Multi-functions and uniform spaces
by John Cecil James Harvey

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

Abstract:
Let (Y,V) be a uniform space, X be a topological space, and F:X → Y be a multi-function.

Chapter 1 introduces the concepts of F being quasi-continuous or uniformly continuous and examines some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions. A typical result is that if F is compact-valued then F is quasi-continuous if and only if F is continuous.

Chapter 2 is concerned with uniform convergence. A typical result if that if {Fd | d □ D} is a net of uniformly (quasi-) continuous multi-functions, which converges uniformly to F, then F is uniformly (quasi-) continuous .

Chapter 3 is concerned with the structure of P(Y), the power set of Y, with the Hausdorff uniformity and its induced topology. A one-one relationship is established between continuous single-valued functions into P(Y) with this topology and quasi-continuous multi-functions into Y.

Chapter 4 is about the semi-group of quasi-continuous real valued multi-functions. A typical result is that {x □ X | F(x) is unbounded) is open in X.

Chapter 5 is a listing of several properties of the set, CM(X,Y), of all continuous multi-functions from X into Y, with various topologies.

MULTI-FUNCTIONS AND UNIFORM SPACES

by

JOHN CECIL JAMES HARVEY JR.

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematics

Approved:

Louis C. Barrett
Head, Major Department

Byron L. McAllister
Chairman, Examining Committee

[signature]
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

June, 1971

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. McAllister for his patience and aid in finishing this project. I would also like to express my appreciation to NASA for their support during the academic year 1969-1970 through grant 8-600-705.

TABLE OF CONTENTS

ABSTRACT

Let $(Y,\mathcal{V})$ be a uniform space, X be a topological space, and $F:X \to Y$ be a multi-function.

Chapter 1 introduces the concepts of F being quasi-continuous or uniformly continuous and examines some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions. A typical result is that if F is compact-valued then F is quasi-continuous if and only if F is continuous.

Chapter 2 is concerned with uniform convergence. A typical result if that if $\{F_d \mid d \in D\}$ is a net of uniformly (quasi-) continuous multi-functions, which converges uniformly to F, then F is uniformly (quasi-) continuous.

Chapter 3 is concerned with the structure of P(Y), the power set of Y, with the Hausdorff uniformity and its induced topology. A one-one relationship is established between continuous single-valued functions into P(Y) with this topology and quasi-continuous multi-functions into Y.

Chapter 4 is about the semi-group of quasi-continuous real valued multi-functions. A typical result is that $\{x \in X \mid F(x) \text{ is unbounded}\}$ is open in X.

Chapter 5 is a listing of several properties of the set, $C_M(X,Y)$, of all continuous multi-functions from X into Y, with various topologies.

INTRODUCTION

Several authors have investigated properties of multi-functions satisfying various types of continuity. Our purpose in this paper will be to define a concept of a uniformly continuous multi-function. Then we will establish several properties of the uniformly continuous multi-functions.

In this work we shall assume that the definitions of the common topological terms are known. Among the symbols to be used are the following.

For a topological space X and a subset $A \subset X$, $\overline{A}$ is the closure of A, A° is the interior of A, $\partial(A)$ is the boundary of A, and X - A is the complement of A in X. $\emptyset$ will denote the empty set. R will denote the real numbers and if $a, b \in R$, $a \leq b$, then $[a,b] = \{x \mid a \leq x \leq b\}$.

By a uniform space we mean a pair $(X,\mathcal{U})$, where X is a topological space and $\mathcal{U}$ is a uniformity as defined in Kelley [9].

For a uniform space $(X,\mathcal{U})$, for any $x \in X$, $A \subset X$ and $U \in \mathcal{U}$; $U(x) = \{y \mid (x,y) \in U\}$; and $U[A] = \bigcup \{U(x) \mid x \in A\}$.

Note $\overline{A} = \bigcap \{U[A] \mid U \in \mathcal{U}\}$.

F is a <u>multi-function</u> <u>from</u> X <u>into</u> Y, $F:X \to Y$, if for each $x \in X$, F assigns a subset $F(x) \subset Y$. (Many authors require that for every $x \in X$, $F(x) \neq \emptyset$ and $F(x) = \overline{F(x)}$). The closedness of F(x) is desireable if we are trying to establish separation properties, see Smithson [15].

For a multi-function $F:X \to Y$, X and Y topological spaces, we will say that F is <u>compact-valued</u> (<u>closed-valued</u>, <u>open-valued</u>) if F(x) is compact (closed, open) for every $x \in X$.

F is <u>upper semi-continuous</u> (<u>usc</u>) <u>at</u> <u>x</u> $\in$ X if for every open set O in Y, if $F(x) \subset O$, then there is a neighborhood N of x such that $F(x') \subset O$ for every $x' \in N$.

F is <u>lower semi-continuous</u> (<u>lsc</u>) <u>at</u> $x \in X$ provided that for every open set O in Y, if $F(x) \cap O \neq \emptyset$, there is a neighborhood N of x such that $F(x') \cap O \neq \emptyset$ for every $x' \in N$.

F is <u>continuous</u> <u>at</u> $x \in X$, provided that F is both usc and lsc at x. F is <u>continuous</u> (<u>usc</u>, <u>lsc</u>) if for every $x \in X$, F is continuous (usc, lsc) at x.

We shall need the following two theorems:

0.1 Theorem.

If i) $(x, \mathcal{U})$ is a uniform space,

ii) A is a compact subset of X,

iii) O is an open set in X, and

iv) $A \subset O$,

then there is a $V \in \mathcal{U}$ such that $V[A] \subset O$.

Proof: Kelley [9] Theorem 33, Chapter 36.

0.2 Definition.

A uniform space $(X, \mathcal{U})$ is a uniformly locally connected space if for each $U \in \mathcal{U}$, there is a $W \in \mathcal{U}$ such that $W \subset U$ and $W(x)$ is connected for every $x \in X$.

0.3 Theorem.

If i) $(X, \mathcal{U})$ is a uniformly locally connected space,

ii) $A \subset X$ such that $\partial(A)$ is compact,

iii) O is an open set in X, and

iv) $\overline{A} \subset O$,

then there is a $U \in \mathcal{U}$ such that $U[\overline{A}] \subset O$.

Proof: By Theorem 0.1 since $\partial(A)$ is compact, there is a $V \in \mathcal{U}$ such that $V[\partial(A)] \subset O$. There is a $W \in \mathcal{U}$, such that W is symmetric and $W^2 \subset V$. Also there is a $U \in \mathcal{U}$, such that $U \subset W$ and $U(x)$ is connected for each $x \in X$. Then $U[\overline{A}] \subset O$. For; let $x \in \overline{A}$. If $x \in \partial(A)$, $U(x) \subset V(x) \subset O$. If $U(x) \cap \partial(A) \neq \emptyset$, there is a $y \in \partial(A)$ such that $y \in U(x) \subset W(x)$. Since W is symmetric, $x \in W(y)$. Thus $U(x) \subset W[W(y)] = W^2(y) \subset V(y) \subset O$.

Suppose $U(x) \cap \partial(A) = \emptyset$. Let $M_1 = U(x) \cap (X-\overline{A})$ and

$M_2 = U(x) \cap \overline{A} = U(x) \cap (A^\circ \cup \partial(A)) =$

$(U(x)) \cap A^\circ) \cup (U(x) \cap \partial(A)) = U(x) \cap A^\circ.$

But $U(x) = M_1 \cup M_2$ and $x \in M_2$; therefore, since $U(x)$ is connected, $M_1 = \emptyset$. Thus $U(x) = M_2 \subset A \subset O$.

CHAPTER 1: QUASI-CONTINUOUS AND UNIFORMLY CONTINUOUS MULTI-FUNCTIONS.

In studying continuous multi-functions into a uniform space it is apparent that a different definition of continuity would be useful since, e.g., the log function, log: $C \to C$, is not a continuous multi-function in the usual sense. Ratner, in his dissertation, defined concepts which he called metrically continuous and uniformly continuous, which for metric spaces are the same as the concepts of quási-continuous and uniformly continuous.

1.1 Definition.

Let i) X be a topological space,

ii) $(Y, \mathcal{V})$ be a uniform space, and

iii) $F: X \to Y$ be a multi-function,

then F is <u>quasi-continuous</u> (<u>q.c.</u>) if both of the following are satisfied:

1) For each $V \in \mathcal{V}$, and each $x \in X$, there is a neighborhood N of x such that $F(x') \subset V[F(x)]$ for each $x' \in N$.

2) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood N of x such that $F(x') \cap V(y) \neq \emptyset$ for each $x' \in N$ and each $y' \in F(x)$.

If a multi-function satisfies condition 1, we will say that it is <u>quasi-upper semi-continuous</u> (qusc), if it satisfies 2, we will say that it is <u>quasi-lower semi-continuous</u> (qlsc).

1.2 Lemma.

Let i) X be a topological space,

ii) $(Y, \mathcal{V})$ be a uniform space, and

iii) $F: X \to Y$ be a multi-function.

Then the following are equivalent:

1) F is quasi-lower semi-continuous.

2) For each $x \in X$ and each $V \in \mathcal{V}$, there is a neighborhood N of x such that $F(x) \subset V[F(x')]$ for each $x' \in N$.

Proof: Suppose that F is qlsc and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$, such that W is symmetric and $W \subset V$. Since F is qlsc there is a neighborhood N of x such that for every $x' \in N$, $F(x') \cap W(y) \neq \emptyset$ for every $y \in F(x)$. Since W is symmetric, if $x' \in N$, for each $y \in F(x)$, $y \in W[F(x')] \subset V[F(x')]$ thus $F(x) \subset V[F(x')]$.

Now suppose condition 2 holds. If $x \in X$ and $V \in \mathcal{V}$, let $W \in \mathcal{V}$ such that W is symmetric and $W \subset V$. There is a neighborhood N of x such that $F(x) \subset W[F(x')]$ for each $x' \in N$. So for any $y \in F(x)$ and any $x' \in N$, $y \in W[F(x')]$. Since W is symmetric there is a $y' \in F(x')$ such that $y' \in W(y)$. Thus $\emptyset \neq F(x') \cap W(y) \subset F(x') \cap V(y)$.

Since y was arbitrary, F is qlsc.

1.3 Theorem.

Let i) X be a topological space,

ii) $(Y,\mathcal{V})$ be a uniform space, and

iii) $F:X \to Y$ be a multi-function,

then the following are equivalent:

1) F is quasi-continuous

2) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood N of x such that for any $x' \in N$, $F(x') \subset V[F(x)]$ and $F(x) \subset V[F(x')]$.

3) For each $V \in \mathcal{V}$ and each $x \in X$, there is a neighborhood N of x such that for any two points $x_1, x_2 \in N$, $F(x_1) \subset V[F(x_2)]$.

Proof: We will show that 1 implies 2, 2 implies 3 and 3 implies 1.

Suppose 1 holds and that $x \in X$ and $V \in \mathcal{V}$ are given. Since F is qusc there is a neighborhood N_1 of x such that $F(x') \subset V[F(x)]$ for each $x' \in N$. Since F is qlsc and by lemma 1.2 there is a neightborhood N_2 of x such that $F(x) \subset V[F(x')]$ for each $x' \in N_2$. Let $N = N_1 \cap N_2$.

Suppose 2 holds, and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$, such that W is symmetric and $W^2 \subset V$.

There is a neighborhood N of x such that for any $x' \in N$, $F(x') \subset W[F(x)]$ and $F(x) \subset W[F(x')]$, let x_1, $x_2 \in N$. Then $F(x_1) \subset W[F(x)]$ and $F(x) \subset W[F(x_2)]$ so $F(x_1) \subset W[F(x)] \subset W \circ W[F(x_2)] = W^2[F(x_2)] \subset V[F(x_2)]$.

Suppose 3 holds and that $x \in X$ and $V \in \mathcal{V}$ are given. There is a neighborhood N of x such that for any pair x_1, $x_2 \in N$, $F(x_1) \subset V[F(x_2)]$. So for any $x' \in N$, $F(x') \subset V[F(x)]$. Thus F is qusc. Also $F(x) \subset V[F(x')]$ for each $x' \in N$, so by lemma 1.3 F is qlsc.

1.4 Definition.

Let i) $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces, and

ii) $F:X \to Y$ be a multi-function.

Then F is <u>uniformly</u> <u>continuous</u> (uc) if F satisfies either of the following conditions:

1) For every $V \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that for each pair $(x,x') \in U$, $F(x') \subset V[F(x)]$.

2) For every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \cap V(y) \neq \emptyset$ for every $y \in F(x)$.

1.5 Lemma.

If i) $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces, and

ii) $F:X \to Y$ is a multi-function,

then conditions 1 and 2 of definition 1.4 are equivalent.

Proof: Suppose that F satisfies condition 1. Let $V \in \mathcal{V}$ be given. There is a $W \in \mathcal{V}$ such that W is symmetric and $W \subset V$. There is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \subset W[F(x)]$. Without loss of generality we may suppose that U is symmetric. Then $(x,x') \in U$ implies that $F(x) \subset W[F(x')]$. For any $y \in F(x)$, $y \in W[F(x')]$ so there is a $y' \in F(x')$ such that $(y,y') \in W$. But then $F(x') \cap W(y) \neq \emptyset$. So F satisfies condition 2.

Suppose F satisfies condition 2. Let $V \in \mathcal{V}$. There is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \cap V(y) \neq \emptyset$ for each $y \in F(x)$. Then for each $y \in F(x)$ there is a $y' \in F(x')$ such that $(y,y') \in V$. Without loss of generality we may assume that V and U are symmetric. Then for $(x,x') \in U$ and for each $y \in F(x)$ there is a $y' \in F(x')$ such that $(y,y') \in V$. Then also $(y',y) \in V$ so $y \in V(y') \subset V[F(x')]$. Thus $F(x) \subset V[F(x')]$. Since U was symmetric we also have $(x',x) \in U$ so $F(x') \subset V[F(x)]$.

Using our definition of continuity, if $F:Y \to X$ and $G:Z \to Y$ are continuous multi-functions then $F \circ G:Z \to X$ is a continuous multi-function. However the composition of two closed valued continuous multi-functions need not be closed valued.

1.6 Example.

Let $X = Z = [0,1]$, $Y = [1,\infty)$.

