = n$ for all $x'$ in some neighborhood of $x$;

then the following are equivalent:

1) F is continuous at x.

2) F is lower semi-continuous at x.

3) F is quasi-lower semi-continuous at x.

4) F is quasi-continuous at x.

Proof: By previous work we need only show that 2 implies 1 or that F is usc at x. Let $\{y_1, y_2, \ldots, y_n\} = F(x)$. Let O be an open set in Y such that $F(x) \subset O$. Let $\{U_1, U_2, \ldots, U_n\}$ be a collection of open set in Y such that

i) $y_i \in U_i$, $i=1, 2, \ldots, n$,

ii) $U_i \cap U_j = \emptyset$, $i \neq j$, and

iii) $\bigcup_{i=1}^{n} U_i \subset O$.

Since F is lsc at x there is a neighborhood N of x such that if $x' \in N$, $F(x') \cap U_i \neq \emptyset$, $i = 1, 2, \ldots, n$. If $x' \in N$, for each i, there is a $z_i \in F(x') \cap U_i$. Then

$$F(x') = \{z_1, z_2, \ldots, z_n\} \subset \bigcup_{i=1}^{n} U_i \subset O.$$

Note: A similar theorem was proved by Smithson [15] Lemma 1, page 449.

1.18 Theorem.

If i) X, Y are topological spaces,

ii) $x \in X$, and

iii) $F:X \to Y$ is a multi-function such that

$\#F(x) = 1$,

then the following are equivalent:

1) F is continuous at x.

2) F is upper semi-continuous at x and there is a neighborhood N of x such that if $x' \in N$ then $F(x') \neq \emptyset$.

Proof: Suppose O is an open set in Y and $F(x) \cap O \neq \emptyset$. Then $F(x) \subset O$ so there is a neighborhood $N_1$ of x such that $F(x') \subset O$ for each $x' \in N_1$. Let $N' = N_1 \cap N$.

1.19 Theorem.

If i) X,Y are topological spaces,

ii) $x \in X$, and

iii) $F:X \to Y$ is multi-function such that

$F(x) = Y$,

then the following are equivalent:

1) F is continuous at x.

2) F is lower semi-continuous at x.

1.20 Theorem.

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ a uniform space,

iii) $x \in X$, and

iv) $F: X \to Y$ is a multi-function such that

$F(x) = Y$,

then the following are equivalent:

1) F is quasi-continuous at x.

2) F is quasi-lower semi-continuous at x.

1.21 Theorem.

If i) $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces,

ii) $F: X \to Y$ is quasi-continuous, and

iii) X is compact,

then F is uniformly continuous.

Proof: Similar to the proof for single-valued functions.

1.22 Corollary.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ are uniform spaces,

ii) X is compact, and

iii) $F: X \to Y$ is compact valued,

then the following are equivalent:

1) F is continuous.

2) F is uniformly continuous.

1.23 Definition.

Let i) X, Y be topological spaces, and

ii) $F:X \to Y$ be a multi-function,

then $\overline{F}$ will denote the multi-function from X into Y defined by $\overline{F}(x) = \overline{F(x)}$ and $G(F) = \{(x,y) \mid y \in F(x)\}$.

Franklin [7] characterized regular spaces by the fact that $G(\overline{F}) = \overline{G(F)}$ (as a subset of $X \times Y$) for any usc function into a regular space and normal spaces by the fact that $\overline{F}$ is usc for any usc F into a normal space, Sec. 2 p. 16-20.

1.24 Theorem.

If i) X is a topological space,

ii) $(Y, \mathcal{V})$ is a uniform space,

iii) $F:X \to Y$ is a multi-function, and

iv) $\overline{F}:X \to Y$ is defined as in 1.20;

then the following are equivalent:

1) F is quasi-continuous.

2) $\overline{F}$ is quasi-continuous.

