Lower semi-continuous multifunctions and properties of the l and k topology
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Abstract:
Let c be a collection of subsets of a topological space X. A binary relation R can be defined on X into c by (x,C) □ R if and only if x □ C. Define R+ , R- : P(X) -> P(c) by R+(A) = {C &isin c | C □ A} and R-(A) = {C □ c | C ∩ A ≠ □}. The smallest topology on c which makes R open (or closed) is called the λ (or κ) topology. We call λc (or κc) the space c with the λ (or κ) topology.

This thesis is divided into three main parts. The first (chapter III) examines properties of λc. For example if c is the family of nonvoid closed subsets of X, then λc compact ⇔ X compact, λc second countable ⇔ X second countable, λc first countable ⇒ X first countable, X and Y are homeomorphic ⇔ λcx and λcy are homeomorphic. These results are extended to more general families of subsets of X. If β □ λc is connected, and if each elements of β is connected in X, then □β is connected. Hence if c is a cover of X by connected sets, then X in connected if λc is.

The second part (chapter IV) deals with lower semi-continuous (l.s.c.) multifunctions. Let f^+ and f^- be Berge's upper inverses. If f has closed point values, then a single-valued function F:X → λc (c nonvoid closed subsets) can be defined which is continuous if and only if f is l.s.c. This simple results is used to obtain a homeomorphism of X to its graph regarded as embedded in λ(cx × cy) (where cx(or cy) are nonvoid closed subsets of X (or Y)). A characterization of l.s.c. in terms of accumulation points is given and conditions are examined under which f is l.s.c. Chapter V deals with properties of the κ topology. If c is the family of all open subsets of X, a fixed point theorem for continuous single-valued functions on κc into κc is presented.

LOWER SEMI-CONTINUOUS MULTIFUNCTIONS AND PROPERTIES OF THE λ AND κ TOPOLOGY

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ABSTRACT

Let $\mathcal{C}$ be a collection of subsets of a topological space X. A binary relation R can be defined on X into $\mathcal{C}$ by $(x,C) \in R$ if and only if $x \in C$. Define R_+, $R_- : P(X) \to P(\mathcal{C})$ by $R_+(A) = \{C \in \mathcal{C} \mid C \subseteq A\}$ and $R_-(A) = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\}$. The smallest topology on $\mathcal{C}$ which makes R open (or closed) is called the λ (or κ) topology. We call $\lambda\mathcal{C}$ (or $\kappa\mathcal{C}$) the space $\mathcal{C}$ with the λ (or κ) topology.

This thesis is divided into three main parts. The first (chapter III) examines properties of $\lambda\mathcal{C}$. For example if $\mathcal{C}$ is the family of nonvoid closed subsets of X, then $\lambda\mathcal{C}$ compact $\Leftrightarrow$ X compact, $\lambda\mathcal{C}$ second countable $\Leftrightarrow$ X second countable, $\lambda\mathcal{C}$ first countable $\Rightarrow$ X first countable, X and Y are homeomorphic $\Leftrightarrow$ $\lambda\mathcal{C}_X$ and $\lambda\mathcal{C}_Y$ are homeomorphic. These results are extended to more general families of subsets of X. If $\mathcal{B} \subseteq \lambda\mathcal{C}$ is connected, and if each elements of $\mathcal{B}$ is connected in X, then $\bigcup \mathcal{B}$ is connected. Hence if $\mathcal{C}$ is a cover of X by connected sets, then X in connected if $\lambda\mathcal{C}$ is.

The second part (chapter IV) deals with lower semi-continuous (l.s.c.) multifunctions. Let f^+ and f^- be Berge's upper inverses. If f has closed point values, then a single-valued function $F: X \to \lambda\mathcal{C}$ ($\mathcal{C}$ nonvoid closed subsets) can be defined which is continuous if and only if f is l.s.c. This simple results is used to obtain a homeomorphism of X to its graph regarded as embedded in $\lambda(\mathcal{C}_X \times \mathcal{C}_Y)$ (where $\mathcal{C}_X$ (or $\mathcal{C}_Y$) are nonvoid closed subsets of X (or Y)). A characterization of l.s.c. in terms of accumulation points is given and conditions are examined under which f is l.s.c.

Chapter V deals with properties of the κ topology. If $\mathcal{C}$ is the family of all open subsets of X, a fixed point theorem for continuous single-valued functions on $\kappa\mathcal{C}$ into $\kappa\mathcal{C}$ is presented.

CHAPTER I

INTRODUCTION

The study of hyperspaces and of multiple-valued functions (or multifunctions) has occupied topologists for over half a century. In 1914 Felix Hausdorff [5] initiated research on an area of mathematics which has expanded ever since. The problem which Hausdorff attacked dealt with the following ideas: If we let d be a metric on a space X, a "distance" between two subsets of X can be defined in various ways, e.g. if $A,B \subseteq X$, let

$$D_1(A,B) = \max \{d(a,b) \mid a \in A,\ b \in B\} \text{ or}$$

$$D_2(A,B) = \min \{d(a,b) \mid a \in A,\ b \in B\}.$$

These distances unfortunately do not form a metric for the space of subsets, and Hausdorff remedied this defect by letting

$$D(A,B) = \max \{\sup_{a \in A} D_2(\{a\},B),\ \sup_{b \in B} D_2(A,\{b\})\}.$$

This function applied to the family of all closed nonvoid subsets of X yields a metric space.

Not too many years after Hausdorff's work was published, the research of Leopold Vietoris [16] gave a new approach to this topic. Again starting with nonvoid, closed subsets, Vietoris obtained a topology on this family by defining a neighborhood of a closed set M to be the class of all closed subsets of X contained in the union of a given finite number of open subsets of X and intersecting each one of these open sets, provided M itself belongs to this class.

It can be shown that if $\mathcal{C}$ is the family of non-empty compact subsets of a bounded metric space, the topology induced on $\mathcal{C}$ by the Hausdorff metric above is the same as the mentioned Vietoris topology.

In 1950 Ernest Michael [10] presented what might justly be called the definitive work on the topic of topologies on spaces of subsets. The Vietoris (here called the "finite") topology is also applied to the topic of multifunctions. Of course, other topologists had already published research on hyperspaces, e.g Kelley [6], Kuratowski [7], etc. Part of the interest of Michael's work, however, lies in the fact that he enlarged the family of subsets from nonvoid closed subsets to the family of all subsets of X. Furthermore continuity and uniform continuity of mappings to these hyperspaces was considered.

C. Kuratowski [8] broadened the subject area, devoting considerable space to the fundamental properties of semi-continuous mappings. These mappings, usually considered under the heading of upper or lower semi-continuity, are correspondences f between the respective power sets of two topological spaces X and Y, i.e. $f:P(X) \to P(Y)$, where an upper semi-continuous map takes a closed subset of Y to a closed subset of X, whereas a lower semi-continuous map assigns to an open subset of Y an open subset of X. The two concepts coincide whenever f is a single-valued function. Kuratowski called the space of non-empty closed subsets with the Vietoris topology the exponential topology 2^X. He also drew attention to the κ and λ topologies, where the κ-topology on a hyperspace is generated by sets of the form $\{ A \subseteq G \mid G$ is an open subset of X, A is an element of the hyperspace$\}$. (The λ topology is defined in chapter II).

The κ topology had been applied to closed nonvoid subsets of compact T_2 spaces X by V.I. Ponomarev [12]. The restrictions placed on X produce several interesting results. Closely tied to κ-spaces are the already defined upper semi-continuous functions. The present paper will, among other projects, generalize the κ-topology to families of subsets not necessarily closed.

A compendium on continuous multivalued functions was published by W.L Strother [14]. His notation unfortunately does not adhere to the more popular terminology already mentioned.

The notation which will be used in this paper is partially based on that introduced by Claude Berge [1]. Similar symbolism is used by E. Čech [2] in his encyclopedic volume on topological spaces.

In 1965 G.T. Whyburn gave investigations on the topic of continuity of multifunctions if not a new direction, then at least a renewed impetus. The summary of conditions on spaces X and Y for which $f:X \rightarrow Y$ is upper semi-continuous is probably the most helpful single page to any student of this fascinating topic. One should point out again, that upper semi-continuity is stressed here. R.E. Smithson [13], on the other hand, gave a characterization of lower semi-continuity analogous to that of upper semi-continuity given by Whyburn.

The research on hyperspaces and multifunctions continues. For the most part, however, the emphasis is on spaces on non-empty closed subsets with the Vietoris topology and on upper semi-continuous functions. The most readily adaptable and usable topological concepts lie in this area; yet the problems of lower semi-continuity also need to be solved. Whereas Ponomarev has worked out many details of the κ-topology, the author will endeavor to answer some questions on the λ-topology.

CHAPTER II

BINARY RELATIONS, MULTIFUNCTIONS AND THE λ - TOPOLOGY.

If X and Y are two sets, the set $X \times Y$ of all ordered pairs (x,y) with $x \in X$, $y \in Y$ is called the cartesian product of X with Y. Any subset R of $X \times Y$ is said to be a (binary) relation in X into Y. If R is a relation, we let $\hat{R}$ denote the set of all ordered pairs (y,x) such that $(x,y) \in R$. Thus $\hat{R}$ is a relation in Y into X.

The set of all first elements of R is called the domain, the set of second elements the range of R (Note that the domain of $\hat{R}$ is the range of R). If the domain of R is X we say R is on X, and R is onto Y provided the range of R is Y.

A relation R in X into Y is said to be single-valued provided that if $(x,y) \in R$ and $(x,z) \in R$ the $y=z$. Single-valued relations are generally conceded to be adequate set-theoretic models of the intuitive notion of a function. Hence by a function on X into Y we shall mean a single-valued binary relation on X into Y.

Associated with each relation R on X into Y is a function f (called the related function) on X into P(Y), the family of all subsets of Y, uniquely determined by the formula

$$f(x) = \{y \in Y \mid (x,y) \in R\}.$$

Furthermore, given a function f on X into P(Y), a relation R on X into Y is uniquely determined by the rule

$$(x,y) \in R \Leftrightarrow y \in f(x).$$

It is clear that f and R uniquely determine each other; from now on we shall call this related function a multifunction on X into Y. Also, since most results obtained will use a relation R on X onto the range of R, we shall assume (unless specifically mentioned) that R is on X onto Y. (Abusing the language somewhat, we shall also say that f is on X onto Y).

The multifunction $f: X \to Y$ gives rise to several other functions mapping either $P(X)$ to $P(Y)$ or $P(Y)$ to $P(X)$. Let $A \subseteq X$. We let

$$f_-(A) = \{y \in Y \mid \exists\, x \in A \text{ with } y \in f(x)\}.$$

It can easily be shown that

$$f_-(A) = \bigcup_{x \in A} f(x).$$

Let: $f_+(A) = Y - f_-(X-A)$.

