Lower semi-continuous multifunctions and properties of the l and k topology
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A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
Let c be a collection of subsets of a topological space X. A binary relation R can be defined on X into c by (x,C) □ R if and only if x □ C. Define R+ , R- : P(X) -> P(c) by R+(A) = {C &isin c | C □ A} and R-(A) = {C □ c | C ∩ A ≠ □}. The smallest topology on c which makes R open (or closed) is called the λ (or κ) topology. We call λc (or κc) the space c with the λ (or κ) topology.

This thesis is divided into three main parts. The first (chapter III) examines properties of λc. For example if c is the family of nonvoid closed subsets of X, then λc compact ⇔ X compact, λc second countable ⇔ X second countable, λc first countable ⇒ X first countable, X and Y are homeomorphic ⇔ λcx and λcy are homeomorphic. These results are extended to more general families of subsets of X. If β □ λc is connected, and if each elements of β is connected in X, then □β is connected. Hence if c is a cover of X by connected sets, then X in connected if λc is.

The second part (chapter IV) deals with lower semi-continuous (l.s.c.) multifunctions. Let f^+ and f^- be Berge's upper inverses. If f has closed point values, then a single-valued function F:X → λc (c nonvoid closed subsets) can be defined which is continuous if and only if f is l.s.c. This simple results is used to obtain a homeomorphism of X to its graph regarded as embedded in λ(cx × cy) (where cx(or cy) are nonvoid closed subsets of X (or Y)). A characterization of l.s.c. in terms of accumulation points is given and conditions are examined under which f is l.s.c. Chapter V deals with properties of the κ topology. If c is the family of all open subsets of X, a fixed point theorem for continuous single-valued functions on κc into κc is presented.

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AND PROPERTIES OF THE λ AND κ TOPOLOGY

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ABSTRACT

Let $\mathcal{C}$ be a collection of subsets of a topological space X. A binary relation R can be defined on X into $\mathcal{C}$ by $(x,C) \in R$ if and only if $x \in C$. Define R_+, $R_- : P(X) \to P(\mathcal{C})$ by $R_+(A) = \{C \in \mathcal{C} \mid C \subseteq A\}$ and $R_-(A) = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\}$. The smallest topology on $\mathcal{C}$ which makes R open (or closed) is called the λ (or κ) topology. We call $\lambda\mathcal{C}$ (or $\kappa\mathcal{C}$) the space $\mathcal{C}$ with the λ (or κ) topology.

This thesis is divided into three main parts. The first (chapter III) examines properties of $\lambda\mathcal{C}$. For example if $\mathcal{C}$ is the family of nonvoid closed subsets of X, then $\lambda\mathcal{C}$ compact $\Leftrightarrow$ X compact, $\lambda\mathcal{C}$ second countable $\Leftrightarrow$ X second countable, $\lambda\mathcal{C}$ first countable $\Rightarrow$ X first countable, X and Y are homeomorphic $\Leftrightarrow$ $\lambda\mathcal{C}_X$ and $\lambda\mathcal{C}_Y$ are homeomorphic. These results are extended to more general families of subsets of X. If $\mathcal{B} \subseteq \lambda\mathcal{C}$ is connected, and if each elements of $\mathcal{B}$ is connected in X, then $\bigcup\mathcal{B}$ is connected. Hence if $\mathcal{C}$ is a cover of X by connected sets, then X in connected if $\lambda\mathcal{C}$ is.

The second part (chapter IV) deals with lower semicontinuous (l.s.c.) multifunctions. Let f^+ and f^- be Berge's upper inverses. If f has closed point values, then a single-valued function $F:X \to \lambda\mathcal{C}$ ($\mathcal{C}$ nonvoid closed subsets) can be defined which is continuous if and only if f is l.s.c. This simple results is used to obtain a homeomorphism of X to its graph regarded as embedded in $\lambda(\mathcal{C}_X \times \mathcal{C}_Y)$ (where $\mathcal{C}_X$(or $\mathcal{C}_Y$) are nonvoid closed subsets of X (or Y)). A characterization of l.s.c. in terms of accumulation points is given and conditions are examined under which f is l.s.c.