Proof: Suppose $F$ is quasi-continuous and $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$. Then there is a neighborhood $N$ of $x$ such that for any pair $x_1, x_2 \in N$, $F(x_1) \subset W[F(x_2)]$. Then $\overline{F}(x_1) \subset W[F(x_2)]$ since $\overline{F}(x_1) = \bigcap\{U[F(x_1)] \mid U \in \mathcal{V}\}$. Thus if $x_1, x_2 \in N$,

$$\overline{F}(x_1) \subset W[F(x_1)] \subset W^2[F(x_2)] \subset W^2[\overline{F}(x_2)] \subset V[\overline{F}(x_2)].$$

So $\overline{F}$ is quasi-continuous.

Suppose $\overline{F}$ is qc and $x \in X$ and $V \in \mathcal{V}$ are given. Then there is a $W \in \mathcal{V}$ such that $W^2 \subset V$. There is a neighborhood $N$ of $x$ such that if $x_1, x_2 \in N$, $\overline{F}(x_1) \subset W[\overline{F}(x_2)]$. Thus if $x_1, x_2 \in N$,

$$F(x_1) \subset \overline{F}(x_1) \subset W[\overline{F}(x_2)] \subset W^2[F(x_2)] \subset V[F(x_2)].$$

1.25 Theorem.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ are uniform spaces, and

ii) $F:X \to Y$ is a multi-function;

then the following are equivalent:

1) $F$ is uniformly continuous.

2) $\overline{F}$ is uniformly continuous.

Proof: Suppose $F$ is uc and $V \in \mathcal{V}$ is given. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and there is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \subset W[F(x)]$. If $(x_1,x_2) \in U$, then

$\overline{F}(x_2) \subset W[F(x_2)] \subset W^2[F(x_1)] \subset W^2[\overline{F}(x_1)] \subset V[\overline{F}(x_1)]$.

If $\overline{F}$ is uc and $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^2 \subset V$. Then there is a $U \in \mathcal{U}$ such that for each pair $(x_1,x_2) \in U$, $\overline{F}(x_2) \subset W[\overline{F}(x_1)]$. If $(x_1, x_2) \in U$, then $F(x_2) \subset \overline{F}(x_2) \subset W[\overline{F}(x_1)] \subset V[F(x_1)]$.

1.26 Theorem.

If i) $(X,\mathcal{U})$ is a uniform space,

ii) $(Y,\mathcal{V})$ is a uniformly locally connected space,

iii) $F:X \to Y$ is uniformly continuous, and

iv) for each $x \in X$, $\partial(F(x))$ is compact,

then $\overline{F}$ is continuous.

Proof: Let 0 be open in Y and $x \in X$ such that $\overline{F}(x) \subset 0$. Since $\partial(F(x))$ is compact there is a $V \in \mathcal{V}$ such that $V[\overline{F}(x)] \subset 0$. By Theorem 1.25 there is a $U \in \mathcal{U}$ such that for each pair $(x_1, x_2) \in U$, $\overline{F}(x_2) \subset V[\overline{F}(x_1)]$. Let $N = U(x)$. If $x' \in N$, then $(x, x') \in U$. Thus $F(x') \subset V[F(x)] \subset 0$. Thus F is usc. Since F is obviously lsc, F is continuous.

1.27 Theorem.

If i) X is a topological space,

ii) $(Y,\mathcal{V})$ is a uniformly locally connected space,

iii) $F:X \to Y$ is quasi-continuous, and

iv) for each $x \in X$, $\partial(F(x))$ is compact,

then $\overline{F}$ is continuous.

The next three theorems will establish some facts about the behavior of multi-functions into product spaces. In the following three theorems let

i) $X$ be a topological space,

ii) for every $a \in A$, $(Y_a, \mathcal{V}_a)$ be a uniform space,

iii) $(Y, \mathcal{V})$ be the uniform space where $Y = \prod_{a \in A} Y_a$ and $\mathcal{V}$ is the product uniformity,

iv) for every $a \in A$, $F_a:X \to Y_a$ be a multi-function,

v) $F:X \to Y$ be the multi-function defined for every $x \in X$ by $F(x) = \prod_{a \in A} F_a(x)$, and

vi) for every $a \in A$, $pr_a:Y \to Y_a$ is the projection map.

