Abstract:
The following paper is a further application of the method of suc-cessive corrections. The analysis of thin-walled cylindrical vessels is shown to be easily made by an application of the Hardy Cross moment and shear distribution procedures.

The paper is divided into two sections. The first section consists of an analysis of thin-walled cylindrical shells and flat-plate heeds Which is made by separating the membrane stresses from the bending stresses. such a separation results in the deflection equation for thin-walled shells being (Formula not captured by OCR) where r' is the membrane deflection, and the remainder of the expression is deflection due to bending.

Tbe second auction of the paper shows the method and resulting ex- pression for the fixed-end moments and shears, the carry-over factor and the distribution factors which are necessary for the application of successive corrections.

Examples are worked by both methods to demonstrate the practicability of the method introduced.

THE ANALYSIS OF THIN-WALLED PRESSURE VESSELS
BY
RELAXATION METHODS

by

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Table of Contents

Appendix

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The paper is divided into two sections. The first section consists of an analysis of thin-walled cylindrical shells and flat-plate heads which is made by separating the membrane stresses from the bending stresses. Such a separation results in the deflection equation for thin-walled shells being

$$v = e^{Bx} (A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx} (C_1 \cos Bx + D_1 \sin Bx) + r'$$

where r' is the membrane deflection, and the remainder of the expression is deflection due to bending.

The second section of the paper shows the method and resulting expression for the fixed-end moments and shears, the carry-over factor and the distribution factors which are necessary for the application of successive corrections.

Examples are worked by both methods to demonstrate the practicability of the method introduced.

INTRODUCTION

During the past twenty years a great deal of interest has been shown in numerical methods of analysis, particularly for engineering problems which do not lend themselves readily to rigid mathematical analysis. This interest was stimulated in this country almost entirely by Professor Hardy Cross(1) who presented the procedures for moment distribution. Numerical methods of analysis had been used before the presentation of this work but a new trend, of accenting the physical rather than the mathematical aspects, was introduced by this method.

The numerical methods of successive corrections has now been applied to such engineering fields as hydraulics, electricity, vibrations, thermodynamics and stress analysis. The wide use of the method attests to its practicability.

This thesis is a further application of the relaxation method. The work is based on the Hardy Cross methods of moments and shear distribution and results in a simplification of the conventional method of thin-walled pressure vessel analysis.

The general equation for the deflection at any point in a thin-walled cylindrical shell is found to be

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'$$

where A_1, B_1, C_1 and D_1 are constants resulting from the solution of a differential equation. These constants are shown to be dependent only on the manner of supporting the ends of the shell.

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(1) Hardy Cross, Analysis of Continuous Frames by Distributing Fixed-End Moments, Transactions A. S. C. E., 1932, pp 1-156.

The conventional method of analysis consists of equating the deflection of the shell to that of its restraining members at the joints. Since there at least four unknown constants in these expressions, it is necessary to equate also the moment or shearing force equations. This means that it is necessary to solve at least four simultaneous equations, and since the restraint on the shell is dependent on the stiffness of the restraining member it is necessary to solve these four equations for each vessel.

By applying artificial restraints to the ends of the shell to eliminate any rotation or deflection of these points, there is only one set of boundary conditions for all shells. This eliminates a major portion of the work from the solution. The artificial restraints are then removed by successive corrections.

In the analysis of thin-walled cylindrical shells the membrane stresses are separated from the bending stresses. This is necessary since the balancing of moments and shears is independent of the membrane stresses.

SECTION I

ANALYSIS OF STRESSES IN THIN-WALLED PRESSURE VESSELS

INTRODUCTION - A thin-walled vessel is one in which the wall thickness is small in comparison with the diameter of the vessel. The degree to which this condition must be satisfied is such that the assumption - the unit stresses within the material, neglecting those produced by bending, are uniform across the longitudinal cross-section of the vessel - can be made with little error.

A thin-walled shell, free from the restraining effects of heads or stiffeners and subjected to an internal fluid pressure, expands uniformly in the radial and longitudinal directions and the stresses induced are uniformly distributed over any cross-section. These stresses are the membrane stresses and are the major stresses to be considered in a pressure vessel at points where there are no discontinuities. For instance, a vessel with a neutral surface in the form of a sphere and with no discontinuities such as riveted or reinforced joints would have no stresses other than membrane stresses. A soap bubble is a perfect example of such a stress state.

A pressure vessel with a thin-walled cylindrical shell such as is to be considered here, must have at least one end closed. Whether the closure is formed by hydraulic pressure and a junction with some other structure or by a head, a discontinuity will exist at the joint. This discontinuity in the uniformity of the structure will produce shearing and bending stresses which are not uniformly distributed over the longitudinal cross-section.

For the application of successive approximations it is necessary to consider the stresses in the vessel as being membrane stresses produced by the internal fluid pressure, superimposed on bending stresses produced by restraining moments and shears.

MEMBRANE STRESSES - The magnitude of the unit stress acting on a longitudinal cross-section of a cylindrical vessel subjected to an internal fluid pressure p is

$$S_1 = \frac{p\,r}{t} \qquad (a)$$

where S_1 is the unit stress acting normal to the longitudinal cross-section, r is the radius of the vessel and t is the wall thickness.

The magnitude of the unit stress acting on a transverse cross-section is given by the expression

$$S_2 = \frac{p\,r}{2t} \qquad (b)$$

where S_2 is the unit stress acting normal to the transverse cross-section of the vessel.

For convenience the stresses S_1 and S_2 will hereafter be referred to as the circumferential membrane stress and the longitudinal membrane stress respectively.

DEFORMATIONS DUE TO MEMBRANE STRESSES - The membrane stresses found above will be accompanied by deformations of the heads and shell of the vessel. In the heads the stresses will produce a change in the diameter at the junction with the shell. However, in a vessel with flat heads as will be considered here, this change will be negligible in comparison with the other deformations and may be considered zero without introducing any appreciable errors. The deformations of the shell consist of an elongation in the longitudinal direction and a change in the shell radius. As the deformation in the longitudinal direction has no bearing on the work to follow, it may be omitted and only the radial deflection found.

THE CHANGE IN SHELL RADIUS DUE TO MEMBRANE STRESSES - The membrane stresses act on a differential particle of the cylindrical shell in the manner shown in Figure 1. The X-axis and Y-axis represent the longitudinal and transverse directions respectively and the Z-axis represents the direction normal to the shell surface.

By Poisson's Ratio

$$S_y = S_1 - uS_2$$

where u is Poisson's ratio for the material of the shell and S_y is the total stress on the circumferential direction.

Also by Young's Modulus

$$S_y = E\, e_y$$

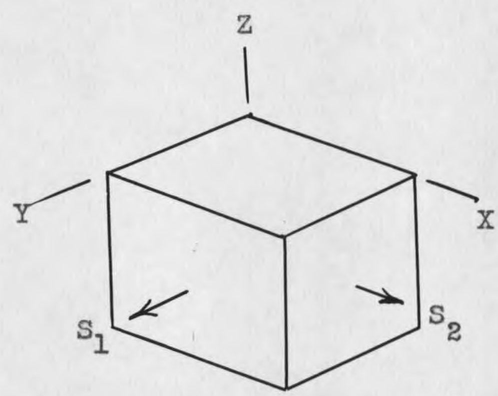


Figure 1

where e_y is the unit distortion in the circumferential direction produced by the stress S_y. Equating these expressions and solving for e_y gives

$$e_y = \frac{S_1 - uS_2}{E}$$

or, using the values for S_1 and S_2 given by equations (a) and (b)

$$e_y = \frac{pr}{2Et}(2 - u)$$

Now by letting ΔC be the change in the length of the circumference of the shell and denoting the change in the radius of the neutral surface of the shell by r', it is evident that

$$\Delta C = 2\pi r e_y = \frac{\pi p r^2}{Et}(2 - u)$$

and

$$r' = \frac{\Delta C}{2\pi}$$

or

$$r' = \frac{pr^2}{2Et}(2 - u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This expression is derived using the stresses defined by equations (a) and (b) which are based on the assumption that the pressure is uniformly distributed within the vessel. Equation (1) must therefore be limited to vessels with uniform pressure distribution.

MEMBRANE STRESSES AND DEFORMATIONS IN VESSELS WITH NON-UNIFORM PRESSURE DISTRIBUTION - Vessels such as storage tanks and standpipes which contain fluids of relatively high densities have membrane stresses which not only depend on the pressure but also on the manner of support of the vessel. For this reason it is not feasible to write a general equation for the change in radius in a vessel of this type. Hereafter in this paper when it is desirable to consider a vessel containing a non-uniform pressure, the change in radius will be referred to as $f(x)$ rather than r'.

BENDING STRESSES IN CYLINDRICAL THIN-WALLED SHELLS - Figure 2 shows a thin-walled cylindrical shell subjected to an externally applied bending moment of M_o pound inches per unit arc length at one end. Since there are no direct loads, this is a case of pure bending of the shell.

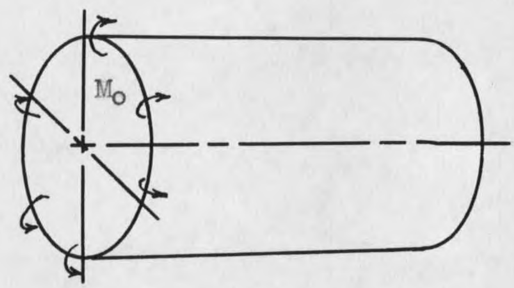


Figure 2

The following analysis will be confined to a longitudinal strip of the shell a unit arc length in width and x units in length. This may be done since both the shell and the loading are symmetrical about the longitudinal axis. This strip is shown in Figure 3. An arbitrary differential length dx, bounded in the unstressed state by planes a and b and in the stressed state by planes a' and b', is represented by the shaded areas.

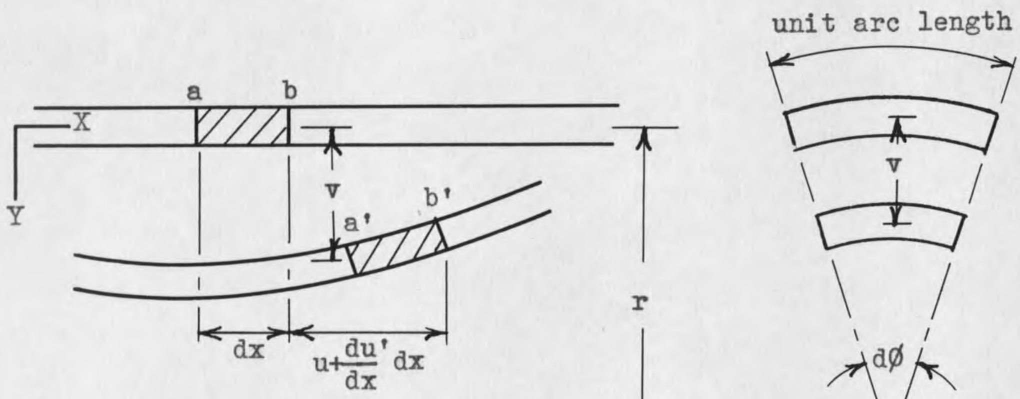


Figure 3

The notation used in the discussion is as follows:

u' = displacement in the X direction

v = displacement in the Y direction

e_x = unit strain in the X direction

e_y = unit strain in the circumferential direction

e_{yx} = unit strain in the X direction caused by the circumferential stress

e_{tx} = unit strain in the X direction caused by the bending stresses

Using this notation it is possible to write the expression for the unit strain in the circumferential direction as

$$e_y = \frac{r d\phi - (r - v) d\phi}{r d\phi} = \frac{v}{r} \qquad (c)$$

The differential section dx is shown again in Figure 4. The plane c'c" is passed parallel to plane a'a" through the intersection of b'b" and the neutral axis. The distance between planes c'c" and a'a" is then the original length of the fibers. Assuming a plane cross-section before bending will be a plane cross-section after bending the final length of the fibers is the distance between planes a'a" and b'b". It is then evident that the change in length of a fiber is the distance between planes b'b" and c'c" measured at the fiber.