(This notation is essentially due to Berge [1]).

If we let $f^+ = (\hat{f})_+$ and $f^- = (\hat{f})_-$

(where $\hat{f}$ is the multifunction related to $\hat{R}$), then

$$f^+, f^-: P(Y) \to P(X).$$

Clearly we have, for $B \subseteq Y$,

$$f^-(B) = \{x \in X \mid f(x) \cap B \neq \emptyset\} \text{ and}$$

$$f^+(B) = \{x \in X \mid f(x) \subseteq B\}.$$

If R is the binary relation associated with the multifunction f, we also denote f^+ by R^+, f^- by R^-, f_- by R_- and f_+ by R_+. The following properties of a multifunction f on X onto Y are easily checked:

1.) If $\mathcal{A} \subseteq P(X)$ then

$$f_-(\bigcup \mathcal{A}) = \bigcup \{f_-(A) \mid A \in \mathcal{A}\}$$

$$f_-(\bigcap \mathcal{A}) \subseteq \bigcap \{f_-(A) \mid A \in \mathcal{A}\}$$

$$f_+(\bigcup \mathcal{A}) \supseteq \bigcup \{f_+(A) \mid A \in \mathcal{A}\}$$

$$f_+(\bigcap \mathcal{A}) = \bigcap \{f_+(A) \mid A \in \mathcal{A}\}.$$

2.) If $A \in P(X), B \in P(X)$ such that $A \subseteq B$ then

$f_+(A) \subseteq f_+(B)$ and

$f_-(A) \subseteq f_-(B)$.

3.) For any $A \in P(X)$

$f_+(A) \subseteq f_-(A)$.

4.) If f is single-valued then

$f^+ = f^-$

5.) If $B \subseteq Y$, $A \subseteq X$,

$$f_-(f^-(B)) \supseteq B \qquad f^-(f_-(A)) \supseteq A$$

$$f_-(f^+(B)) \subseteq B \qquad f^-(f_+(A)) \subseteq A$$

$$f_+(f^-(B)) \supseteq B \qquad f^+(f_-(A)) \supseteq A$$

$$f_+(f^+(B)) \subseteq B \qquad f^+(f_+(A)) \subseteq A$$

6.) If $\mathcal{B} \subseteq P(Y)$

$$f^-(\bigcup \mathcal{B}) = \bigcup \{f^-(B) \mid B \in \mathcal{B}\}$$

$$f^-(\bigcap \mathcal{B}) \subseteq \bigcap \{f^-(B) \mid B \in \mathcal{B}\}$$

$$f^+(\bigcup \mathcal{B}) \supseteq \bigcup \{f^+(B) \mid B \in \mathcal{B}\}$$

$$f^+(\bigcap \mathcal{B}) = \bigcap \{f^+(B) \mid B \in \mathcal{B}\}$$

7.) If $B \in P(Y)$, and $B' \in P(Y)$ such that $B \subseteq B'$ then

$f^-(B) \subseteq f^-(B')$ and

$f^+(B) \subseteq f^+(B')$.

8.) If $B \in P(Y)$

$f^+(B) \subseteq f^-(B)$

These properties will be used in proofs without specific reference.

Assume now that $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ are topological spaces. We let $\mathrm{co}\mathcal{T}$ (resp. $\mathrm{co}\mathcal{S}$)denote the collection of closed sets of X (resp. Y). We shall say that the multifunction $f:X \to Y$ is <u>upper semi-continuous</u> (u.s.c.) if and only if for $B \in \mathrm{co}\mathcal{S}$, $f^-(B) \in \mathrm{co}\mathcal{T}$. f is said to be lower <u>semi-continuous</u> (l.s.c.) provided that for $C \in \mathcal{S}$, $f^-(C) \in \mathcal{T}$. It follows at once from the definitions of f^+ and f^- that f is u.s.c. iff for $C \in \mathcal{S}$, $f^+(C) \in \mathcal{T}$ and that f is l.s.c. iff f for $B \in \mathrm{co}\mathcal{S}$, $f^+(B) \in \mathrm{co}\mathcal{T}$. f is said to be open (or closed) provided that for $A \in \mathcal{T}$ (or $A \in \mathrm{co}\mathcal{T}$), $f_-(A) \in \mathcal{S}$ (or $f_-(A) \in \mathrm{co}\mathcal{S}$). It follows again from the definition that f is open iff $f_+(A) \in \mathrm{co}\mathcal{S}$ for $A \in \mathrm{co}\mathcal{T}$ and f is closed iff $f_+(A) \in \mathcal{S}$ for $A \in \mathcal{T}$.

The following elementary results are immediate.

(i) If $f:X \to Y$, then $\hat{\hat{f}}=f$.

(ii) If $f:X \to Y$ and $g:Y \to Z$ are multifunctions, then $\widehat{gf} = \hat{f}\hat{g}$.

(iii) $f:X \to Y$ is u.s.c. (l.s.c.) iff $\hat{f}$ is closed (open) (See also Čech [2]).

Let $\mathcal{C}$ be a family of subsets of the space X. We can define a binary relation R on X to $\mathcal{C}$ as follows: $(x,C) \in R$ iff $x \in C$, where $C \in \mathcal{C}$. For $A \subseteq X$, R_+ and $R_-:P(X) \to P(\mathcal{C})$

satisfy $R_+(A) = \{C \in \mathcal{C} \mid C \subseteq A\}$, $R_-(A) = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\}$. R^- and R^+ are mappings of $P(\mathcal{C})$ to $P(X)$, where for $\mathcal{A} \subseteq \mathcal{C}$,

$R^-(\mathcal{A}) = \cup\, \mathcal{A}$ and

$R^+(\mathcal{A}) = X - R^-(\mathcal{C} - \mathcal{A}) = \{x \in X \mid \text{if } x \in C, C \in \mathcal{C}, \text{then } C \in \mathcal{A}\}$.

Using these multifunctions, we let $\lambda\,\mathcal{C}$ denote the space $\mathcal{C}$ with the following topology: The open sets are generated by sets of the form $R_-(G)$ for G open in X. With this convention, no distinction will be made between $\mathcal{C}$ and $\lambda\mathcal{C}$ unless ambiguities might occur. R is closed if for any K closed in X, $R_-(K)$ is closed in $\lambda\mathcal{C}$. As above, for K closed in X, $R_-(K)$ is closed iff $R_+(X-K)$ is open. For convenience's sake, we shall denote by λX the space of all nonvoid closed subsets of X and by $\lambda\mathcal{P}$ the space of all nonvoid subsets of X, with the λ topology (see Kuratowski [8]).

Since R_- does not preserve intersections (see 1. above), the sets $R_-(G)$, $G \subseteq X$, G open, do not form a basis for the hyperspace $\lambda\mathcal{C}$ but merely a subbasis. A basis element of $\lambda\mathcal{C}$ thus has the form $\bigcap_{i=1}^{n} R_-(G_i)$, G_i open in X. It is now clear that the λ topology is the smallest topology for which R is open. Also R is u.s.c. provided $R^-(\mathcal{A})$ is open in X, where $\mathcal{A}$ is open in $\lambda\mathcal{C}$. It is of interest to note that R is single-valued iff the elements of $\mathcal{C}$ are mutually disjoint.

Using $\mathcal{C}$ and R as before, another topology may be assigned to $\mathcal{C}$, called the κ-topology (Ponomarev [12]). It is the smallest topology for $\mathcal{C}$ for which R is closed, i.e. such that whenever M is closed in X, $R_-(M)$ is closed. Sets of the form $R_+(G)$, G open in X, form not merely a subbasis but actually a basis for the hyperspace $\kappa\mathcal{C}$. The notation κX and $\kappa\mathcal{P}$ follows that of the λ-topology.

Before considering properties of the κ and λ topologies; a few facts about binary relations R in X into $\mathcal{C} \subseteq P(X)$ are worth mentioning.

(i) $\forall A \subseteq X$, $R_-(A) = \emptyset \Leftrightarrow A \cap \text{domain } R = \emptyset$.

(ii) $\forall A \subseteq X$, $R_+(A) \supseteq \mathcal{C} - \text{range } R$.

(iii) If $\mathcal{A} \subseteq P(X)$, then $\emptyset \in \mathcal{A} \Rightarrow \emptyset \in R_-[\mathcal{A}]$.

This is true for R_+ iff R is onto $\mathcal{C}$.

(iv) R is on X $\Leftrightarrow$ $(\forall\, \mathcal{A} \subseteq P(X),\ \emptyset \notin \mathcal{A} \Rightarrow \emptyset \notin R_-[\mathcal{A}])$.

(v) If $\hat{R}$ is single-valued and R is on X, then

$$\forall\, \mathcal{A} \subseteq P(X),\ \emptyset \notin \mathcal{A} \Rightarrow \emptyset \notin R_+[\mathcal{A}].$$

(vi) If R is on X and $\mathcal{A} \subseteq P(X)$ has the finite intersection property (f.i.p.) then $R_-[\mathcal{A}]$ has the f.i.p.

(vii) $\hat{R}$ single-valued and R on X $\Rightarrow$ R_+ preserves the f.i.p.

(viii) R single-valued $\Rightarrow$ R_- preserves the basis property.

Proof: (v) Assume $\emptyset \notin \mathcal{A}$. If $R_+(A) = \emptyset$ for some $A \in \mathcal{A}$, then $\mathcal{C} = R_-(X-A)$ since $R_+(A) = \mathcal{C} - R_-(X-A)$. Since $A \neq \emptyset$ $\exists\, x \in A$ such that $R_-(\{x\}) \neq \emptyset$. Since R is on X $\exists\, C \in \mathcal{C}$ such that $x \in C$. But $C \in R_-(X-A)$, so C meets X-A. Assume $x' \in C \cap (X-A)$. $C \in R_-(\{x\}) \cap R_-(\{x'\})$ but, because $\hat{R}$ is single-valued,

$R_-(\{x\}) \cap R_-(\{x'\}) = \emptyset$, a contradiction.

(vi) Let $R_-(A_i) \in R_-[\mathcal{a}], i=1,2,\ldots,n$. Assume $\bigcap_{i=1}^{n} R_-(A_i) = \emptyset$, i.e. $\nexists\ C \in \mathcal{C}$ with C meeting each A_i. However, since $\mathcal{a}$ has the f.i.p., $\bigcap_{i=1}^{n} A_i \neq \emptyset$, a contradiction.