Then the following is a familiar theorem for single-valued functions.

"If $X$ is a topological space then $f:X \to Y$ is continuous iff for every $a \in A$, $f_a = pr_a \circ f:X \to Y$ is continuous."

The same theorem is not true for continuous multi-functions. The following example is from Strother [17].

1.28 Example.

Let $X = Y_1 = Y_2 = [-1,1]$, $Y = Y_1 \times Y_2$. Define $F:X \to Y$ by $F(x) = \{(-1,1), (1,-1)\}$ if x is rational; and $\{(-1,-1), (1,1)\}$ if x is irrational.

$pr_1 \circ F(x) = \{-1,1\} = pr_2 \circ F(x)$ for all x, so $pr_1 \circ F$ and $pr_2 \circ F$ are continuous but F is not continuous.

1.29 Theorem.

F is quasi-continuous iff for every $a \in A$, $F_a$ is quasi-continuous.

Proof: Suppose F is qc, let $a \in A$, $V_a \in \mathcal{V}_a$ and $x \in X$ be given. Let $\hat{V}_a = (pr_a^{-1} \times pr_a^{-1})[V_a]$. There is a neighborhood N of x such that if $x_1, x_2 \in N$ then $F(x_1) \subset \hat{V}_a[F(x_2)]$. Then

$F_a(x_1) = pr_a[F(x_1)] \subset pr_a[\hat{V}_a[F(x)]] \subset V_a[F_a(x_2)]$.

Suppose that for each $a \in A$, $F_a$ is qc. Let $V \in \mathcal{V}$ and $x \in X$ be given. Then there is a finite subset $J \subset A$ and $\{V_b \in \mathcal{V}_b \mid b \in J\}$ such that $\hat{V} = \bigcap \{\hat{V}_b \mid b \in J\} \subset V$. Since for each $b \in J$, $F_b$ is qc and J is finite there exists a neighborhood N of x such that if $x_1, x_2 \in N$ and

$b \in J$ then $F_b(x_1) \subset V_b[F_b(x_2)]$.

Then if $x_1, x_2 \in N$,

$F(x_1) = \prod_{a\in A} F_a(x_1) \subset \hat{V}[\prod_{a\in A} F_a(x_2)] \subset V[\prod_{a\in A} F_a(x_2)] =$

$V[F(x_2)]$.

1.30 Corollary.

If for every $a \in A$, $F_a$ is compact-valued, $F$ is continous iff for every $a \in A$, $F_a$ is continuous.

Proof: For every $x \in X$, $F(x)$ is compact so by theorem 1.15 $F$ is continuous iff $F$ is qc iff for every $a \in A$, $F_a$ is qc iff for every $a \in A$, $F_a$ is continuous.

1.31 Theorem.

If $(X, \mathcal{U})$ is a uniform space then the following are equivalent:

1) $F$ is uniformly continuous.

2) For every $a \in A$, $F_a$ is uniformly continuous.

Proof: Suppose $F$ is uc. Let $a \in A$ and $V_a \in \mathcal{V}_a$ be given. There is a $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$ then $F(x_2) \subset \hat{V}_a[F(x_1)]$. Then if $(x_1, x_2) \in U$,

$F_a(x_2) = pr_a[F(x_2) \subset pr_a[\hat{V}_a[F(x_1)]] = V_a[pr_aF(x_1)]$
$= V_a[F_a(x_1)]$. Thus $F_a$ is uc.

Suppose for every $a \in A$, $F_a$ is uc. Let $V \in \mathcal{V}$ be given. As in Theorem 1.19 there exists a finite subset $J \subset A$ and $\{V_b \in \mathcal{V}_b \mid b \in J\}$ such that $\hat{V} = \bigcap_{b \in J} \hat{V}_b \subset V$. Since J is finite there is a $U \in \mathcal{U}$ such that for every $b \in J$, if $(x_1, x_2) \in U$, then $F_b(x_2) \subset V_b[F_b(x_2)]$. If $(x_1, x_2) \in U$ then

$F(x_2) = \prod_{a \in A} F_a(x_1) \subset \hat{V}[\prod_{a \in A} F_a(x_2)] \subset V[F(x_2)]$. Thus

F is uc.