Let y be the distance from the neutral axis to any fiber. Then

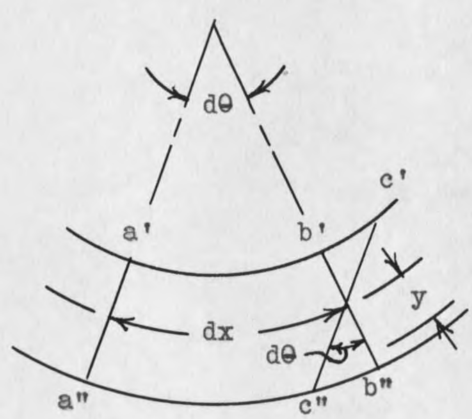


Figure 4

$$e_{tx} = y \frac{d\theta}{dx} \qquad \text{(d)}$$

Using the sign convention that deformations are positive when they increase the length of the particle

$$e_x = e_{yx} + e_{tx}$$

From Poisson's Ratio

$$e_{yx} = -ue_y$$

and using the values given in equations (c) and (d)

$$e_x = -u\frac{v}{r} + y\frac{d\theta}{dx} \qquad \text{(e)}$$

It now becomes necessary to use expressions giving the stresses in terms of the strains. Consider the differential volume shown in Figure 5, with sides dx, dy and dz. As before, let S_x, S_y and S_z be the unit stresses and e_x, e_y and e_z be the unit strains, where S_z and e_z are the stress and the strain respectively in the radial direction and the other notations are as previously used.

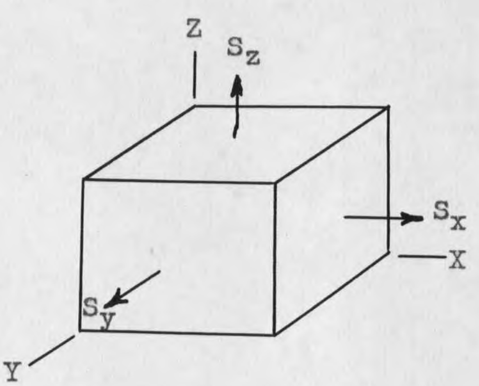


Figure 5

From the definition of Poisson's ratio and of Young's modulus it is possible to write

$$Ee_x = S_x - u(S_y + S_z)$$

$$Ee_y = S_y - u(S_x + S_z)$$

$$Ee_z = S_z - u(S_x + S_y)$$

However, in the case of the thin-walled shell the stress in the radial direction is zero, therefore

$$Ee_x = S_x - uS_y \tag{f}$$

$$Ee_y = S_y - uS_x \tag{g}$$

Solving for S_y from expression (g) and substituting the resulting expression into equation (f)

$$Ee_x = S_x - u(Ee_y + uS_x)$$

and solving for S_x from expression (f) and substituting the resulting expression in equation (g)

$$Ee_y = S_y - u(Ee_x + uS_y)$$

The preceding two expressions can now be solved for S_x and S_y as

$$S_x = \frac{E}{1 - u^2}(e_x + u e_y)$$

$$S_y = \frac{E}{1 - u^2}(e_y + u e_x)$$

The substitution of the value of e_x given in expression (c) and the value of e_y given in expression (e) results in

$$S_x = \frac{E}{1 - u^2}\left(y \frac{d\theta}{dx}\right) \tag{h}$$

$$S_y = \frac{E}{1 - u^2}\left[(1 - u^2)\frac{y}{r} + uy\frac{d\theta}{dx}\right] \tag{i}$$

It is of interest to note that the stress S_x distributed over the transverse cross-section of the shell is similar in form to stress in a straight beam subjected to pure bending and that the stress S_y distributed over the longitudinal cross-section and acting tangent to the arc of the shell is similar in form to that of a curved beam subjected to pure bending.

RESISTING MOMENTS AND SHEARS IN THIN-WALLED CYLINDRICAL SHELLS - Since it is necessary to know the values of the restraining moments and shears at the discontinuity it is necessary to find expressions for these restraints. The expressions must be functions of both membrane stresses and bending stresses. The easiest function to find which has this property is the total deflection.

Let V be the shearing force per unit arc length acting perpendicular to the axis, M be the moment per unit arc length in the plane perpendicular to the radius and $F\phi$ be the normal force per unit length acting perpendicular to the radius of the shell as shown in Figure 6. By applying the

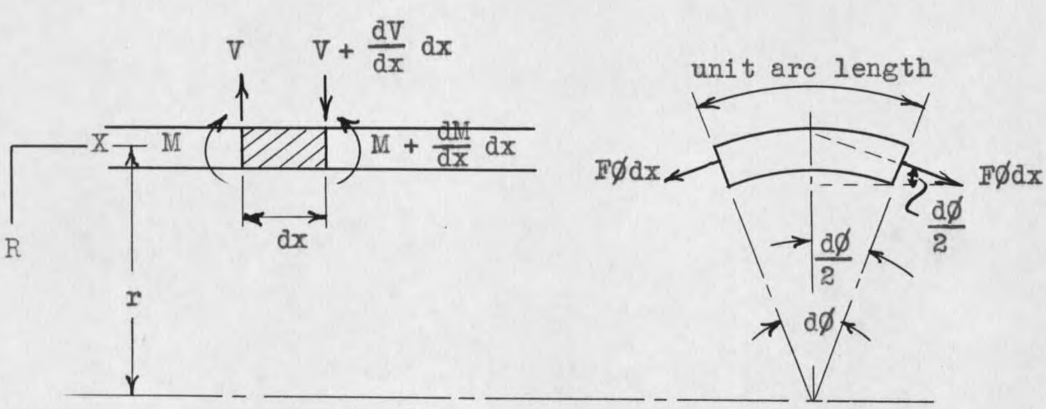


Figure 6

equations of static equilibrium to the differential section it is possible to get relationships between these forces and moments.

The summation of the moments about the left end of the section is

$$M - (M + \frac{dM}{dx}dx) + (V + \frac{dV}{dx}dx)dx - 2F\phi dx \sin\frac{d\phi}{2}\frac{dx}{2} = 0$$

Since it is possible to neglect powers of the differential higher than the first without introducing appreciable errors the equation can be written as

$$V = \frac{dM}{dx} \qquad (j)$$

The summation of the forces acting in the radial direction gives

$$-V + (V + \frac{dV}{dx}\,dx) + 2F\phi\,dx\,\sin\frac{d\phi}{2} = 0$$

but the section is a unit arc length in width and

$$r\,d\phi = 1$$

Since the unit arc length is small in comparison with the radius of the shell, $d\phi$ is small and the difference between $\sin\frac{d\phi}{2}$ and $\frac{d\phi}{2}$ is negligible. By making use of the two foregoing statements in the equation for the summation of the forces in the radial direction

$$\frac{dV}{dx} = \frac{F\phi}{r} \qquad (k)$$

The circumferential force $F\phi$ and the moment M may be written as functions of the circumferential and longitudinal stresses in the following manner

$$F\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} S_y dy$$

$$M = \int_{-\frac{t}{2}}^{\frac{t}{2}} S_x y dy$$

where t is the shell thickness and y is as shown in Figure 4. By using the values of S_y and S_x given in expressions (h) and (i) and performing the indicated integrations

$$F\phi = \frac{E\,v\,t}{r} \qquad (1)$$

$$M = \frac{E\,t^3}{12(1 - u^2)}\,\frac{d\theta}{dx}$$

Since the term $\frac{E\,t^3}{12(1 - u^2)}$ is related only to the physical dimensions of the shell and is a measure of its resistence to bending, it is defined as the "flexural rigidity" and denoted by D. Then

$$M = D\,\frac{d\theta}{dx}$$

Referring to Figure 4 and making the assumption that $d\theta$ is small it is possible to write without introducing appreciable error

$$\tan d\theta = d\theta = \frac{dv}{dx}$$

Then

$$\frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

and

$$M = D\,\frac{d^2v}{dx^2} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

Differentiating this expression with respect to X and making the substitution indicated in expression (j)

$$V = D\,\frac{d^3v}{dx^3} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (3)$$

Again it is interesting to note the similarity of the expressions for the resisting moment and shear in a cylindrical shell and the expressions for the corresponding quantities in a straight beam.

THE DEFLECTION OF A THIN-WALLED CYLINDRICAL SHELL DUE TO BENDING - By differentiating equation (5) and substituting from expression (k) for $\frac{dV}{dx}$

$$\frac{d^4v}{dx^4} = -\frac{T\phi}{Dr}$$

or, by using the value of $T\phi$ given in expression (1) and rearranging the terms

$$\frac{d^4v}{dx^4} - \frac{E\ t\ v}{D\ r^2} = 0$$

This is the differential equation for the deflection of a thin-walled cylindrical shell due to the restraining moment M_o. The solution for this equation is as follows.

Let

$$\frac{E\ t}{Dr^2} = 4B^4$$

$$v = e^{ax}$$

then

$$\frac{d^4v}{dx^4} = a^4e^{ax}$$

and

$$a^4e^{ax} + rB^4e^{ax} = 0$$

From this

$$a = \pm B\sqrt{2i} \quad \text{or} \pm B\ 2\ (\pm\sqrt{i})$$

But

$$\sqrt{i} = \cos 45^o + i \sin 45^o = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

And therefore

$$a = \pm B(1 + i) \text{ or } \pm B(-1 + i)$$

The deflection at any point may then be written as

$$v = Ae^{(B + iB)x} + Be^{(-B + iB)x} + Ce^{(-B - iB)x} + De^{(B - iB)x}$$

where A, B, C and D are arbitrary constants. Factoring this gives

$$v = e^{Bx}(Ae^{iBx} + De^{-iBx}) + e^{-Bx}(Be^{iBx} + Ce^{-iBx})$$

and converting to the trigonometric form

$$v = e^{Bx} \quad (A + D) \cos Bx + (iA - iD) \sin Bx$$
$$+ e^{-Bx} \quad (B + C) \cos Bx + (iB - iC) \sin Bx$$

Now by replacing the combinations of constants appearing above by new arbitrary constants the deflection equation for the case of pure bending appears as

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) \qquad (m)$$

where the constants A_1, B_1, C_1 and D_1 are dependent on the boundary conditions of the shell.

<u>TOTAL DEFLECTION OF THIN-WALLED CYLINDRICAL SHELLS</u> - It has previously been pointed out that the deflection of a thin-walled vessel can be thought of as a deflection due to the membrane stresses superimposed on the deflection caused by restraining moments and shears. The total deflection of any point in the shell then is the sum of the deflection

caused by the membranes stresses and the deflection caused by bending. Letting v_t be the total deflection, the equation may be written as

$$v_t = v + r'$$

in the case of uniform pressure, or as

$$v_t = v + f(x)$$

in the case of non-uniform pressure.

<u>THE EFFECT OF SHELL LENGTH ON THE TOTAL DEFLECTION</u> - The shell length has no effect on the deflection due to membrene stresses as can be seen by examining equation (1).