Define $G:Z \to Y$ by $G(z) = [1,\frac{1}{z}]$; $z \in (0,1]$,

$G(0) = [1,\infty)$;

$F:Y \to X$ by $F(y) = [\frac{1}{x},1]$;

then $F \circ G(0) = (0,1]$ is not closed.

However the following is true.

1.7 Theorem.

Let i) Z and Y are topological spaces,

ii) X is a regular space,

iii) $G:Z \to Y$ is a compact-valued continuous multi-function, and

iv) $F:Y \to X$ is a continuous closed-valued multi-function,

then $F \circ G$ is a closed valued function.

Proof: Let $z \in Z$, we will shown that if $x \notin F \circ G(z)$ then $x \notin \overline{F \circ G(z)}$. Suppose $x \in F(y)$. Since Z is regular, there exists open sets O_1, O_2 in X such that $F(y) \subset O_1$, $x \in O_2$ and $O_1 \cap O_2 = \emptyset$. Since F is continuous, there exists a neighborhood N of y, such that if $y' \in N$, then $F(y') \subset O_1$.

Suppose $x \notin F \circ G(z) = F[G(z)]$. Since $G(z)$ is compact, there exists finite sets $\{N_1, N_2, \ldots, N_n\}$ of open sets in Y and $\{O_{11}, O_{12}, \ldots, O_{1n}\}$ and $\{O_{21}, O_{22}, \ldots, O_{2n}\}$ of open sets in X such that

i) $G(z) \subset \bigcup_{i=1}^{n} N_i$,

ii) $F[N_i] \subset O_{1i}$, $i=1, 2, \ldots, n$,

iii) $x \in O_{2i}$, $i=1, 2, \ldots, n$; and

iv) $O_{1i} \cap O_{2i} = \emptyset$, $i=1, 2, \ldots, n$.

Then

$F \circ G(z) = F[G(z)] \subset F[\bigcup_{i=1}^{n} N_i] \subset \bigcup_{i=1}^{n} F[N_i] \subset \bigcup_{i=1}^{n} O_{1i}$. Now

$x \in \bigcap_{i=1}^{n} O_{2i}$, and $(\bigcup_{i=1}^{n} O_{1i}) \cap (\bigcap_{i=1}^{n} O_{2i}) = \emptyset$.

So $x \notin \overline{F \circ G(z)}$.

The above shows that if F is a closed-valued continuous multi-function into a regular space and K is a compact subset of the domain of F then F[K] is closed.

1.8 Lemma.

If i) Z, Y are topological spaces,

ii) $(X,\mathcal{U})$ is a uniform space,

iii) $G:Z \to Y$ is continuous compact-valued multifunction, and

iv) $F:Y \to X$ is quasi-continuous,

then $F \circ G$ is quasi-continuous

Proof: Let $W \in \mathcal{U}$, $z \in Z$. For each $y \in G(z)$ there is, by Theorem 1.3, an open neighborhood M of y such that if $y', y'' \in M$, then $F(y') \subset W[F(y'')]$. Since $G(z)$ is compact there is a finite set of open sets $\{M_1, \ldots, M_n\}$ and a set of points $\{y_1, y_2, \ldots, y_n\}$ such that

i) $y_i \in M_i \cap G(z)$, $i=1, 2, \ldots, n$,

ii) $G(z) \subset \bigcup_{i=1}^{n} M_i$, and

iii) if $y', y'' \in M_i$, $F(y') \subset W[F(y'')]$ for $i=1, 2, \ldots, n$.

There is a neighborhood N of z such that if $z' \in N$ then $G(z') \subset \bigcup_{i=1}^{n} M_i$ and $G(z') \cap M_i \neq \emptyset$ $i=1, \ldots, n$. Let $z' \in N$. Then $F \circ G(z') = F[G(z')] \subset F[\bigcup_{i=1}^{n} M_i] \subset \bigcup_{i=1}^{n} F[M_i]$ $\subset \bigcup_{i=1}^{n} W[F(y_i)] \subset W[F \circ G(z)]$.

For each $i=1, \ldots, n$, there is a point $y'_i \in G(z') \cap M_i$.

Then $F \circ G(z) \subset F[\bigcup_{i=1}^{n} M_i] \subset \bigcup_{i=1}^{n} W[F(y'_i)] \subset W[F \circ G(z')]$.

Thus $F \circ G$ is quasi-continuous by Theorem 1.3.

To show that the requirement of $G(z)$ being compact for each $z \in Z$ is essential consider the following:

1.9 Example

Let $X = Y = Z = [0,\infty)$.

Define $G:Z \to Y$ by $G(z) = [0,\frac{1}{z}]$ $z \in (0,\infty)$,

$= [0,\infty)$ $z = 0$,

$F:Y \to X$ by $F(y) = \{y\}$

Then G is continuous and F is quasi-continuous, but $F \circ G$ is not quasi-continuous.

1.10 Example.

Let $X = Y = Z = [0,\infty)$.

Define $G:Z \to Y$ by $G(z) = \{z + k \mid k = 0, 1, 2, 3, \ldots\}$

$F:Y \to X$ by $F(y) = \{y^2\}$

Then G and F are both quasi-continuous but $F \circ G$ is not quasi-continuous.

For uniformly continuous multi-functions we achieve a better result.

1.11 Theorem.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, and ii) $F:Y \to X$ and $G:Z \to Y$ are uniformly continuous, then $F \circ G: Z \to X$ is uniformly continuous.

Proof: Let $U \in \mathcal{U}$. Then there is a symmetric $V \in \mathcal{V}$ such that for any pair $(y,y') \in V$, $F(y') \subset U[F(y)]$, and $W \in \mathcal{W}$ such that if $(z,z') \in W$, $G(z') \subset V[G(z)]$. Let $(z,z') \in W$ and $x' \in F \circ G(z')$. There is a $y' \in G(z')$ such that $x' \in F(y')$. Since $G(z') \subset V[G(z)]$ and V is symmetric there is a $y \in G(z)$ such that $(y,y') \in V$. Then $x' \in F(y') \subset V[F(y)]$. Therefore $F \circ G(z') \subset V[F \circ G(z)]$.

The next few theorems will establish some relationships among continuous, quasi-continuous, and uniformly continuous multi-functions.

In Ratner's dissertation [13] he established theorems of the following sort to establish the connection between the definitions of upper semi-continuity and lower semi-continuity for multi-functions and the traditional definitions for single valued functions.

"If $F:X \to R$ is a lower semi-continuous multi-function, such that for every $x \in X$, $F(x)$ is

> bounded above, and $f: X \to R$ is defined by $f(x) = \text{lub}\{F(x)\}$ then f is a lower semi-continuous single-valued function."

The following theorem achieves a similar result for uniformly continuous multi-functions.

1.12 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $f, g: X \to R$ are functions such that $f(x) \leq g(x)$, for every $x \in X$, and

iii) $F: X \to R$ is a multi-function such that for every $x \in X$, $F(x) = [f(x), g(x)]$,

then F is uniformly continuous iff f and g are uniformly continuous.

Proof: Suppose f and g are both uniformly continuous. Let $r > 0$ be given. Let

$V_r = \{(r_1, r_2) \in R \times R \mid |r_1 - r_2| < r\}$. Then there exists $U \in \mathcal{U}$, such that if $(x_1, x_2) \in U$ then

$f(x_2) \in V_r(f(x_1))$ and $g(x_2) \in V_r(g(x_1))$.

Then $F(x_2) = [f(x_2), g(x_2)]$

$$\subset (f(x_1) - r, g(x_1) + r)$$

$$= V_r[F(x_1)].$$

Suppose F is a uniformly continuous multi-function.

Let $r > 0$ be given. There exists a symmetric $U \in \mathcal{U}$, such that if $(x_1, x_2) \in U$, then $F(x_2) \subset V_r[F(x_1)]$. Then $[f(x_2), g(x_2)] \subset V_r[F(x_1)] = (f(x_1) - r, g(x_1) + r)$, so $f(x_2) > f(x_1) - r$. But $(x_2, x_1) \in U$ if $(x_1, x_2) \in U$ so that $F(x_1) \subset V_r[F(x_2)]$. Thus $f(x_1) > f(x_2) - r$, then $f(x_1) - r < f(x_2) < f(x_1) + r$. So $f(x_2) \in V_r(f(x_1))$. Similarly $g(x_2) \in V_r(g(x_1))$.

1.13 Lemma.

If i) X is a topological space,

ii) $(Y, \mathcal{V})$ is a uniform space,

iii) $x \in X$, and

iv) $F: X \to Y$ is a multi-function

such that $F(x)$ is compact;

the the following are equivalent:

1) F is lower semi-continuous at x.

2) F is quasi-lower semi-continuous at x.

Proof: Obviously 2 implies 1. For the converse suppose 1 is true, so that given any O open in Y such that $F(x) \cap O \neq \emptyset$, there is a neighborhood N_1 of x such that $F(x') \cap O \neq \emptyset$ for every $x' \in N_1$. Let $V \in \mathcal{V}$ be given. Then there is a $W \in \mathcal{V}$, such that W is symmetric and $W^2 \in V$. Since $F(x)$ is compact there is a finite set

$\{y_1, y_2, \ldots, y_n\}$, $F(x) \subset \bigcup_{i=1}^{n} W(y_i)$. Since F is lsc at x there is a neighborhood N_i of x such that $F(x') \cap W(y_i) \neq \emptyset$ for every $i=1, \ldots, n$ and for every $x' \in N_i$. Now if $y \in F(x)$ and $x' \in \bigcap_{i=1}^{n} N_i$ there is a y_i such that $y \in W(y_i)$, and then $\emptyset \neq F(x') \cap W(y_i) \subset F(x') \cap W^2(y) \subset F(x') \cap V(y)$. So F is qlsc.

1.14 Lemma.

i) X is a topological space,

ii) $(Y, \mathcal{U})$ is uniform space,

iii) $x \in X$, and

iv) $F: X \to Y$ is a multi-function such that $F(x)$ is compact,

then the following are equivalent:

1) F is upper semi-continuous at x.

2) F is quasi upper semi-continuous at x.

Proof: 1 obviously implies 2. For the converse use Theorem 0.1.

1.15 Theorem

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ is a uniform space,

iii) $x \in X$, and

iv) $F:X \to Y$ is a multi-function such that $F(x)$ is compact;

then the following are equivalent;

1) F is continuous at x.

2) F is quasi-continuous at x.

Proof: Lemmas 1.13 and 1.14.

1.16 Corollary.

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ is a uniform space, and

iii) $F:X \to Y$ is a compact-valued multi-function;

then the following are equivalent:

1) F is continuous.

2) F is quasi-continuous.

For a finite set S let #S denote the number of elements in S.

1.17 Theorem.

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ is a T_2 uniform space, and

iii) $F:X \to Y$ is a multi-function such that

$\#F(x') = n$ for all x' in some neighborhood of x;

then the following are equivalent:

1) F is continuous at x.

2) F is lower semi-continuous at x.

3) F is quasi-lower semi-continuous at x.

4) F is quasi-continuous at x.

Proof: By previous work we need only show that 2 implies 1 or that F is usc at x. Let $\{y_1, y_2, \ldots, y_n\} = F(x)$. Let O be an open set in Y such that $F(x) \subset O$. Let $\{U_1, U_2, \ldots, U_n\}$ be a collection of open set in Y such that

i) $y_i \in U_i$, $i=1, 2, \ldots, n$,

ii) $U_i \cap U_j = \emptyset$, $i \neq j$, and

iii) $\bigcup_{i=1}^{n} U_i \subset O$.

Since F is lsc at x there is a neighborhood N of x such that if $x' \in N$, $F(x') \cap U_i \neq \emptyset$, $i = 1, 2, \ldots, n$. If $x' \in N$, for each i, there is a $z_i \in F(x') \cap U_i$. Then

$$F(x') = \{z_1, z_2, \ldots, z_n\} \subset \bigcup_{i=1}^{n} U_i \subset O.$$

Note: A similar theorem was proved by Smithson [15] Lemma 1, page 449.

1.18 Theorem.

If i) X, Y are topological spaces,

ii) $x \in X$, and

iii) $F:X \to Y$ is a multi-function such that

$\#F(x) = 1$,

then the following are equivalent:

1) F is continuous at x.

2) F is upper semi-continuous at x and there is a neighborhood N of x such that if $x' \in N$ then $F(x') \neq \emptyset$.

Proof: Suppose O is an open set in Y and $F(x) \cap O \neq \emptyset$. Then $F(x) \subset O$ so there is a neighborhood N_1 of x such that $F(x') \subset O$ for each $x' \in N_1$. Let $N' = N_1 \cap N$.

1.19 Theorem.

If i) X,Y are topological spaces,

ii) $x \in X$, and

iii) $F:X \to Y$ is multi-function such that

$F(x) = Y$,

then the following are equivalent:

1) F is continuous at x.

2) F is lower semi-continuous at x.

1.20 Theorem.

If i) X is a topological space,

ii) $(Y,\mathcal{U})$ a uniform space,

iii) $x \in X$, and

iv) $F:X \to Y$ is a multi-function such that

$F(x) = Y$,

then the following are equivalent:

1) F is quasi-continuous at x.

2) F is quasi-lower semi-continuous at x.

1.21 Theorem.

If i) $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces,

ii) $F:X \to Y$ is quasi-continuous, and

iii) X is compact,

then F is uniformly continuous.

Proof: Similar to the proof for single-valued functions.

1.22 Corollary.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ are uniform spaces,

ii) X is compact, and

iii) $F:X \to Y$ is compact valued,

then the following are equivalent:

1) F is continuous.

2) F is uniformly continuous.

1.23 Definition.

Let i) X, Y be topological spaces, and

ii) $F:X \to Y$ be a multi-function,

then $\overline{F}$ will denote the multi-function from X into Y defined by $\overline{F}(x) = \overline{F(x)}$ and $G(F) = \{(x,y) \mid y \in F(x)\}$.

Franklin [7] characterized regular spaces by the fact that $G(\overline{F}) = \overline{G(F)}$ (as a subset of $X \times Y$) for any usc function into a regular space and normal spaces by the fact that $\overline{F}$ is usc for any usc F into a normal space, Sec. 2 p. 16-20.

1.24 Theorem.

If i) X is a topological space,

ii) $(Y, \mathcal{V})$ is a uniform space,

iii) $F:X \to Y$ is a multi-function, and

iv) $\overline{F}:X \to Y$ is defined as in 1.20;

then the following are equivalent:

1) F is quasi-continuous.