# CHAPTER 2: UNIFORM CONVERGENCE OF MULTI-FUNCTIONS

For two topological spaces X, Y let $F_M(X,Y)$ denote the set of all multi-functions from X into Y.

2.1 Definition.

Let i) X be a topological space,

ii) $(Y,\mathcal{V})$ be a uniform space,

iii) $\{F_d \mid d \in D\}$ be a net in $F_M(X,Y)$ and

iv) $F \in F_M(X,Y)$.

The family $\{F_d : d \in D\}$ <u>converges</u> <u>uniformly</u> <u>to</u> F, written $F_d \to F_{(un)}$ if for every $V \in \mathcal{V}$, there is a $d_o \in D$ such that for every $d \geq d_o$, if $x \in X$, $F_d(x) \subset V[F(x)]$ and $F(x) \subset V[F_d(x)]$.

2.2 Definition.

Let i) X be a topological space,

ii) $(Y,\mathcal{U})$ be a uniform space, and

iii) $\mathfrak{F}$ be a filter in $F_M(X,Y)$

a) The filter $\mathfrak{F}$ <u>converges</u> <u>uniformly</u> <u>to</u> a multi-function $F: X \to Y$, written $\mathfrak{F} \to F_{(un)}$, if for every $V \in \mathcal{V}$, there is a $B \in \mathfrak{F}$ such that for every $x \in X$, if $H \in B$,

$H(x) \subset V[F(x)]$ and $F(x) \subset V[H(x)]$.

b) $\mathfrak{F}$ will denote the filter generated by the collection of the sets of the form $\overline{H} = \{\overline{F}:X \to Y \mid F \in H\}$, $H \in \mathfrak{F}$.

NOTE. Throughout the remainder of this chapter we will assume that X is a topological space and $(Y,\mathcal{V})$ is a uniform space.

2.3 Theorem.

If i) $\{F_d \mid d \in d\}$ is a net in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$

then the following are equivalent:

1) $F_d \to F_{(un)}$

2) $\overline{F}_d \to \overline{F}_{(un)}$

3) $\overline{F}_d \to F_{(un)}$

4) $F_d \to \overline{F}_{(un)}$

Proof: Suppose $F_d \to F_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$ and $x \in X$ then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Then

$\overline{F}_d(x) \subset W[F_d(x)] \subset W^2[F(x)] \subset V[\overline{F}(x)]$ and

$\overline{F}(x) \subset W[F(x)] \subset W^2[F_d(x)] \subset V[\overline{F}_d(x)]$.

Suppose $\overline{F}_d \to \overline{F}_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$ and $x \in X$ then $\overline{F}_d(x) \subset W[\overline{F}(x)]$ and $\overline{F}(x) \subset W[\overline{F}_d(x)]$. Then

$F_d(x) \subset \overline{F}_d(x) \subset W[\overline{F}(x)] \subset W^2[F(x)] \subset V[F(x)]$, and

$F(x) \subset \overline{F}(x) \subset W[\overline{F}_d(x)] \subset W^2[F_d(x)] \subset V[F_d(x)]$.

Thus 1 and 2 are equivalent. That 3 and 4 are also equivalent follows from the fact $\overline{\overline{F}}_d = \overline{F}_d$ and $\overline{\overline{F}} = \overline{F}$.

The following theorem is equivalent to the one above only it is stated in terms of filters.