However, this is not so in the case of the deflection due to bending. Since x is measured along the longitudinal axis of the shell the deflection is a function of shell length. Note that the first term to the right of the equation sign in equation (m) increases as the shell length increases. It is apparent that if the deflection is not to become unduly large in a shell whose length approaches infinity the constants A_1 and B_1 must be extremely small. Since these constants do not depend on the physical dimensions of the shell, it can be concluded that the first term may be neglected in shells of ordinary length without introducing appreciable error. The limiting value of BL is approximately 5.5 for an error of less than 1 per cent.

On the other hand, in a very short shell all of the constants must be small since the deflection can reasonably be expected to be small. In such a shell the first term may account for a significant portion of the deflection and may not be neglected.

In the succeeding discussions long shells or short shells will be referred to. A long shell will be defined as one whose total deflection is given by

$$v = e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r' \; . \; . \; . \; . \; . \; . \; (4)$$

and a short shell will be defined as one whose deflection is given by

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'. \; (5)$$

STRESSES DUE TO BENDING IN FLAT CIRCULAR HEADS - The stresses due to bending in flat circular heads may be found by considering a flat circular plate with a moment of M_o pound inches per unit arc length applied at the edge. Since both the load and the plate are symmetrical about the center, the deflection of all points equally distant from the center is the same. This allows the analysis to be made on a diametrical section.

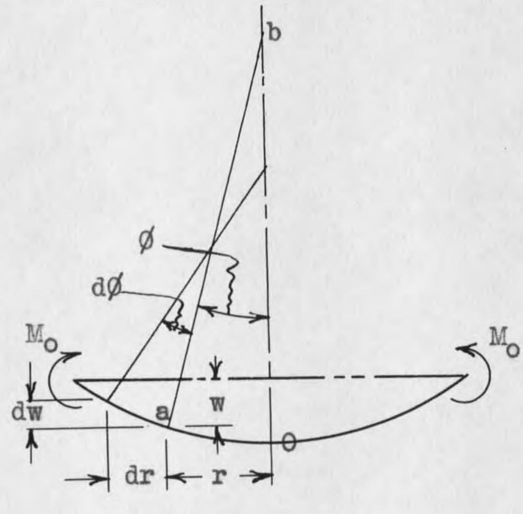


Figure 7

Let the deflection of any point on the neutral plane be denoted by w and the radial distance of the point from the center be denoted by r as shown in Figure 7. Then assuming only small deflections, the slope at any point in the neutral surface may be written as

$$\phi = -\frac{dw}{dr} \qquad (n)$$

Now let r_n be the radius of curvature in the diametrical plane and r_t be the radius of curvature normal to the diametrical plane. Further, assuming that for a small change $d\phi$ in the angle ϕ

$$dr = \text{arc length described} = r_n d\phi$$

it is possible to write

$$\frac{d\phi}{dr} = \frac{1}{r_n} = -\frac{d^2w}{dr^2} \qquad (o)$$

Since the line ab, when rotated about the line ob, describes a cone it must be the radius of curvature in the plane normal to the diametrical section, and

$$\overline{ab} = r_t$$

Again assuming only small deflections

$$\tan\phi = \phi = \frac{r}{r_t} = -\frac{dw}{dr}$$

or

$$\frac{1}{r_t} = -\frac{1}{r}\frac{dw}{dr} \qquad (p)$$

If the deflection of the plate is small in comparison to the thickness t, the assumption can be made that the neutral plane of the plate is the centroidal plane and as in the case of a beam, the unit strains are proportional to the distance from the center. Denoting the unit strain in the radial direction by e_r and the unit strain in the direction normal to the radius as e_t and letting y be the distance from the center

$$e_r = \frac{y}{r_n}$$

and

$$e_t = \frac{y}{r_t}$$

By applying Hooke's law the stresses can now be written as

$$S_r = \frac{Ey}{1 - u^2}\left(\frac{1}{r_n} + u\frac{1}{r_t}\right) \qquad (q)$$

and

$$S_t = \frac{Ey}{1 - u^2}\left(\frac{1}{r_t} + u\frac{1}{r_n}\right) \qquad (r).$$

where S_r and S_t are the unit stresses in the radial and circumferential directions respectively.

The maximum stresses in the head are found by replacing y in the equations above by t/2.

RESISTING MOMENTS AND SHEARS IN FLAT CIRCULAR HEADS - The resisting moments may be written as functions of the stresses by using the relationships

$$M_r = \int_{-\frac{t}{2}}^{\frac{t}{2}} S_r\, y\, dy$$

and

$$M_t = \int_{-\frac{t}{2}}^{\frac{t}{2}} S_t\, y\, dy$$

where M_r and M_t are the resisting moments acting on the circumferential and radial cross-sections respectively. Substituting expressions (q) and (r) for the stresses S_r and S_t and integrating, the moments are found in terms of the curvatures as

$$M_r = D(\frac{1}{r_n} + u\frac{1}{r_t})$$

and

$$M_t = D(\frac{1}{r_t} + u\frac{1}{r_n})$$

where D is the flexural rigidity defined previously.

A more convenient form for these moments may be had by substituting the values for the curvatures given in equations (o) and (p). Then

$$M_r = -D(\frac{d^2w}{dr^2} + \frac{u}{r}\frac{dw}{dr}) \quad . . . . (6)$$

and

$$M_t = -D(\frac{1}{r}\frac{dw}{dr} + u\frac{d^2w}{dr^2}) \quad . . . . (7)$$

The shearing forces present act only on the circumferential cross-section and are composed of the load within the circular section of radius r divided by the length of the cross-section or

$$V = \frac{p\,r}{2} \quad . . . . (8)$$

where V is the shearing force per unit arc length of the circumference at the point under consideration.

DEFLECTION OF FLAT CIRCULAR HEADS UNDER UNIFORM LOAD - The deflection of a flat circular head under a uniform fluid pressure may be found by considering a pie-shaped slice of the head as shown in Figure 8. There are no shearing forces acting on the sides of such a slice due to the symmetry of the slice and the loading, and all of the forces which act on a small element are shown in the figure with the notations previously used.

Applying the equations of static equilibrium to the section and taking moments about point c, assuming that the angle $d\phi$ is small

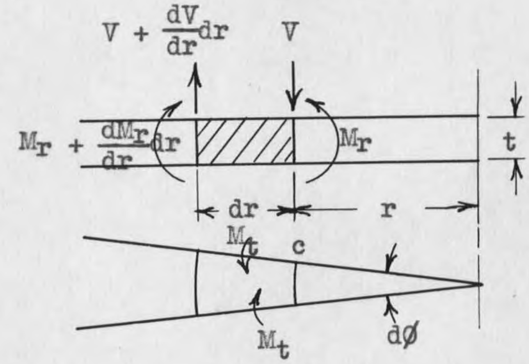


Figure 8

$$M_c = (M_r + \frac{dM_r}{dr}dr)(r + dr)d\phi$$
$$- M_r\, r\, d\phi - M_t\, dr\, d\phi$$
$$+ (V + \frac{dV}{dr}\, dr)(r + dr)d\phi\, dr$$
$$+ p\left[r d\phi\, dr + \frac{r + dr}{2}\, dr\, d\phi\right] \frac{dr}{2} = 0$$

By neglecting differential quantities of higher order than the first this equation becomes

$$M_r\, dr + r\,\frac{dM_r}{dr} + V\, r - M_t = 0$$

The substitution of the values for M_r and M_t given by equations (6) and (7) gives the differential equation for the head deflection as

$$\frac{d^3w}{dr^3} + \frac{1}{r}\,\frac{d^2w}{d^2r} - \frac{1}{r^2}\,\frac{dw}{dr} = \frac{p\, r}{2\, D} \qquad (s)$$

The equation may be solved by noting that it can be written as

$$\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}(r\frac{dw}{dr})\right] = \frac{p\, r}{2\, D}$$

and integrating to get

$$\frac{1}{r}\,\frac{d}{dr}(r\,\frac{dw}{dr}) = \frac{p\, r^2}{r\, D} + C_1$$

where C_1 is a constant of integration dependent on the boundary conditions of the plate. Another integration gives

$$r \frac{dw}{dr} = \frac{p\, r^4}{16\, D} + \frac{C_1\, r^2}{2} + C_2$$

Since the plate and its loading are symmetrical about the center, the slope at the center must be zero, therefore

$$- \phi = \frac{dw}{dr} = 0 = C_2$$

Integrating again, the deflection is given as

$$w = \frac{p\, r^4}{64\, D} + \frac{C_1\, r^2}{4} + C_3 \quad \ldots\ldots\ldots\ldots \quad (9)$$

where C_3 is a constant of integration like C_1.

EXAMPLE 1 - A pressure vessel consists of a thin-walled cylindrical shell 20 inches in diameter, 20 inches long, and $\frac{1}{8}$ inch in wall thickness, and two circular flat heads $\frac{3}{8}$ inches in thickness. The internal pressure is 30 psi. For the material of the shell and heads Young's modulus is 30,000,000 psi and Poisson's ratio is 0.25. It is desired to find the moments existing at the joints.

Solution:

$$4B^4 = \frac{Et}{D_s a^2} = \frac{30 \times 10^6 \times 0.125}{5,208 \times 10^2}$$

$$B^2 = 1.3416$$

$$B = 1.1583$$

$$BL = 20 \times 1.1583 = 23.166$$

Since BL is larger than 5.5 this is a long shell and the total deflection is given by equation (4) as

$$v = e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'$$

Assuming that the radial distortion of the heads is negligible, $v = 0$ at $x = 0$. In the deflection equation then, $C_1 = -r'$. The deflection is then

$$v = e^{-Bx}(-r' \cos Bx + D_1 \sin Bx) + r'.$$

Differentiating this gives

$$\frac{dv}{dx} = -Be^{-Bx}(-r' \cos Bx + D_1 \sin Bx) + Be^{-Bx}(r' \sin Bx + D_1 \cos Bx)$$

Denoting the rotation of the shell at the joints where $x = 0$ as ϕ_s, and noting that $\phi_s = \frac{dv}{dx}$

$$\phi_s = B(r' + D_1)$$

The second derivation of the deflection equation gives

$$\frac{d^2v}{dx^2} = -2B^2e^{-Bx}(r' \sin Bx + D_1 \cos Bx)$$

Now denoting the moment at the end of the shell where $x = 0$ by M_s, denoting the flexural rigidity of the shell as D_s, and employing equation (2)

$$M_s = D_s(-2B^2D_1)$$

Now considering the head the following equations can be written

$$w = \frac{pr^4}{64D_h} + \frac{C_1r^2}{4} + C_3$$

$$\frac{dw}{dr} = \frac{pr^3}{16D_h} + \frac{C_1r}{2}$$

$$\frac{d^2w}{dr^2} = \frac{3pr^2}{16D_h} + \frac{C_1}{2}$$

where D_h denotes the flexural rigidity of the heads.