(viii) Assume $\mathcal{C} \subseteq P(X)$ has the basis property. Let $C \in \mathcal{C}$ with $C \in R_-(A) \cap R_-(B)$. (Without loss of generality let $C \neq \emptyset$). Because $\hat{R}$ is single-valued $A \cap B \neq \emptyset$ and $\exists\ x \in C$ such that $x \in A \cap B$. Hence $\exists\ D \in \mathcal{a}$ such that $x \in D \subseteq A \cap B$, and thus $C \in R_-(D) \subseteq R_-(A) \cap R_-(B)$.

CHAPTER III

PROPERTIES OF THE λ TOPOLOGY

Let X be a topological space and let $\mathcal{C}$ be a family of subsets of X.

Proposition III .1 : If $\mathcal{C}$ includes all singleton subsets of X, then X can be embedded as a subspace of $\lambda\mathcal{C}$.

Proof: Use the map $x \rightarrow \{x\}$.

Corollary III 1. If X is T_1, then X can be embedded as a subspace of λX.

Proof: In a T_1 space $\{x\} = \text{cl}\,\{x\}$

Proposition III .2 : If a T_1 space X has more than one point, then λX is T_0 but not T_1.

Before proceeding with the proof, we list the following lemmas.

Lemma III.2.1 : If C and D belong to λX and if $C \in \text{cl}\,\{D\}$, then $C \cap D \neq \emptyset$.

Proof of the lemma: If C does not meet D, then the open set X - D contains C but does not meet D. So $C \notin \text{cl}\,\{D\}$. The converse does not hold, since if $D \subseteq C$, then X - D is open and C meets X - D, but D does not.

Lemma III.2.2 : If $C \subseteq D$, then $C \in \text{cl}\,\{D\}$.

Note: If $\mathcal{A} \subseteq \mathcal{C}$, $\text{cl}\,\mathcal{A} = \{C \in \mathcal{C} \mid$ each open subset G of X that meets C also meets $\bigcup \mathcal{A}$ }. Thus for $\mathcal{A} = \{A\}$, $\mathrm{cl}\{A\} = \{C \in \mathcal{C} \mid C \cap G \neq \emptyset,\ G$ open in $X, \Rightarrow A \cap G \neq \emptyset\}$.

Proof of proposition III.2 : Let A and B be elements of λX and assume there is a point in A not in B. Then $\mathrm{cl}\{A\} \neq \mathrm{cl}\{B\}$, since $X - B$ is an open set with $A \cap (X-B) \neq \emptyset$. Hence λX is T_0. Let $A \in \lambda X$, $A \neq X$. Then $A \in \mathrm{cl}\{X\}$ (lemma III. 2.2), but $A \notin \{X\}$, so that λX is not T_1.

In general we shall use collections $\mathcal{C}$ which do not include $\emptyset$. In case $\mathcal{C}$ does include $\emptyset$, we have the following

<u>Proposition III.3</u> : Let $\emptyset \in \mathcal{C}$, then $\lambda\mathcal{C}$ is compact and connected.

Proof: Let $\mathcal{K}$ be a family of closed subsets of $\lambda\mathcal{C}$ with f.i.p. Since $\emptyset$ belongs to every closed set in $\lambda\mathcal{C}$, $\emptyset \in \bigcap \mathcal{K}$, so that $\lambda\mathcal{C}$ is compact. If K is closed in X, then $\emptyset \in R_+(K) \subseteq \lambda\mathcal{C}$. Hence $\lambda\mathcal{C}$ is connected. (It is of interest to note that $\{\emptyset\}$ is closed in $\lambda\mathcal{C}$ if $\emptyset \in \mathcal{C}$).

<u>Remark</u> : For the remainder of this chapter we shall assume that $\emptyset \notin \mathcal{C}$. In that case, if X has the indiscrete (trivial) topology, then $\lambda\mathcal{C}$ also has the indiscrete topology. Since the indiscrete topology is of limited interest, we shall assume that the underlying spaces X, Y, etc. have a non-trivial topology.

The following theorems deal with relationships between properties of X and properties of $\lambda\mathcal{C}$.

Theorem III.4 : If $\mathcal{C}$ contains all singleton subsets of X, then (i) X is compact <=> $\lambda\mathcal{C}$ is compact and (ii) X is sequentially compact <=> $\lambda\mathcal{C}$ is sequentially compact.

Proof : (i) Let $\{R_-(G_\alpha)\}_\alpha$ be a cover for $\lambda\mathcal{C}$ by subbasic open sets. Then $\{G_\alpha\}_\alpha$ covers X, since for every $x \in X$, $\{x\} \in R_-(G_\alpha)$ for some α. Because X is compact, $\{G_i\}_{i=1}^n$ covers X and $\{R_-(G_i)\}_{i=1}^n$ covers $\lambda\mathcal{C}$, since every element of $\mathcal{C}$ must meet at least one G_i. Conversely, let $\{G_\alpha\}_\alpha$ be an open cover for X. Since every $C \in \mathcal{C}$ meets at least one G_α, i.e. $C \in R_-(G_\alpha)$, $\{R_-(G_\alpha)\}_\alpha$ is an open cover for $\lambda\mathcal{C}$. Hence $\{R_-(G_i)\}_{i=1}^n$ is also a cover. Because $\{x\} \in \mathcal{C}, \forall x \in X$, every $\{x\} \in R_-(G_j)$ for some $j \in \{1,2,\dots,n\}$, and hence $x \in G_j$, so that $\{G_i\}_{i=1}^n$ forms a finite subcover of $\{G_\alpha\}_\alpha$.

(ii) Suppose X is sequentially compact and let $A_1, A_2, \dots$ be a sequence of elements of $\mathcal{C}$. Choose $a_i \in A_i$, $i=1,2,\dots$. Because X is sequentially compact, there exists a convergent subsequence $a_{n1}, a_{n2}, \dots$ converging to $p \in X$. By hypothesis $\{p\} \in \mathcal{C}$ and $\{A_{nj}\}_{j=1}^\infty$, converges to $\{p\}$, hence $\lambda\mathcal{C}$ is sequentially. Conversely, if $\{a_\alpha\}$ is a sequence in X, $\{\{a_\alpha\}\}$ is a sequence in $\mathcal{C}$, hence there is a subsequence $\{\{a_i\}\}_{i=1}^\infty$ converging to $C \in \mathcal{C}$. Clearly every point $p \in C$ is a limit point of this subsequence, so that X is sequentially compact.

Proposition III.5 : Let X be a compact T_2 space, $\mathcal{B} \subseteq \lambda\mathcal{C}$. In general it is not true that if $\bigcup \mathcal{B}$ is compact then $\mathcal{B}$ is compact.

Proof : We prove this proposition by giving a counterexample. Let $X = [0,1] \times [0,1]$ with the usual relative topology. Let $B_n = \{ (x,y) \in X \mid 0 \leq y \leq \frac{1}{2} , x = \frac{1}{n} \}$, where $n = 1,2,\ldots.$, and let $B_o = \{\{0\} \times [0,1] \}$ (Figure 1). Then if $\mathcal{B} = \{B_n\}_{n=o}^{\infty}$, $\bigcup \mathcal{B}$ is compact, whereas $\mathcal{B}$ is not compact

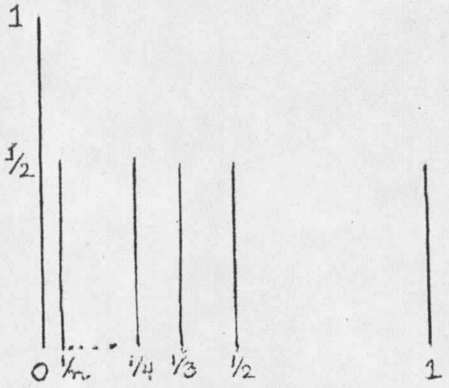

Figure 1

Example of a noncompact subspace of $\lambda\mathcal{C}$

Proposition III .6 : If X is compact and T_2, and if $\mathcal{B}$ is a compact subset of $\lambda\mathcal{C}$, then it does not always follow that $\bigcup \mathcal{B}$ is compact in X

Proof (by counterexample): Let X be any closed bounded subset of the euclidean plane properly containing $[0,1] \times [0,1]$. Let $B_n = \{ (x,y) \in X \mid 0 \leq y \leq 1,\ x = \frac{1}{n} \}$, where

$n = 1,2,3,\ldots,$ and let $B_o = \{(0, \frac{1}{2})\}$. Then $\mathcal{B} = \{B_n\}_{n=o}^{\infty}$ is compact but $\cup\mathcal{B}$ is not.

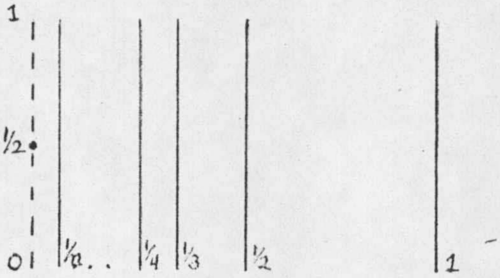

Figure 2

Example for Proposition III.6

Note: Clearly $\mathcal{B}$ is compact in $\lambda\mathcal{C}$ if and only if for every collection $\mathcal{E}$ of open sets of X such that $\cup\mathcal{E}$ meets each element of $\mathcal{B}$, there is a finite subcollection $\mathcal{E}'$ of $\mathcal{E}$ such that $\cup\mathcal{E}'$ meets every element of $\mathcal{B}$.

<u>Theorem III.7</u>: If $\mathcal{C}$ contains all singletons, then X is second countable if and only if $\lambda\mathcal{C}$ is second countable.

Proof : Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a countable base for X. Let $\mathcal{B}$ be the family of all finite intersections of sets of the form $R_-(E_i)$ with $E_i \in \mathcal{E}$. Then $\mathcal{B}$ is a countable base for $\lambda\mathcal{C}$. Conversely if $\lambda\mathcal{C}$ has a countable base, then there is also a countable base of the form $R_-(E_1^j) \cap \ldots \cap R_-(E_{n_j}^j)$, $j=1,2,\ldots$. Let O be open in X, $x \in O$. Then $\{x\} \in R_-(O)$ and $\{x\} \in R_-(E_1^j) \cap \ldots \cap R_-(E_n^j) \subseteq R_-(O)$, for some j. Thus $x \in E_1^j \cap \ldots \cap E_n^j \subseteq O$. Hence the E_i^j together with all their finite intersections form a countable base for X.

Theorem III. 8 : Let X be first countable, $\mathcal{C}$ a family of countable subsets of X. Then $\lambda\mathcal{C}$ is first countable

Proof : Let $A \in \mathcal{C}$, $A = \{x_1, x_2, \dots\}$. For each x_i, let $\{O_{ij}\}, j=1,2,\dots,$ be a countable point base. Form $\{R_-(O_{ij}) \mid i=1,2,\dots;\ j=1,2,\dots\}$. Let $A \in R_-(G)$, G open. Then for some k, $x_k \in A \cap G$, hence there is a t such that $O_{kt} \subseteq G$ and $x_k \in O_{kt}$. Then $A \in R_-(O_{kt}) \subseteq R_-(G)$. Thus finite intersections of the $R_-(O_{ij})$ form the required local base.