Chapter V deals with properties of the κ topology. If $\mathcal{C}$ is the family of all open subsets of X, a fixed point theorem for continuous single-valued functions on $\kappa\mathcal{C}$ into $\kappa\mathcal{C}$ is presented.

CHAPTER I

INTRODUCTION

The study of hyperspaces and of multiple-valued functions (or multifunctions) has occupied topologists for over half a century. In 1914 Felix Hausdorff [5] initiated research on an area of mathematics which has expanded ever since. The problem which Hausdorff attacked dealt with the following ideas: If we let d be a metric on a space X, a "distance" between two subsets of X can be defined in various ways, e.g. if $A,B \subseteq X$, let

$$D_1(A,B) = \max \{d(a,b) \mid a \in A,\ b \in B\} \text{ or}$$

$$D_2(A,B) = \min \{d(a,b) \mid a \in A,\ b \in B\}.$$

These distances unfortunately do not form a metric for the space of subsets, and Hausdorff remedied this defect by letting

$$D(A,B) = \max \{\sup_{a \in A} D_2(\{a\},B),\ \sup_{b \in B} D_2(A,\{b\})\}.$$

This function applied to the family of all closed nonvoid subsets of X yields a metric space.

Not too many years after Hausdorff's work was published, the research of Leopold Vietoris [16] gave a new approach to this topic. Again starting with nonvoid, closed subsets, Vietoris obtained a topology on this family by defining a neighborhood of a closed set M to be the class of all closed subsets of X contained in the union of a given finite number of open subsets of X and intersecting each one of these open sets, provided M itself belongs to this class.

It can be shown that if $\mathcal{C}$ is the family of non-empty compact subsets of a bounded metric space, the topology induced on $\mathcal{C}$ by the Hausdorff metric above is the same as the mentioned Vietoris topology.

In 1950 Ernest Michael [10] presented what might justly be called the definitive work on the topic of topologies on spaces of subsets. The Vietoris (here called the "finite") topology is also applied to the topic of multifunctions. Of course, other topologists had already published research on hyperspaces, e.g Kelley [6], Kuratowski [7], etc. Part of the interest of Michael's work, however, lies in the fact that he enlarged the family of subsets from nonvoid closed subsets to the family of all subsets of X. Furthermore continuity and uniform continuity of mappings to these hyperspaces was considered.

C. Kuratowski [8] broadened the subject area, devoting considerable space to the fundamental properties of semi-continuous mappings. These mappings, usually considered under the heading of upper or lower semi-continuity, are correspondences f between the respective power sets of two topological spaces X and Y, i.e. $f:P(X) \rightarrow P(Y)$, where an upper semi-continuous map takes a closed subset of Y to a closed subset of X, whereas a lower semi-continuous map assigns to an open subset of Y an open subset of X. The two concepts coincide whenever f is a single-valued function. Kuratowski called the space of non-empty closed subsets with the Vietoris topology the exponential topology 2^X. He also drew attention to the κ and λ topologies, where the κ-topology on a hyperspace is generated by sets of the form $\{ A \subseteq G \mid G$ is an open subset of X, A is an element of the hyperspace$\}$. (The λ topology is defined in chapter II).

The κ topology had been applied to closed nonvoid subsets of compact T_2 spaces X by V.I. Ponomarev [12]. The restrictions placed on X produce several interesting results. Closely tied to κ-spaces are the already defined upper semi-continuous functions. The present paper will, among other projects, generalize the κ-topology to families of subsets not necessarily closed.

A compendium on continuous multivalued functions was published by W.L Strother [14]. His notation unfortunately does not adhere to the more popular terminology already mentioned.

The notation which will be used in this paper is partially based on that introduced by Claude Berge [1]. Similar symbolism is used by E. Čech [2] in his encyclopedic volume on topological spaces.

In 1965 G.T. Whyburn gave investigations on the topic of continuity of multifunctions if not a new direction, then at least a renewed impetus. The summary of conditions on spaces X and Y for which $f: X \to Y$ is upper semi-continuous is probably the most helpful single page to any student of this fascinating topic. One should point out again, that upper semi-continuity is stressed here. R.E. Smithson [13], on the other hand, gave a characterization of lower semi-continuity analogous to that of upper semi-continuity given by Whyburn.