2) $\overline{F}$ is quasi-continuous.

Proof: Suppose F is quasi-continuous and $x \in X$ and $V \in \mathcal{V}$ are given. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$. Then there is a neighborhood N of x such that for any pair $x_1, x_2 \in N$, $F(x_1) \subset W[F(x_2)]$. Then $\overline{F}(x_1) \subset W[F(x_2)]$ since $\overline{F}(x_1) = \bigcap \{U[F(x_1)] \mid U \in \mathcal{V}\}$. Thus if $x_1, x_2 \in N$,

$$\overline{F}(x_1) \subset W[F(x_1)] \subset W^2[F(x_2)] \subset W^2[\overline{F}(x_2)] \subset V[\overline{F}(x_2)].$$

So $\overline{F}$ is quasi-continuous.

Suppose $\overline{F}$ is qc and $x \in X$ and $V \in \mathcal{V}$ are given. Then there is a $W \in \mathcal{V}$ such that $W^2 \subset V$. There is a neighborhood N of x such that if $x_1, x_2 \in N$, $\overline{F}(x_1) \subset W[\overline{F}(x_2)]$. Thus if $x_1, x_2 \in N$,

$$F(x_1) \subset \overline{F}(x_1) \subset W[\overline{F}(x_2)] \subset W^2[F(x_2)] \subset V[F(x_2)].$$

1.25 Theorem.

If i) $(X,\mathcal{U})$, $(Y,\mathcal{V})$ are uniform spaces, and

ii) $F: X \to Y$ is a multi-function;

then the following are equivalent:

1) F is uniformly continuous.

2) $\overline{F}$ is uniformly continuous.

Proof: Suppose F is uc and $V \in \mathcal{V}$ is given. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and there is a $U \in \mathcal{U}$ such that for any pair $(x,x') \in U$, $F(x') \subset W[F(x)]$. If $(x_1,x_2) \in U$, then

$\overline{F}(x_2) \subset W[F(x_2)] \subset W^2[F(x_1)] \subset W^2[\overline{F}(x_1)] \subset V[\overline{F}(x_1)]$.

If $\overline{F}$ is uc and $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^2 \subset V$. Then there is a $U \in \mathcal{U}$ such that for each pair $(x_1, x_2) \in U$, $\overline{F}(x_2) \subset W[\overline{F}(x_1)]$. If $(x_1, x_2) \in U$, then $F(x_2) \subset \overline{F}(x_2) \subset W[\overline{F}(x_1)] \subset V[F(x_1)]$.

1.26 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $(Y, \mathcal{V})$ is a uniformly locally connected space,

iii) $F: X \to Y$ is uniformly continuous, and

iv) for each $x \in X$, $\partial(F(x))$ is compact,

then $\overline{F}$ is continuous.

Proof: Let O be open in Y and $x \in X$ such that $\overline{F}(x) \subset O$. Since $\partial(\overline{F}(x))$ is compact there is a $V \in \mathcal{V}$ such that $V[\overline{F}(x)] \subset O$. By Theorem 1.25 there is a $U \in \mathcal{U}$ such that for each pair $(x_1, x_2) \in U$, $\overline{F}(x_2) \subset V[\overline{F}(x_1)]$. Let $N = U(x)$. If $x' \in N$, then $(x, x') \in U$. Thus $F(x') \subset V[F(x)] \subset O$. Thus F is usc. Since F is obviously lsc, F is continuous.

1.27 Theorem.

If i) X is a topological space,

ii) $(Y, \mathcal{V})$ is a uniformly locally connected space,

iii) $F:X \to Y$ is quasi-continuous, and

iv) for each $x \in X$, $\partial(F(x))$ is compact,

then $\overline{F}$ is continuous.

The next three theorems will establish some facts about the behavior of multi-functions into product spaces. In the following three theorems let

i) X be a topological space,

ii) for every $a \in A$, $(Y_a, \mathcal{V}_a)$ be a uniform space,

iii) $(Y, \mathcal{V})$ be the uniform space where $Y = \prod_{a \in A} Y_a$ and $\mathcal{V}$ is the product uniformity,

iv) for every $a \in A$, $F_a : X \to Y_a$ be a multi-function,

v) $F:X \to Y$ be the multi-function defined for every $x \in X$ by $F(x) = \prod_{a \in A} F_a(x)$, and

vi) for every $a \in A$, $pr_a : Y \to Y_a$ is the projection map.

Then the following is a familiar theorem for single-valued functions.

"If X is a topological space then $f:X \to Y$ is continuous iff for every $a \in A$, $f_a = pr_a \circ f : X \to Y$ is continuous."

The same theorem is not true for continuous multi-functions. The following example is from Strother [17].

1.28 Example.

Let $X = Y_1 = Y_2 = [-1,1]$, $Y = Y_1 \times Y_2$. Define $F:X \to Y$ by $F(x) = \{(-1,1), (1,-1)\}$ if x is rational; and $\{(-1,-1), (1,1)\}$ if x is irrational.

$pr_1 \circ F(x) = \{-1,1\} = pr_2 \circ F(x)$ for all x, so $pr_1 \circ F$ and $pr_2 \circ F$ are continuous but F is not continuous.

1.29 Theorem.

F is quasi-continuous iff for every $a \in A$, F_a is quasi-continuous.

Proof: Suppose F is qc, let $a \in A$, $V_a \in \mathcal{V}_a$ and $x \in X$ be given. Let $\hat{V}_a = (pr_a^{-1} \times pr_a^{-1})[V_a]$. There is a neighborhood N of x such that if $x_1, x_2 \in N$ then $F(x_1) \subset \hat{V}_a[F(x_2)]$. Then

$F_a(x_1) = pr_a[F(x_1)] \subset pr_a[\hat{V}_a[F(x)]] \subset V_a[F_a(x_2)]$.

Suppose that for each $a \in A$, F_a is qc. Let $V \in \mathcal{V}$ and $x \in X$ be given. Then there is a finite subset $J \subset A$ and $\{V_b \in \mathcal{V}_b \mid b \in J\}$ such that $\hat{V} = \bigcap \{\hat{V}_b \mid b \in J\} \subset V$. Since for each $b \in J$, F_b is qc and J is finite there exists a neighborhood N of x such that if $x_1, x_2 \in N$ and

$b \in J$ then $F_b(x_1) \subset V_b[F_b(x_2)]$.

Then if $x_1, x_2 \in N$,

$F(x_1) = \prod_{a \in A} F_a(x_1) \subset \hat{V}[\prod_{a \in A} F_a(x_2)] \subset V[\prod_{a \in A} F_a(x_2)] =$

$V[F(x_2)]$.

1.30 Corollary.

If for every $a \in A$, F_a is compact-valued, F is continuous iff for every $a \in A$, F_a is continuous.

Proof: For every $x \in X$, $F(x)$ is compact so by theorem 1.15 F is continuous iff F is qc iff for every $a \in A$, F_a is qc iff for every $a \in A$, F_a is continuous.

1.31 Theorem.

If $(X, \mathcal{U})$ is a uniform space then the following are equivalent:

1) F is uniformly continuous.

2) For every $a \in A$, F_a is uniformly continuous.

Proof: Suppose F is uc. Let $a \in A$ and $V_a \in \mathcal{V}_a$ be given. There is a $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$ then $F(x_2) \subset \hat{V}_a[F(x_1)]$. Then if $(x_1, x_2) \in U$,

$F_a(x_2) = pr_a[F(x_2) \subset pr_a[\hat{V}_a[F(x_1)]] = V_a[pr_a F(x_1)]$
$= V_a[F_a(x_1)]$. Thus F_a is uc.

Suppose for every $a \in A$, F_a is uc. Let $V \in \mathcal{V}$ be given. As in Theorem 1.19 there exists a finite subset $J \subset A$ and $\{V_b \in \mathcal{V}_b \mid b \in J\}$ such that $\hat{V} = \bigcap_{b \in J} \hat{V}_b \subset V$. Since J is finite there is a $U \in \mathcal{U}$ such that for every $b \in J$, if $(x_1, x_2) \in U$, then $F_b(x_2) \subset V_b[F_b(x_2)]$. If $(x_1, x_2) \in U$ then

$$F(x_2) = \prod_{a \in A} F_a(x_1) \subset \hat{V}[\prod_{a \in A} F_a(x_2)] \subset V[F(x_2)].$$ Thus F is uc.

CHAPTER 2: UNIFORM CONVERGENCE OF MULTI-FUNCTIONS

For two topological spaces X, Y let $F_M(X,Y)$ denote the set of all multi-functions from X into Y.

2.1 Definition.

Let i) X be a topological space,

ii) $(Y,\mathcal{V})$ be a uniform space,

iii) $\{F_d \mid d \in D\}$ be a net in $F_M(X,Y)$ and

iv) $F \in F_M(X,Y)$.

The family $\{F_d : d \in D\}$ <u>converges uniformly to</u> F, written $F_d \to F_{(un)}$ if for every $V \in \mathcal{V}$, there is a $d_o \in D$ such that for every $d \geq d_o$, if $x \in X$, $F_d(x) \subset V[F(x)]$ and $F(x) \subset V[F_d(x)]$.

2.2 Definition.

Let i) X be a topological space,

ii) $(Y,\mathcal{U})$ be a uniform space, and

iii) $\mathfrak{F}$ be a filter in $F_M(X,Y)$

a) The filter $\mathfrak{F}$ <u>converges uniformly to</u> a multi-function $F:X \to Y$, written $\mathfrak{F} \to F_{(un)}$, if for every $V \in \mathcal{V}$, there is a $B \in \mathfrak{F}$ such that for every $x \in X$, if $H \in B$,

$H(x) \subset V[F(x)]$ and $F(x) \subset V[H(x)]$.

b) $\mathfrak{F}$ will denote the filter generated by the collection of the sets of the form $\overline{H} = \{\overline{F}: X \to Y \mid F \in H\}$, $H \in \mathfrak{F}$.

NOTE. Throughout the remainder of this chapter we will assume that X is a topological space and $(Y, \mathcal{V})$ is a uniform space.

2.3 Theorem.

If i) $\{F_d \mid d \in d\}$ is a net in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$

then the following are equivalent:

1) $F_d \to F_{(un)}$

2) $\overline{F}_d \to \overline{F}_{(un)}$

3) $\overline{F}_d \to F_{(un)}$

4) $F_d \to \overline{F}_{(un)}$

Proof: Suppose $F_d \to F_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$ and $x \in X$ then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Then

$\overline{F}_d(x) \subset W[F_d(x)] \subset W^2[F(x)] \subset V[\overline{F}(x)]$ and

$\overline{F}(x) \subset W[F(x)] \subset W^2[F_d(x)] \subset V[\overline{F}_d(x)]$.

Suppose $\overline{F}_d \to \overline{F}_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$ and $x \in X$ then $\overline{F}_d(x) \subset W[\overline{F}(x)]$ and $\overline{F}(x) \subset W[\overline{F}_d(x)]$. Then $F_d(x) \subset \overline{F}_d(x) \subset W[\overline{F}(x)] \subset W^2[F(x)] \subset V[F(x)]$, and $F(x) \subset \overline{F}(x) \subset W[\overline{F}_d(x)] \subset W^2[F_d(x)] \subset V[F_d(x)]$.

Thus 1 and 2 are equivalent. That 3 and 4 are also equivalent follows from the fact $\overline{\overline{F}}_d = \overline{F}_d$ and $\overline{\overline{F}} = \overline{F}$.

The following theorem is equivalent to the one above only it is stated in terms of filters.

2.4 Theorem.

If i) $\mathfrak{F}$ is a filter in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$

then the following are equivalent:

1) $\mathfrak{F} \to F_{(un)}$

2) $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$

3) $\overline{\mathfrak{F}} \to F_{(un)}$

4) $\mathfrak{F} \to \overline{F}_{(un)}$

Proof: Suppose $\mathfrak{F} \to F_{(un)}$ and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$, such that $W^2 \subset V$ and $H \in \mathfrak{F}$ such that if $G \in H$, then $G(x) \subset W[F(x)]$ and $F(x) \subset W[G(x)]$ for every $x \in X$. Then if $G' \in \overline{H} \in \overline{\mathfrak{F}}$, there is a $G \in H$ such that $\overline{G} = G'$. Thus if $x \in X$, $G'(x) = \overline{G}(x) \subset W[G(x)] \subset W^2[F(x)] \subset V[\overline{F}(x)]$ and $\overline{F}(x) \subset W[F(x)] \subset W^2[G(x)] \subset V[\overline{G}(x)]$.

Thus $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$.

Suppose $\overline{\mathfrak{F}} \to \overline{F}_{(un)}$; and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$, such that $W^2 \subset V$ and $H' \in \overline{\mathfrak{F}}$ such that if $G' \in H'$, and $x \in X$ then $G'(x) \subset W[\overline{F}(x)]$ and $\overline{F}(x) \subset W[G'(x)]$. There is a $H \in \mathfrak{F}$ such that $\overline{H} \subset H'$. If $G \in H$ and $x \in X$ then $\overline{G} \in \overline{H}$ so $G(x) \subset \overline{G}(x) \subset W[\overline{F}(x)] \subset W^2[F(x)] \subset V[F(x)]$ and $F(x) \subset \overline{F}(x) \subset W[\overline{G}(x)] \subset W^2[G(x)] \subset V[G(x)]$. Thus $\mathfrak{F} \to F_{(un)}$.

The equivalence of 3 and 4 to the others is again proved by observing that $\overline{\overline{\mathfrak{F}}} = \overline{\mathfrak{F}}$.

2.5 Theorem.

If i) $\{F_d \mid d \in D$ is a net in $F_M(X,Y)$,

ii) $F, G \in F_M(X,Y)$,

iii) $F_d \to F_{(un)}$, and

iv) $F_d \to G_{(un)}$

then $\overline{F} = \overline{G}$.

Proof: Let $x \in X$, and $V \in \mathcal{V}$. There is a $W \in \mathcal{V}$ such that $W^2 \subset V$ and $d_o \in D$ such that if $d \geq d_o$, $F_d(x) \subset W[G(x)]$ and $F(x) \subset W[F_d(x)]$. Thus $F(x) \subset W[F_d(x) \subset W^2[F(x)] \subset V[G(x)]$. Then $F(x) \subset \bigcap\{V[G(x)] \mid V \in \mathcal{V}\} = \overline{G}(x)$. Similarly $G(x) \subset \overline{F}(x)$, so $\overline{F}(x) = \overline{G}(x)$.