Theorem III. 9 : Let X be a metric space, $\mathcal{C}$ a family of Lindelöf subsets. Then $\lambda\mathcal{C}$ is first countable.

Proof : Let $A \in \mathcal{C}$. We wish to exhibit a countable neighborhood base for the neighborhood system of A. For each $x \in X$ we form $N_{\frac{1}{i}}(x)$, $i=1,2,\dots$. Because A is Lindelöf , $\mathcal{D} = \{N_{\frac{1}{i}}(x^{i}_{k}) \mid k=1,2,\dots;\ i=1,2,\dots\}$, with $x^{i}_{k} \in A$, forms a cover for A. We shall show that $R_-[\mathcal{D}]$ is a neighborhood basis for A: For let $\mathcal{H}$ be a neighborhood of A. Then there are open sets $H_1, H_2, \dots, H_n$ of X such that $A \in \bigcap_{i=1}^{n} R_-(H_i) \subseteq \mathcal{H}$. Let H be any one of the H_i. Choose $x \in H \cap A$ such that $\inf\limits_{y \in G \cap A} d(x,y) = \rho > 0$. There exists an integer n such that $\frac{1}{n} < \frac{\rho}{2}$. Since $\{N_{\frac{1}{n}}(x^{n}_{k}) \mid k=1,2,\dots\}$ covers A, $\exists\, x^{n}_{k}$ such that $N_{\frac{1}{n}}(x^{n}_{k}) \subseteq H$. Hence $R_-(N_{\frac{1}{n}}$

$(x_{n_k})) \subseteq R_-(H)$ and the theorem follows.

Corollary III. 9 : Let X be a metric space, $\mathcal{C}$ a family of compact subsets. Then $\lambda\mathcal{C}$ is first countable.

Theorem III. 10 : If $\mathcal{C}$ contains all singleton subsets of X and if $\lambda\mathcal{C}$ is first countable, then X is first countable.

Proof : Let $x \in X$ and let N be any open neighborhood of x. Clearly $R_-(N)$ is an open neighborhood of $\{x\}$. Because $\lambda\mathcal{C}$ is first countable, there exist open subsets $N_1, \ldots, N_t$ of X such that $\{x\} \in \bigcap_{i=1}^{t} R_-(N_i) \subseteq R_-(N)$. The sets N_i together with all finite intersections thus form a countable neighborhood basis.

Proposition III. 11 : (i) If $X \in \mathcal{C}$, then $\lambda\mathcal{C}$ is separable.

(ii) λX is always separable.

(iii) $\lambda\mathcal{C}$ is separable $\not\Rightarrow$ X is separable

Proof : (i) $X \in R_-(G)$ for any open set G of X, hence $\{X\}$ is dense in $\lambda\mathcal{C}$.

(ii) Follows from (i).

(iii) Let X be a nonseparable space and assume $X \in \mathcal{C}$. Apply (i).

Proposition III. 12 : The family of all finite subsets of a T_1 space X is dense in λX.

Proof : The proof is similar to that for the finite topology (see [10]).

In the next two theorems we give a characterization of λX in terms of the topology of X.

Theorem III. 13 : If X is a T_1 space, the elements $\{x\}$, $x \in X$, are the only points of λX without limit points in λX.

Proof : Let $A \in \lambda X$, $A \neq \{x\}$. Then $A \in R_-(X - \{x\})$ and $\{x\} \notin R_-(X - \{x\})$, hence A is not a limit point of $\{\{x\}\}$ (see lemma III. 2.2). If $x \in B \in \lambda X$ and B is not a singleton, then $\{x\}$ is a limit point of $\{B\}$. This is also true for any $\lambda\mathcal{C}$ where $\mathcal{C}$ includes all the singleton subsets of X.

Theorem III. 14 : Let X and Y be T_1 spaces. Then X is homeomorphic to Y if and only if λX is homeomorphic to λY.

Proof : If X is homeomorphic to Y, the result follows immediately. Suppose that λX is homeomorphic to λY. Since the property of having (or not having) a limit point is preserved under homeomorphisms, the singletons $\{x\}$, $x \in X$, must map to singletons $\{y\}$, $y \in Y$, so that X and Y are homeomorphic.

The following is an example of a λ space. Let X be the segment $[0,1]$, and let $\mathcal{C}$ be the family of all nonvoid, closed connected subsets of X. With the interval $C = [a,b] \in \mathcal{C}$, we associate the point $\left(\frac{a+b}{2}, \frac{b-a}{2}\sqrt{3}\right)$. This association is a one-to-one correspondence of $\mathcal{C}$ onto the equilateral triangle T with vertices (0,0) , (1,0) and ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$). We shall topologize T so as to make it homeomorphic to $\lambda \mathcal{C}$. Let G be an open connected subset of X. R_{-} (G) = $\{ C \in \mathcal{C} \mid$ C meets G$\}$, and each $C \in R_{-}$ (G) uniquely determines a point of T. The open set corresponding to R_{-} (G) is indicated in figure 3. (shaded area, heavy lines included).

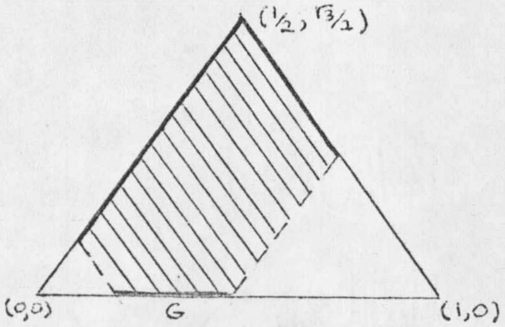

Figure 3

Subbasic open set of $\lambda \mathcal{C}$ for subcontinuum G of [0,1]

Typical basic open and closed subset of $\lambda \mathcal{C}$ are shown in figures 4 and 5.

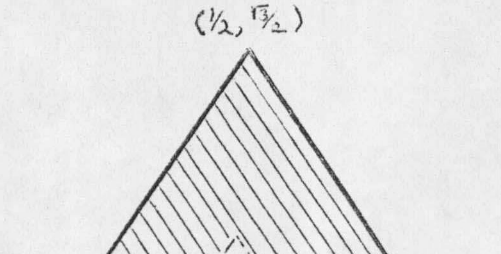

Figure 4

Basic open set of $\lambda \mathcal{C}$

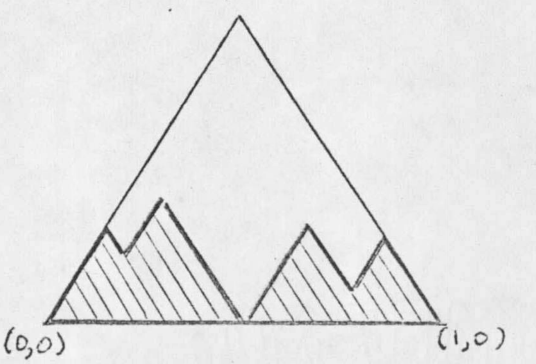

Figure 5

Basic closed sets of λC

λC is neither regular nor normal, since X lies in every open subset of λC. However deleting X, or even an entire neighborhood of X, will not make X regular (or normal). For let $P \in \lambda C$, $\mathfrak{F}$ a closed subset of λC such that $P \notin \mathfrak{F}$ (see figure 6), then every open set containing $\mathfrak{F}$ also concontains P.

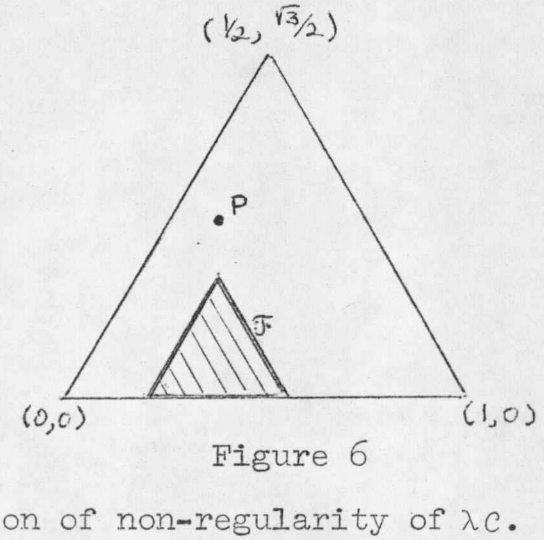

Figure 6

Illustration of non-regularity of λC.

We shall next study the question of connectedness of $\lambda\mathcal{C}$.

<u>Proposition III. 15</u> : Let $\mathcal{B} \subseteq \lambda\mathcal{C}$ be such that $\mathcal{B}$ is connected in $\lambda\mathcal{C}$ and let each element of $\mathcal{B}$ be connected in X. Then $R^-(\mathcal{B}) = \bigcup \mathcal{B}$ is connected.

Proof : Suppose $\bigcup \mathcal{B}$ is not connected, i.e. there exist open subsets H' and K' of X such that $\bigcup \mathcal{B} = H \cup K$, where $H = H' \cap \bigcup \mathcal{B}$, $K = K' \cap \bigcup \mathcal{B}$, and $H \cap K = \emptyset$. Then $\mathcal{H} = R_-(H') \cap \mathcal{B}$ and $\mathcal{K} = R_-(K') \cap \mathcal{B}$ are relatively open subsets of $\mathcal{B}$ and $\mathcal{B} = \mathcal{H} \cup \mathcal{K}$, with $\mathcal{H} \cap \mathcal{K} = \emptyset$, a contradiction.

The following example shows that if not all the elements of $\mathcal{B}$ are connected, the conclusion of the theorem does not hold. Let $X = \{a,b\}$, let the topology $\mathcal{L}$ on X be $\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, and let $\mathcal{C} = \mathcal{L} - \{\emptyset\}$. The open sets of $\lambda\mathcal{C}$ are $\{\{a\}, \{a,b\}\}$; $\{\{b\}, \{a,b\}\}$; $\mathcal{L}$; $\{\{a,b\}\}$. Clearly $\mathcal{C}$ is connected, but $R^-(\mathcal{C}) = \bigcup \mathcal{C} = X$ is not.

<u>Corollary 15</u> : If $\mathcal{A} \subseteq \mathcal{C}$ is a covering of X by connected sets and if $\mathcal{A}$ is connected in $\lambda\mathcal{C}$, then X is connected.

<u>Proposition III. 16</u> : Let $\mathcal{C}$ be any collection of subsets of X <u>including</u> X. Then $\lambda\mathcal{C}$ is connected.