The research on hyperspaces and multifunctions continues. For the most part, however, the emphasis is on spaces on non-empty closed subsets with the Vietoris topology and on upper semi-continuous functions. The most readily adaptable and usable topological concepts lie in this area; yet the problems of lower semi-continuity also need to be solved. Whereas Ponomarev has worked out many details of the κ-topology, the author will endeavor to answer some questions on the λ-topology.

CHAPTER II

BINARY RELATIONS, MULTIFUNCTIONS AND THE λ - TOPOLOGY.

If X and Y are two sets, the set $X \times Y$ of all ordered pairs (x,y) with $x \in X$, $y \in Y$ is called the cartesian product of X with Y. Any subset R of $X \times Y$ is said to be a (binary) relation in X into Y. If R is a relation, we let $\hat{R}$ denote the set of all ordered pairs (y,x) such that $(x,y) \in R$. Thus $\hat{R}$ is a relation in Y into X.

The set of all first elements of R is called the domain, the set of second elements the range of R (Note that the domain of $\hat{R}$ is the range of R). If the domain of R is X we say R is on X, and R is onto Y provided the range of R is Y.

A relation R in X into Y is said to be single-valued provided that if $(x,y) \in R$ and $(x,z) \in R$ the $y=z$. Single-valued relations are generally conceded to be adequate set-theoretic models of the intuitive notion of a function. Hence by a function <u>on</u> X <u>into</u> Y we shall mean a single-valued binary relation on X into Y.

Associated with each relation R on X into Y is a function f (called the related function) on X into P(Y), the family of all subsets of Y, uniquely determined by the formula

$$f(x) = \{y \in Y \mid (x,y) \in R\}.$$

Furthermore, given a function f on X into P(Y), a relation R on X into Y is uniquely determined by the rule

$$(x,y) \in R \Leftrightarrow y \in f(x).$$

It is clear that f and R uniquely determine each other; from now on we shall call this related function a <u>multi-function</u> on X into Y. Also, since most results obtained will use a relation R on X onto the range of R, we shall assume (unless specifically mentioned) that R is <u>on</u> X <u>onto</u> Y. (Abusing the language somewhat, we shall also say that f is <u>on</u> X <u>onto</u> Y).

The multifunction $f: X \to Y$ gives rise to several other functions mapping either P(X) to P(Y) or P(Y) to P(X). Let $A \subseteq X$. We let

$$f_-(A) = \{y \in Y \mid \exists\, x \in A \text{ with } y \in f(x)\}.$$

It can easily be shown that

$$f_-(A) = \bigcup_{x \in A} f(x).$$

Let $f_+(A) = Y - f_-(X-A)$.

(This notation is essentially due to Berge [1]).

If we let $f^+ = (\hat{f})_+$ and $f^- = (\hat{f})_-$

(where $\hat{f}$ is the multifunction related to $\hat{R}$), then

$$f^+, f^-: P(Y) \to P(X).$$

Clearly we have, for $B \subseteq Y$,

$$f^-(B) = \{x \in X \mid f(x) \cap B \neq \emptyset\} \text{ and}$$

$$f^+(B) = \{x \in X \mid f(x) \subseteq B\}.$$

If R is the binary relation associated with the multi-function f, we also denote f^+ by R^+, f^- by R^-, f_- by R_- and f_+ by R_+. The following properties of a multifunction f on X onto Y are easily checked:

1.) If $\mathcal{A} \subseteq P(X)$ then

$$f_-(\cup \mathcal{A}) = \cup\{f_-(A) \mid A \in \mathcal{A}\}$$

$$f_-(\cap \mathcal{A}) \subseteq \cap\{f_-(A) \mid A \in \mathcal{A}\}$$

$$f_+(\cup \mathcal{A}) \supseteq \cup\{f_+(A) \mid A \in \mathcal{A}\}$$

$$f_+(\cap \mathcal{A}) = \cap\{f_+(A) \mid A \in \mathcal{A}\}.$$

2.) If $A \in P(X), B \in P(X)$ such that $A \subseteq B$ then

$f_+(A) \subseteq f_+(B)$ and

$f_-(A) \subseteq f_-(B)$.

3.) For any $A \in P(X)$

$f_+(A) \subseteq f_-(A)$.