2.6 Theorem.

If i) $\mathfrak{F}$ is a filter in $F_M(X,Y)$,

ii) $F, G \in F_M(X,Y)$,

iii) $\mathfrak{F} \to F_{(un)}$, and

iv) $\mathfrak{F} \to G_{(un)}$

then $\overline{F} = \overline{G}$.

The next few theorems establish that uniform convergence preserves quasi-continuity and uniform continuity.

Let i) $\{F_d \mid d \in D\}$ be a net in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$.

2.7 Lemma.

If i) for every $d \in D$, F_d is quasi-upper semi-continuous, and

ii) $F_d \to F$ (un),

then F is quasi-upper semi-continuous.

Proof: Let $V \in \mathcal{V}$ be given. There is a $W \in \mathcal{V}$ such that $W^3 \subset V$ and a $d_o \in D$ such that if $d \geq d_o$ and $x \in X$, then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Since F_{d_o} is qusc, there is a neighborhood N of x such that if $x' \in N$ then $F_{d_o}(x') \subset W[F_{d_o}(x)]$. Let $x' \in N$, then $F(x') \subset W[F_{d_o}(x')] \subset W^2[F_{d_o}(x)] \subset W^3[F(x)] \subset V[F(x)]$. Thus F is qusc at x. Since x was arbitrary in X, F is qusc.

2.8 Lemma.

If i) for every $d \in D$, F_d is quasi-lower semi-continuous, and

ii) $F_d \to F$ (un).

Then F is quasi-lower semi-continuous.

Proof: Similar to the proof above using Lemma 1.2 to characterize qlsc.

2.9 Corollary.

If i) for every $d \in D$, F_d is quasi-continuous, and

ii) $F_d \to F$ (un);

then F is quasi-continuous.

2.10 Theorem.

If i) $(Y, \mathcal{U})$ is a complete space ,

ii) for every $d \in D$, F_d is continuous and compact valued, and

iii) $F_d \to F_{(un)}$;

then $\bar{F}$ is continuous.

Proof: From 2.9, $\bar{F}$ is qc so we need only show that $\bar{F}$ is compact valued. Let $x \in X$. If $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^4 \subset V$ and $d_o \in D$ such that if $d' \geq d_o$, $F_{d'}(x) \subset W[F(x)]$ and $F(x) \subset W[F_{d'}(x)]$. Then $F_{d_o}(x)$ is compact so there is a finite subset $\{y'_1, y'_2 \ldots, y'_n\}$ of $F_{d_o}(x)$ such $F_{d_o}(x) \subset \bigcup_{i=1}^{n} W(y_i)$. For each $i = 1, \ldots, n$, $F_{d_o}(x) \subset W[F(x)]$ so there is a $y_i \in F(x)$ such that $y'_i \in W(y_i)$. Now

$$F(x) \subset W[F_{d_o}(x)] \subset W[\bigcup_{i=1}^{n} W(y'_i)] = \bigcup_{i=1}^{n} W^2(y'_i) \subset \bigcup_{i=1}^{n} W^3(y_i).$$

Thus $\bar{F}(x) \subset W[F(x)] \subset \bigcup_{i=1}^{n} W^4(y_i) \subset \bigcup_{i=1}^{n} V(y_i)$.

Since $F(x)$ is a complete totally bounded subset of a complete space, $F(x)$ is compact.

A theorem similar to the above was proved by

Smithson [16].

Let i) $\mathfrak{F}$ be a filter in $F_M(X,Y)$, and

ii) $F \in F_M(X,Y)$.

Then the following are a restatement of the previous theorems in terms of filters rather than nets.

2.11 Lemma.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-upper semi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-upper semi-continuous.

2.12 Lemma.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-lower semi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-lower semi-continuous.

2.13 Corollary.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is quasi-continuous, and

ii) $\mathfrak{F} \to F_{(un)}$;

then F is quasi-continuous.

2.14 Theorem.

If i) there is an $H \in \mathfrak{F}$ such that for every $G \in H$, G is continuous and compact valued, and

ii) $\mathfrak{F} \to F_{(un)}$;

then $\overline{F}$ is continuous.

The next theorems establish similar facts for uniform continuity.

2.15 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $\{F_d \mid d \in D\}$ is a net of uniformly continuous multi-functions, and

iii) $F_d \to F_{(un(}$;

then F is uniformly continuous.

Proof: If $\overline{V} \in \mathcal{V}$, there is a $W \in \mathcal{V}$ such that $W^3 \in V$ and $d_o \in D$ such that if $x \in X$ and $d \geq d_o$ then $F_d(x) \subset W[F(x)]$ and $F(x) \subset W[F_d(x)]$. Since F_{d_o} is uc, there is a $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$ then $F_{d_o}(x_2) \subset W[F_{d_o}(x_1)]$. Then if $(x_1, x_2) \in U$, $F(x_2) \subset W[F_{d_o}(x_2)] \subset W^2[F_{d_o}(x_1)] \subset W^3[F(x_1)] \subset V[F(x_1)]$. Thus F is uc.

2.16 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space,

ii) $\mathfrak{F}$ is a filter in $F_M(X,Y)$,

iii) there is an $H \in \mathfrak{F}$ such that every $G \in H$ is uniformly continuous, and

iv) $\mathfrak{F} \to F_{(un)}$

then F is uniformly continuous.

CHAPTER 3: THE HYPERSPACE OF A UNIFORM SPACE

Let $(Y, \mathcal{V})$ be a uniform space, $P(Y) = \{A \mid A \subset Y\}$; and $\hat{\mathcal{V}}$ shall be the uniformity induced on $P(Y)$ by the collection of all of the sets of the form,
$\hat{V} = \{(A,B) \in P(Y) \times P(Y) \mid A \subset V[B] \text{ and } B \subset V[A]\}$ for $V \in \mathcal{V}$. $\hat{\mathcal{V}}$ is the Hausdorff uniformity. The following definitions are motivated by convergence in $P(Y)$ with respect to either the topology induced by the Hausdorff uniformity or the finite topology; see Michaels [12].

3.1 Definition.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then B_d <u>converges to</u> A <u>with respect to the finite topology</u> on $P(Y)$, written $B_d \to A_{(f.t.)}$, if for any open set O in Y

i) if $A \subset O$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \subset O$; and ii) if $A \cap O \neq \emptyset$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \cap O \neq \emptyset$.

3.2 Definition.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then B_d <u>converges to</u> A <u>with respect to the topology induced</u>

by the Hausdorff uniformity on P(Y), written $B_d \to A_{(u.t.)}$, if for every $V \in \mathcal{V}$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \subset V[A]$ and $A \subset V[B_d]$.

For filters we have:

3.3 Definition.

If i) $\mathfrak{F}$ is a filter in P(Y), and

ii) $A \subset P(Y)$;

then $\mathfrak{F}$ converges to A with respect to the finite topology on P(Y), written $\mathfrak{F} \to A_{(f.t.)}$, if for any O open in Y i) if $A \subset O$, there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \subset O$, and ii) if $A \cap O \neq \emptyset$, there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \cap O \neq \emptyset$.

3.4 Definition.

If i) $\mathfrak{F}$ is a filter in P(Y), and

ii) $A \in P(Y)$;

then $\mathfrak{F}$ converges to A with respect to the topology induced by the Hausdorff uniformity on P(Y), written $\mathfrak{F} \to A_{(u.t.)}$ if for every $V \in \mathcal{V}$ there is an $H \in \mathfrak{F}$ such that if $B \in H$, $B \subset V[A]$ and $A \subset V[B]$.

The two concepts of convergence are distinct but they do coincide on the set of compact subsets, see Michaels

[12].

3.5 Example.

Let $D = \{1, 2, 3, 4, . . .\}$, $Y = [0,\infty)$,

$A = \{0, 1, 2, 3, . . .\}$; and $B_d = \{\frac{1}{d}, 1 + \frac{1}{d}, 2 + \frac{1}{d}, . . .\}$

for each $d \in D$. Then $B_d \to A_{(u.t.)}$ but $B_d \not\to A_{(f.t.)}$.

3.6 Example.

Let $D = \{1, 2, 3, 4, . . .\}$, $Y = A = [0,\infty)$

$B_d = [0,d]$, for every $d \in D$.

Then $B_d \not\to A_{(u.t.)}$ but $B_d \to A_{(f.t.)}$.

3.7 Theorem.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$, and

ii) $A \in P(Y)$;

then the following are equivalent:

1) $B_d \to A_{(u.t.)}$

2) $\overline{B}_d \to \overline{A}_{(u.t.)}$

3) $\overline{B}_d \to A_{(u.t.)}$

4) $B_d \to \overline{A}_{(u.t.)}$

Proof: Similar to the proof of 2.3.

3.8 Definition.

Let $\mathfrak{F}$ be a filter on $P(Y)$. For $H \in \mathfrak{F}$ let $\overline{H} = \{\overline{B} \mid B \in H\}$. Then $\overline{\mathfrak{F}}$ will denote the filter generated by the set $\{\overline{H} \mid H \in \mathfrak{F}\}$.

3.9 Theorem.

If i) $\mathfrak{F}$ is a filter on $P(Y)$, and

ii) $A \in P(Y)$;

then the following are equivalent:

1) $\mathfrak{F} \to A_{(u.t.)}$

2) $\overline{\mathfrak{F}} \to \overline{A}_{(u.t.)}$

3) $\overline{\mathfrak{F}} \to A_{(u.t.)}$

4) $\mathfrak{F} \to \overline{A}_{(u.t.)}$

Proof: First we prove that: if $\mathfrak{F}$ is a filter on $P(Y)$, $\overline{\mathfrak{F}}$ is a filter on $P(Y)$. Let $H'_1, H'_2 \in \overline{\mathfrak{F}}$ then there are $H_1, H_2 \in \mathfrak{F}$ such that $\overline{H}_1 \subset H'_1$ and $\overline{H}_2 \subset H'_2$. There is a $B \in P(Y)$ such that $B \in H_1 \cap H_2$ so $\overline{B} \in \overline{H}_1 \cap \overline{H}_2 \subset H'_1 \cap H'_2$.

The rest of the properties of a filter follow from the definition of $\overline{\mathfrak{F}}$. The remainder of the proof is continued as in 2.4.

3.10 Theorem.

If i) $\{B_d \mid d \in D\}$ is a net in $P(Y)$,

ii) $A, A' \in P(Y)$,

iii) $B_d \to A_{(u.t.)}$, and

iv) $B_d \to A'_{(u.t.)}$

then $\overline{A} = \overline{A'}$.

Proof: See the proof of Theorem 2.5.

3.11 Theorem.

If i) $\mathfrak{F}$ is a filter in $P(Y)$

ii) $A, A' \in P(Y)$

iii) $\mathfrak{F} \to A_{(u.t.)}$, and

iv) $\mathfrak{F} \to A'_{(u.t.)}$

then $\overline{A} = \overline{A'}$.

Proof: See the proof of Theorem 2.6.

3.12 Definition.

Let i) $(Y, \mathcal{V})$ be a uniform space, and

ii) $\{B_d \mid d \in D\}$ be a net in $P(Y)$.

$\{B_d \mid d \in D\}$ is a <u>Cauchy net of subsets</u> of Y if for each $V \in \mathcal{V}$, there is a $d_o \in D$, such that if $d_1, d_2 \geq d_o$ then $B_{d_1} \subset V[B_{d_2}]$.

Notice that if $B_d \rightarrow A_{(u.t.)}$ then $\{B_d \mid d \in D\}$ is a Cauchy net.

The following notation and definitions are adopted from Kuratowski [11], Chapter 11, Sec. 29, Page 335-337.

3.13 Definition.

Let i) X be a topological space, and

ii) $\{B_d \mid d \in D\}$ be a net in P(Y).

Ls B_d = $\{x \in X \mid$ for every O open in X and $d_o \in D$, if $x \in O$, there is a $d \geq d_o$ such that $B_d \cap O \neq \emptyset\}$. Ls B_d = $\{x \in X \mid$ for every O open in X, if $x \in O$ there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \cap O \neq \emptyset\}$.

Then if Li B_d = Ls B_d we say that Lim B_d exists and Lim B_d = Li B_d = Ls B_d.

This definition is the one usually used in defining convergence of sets.

3.14 Theorem.

If i) $(Y, \mathcal{U})$ is a uniform space, and

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subsets of Y;

then Li B_d = Ls B_d.

Proof: It is known that Li $B_d \subset$ Ls B_d, so it suffices to show that Ls $B_d \subset$ Li B_d. If Ls $B_d = \emptyset$, there is nothing to show. If $x \in$ Ls B_d, let O be open, $x \in O$. There is a $W \in \mathcal{V}$, such that W is symmetric and $W^2(x) \subset O$ and $d_o \in D$, such that if $d_1, d_2 \geq d_o$ then $B_{d1} \subset W[B_{d2}]$. Let $d \geq d_o$, there is a $d' \geq d_o$ such that $B_{d'} \cap W(x) \neq \emptyset$. Thus there is an $x' \in B_{d'} \cap W(x)$. Since $B_{d'} \subset W[B_d]$ and W is symmetric there exists $x_o \in W(x') \cap B_d$. Then

$x_o \in W(x') \cap B_d \subset W^2(x) \cap B_d \subset O \cap B_d$, hence $x \in$ Li B_d.

3.15 Theorem.

If i) $(Y, \mathcal{V})$ is a uniform space,

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subsets in Y, and

iii) $B_d \to B_{(u.t.)}$; $B \in P(Y)$;

then i) Lim $B_d = \overline{B}$ and

ii) $B = \emptyset$ iff there is a $d_o \in D$ such that if $d \geq d_o$, $B_d = \emptyset$.

Proof: Suppose $B \neq \emptyset$, there is a $y \in B$. Then if $V = Y \times Y \in \mathcal{V}$, there is a $d_o \in D$ such that if $d \geq d_o$,

$y \in B \subset V[B_d]$. Thus $B_d \neq \emptyset$ for $d \geq d_o$. If $B = \emptyset$, there is a $d_o \in D$ such that if $d \geq d_o$, $B_d \subset V[B] = \emptyset$.