Proof : X belongs to every open set in $\lambda\mathcal{C}$. Thus $\lambda\mathcal{C}$ is not the union of two disjoint nonempty open sets.

Suppose that $X \in \lambda\mathcal{C}$ and that $\lambda\mathcal{C} - \{X\}$ is disconnected. Then by III. 16 X is a cutpoint of $\lambda\mathcal{C}$ and hence $\{X\}$ must be either open or closed (see [9]). Unless $\mathcal{C}$ consists of only one element. $\mathrm{cl}\{X\} \neq \{X\}$ so that $\{X\}$ must be open. We thus have proven the following

<u>Proposition III. 17</u> : If $X \in \mathcal{C}$ and $\lambda\mathcal{C} - \{X\}$ is disconnected, then $\{X\}$ is open in $\lambda\mathcal{C}$.

Under the hypotheses of III. 17, $\{X\}$ contains some subbasic open set $R_-(G)$ of $\lambda\mathcal{C}$, and $R_-(G)$ does not meet $\lambda\mathcal{C} - \{X\}$. Hence $G \cap \bigcup (\mathcal{C} - \{X\}) = \emptyset$, i.e. $\mathrm{int}\,(X - \bigcup(\mathcal{C} - \{X\})) \neq \emptyset$. But this means that $\bigcup (\mathcal{C} - \{X\})$ is not dense in X, so that $\mathcal{C} - \{X\}$ cannot cover X.

<u>Proposition III. 18</u> : Let $X = A \cup B$, where A and B are nonvoid open subsets of X and $A \cap B = \emptyset$. Suppose $\mathcal{C}$ is a collection of subsets of X including X itself, and that all elements other than X of $\mathcal{C}$ are connected. Then $\{X\}$ is a cutpoint of $\lambda\mathcal{C}$.

Proof : Since every element of $\mathcal{C}$ (other than X) lies in either A or in B, we have immediately that $\lambda\mathcal{C} - \{X\} = R_-(A) \cup R_-(B)$. Clearly neither $R_-(A)$ nor $R_-(B)$ are empty. Because every element of $\mathcal{C} - \{X\}$ is connected, no C can meet both A and B, so that $R_-(A) \cap R_-(B) = \emptyset$.

<u>Proposition III. 19</u> If $\mathcal{P}$ is the family of all nonvoid subset of X, $\mathcal{C}$ the family of all nonvoid

closed subsets, and if $F : \lambda\mathcal{P} \to \lambda\mathcal{C}$ is defined by $f(A) = \mathrm{cl}\, A$, then f is a retraction.

Proof : The only difficulty lies in showing that f is continuous. Let R be the relation on X to $\mathcal{C}$ and R' the relation on X to $\mathcal{P}$ (see page 8). Let $R_-(G)$ be a subbasic open neighborhood of $\mathrm{cl}\, A$ in $\lambda\mathcal{C}$. Since G is open in X, $G \cap \mathrm{cl}\, A \neq \emptyset$ implies $G \cap A \neq \emptyset$. Hence $R'_-(G)$ is a neighborhood of A in $\lambda\mathcal{P}$. Also $f(R'_-(G)) \subseteq R_-(G)$, for if $K \in R'_-(G)$, then K meets G, so that $f(K) = K$ also meets G and $f(K) \in R_-(G)$.

In chapter I it was pointed out that the Hausdorff metric generates the finite topology on the family of all compact, closed nonvoid subsets of X. Let d be a metric on X. For subsets A and B of X we define $D(A,B) = \inf \{d(a,b) \mid a \in A,\ b \in B\}$. D is only a "generalized metric" in the sense of Alexandrof & Hopf (the triangle inequality does not hold and X is at zero distance from every nonempty subset of X). We further define an ϵ neighborhood of $A \subseteq X$ to be $\mathfrak{N}_\epsilon(A) = \{B \subseteq X \mid D(A,B) < \epsilon\}$. A paratopology on a space is a collection of subsets closed under unions (but not necessarily under finite intersections) (see [11]).

Theorem III. 20 : The λ paratopology on $\mathcal{C}$ is the paratopology induced on $\mathcal{C}$ by the D function defined above.

Proof : We must show first that given $A \in \lambda\mathcal{C}$, $\epsilon > 0$, there is an open subset G of X such that $A \in R_-(G) \subseteq \mathfrak{M}_\epsilon(A)$. Consider the subset $G = M_\epsilon(A) = \{x \in X \mid d(A,x) < \epsilon\}$ of X. Clearly G is open, and $A \in R_-(G)$ since $A \subseteq G$. Let $K \in R_-(G)$; then $K \cap G \neq \emptyset$, so that $D(K,A) < \epsilon$ and hence $K \in \mathfrak{M}_\epsilon(A)$. Thus $A \in R_-(G) \subseteq \mathfrak{M}_\epsilon(A)$. We next show that if $A \in \mathcal{C}$ and $G \subseteq X$ with $A \in R_-(G)$, then $\exists\, B \in \mathcal{C}$ and $\epsilon > 0$ such that $A \in \mathfrak{M}_\epsilon(B) \subseteq R_-(G)$. Fix $a \in A \cap G$ and let $\epsilon = \inf_{b \in X-G} d(a,b) > 0$. Since $a \in A$, $A \in \mathfrak{M}_\epsilon(\{a\})$. Let $K \in \mathfrak{M}_\epsilon(\{a\})$. Then $D(K,\{a\}) < \epsilon$ so that $K \cap G \neq \emptyset$ and $K \in R_-(G)$. Thus $A \in \mathfrak{M}_\epsilon(\{a\}) \subseteq R_-(G)$.

CHAPTER IV

LOWER SEMI-CONTINUOUS MULTIFUNCTIONS

In this chapter we consider multifunctions $f: X \to Y$ and examine what restrictions placed on X or on Y make f l.s.c. as well as what conditions are needed on a l.s.c. function to make it preserve certain properties of topological spaces.

If f is a multifunction with f(x) closed in Y for every $x \in X$ (we then say that f has closed point values), then f gives rise to a (single-valued) function F mapping X <u>into</u> λY.

<u>Proposition IV.1</u>: Let f be a multifunction on a space X onto a T_1 space Y such that for each $x \in X$, f(x) is closed in Y. Let $F: X \to \lambda Y$ be defined by $F(x) = f(x)$. Then f is l.s.c. if and only if F is continuous.

Proof: Suppose f is l.s.c. Let $R_+(K)$, where K is closed in X, be an element of the closed subbase of λY (where R is the binary relation on X to $\mathcal{C} = P(X)$ defined on page 8). Then $F^{-1}[R_+(K)] = \{x \in X \mid f(x) \subseteq K\} = f^+(K)$. Since f is l.s.c., $f^+(K)$ is closed. That is, $F^{-1}[R_+(K)]$ is closed. Thus F is continuous. Now suppose F is continuous, and let K be closed in Y. Then $R_+(K)$ is closed in λY, so that $F^{-1}[R_+(K)]$ is closed in X. But $f^+(K) = F^{-1}[R_+(K)]$ and hence $f^+(K)$ is closed.

<u>Theorem IV.2</u>: Let f be a l.s.c. multifunction on X onto a T_1 space Y with f(x) closed and connected for each $x \in X$. Then f_- preserves connectedness.

Proof 1: Let A be any connected set in X and assume $f_-(A)$ is not connected. Hence there exist open sets H and K in Y such that $f_-(A) = (f_-(A) \cap H) \cup (f_-(A) \cap K)$ with $(f_-(A) \cap H) \cap (f_-(A) \cap K) = \emptyset$. Since f is l.s.c., $f^-(H)$ and $f^-(K)$ are open in X. Clearly $A = (f^-(H) \cap A) \cup (f^-(K) \cap A)$ and neither $f^-(H) \cap A$ nor $f^-(K) \cap A$ is empty. Assume $x \in (f^-(H) \cap A) \cap (f^-(K) \cap A)$. Then $f(x)$ meets both H and K; however, since $f(x)$ is connected this is not possible. Hence $(f^-(H) \cap A) \cap (f^-(K) \cap A) = \emptyset$. I.e. A is not connected, a contradiction. We also give the following proof which makes use of proposition IV.1.

Proof 2: Let A be a connected subset of X. By IV.1, F is continuous, hence $F[A]$ is connected in λY. But then $\cup F[A]$ $= \bigcup_{x \in A} f(x) = f_-(A)$ is connected in X by proposition III.15.

Corollary IV.2.1: Let f be an open multifunction on X onto Y with closed connected point inverses. If B is a connected subset of Y, then $f^-(B)$ is a connected subset of X.

Corollary IV.2.2: Let f be a l.s.c. multifunction on a T_1 space X onto a T_1 space Y such that for each $x \in X$, $f(x)$ is connected. If X is connected, then Y is connected.

If f is a closed multifunction on X onto Y, we can further define a (single-valued) function $\bar{\bar{F}}: \lambda X \to \lambda Y$ by $\bar{\bar{F}}(A) = f_-(A)$, where A is a subset of X.

In the following lemmas R denotes a binary relation on X

(resp. Y) to λX (resp. λY) (see also p.9).

Lemma IV.3.1: Let $B \subseteq Y$. Then

(i) $\bar{F}^{-1}[R_+(B)] = R_+(f^+(B))$ and

(ii) $\bar{F}^{-1}[R_-(B)] = R_-(f^-(B))$.

Proof: (i) $\bar{F}^{-1}[R_+(B)] = \{A \in \lambda X \mid \bar{F}(A) \subseteq B\} = \{A \in \lambda X \mid f_-(A) \subseteq B\} = \{A \in \lambda X \mid A \subseteq f^+(B)\} = R_+(f^+(B))$.

(ii) $\bar{F}^{-1}[R_-(B)] = \{A \in \lambda X \mid \bar{F}(A) \cap B \neq \emptyset\} = \{A \in \lambda X \mid f_-(A) \cap B \neq \emptyset\} = \{A \in \lambda X \mid A \cap f^-(B) \neq \emptyset\} = R_-(f^-(B))$.

Lemma IV.3.2: Let X be a T_1 space and let $K \subseteq X$.

(i) $R_+(K)$ is closed in λX if and only if K is closed in X.

(ii) $R_-(K)$ is open in λX if and only if K is open in X.

Proof: (i) $\leftarrow$: Follows from the definition.