2.4 Theorem.

If i) $\mathfrak{F}$ is a filter in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$

then the following are equivalent:

1) $\mathfrak{F} \to F_{(un)}$

2) $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$

3) $\overline{\mathfrak{F}} \to F_{(un)}$

4) $\mathfrak{F} \to \overline{F}_{(un)}$

Proof: Suppose $\mathfrak{F} \to F_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$, such that $W^2 \subset V$ and $H \in \mathfrak{F}$ such that if $G \in H$, then $G(x) \subset W[F(x)]$ and $F(x) \subset W[G(x)]$ for every $x \in X$. Then if $G' \in \overline{H} \in \overline{\mathfrak{F}}$, there is a $G \in H$ such that $\overline{G} = G'$. Thus if $x \in X$, $G'(x) = \overline{G}(x) \subset W[G(x)] \subset W^2[F(x)] \subset V[\overline{F}(x)]$ and $\overline{F}(x) \subset W[F(x)] \subset W^2[G(x)] \subset V[\overline{G}(x)]$.

Thus $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$.

Suppose $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$; and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$, such that $W^2 \subset V$ and $H' \in \overline{\mathfrak{F}}$ such that if $G' \in H'$, and $x \in X$ then $G'(x) \subset W[\overline{F}(x)]$ and $\overline{F}(x) \subset W[G'(x)]$. There is a $H \in \mathfrak{F}$ such that $\overline{H} \subset H'$. If $G \in H$ and $x \in X$ then $\overline{G} \in \overline{H}$ so $G(x) \subset \overline{G}(x) \subset W[\overline{F}(x)] \subset W^2[F(x)] \subset V[F(x)]$ and $F(x) \subset \overline{F}(x) \subset W[\overline{G}(x)] \subset W^2[G(x)] \subset V[G(x)]$. Thus $\mathfrak{F} \to F_{(un)}$.

The equivalence of 3 and 4 to the others is again proved by observing that $\overline{\overline{\mathfrak{F}}} = \overline{\mathfrak{F}}$.

2.5 Theorem.

If i) $\{F_d \mid d \in D$ is a net in $F_M(X,Y)$,

ii) $F, G \in F_M(X,Y)$,

iii) $F_d \to F_{(un)}$, and

iv) $F_d \to G_{(un)}$

then $\overline{F} = \overline{G}$.

Proof: Let $x \in X$, and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$, $F_d(x) \subset W[G(x)]$ and $F(x) \subset W[F_d(x)]$. Thus $F(x) \subset W[F_d(x) \subset W^2[F(x)] \subset V[G(x)]$. Then $F(x) \subset \bigcap \{V[G(x)] \mid V \in \mathcal{V}\} = \overline{G}(x)$. Similarly $G(x) \subset \overline{F}(x)$, so $\overline{F}(x) = \overline{G}(x)$.

2.6 Theorem.

If i) $\mathfrak{F}$ is a filter in $F_M(X,Y)$,

ii) $F, G \in F_M(X,Y)$,

iii) $\mathfrak{F} \to F_{(un)}$, and

iv) $\mathfrak{F} \to G_{(un)}$

then $\overline{F} = \overline{G}$.

The next few theorems establish that uniform convergence preserves quasi-continuity and uniform continuity.

Let i) $\{F_d \mid d \in D\}$ be a net in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$.

2.7 Lemma.

If i) for every $d \in D$, $F_d$ is quasi-upper semi-continuous, and

ii) $F_d \to F_{(un)}$,

then F is quasi-upper semi-continuous.

Proof: Let $V \in \mathcal{V}$ be given. There is a $W \in \mathcal{V}$ such that $W^3 \subset V$ and a $d_o \in D$ such that if $d \geq d_o$ and $x \in X$, then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Since $F_{d_o}$ is qusc, there is a neighborhood N of x such that if $x' \in N$ then $F_{d_o}(x') \subset W[F_{d_o}(x)]$. Let $x' \in N$, then $F(x') \subset W[F_{d_o}(x')] \subset W^2[F_{d_o}(x)] \subset W^3[F(x)] \subset V[F(x)]$. Thus F is qusc at x. Since x was arbitrary in X, F is qusc.

2.8 Lemma.

If i) for every $d \in D$, $F_d$ is quasi-lower semi-continuous, and

ii) $F_d \to F_{(un)}$.

Then F is quasi-lower semi-continuous.

Proof: Similar to the proof above using Lemma 1.2 to characterize qlsc.