Letting a be the maximum radius in the heads and ϕ_h be the rotation of the heads at the joints, and applying expression (n)

$$\phi_h = \frac{dw}{dr} = - \frac{pa^3}{16D_h} - \frac{C_1 a}{2}$$

Denoting the moment in the radial direction at the edge of the heads by M_r and using equation (6)

$$M_r = - D_h\left[\frac{pa^2}{16D_h}(3 + u) + \frac{C_1}{2}(1 + u)\right]$$

The four equations ϕ_s, M_s, ϕ_h and M_r contain two unknown constants. However, by assuming a rigid connection between the shell and the heads it is possible to find one constant in terms of the other since $\phi_s = \phi_h$, or

$$B(r' + D_1) = \frac{pa^3}{16D_h} + \frac{C_1 a}{2}$$

then

$$D_1 = \frac{p\, a^3}{16BD_h} + \frac{C_1\, a}{2\; B} - r'$$

Since the joint must be in equilibrium

$$M_s = - M_r$$

or

$$- 2D_s B^2 D_1 = D_h\left[\frac{pa^2}{16D_h}(3 + u) + \frac{C_1}{2}(1 + u)\right]$$

and using the value of D_1 found above

$$- 2D_sB^2 \left[\frac{pa^3}{16BD_h} + \frac{C_1a}{2\ B} - r'\right] = D_h \left[\frac{pa^2}{16D_h} (3 + u) + \frac{C_1}{2} (1 + u)\right]$$

and solving for C_1

$$C_1 = \frac{- 4D_sB^2\left[\frac{pa^3}{16BD_h} - r'\right] - \frac{pa^2}{8} (3 + u)}{2D_sBa + D_h(1 + u)}$$

The existing moments can now be found. These are: the moment in the shell, the radial moment in the head and the circumferential moment in the head. Substituting the value just found for C_1 into the expression for M_r gives

$$Mr = - D_h\left[\frac{pa^2}{16D_h} (3 + u) - \frac{2D_sB^2\left(\frac{pa^3}{16BD_h} - r'\right) + \frac{pa^2}{16} (3 + u)}{2D_sBa + D_h(1 + 7)} (1 + u)\right]$$

and for the circumferential moment in the head M_t

$$M_t = - D_h\left[\frac{pa^2}{16D_h}(3u + 1) - \frac{2D_sB^2\left(\frac{pa^3}{16BD_h} - r'\right) + \frac{pa^2}{16} (3 + u)}{2D_sBa + D_h(1 + u)} (1 + u)a\right]$$

The values of the constants to be applied to these equations are:

$$D_s = \frac{E\ t^3}{12(1 - u^2)} = \frac{30 \times 10^6 \times 0.125^3}{12(1 - 0.25^2)} = 5,208$$

$$D_h = \frac{E\ t^3}{12(1 - u^2)} = \frac{30 \times 10^6 \times 0.375^3}{12(1 - 0.25^2)} = 140,625$$

$$r' = \frac{pa^2}{2Et} (2 - u) = \frac{30 \times 10^2(2 - 0.25)}{2 \times 30 \times 10^6 \times 0.125} = 0.0007''$$

Using these values in the moment expressions above give

M_r = <u>158 # - inches/inch</u>

M_t = <u>123 # - inches/inch</u>

<u>EXAMPLE 2</u> - A thin-walled pressure vessel is made of steel with Poisson's ratio of 0.25 and $E = 30 \times 10^6$. The shell is ½" thick, 10" long and 42¾" in diameter. If the internal pressure is 300 psi and the shell is restrained against radial translation at the ends but is otherwise free, find the transverse shear at the ends.

Solution:

$$D = \frac{Et^3}{12(1 - u^2)} = \frac{30 \times 10^6 \times 0.5^3}{12(1 - 0.25^2)} = 333,333$$

$$4B^4 = \frac{Et}{Dr^2} = \frac{30 \times 10^6 \times 0.25}{333,333 \times 21.375^2}$$

$$B = 0.280$$

$$B^2 = 0.0784$$

$$B^3 = 0.02195$$

$$BL = 0.280 \times 10 = 2.80$$

$$r' = \frac{Pr^2}{2Et}(2 - u) = \frac{300 \times 21.375^2}{2 \times 30 \times 10^6}(2 - 0.25) = 0.0039978"$$

This is a short shell and the deflection is defined by

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'$$

Since all radial deflection is prevented at the ends of the shell the following two equations may be written

$$0 = A_1 + C_1 + r'$$

$$0 = e^{BL} \cos BL\, A_1 + e^{BL} \sin BL\, B_1 + e^{-BL} \cos BL\, C_1 + e^{-BL} \sin BL\, D_1 + r'$$

The equation for the moment existing in the shell is

$$M = 2DB^2e^{Bx}(-A_1 \sin Bx + B_1 \cos Bx) + 2DB^2(C_1 \sin Bx - D_1 \cos Bx)e^{-Bx}$$

Since the shell is free to rotate at the ends, the moments at these points must be zero, and

$$0 = 2DB^2 \ (B_1 - D_1)$$

$$0 = 2DB^2e^{BL}(-A_1 \sin BL + \cos BL) + 2DBe^{2-BL}(C_1 \sin BL - D_1 \cos BL)$$

There are now four equations containing the four unknown constants. The constants are then

$$A_1 = \frac{r'(e^{-2BL} - \cos 2\ BL + 2 \sinh BL \cos BL)}{2 \cos 2\ BL - 2 \cosh 2\ BL} = 0.00024100$$

$$B_1 = \frac{r'(2 \cosh BL \sin BL - \sin 2\ BL)}{2 \cos 2\ BL - 2 \cosh 2\ BL} = 0.000093082$$

$$C_1 = \frac{r'(e^{2BL} - \cos 2BL - 2 \sinh BL \cosh BL)}{2 \cos 2\ BL - 2 \cosh 2\ BL} = 0.00423895$$

$$D_1 = \frac{r'(2 \cosh BL \sin BL - \sin 2\ BL}{2 \cos 2\ BL - 2 \cosh 2\ BL} = 0.000093082$$

The equation for the transverse shear is

$$V = 2DB^3e^{Bx}\left[-A_1(\cos Bx + \sin Bx) + B_1(\cos Bx - \sin Bx)\right]$$
$$+ 2DB^3e^{-Bx}\left[C_1(\cos Bx - \sin Bx) + D_1(\cos Bx + \sin Bx)\right]$$

At the end where $x = 0$

$$V = 2DB^3 \ \ -A_1 + B_1 + C_1 + D_1$$

$$= 2 \times 333{,}333 \times 0.02195 \left[-0.00024100 - 0.000093082 - 0.00423895 - 0.000093082\right]$$

$$= \underline{\underline{69 \ \#/in.}}$$

It should be noted that the major portion of the work involved in this solution is in the determination of the expressions for the constants A_1, B_1, C_1 and D_1. The expressions above were obtained by solving a fourth order determinant.

SECTION II

SUCCESSIVE CORRECTIONS APPLIED TO THIN-WALLED PRESSURE VESSELS

INTRODUCTION - In the preceeding analytical analysis the ends of the shell rotated and translated simultaneously in going from the unstressed to the stressed equilibrium positions. These rotations and translations were unknown and depended on the physical properties of the shell and its restraining members. Because of this it was necessary to solve the equations for the deflections of the shell simultaneously with those of its restraining members.

A simplified analysis can be made by applying artificial restraints at the ends of the members to prevent rotation and translation. The deflection of each of the members is then known at a sufficient number of points to allow the moments and shears in any member to be found independently of the other members. The artificial restraints can then be removed by an application of the method of successive corrections.

THE METHOD OF SUCCESSIVE CORRECTIONS - Figure 9(a) is a diagram of the longitudinal cross-section of the neutral surface of an unstressed pressure vessel. The points a and b represent the joints between the shell

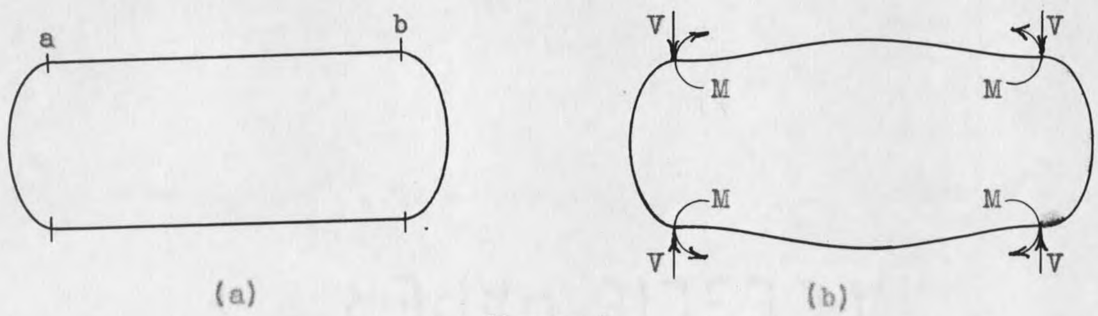


Figure 9

and the heads. Figure 9(b) shows the same vessel subjected to an internal fluid pressure and artificial restraining moments and forces which prevent any rotation or translation of the joints. These restraints are removed one at a time while holding the other joints fixed with the remaining restraints.

Since the restraints are applied to the joint and not to any individual member and since the joint must be in equilibrium, the restraints will be distributed among all of the members at the joint. The amount of moment and the amount of shear which each member will assume is called the fixed-end moment and the fixed-end shear respectively.

If the restraining moment is removed from one of the joints of the vessel in Figure 9(b), the joint will be statically unbalanced. This unbalance will cause the joint to rotate until equilibrium conditions are again attained. The moments existing in the members at the joint will change, one increasing and the other decreasing until they are of equal magnitude and opposite in sign. The amount of moment change in each member will be proportional to the stiffness of the member. The sum of the individual changes in all of the members will be equal to the amount of external moment removed.

Since the rotation of one joint changes the deflection curve of the neutral plane of the vessel between that joint and the adjacent joints, there occurs simultaneously with the change in moments at the rotated joint, a change in the external force required at these joints if they are not to translate, and a change in the external moment required at the adjacent joints if they are not to rotate. This condition is shown in

Figure 10(a) which has the restraining moment removed from the left-hand end.

Although the removal of the restraining moment at one end increases the restraints at the other end and a change in the shear at the rotated end, the changes are always less than the amount of restraint removed and the vessel in Figure 10(a) is nearer to the condition of unrestrained equilibrium than the vessel in Figure 9(b).

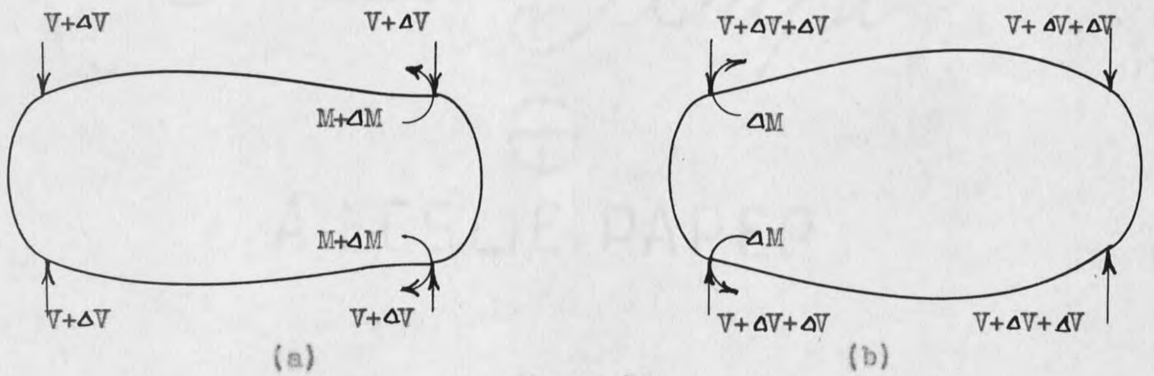


Figure 10

Figure 10(b) shows the vessel and its restraints after removal of restraining moments at both the left and right ends. The change in the restraints due to these removals is denoted by ΔV and ΔM. The removal of the restraints is accomplished by allowing the joint to rotate or translate to its equilibrium position by applying forces or moments equal in magnitude to the artificial restraining forces or moments, but opposite in direction, at the point of restraint.

The removal of a restraining force at a joint causes a change in the deflection of the neutral plane as does the removal of a restraining moment and creates a change in the restraining moment at that joint and a change in the rotation and deflection restraints at the adjacent joints.