If $y \in B$, and O is any open set in Y such that $y \in O$, there is a symmetric $V \in \mathcal{V}$ such that $V(y) \subset O$, and $d_o \in D$ such that if $d \geq d_o$, then $B \subset V[B_d]$ and $B_d \subset V[B]$. Then there is a $y' \in B_d$ such that $y' \in V(y)$. Thus $y' \in V(y) \cap B_d \subset B_d \cap O \neq \emptyset$. Hence $y \in \text{Li } B_d = \text{Lim } B_d$, so $B \subset \text{Lim } B_d$.

If $y \in \text{Lim } B_d$, we will show that $B_d \to B \cup \{y\}_{(u.t.)}$, so that $y \in B \cup \{y\} \subset \overline{B}$. Let $V \in \mathcal{V}$, there is a symmetric $W \in \mathcal{V}$, such that $W \subset V$, and $d_1 \in D$ such that if $d \geq d_1$, $B_d \cap W(y) \neq \emptyset$, so $y \in W[B_d]$. There is a $d_2 \in D$ such that if $d \geq d_2$, $B_d \subset W[B]$ and $B \subset W[B_d]$. Let $d' = d_1 \vee d_2$. If $d \geq d'$ then $B_d \subset W[B] \subset W[B \cup \{y\}]$ and $B \cup \{y\} \subset W[B_d]$.

Therefore $\text{Lim } B_d = B$.

NOTE: The above does not say that if $\{B_d \mid d \in D\}$ is a Cauchy net that $\text{Lim } B_d \neq \emptyset$ or that if $B = \text{Lim } B_d$ that $B_d \to B_{(un)}$. However Kuratowski [11] shows that if $\{B_d \mid d \in D\}$ is a cauchy net of subsets of a compact metric space then $B_d \to \text{Lim } B_{d(un)}$; Vol. I, Chapter III, Par. 33, p. 407.

The next few theorems will serve to characterize continuous, quasi-continuous and uniformly continuous multi-functions in terms of nets, similar results are also obtainable in terms of filters.

3.16 Theorem.

If i) X, Y are topological spaces, and

ii) $F: X \to Y$ is a multi-function;

then the following are equivalent:

1) F is continuous

2) For every net $\{x_d \mid d \in D\}$ in X if $x_d \to x$ then $F(x_d) \to F(x)_{(f.t.)}$.

Proof: Let F be continuous; $x_d \to x$ in X; and O be open in Y. If $F(x) \subset O$ there exists a neighborhood N of x such that $F(x') \subset O$ if $x' \in N$. There exists $d_o \in D$ such that if $d \geq d_o$, $x_d \in N$. Therefore if $d \geq d_o$, $F(x_d) \subset O$. If $F(x_d) \cap O \neq \emptyset$ there exists a neighborhood N of X such that if $x' \in N$ then $F(x') \cap O \neq \emptyset$. There exists a $d_o \in D$ such that if $d \geq d_o$ then $x_d \in N$. Therefore if $d \geq d_o$, $F(x_d) \cap O \neq \emptyset$. Thus $F(x_d) \to F(x)_{(f.t.)}$.

Suppose F is not continuous. Then there exists an $x \in X$ and an open set $O \subset Y$ such that either i) $F(x) \subset O$ but there exists a net $\{x_d \mid d \in D\}$ such that $F(x_d) \not\subset O$ and $x_d \to x$ or ii) $F(x) \cap O \neq \emptyset$ but there exists a net $\{x_d \mid d \in D\}$ such that $F(x_d) \cap O = \emptyset$ and $x_d \to x$. (Use the family of neighborhoods of x as an indexing set.)

3.15 Theorem.

If i) X is a topological space,

ii) $(Y, \mathcal{V})$ is a uniform space, and

iii) $F: X \to Y$ is a multi-function;

then the following are equivalent:

1) F is quasi-continuous.

2) If $\{x_d \mid d \in D\}$ is a net in X and $x_d \to x$, then $F(x_d) \to F(x)_{(u.t.)}$.

Proof: Suppose F is qc, and $x_d \to x$ in X. Let $V \in \mathcal{V}$ be given. There exists a neighborhood N of x such that if $x_1 \in N$ then $F(x_1) \subset V[F(x)]$ and $F(x) \subset V[F(x_1)]$, and $d_0 \in D$ such that if $d_1 \leq d$ then $x_d \in N$. If $d \geq d_1$, $F(x_d) \subset V[F(x)]$ and $F(x) \subset V[F(x_d)]$, thus $F(x_d) \to F(x)_{(u.t.)}$.

If F is not qc there is an $x \in X$ and a $V \in \mathcal{V}$, such that either i) there exists a net $\{x_d \mid d \in D\}$ such that for every $d \in D$, $F(x_d) \not\subset V[F(x)]$ but $x_d \to x$; or ii) there exists a net $\{x_d \mid d \in D\}$ such that for every $d \in D$, $F(x) \not\subset V[F(x_d)]$ but $x_d \to x$, (Use the neighborhood system of x for D.).

3.18 Lemma.

If i) $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces,

ii) $F: X \to Y$ is uniformly continuous, and

iii) $\{x_d \mid d \in D\}$ is a Cauchy net in X;

then $\{F(x_d) \mid d \in D\}$ is a Cauchy net of subsets in Y.

Proof: Given $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$, then $F(x_2) \subset V[F(x_1)]$. Then there exists $d_0 \in D$ such that if $d_1, d_2 \geq d_0$ then $(x_{d_1}, x_{d_2}) \in U$. Then if $d_1, d_2 \geq d_0$, $F(x_{d_2}) \subset V[F(x_{d_1})]$.

3.19 Theorem.

If i) $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces,

ii) X is totally bounded, and

iii) $F: X \to Y$ is a multi-function;

then the following are equivalent:

1) F is uniformly continuous

2) If $\{x_d \mid d \in D\}$ is a Cauchy net in X, then $\{F(x_d) \mid d \in D\}$ is a Cauchy net of subset of Y.

Proof: Suppose F is not uc, then there is a $V \in \mathcal{V}$ such that for every $U \in \mathcal{U}$ there is a pair $(x, y) \in U$ such that $F(y) \not\subset V[F(x)]$. Let $\{U_a \mid a \in A\}$ be a basis of $\mathcal{U}$ consisting of symmetric members of $\mathcal{U}$, such that $a \geq a'$ iff $U_a \subset U_{a'}$.

There exist $\{(x_a, y_a) \mid a \in A\}$ a net of pairs in $X \times X$ such that $(x_a, y_a) \in U_a$ and $F(y_a) \not\subset V[F(x_a)]$. X is totally bounded so there exists a cofinal subset A' of A such that $\{(x_a, y_a) \mid a \in A'\}$ is Cauchy in the product uniformity. Define $\nabla = A' \times \{0,1\}$ with lexicographic order. Define $S = \{s_{(a,i)} \mid (a,i) \in \nabla\}$ where $s_{(a,0)} = x_a$, $s_{(a,1)} = y_a$. S is a Cauchy net. If $U \in \mathcal{U}$, there is a $W \in \mathcal{U}$ such that $W^2 \subset U$. Since $\{(x_a, y_a) \mid a \in A'\}$ is Cauchy there exist $a_0 \in A'$ such that $U_{a_0} \subset W$ and if $a_1, a_2 \geq a_0$, then $((x_{a_1}, y_{a_1}), (x_{a_2}, y_{a_2})) \in W \times W$. Now if (a_1, i), $(a_2, i) \geq (a_0, 0)$ then $(s_{(a_1, i)}, s_{(a_2, i)}) \in U$, $(x_{a_1}, x_{a_2}) \in W \subset U$ and $(y_{a_1}, y_{a_2}) \in W \subset U$. Thus

$(x_{a_1}, y_{a_2}) \in U_{a_2} \subset U_{a_0} \subset W$, and U_{a_1} is symmetric. So $(x_{a_1}, y_{a_2}) \in W \circ U_{a_2} \subset W \circ W = W^2 \subset U$. But $\{F(s_{(a,i)}) \mid (a,i) \in \nabla\}$ is not a Cauchy net of subsets of Y, Since for every $a \in A$,

$F(s_{(a,1)}) = F(y_a) \not\subset V[F(x_a)] = V[F(s_{(a,0)})]$.

It is known that if $F:X \to Y$ is a multifunction and $f:X \to P(Y)$ is the function defined by $f(x) = F(x)$, then F is continuous multi-function iff f is continuous when P(Y) has the finite topology. The previous theorem show that $F:X \to Y$ is a quasi-continuous iff $f:X \to P(Y)$ is continuous when P(Y) has the topology induced by the Hausdorff uniformity, and that F is uniformly continuous iff f is uniformly continuous when P(Y) has the Hausdorff uniformity.

3.20 Definition.

If $(Y,\mathcal{V})$ is a uniform space, then Y is a uniformly locally compact space if for each $V \in \mathcal{V}$, there is a W such that $W \subset V$ and $W(y)$ is compact for every $y \in Y$.

3.21 Lemma.

Let i) $(Y,\mathcal{V})$ be a uniformly locally compact space, and

ii) $\{B_d \mid d \in D\}$ be a Cauchy net of non-empty subsets of Y;

then $\text{Lim } B_d \neq \emptyset$.

Proof: By the previous lemma, we need only show that $\text{Ls } B_d \neq \emptyset$. If $W \in \mathcal{U}$ is given, such that $W(y)$ is compact for all $y \in Y$, we may assume W is symmetric (the closed symmetric members of $\mathcal{V}$ form a basis of the uniformity). There is a $d_o \in D$ such that if $d' \geq d_o$ then $B_{d_o} \subset W[B_{d'}]$ and $B_{d'} \subset W[B_{d_o}]$. Let $y_o \in B_d$, then for each $d' \in D$, $d' \geq d_o$, there is a $y_{d'} \in B_{d'}$ such that $y_{d'} \in W(y_o)$. So $\{y_{d'} \mid d' \geq d_o\}$ is a net in $W(y_o)$. $W(y_o)$ is compact, so there exists $y \in W(y_o)$ and a subnet $\{y_{d''} \mid d'' \geq d_o\}$ such that $y_{d''} \to y$. Then $y \in \text{Ls } B_d$.

3.22 Lemma.

Let i) $(Y, \mathcal{V})$ be a uniformly locally compact space,

ii) $\{B_d \mid d \in D\}$ a Cauchy net of subsets of Y.

iii) $A = \text{Lim } B_d$, and

iv) $V \in \mathcal{V}$;

then there exists a $d_o \in D$ such that if $d \geq d_o$ then

$B_d \subset V[A]$.

Proof: Suppose not, then there exists $V \in \mathcal{V}$, such that for all $d \in D$, there exists $d' \geq d$ with $B_{d'} \not\subset V[A]$. Let $W \in \mathcal{V}$ such that W is symmetric, $W^2 \subset V$, and $W(x)$ is compact for every $x \in X$. There exists $d_0 \in D$ such that if $d_1, d_2 \geq d_0$ then $B_{d_1} \subset W[B_{d_2}]$. But there exists $d' \geq d_0$ such that $B_{d'} \not\subset W^2[A]$. So there exists $y_{d'} \in B_{d'}$ such that $y_{d'} \notin W^2[A]$. Thus $W(y_{d'}) \cap W[A] = \emptyset$. But as above, there exists $y_0' \in W(y_{d'})$ such that $y_0' \in \text{Ls } B_d$. So $Y_{d'} \in W(y_{o'}) \subset W[\text{Ls } B_d] = W[A] \subset V[A]$ (contradiction).

3.23 Lemma.

If i) $(Y, \mathcal{U})$ is a uniform space,

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subsets of Y,

iii) $A = \text{Lim } B_d$, and

iv) $V \in \mathcal{V}$;

then there exists $d_0 \in D$, such that if $d \geq d_0$, then $A \subset V[B_d]$.

Proof: Let $W \in \mathcal{V}$, such that W is symmetric, $W^2 \subset V$ and $d_0 \in D$ such that if $d_1, d_2 \geq d_0$, then $B_{d_1} \subset W[B_{d_2}]$. If

$y \in A$, there exists $d' \geq d_o$, such that $B_{d'} \cap W(y) \neq \emptyset$. Since W is symmetric $y \in W[B_{d'}]$. If $d \geq d_o$ then $y \in W[B_{d'}] \subset W^2[B_d] \subset V[B_d]$. Since d is independent of the choice of y, $A \subset V[B_d]$, for all $d \geq d_o$.

3.24 Definition.

If $(Y, \mathcal{U})$ is a uniform space, Y is <u>hypercomplete</u> if for every Cauchy net of subsets of Y, $\{B_d \mid d \in D\}$ there exists $A \subset Y$ such that $B_d \to A_{(u.t.)}$.

NOTE: By Theorem 3.7 we could assume that A is closed.

3.25 Theorem.

If $(Y, \mathcal{V})$ is a uniformly locally compact space then Y is hypercomplete.

Proof: Let $\{B_d \mid d \in D\}$ be a Cauchy net of subsets of Y. If $B_d = \emptyset$ for all $d \in D$, then $B_d \to \emptyset_{(ut)}$. If $B_d \neq \emptyset$ for every $d \in D$, then let $A = \text{Lim } B_d$. Let $V \in \mathcal{V}$, by 3.18 there exists $d_1 \in D$, such that if $d \geq d_1$ then $B_d \subset V[A]$; By 3.19 there exists $d_2 \in D$ such that if $d \geq d_2$ then $A \subset V[B_d]$. Let $d_o = d_1 \vee d_2$.

NOTE: We also have that if $B_d \to B_{(u.t.)}$ then $B = \text{Lim } B_d$.

The following theorems will characterize hypercomplete uniform spaces in terms of multi-functions.

3.26 Lemma.

If i) Y is a hypercomplete space, and

ii) $\{B_d \mid d \in D\}$ is a Cauchy net of subset of Y;

then $B_d \to \text{Lim } B_{d\ (u.t.)}$.

Proof: By 3.13 Lim B_d exists. Since Y is hypercomplete, there exists an $A \in Y$ such that $B_d \to A_{(u.t.)}$. Then by 3.14 Lim $B_d = \overline{A}$. Thus $B_d \to \overline{A} = \text{Lim } B_{(u.t.)}$ by 3.7.

3.27 Theorem.

If (Y, V) is a uniform space then the following are equivalent:

1) Y is hypercomplete.

2) $2^Y = \{A \subset Y \mid A \neq \emptyset \text{ and } A = \overline{A}\}$ is complete with respect to the Hausdorff uniformity.

