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Hyperbolicity of the fixed point set for the simple genetic algorithm

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ABSTRACT

We study an infinite population model for the genetic algorithm, where the iteration of the algorithm corresponds to an iteration of a map G. The map G is a composition of a selection operator and a mixing operator, where the latter models effects of both mutation and crossover. We examine the hyperbolicity of fixed points of this model. We show that for a typical mixing operator all the fixed points are hyperbolic.

1. Introduction

In this paper we study a dynamical systems model of the genetic algorithm (GA). This model was introduced by Vose [15] and further extended in [10,14]. A practical implementation of the genetic algorithm seeks solutions in a finite search space which we denote $\Omega = \{1, 2, \ldots, n\}$. Each element of Ω can be thought of as a "species" with a given fitness value; the goal of the algorithm is to maximize the fitness. Usually there are multiple species with high fitness value and n is large. In order to avoid suboptimal solutions the GA algorithm uses mutation and crossover operations to maintain diversity in the pool of r individuals, representing the n species. The infinite population model considers an infinite population of individuals represented by the probability distribution over Ω,

$$p = (p_1, p_2, \ldots, p_n) \tag{1}$$

where p_i is the proportion of the ith species in the population. An update of the genetic algorithm consists of mutation, selection and crossover steps and is represented in the infinite population model as an iteration of a fixed function G.

Although the precise correspondence between behavior of such infinite population genetic algorithm and the behavior of the GA for finite population sizes has not been established in detail, the infinite population model has the advantage of being a well-defined dynamical system. Therefore, the techniques of dynamical systems theory can be used to formulate and answer some fundamental questions about the GA.

The best behaved finite population GA's viewed as stochastic maps will share the convergence properties with discrete irreducible Markov processes: convergence to a unique stationary probability distribution. Since such distributions correspond to fixed points of the infinite population model, fixed points will be fundamental objects of interest in our study. The behavior of the map G in the neighborhood of a fixed point x is determined by the eigenvalues of the linearization $DG(x)$. If all the eigenvalues have absolute value less than one, then all iterates starting near x converge to x. If there is at least one eigenvalue with absolute value greater than one, then almost all iterates will diverge from x [11]. Such a classification based on linear approximation is possible only if no eigenvalues lie on the unit circle in the complex plane. Fixed points x, for which $DG(x)$ has this property, are called *hyperbolic*. If at least one eigenvalue of $DG(x)$ has modulus 1, the fixed point is *non-hyperbolic*.

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It is easy to see that hyperbolicity is an open condition, i.e. if a fixed point is hyperbolic, then all small perturbations of the map G will still admit a fixed point with eigenvalues off the unit circle. On the other hand, non-hyperbolic fixed points can disappear under arbitrarily small perturbations. It is clear that they must be present for some maps G, since a fixed point is not hyperbolic when it undergoes a bifurcation. Therefore the best result one can hope for is that all fixed points should be hyperbolic for a dense set of admissible maps G.

When considering the suitability of a class of models to represent a given phenomena, one question that needs to be addressed is whether the class is rich enough, or whether it is so constrained by its structure, that its dynamics is confined to a narrow range of possibilities. Hyperbolicity is generic for a space of all smooth maps on a compact manifold. However, the GA map is a very specific quadratic map defined on a very specific manifold with boundary — a simplex. Therefore the question whether the class of GA maps is rich enough in a sense that hyperbolicity is still generic in this much smaller class of maps, is both interesting and nontrivial.

Vose and Eberlein [14] considered a class of mappings G that were a composition of a fixed mutation and crossover maps, and a proportional selection map. The set of fitness functions that correspond to the proportional selection was parameterized by the positive orthant of the appropriate dimension. They have shown that for an open and dense set of such fitness functions, the corresponding map G has hyperbolic fixed points.

In this contribution we will take a different path. We consider a class of mappings $G = M \circ F$ where F is arbitrary, but fixed, selection map and M is a mixing map from a class described in Definition 2. This class is broad enough to include all mixing maps formed by a composition of the mutation and crossover maps described in monographs by Reeves and Rowe [10] and Vose [14]. We show that for an open and dense set of mixing maps, the corresponding map G has hyperbolic fixed points.