However, as in the case of a moment removal the restraints which are created are always smaller than the restraint removed and the vessel continues to approach the condition of unrestrained equilibrium. In the case of most pressure vessels the restraints remaining after removing each of the restraints at a joint twice, will be negligible.

This process of successive approximations is simple to carry out as will be demonstrated in the succeeding pages. The work remains entirely numerical as long as only one restraint is removed at a time and the other joints remain fixed in position.

In the succeeding work the algebraic signs of the quantities have been ignored except in the case of non-uniform pressure distribution. This enable the solutions to be carried out from the physical rather than the mathematical standpoint.

FIXED-END MOMENTS IN LONG THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO NON-UNIFORM PRESSURE DISTRIBUTION - The amount of bending moment at the ends of a long thin-walled cylindrical shell subjected to a non-uniform pressure distribution can be found by solving the general deflection equation. This equation is

$$v = e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + f(x)$$

Since under fixed-end conditions $v = 0$ at $x = 0$ with the origin considered to be at one end of the shell

$$C_1 = -f(0)$$

where f(o) is the change in radius due to the membrane stresses at $x = 0$.

Letting ϕ be the rotation at the end of the shell

$$\phi = \frac{dy}{dx}$$

or

$$\phi = -Be^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + Be^{-Bx}(-C_1 \sin Bx + D_1 \cos Bx) + f'(x)$$

where $f'(x)$ is the first derivative of the membrane deflection.

Under fixed-end conditions $\phi = 0$ at $x = 0$ and

$$0 = -B(C_1 - D_1) + f'(o)$$

where $f'(o)$ is the derivative of the membrane deflection at $x = 0$.

Substituting the value of C_1 into the above equation and solving for D_1 gives

$$D_1 = -f(o) - \frac{f'(o)}{B}$$

Another derivation of the deflection equation gives

$$\frac{d^2y}{dx^2} = -2B^2e^{-Bx}(-C_1 \sin Bx + D_1 \cos Bx) + f''(x)$$

By equation (8) the fixed-end moment at $x = 0$ may then be written as

$$FEM = 2DB^2\left[f(o) + \frac{f'(o)}{B}\right] + Df''(o) \quad . \; . \; . \; . \; . \; . \; . \; (9)$$

and the fixed-end moment at $x = L$ as

$$FEM = 2DB^2e^{-BL}\left[\left(f(o) + \frac{f'(o)}{B}\right)\cos BL + f(o)\sin BL\right] + Df''(L) \quad . \; . \; . \; (10)$$

where $f''(o)$ and $f''(L)$ are the second derivatives of the membrane deflection at $x = 0$ and $x = L$ respectively.

FIXED-END MOMENTS IN LONG THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO UNIFORM PRESSURE DISTRIBUTION - The fixed-end moments in long thin-walled cylindrical shells can be obtained directly from equations (9) and (10). The uniform pressure distribution substitutes a constant membrane deflection r' for the variable $f(x)$. Since the derivatives of a constant are zero, equations (9) and (10) become

$$FEM = 2DB^2r' \quad . . . . (11)$$

By substituting the expressions for D and r' and rearranging the terms, this equation can be written

$$FEM = \frac{pr^2}{4}(2 - \nu)\frac{1}{\sqrt{3(1 - \nu^2)}} \quad . . . . (11a)$$

FIXED-END MOMENTS IN SHORT THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO NON-UNIFORM DISTRIBUTION - The fixed-end moments in short thin-walled shells can be found by a procedure similar to that used in finding the fixed-end moments in long thin-walled shells. However, as will be seen the resulting expression is much more cumbersome for the general case.

The deflection equation for a short shell is given by equation (8) as

$$y = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + f(x)$$

and since the slope at any point in the shell is equal to the differential of the deflection equation

$$\phi = Be^{Bx}\left[A_1(\cos Bx - \sin Bx) + B_1(\cos Bx + \sin Bx)\right]$$
$$+ Be^{-Bx}\left[C_1(\cos Bx + \sin Bx) - D_1(\cos Bx - \sin Bx)\right] + f'(x)$$

The boundary conditions for the fixed-end condition of a short vessel are the same as those for a long vessel, that is, $v = 0$ when $x = 0$; $\phi = 0$ when $x = 0$; $v = 0$ when $x = L$; and $\phi = 0$ when $x = L$. Using these conditions in the two equations written above it is possible to write the following four equations.

$$A_1 + C_1 = -f(o)$$

$$A_1 + B_1 - C_1 + D_1 = -\frac{f'(o)}{B}$$

$$A_1e^{BL}\cos BL + B_1e^{BL}\sin BL + C_1e^{-BL}\cos BL + D_1e^{-BL}\sin BL = -f(L)$$

$$A_1e^{BL}(\cos BL - \sin BL) + B_1e^{BL}(\cos BL + \sin BL)$$
$$- C_1e^{-BL}(\cos BL + \sin BL) + D_1e^{-BL}(\cos BL - \sin BL) = -\frac{f'(L)}{B}$$

From these equations it is evident that all of the constants A_1, B_1, C_1 and D_1 will have a common denominator equal to the value of a determinant made up of all of the terms which appear to the left of the equal signs. The numerators for each of the constants will be the values of the determinants composed of the terms to the left of the equal signs with the terms to the right of the equal signs replacing the coefficients of the constant being solved for. Denoting the denominator determinant by m and the numerator determinants of A_1, B_1, C_1 and D_1 by a, b, c, and d respectively

$$m = \begin{vmatrix} +1 & 0 & +1 & 0 \\ +1 & +1 & -1 & +1 \\ +e^{BL}\cos BL & +e^{BL}\sin BL & +e^{-BL}\cos BL & +e^{-BL}\sin BL \\ +e^{BL}(\cos BL - \sin BL) & +e^{BL}(\cos BL + \sin BL) & -e^{-BL}(\cos BL + \sin BL) & +e^{-BL}(\cos BL - \sin BL) \end{vmatrix}$$

$$a = \begin{vmatrix} -f(0) & 0 & +1 & 0 \\ -\frac{f'(0)}{B} & +1 & -1 & +1 \\ -f(L) & +e^{BL}\sin BL & +e^{-BL}\cos BL & +e^{-BL}\sin BL \\ -\frac{f'(L)}{B} & +e^{BL}(\cos BL + \sin BL) & -e^{-BL}(\cos BL + \sin BL) & +e^{-BL}(\cos BL - \sin BL) \end{vmatrix}$$

$$b = \begin{vmatrix} +f(0) & +1 & +1 & 0 \\ +\frac{f'(0)}{B} & +1 & -1 & +1 \\ +f(L) & +e^{BL}\sin BL & +e^{-BL}\cos BL & +e^{-BL}\sin BL \\ +\frac{f'(L)}{B} & +e^{BL}(\cos BL - \sin BL) & -e^{-BL}(\cos BL + \sin BL) & +e^{-BL}(\cos BL - \sin BL) \end{vmatrix}$$

$$c = \begin{vmatrix} -f(0) & +1 & 0 & 0 \\ -\frac{f'(0)}{B} & +1 & +1 & +1 \\ -f(L) & +e^{BL}\cos BL & +e^{BL}\cos BL & +e^{-BL}\sin BL \\ -\frac{f'(L)}{B} & +e^{BL}(\cos BL - \sin BL) & +e^{BL}(\cos BL + \sin BL) & +e^{-BL}(\cos BL - \sin BL) \end{vmatrix}$$

$$d = \begin{vmatrix} +f(0) & +1 & 0 & +1 \\ +\frac{f'(0)}{B} & +1 & +1 & -1 \\ +f(L) & +e^{BL}\cos BL & +e^{BL}\sin BL & +e^{-BL}\cos BL \\ +\frac{f'(L)}{B} & +e^{BL}(\cos BL - \sin BL) & +e^{BL}(\cos BL + \sin BL) & -e^{-BL}(\cos BL + \sin BL) \end{vmatrix}$$

The solutions of these determinants are found to be

$$m = 2 \cosh 2BL + 2BL - 4$$

$$a = f(0)(\sin 2BL - \cos 2BL + 2 - e^{-2BL}) + \frac{f'(0)}{B}(2 \sin^2 BL)$$
$$+ f(L)(-2 \sinh BL \cos BL - 2 \cosh BL \sin BL) + \frac{f'(L)}{B}(2 \sinh BL \sin BL)$$

$$b = f(0)(e^{-2BL} - \sin 2BL - \cos 2BL) + \frac{f'(0)}{B}(1 - \sin 2BL - e^{-2BL})$$
$$+ f(L)(2 e^{-BL} \sin BL + 2 \sinh BL \cos BL - 2 \sinh BL \sin BL)$$
$$+ \frac{f'(L)}{B}(2 e^{-BL} \sin BL - 2 \sinh BL \cos BL)$$

$$c = f(0)(2 - \sin 2BL - \cos 2BL - e^{2BL}) + \frac{f'(0)}{B}(-2 \sin^2 BL)$$
$$+ f(L)(2 \sinh BL \cos BL + 2 \cosh BL \sin BL) + \frac{f'(L)}{B}(-2 \sinh BL \sin BL)$$

$$d = f(0)(\cos 2BL - \sin 2BL - e^{2BL}) + \frac{f'(0)}{B}(\sin 2BL + 1 - e^{2BL})$$
$$+ f(L)(2 \sinh BL \cos BL + 2 \sinh BL \sin BL + 2 e^{BL} \sin BL)$$
$$+ \frac{f'(L)}{B}(2 \sinh BL \cos BL - 2 e^{BL} \sin BL)$$

The constants are now evaluated as $A_1 = a/m$, $B_1 = b/m$, $C_1 = c/m$ and $D_1 = d/m$.

Denoting the fixed-end moments at the ends $x = 0$ and $x = L$ as FEM_0 and FEM_L respectively and using equation (2) for the moment at any point in the shell

$$M = D\frac{d^2 y}{dx^2} = 2DB^2 e^{Bx}(-A_1 \sin Bx + B_1 \cos Bx)$$
$$+ 2DB^2 e^{-Bx}(C_1 \sin Bx - D_1 \cos Bx) + D f''(x)$$

$$FEM_0 = 2DB^2\left[\frac{b - d}{m}\right] + D f''(0)$$

$$= 2DB^2\left[J_1 f(0) + J_2 \frac{f'(0)}{B} + J_3 f(L) + J_4 \frac{f'(L)}{B}\right] + D f''(0) \quad . \; . \; (12)$$

$$FEM_L = 2DB^2 e^{BL}(-A_1 \sin BL + B_1 \cos BL)$$

$$+ 2DB^2 e^{-BL}(C_1 \sin BL - D_1 \cos BL) + D f''(L)$$

$$= 2DB^2\left[J_3 f(0) - J_4 \frac{f'(0)}{B} + J_1 f(L) - J_2 \frac{f'(L)}{B}\right] + D f''(L) \quad . \; . \; (13)$$

where

$$J_1 = \frac{\cosh 2BL - \cos 2BL}{\cosh 2BL + \cos 2BL - 2}$$

$$J_2 = \frac{\sinh 2BL - \sin 2BL}{\cosh 2BL + \cos 2BL - 2}$$

$$J_3 = \frac{-4 \sinh BL \sin BL}{\cosh 2BL + \cos 2BL - 2}$$

$$J_4 = \frac{2 \cosh BL \sin BL - 2 \sinh BL \cos BL}{\cosh 2BL + \cos 2BL - 2}$$

The values of the J functions above are given in Table I of the appendix.