4.) If f is single-valued then

$f^+ = f^-$

5.) If $B \subseteq Y$, $A \subseteq X$,

$$f_-(f^-(B)) \supseteq B \qquad f^-(f_-(A)) \supseteq A$$

$$f_-(f^+(B)) \subseteq B \qquad f^-(f_+(A)) \subseteq A$$

$$f_+(f^-(B)) \supseteq B \qquad f^+(f_-(A)) \supseteq A$$

$$f_+(f^+(B)) \subseteq B \qquad f^+(f_+(A)) \subseteq A$$

6.) If $\mathcal{B} \subseteq P(Y)$

$$f^-(\cup \mathcal{B}) = \cup\{f^-(B) \mid B \in \mathcal{B}\}$$

$$f^-(\cap \mathcal{B}) \subseteq \cap\{f^-(B) \mid B \in \mathcal{B}\}$$

$$f^+(\cup \mathcal{B}) \supseteq \cup\{f^+(B) \mid B \in \mathcal{B}\}$$

$$f^+(\cap \mathcal{B}) = \cap\{f^+(B) \mid B \in \mathcal{B}\}$$

7.) If $B \in P(Y)$, and $B' \in P(Y)$ such that $B \subseteq B'$ then

$f^-(B) \subseteq f^-(B')$ and

$f^+(B) \subseteq f^+(B')$.

8.) If $B \in P(Y)$

$f^+(B) \subseteq f^-(B)$

These properties will be used in proofs without specific reference.

Assume now that $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ are topological spaces. We let $co\mathcal{T}$ (resp. $co\mathcal{S}$) denote the collection of closed sets of X (resp. Y). We shall say that the multifunction $f:X \to Y$ is upper semi-continuous (u.s.c.) if and only if for $B \in co\mathcal{S}$, $f^-(B) \in co\mathcal{T}$. f is said to be lower semi-continuous (l.s.c.) provided that for $C \in \mathcal{S}$, $f^-(C) \in \mathcal{T}$. It follows at once from the definitions of f^+ and f^- that f is u.s.c. iff for $C \in \mathcal{S}$, $f^+(C) \in \mathcal{T}$ and that f is l.s.c. iff f for $B \in co\mathcal{S}$, $f^+(B) \in co\mathcal{T}$. f is said to be open (or closed) provided that for $A \in \mathcal{T}$ (or $A \in co\mathcal{T}$), $f_-(A) \in \mathcal{S}$ (or $f_-(A) \in co\mathcal{S}$). It follows again from the definition that f is open iff $f_+(A) \in co\mathcal{S}$ for $A \in co\mathcal{T}$ and f is closed iff $f_+(A) \in \mathcal{S}$ for $A \in \mathcal{T}$.

The following elementary results are immediate.

(i) If $f:X \to Y$, then $\hat{\hat{f}}=f$.

(ii) If $f:X \to Y$ and $g:Y \to Z$ are multifunctions, then $\widehat{gf} = \hat{f}\hat{g}$.

(iii) $f:X \to Y$ is u.s.c. (l.s.c.) iff $\hat{f}$ is closed (open) (See also Čech [2]).

Let $\mathcal{C}$ be a family of subsets of the space X. We can define a binary relation R on X to $\mathcal{C}$ as follows: $(x,C) \in R$ iff $x \in C$, where $C \in \mathcal{C}$. For $A \subseteq X$, R_+ and $R_-:P(X) \to P(\mathcal{C})$

satisfy $R_+(A) = \{C \in \mathcal{C} \mid C \subseteq A\}$, $R_-(A) = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\}$. R^- and R^+ are mappings of $P(\mathcal{C})$ to $P(X)$, where for $\mathcal{a} \subseteq \mathcal{C}$,

$R^-(\mathcal{a}) = \bigcup \mathcal{a}$ and

$R^+(\mathcal{a}) = X - R^-(\mathcal{C}-\mathcal{a}) = \{x \in X \mid \text{if } x \in C, C \in \mathcal{C}, \text{then } C \in \mathcal{a}\}$.