3) If i) (X, U) is a uniform space,

ii) X' is a dense subset of X

iii) $F: X' \to Y$ is a uniformly continuous multi-function such that for every

$x' \in X'$, $F(x') \neq \emptyset$;

then there exists $G: X \to Y$ such that

i) G is uniformly continuous

ii) for every $x' \in X'$, $G(x') = F(x')$, and

iii) $\overline{G}$ is unique.

Proof: That 1 was equivalent to 2 was shown in theorems 3.7 and 3.8. Then we need only show that 1 implies 3 and 3 implies 1.

Suppose 1 is true, and that $(X, \mathcal{U})$, X' and $F: X' \to Y$ are as in 3. We define $G: X \to Y$ as follows. If $x \in X$, let $\{x_d \mid d \in D\}$ be a net in X' such that $x_d \to x$. Define $G(x) = \text{Lim } F(x_d)$.

1) $F(x_d) \to G(x)_{(u.t.)}$ by Lemma 3.25.

2) $G(x)$ is independent of the choice of the net. Suppose $\{x_d \mid d \in D\}$ and $\{x_c \mid c \in C\}$ are nets in X' such that $x_d \to x$ and $x_c \to x$. Then we must show $\text{Lim } F(x_d) = \text{Lim } F(x_c)$. Let $V \in \mathcal{V}$, there exists a $W \in \mathcal{V}$, such that $W^3 \subset V$ and $U \in \mathcal{U}$ such that if $(x_1, x_2) \in U$ and $x_1, x_2 \in X'$ then $F(x_2) \subset W[F(x_1)]$. Since $F(x_d) \to \text{Lim } F(x_d)_{(u.t.)}$ and $F(x_c) \to \text{Lim } F(x_c)_{(u.t.)}$ there exist $d_0 \in D$ and $c_0 \in C$ such that if $d' \geq d_0$ and

$c' \geq c_o$ then i) $F(x_{d'}) \subset W[\text{Lim } F(x_d)]$, and

ii) $\text{Lim } F(x_c) \subset W[F(x_{c'})]$.

Since $x_d \to x$ and $x_c \to x$, there exists $d_1 \in D$ and $c_1 \in C$, $d_1 \geq d_o$, $c_1 \geq c_o$ such that if $d' \geq d_1$ and $c' \geq c_1$ then $(x_{d'}, x_{c'}) \in U$, so $F(x_{c'}) \subset W[F(x_{d'})]$. Thus

$\text{Lim } F(x_c) \subset W[F(x_c\ )] \subset W^2[F(x_{d_1})] \subset W^3[\text{Lim } F(x_d)]$

$\subset V[\text{Lim } F(x_d)]$.

Then $\text{Lim } F(x_c) \subset \overline{\text{Lim } F(x_d)} = \text{Lim } F(x_d)$. Similarly $\text{Lim } F(x_d) \subset \text{Lim } F(x_c)$, so that $\text{Lim } F(x_d) = \text{Lim } F(x_c)$.

3) G is uniformly continuous.

Let $V \in \mathcal{V}$, there is a $W \in \mathcal{V}$, such that $W^3 \subset V$. Then there exists a $U \in \mathcal{U}$ such that if $x'_1, x'_2 \in X'$ and $(x'_1, x'_2) \in U$, $F(x'_2) \in W[F(x'_1)]$. There exists a $U_o \in \mathcal{U}$ such that $U_o^3 \subset U$. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in U_o$. Choose nets $\{x_d \mid d \in D\}$ and $\{x_c \mid c \in C\}$ in X' such that $x_d \to x_1$ and $x_c \to x_2$. Then there exist $c_o \in C$ and $d_o \in D$ such that if $c' \geq c_o$ and $d' \geq d_o$ then $G(x_2) \subset W[F(x_{c'})]$ and $F(x_{d'}) \subset W[G(x_1)]$. Choose $c' \geq c_o$, $d' \geq d_o$ so that $(x_{d'}, x_1) \in U_o$ and $(x_2, x_{c'}) \in U_o$. Then $(x_{c'}, x_{d'}) \in U_o^3$.

So $G(x_2) \subset W[F(x_{c'})] \subset W^2[F(x_{d'})] \subset W^3[G(x_1)] \subset V[G(x_1)]$.

Therefore G is uc.

4) Define $H: X \to Y$ by

i) if $x' \in X'$, $H(x') = F(x')$ and

ii) if $x \in X - X'$, $H(x) = G(x)$.

Then $\overline{H}(x) = G(x)$ for all $x \in X$. Thus H is uc since $\overline{H}$ is uc.

5) $\overline{H}$ is unique.

Suppose $H_1: X \to Y$ is a uc multi-function and for every $x' \in X$, $H_1(x') = F(x')$. Then let $x \in X$ and $\{x_d \mid d \in D\}$ be a net in X' such that $x_d \to x$.

$H(x_d) = H_1(x_d) \to H_1(x)_{(u.t.)}$. Thus $H(x_d) \to H_1(x)_{(u.t.)}$.

But then $\overline{H_1(x)} = \overline{H(x)}$ so $\overline{H}_1 = \overline{H} = G$.

NOTE: The technique of the above proof is essentially the same as that used by Ratner [13] in proving Theorem 48, p. 40.

We must now show that 3 implies 1. Let $\{B_d \mid d \in D\}$ be a Cauchy net of subsets of Y.

$(D, \leq)$ is a directed set. Let $C = D \cup \{\infty\}$. Extend the order $\leq$ to C by, if $d \in D$, then $d \leq \infty$. For each $V \in \mathcal{V}$, there exists a $d_o \in D$ such that if $d_1, d_2 \geq d$,

$B_{d_1} \subset V[B_{d_2}]$. Define a uniformity for C as follows:

For each $V \in \mathcal{V}$, choose d(V) as above. Let

$\tilde{V}_1 = \{(d_1, d_2) \in D \times D \mid B_{d_1} \subset V[B_{d_2}] \text{ and } B_{d_2} \subset V[B_{d_1}]\}$, and

$\tilde{V}_0 = \{(d_1, \infty) \text{ or } (\infty, d_1) \mid d_1 \geq d(V)\}$. Let $\overline{V} = \tilde{V}_0 \cup \tilde{V}_1$.

$\{\overline{V} \mid V \in \mathcal{V}\}$ forms a subbase for a uniformity $\overline{\mathcal{V}}$ on C and D is dense in C. Define $F: D \to Y$ by $F(d) = B_d$. F is uc since if $(d_1, d_2) \in (D \times D) \cap \overline{V}$ then $F(d_2) \subset V[F(d_1)]$. Then F can be extended to a uc function $G: C \to Y$. Let $A = G(\infty)$. Then $\{d \in D\}$ is a net in C and $d \to \infty$ so $F(d) \to A_{(u.t.)}$, Theorem 3.16. Thus Y is hypercomplete.

CHAPTER 4: THE SEMI-GROUP OF QUASI-CONTINUOUS REAL-VALUED MULTI-FUNCTIONS.

Let X be a topological space. Then we shall use the following notations:

$Q_M(X) = \{F \in F_M(X,R) \mid F$ is quasi-continuous and for every $x \in X$, $F(x) \neq \emptyset\}$;

$C_M(X) = \{F \in F_M(X,R) \mid F$ is continuous and for every $x \in X$, $F(x) \neq \emptyset\}$;

$K_M(X) = \{F \in C_M(X) \mid F$ is compact-valued$\}$;

$C(X) = \{f: X \to R \mid f$ is a continuous function$\}$.

For convenience in notation we let for $r > 0$,
$V_r = \{(r_1, r_2) \in R \times R \mid |r_1 - r_2| < r\}$.

A familiar theorem is that $(C(X), +)$, where $(f + G)(x) = f(x) + g(x)$, is a topological group. Day in [5] showed that the set $(K_M(X); +)$ is a topological semi-algebra. For quasi-continuous multi-functions we achieve a less restrictive result on $Q_M(X)$.

4.1 Definition.

If F_1 and $F_2 \in Q_M(X)$

$F_1 + F_2$ is defined by
$(F_1 + F_2)(x) = \{y_1 + y_2 \mid y_i \in F_i, i=1, 2\}$.

4.2 Theorem.

If F_1, $F_2 \in Q_M(X)$ then $F_1 + F_2 \in Q_M(X)$.

Proof: Define $\nabla: X \to X \times X$ by $\nabla(x) = (x,x)$. Then $F_1 \times F_2 : X \times X \to R \times R$ is qc by theorem 1.29 and $+ : R \times R \to R$ is uc.

Thus $F_1 + F_2 = + \circ (F_1 \times F_2 \circ \nabla)$ is qc by theorems 1.8 and 1.10

4.3 Theorem.

If i) $(X, \mathcal{U})$ is a uniform space, and

ii) F_1, F_2 are uniformly continuous;

then $F_1 + F_2$ is uniformly continuous.

Proof: Similar to the above.

4.4 Definition.

For K compact in X and $r > 0$, let

$W(K, r) = \{(F, G) \in Q_M(X) \times Q_M(X) \mid$ for every $x \in X$, $F(x) \subset V_r[G(x)]$, and $G(x) \subset V_r[F(x)]$.

Let $\hat{\mathcal{U}}$ be the uniformity induced on $Q_M(X)$ by $\{W(K, r), K$ compact in $X, r > 0\}$.

Let $\tau(H)$ be the topology induced on $Q_M(X)$ by $\hat{\mathcal{U}}$.

4.5 Theorem.

$(Q_M(X), +, \tau(H))$ is a topological semi-group.

Proof: We shall show that + is uniformly continuous with respect to $\hat{\mathcal{U}}$. Let F_1, G_1, K_1 and r be given. Let $r_o = r/2$.

If $(F_1, F_2) \in W(K, r_o)$ and $(G_1, G_2) \in W(K, r_o)$, and

$$
\begin{aligned}
x \in K,\ (F_1 + G_1)(x) &= F_1(x) + G_1(x) \\
&\subset V_{r_o}[F_2(x)] + V_{r_o}[G_2(x)] \\
&\subset V_{2r_o}[F_2(x) + G_2(x)] \\
&\subset V_r[(F_2 + G_2)(x)].
\end{aligned}
$$

Similarly $(F_2 + G_2)(x) \subset V_r[(F_1 + G_1)(x)]$. Thus $(F_1 + G_1, F_2 + G_2) \in W(K,r)$.

One notes that similar results cannot be obtained for $Q_M(X)$ and a multiplication defined in a pointwise manner, since the space would not be closed under this operation.

4.6 Example.

Let $X = [0,1]$ define $F:X \to R$ by $F(x) = \{x + k \mid k = 0, 1, 2, 3, \ldots\}$. Then $F^2(x) = \{y_1 y_2 \mid y_1, y_2 \in F(x)$ is not qc.

To examine some of the properties of $(Q_M(X), \tau(H))$ we will use the following notation. For $x_o \in X$ and $F_o \in Q_M(X)$ let:

1) $B_{x_o} = \{F \in Q_M(X) \mid F(x_o)$ is bounded below$\}$.

2) $B^{x_o} = \{F \in Q_M(X) \mid F(x_o)$ is bounded above$\}$.

3) $B_{F_o} = \{x \in X \mid F_o(x)$ is bounded below$\}$.

4) $B^{F_o} = \{x \in X \mid F_o(x)$ is bounded above$\}$.

5) $C_{x_o} = Q_M(X) - B_{x_o}$

6) $C^{x_o} = Q_M(X) - B^{x_o}$

7) $C_{F_o} = X - B_{F_o}$

8) $C^{F_o} = X - B^{F_o}$

4.7 Lemma.

For every $x \in X$, B_x, B^x, C_x, and C^x are open in $(Q_M(X), \tau(H))$.

Proof: Let $x_o \in X$ and $F \in B_{x_o}$. Suppose $(F,G) \in W\{x_o\}, 1)$. Since $F \in B_{x_o}$, there exists $M \in R$ such that $F(x_o) \subset [M,\infty)$. Then $G(x_o) \subset V_1[F(x_o)] \subset [M-1,\infty)$. Thus $G \in B_{x_o}$. Since $\{G \mid (F,G) \in W(\{x_o\}, 1)\}$ is a neighborhood of F, B_{x_o} is open. In a similar fashion B^{x_o} is also open.

Let $x_o \in X$ and $F \in C_{x_o}$. Suppose $(F,G) \in W(\{x_o\}, 1)$. For $M \in R$, there exists $r_o \in F(x_o)$ such that $r_o \leq M-1$. Since $F(x_o) \subset V_1[G(x_o)]$ there exists $r' \in G(x_o)$ such that $|r_o - r'| < 1$. Then $r' < M$. Thus G is not bounded below. Therefore C_{x_o} is open. Similarly C^{x_o} is open.

4.8 Corollary.

If $X \neq \emptyset$, then $Q_M(X)$ is not connected.

Proof: Let $x_o \in X$. Define $F_1: X \to R$ by $F_1(x) = R$ for every $x \in X$, and $F_2: X \to R$ by $F_2(x) = \{0\}$ for every $x \in X$. Then $F_2 \in B^{x_o}$ and $F_1 \in C^{x_o}$.

4.9 Theorem.

If $F \in Q_M(X)$, then B_F, B^F, C_F and C^F are open in X.

Proof: If $x \in B_F$, there exists $M \in R$ such that $F(x) \subset [M,\infty)$. Then there is a neighborhood N of x such that $F[N] \subset [M-1,\infty)$. Thus $N \subset B_F$. Therefore B_F is open. Similarly B^F is open.

If $x \in C_F$, there exists a neighborhood N of x such that if $x' \in N$ then $F(x) \subset V_1[F(x')]$. Let $x' \in N$. If $M \in R$, there exists $r \in F(x)$ such that $r < M-1$. Then there exists $r' \in F(x')$ such that $| r = r' | < 1$ so $r' < M$.

Thus $N \subset C_F$.

4.10 Corollary.

If i) X is connected, and

ii) for some $x \in X$, $F(x)$ is bounded below (above),

then for every $x \in X$, $F(x)$ is bounded below (above).

4.11 Corollary.

If i) X is connected, and

ii) for some $x \in X$, $\overline{F}(x)$ is compact;

then $\overline{F}$ is compact valued.

4.12 Corollary:

If i) X is connected, and

ii) for some $x \in X$, $f(x) = \text{lub}\{F(x)\} < +\infty$;

then $f:X \to R$ is defined for every $x \in X$, and is continuous.