FIXED-END MOMENTS IN SHORT THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO UNIFORM PRESSURE DISTRIBUTION - As in the case of the long shell, the application of a uniform pressure distribution reduces the expressions for the fixed-end moments. The variable f(x) is replaced in the original equation by the constant P' and the final expression becomes

$$FEM = 2\,DB^2(J_1 + J_3)\,P' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (14)$$

FIXED-END SHEAR IN LONG THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO NON-UNIFORM PRESSURE DISTRIBUTION - The shear at any point in a thin-walled cylindrical shell is given by equation (5) as

$$V = D\frac{d^3y}{dx^3}$$

and since the fixed-end conditions are the same as those used for finding the fixed-end moments, the expressions for the fixed-end shears may be obtained by using the values of the constants as found in the derivation of the fixed-end moment expressions.

Since

$$\frac{d^3y}{dx^3} = 2B^3e^{-Bx}\left[C_1(\cos Bx - \sin Bx) + D_1(\cos Bx + \sin Bx)\right] + f'''(x)$$

the shear existing at the end x = 0 becomes

$$FES_0 = -2DB^3\left[2\,f(0) + \frac{f'(0)}{B}\right] + D\,f'''(0)\;.\;.\;.\;.\;.\;.\;(15)$$

and the shear at the end x = L becomes

$$FES_L = -2DB^3e^{-BL}\left[2\,f(0)\,\cos BL + \frac{f'(0)}{B}(\cos BL + \sin BL)\right] + D\,f'''(L)\,.(16)$$

where $f'''(0)$ and $f'''(L)$ denote the third derivatives of the membrane deflection at the ends x = 0 and x = L respectively.

<u>FIXED-END SHEAR IN LONG THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO UNIFORM PRESSURE DISTRIBUTION</u> - The fixed-end shear existing in a long thin-walled cylindrical shell under uniform pressure is found by substituting r' for f(x) in the above equations. This gives for the fixed-end shear

$$FES = 4DB^3\,r' = 2BFEM\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(17)$$

<u>FIXED-END SHEAR IN SHORT THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO NON-UNIFORM PRESSURE DISTRIBUTION</u> - The fixed-end shear in the short cylindrical shell may be had by using the constants as found in the derivation of the fixed-end moment expression in the short cylindrical shell.

Equation (3) gives

$$V = 2DB^3e^{Bx}\left[- A_1(\cos Bx + \sin Bx) + B_1(\cos Bx - \sin Bx)\right]$$
$$+ 2DB^3e^{-Bx}\left[C_1(\cos Bx - \sin Bx) + D_1(\cos Bx + \sin Bx)\right] + D\, f'''(x)$$

Letting FES_o and FES_L be the fixed-end shears at $x = 0$ and $x = L$ respectively

$$FES_o = 2DB^3\left[- A_1 + B_1 + C_1 + D_1\right] + D\, f'''(0)$$

$$= 2DB^3\left[J_5\, f(0) - J_1\frac{f'(0)}{B} + J_6\, f(L) + J_3\frac{f'(L)}{B}\right] + D\, f'''(0) \quad . (18)$$

$$FES_L = 2DB^3e^{BL}\left[- A_1(\cos BL + \sin BL) + B_1(\cos BL - \sin BL)\right]$$
$$+ 2DB^3e^{-BL}\left[C_1(\cos BL - \sin BL) + D_1(\cos BL + \sin BL)\right] + D\, f'''(L)$$

$$= 2DB^3\left[- J_5\, f(0) - J_3\frac{f'(0)}{B} - J_6\, f(L) - J_1\frac{f'(L)}{B}\right] + D\, f'''(L) . (19)$$

where

$$J_5 = \frac{-2\sinh 2BL - 2\sin 2BL}{\cosh 2BL + \cos 2BL - 2}$$

$$J_6 = \frac{4\sinh BL\cos BL + 4\cosh BL\sin BL}{\cosh 2BL + \cos 2BL - 4}$$

The values of J_5 and J_6 may be found in Table I of the appendix.

FIXED-END SHEAR IN SHORT CYLINDRICAL THIN-WALLED SHELLS SUBJECTED TO UNIFORM PRESSURE DISTRIBUTION - In the case of uniform pressure distribution where the constant membrane displacement is used in place of the general displacement, equations (18) and (19) become

$$FES = 2DB^3(J_5 + J_6)r' \quad . . . . (20)$$

CARRY-OVER FACTORS FOR SHORT THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO NON-UNIFORM PRESSURE DISTRIBUTION - The carry-over factor for a short thin-walled cylindrical shell can be found by using the equation for the deflection of such a shell and placing a unit moment at one end. The moment at the other end is then the moment carry-over factor or the amount of moment produced at one end by a unit change in the moment at the other end.

Since the only load on the shell is a unit bending moment at one end the membrane terms are non-existent and

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx)$$

Using the boundary conditions that $v = 0$ at $x = 0$, $\frac{d^2v}{dx^2} = 1$ at $x = 0$, $v = 0$ at $x = L$ and $\phi = 0$ at $x = L$, the following equations can be written

$$A_1 + C_1 = 0$$

$$+ B_1 2B^2 - D_1 2B^2 = 1$$

$$A_1 e^{BL} \cos BL + B_1 e^{BL} \sin BL + C_1 e^{-BL} \cos BL + D_1 e^{-BL} \sin BL = 0$$

$$A_1 e^{BL}(\cos BL - \sin BL) + B_1 e^{BL}(\cos BL + \sin BL) - C_1 e^{-BL}(\cos BL + \sin BL) + D_1 e^{-BL}(\cos BL - \sin BL) = 0$$

As before let the denominator of the constants be m and the numerators of A_1, B_1, C_1 and D_1 be a, b, c and d, respectively, then

$$m = 4B^2(\sin 2BL - \sinh 2BL$$

$$a = - 2 \sin^2 BL$$

$$b = e^{-2BL} + \sin 2BL - 1$$

$$c = 2 \sin^2 BL$$

$$d = -1 \sin 2BL + e^{2BL}$$

The constants are: $A_1 = a/m$, $B_1 = b/m$, $C_1 = c/m$ and $D_1 = d/m$. The substitution of these values into the moment equation

$$M = 2DB^2e^{BL}(-A_1 \sin BL + B_1 \cos BL) + 2DB^2e^{-BL}(C_1 \sin BL - D_1 \cos BL)$$

results in the carry-over factor being

$$C = -\frac{2 \cosh BL \sin BL - 2 \sinh BL \cos BL}{\sin 2BL - \sinh 2BL} \quad \ldots\ldots\ldots (21)$$

The values of C are tabulated in Table II of the appendix.

THE EFFECT OF A UNIT ANGLE CHANGE ON THE MOMENT AT THE END OF A LONG THIN-WALLED CYLINDRICAL SHELL - Since the rotations of all members at the joint are the same, the distribution factors may be calculated from this angle change. For this reason it is necessary to have the relationship between the rotation and the moment at the end of the shell. This may be done by assuming boundary conditions of $v = 0$ at $x = 0$, $\phi = 1$ at $x = 0$, and solving for the moment created.

Since

$$v = e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx)$$

and

$$\phi = -Be^{-Bx}\, C_1(\cos Bx - \sin Bx) + D_1(\cos Bx + \sin Bx)$$

the following equations may be written

$$0 = C_1 \text{ or } C_1 = 0$$

and

$$1 = -B\, D_1 \text{ or } D_1 = -1/B$$

Then

$$M' = D\frac{d^2v}{dx^2} = 2DB^2\ (-D_1) = +\ 2DB \quad . \ . \ . \ . \ . \ . \ . \ . \ (22)$$

The distribution factor for the moments at any joint may then be had by calculating the effect of a unit angle change on the moment in the joining members and distributing this moment unbalance in the relation

$$f_s = \frac{M'_s}{M'_s + M'_h}$$

$$f_h = \frac{M'_h}{M'_s + M'_h}$$

where f is the moment distribution factor for the member indicated by the subscript.

<u>THE EFFECT OF A UNIT ANGLE CHANGE ON THE MOMENT AT THE END OF A SHORT THIN-WALLED CYLINDRICAL SHELL</u> - The boundary conditions for this case may be set up as, at $x = 0$, $v = 0$; at $x = L$, $v = 0$; at $x = 0$, $\phi = 1$; and at $x = L$, $\phi = 0$. This allows the following equations to be written.

$$A_1 + C_1 = 0$$

$$A_1 + B_1 - C_1 + D_1 = \frac{1}{B}$$

$$A_1^{BL} \cos BL + B_1e^{BL} \sin BL + C_1e^{-BL} \cos BL + D_1e^{-BL} \sin BL = 0$$

$$A_1e^{BL}(\cos BL - \sin BL) + B_1e^{BL}(\cos BL + \sin BL) - C_1e^{-BL}(\cos BL + \sin BL) + D_1e^{-BL}(\cos BL - \sin BL) = 0$$

As before the constants may be evaluated by using determinants. Letting m be the denominator and a, b, c and d be the numerators of A_1, B_1, C_1 and D_1 respectively.

$$m = 2 \cosh 2BL + 2 \cos 2BL - 4$$

$$a = \frac{-2 \sin^2 BL}{B}$$

$$b = \frac{e^{-2BL} + \sin 2BL - 1}{B}$$

$$c = \frac{2 \sin^2 BL}{B}$$

$$d = \frac{e^{2BL} - \sin 2BL - 1}{B}$$

Since $A_1 = a/m$, $B_1 = b/m$, $C_1 = c/m$ and $D_1 = d/m$ as before, the expression for the resulting moment at the joint is found as

$$M = 2DB^2 e^{Bx}(-A_1 \sin Bx + B_1 \cos Bx) + 2DB^2 e^{-Bx}(C_1 \sin Bx - D_1 \cos Bx)$$

or

$$M' = 2DB\, J_2 \quad \ldots\ldots\ldots\ldots\ldots \quad (23)$$

The distribution factors are found as for long shells.

THE EFFECT OF A UNIT MOMENT CHANGE ON SHEAR AT THE END OF A LONG THIN-WALLED CYLINDRICAL SHELL - In finding the change in the shearing forces caused by balancing the bending moments it is necessary to know the relationship between a unit moment change and the corresponding change in shear. This may be obtained by placing a unit moment on the end of the vessel and solving for the resulting shear. The boundary conditions are then, at $x = 0$, $v = 0$, and at $x = 0$, $\frac{d^2v}{dx^2} = 1$. Then

$$v = e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx)$$

and

$$C_1 = 0$$

$$\frac{d^2v}{dx^2} = 2B^2e^{-Bx}(C_1 \sin Bx + D_1 \cos Bx)$$

$$D_1 = -\frac{1}{2B^2}$$

and the shear is then

$$V_1 = B \quad . . . . (24)$$

THE EFFECT OF A UNIT MOMENT CHANGE ON SHEAR IN A SHORT THIN-WALLED CYLINDRICAL SHELL - Since the constants have already been evaluated for a unit moment change in short shells it is easy to find the relationship between the shear and the moment. The constants as found in the solution for the carry-over factor when substituted into the equation for the shear give

$$V_1 = \frac{B(\cosh 2BL - \cos 2BL)}{\sin 2BL - \sinh 2BL} = BK_1 \quad . . . . (25)$$

The values of K_1 are given in Table II of the appendix.