We now proceed to describe in more detail the infinite population model.

2. The infinite population genetic algorithm

The genetic algorithm searches for solutions in the search space $\Omega = \{1, 2, \dots n\}$; each element of Ω can be thought of as a type of individual. We consider a total population of size r with $r \gg n$. We represent such a population as an *incidence vector*:

$$v = (v_1, v_2, \dots, v_n)^T$$

where v_k is the number of times the individual of type k appears in the population. It follows that $\sum_k v_k = r$. We also identify a population with the *population incidence vector*

$$p = (p_1, p_2, \dots, p_n)^T$$

where $p_k = \frac{v_k}{r}$ is the proportion of the kth individual in the population. The vector p can be viewed as a probability distribution over Ω. In this representation, the iterations of the genetic algorithm yield a sequence of vectors $p \in \Lambda^r$ where

$$\Lambda^r := \left\{ (p_1, p_2, \dots, p_n)^T \in \mathbb{R}^n \mid p_k = \frac{v_k}{r} \text{ and } v_k \in \{0, \dots, r\} \text{ for all } k \in \{1, \dots, n\} \text{ with } \sum_k v_k = r \right\}.$$

We define

$$\Lambda := \left\{ x \in \mathbb{R}^n \,\middle|\, \sum x_k = 1 \text{ and } x_k \geq 0 \text{ for all } k \in \{1, \dots, n\} \right\}.$$

Note that $\Lambda^r \subset \Lambda \subset \mathbb{R}^n$, where Λ is the unit simplex in $\mathbb{R}^n$. Not every point $x \in \Lambda$ corresponds to a population incidence vector $p \in \Lambda^r$, with fixed population size r, since p has non-negative rational entries with denominator r. However, as the population size r gets arbitrarily large, Λ^r "becomes dense" in Λ, that is, $\cup_{r \geq N} \Lambda^r$ is dense in Λ for all N. Thus Λ may be viewed as a set of admissible states for infinite populations. We will use p to denote an arbitrary point in Λ^r and x to denote an arbitrary point in Λ. Thus p always represents a population incidence vector in a finite population and x the corresponding quantity in infinite population, which is the probability distribution over Ω. Unless otherwise indicated, $x \in \Lambda$ is a column vector.

Let $G(x)$ represent the action of the genetic algorithm on $x \in \Lambda$. The map G is a composition of three maps: selection, mutation, and crossover. We will now describe each of these in turn.

We let $F : \Lambda \to \Lambda$ represent the selection operator. The kth component, $F_k(x)$, represents the probability that an individual of type k will result if selection is applied to $x \in \Lambda$. As an example, consider proportional selection where the probability of an individual $k \in \Omega$ being selected is

$$Pr[k|x] = \frac{x_k f_k}{\sum_{j \in \Omega} x_j f_j},$$

where $x \in \Lambda$ is the population incidence vector, and f_k, the kth entry of the vector f, is the fitness of $k \in \Omega$. Define $\mathrm{diag}(f)$ as the diagonal matrix with entries from f along the diagonal and zeros elsewhere. Then, for $F : \Lambda \to \Lambda$, proportional selection is defined as

$$F(x) = \frac{\text{diag}(f)x}{f^T x}.$$

We restrict our choice of selection operators, F, to those which are $\mathcal{C}^1$, that is, selection operators with continuous first derivative. We note that by Theorem 10.2 in Vose [14], ranking, tournament and proportional selection operators are focused, which, by the definition of this term, means that they are continuously differentiable in Λ.

We let $U : \Lambda \to \Lambda$ represent mutation. Here U is an $n \times n$ real valued matrix with ijth entry $u_{ij} > 0$ for all i, j, and where u_{ij} represents the probability that item $j \in \Omega$ mutates into $i \in \Omega$. That is, $(Ux)_k := \sum_i u_{ki} x_i$ is the probability an individual of type k will result after applying mutation to population x.