4.13 Corollary.

If i) X is connected, and

ii) for some $x \in X$ $g(x) = \text{glb}\{F(x)\} > -\infty$;

then $g:X \to R$ is defined for every $x \in X$ and is continuous.

If we only assume that $F \in C_M(x)$ then B_F and B^F are open but C^F and C_F need not be. See example 1.

An interesting fact about $Q_M(X)$ is the following.

4.14 Theorem.

If i) X is connected, and

ii) $x, y \in X$;

then $B_x = B_y$ and $B^x = B^y$.

Proof: If X is connected and $F \in Q_M(X)$ then either $X = C_F$ or $X = B_F$. Then $F \in B_X$ iff $x \in B_F$ iff $B_F = X$ iff $y \in B_F$ iff $F \in B_y$. Similarly $B^x = B^y$.

4.15 Corollary.

If i) A is a component of X, and

ii) $x, y \in A$;

then $B_x = B_y$ and $B^x = B^y$.

4.16 Definition.

A space X is pseudo-compact if for every $f \in C(X)$, $f[X]$ is bounded.

4.17 Theorem.

The following are equivalent;

1) X is pseudo-compact

2) For every $F \in Q_M(X)$ such that $F(x)$ is bounded for every $x \in X$, $F[X]$ is bounded.

Proof: Suppose X is pseudo-compact and $F \in Q_M(X)$ such that for every $x \in X$, $F(x)$ is bounded. Let $g(x) = \text{glb}\,\{F(x)\}$ and $f(x) = \text{lub}\,\{F(x)\}$, then f and g are continuous. $f[X]$ and $g[X]$ are bounded so there exists an M such that $f[X] \cup g[X] \subset [-M,M]$. Then $F[X] \subset [-M,M]$.

CHAPTER 5: SPACES OF MULTI-FUNCTIONS

Let X and Y be topological spaces; then

i) $C(X,Y) = \{f:X \rightarrow Y \mid f \text{ is a continuous function}\}$;

ii) $C_M(X,Y) = \{F \in F_M(X,Y) \mid F \text{ is continuous}\}$;

iii) $\overline{C}_M(X,Y) = \{F \in C_M(X,Y) \mid F = \overline{F}\}$;

iv) $\mathfrak{F}_1(X) = \{\{x\}\} \mid x \in X\}$;

v) $K_M(X,Y) = \{F \in C_M(X,Y) \mid F \text{ is compact-valued}\}$.

5.1 Definition.

Let S be a collection of subsets of X such that $\bigcup S = X$. We define the $(S,0)$ topology on $C_M(X,Y)$ to be the one generated by the subbase of all of the sets of the form

$\kappa(A,0) = \{F \in C_M(X,Y) \mid F(x) \subset 0, \text{ for every } x \in A\}$ or

$\lambda(A,0) = \{F \in C_M(X,Y) \mid F(x) \cap 0 \neq \emptyset \text{ for every } x \in A\}$ where

$A \in S$ and 0 is open in Y. If S is the family of compact subsets of X, the $(S,0)$ topology will be referred to as the compact-open (or c-o) topology. J. M. Day studied the c-o topology on spaces of relations in her dissertation [5]. Smithson examined this topology in [15].

The following are a list of separation properties for $C_M(X,Y)$ and $\overline{C}_M(X,Y)$ and $K_M(X,Y)$ with the $(S,0)$ topologies.

5.2 Lemma.

If i) Y is a T_1 space, and

ii) $\mathfrak{F}_1(X) \subset S$,

then $C_M(X,Y)$ with $(S,0)$ topology is a T_0 space.

Proof: If $F \neq G \in C_M(X,Y)$, there exists $x_0 \in X$, such that $F(x_0) \neq G(x_0)$. We suppose $G(x_0) \not\subset F(x_0)$. So there exists $y_0 \in G(x_0)$, $y_0 \notin F(x_0)$. Then $G \notin \kappa(\{x\}, Y - \{y_0\})$, $F \in \kappa(\{x\}, Y - \{y_0\})$.

5.3 Lemma.

If $\mathfrak{F}_1(X) \subset S$, then $\overline{C}_M(X,Y)$ with $(S,0)$ topology is always T_0.

Proof: If $F, G \in \overline{C}_M(X,Y)$, $F \neq G$, we may suppose there exists x_0 such that $F(x_0) \not\subset G(x_0)$. Then $F \in \lambda(\{x\}, Y - G(x_0))$ but $G \notin \lambda(\{x\}, Y - G(x_0))$.

5.4 Lemma.

If i) Y is a T space, and

ii) $\mathfrak{F}_1(X) \subset S$;

then $\overline{C}_M(X,Y)$ with the $(S,0)$ topology is a T_1 space.

Proof: Let $F, G \in \overline{C}_M(X,Y)$. If $F \neq G$, there exists $x_0 \in X$ such that $F(x_0) \neq G(x_0)$. Suppose $F(x_0) \not\subset G(x_0)$, then there exists $y_o \in F(x_o)$, $y_o \notin G(x_o)$. Then

$G \in \kappa(\{x_o\}, Y - y_o)$, $F \notin \kappa(\{x_o\}, Y - \{y_o\})$;

$F \in \lambda(\{x_o\}, Y - G(x_o))$ and $G \notin \lambda(\{x_o\}, Y - G(x_o))$.

5.5 Lemma.

If i) Y is a T_1 regular space, and

ii) $\mathfrak{F}_1(X) \subset S$;

then $\overline{C}_M(X,Y)$ with the $(S,0)$ topology is a T_2 space.

Proof: Let F, G $\in C_M(X,Y)$, $F \neq G$. Suppose there exists $x_o \in X$ such that $F(x_o) \not\subset G(x_o)$. Then there exists $y_o \in F(x_o)$, $y_o \notin G(x_o)$. So there exists U_1, U_2 open sets in Y such that $y_o \in U_1$, $G(x_o) \subset U_2$, and $U_1 \cap U_2 = \emptyset$. Then $F \in \lambda(\{x_o\}, U_1)$ and $G \in \kappa(\{x_o\}, U_2)$.

5.6 Lemma.

If i) Y is a T_2 space, and

ii) $\mathfrak{F}_1(X) \subset S$;

then $K_M(X,Y)$ with $(S,0)$ topology is a T_2 space.

Proof: If F, G $\in K_M(X,Y)$, $F \neq G$, suppose $x_o \in X$ such that $F(x_o) \not\subset G(x_o)$, so there exists $y_o \in F(x_o)$, $y_o \notin G(x_o)$. Since $G(x_o)$ is compact and Y is T_2, there exists U_1, U_2

open sets in Y such that $y_0 \in U_1$, $G(x_0) \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Then $F \in \lambda(\{x_0\}, U_1)$ and $G \in \kappa(\{x_0\}, U_2)$.

5.7 Lemma.

If Y is a T_1 normal space then $\overline{C}_M(X,Y)$ with the c-o topology is a T_1 regular space.

Proof: By 5.5 we have $\overline{C}_M(X,Y)$ is T_2. If $F \in \overline{C}_M(X,Y)$ we will show that there is a basis of closed neighborhoods of F. If $A \in \check{\mathcal{S}}$, O is open in Y, and $F \in \kappa(A,O)$ then for each $x \in A$, $F(x) \subset O$ so there exists O_1 open in Y such that $F(x) \subset O_1 \subset \overline{O}_1 \subset O$. There exists an open neighborhood U of x such that $F[U] \subset O_1$. Since A is compact there exist finite collections, i) $\{O_1, O_2, \ldots, O_n\}$ of open sets in Y, and ii) $\{U_1, U_2, \ldots, U_n\}$ of open sets in X, such that

i) $F[U_i] \subset O_i \subset \overline{O}_i \subset O \quad i=1, \ldots, n$, and

ii) $A \subset \bigcup_{i=1}^{n} U_i$.

Then $F \in \kappa(A, \bigcup_{i=1}^{n} O_i)$ and $\bigcup_{i=1}^{n} \overline{O}_i \cup O$.

Now $\kappa(A, \bigcup_{i=1}^{n} O_i) \subset \overline{C}_M(X,Y) - \lambda(A, Y - \bigcup_{i=1}^{n} \overline{O}_i) \subset \kappa(A,O)$.

Thus if $F \in \kappa(A,O)$ there is a closed neighborhood N of F such that $F \in N \subset \kappa(A,O)$.

If $F \in \lambda(A,O)$ and $x \in A$, $F(x) \cap O \neq \emptyset$, so there exist $y \in F(x) \cap O$ and O_1 open in Y such that $y \in O_1 \subset \overline{O}_1 \subset O$. Since A is compact there exists finite collections, i) $\{O_1, O_2, \ldots, O_n\}$ of open sets in Y, and ii) $\{U_1, U_2, \ldots, U_n\}$ of open sets in X, such that

i) $A \subset \bigcup_{i=1}^{n} U_i$

ii) if $x \in U_i$, $F(x) \cap O_i \neq \emptyset$; and

iii) $\bigcup_{i=1}^{n} \overline{O}_i \subset O$.

Then $F \in \lambda(A, \bigcup_{i=1}^{n} O_i)$ and

$$\lambda(A, \bigcup_{i=1}^{n} O_i) \subset \overline{C}_M(X,Y) - \kappa(A, Y - \bigcup_{i=1}^{n} \overline{O}_i) \subset \lambda(A,O).$$

Thus if $F \in \lambda(A,O)$ there is a closed neighborhood N of F such that $F \in N \subset \lambda(A,O)$.

If N is a neighborhood of F in $\overline{C}_M(X,Y)$ there exist compact sets $A_1, A_2, \ldots, A_n$ and open sets $O_1, O_2, \ldots, O_n$ in Y such that

$$F \in [\bigcap_{i=1}^{k} \kappa(A_i, O_i)] \cap [\bigcap_{k+1}^{n} \lambda(A_i, O_i)] \subset N.$$

Then for each $i = 1, \ldots, n$, there exists a closed neighborhood N_i of F such that

i) $F \in N_i \subset \kappa(A_i, O_i)$ $i = 1, \ldots, k$, or

ii) $F \in N_i \subset \lambda(A_i, O_i)$ $i = k+1, \ldots, n$.

Then $F \in \bigcap_{i=1}^{n} N_i \subset N$. Thus $\overline{C}_M(X,Y)$ is regular.

5.8 Lemma.

If Y is a T_1 regular space, then $K_M(X,Y)$ with the c-o topology is a T_1 regular space.

Proof: Proceed as in the previous theorem using the fact that F(x) is compact. Thus if O is open in Y and $F(x) \subset O$ then there exists O_1 open in Y such that $F(x) \subset O_1 \subset \overline{O}_1 \subset O$.

Smithson in [15] has investigated some of the above properties. If Y is T_1, the converses to 5.5 and 5.7 also hold, see Michael [12] and Kuratowski [11].

The next theorems show that some of the theory of single-valued function spaces caries over to spaces of multi-functions.

5.9 Definition.

A topology τ on $C_M(X,Y)$ is <u>jointly-continuous</u> if $\omega: C_M(X,Y) \times X \to Y$ defined by $\omega(F,x) = F(x)$ is a continuous multi-function.

5.10 Theorem

A topology τ is jointly-continuous on $C_M(X,Y)$ iff for every topological space Z, if $G^*:Z \rightarrow C_M(X,Y)$ is a continuous single valued function then $G:Z \times X \rightarrow Y$, defined by $G(z,x) = G^*(z)(x)$ is a continuous multi-function.

Proof: Similar to the proof in [2].

5.11 Theorem.

If X is a locally compact space, then the c-o topology on $C_M(X,Y)$ is jointly-continuous.

Proof: Let $F \in C_M(X,Y)$, $x \in X$, and O be open in Y. If $F(x) \subset O$, since F is continuous there is a neighborhood N of x such that $F[N] \subset O$. Since X is locally compact we may assume that N is compact. Thus $(F,x) \in \kappa(N,O) \times N$ and $\omega(\kappa(N,O) \times N) \subset O$.

If $F(x) \cap O \neq \emptyset$, since F is continuous there is a compact neighborhood N of x such that $F(x') \cap O \neq \emptyset$, for all $x' \in N$. Thus $(F,x) \in \lambda(N,O) \times N$ and $\omega(F', x') \cap O \neq \emptyset$ for every $(F', x') \in \lambda(N,O) \times N$. Thus ω is a continuous multi-function.

5.12 Definition.

A topology u on $C_M(X,Y)$ is proper if for every topological space Z, if $G:Z \times X \to Y$ is a continuous multi-function, and $G^*:Z \to C_M(X,Y)$ is defined by

$G^*(z)(x) = G(z,x)$ then G^* is a continuous single valued function.

5.13 Theorem.

The compact-open topology on $C_M(X,Y)$ is always proper.

Proof: Let Z be a topological space. It will suffice to show that for any compact $A \subset X$, and any open set $O \subset Y$, and any continuous multi-function $G:Z \times X \to Y$, $G^{*-1}(\kappa(A,O))$ and $G^{*-1}(\lambda(A,O))$ are open sets in $Z \times X$.

$G^{*-1}(\kappa(A,O)) = \{z \in Z \mid G(z,x) \subset O \text{ for every } x \in A\}$.

If $z \in G^{*-1}(\kappa(A,O))$ and $x \in A$, there exists a neighborhood N of z and a neighborhood M of x such that $G[N \times M] \subset O$. Since A is compact, there exists finite collections, i) $\{M_1, \ldots M_n\}$ of neighborhood of points of A, and ii) $\{N_i, \ldots, N_n\}$ of neighborhoods of z; such that

i) $A \subset \bigcup_{i=1}^{n} M_i$, and

ii) $G[N_i \times M_i] \subset O$, $i=1, \ldots, n$.

Then $\bigcap_{i=1}^{n} N_i$ is a neighborhood of z. If $x \in A$ and

$z' \in \bigcap_{i=1}^{n} N_i$, $G^*(z')(x) = G(z',x) \subset O$. Thus

$\bigcap_{i=1}^{n} N_i \subset G^{*-1}(\kappa(A,O)$. Therefore $G^{*-1}(\kappa(A,O)$ is open.

If $z \in G^{*-1}(\lambda(A,))$ and $x \in A$, there exists a neighborhood M of x such that if $(z',x') \in N \times M$ then $G(z',x') \cap O \neq \emptyset$. Then as above there exist finite collections, i) $\{M_1, \ldots, M_n\}$ of neighborhoods of points of A, and ii) $\{N_1, \ldots, N_n\}$ of neighborhoods of z; such that;

1) $A \subset \bigcup_{i=1}^{n} M_i$ and

2) if $(z',x') \in N_i \times M_i$ then $G(z',x') \cap O \neq \emptyset$.