THE EFFECT OF A UNIT DISPLACEMENT ON THE SHEAR AT THE END OF A LONG THIN-WALLED CYLINDRICAL SHELL - In order to find the shear distribution factors between members at a joint it is necessary to know how much the shear will change with a unit displacement since all members will displace the same amount. This case has the boundary conditions of $v = 1$ at $x = 0$, and $\phi = 0$ at $x = 0$. Then

$$C_1 = 1$$

$$D_1 = 1$$

and

$$V_2 = 4B^3D \quad \dots\dots\dots\dots\dots \quad (26)$$

THE EFFECT OF A UNIT DISPLACEMENT ON THE SHEAR IN A SHORT THIN-WALLED CYLINDRICAL SHELL - In the case of a unit displacement in short shells the boundary conditions are $v = 1$ at $x = 0$, $v = 0$ at $x = L$, $\emptyset = 0$ at $x = 0$, and $\emptyset = 0$ at $x = L$. From these the following equations may be written

$$A_1 + C_1 = 1$$

$$A_1 + B_1 - C_1 + D_1 = 0$$

$$A_1e^{BL}\cos BL + B_1e^{BL}\sin BL + C_1e^{-BL}\cos BL + D_1e^{-BL}\sin BL = 0$$

$$A_1e^{BL}(\cos BL - \sin BL) + B_1e^{BL}(\cos BL + \sin BL) - C_1e^{-BL}(\cos BL + \sin BL) + D_1e^{-BL}(\cos BL - \sin BL) = 0$$

Using the same notation as before,

$$m = 2\cosh 2BL + 2\cos 2BL - 4$$

$$a = e^{-2BL} + \cos 2BL - \sin 2BL - 2$$

$$b = -e^{-2BL} + \cos 2BL + \sin 2BL$$

$$c = e^{2BL} + \sin 2BL + \cos 2BL - 2$$

$$d = e^{2BL} + \sin 2BL - \cos 2BL$$

These values for the constants when used with the equation for the shear give

$$V = 2DB^3e^{Bx}\left[-A_1(\cos Bx + \sin Bx) + B_1(\cos Bx - \sin Bx)\right] + 2DB^3e^{-Bx}\left[C_1(\cos Bx - \sin Bx) + D_1(\cos Bx + \sin Bx)\right]$$

or at $x = 0$

$$V = 2DB^3\left[-A_1 + B_1 + C_1 + D_1\right]$$

then

$$V_2 = \frac{4DB^3 \quad \sinh 2BL + \sin 2BL}{\cosh 2BL + \cos 2BL - 2} = 4DB^3K_2 \quad . \; . \; . \; . \; . \; . \quad (27)$$

The values of the constant K_2 are given in Table II of the appendix.

THE EFFECT OF A UNIT CHANGE IN SHEAR AT ONE END OF THE MOMENTS AT BOTH ENDS OF A SHORT THIN-WALLED CYLINDRICAL SHELL - Since the carry-over factor in the case of a unit translation is not necessarily the same as the carry-over for a unit rotation it is necessary to have a means of finding the moment change caused by a change in shear. For this case the boundary conditions are $\phi = 0$ at $x = 0$, $\phi = 0$ at $x = L$, $V = 1$ at $x = 0$, and $V = 0$ at $x = L$. This gives the following four equations.

$$A_1 + B_1 - C_1 + D_1 = 0$$

$$-A_1 + B_1 + C_1 + D_1 = \frac{1}{2DB^3}$$

$$A_1e^{BL}\cos BL + B_1e^{BL}\sin BL + C_1e^{-BL}\cos BL + D_1e^{-BL}\sin BL = 0$$

$$A_1e^{BL}(\cos BL - \sin BL) + B_1e^{BL}(\cos BL + \sin BL) - C_1e^{-BL}(\cos BL + \sin BL) + D_1e^{-BL}(\cos BL - \sin BL) = 0$$

Using determinants and the notation as before

$$m = 4 \sinh 2BL + 4 \sin 2BL$$

$$a = \frac{1}{2DB^3}\left(e^{-2BL} + \cos 2BL - \sin 2BL - 2\right)$$

$$b = \frac{1}{2DB^3}\left(\sin 2BL + \cos 2BL - e^{-2BL}\right)$$

$$c = \frac{1}{2DB^3}\left(e^{2BL} + \cos 2BL + \sin 2BL - 2\right)$$

$$d = \frac{1}{2DB^3}\left(e^{2BL} + \sin 2BL - \cos 2BL\right)$$

These values for the constants when substituted into the equation for the moment at the end $x = 0$ and the end $x = L$ respectively give

$$M_0' = \frac{\cos 2BL - \cosh 2BL}{2B(\sin 2BL + \sinh 2BL)} = \frac{K_3}{B} \quad . . . . (28)$$

and

$$M_L' = \frac{2 \sinh BL \sin BL}{B(\sin 2BL + \sinh 2BL)} = \frac{K_4}{B} \quad . . . . (29)$$

The values of K_3 and K_4 are tabulated in Table II of the appendix.

FIXED-END MOMENTS AND SHEARS IN FLAT CIRCULAR HEADS - The fixed-end moments in a flat plate when used as the head of a cylindrical pressure vessel can be found in a manner similar to that used for the fixed-end moments in the shell. The head has a fixed-end shearing force however, which can be found much more simply than that of the shell as it is the membrane stress transmitted to the shell and given by

$$\frac{\text{total force}}{\text{shearing area}} = \frac{\pi r^2 p}{2h\pi r} = \frac{p\,r}{2\,h} \quad . . . . (30)$$

If the thickness of the head h is replaced by the shell thickness t, this will be recognized as the longitudinal membrane stress in the shell.

The deflection is given by the equation

$$w = \frac{p\,r^4}{64\,D} - \frac{c_1 r^2}{4} + c_3$$

and

$$-\phi = \frac{dw}{dr} = \frac{p\ r^3}{16\ D} + \frac{c_1 r}{2}$$

but ϕ is zero when r is a maximum. By letting the radius of the vessel be 'a'

$$0 = \frac{p\ a^3}{16\ D} + \frac{c_1 a}{2}$$

or

$$c_2 = \frac{-p\ a^2}{8\ D}$$

The deflection equation is then

$$w = \frac{p\ r^4}{64\ D} - \frac{p\ r^2\ a^2}{32\ D} + c_3$$

and since the deflection is zero when r = a

$$0 = \frac{-p\ a^4}{64\ D} + c_3$$

or

$$c_3 = \frac{p\ a^4}{64\ D}$$

The final deflection equation can now be written as

$$w = \frac{p\ r^4}{64\ D} - \frac{p\ a^2 r^2}{32\ D} + \frac{p\ a^4}{64\ D}$$

and

$$\frac{dw}{dr} = \frac{p\ r^3}{16\ D} - \frac{p\ a^2 r}{16\ D}$$

and

$$\frac{d^2w}{dr^2} = \frac{3\ p\ r^2}{16\ D} - \frac{p\ a^2}{16\ D}$$

The equations for the fixed-end moments in the plate are then given by equations (6) and (7) which give, using the values above

$$FEM_p = \frac{3\ p\ a^2}{16} - \frac{p\ a^2}{16} = \frac{p\ r^2}{8} \quad \ldots\ldots\ldots\ldots \quad (31)$$

and

$$FEM_t = u\,\frac{p\ r^2}{8} \quad \ldots\ldots\ldots\ldots\ldots \quad (32)$$

where r is the radius of the vessel.

<u>THE EFFECT OF A UNIT ANGLE CHANGE ON THE MOMENTS AT THE EDGE OF A FLAT CIRCULAR HEAD</u> - The distribution factors for cylindrical pressure vessels with flat heads can be obtained by placing a unit angle change on the joint between the shell and the head and solving for the resulting change in the moments in the plates and the shell. Since in the case of vessels with flat heads, the translation of the joint is negligible; there is no shear distribution factor.

Considering the plate we have

$$w = \frac{p\ r^4}{64\ D} + \frac{c_1 r^2}{4} + c_3$$

and

$$\frac{dw}{dr} = \frac{p\ r^3}{16\ D} + \frac{c_1 r}{2}$$

By arbitrarily placing a unit rotation at the edge of the plate and using the same notations as before

$$1 = \frac{p\ a^3}{16\ D} + \frac{c_1 a}{2}$$

and

$$c_1 = \frac{2}{a} \quad \frac{p\ a^2}{8\ D}$$

Therefore

$$\frac{dw}{dr} = \frac{p\,r^3}{16\,D} + \frac{r}{2}\;\frac{2}{a} - \frac{p\,a^2}{8\,D}$$

and

$$\frac{d^2w}{dr^2} = \frac{3\,p\,r^2}{16\,D} + \frac{1}{2}\;\frac{2}{a} - \frac{p\,a^2}{8\,D}$$

$$= \frac{p\,a^2}{8\,D} + \frac{1}{a}$$

Now designating the flexural rigidity of the plate as D_h and noting that the pressure is zero, the resulting moments are, where r is the radius of the vessel,

$$M_r' = M_t' = D_h\frac{(1 + u)}{r} \quad \ldots\ldots\ldots\ldots\ldots (33)$$

<u>MOMENT DISTRIBUTION FACTORS FOR VESSELS WITH LONG THIN-WALLED CYLINDRICAL SHELLS AND FLAT HEADS</u> - The moment distribution factor f is

$$f_s = \frac{M_s'}{M_s' + M_h'}$$

where M' denotes the moment caused by a unit angle change. For a long shell by equation (22)

$$M' = 2DB$$

and for a flat plate head by equation (33)

$$M_r' = M_t' = D_h\,\frac{(1 + u)}{r}$$

then denoting the distribution factor for the shell by f_s and the distribution factor for moments in the head in the radial and tangential directions by f_r and f_t respectively

$$f_s = \frac{2D_sB}{2D_sB + \frac{D_h}{r}(1 + u)} \quad \ldots \ldots \ldots \ldots \ldots (34)$$

and

$$f_r = f_t = \frac{\frac{D_h}{r}(1 + u)}{2D_sB + \frac{D_h}{r}(1 + u)} \quad \ldots \ldots \ldots \ldots \ldots (35)$$

These equations when inverted give a simpler solution or

$$\frac{1}{f_s} = 1 + \frac{D_h (1 + u)}{2D_sB} \quad \ldots \ldots \ldots \ldots \ldots (34a)$$

and

$$\frac{1}{f_r} = \frac{1}{f_t} = 1 + \frac{2D_sB}{\frac{D_h}{r}(1 + u)} \quad \ldots \ldots \ldots \ldots \ldots (35a)$$

In the foregoing expressions D_s and D_h denote the flexural rigidity of the shell and head respectively.

MOMENT DISTRIBUTION FACTORS FOR VESSELS WITH SHORT THIN-WALLED CYLINDRICAL SHELLS AND FLAT HEADS - The effect of a unit angle change in the moment at the end of a short thin-walled cylindrical shell is given by equation (23) as

$$M' = 2D_sB\, J_2$$

and for the flat head by equation (33) as

$$M_r' = M_t' = D_h \frac{(1 + u)}{r}$$

The distribution factors are then

$$f_s = \frac{2D_sB\, J_2}{2D_sBJ_2 + \frac{D_h}{r}(1 + u)} \quad \ldots \ldots \ldots \ldots \ldots (36)$$

and

$$f_r = f_t = \frac{\frac{Dh}{r}(1 + u)}{2D_sBJ_2 + \frac{Dh}{r}(1 + u)} \quad \ldots\ldots\ldots\ldots \quad (37)$$

Again the reciprocals are simpler

$$\frac{1}{f_s} = 1 + \frac{\frac{Dh}{r}(1 + u)}{2D_sBJ_2} \quad \ldots\ldots\ldots\ldots \quad (36a)$$

$$\frac{1}{f_r} = \frac{1}{f_t} = 1 + \frac{2D_sBJ_2}{\frac{Dh}{r}(1 + u)} \quad \ldots\ldots\ldots\ldots \quad (37a)$$

EXAMPLE 3 - Solve Example 1 by means of successive corrections.