Using these multifunctions, we let $\lambda\mathcal{C}$ denote the space $\mathcal{C}$ with the following topology: The open sets are generated by sets of the form $R_-(G)$ for G open in X. With this convention, no distinction will be made between $\mathcal{C}$ and $\lambda\mathcal{C}$ unless ambiguities might occur. R is closed if for any K closed in X, $R_-(K)$ is closed in $\lambda\mathcal{C}$. As above, for K closed in X, $R_-(K)$ is closed iff $R_+(X-K)$ is open. For convenience's sake, we shall denote by λX the space of all nonvoid closed subsets of X and by $\lambda\mathcal{P}$ the space of all nonvoid subsets of X, with the λ topology (see Kuratowski [8]).

Since R_- does not preserve intersections (see 1. above), the sets $R_-(G)$, $G \subseteq X$, G open, do not form a basis for the hyperspace $\lambda\mathcal{C}$ but merely a subbasis. A basis element of $\lambda\mathcal{C}$ thus has the form $\bigcap_{i=1}^{n} R_-(G_i)$, G_i open in X. It is now clear that the λ topology is the smallest topology for which R is open. Also R is u.s.c. provided $R^-(\mathcal{a})$ is open in X, where $\mathcal{a}$ is open in $\lambda\mathcal{C}$. It is of interest to note that R is single-valued iff the elements of $\mathcal{C}$ are mutually disjoint.

Using $\mathcal{C}$ and R as before, another topology may be assigned to $\mathcal{C}$, called the κ-topology (Ponomarev [12]). It is the smallest topology for $\mathcal{C}$ for which R is closed, i.e. such that whenever M is closed in X, $R_-(M)$ is closed. Sets of the form $R_+(G)$, G open in X, form not merely a subbasis but actually a basis for the hyperspace $\kappa\mathcal{C}$. The notation κX and $\kappa\mathcal{P}$ follows that of the λ-topology.

Before considering properties of the κ and λ topologies, a few facts about binary relations R in X into $\mathcal{C} \subseteq P(X)$ are worth mentioning.

(i) $\forall A \subseteq X$, $R_-(A) = \emptyset \iff A \cap \text{domain } R = \emptyset$.

(ii) $\forall A \subseteq X$, $R_+(A) \supseteq \mathcal{C} - \text{range } R$.

(iii) If $\mathcal{A} \subseteq P(X)$, then $\emptyset \in \mathcal{A} \Rightarrow \emptyset \in R_-[\mathcal{A}]$.

This is true for R_+ iff R is onto $\mathcal{C}$.

(iv) R is on X $\iff$ ($\forall \mathcal{A} \subseteq P(X)$, $\emptyset \notin \mathcal{A} \Rightarrow \emptyset \notin R_-[\mathcal{A}]$).

(v) If $\hat{R}$ is single-valued and R is on X, then

$\forall \mathcal{A} \subseteq P(X)$, $\emptyset \notin \mathcal{A} \Rightarrow \emptyset \notin R_+[\mathcal{A}]$.

(vi) If R is on X and $\mathcal{A} \subseteq P(X)$ has the finite intersection property (f.i.p.) then $R_-[\mathcal{A}]$ has the f.i.p.

(vii) $\hat{R}$ single-valued and R on X $\Rightarrow$ R_+ preserves the f.i.p.

(viii) R single-valued $\Rightarrow$ R_- preserves the basis property.

Proof: (v) Assume $\emptyset \notin \mathcal{A}$. If $R_+(A) = \emptyset$ for some $A \in \mathcal{A}$, then $\mathcal{C} = R_-(X-A)$ since $R_+(A) = \mathcal{C} - R_-(X-A)$. Since $A \neq \emptyset$ $\exists$ $x \in A$ such that $R_-(\{x\}) \neq \emptyset$. Since R is on X $\exists$ $C \in \mathcal{C}$ such that $x \in C$. But $C \in R_-(X-A)$, so C meets X-A. Assume $x' \in C \cap (X-A)$. $C \in R_-(\{x\}) \cap R_-(\{x'\})$ but, because $\hat{R}$ is single-valued,

$R_-(\{x\}) \cap R_-(\{x'\}) = \emptyset$, a contradiction.

(vi) Let $R_-(A_i) \in R_-[\mathcal{A}], i=1,2,\ldots,n$. Assume $\bigcap_{i=1}^{n} R_-(A_i) = \emptyset$, i.e. $\nexists$ $C \in \mathcal{C}$ with C meeting each A_i. However, since $\mathcal{A}$ has the f.i.p., $\bigcap_{i=1}^{n} A_i \neq \emptyset$, a contradiction.