(i) $\rightarrow$: This will follow if we prove that $R_+(\mathrm{cl}K) = \mathrm{cl}\, R_+(K)$. Since $R_+(K)$ is contained in the closed set $R_+(\mathrm{cl}K)$, $\mathrm{cl}\, R_+(K) \subseteq R_+(\mathrm{cl}K)$. Let $H \in R_+(\mathrm{cl}K)$ and let $\bigcap_{i=1}^{n} R_-(O_i)$ be any basic neighborhood of H in λX. Since $H \subseteq \mathrm{cl}K$ and since for $i=1,2,\ldots,n$ $H \cap O_i \neq \emptyset$, we have $O_i \cap K \neq \emptyset$, $i=1,2,\ldots,n$. Choose $x_i \in O_i \cap K$ and let $A = \{x_1,x_2,\ldots,x_n\}$. Then $A \in \lambda X$, $A \in \bigcap_{i=1}^{n} R_-(O_i)$ and $A \in R_+(K)$. Thus $H \in \mathrm{cl}\, R_+(K)$ and $R_+(\mathrm{cl}K) = \mathrm{cl}\, R_+(K)$.

(ii) $\leftarrow$: Follows from the definition.

(ii) $\rightarrow$: This will follow if we prove that $R_-(\mathrm{int}K) = \mathrm{int}\, R_-(K)$. Let $H \in \mathrm{int}\, R_-(K)$, but assume that $H \notin R_-(\mathrm{int}K)$. Thus $H \cap \mathrm{int}K = \emptyset$ so that $H \subseteq \mathrm{cl}(X-K)$. Let $\bigcap_{i=1}^{n} R_-(O_i)$ be a basic open neighborhood of H in λX. Then $O_i \cap (X-K) \neq \emptyset$ for $i=1,2,\ldots,n$. Let $x_i \in O_i \cap (X-K)$ and let $A = \{x_1, x_2, \ldots, x_n\}$. Clearly $A \in \lambda X$, $A \in \bigcap_{i=1}^{n} R_-(O_i)$ and $A \subseteq X-K$. Hence $H \in \mathrm{cl}\, R_+(X-K)$, a contradiction, since $\lambda X - \mathrm{int}\, R_-(K) = \mathrm{cl}(\lambda X - R_-(K)) = \mathrm{cl}\, R_+(X-K)$. Since evidently $R_-(\mathrm{int}K) \subseteq \mathrm{int}\, R_-(K)$, we now have $R_-(\mathrm{int}K) = \mathrm{int}\, R_-(K)$. (This proof is patterned after that given by C. Kuratowski [8] for the exponential topology).

Proposition IV.3: Let f be a closed multifunction on a T_1 space X onto a T_1 space Y. Then $\bar{\bar{F}}: \lambda X \to \lambda Y$ is continuous if and only if f is l.s.c.

Proof: For $B \subseteq Y$, $\bar{\bar{F}}^{-1}[R_-(B)] = R_-(f^-(B))$ (lemma IV.3.1). Hence if $R_-(B)$ is open and $\bar{\bar{F}}$ is continuous, $R_-(f^-(B))$ is open in λX and thus $f^-(B)$ is open in X (lemma IV.3.2), i.e. f is l.s.c. On the other hand if $f^-(B)$ is open, where B is open in Y, then $R_-(f^-(B))$ is open, i.e. $\bar{\bar{F}}$ is continuous.

Remark: IV.1 - IV.3 are valid for arbitrary spaces X and Y if we take F as $f: X \to \lambda P(Y)$ and $\bar{\bar{F}}$ as $f_-: \lambda P(X) \to \lambda P(Y)$, without assuming that f is closed. The same proofs can be used.

Consider now a relation $f: X \to Y$ as a subset of $X \times Y$. Let $g: X \to X \times Y$ be defined by $g(x) = \{(x,y) \mid y \in f(x)\}$. Then g is the inverse of the restriction to f of the projection $p: X \times Y \to X$. Clearly g is open.

Lemma IV.4.1: Let g be the function defined by $g(x) = \{(x,y) \mid y \in f(x)\} \subseteq X \times Y$. Then g is l.s.c. (or u.s.c.) if and only if f is l.s.c. (or u.s.c.)

Proof: Assume f is l.s.c. and let $U\times V$ be a basic open set in $X\times Y$. $g^{-}(U\times V) = \{x \mid f(x) \text{ meets } V,\ x\in U\} = U\cap f^{-}(V)$ which is open in X. Hence g is l.s.c. Conversely, if g is l.s.c., $f^{-}(V) = X\cap f^{-}(V) = g^{-}(X\times V)$, so that f is l.s.c. The proof for u.s.c. is essentially the same.

When f is a single-valued and continuous binary relation, the existence of a homeomorphism of $f\subseteq X\times Y$ to X is well known (see [4]). Since g(x) is a subset of $f\subsetneq X\times Y$, it is reasonable to inquire under what conditions $\{g(x) \mid x\in X\}$ considered as a family of closed subsets of $\lambda(X\times Y)$ is homeomorphic to X. The following diagram shows the situation when f is l.s.c. on a T_1 space X onto a T_1 space Y.

$$\begin{tikzcd}
X \arrow[r, "g \text{ l.s.c.}"'] \arrow[ddr, dashed, "\begin{array}{c}\exists G\\ \text{continuous}\end{array}"'] & f \arrow[d, phantom, sloped, "\supseteq"] \\
& X\times Y \arrow[d, phantom, sloped, "\supseteq", "\text{(via singletons)}"'] \\
& g[X] \subseteq \lambda(X\times Y)
\end{tikzcd}$$

Figure 7

A homeomorphism G of X and g[X]

<u>Theorem IV.4</u>: Let f be a closed l.s.c. multifunction on a T_1 space X onto a T_1 space Y. Then there is a homeomorphism G of X onto g[X], where $g(x) = \{(x,y) \mid y\in f(x)\}\subseteq X\times Y$.

Proof: We define G by $G(x) = g(x)$ for $x\in X$. The continuity of G follows at once from proposition IV.1. G is clearly one to one onto $g[X]$. It remains to show that G is closed (or open). Let U be a closed subset of X. Then $G[U] = \{g(x) \mid x \in U\}$, and $R_+(U \times f_-(U)) \cap G[X] = \{g(x) \mid g(x) \subseteq U \times f_-(U)\} = G[U]$.

It is sometimes desirable to characterize l.s.c. in terms of l.s.c. at a point. We say $f\colon X \to Y$ is l.s.c. at $x \in X$ if and only if for every open set V in Y such that $V \cap f(x) \neq \emptyset$ there exists an open neighborhood U of x such that $f(u) \cap V \neq \emptyset$ for each $u \in U$ (see[2]). If f is l.s.c. on X onto Y, we can define a single-valued map $F\colon X \to \lambda\mathcal{C}$, where $\mathcal{C}$ consists of all sets $f(x)$, $x \in X$. (see [3]).

<u>Proposition IV.5</u>: Let $F\colon X \to \lambda\mathcal{C}$ be defined by $F(x) = f(x)$, where $\mathcal{C} = \{f(x) \mid x \in X\}$, and f is a multifunction on X onto Y. Then f is l.s.c. if and only if F is continuous.

Proof: The proof is analogous to the proof of proposition IV.1.

The next few theorems deal with some sufficient conditions for a multifunction $f\colon X \to Y$ to be l.s.c.

<u>Proposition IV.6</u>: Let f be a multifunction on X onto Y such that $f^-(\{y\})$ is open in X for each $y \in Y$. Then f is l.s.c.

Proof: Let B be an open subset of Y. If $y \in B$, then $f^-(\{y\}) \subseteq f^-(B)$; hence $f^-(B) = \bigcup_{y \in B} f^-(\{y\})$ is open.

<u>Definition IV.7</u>: Let f be a multifunction on X onto Y. We shall say that f is <u>inverse regular</u> (i.r.)

provided that for every $y \in Y$ and for each closed subset B of Y, with $y \notin B$, there are subsets U and V of Y such that $B \subseteq U$, $y \in V$, $U \cap V = \emptyset$ and $f^-(U)$ and $f^-(V)$ are open in X.

Corollary IV.7: Let f be a l.s.c. multifunction on X onto a regular space Y. Then f is i.r.

Proof: This follows from the definition of i.r.

Theorem IV.8: Let f be i.r. on X onto Y. Then f is l.s.c.

Proof: Let B be an open subset of Y and assume that $f^-(B)$ is not open in X. Thus there is a point $p \in f^-(B)$ such that p is an accumulation point of $X - f^-(B)$. Assume $y \in f(p) \cap B$. Since f is i.r., there are subsets U and V of Y such that $y \in U$, $Y - B \subseteq V$, and $f^-(U)$ and $f^-(V)$ are open in X. In particular, $p \in f^-(U) \subseteq f^-(B)$ so that $f^-(B)$ is a neighborhood of p which fails to meet $X - f^-(B)$, a contradiction.

We next give a simple example of a l.s.c. multifunction. Let R be the set of real numbers and let $\alpha\colon R \to R$ be defined by

$$\alpha(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leqslant 0 \end{cases} \qquad \text{(dotted lines, figure 8)}$$

where $x \in R$. Let $f\colon R \to R^2$ be defined by $f(x) = \{(x,y) \in R^2 \mid y \leqslant \alpha(x)\}$ (solid lines, figure 8).

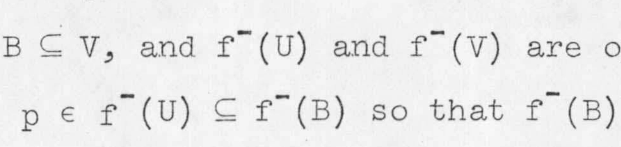

Figure 8

A l.s.c. multifunction $f\colon R \to R^2$

Let K be the closed subset of R^2 shown in figure 8. Then $f^-(K)$ is not closed in R so that f is not u.s.c. If $B \subseteq R^2$ is such that for each $b \in B$, $b > \alpha(x)$, $x \in R$, then $f^-(B) = \emptyset$. Also f is i.r. onto the range of f.

By contrast, we may define an u.s.c. mapping $f: R \to R^2$ as follows. Let $\alpha: R \to R$ be given by

$$\alpha(x) = \begin{cases} 1 \text{ if } x \leqslant 0 \\ 0 \text{ if } x > 0 \end{cases}$$

Then f is defined by $f(x) = \{(x,y) \in R^2 \mid y \leqslant \alpha(x)\}$. f is u.s.c but not l.s.c.

Definition IV.9: A multifunction f on X onto Y is said to be strongly inverse regular (s.i.r.) provided that for every $y \in Y$ and for each closed subset B of Y with $y \notin B$, there are disjoint open subsets U and V of Y with $B \subseteq U$, $y \in V$, and such that $f^-(U)$ and $f^-(V)$ are open and disjoint in X.

Corollary IV.9: If f is a s.i.r. multifunction on X onto Y, then f is l.s.c.

Proof: If f is s.i.r., then f is i.r. The corollary then follows from IV.8.