Let crossover, $C : \Lambda \to \Lambda$, be defined by

$$C(x) = (x^T C_1 x, \ldots, x^T C_n x)$$

for $x \in \Lambda$, where $C_1, \ldots, C_n$ is a sequence of symmetric non-negative $n \times n$ real valued matrices. Here $C_k(x)$ represents the probability that an individual k is created by applying crossover to population x.

Definition 1. Let $Mat_n(\mathbb{R})$ represent the set of $n \times n$ matrices with real valued entries. An operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is *quadratic* if there exist matrices $A_1, A_2, \ldots, A_n \in Mat_n(\mathbb{R})$ such that $A(x) = (x^T A_1 x, \ldots, x^T A_n x)$. We denote a quadratic operator with its corresponding matrices as $A = (A_1, \ldots, A_n)$.

Thus, the crossover operator, $C = (C_1, \ldots, C_n)$, is a quadratic operator [13].

We combine mutation and crossover to obtain the mixing operator $M := C \circ U$. The kth component of the mixing operator

$$M_k(x) = x^T (U^T C_k U) x$$

represents the probability that an individual of type k will result after applying mutation and crossover to population x. Since C_k is symmetric, M_k is symmetric. Further, since C_k is non-negative and U is positive for all k, M_k is also positive for all k. Additionally, it is easy to see check that since $\sum_{k=1}^n [M_k]_{ij} = 1$, $M : \Lambda \to \Lambda$, mixing is also a quadratic operator [13]. Here $[M_k]_{ij}$ denotes the ijth entry of the matrix M_k. This motivates the following general definition of a mixing operator.

Definition 2. Let $Mat_n(\mathbb{R})$ represent the set of $n \times n$ matrices with real valued entries. We call a quadratic operator, $M = (M_1, \ldots, M_n)$, a *mixing operator* if the following properties hold:

(1) $M_k \in Mat_n(\mathbb{R})$ is symmetric for all $k = 1, \ldots, n$;
(2) $[M_k]_{ij} > 0$ for all $i, j \in \{1, \ldots, n\}$, and for all $k = 1, \ldots, n$;
(3) $\sum_{k=1}^n [M_k]_{ij} = 1$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, n$.

Let $\mathcal{M}$ be the set of quadratic operators M satisfying (1)–(3). It is easy to see that (3) implies that $M \in \mathcal{M}$ maps Λ to Λ. We define a norm, $\|\cdot\|$, on $\mathcal{M}$ by considering $M \in \mathcal{M}$ as a vector in $\mathbb{R}^{n^3}$ and using the Euclidean norm in $\mathbb{R}^{n^3}$. For an alternative norm on the set of quadratic operators, see [12].

Definition 3. Given $M \in \mathcal{M}$, the map

$$G := M \circ F \tag{2}$$

is the complete operator for the genetic algorithm, or a *GA map*.

Observe that the map F is defined only on $\Lambda \subset \mathbb{R}^{n+}$, where $\mathbb{R}^{n+} = \{x \in \mathbb{R} \mid x_i \geq 0\}$ is the non-negative cone in $\mathbb{R}^n$. For convenience we extend the domain of definition of F to $\mathbb{R}^{n+} \setminus \{0\}$. The extension of F is denoted $\tilde{F}$ and is defined by

$$\tilde{F}(u) := F\left(\frac{u}{\sum_i u_i}\right).$$

Thus $\tilde{F}|_\Lambda = F$, and for $x \in \Lambda$, $D\tilde{F}(x)|_\Lambda = DF(x)$, the Jacobian of F. Because $\tilde{F} : \mathbb{R}^{n+} \to \Lambda$, it is clear that the map G is extends to a map $\tilde{G} : \mathbb{R}^{n+} \to \Lambda$ and the preceding remarks apply to $\tilde{G}$ as well. In order to simplify the notation we will use symbols F and G for these extended functions.

Recall that if $f(x) = x$, a point x is called a fixed point of f.

Definition 4. A property is *typical*, or *generic*, in a set S, if it holds for an open and dense set of parameter values in S.