2.9 Corollary.

If i) for every $d \in D$, $F_d$ is quasi-continuous, and

ii) $F_d \to F_{(un)}$;

then F is quasi-continuous.

2.10 Theorem.

If i) $(Y, \mathcal{U})$ is a complete space ,

ii) for every $d \in D$, $F_d$ is continuous and compact valued, and

iii) $F_d \to F_{(un)}$;

then $\overline{F}$ is continuous.

Proof: From 2.9, $\overline{F}$ is qc so we need only show that $\overline{F}$ is compact valued. Let $x \in X$. If $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^4 \subset V$ and $d_o \in D$ such that if $d' \geq d_o$, $F_{d'}(x) \subset W[F(x)]$ and $F(x) \subset W[F_{d'}(x)]$. Then $F_{d_o}(x)$ is compact so there is a finite subset $\{y'_1, y'_2 \ldots, y'_n\}$ of $F_{d_o}(x)$ such $F_{d_o}(x) \subset \bigcup_{i=1}^{n} W(y_i)$. For each $i = 1, \ldots, n$, $F_{d_o}(x) \subset W[F(x)]$ so there is a $y_i \in F(x)$ such that $y'_i \in W(y_i)$. Now

$$F(x) \subset W[F_{d_o}(x)] \subset W[\bigcup_{i=1}^{n} W(y'_i)] = \bigcup_{i=1}^{n} W^2(y'_i) \subset \bigcup_{i=1}^{n} W^3(y_i).$$

Thus $\overline{F}(x) \subset W[F(x)] \subset \bigcup_{i=1}^{n} W^4(y_i) \subset \bigcup_{i=1}^{n} V(y_i)$.

Since $\overline{F}(x)$ is a complete totally bounded subset of a complete space, $\overline{F}(x)$ is compact.

A theorem similar to the above was proved by

Smithson [16].

Let i) $\mathfrak{F}$ be a filter in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$.

Then the following are a restatement of the previous theorems in terms of filters rather than nets.

2.11 Lemma.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-upper semi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-upper semi-continuous.

2.12 Lemma.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-lower semi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-lower semi-continuous.

2.13 Corollary.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-continuous.

2.14 Theorem.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, $G$ is continuous and compact valued, and

ii) $\mathfrak{F} \to F_{(un)}$;

then $\overline{F}$ is continuous.

The next theorems establish similar facts for uniform continuity.

2.15 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $\{F_d \mid d \in D\}$ is a net of uniformly continuous multi-functions, and

iii) $F_d \to F_{(un(}$;

then F is uniformly continuous.

Proof: If $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^3 \in V$ and $d_o \in D$ such that if $x \in X$ and $d \geq d_o$ then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Since $F_{d_o}$ is uc, there is a $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$ then $F_{d_o}(x_2) \subset W[F_{d_o}(x_1)]$. Then if $(x_1, x_2) \in U$, $F(x_2) \subset W[F_{d_o}(x_2)] \subset W^2[F_{d_o}(x_1)] \subset W^3[F(x_1)] \subset V[F(x_1)]$. Thus F is uc.

2.16 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $\mathfrak{F}$ is a filter in $F_M(X,Y)$,

iii) there is an $H \in \mathfrak{F}$ such that every $G \in H$ is uniformly continuous, and

iv) $\mathfrak{F} \to F_{(un)}$

then F is uniformly continuous.

# CHAPTER 3: THE HYPERSPACE OF A UNIFORM SPACE

Let $(Y, \mathcal{V})$ be a uniform space, $P(Y) = \{A \mid A \subset Y\}$; and $\hat{\mathcal{V}}$ shall be the uniformity induced on $P(Y)$ by the collection of all of the sets of the form,
$\hat{V} = \{(A,B) \in P(Y) \times P(Y) \mid A \subset V[B]$ and $B \subset V[A]\}$ for $V \in \mathcal{V}$. $\hat{\mathcal{V}}$ is the Hausdorff uniformity. The following definitions are motivated by convergence in $P(Y)$ with respect to either the topology induced by the Hausdorff uniformity or the finite topology; see Michaels [12].