Solution: The fixed-end moments for the shell by equation (11) is

$$FEM = 2DB^2r' = 2 \times 5,208 \times 1.3416 \times 0.0007 = 9.8 \text{ in-\#/in.}$$

The fixed-end moments in the heads are by equations (31) and (32)

$$FEM_r = \frac{Pr^2}{8} = \frac{30 \times 10^2}{8} = 375 \text{ in-\#/in.}$$

$$FEM_t = \frac{u\,Pr^2}{8} = \frac{30 \times 10^2 \times 0.25}{8} = 94 \text{ in-\#/in.}$$

The distribution of the moment unbalance at the joints by equation (34a) is

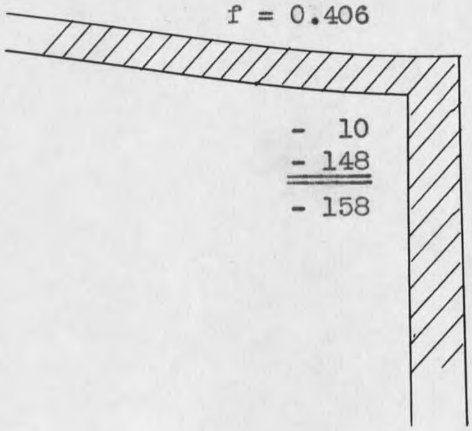
$$\frac{1}{f_s} = 1 + \frac{\frac{Dh}{r}(1 + u)}{2D_sB} = 1 + \frac{\frac{140,625}{10} \times (1 + 0.25)}{2 \times 5,208 \times 1.1683} = 2.457$$

$$f_s = 0.406$$

Since there are only two members at the joint the remainder of the moment unbalance must be distributed to the head or

$$f_r = f_t = 1 - 0.406 = 0.594$$

The moment distribution is now carried out as follows, assuming clockwise resisting moments are positive.



	M_s	M_r	M_t
f =	0.406	0.594	0.594
	- 10	+ 375	+ 94
	- 148	- 217	- 217
	- 158	+ 158	- 123

EXAMPLE 4 - Solve Example 2 by successive corrections.

Solution:

The fixed-end moment for the shell is given by equation (14) as

$$FEM = 2DB^2(J_1 + J_3)r' = 2 \times 333{,}333 \times 0.0784)(1.003 - 0.082)$$
$$(0.0039978) = 199\#\text{-in/in.}$$

The fixed-end shear in the shell is given by equation (20) as

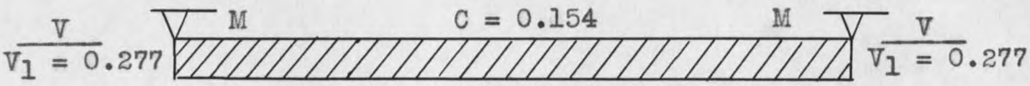
$$FES = 2DB^3(J_5 + J_6)r' = 2 \times 333{,}333 \times 0.02195\ (-2.009 - 0.148)$$
$$\times\ 0.0039978 = 133\#/\text{in}^2/\text{in.}$$

The shear correction factor is given by equation (25) as

$$V' = BK_1 = 0.280 \times 0.9897 = 0.277$$

The moment carry-over is given by equation (21) and Table II as $c = 0.154$.

The distribution is then



V	M	M	V
+ 133	+ 199	- 199	+ 133
	+ 31	+ 199	- 54
- 63	- 230	- 35	
	+ 5	+ 35	- 10
- 1	- 5	+ 1	
		- 1	
+ 69	0	0	+ 69

CONCLUSION - The relaxation method eliminates most of the tedious work involved in thin-walled pressure vessel analysis by introducing an artificial set of boundary conditions and then removing the artificial conditions by the Hardy Cross methods of moment and shear distribution. The artificial boundary conditions are introduced so that the fixed-end moments and shears may be easily computed using either formulas or tables.

Tables of the necessary constants are provided as a further aid to computation. This method is indicated as being valuable since the constants depend only on shell length and stiffness and will apply to all vessels with thin-walled cylindrical shells.

This paper does not cover all of the calculations which are necessary for the analysis of all types of thin-walled pressure vessels. The vessels treated are the simplest to analyze but they demonstrate the procedures and method of attack without confusing the reader to an unnecessary degree. On the other hand, the power of the method is not shown to the best advantage as the savings in time and labor are proportional to the complexity of the problem being treated.

Further applications can be made to vessels with eliptical shells and shells with non-uniform diameter in the longitudinal cross-section, as well as to vessels with hemispherical and eliptical heads. When this has been done the field of pressure vessel analysis will have been simplified and the true value of the application of relaxation methods to pressure vessels will have been realized.

APPENDIX

TABLE I

CONSTANTS FOR FIXED-END MOMENTS AND SHEARS IN SHORT SHELLS

BL	J_1	J_2	J_3	J_4	J_5	J_6
1.0	+ 3.104	+ 2.020	- 2.939	+ 0.986	- 6.740	+ 5.745
1.2	2.232	1.699	1.996	.803	4.356	3.170
1.4	1.731	1.480	1.413	.676	3.211	1.840
1.6	1.430	1.325	1.023	.570	2.624	1.080
1.8	1.246	1.215	0.743	.479	2.315	0.612
2.0	+ 1.134	+ 1.138	- 0.535	+ 0.400	- 2.152	+ 0.310
2.2	1.068	1.084	.375	.329	2.070	.111
2.4	1.031	1.049	.251	.265	2.031	- 0.019
2.6	1.012	1.027	.155	.207	2.014	.101
2.8	1.003	1.014	.082	.156	2.009	.148
3.0	+ 1.000	+ 1.007	- 0.028	+ 0.113	- 2.008	- 0.169
3.2	1.000	1.003	+ 0.010	.077	2.007	.173
3.4	1.001	1.001	.034	.048	2.007	.163
3.6	1.001	1.001	.048	.025	2.007	.147
3.8	1.002	1.001	.055	.008	2.006	.126
4.0	+ 1.002	+ 1.001	+ 0.056	- 0.004	- 2.004	- 0.103
4.2	1.001	1.001	.052	.011	2.003	.082
4.4	1.001	1.001	.047	.016	2.002	.062
4.6	1.001	1.001	.040	.018	2.001	.044
4.8	1.001	1.000	.033	.019	2.001	.030
5.0	+ 1.000	+ 1.000	+ 0.026	- 0.017	- 2.000	- 0.018
5.2	1.000	1.000	.019	.015	2.000	.009
5.4	1.000	1.000	.014	.013	2.000	.002
5.6	1.000	1.000	.009	.010	2.000	+ 0.002

TABLE II

CONSTANTS FOR DISTRIBUTIONS AND CARRY-OVER IN SHORT SHELLS

BL	K_1	K_2	K_3	K_4	C
1.0	- 1.538	+ 5.370	- 0.461	+ 0.436	+ 0.488
1.2	1.314	2.139	.512	.458	.472
1.4	1.170	1.606	.538	.440	.457
1.6	1.080	1.312	.545	.390	.430
1.8	1.026	1.157	.538	.321	.395
2.0	- 0.997	+ 1.076	- 0.527	+ 0.249	+ 0.352
2.2	.985	1.035	.516	.181	.303
2.4	.983	1.015	.508	.124	.252
2.6	.985	1.007	.502	.077	.202
2.8	.990	1.004	.500	.041	.154
3.0	- 0.994	+ 1.004	- 0.498	+ 0.014	+ 0.112
3.2	.997	1.004	.498	- 0.005	.077
3.4	.999	1.004	.499	.017	.047
3.6	1.000	1.003	.499	.024	.025
3.8	1.001	1.003	.499	.027	.008
4.0	- 1.001	+ 1.002	- 0.500	- 0.028	- 0.004
4.2	1.001	1.002	.500	.026	.011
4.4	1.000	1.001	.500	.023	.016
4.6	1.000	1.001	.500	.020	.018
4.8	1.000	1.000	.500	.016	.019
5.0	- 1.000	+ 1.000	- 0.500	- 0.013	- 0.017
5.2	1.000	1.000	.500	.010	.015
5.4	1.000	1.000	.500	.007	.013
5.6	1.000	1.000	.500	.005	.010

SUMMARY OF FORMULII FOR THE RELAXATION METHOD

General Formulii Applying to Thin-Walled Cylindrical Shells and Flat Heads

$$D = \frac{Et^3}{12(1 - u^2)}$$

$$4B^4 = \frac{Et}{D\ r^2}$$

$$r' = \frac{pr^2}{2Et}(2 - u)$$

Formulii Applying to Long Thin-Walled Cylindrical Shells Under Uniform Pressure Distribution

$$FEM = 2DB^2 r'$$

$$FES = 4DB^3 r'$$

$$M' = 2DB$$

$$V_1 = B$$

$$V_2 = 4DB^3$$

Formulii Applying to Short Thin-Walled Cylindrical Shells Under Uniform Pressure Distribution

$$FEM = 2DB^2(J_1 + J_3)r'$$

$$FES = 2DB^3(J_5 + J_6)r'$$

$$M' = 2DB\ J_2$$

$$V_1 = B\ K_4$$

$$V_2 = 4DB^3 K_1$$

$$M_0' = \frac{K_2}{B}$$

$$M_L' = \frac{K_3}{B}$$

Formulii Applying to Flat Circular Heads Under Uniform Pressure Distribution

$$FEM_r = \frac{Pr^2}{8}$$

$$FEM_t = u\,\frac{Pr^2}{8}$$

$$FES = \frac{Pr}{2h}$$

$$M_r' = M_t' = D\,\frac{(1 + u)}{r}$$

Formulii Applying to Distribution of Moments in Vessels with Long Thin-Walled Cylindrical Shells and Flat Heads

$$f_s = \frac{2D_sB}{2D_sB + \frac{Dh}{r}(1 + u)}$$

$$f_h = \frac{\frac{D_h}{r}(1 + u)}{2D_sB + \frac{D_h}{r}(1 + u)}$$

Formulii Applying to Distribution of Moment in Vessels With Thin-Walled Cylindrical Shells and Flat Heads

$$f_s = \frac{2D_sBJ_2}{2D_sBJ_2 + \frac{Dh}{r}(1 + u)}$$

$$f_h = \frac{\frac{Dh}{r}(1 + u)}{2D_sBJ_2 + \frac{Dh}{r}(1 + u)}$$

LIST OF ABBREVIATIONS

C = Moment carry-over factor for short shells.

D = Flexural Rigidity

D_s = Flexural Rigidity of the shell

D_h = Flexural Rigidity of the head

E = Young's Modulus

e_y, e_x ... = Unit strains

F = Normal force

FEM = Fixed-end moment

FEM_r = Fixed-end moment in a diametrical plane of a flat head

FEM_t = Fixed-end moment in a plane perpendicular to a diameter of a flat head

FES = Fixed-end shear

f_s = Moment distribution factor for the shell

f_r = Moment distribution factor in a diametrical plane of a flat head

f_t = Moment distribution factor in a plane perpendicular to a diameter of a flat head

h = Head thickness

L = Shell length

M = Moment

M_r = Moment in a diametrical plane of a flat head

M_t = Moment in a plane perpendicular to a diameter of a flat head

M' = Moment resulting from a unit angle change

M_o' = Moment at a joint resulting from a unit shear change at the joint

M_L' = Moment at a joint resulting from a unit shear change at an adjacent joint

p = Unit pressure

r = Vessel radius

r' = Membrane deflection of a shell

s = Stress

t = Shell thickness

u = Poisson's Ratio

v = Shell deflection

V = Shear

V_1 = Change in shear due to a unit moment change

V_2 = Change in shear due to a unit displacement

w = Head deflection

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