To prove the next theorem we also need to introduce the concept of a directed family of sets ([15]). We call a nonvoid family $\mathcal{Q}$ of nonvoid subsets of X a directed family in X provided that for any A_1 and A_2 in $\mathcal{Q}$ $\exists A_3 \in \mathcal{Q}$ with $A_3 \subseteq A_1 \cap A_2$. $\mathcal{Q}$ is said to converge to $p \in X$ if every open set O, with $p \in O$, contains some $A \in \mathcal{A}$. A directed family $\mathcal{A}'$ is said to be a <u>directed underfamily</u> of $\mathcal{A}$ provided that whenever $A \in \mathcal{A}$, $\exists A' \in \mathcal{A}'$ with $A' \subseteq A$.

<u>Theorem IV.10</u>: If f is a s.i.r. multifunction on X onto Y with closed point values, then f_- maps compact sets to closed sets.

Proof: Let C be a compact subset of X and assume $f_-(C)$ is not closed. Thus $\exists y_0 \in \text{cl } f_-(C) - f_-(C)$. Let $\mathcal{B} = \{O \cap f_-(C) \mid y_0 \in O$, O open in Y$\}$. $\mathcal{B}$ is a directed family and so is $\mathcal{A} = \{f^-(B) \cap C \mid B \in \mathcal{B}\}$. Since C is compact, there is a directed underfamily $\mathcal{A}'$ of $\mathcal{A}$ converging to some point $c \in C$ (see [15]). Since $c \in C$, $f(c) \subseteq f_-(C)$ and $y_0 \notin f(c)$. Because f is s.i.r. there are open subsets U and V of Y with $U \cap V = \emptyset$, $f(c) \subseteq U$, $y_0 \in V$, $f^-(U)$ and $f^-(V)$ open and disjoint in X. Clearly $c \in f^-(U)$. Also $A = f^-(V \cap f_-(C)) \cap C \in \mathcal{A}$. Since $\mathcal{A}'$ converges to c, and since $f^-(U)$ is an open set containing c, $\exists A' \in \mathcal{A}'$ such that $A' \subseteq f^-(U)$. Because $A \in \mathcal{A}$, $\exists A'' \in \mathcal{A}'$ such that $A'' \subseteq A$. But A'' and A' must meet, which is not possible since $f^-(U) \cap A = \emptyset$. Hence $f_-(C)$ must be closed.

We conclude this chapter with a characterization of lower semi-continuity in terms of accumulation points.

<u>Theorem IV.11</u>: Let f be a multifunction on X onto Y. Then f is l.s.c. if and only if whenever p is an accumulation point of $A \subseteq X$, then every point of $f(p)$ is an adherent point of $f_-(A)$.

Proof: Assume f to be l.s.c. Let p be an accumulation point of $A \subseteq X$. Without loss of generality assume $p \notin A$. Suppose $y \in f(p) - f_-(A)$ and assume there is an open neighborhood O of y such that $O \cap f_-(A) = \emptyset$. Because f is l.s.c., $f^-(O)$ is open. Clearly $p \in f^-(O)$, hence $f^-(O) \cap A \neq \emptyset$. Let $x \in f^-(O) \cap A$; then $f(x)$ meets O and $f(x) \subseteq f_-(A)$, a contradiction of the above assumption. Conversely let V be an open subset of Y and suppose $f^-(V)$ is not open in X. Thus $\exists p \in f^-(V)$ and p is an accumulation point of $X - f^-(V) = f^+(Y - V)$. According to the hypothesis every point of $f(p)$ is an adherent point of $f_-(f^+(Y - V)) \subseteq Y - V$. But there is a point $y \in f(p) \cap V$, and y cannot be an adherent point of $f_-(f^+(Y - V))$, a contradiction.

CHAPTER V

PROPERTIES OF THE κ TOPOLOGY

If X is a T_1 space (with a nontrivial topology), the space κX is a compact and connected T_0 space (see [12]). It is natural to inquire what further topological properties κX has and, more generally, if $\mathcal{C}$ is a family of subsets of X, what can be said about $\kappa\mathcal{C}$. Some of these questions will be answered in this chapter. (All underlying spaces are supposed to have a nontrivial topology).

Theorem V.1: If $\mathcal{C}$ contains all singleton subsets of X, then X is second countable provided $\kappa\mathcal{C}$ is second countable.

Proof: Since there is a countable base for $\kappa\mathcal{C}$, there is a countable base of the form $R_+(G_1)$, $R_+(G_2)$, ..., where G_i is an open subset of X, i=1,2,... Let x be a point of an open subset O of X. Then $\{x\} \in R_+(O)$. For some i, $\{x\} \in R_+(G_i) \subseteq R_+(O)$, and then $x \in G_i \subseteq O$. Thus $\{G_i\}_{i=1}^{\infty}$ forms a countable basis for X.

Theorem V.2: Let X be a T_1 space. If κX is second countable, then X is compact.

Proof: We know from V.1 that X is second countable. By hypothesis X is T_1. Hence we need only show that every countably infinite subset of X has a limit point. Suppose H is a countably infinite subset of X without limit points. Then H is closed and discrete in the relative topology. κH considered as a subspace of κX is also second countable. Let $\{O_\alpha\}_\alpha$ be any basis for κH. If $E \in \kappa H$, then $E \in R_+(E)$ and $R_+(E)$ is open in κH. Hence for some α, $E \in O_\alpha \subseteq R_+(E)$. Let E_1 and E_2 be elements of κH such that $E_1 \neq E_2$. Suppose $E_1 \in O_{\alpha_1} \subseteq R_+(E_1)$ and $E_2 \in O_{\alpha_2} \subseteq R_+(E_2)$. Since also $R_+(E_1) \neq R_+(E_2)$ we have $O_{\alpha_1} \neq O_{\alpha_2}$. But H has an uncountable number of subsets, so that $\{O_\alpha\}_\alpha$ cannot be a countable basis for κH, a contradiction. Hence X is compact. (For an analogous proof for the exponential topology see [10]).

Theorem V.3: Let $\mathcal{C}$ be a family of subsets of X with $\{x\} \in \mathcal{C}$, for $x \in X$. Then X is separable if and only if $\kappa\mathcal{C}$ is separable.

Proof: Assume X is separable and let A be a countable dense subset of X, say $A = \{a_1, a_2, \ldots\}$. Let $\mathcal{A} = \{\{a_1\}, \{a_2\}, \ldots\} \subseteq \kappa\mathcal{C}$. Let $\mathcal{B}$ be an arbitrary open subset of $\kappa\mathcal{C}$, say $\mathcal{B} = \bigcup_\alpha R_+(G_\alpha)$, with G_α open in X. Since each G_α contains some point $a_i \in A$, $\mathcal{B} \cap \mathcal{A} \neq \emptyset$, so that $\mathcal{A}$ is dense in $\kappa\mathcal{C}$. Conversely let $\mathcal{A} = \{A_1, A_2, \ldots\}$ be a countable dense subset of $\kappa\mathcal{C}$. From each A_i we choose an arbitrary element a_i and form $A = \{a_1, a_2, \ldots\}$. We now show that A is a countable dense subset of X. Clearly A is countable. Let O be any open subset of X. Then $R_+(O)$ is open in $\kappa\mathcal{C}$ and hence $\mathcal{A} \cap R_+(O) \neq \emptyset$. Assume $A_k \in \mathcal{A} \cap R_+(O)$. Then $A_k \subseteq O$, and since $a_k \in A_k$, we have $A \cap O \neq \emptyset$. Therefore X is separable.

Proposition V.4: Let $f\colon \kappa\mathcal{P} \to \kappa X$ be defined by $f(A) = \mathrm{cl}A$, $A \subseteq X$. Assume X is normal. Then f is a retraction.

Proof: Let $R_+(G)$ be an open neighborhood of $\mathrm{cl}A$ in κX. All we need to show is that there is a neighborhood $\mathcal{H}$ of A in $\kappa\mathcal{P}$ such that $f[\mathcal{H}] \subseteq R_+(G)$, or equivalently, that there is an open subset H of X such that $\mathrm{cl}A \subseteq H \subseteq \mathrm{cl}H \subseteq G$. The last statement, however, is a direct consequence of the assumption that X is normal.

Proposition V.5: Let $\mathcal{C}$ be a family of subsets of X including all singletons. Then $\kappa\mathcal{C}$ has no isolated points if and only if X has no isolated points.

Proof: Assume X has no isolated points. Let A be an isolated point of $\kappa\mathcal{C}$. Thus $\{A\}$ is open, hence $\{A\} = R_+(G)$ for some G open in X, i.e. A is the only element of $\mathcal{C}$ in $R_+(G)$. Hence since $\mathcal{C}$ includes all singletons, $G = A$. However if $a \in A$, $a' \in A$ with $a \neq a'$, then $\{a\}$ and $\{a'\}$ are distinct elements of $R_+(G)$; hence A must be a singleton, say $A = \{a\}$, and then also $G = \{a\}$. But X has no isolated points, a contradiction. Conversely, if $\kappa\mathcal{C}$ has no isolated points and $\{x\} \subseteq X$ is open, then $R_+(\{x\}) = \{\{x\}\}$ is open in $\kappa\mathcal{C}$, again a contradiction. This completes the proof of the proposition.

The following example of a κ space (which is neither normal nor regular) is patterned after the example of a λ space given on pages 19 - 21. Let $X = [0,1]$ and let $\mathcal{C}$ be the family of all nonvoid, closed connected subsets of X (i.e. $\mathcal{C}$ consists of the singletons and the closed subintervals of $[0,1]$). With each $C = [a,b] \in \mathcal{C}$, associate the point $\left(\frac{a+b}{2}, \frac{b-a}{2}\sqrt{3}\right)$.

This association gives a one to one correspondence of $\mathcal{C}$ onto the triangle with vertices (0,0), (1,0) and $(\frac{1}{2},\frac{\sqrt{3}}{2})$. We shall topologize the triangle to make it homeomorphic with $\kappa\mathcal{C}$ (see also page 20). Let G be an open connected subset of X and let K be a closed connected subset of X. Figures 9 (resp. 10) show what typical basic open (resp. closed) subsets of $\kappa\mathcal{C}$ look like (shaded areas and heavy lines are included in the sets).

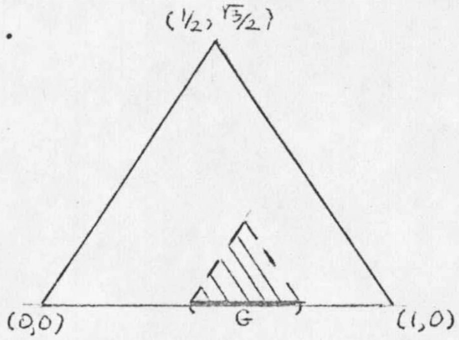

Figure 9

A basic open subset $R_+(G)$ of $\kappa\mathcal{C}$

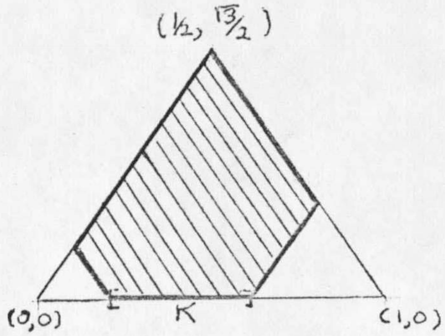

Figure 10

A basic closed set $R_-(K)$ of $\kappa\mathcal{C}$

If P is a point of $\kappa\mathcal{C}$ such that $P \notin R_-(K)$ (of figure 10), then every open set containing $R_-(K)$ also contains P. Thus $\kappa\mathcal{C}$ is not regular.