3. Main results

In what follows we will fix a selection operator F and discuss how changes in mixing operator M affects finiteness and hyperbolicity of the fixed points of the GA map $G = M \circ F$. However, we will require some generic properties from the map F. In particular we assume that there is a class $\mathcal{F}$ of selection operators with a generic subset $\mathcal{F}_0 \subset \mathcal{F}$ such that for every $F \in \mathcal{F}_0$ there is a generic subset $B_F \subset \Lambda$ with the property that rank $(DF(x)) = n - 1$ for all $x \in B_F$. For future reference we define B_F precisely

$$B_F := \{x \in \Lambda \mid \text{rank}\, DF(x) = n - 1\}. \tag{3}$$

Note that the assumption rank $(DF(x)) = n - 1$ is equivalent to maximal rank condition since the range of F is the $n - 1$ dimensional space Λ. This assumption is valid generically for proportional selection [14]. For the rest of the paper we fix $F \in \mathcal{F}_0$.

Theorem 5. *Let $G = M \circ F$ be a GA map with $F \in \mathcal{F}_0$. For a typical mixing operator $M \in \mathcal{M}$, G is hyperbolic.*

To prove the Theorem 5, we will need the following three propositions.

Proposition 6. *There is a generic set $\mathcal{M}_F \subset \mathcal{M}$, such that for all $M \in \mathcal{M}_F$*

(a) *the fixed point set of $G = M \circ F$ is finite;*
(b) 1 *is not an eigenvalue of $DG(x)$ for any fixed point x;*
(c) *all fixed points of G are in the set B_F.*

The proof of part (a) is presented in detail in [3] and relies heavily on the notion of transversality. References for the relevant background material include [1,4–6]. For convenience, we provide a brief sketch of the proof of part (a):

Let $\mathbb{R}^n_0 := \{x \in \mathbb{R}^n \mid \sum_i x_i = 0\}$ and note that the tangent space to Λ at any $x \in \Lambda$ is isomorphic to $\mathbb{R}^n_0$. We define $ev(M, x) : \mathcal{M} \times \Lambda \to \mathbb{R}^n_0$ by $ev(M, x) := \rho_M(x)$ for $M \in \mathcal{M}$ and $x \in \Lambda$ where $\rho_M(x) := M(F(x)) - x$. First, we show that rank of the derivative of ev is $n - 1$. It thereby follows by definition, that ev is transversal to $\{0\}$, $ev \pitchfork \{0\}$. Then by the Transversal Density Theorem and the Openness of Transversal Intersection Theorem [1], the set of mixing operators $\mathcal{M}_{\{0\}} := \{M \in \mathcal{M} \mid \rho_M \pitchfork \{0\}\}$ is open and dense in $\mathcal{M}$. Finally, we show that $M_{\{0\}}$ corresponds to the set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many solutions. That is, for all $M \in \mathcal{M}_{\{0\}}$, $(M \circ F - I)(x) = 0$ has finitely many solutions and thus $G = M \circ F$ has finitely many fixed points.

(b) For $M \in \mathcal{M}$, let $G^M := M \circ F$. By definition, since $\rho_M \pitchfork \{0\}$, we know that $DG^M(x) - I : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is surjective. Thus $DG^M(x) - I$ is a linear isomorphism on a finite dimensional vector space [2]. Thus, for all $v \neq 0$, $(DG^M(x) - I)v \neq 0$ and 1 cannot be an eigenvalue of $DG^M(x)$ and for all fixed points x_i for $i = 1, \ldots, j$ of $G = M \circ F$, x_i is non-degenerate.

(c) Select $M_0 \in \mathcal{M}_{\{0\}}$ and x_0, one of its finite number of fixed points. Then we have $ev(M_0, x_0) = 0$ by definition of ev function. In the proof of Lemma 4.1 [3] it is shown that $\frac{dev}{dM}$ has rank $n - 1$, which means that it is surjective.

By Implicit Function Theorem there exists a differentiable map h such that $ev(h(x), x) \equiv 0$ in the neighborhood of x_0. Since h is continuous, for every open neighborhood N_{M_0} of M_0 there is an open neighborhood of x_0 which maps by h to N_{M_0}. Since B_F is open and dense, there is in open and dense set of M's in the neighborhood N_{M_0} for which the fixed point x is in the set B_F. Since G has only finitely many fixed points and the finite intersection of open and dense sets is still open and dense, there is a generic set $\mathcal{M}_F \subset \mathcal{M}_{\{0\}}$ such that for every $M \in \mathcal{M}_F$ the finite fixed point set is in B_F.