(viii) Assume $\mathcal{C} \subseteq P(X)$ has the basis property. Let $C \in \mathcal{C}$ with $C \in R_-(A) \cap R_-(B)$. (Without loss of generality let $C \neq \emptyset$). Because $\hat{R}$ is single-valued $A \cap B \neq \emptyset$ and $\exists$ $x \in C$ such that $x \in A \cap B$. Hence $\exists$ $D \in \mathcal{A}$ such that $x \in D \subseteq A \cap B$, and thus $C \in R_-(D) \subseteq R_-(A) \cap R_-(B)$.

CHAPTER III

PROPERTIES OF THE λ TOPOLOGY

Let X be a topological space and let $\mathcal{C}$ be a family of subsets of X.

Proposition III .1 : If $\mathcal{C}$ includes all singleton subsets of X, then X can be embedded as a subspace of $\lambda\mathcal{C}$.

Proof: Use the map $x \to \{x\}$.

Corollary III 1. If X is T_1, then X can be embedded as a subspace of λX.

Proof: In a T_1 space $\{x\} = cl\{x\}$

Proposition III .2 : If a T_1 space X has more than one point, then λX is T_0 but not T_1.

Before proceeding with the proof, we list the following lemmas.

Lemma III.2.1 : If C and D belong to λX and if $C \in cl\{D\}$, then $C \cap D \neq \emptyset$.

Proof of the lemma: If C does not meet D, then the open set X - D contains C but does not meet D. So $C \notin cl\{D\}$. The converse does not hold, since if $D \subseteq C$, then X - D is open and C meets X - D, but D does not.

Lemma III.2.2 : If $C \subseteq D$, then $C \in cl\{D\}$.

Note: If $\mathcal{A} \subseteq \mathcal{C}$, $cl\,\mathcal{A} = \{C \in \mathcal{C} \mid$ each open subset G of X that meets C also meets $\cup \mathcal{A}$ }. Thus for $\mathcal{A} = \{A\}$, $cl\{A\} = \{C \in \mathcal{C} \mid C \cap G \neq \emptyset,\ G \text{ open in } X, \Rightarrow A \cap G \neq \emptyset\}$.

Proof of proposition III.2 : Let A and B be elements of λX and assume there is a point in A not in B. Then $cl\{A\} \neq cl\{B\}$, since X - B is an open set with $A \cap (X-B) \neq \emptyset$. Hence λX is T_0. Let $A \in \lambda X$, $A \neq X$. Then $A \in cl\{X\}$ (lemma III. 2.2), but $A \notin \{X\}$, so that λX is not T_1.

In general we shall use collections $\mathcal{C}$ which do not include $\emptyset$. In case $\mathcal{C}$ does include $\emptyset$, we have the following

<u>Proposition III.3</u> : Let $\emptyset \in \mathcal{C}$, then $\lambda\mathcal{C}$ is compact and connected.

Proof: Let $\mathcal{K}$ be a family of closed subsets of $\lambda\mathcal{C}$ with f.i.p. Since $\emptyset$ belongs to every closed set in $\lambda\mathcal{C}$, $\emptyset \in \cap \mathcal{K}$, so that $\lambda\mathcal{C}$ is compact. If K is closed in X, then $\emptyset \in R_+(K) \subseteq \lambda\mathcal{C}$. Hence $\lambda\mathcal{C}$ is connected. (It is of interest to note that $\{\emptyset\}$ is closed in $\lambda\mathcal{C}$ if $\emptyset \in \mathcal{C}$).

<u>Remark</u> : For the remainder of this chapter we shall assume that $\emptyset \notin \mathcal{C}$. In that case, if X has the indiscrete (trivial) topology, then $\lambda\mathcal{C}$ also has the indiscrete topology. Since the indiscrete topology is of limited interest, we shall assume that the underlying spaces X, Y, etc. have a non-trivial topology.

The following theorems deal with relationships between properties of X and properties of $\lambda\mathcal{C}$.

Theorem III.4 : If $\mathcal{C}$ contains all singleton subsets of X, then (i) X is compact <=> $\lambda\mathcal{C}$ is compact and (ii) X is sequentially compact <=> $\lambda\mathcal{C}$ is sequentially compact.