3.1 Definition.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then $B_d$ <u>converges to</u> $A$ <u>with respect to the finite topology on</u> $P(Y)$, written $B_d \to A_{(f.t.)}$, if for any open set $O$ in $Y$
i) if $A \subset O$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \subset O$; and ii) if $A \cap O \neq \emptyset$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \cap O \neq \emptyset$.

3.2 Definition.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then $B_d$ <u>converges to</u> $A$ <u>with respect to the topology induced</u>

by the Hausdorff uniformity on P(Y), written $B_d \to A_{(u.t.)}$, if for every $V \in \mathcal{V}$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \subset V[A]$ and $A \subset V[B_d]$.

For filters we have:

3.3 Definition.

If i) $\mathfrak{F}$ is a filter in P(Y), and

ii) $A \subset P(Y)$;

then $\mathfrak{F}$ converges to A with respect to the finite topology on P(Y), written $\mathfrak{F} \to A_{(f.t.)}$, if for any O open in Y i) if $A \subset O$, there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \subset O$, and ii) if $A \cap O \neq \emptyset$, there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \cap O \neq \emptyset$.

3.4 Definition.

If i) $\mathfrak{F}$ is a filter in P(Y), and

ii) $A \in P(Y)$;

then $\mathfrak{F}$ converges to A with respect to the topology induced by the Hausdorff uniformity on P(Y), written $\mathfrak{F} \to A_{(u.t.)}$ if for every $V \in \mathcal{V}$ there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \subset V[A]$ and $A \subset V[B]$.

The two concepts of convergence are distinct but they do coincide on the set of compact subsets, see Michaels

[12].

3.5 Example.

Let $D = \{1, 2, 3, 4, \ldots\}$, $Y = [0,\infty)$,

$A = \{0, 1, 2, 3, \ldots\}$; and $B_d = \{\frac{1}{d}, 1 + \frac{1}{d}, 2 + \frac{1}{d}, \ldots\}$

for each $d \in D$. Then $B_d \to A_{(u.t.)}$ but $B_d \not\to A_{(f.t.)}$.

3.6 Example.

Let $D = \{1, 2, 3, 4, \ldots\}$, $Y = A = [0,\infty)$

$B_d = [0,d]$, for every $d \in D$.

Then $B_d \not\to A_{(u.t.)}$ but $B_d \to A_{(f.t.)}$.

3.7 Theorem.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then the following are equivalent:

1) $B_d \to A_{(u.t.)}$

2) $\overline{B}_d \to \overline{A}_{(u.t.)}$

3) $\overline{B}_d \to A_{(u.t.)}$

4) $B_d \to \overline{A}_{(u.t.)}$

Proof: Similar to the proof of 2.3.

3.8 Definition.

Let $\mathfrak{F}$ be a filter on $P(Y)$. For $H \in \mathfrak{F}$ let $\overline{H} = \{\overline{B} \mid B \in H\}$. Then $\overline{\mathfrak{F}}$ will denote the filter generated by the set $\{\overline{H} \mid H \in \mathfrak{F}\}$.

3.9 Theorem.

If i) $\mathfrak{F}$ is a filter on $P(Y)$, and

ii) $A \in P(Y)$;

then the following are equivalent:

1) $\mathfrak{F} \to A_{(u.t.)}$

2) $\overline{\mathfrak{F}} \to \overline{A}_{(u.t.)}$

3) $\overline{\mathfrak{F}} \to A_{(u.t.)}$

4) $\mathfrak{F} \to \overline{A}_{(u.t.)}$

Proof: First we prove that: if $\mathfrak{F}$ is a filter on $P(Y)$, $\overline{\mathfrak{F}}$ is a filter on $P(Y)$. Let $H'_1, H'_2 \in \overline{\mathfrak{F}}$ then there are $H_1, H_2 \in \mathfrak{F}$ such that $\overline{H}_1 \subset H'_1$ and $\overline{H}_2 \subset H'_2$. There is a $B \in P(Y)$ such that $B \in H_1 \cap H_2$ so $\overline{B} \in \overline{H}_1 \cap \overline{H}_2 \subset H'_1 \cap H'_2$.