Thus if $z' \in \bigcap_{i=1}^{n} N_i$ and $x' \in A$, $G^*(z')(x') = G(z',x')$ and $G(z',x') \cap O \neq \emptyset$. Thus $\bigcap_{i=1}^{n} N_i \in G^{*-1}(\lambda(A,O)$. Therefore $G^{*-1}(\lambda(A,O)$ is open.

The following theorems have elementary proofs which are similar to the proofs of the same theorems for single-valued function spaces.

5.14 Theorem.

If u is a proper topology on $C_M(X,Y)$ and τ is a jointly-continuous topology on $C_M(X,Y)$, then $u \leq \tau$ (i.e. if O is open in u then O is open in τ).

5.15 Theorem.

If u is a proper topology on $C_M(X,Y)$ and u' is a topology on $C_M(X,Y)$ such that $u' \leq u$ then u' is proper. If τ is jointly-continuous and τ' is a topology on $C_M(X,Y)$ such that $\tau \leq \tau'$, then τ' is jointly-continuous.

5.16 Theorem.

The c-o topology is coarser than every jointly continuous topology. If X is locally compact the c-o topology is the finest of the proper topologies.

The proof of 5.16 can be done by a mapping argument as in 5.14 and 5.15 or it can be done in a similar fashion to the proof Arens gave in [1]. The proof can also be found in Smithson [15].

Using Arens paper, we find in general that there does not exist a minimal jointly-continuous topology for $C_M(X,Y)$. However if $\{\mathcal{U}_a \mid a \in A\}$ is a family of proper topologies in $C_M(X,Y)$ and $\mathcal{U}$ is the topology generated by the union of the $\{\mathcal{U}_a\}$ then $\mathcal{U}$ is a proper topology. Thus there is a maximal proper topology for $C_M(X,Y)$.

5.17 Definition.

Let i) $\{F_d, d \in D\}$ be a net in $C_M(X,Y)$ and

ii) $F \in C_M(X,Y)$.

Then F_d converges continuously to F if for every $x \in X$ and every net $\{x_c \mid c \in C\}$, $x_c \to x$ implies $F_d(x_c) \to F(x)_{(ft)}$ (i.e. if O is open in Y, and

i) If $F(x) \subset O$ then there exists $d_o \in D$ and $c_o \in C$ such that if $d \geq d_o$ and $c \geq c_o$ then $F_d(x_c) \subset O$,

ii) if $F(x) \cap O \neq \emptyset$ there exists $d_o \geq D$ and $c_o \geq C$ such that if $d \geq d_o$ and $c \geq c_o$ then $F_d(x_c) \cap O \neq \emptyset$.

5,18 Theorem.

A topology τ on $C_M(X,Y)$ is jointly-continuous iff for every net $\{F_d \mid d \in D\}$ and every $F \in C_M(X,Y)$, if $F_d \to F_{(\tau)}$

then F_d converges continuously to F.

Proof: The following is similar to the technique used in [2]. If τ is a jointly-continuous topology then $\omega: C_M(X,Y) \times X \to Y$ is continuous. Suppose $\{F_d \mid d \in D\}$ is a net in $C_M(X,Y)$ and $\{x_c, c \in C\}$ is a net in X. If $F_d \to F_{(\tau)}$ and $x_c \to x$ then $(F_d, x_c) \to (F,x)$ in the product topology. Thus $\omega(F_d, x_c) \to (F(x)_{(f.t.)}$, Theorem 3.16.

Suppose $F_d \to F_{(\tau)}$ implies F_d converges continuously to F. Let Z be a topological space and $G^*: Z \to C_M(X,Y)$ be a continuous single-valued function. Define $G: Z \times X \to Y$ by $G(z,x) = G^*(z)\ (x)$. Then G is a multi-function. If $\{z_d \mid d \in D\}$ is a net in Z converging to z, and $\{x_c \mid c \in C\}$ is a net in X converging to x, then (x_d, x_c) converges to (z,x) in $Z \times X$ and $G^*(z_d) \to G^*(z)_{(\tau)}$ since G^* is continuous.

Thus $G(z_d, x_c) \to G(z, x)_{(f.t.)}$.

Therefore G is continuous, Theorem 3.16.

5.19 Theorem.

A topology u on $C_M(X,Y)$ is proper iff for every net $\{F_d; d \in D\}$ and F in $C_M(X,Y)$ if F_d converges continuously to F then $F_d \to F_{(u)}$.

Proof: Suppose u is proper and $\{F_d \mid d \in D\}$ us a net in $C_M(X,Y)$ such that F_d converges continuously to F. Let $Z = D \cup \{\infty\}$ with the added order that $\infty \geq d$ for every $d \in D$. Define $G:Z \times X \to Y$ by $G(d,x) = F_d(x)$ if $d \in D$ and $G(\infty,x) = F(x)$. If $x_c \to x$ in X then $G(d,x_c) = F_d(x_c) \to G(\infty,x) = F(x)_{(f.t.)}$. Thus G is continuous, Theorem 3.16. Then $G^*:Z \to C_M(X,Y)$ is defined by $G^*(d) = F_d$ if $d \in D$, and $G^*(\infty) = F$. Since G^* is continuous and $\{d \in D\} \to \infty$, $F_d \to F_{(u)}$.

Suppose continuous convergence in $C_M(X,Y)$ implies convergence with respect to the topology u. Let

i) Z be a topological space,

ii) $G:Z \times X \to Y$ ne a continuous multi-function, and

iii) $G^*:Z \to C_M(X,Y)$ be defined by $G^*(z)\ (x) = G(z,x)$.

Let $\{z_c \mid c \in C\}$ be a net in z such that $z_c \to z \in Z$ and $\{x_d \mid d \in D\}$ be a net in X such that $x_d \to x \in X$. Then $G(z_d, x_c) \to G(z,x)$ so $G^*(z_c)$ converges continuously to $G^*(z)$. Thus G^* is a continuous single-valued function.

We have shown that if X is a locally compacts space, then for argitrary topological spaces Z and Y, $G: Z \times X \to Y$ is a continuous multi-function iff $G^*: Z \to C_M(X,Y)$, $C_M(X,Y)$ endowed with the c-o topology, is a continuous single valued function.

5.20 Lemma.

If i) X and Z are locally compact spaces, and

ii) Y is an arbitrary space;

then $\{\kappa(K_1 \times K_2, O), \lambda(K_1 \times K_2, O)$ K_1 compact in Z, K_2 compact in X and O open in Y$\}$ forms a subbase for the c-o topology on $C_M(X \times Z, Y)$.

Proof: Let C be a compact subset of $Z \times X$, and O be an open set in Y. If $F \in \kappa(C,O)$, and $(z,x) \in C$, $F(z,x) \subset O$. Then there exists a compact neighborhood K_1 of z and a compact neighborhood K_2 of x such that $F(K_1 \times K_2) \subset O$. Thus $F \in \kappa(K_1 \times K_2, O)$. Since C is compact there exist collections $\{K_{1_i} \mid i=1, 2, \ldots, n\}$ of compact subsets of Z and $\{K_{2_i} \mid i=1, \ldots, n\}$ of compact subsets of X such that $C \subset \bigcup_{i=1}^{n} K_{1_i} \times K_{2_i}$ and $F \in \bigcap_{i=1}^{n} \kappa(K_{1_i} \times K_{2_i}, O)$. Then

$\bigcap_{i=1}^{n} \kappa(K_{1_i} \times K_{2_i}, O) \subset \kappa(C,O)$.

If $F \in \lambda(C,O)$ and $(z,x) \in C$, there exists compact neighborhoods K_1 of z and K_2 of x such that if $(z', x') \in K_1 \times K_2$ then $F(z', x') \cap O \neq \emptyset$. Thus $F \in \lambda(K_1 \times K_2, O)$. Since C is compact, there exist finite sets $\{K_{1_i} \mid i=1, \ldots, n\}$ of compact subsets of Z and $\{K_{2_i} \mid i=1, \ldots, n\}$ of compact subsets of X such that $C \subset \bigcup_{i=1}^{n} K_{1_i} \times K_{2_i}$ and $F \in \bigcap_{i=1}^{n} \lambda(K_{1_i} \times K_{2_i}, O)$. Then $\bigcap_{i=1}^{n} \lambda(K_{1_i} \times K_{2_i}, O) \subset \lambda(C,O)$.

5.21 Theorem.

If i) X is a locally compact topological space,

ii) Y and Z are topological spaces, and

iii) $\{\kappa(K_1 \times K_2, O); \lambda(K_1 \times K_2, O)\}$, as defined in Lemma 5.20, forms a subbase for the c-o topology for $C_M(Z \times X, Y)$,

then $C_M(Z \times X, Y)$ and $C(Z, C_M(X,Y))$ are isomorphic spaces.

Proof: The sets of the form $(K_1, \kappa(K_2, O)$ and $(K_1, \lambda(K_2, O))$ are a subbase for the c-o topology on

$C(Z, C_M(X,Y))$. Then we check that the mapping $G \to G^*$ sends $\kappa(K_1 \times K_2, O)$ onto $(K_1, \kappa(K_2, O))$ and $\lambda(K_1 \times K_2, O)$ onto $(K_1, \lambda(K_2, O))$.

$G \in \kappa(K_1 \times K_2, O)$ iff for every $z \in K_1$ and $x \in K_2$ $G(z, x) = G^*(z)(x) \subset O$ iff for every $z \in K_1 G^*(z)(x) \subset O$ for every $x \in K_2$ iff for every $z \in K_1$, $G^*(z) \in \kappa(K_2, O)$ iff $G^* \in (K_1, \kappa(K_2, O))$. And $G \in \lambda(K_1 \times K_2, O)$ iff for every $z \in K_1$, and $x \in K_2$, $G(z,x) \cap O = G^*(z)(x) \cap O \neq \emptyset$ iff for every $z \in K_1$, $G^*(z) \in \lambda(K_2, O)$ iff $G^* \in (K_1, \lambda(K_2, O))$. Since we have a 1-1 correspondence which takes the subbase of one topology onto the subbase of the other topology, we have a homeomorphism.

The next theorem will deal with the continuity of the composition map. Following Dugundji [6] Chapter XII, Section 11, we define ;

if X, Y and Z are topological spaces

$T:K_M(X,Y) \times C_M(Y,Z) \to C_M(X,Z)$ by $T(G,F) = F \circ G$.

5.22 Theorem.

If Y is locally compact, then T is continuous.

Proof: Since T is a single valued function, we only have to consider subbasic open sets of $C_M(X,Z)$. Let K be a compact subset of X and O be open in Z. If $T(G,F) \in \kappa(K,O)$ then $F \circ G(x) = F(G(x)) \subset O$ for all $x \in K$. Thus $G[K] \subset \{y \in Y \mid F(y) \subset O\} = O'$ and O' is open. Berge [3] Chapter VI, Par. 1, Theorem 3 showed that $G[K]$ is compact. Then since Y is locally compact, there exists a compact neighborhood K' of $G[K]$ such that $G[K] \subset K' \subset O'$. Let O'' be open in Y such that $G[K] \subset O'' \subset K'$. Then $G \in \kappa(K,O'')$ and $F \in \kappa(K', O)$. Let $G' \in \kappa(K, O'')$ and $F' \in \kappa(K',O)$, then $T(G',F') = F' \circ G' \in \kappa(K, O)$. If $x \in K$, $G'(x) \subset O'' \subset K'$. If $y \in G'(x)$, $y \in K'$, then $F'(y) \subset O$. Thus $F' \circ G'(x) = F'[G'(x)] \subset F'[K'] \subset O$. Therefore $F' \circ G' = T(G', F') \in \kappa(K, O)$.

If $T(G, F) \in \lambda(K, O)$, and $x \in K$, $G(x) \subset \{y \in Y \mid F(y) \cap O \neq \emptyset\} = O'$, and $G(x) \neq \emptyset$. Since $G[K]$ is compact there exists an open set O'' in Y and a compact set K' in Y such that $G[K] \subset O'' \subset K' \subset O'$. If $F' \in \lambda(K', O)$ and $G' \in \lambda(K, O'')$ and $x \in K$, then there exists $y \in K'$, $y \in G'(x)$. Now $F' \circ G'(x) \cap O \supset F'(y) \cap O \neq \emptyset$ since $y \in K'$.

REFERENCES

1. Arens, R. F., A Topology for Spaces of Transformations, Ann. of Math. Vol. 47, (1946) pp. 480-489.

2. Arens, R. F., and Dugundji, James, Topologies for Function Spaces, Pacific J. Vol. 1 (1951) 5-31.

3. Berge, C., Topological Spaces, MacMillan Co., N.Y. 1963.

4. Bourbaki, N., General Topology part 1, Addison-Wesley Pub. Co. 1966.

5. Day, J. M., The Compact Open Topology for a Space of Relations and . . ., Dissertation, U. of Florida 1964.

6. Dugundji, James, Topology, Allyn and Bacon, Inc., Boston. 1966.

7. Franklin, S. P., Concerning Continuous Relations, Dissertation, University of California at Los Angeles, 1963.

8. Gillman, L., and Jerison, M., Rings of Continuous Functions, D. Van Nostrand Co., Inc. Princeton N.J., 1960.

9. Kelley, J. L., General Topology, D. Van Nostrand Co., Inc., Princeton, N.J., 1955.

10. Kelley, J. L., Hypercomplete Linear Topological Spaces, Michigan Math J. Vol. 5(1958) pp. 235-246.

11. Kuratowski, K., Topology Vol. I and II, Academic Press N.Y., 1966.

12. Michael. E., Topologies on Spaces of Subsets, Trans. Amer. Math Soc. Vol. 71 (1951) pp. 152-158.

13. Ratner, L., Multi-valued Transformations Dissertation, University of California at Los Angeles., 1949.

14. Smithson, R., Changes of Topology and Fixed Points for Multi-valued Functions, Proc. Amer, Math Soc. Vol. 16 (1965) pp. 448-454.

15. ______, Topologies on Sets of Relations to appear J. Nat. Sci. Math Lahore, W. Pakistan.

16. ______, Uniform Convergence for Multi-functions to appear.

17. Strother, W. L., Continuity for Multi-functions and some Applications to Topology. Dissertation, Tulane, 1951.