Proof : (i) Let $\{R_-(G_\alpha)\}_\alpha$ be a cover for $\lambda\mathcal{C}$ by subbasic open sets. Then $\{G_\alpha\}_\alpha$ covers X, since for every $x \in X$, $\{x\} \in R_-(G_\alpha)$ for some α. Because X is compact, $\{G_i\}_{i=1}^n$ covers X and $\{R_-(G_i)\}_{i=1}^n$ covers $\lambda\mathcal{C}$, since every element of $\mathcal{C}$ must meet at least one G_i. Conversely, let $\{G_\alpha\}_\alpha$ be an open cover for X. Since every $C \in \mathcal{C}$ meets at least one G_α, i.e. $C \in R_-(G_\alpha)$, $\{R_-(G_\alpha)\}_\alpha$ is an open cover for $\lambda\,\mathcal{C}$. Hence $\{R_-(G_i)\}_{i=1}^n$ is also a cover. Because $\{x\} \in \mathcal{C}, \forall\, x\in X$, every $\{x\} \in R_-(G_j)$ for some $j \in \{1,2,\ldots,n\}$, and hence $x \in G_j$, so that $\{G_i\}_{i=1}^n$ forms a finite subcover of $\{G_\alpha\}_\alpha$.

(ii) Suppose X is sequentially compact and let $A_1, A_2, \ldots$ be a sequence of elements of $\mathcal{C}$. Choose $a_i \in A_i$, $i=1,2,\ldots$. Because X is sequentially compact, there exists a convergent subsequence $a_{n1}, a_{n2}, \ldots$ converging to $p \in X$. By hypothesis $\{p\} \in \mathcal{C}$ and $\{A_{nj}\}_{j=1}^\infty$, converges to $\{p\}$, hence $\lambda\mathcal{C}$ is sequentially. Conversely, if $\{a_\alpha\}$ is a sequence in X, $\{\{a_\alpha\}\}$ is a sequence in $\mathcal{C}$, hence there is a subsequence $\{\{a_i\}\}_{i=1}^\infty$ converging to $C \in \mathcal{C}$. Clearly every point $p \in C$ is a limit point of this subsequence, so that X is sequentially compact.

<u>Proposition III.5</u> : Let X be a compact T_2 space, $\mathcal{B} \subseteq \lambda\mathcal{C}$. In general it is not true that if $\bigcup \mathcal{B}$ is compact then $\mathcal{B}$ is compact.

Proof : We prove this proposition by giving a counter-example. Let $X = [0,1] \times [0,1]$ with the usual relative topology. Let $B_n = \{ (x,y) \in X \mid 0 \leq y \leq \frac{1}{2} , x = \frac{1}{n} \}$, where $n = 1,2,\ldots.,$ and let $B_o = \{\{0\} \times [0,1] \}$ (Figure 1). Then if $\mathcal{B} = \{B_n\}_{n=o}^{\infty}$, $\bigcup \mathcal{B}$ is compact, whereas $\mathcal{B}$ is not compact

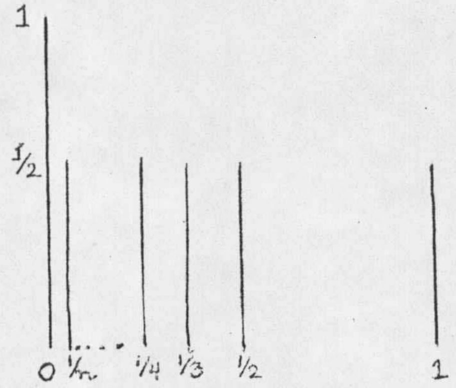

Figure 1

Example of a noncompact subspace of $\lambda\mathcal{C}$

<u>Proposition III .6</u> : If X is compact and T_2, and if $\mathcal{B}$ is a compact subset of $\lambda\mathcal{C}$, then it does not always follow that $\bigcup \mathcal{B}$ is compact in X

Proof (by counterexample): Let X be any closed bounded subset of the euclidean plane properly containing $[0,1] \times [0,1]$. Let $B_n = \{ (x,y) \in X \mid 0 \leq y \leq 1, \ x = \frac{1}{n} \}$, where