Let d be a bounded metric on X and, for any subset A of X, let an ϵ neighborhood of A be defined by $\mathcal{H}_\epsilon(A) = \{B \subseteq X \mid \sup_{b \in B} d(A,b) < \epsilon\}$, where $\epsilon > 0$ and $d(A,b) = \inf_{a \in A} d(a,b)$

Proposition V.6: The κ topology on the family $\mathcal{C}$ of all nonvoid closed subsets of a bounded metric space X is the same as the topology for $\mathcal{C}$ having a basis consisting of sets $\mathcal{H}_\epsilon(A)$, for $\epsilon > 0$ and $A \in \mathcal{C}$.

Proof: The proposition is proved by comparing basic open sets in both topologies. We shall first show that if A and B are closed subsets of X, and if $\epsilon > 0$ is given, then there is an open subset G of X such that $A \in R_+(G) \subseteq \mathcal{H}_\epsilon(B)$. Let $r = \frac{1}{2}\inf_{a \in A} d(a,X-C)$, where $C = \{x \in X \mid d(x,B) < \epsilon\}$. If $G = \{x \in X \mid d(x,A) < r\}$, then $A \in R_+(G)$. Because of the way r was chosen, $G \subseteq C$, so that $R_+(G) \subseteq \mathcal{H}_\epsilon(B)$. Assume now that A is a closed subset and G an open subset of X with $A \in R_+(G)$. We shall show that there exists a closed subset B of X and $\epsilon > 0$ such that $A \in \mathcal{H}_\epsilon(B) \subseteq R_+(G)$. Let $\epsilon' = \inf_{a \in A} d(a,X-G)$ and let $\epsilon = \frac{\epsilon'}{2}$ (clearly $\epsilon' > 0$). If $B = \{x \in X \mid d(A,x) < \epsilon\}$, then $A \in \mathcal{H}_\epsilon(B) \subseteq R_+(G)$.

For any given collection $\mathcal{C}$ of subsets of a space X, consider the mapping $i: \lambda\mathcal{C} \to \kappa\mathcal{C}$ defined by $i(C) = C$. It is natural to inquire which collections $\mathcal{C}$ make this map (which is clearly one to one and onto) continuous. If $\mathcal{C}$ is a disjoint family, we can give a partial answer to the question. In the following proposition let R denote the binary relation defined on pp. 8 and 9 on X into $\lambda\mathcal{C}$ and let R' denote the same relation construed as on X into $\kappa\mathcal{C}$.

Proposition V.7: Let $i: \lambda\mathcal{C} \to \kappa\mathcal{C}$ be defined by $i(\mathcal{C}) = \mathcal{C}$, where $\mathcal{C}$ is a disjoint family of subsets of X.

(i) If R' is u.s.c., then i is continuous.

(ii) If R is l.s.c., then i^{-1} is continuous.

Proof: (i) i is continuous if and only if for each closed subset K of X there is a closed subset L of X such that $R'_{-}(K) = R_{+}(L)$. Let K be given and let $L = R'^{-}(R'_{-}(K))$. Since R' is u.s.c., L is closed in X. It is easy to see, using the disjointness of $\mathcal{C}$, that $R_{+}(L) = R'_{-}(K)$. The proof of (ii) is similar.

Note that if $\mathcal{C}$ is a locally finite family of closed subsets of X, then $\cup R'_{-}(K)$ is closed in X (where K is a closed subset of X), so that in this case R' is always u.s.c.

Proposition V.8: Let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a connected subset of $\kappa\mathcal{C}$ with every A_α connected in X. Then $\underset{\alpha}{\cup} A_\alpha = A$ is connected.

Proof: Assume A is not connected. Hence there are open subsets G and H of X with $(G \cap A) \cup (H \cap A) = A$, $(G \cap A) \cap (H \cap A) = \emptyset$, $G \cap A \neq \emptyset$ and $H \cap A \neq \emptyset$. Since every element A_α is connected, it must lie either in G or in H but not in both. Hence $(R_{+}(G) \cap \mathcal{A}) \cup (R_{+}(H) \cap \mathcal{A}) = \mathcal{A}$ and $(R_{+}(G) \cap \mathcal{A}) \cap (R_{+}(H) \cap \mathcal{A}) = \emptyset$. Clearly also

$R_+(G)\cap\mathcal{A} \neq \emptyset \neq R_+(H)\cap\mathcal{A}$. This is a contradiction of the hypothesis that $\mathcal{A}$ is connected. Hence A must be connected.

Corollary V.8: If $\mathcal{C}$ is a cover of X by connected sets, then X is connected.

We now consider $\kappa\mathcal{L}$, where $\mathcal{L}$ is the family of all open subsets of X (including the null set). Let R be the binary relation on X to $\mathcal{L}$ defined on page 9. Clearly R is closed and l.s.c. A few elementary properties of $\kappa\mathcal{L}$ are given in the following

Proposition V.9: If $\mathcal{L}$ is the family of all open subsets of X, then (i) $\kappa\mathcal{L}$ is connected.

(ii) $\kappa\mathcal{L}$ is compact

(iii) $\kappa\mathcal{L}$ is locally connected

(iv) If $\emptyset \notin \mathcal{A}$ and $\mathcal{A}$ is dense in $\kappa\mathcal{L}$, then $\cup\mathcal{A}$ is dense in X

(v) If A is an open dense subset of X, then $\mathcal{A} = \{A\cap O \mid O \text{ is open in } X\}$ is dense in $\kappa\mathcal{L}$.

Proof: (i) follows from the fact that $\emptyset$ lies in every open subset of $\kappa\mathcal{L}$, and (ii) from the fact that X belongs to every nonvoid closed subset of $\kappa\mathcal{L}$.

(iii) Let $\mathcal{A}$ be an open neighborhood of $A \in \mathcal{L}$. Without loss of generality we may assume that $\mathcal{A} = R_+(G)$. Since $\emptyset$ lies in every open subset of $\kappa\mathcal{L}$, $R_+(G)$ is connected.

(iv) Let O be any open subset of X. Since $\mathcal{A}$ is dense in $\kappa\mathcal{L}$, $\mathcal{A}\cap R_+(O) \neq \emptyset$, i.e. $\exists A \in \mathcal{A}$ such that $A \subseteq O$. Hence $\cup\mathcal{A}\cap O \neq \emptyset$.

(v) Let $R_+(O)$ be a basic open set of $\kappa\mathcal{L}$. By definition, $A \cap O \neq \emptyset$, so that $\mathcal{A} \cap R_+(O) \neq \emptyset$.

We conclude this chapter with some observations on fixed point properties in $\kappa\mathcal{L}$.

<u>Proposition V.10</u>: Let f be a (single-valued) continuous mapping on $\kappa\mathcal{L}$ onto $\kappa\mathcal{L}$. Then X is fixed under f.

Proof: In $\kappa\mathcal{L}$, $A \in cl\{B\}$ is equivalent to $B \subseteq A$. Now because f is onto, X is the image of some $L \in \kappa\mathcal{L}$, i.e. $X = f(L)$. But $L \subseteq X$, so $X \in cl\{L\}$ whence, by continuity of f, $f(X) \in cl\{f(L)\} = cl\{X\}$. But then $X \subseteq f(X)$ which implies $f(X) = X$.

<u>Theorem V.11</u>: Every single-valued continuous map on $\kappa\mathcal{L}$ into $\kappa\mathcal{L}$ has at least one fixed point.

Proof: Assume $f(\emptyset) = E_1$. Clearly $\emptyset \subseteq E_1$. Suppose that for all ordinals $\beta < \alpha$ we have assigned $E_\beta \in \kappa\mathcal{L}$ in such a way that $E_{\beta'} \subseteq E_{\beta''}$ whenever $\beta' < \beta''$. We construct E_α as follows: If $\alpha = \alpha'+1$, let $E_\alpha = f(E_{\alpha'})$. If α is of the second kind, let $E_\alpha = \bigcup_{\alpha' < \alpha} E_{\alpha'}$. We must show that for $\beta < \beta' \leqslant \alpha$ we have $E_\beta \subseteq E_{\beta'}$. The only missing case occurs when $\alpha = \beta'$. In that case if α is of the second kind, then $E_\alpha = \bigcup_{\beta < \alpha} E_\beta$, so that $E_\beta \subseteq E_\alpha$. If $\alpha = \alpha'+1$, then $E_\beta \subseteq E_{\alpha'}$. We shall show that $E_{\alpha'} \subseteq E_\alpha$. By construction, $E_\alpha = f(E_{\alpha'})$. If $\alpha' = \alpha''+1$, then $E_{\alpha''} \subseteq E_{\alpha'}$ so that $E_{\alpha'} \in cl\{E_{\alpha''}\}$, and $E_\alpha = f(E_{\alpha'}) \in cl\{f(E_{\alpha''})\}$. Hence $E_\alpha \in cl\{E_{\alpha'}\}$, i.e. $E_{\alpha'} \subseteq E_\alpha$. If, however, α' is of the second kind, then

$E_{\alpha'} = \bigcup_{\beta<\alpha'} E_{\beta}$, so that for any $\beta < \alpha'$, $E_{\beta} \subseteq E_{\alpha'}$. Hence $E_{\alpha'} \in$ cl$\{E_{\beta}\}$, which implies $f(E_{\alpha'}) \in$ cl$\{f(E_{\beta})\}$, that is $E_{\alpha'+1} = E_{\alpha}$ $\in$ cl$\{E_{\beta+1}\}$ for each $\beta < \alpha'$. Thus $E_{\beta} \subseteq E_{\beta+1} \subseteq E_{\alpha}$. The increasing, transfinite sequence of open sets E_{β} thus constructed must terminate at some ordinal γ whose cardinality does not exceed the least cardinal $\aleph$ such that X has a base of cardinality $\aleph$. Thus $E_{\gamma} = E_{\gamma+1}$, i.e. $f(E_{\gamma}) = E_{\gamma+1} = E_{\gamma}$, i.e. E_{γ} is a fixed point under f. (this proof roughly parallels one for κX given in [12].)

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