The rest of the properties of a filter follow from the definition of $\overline{\mathfrak{F}}$. The remainder of the proof is continued as in 2.4.

3.10 Theorem.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$,

ii) $A, A' \in P(Y)$,

iii) $B_d \to A_{(u.t.)}$, and

iv) $B_d \to A'_{(u.t.)}$

then $\overline{A} = \overline{A'}$.

Proof: See the proof of Theorem 2.5.

3.11 Theorem.

If i) $\mathfrak{F}$ is a filter in $P(Y)$

ii) $A, A' \in P(Y)$

iii) $\mathfrak{F} \to A_{(u.t.)}$, and

iv) $\mathfrak{F} \to A'_{(u.t.)}$

then $\overline{A} = \overline{A'}$.

Proof: See the proof of Theorem 2.6.

3.12 Definition.

Let i) $(Y, \mathcal{V})$ be a uniform space, and

ii) $\{B_d \mid d \in D\}$ be a net in $P(Y)$.

$\{B_d \mid d \in D\}$ is a <u>Cauchy net of subsets</u> of Y if for each $V \in \mathcal{V}$, there is a $d_0 \in D$, such that if $d_1, d_2 \geq d_0$ then $B_{d_1} \subset V[B_{d_2}]$.

Notice that if $B_d \to A_{(u.t.)}$ then $\{B_d \mid d \in D\}$ is a Cauchy net.

The following notation and definitions are adopted from Kuratowski [11], Chapter 11, Sec. 29, Page 335-337.

3.13 Definition.

Let i) X be a topological space, and

ii) $\{B_d \mid d \in D\}$ be a net in P(Y).

Ls $B_d = \{x \in X \mid$ for every O open in X and $d_o \in D$, if $x \in O$, there is a $d \geq d_o$ such that $B_d \cap O \neq \emptyset\}$. Ls $B_d = \{x \in X \mid$ for every O open in X, if $x \in O$ there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \cap O \neq \emptyset\}$.

Then if Li $B_d$ = Ls $B_d$ we say that Lim $B_d$ exists and Lim $B_d$ = Li $B_d$ = Ls $B_d$.

This definition is the one usually used in defining convergence of sets.

3.14 Theorem.

If i) $(Y, \mathcal{U})$ is a uniform space, and

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subsets of Y;

then Li $B_d$ = Ls $B_d$.

Proof: It is known that Li $B_d \subset$ Ls $B_d$, so it suffices to show that Ls $B_d \subset$ Li $B_d$. If Ls $B_d = \emptyset$, there is nothing to show. If $x \in$ Ls $B_d$, let O be open, $x \in O$. There is a $W \in \mathcal{V}$, such that W is symmetric and $W^2(x) \subset O$ and $d_o \in D$, such that if $d_1, d_2 \geq d_o$ then $B_{d1} \subset W[B_{d2}]$. Let $d \geq d_o$, there is a $d' \geq d_o$ such that $B_{d'} \cap W(x) \neq \emptyset$. Thus there is an $x' \in B_{d'} \cap W(x)$. Since $B_{d'} \subset W[B_d]$ and W is symmetric there exists $x_o \in W(x') \cap B_d$. Then

$x_o \in W(x') \cap B_d \subset W^2(x) \cap B_d \subset O \cap B_d$, hence $x \in$ Li $B_d$.

3.15 Theorem.

If i) $(Y, \mathcal{V})$ is a uniform space,

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subsets in Y, and

iii) $B_d \to B_{(u.t.)}$; $B \in P(Y)$;

then i) Lim $B_d = \overline{B}$ and

ii) $B = \emptyset$ iff there is a $d_o \in D$ such that if $d \geq d_o$, $B_d = \emptyset$.

Proof: Suppose $B \neq \emptyset$, there is a $y \in B$. Then if $V = Y \times Y \in \mathcal{V}$, there is a $d_o \in D$ such that if $d \geq d_o$,