Proposition 7. *Let $G = M \circ F$ be a GA map. The set of mixing operators M, for which the fixed points of G are hyperbolic, forms an open set in $\mathcal{M}$.*

The proof of this proposition is straightforward. If a GA map G has hyperbolic fixed points, since Λ is compact there can be only finitely many fixed points in Λ. Consider one such fixed point x and let $\det(DG(x))$ denote the determinant of $DG(x)$. Since

$$\det(DG(x) - \lambda I) = \det([DM \circ F(x)]DF(x) - \lambda I)$$

is a continuous function of M, if the spectrum of $DG(x)$ does not intersect the unit circle, then there is a $\delta_0 = \delta_0(x) > 0$ such that the spectrum of $DG_{M'}$ corresponding to any M' with $||M - M'|| < \delta_0(x)$ will not intersect the unit circle. Since there are finitely many fixed points, there is a minimal $\delta = min_x \delta_x$. Then all maps $G_{M'}$ corresponding to M' with $||M - M'|| \leq \delta$ are hyperbolic.

More challenging is the proof of the following proposition.

Proposition 8. *Let $G = M \circ F$ be a GA map with $F \in \mathcal{F}_0$. The set of mixing operators for which the fixed points of G are hyperbolic, forms a dense set in $\mathcal{M}$.*

To prove Proposition 8 we first observe that a small perturbation of a given map G yields a map with a finitely many fixed points. The key step in the proof of Proposition 8 is a construction of a perturbation that preserves the given fixed point (i.e. the perturbed map has the same fixed point x as the original map) with the property that x is hyperbolic for the perturbed map. This procedure can be applied successively to finitely many equilibria by progressively choosing smaller perturbations in order not to disturb equilibria that are already hyperbolic.

In Section 4 we describe the set of perturbations of the mixing operators that will be used to prove Proposition 8. In Section 5 we describe how such a perturbation affects the characteristic polynomial of $DG(x)$. Finally, in Section 6 we prove Proposition 8.

4. The class of perturbations

We now describe the set of perturbations of $M \in \mathcal{M}$. In particular, we are interested in perturbations of M that are still elements of the set $\mathcal{M}$, and additionally have the property that they preserve the fixed point of interest.

Let $G = M \circ F$ be a GA map (2) with a fixed point x. Let $\mathcal{Q}(x)$ represent a class of quadratic operators $Q = (Q_1, \ldots, Q_n)$ for which the following properties hold:

(1) $Q_k \in Mat_n(\mathbb{R})$ is symmetric for all $k = 1, \ldots, n$;
(2) $\sum_k Q_k = 0$;
(3) $[F(x)]^T Q_k F(x) = 0$ for all $k = 1, \ldots, n$ where x is the fixed point.

Definition 9. Let $M \in \mathcal{M}$ with $G = M \circ F$ a GA map with a fixed point x. Let $\mathcal{P}(x, M) \subset \mathcal{Q}(x)$ be defined as follows:

$$\mathcal{P}(x, M) := \{Q \in \mathcal{Q}(x) | [M_k \pm Q_k]_{ij} > 0 \text{ for all } i, j, k = 1, \ldots, n\}.$$

We call $P \in \mathcal{P}(x, M)$ an *admissible* perturbation.

The requirement (3) seems to be very strong and one can question whether the class of admissible perturbations $\mathcal{P}(x, M)$ is nonempty. In Lemma 15 we will show that this class is not empty for any x and M by explicitly constructing a perturbation $P \in \mathcal{P}(x, M)$.

Lemma 10. *Let $G = M \circ F$ be a GA map (2) with a fixed point x. Given $Q \in \mathcal{Q}(x)$, there exists $\bar{\epsilon}$ such that for all $0 \le \epsilon \le \bar{\epsilon}$, $\epsilon Q \in \mathcal{P}(x, M)$.*

Proof. Let $Q \in \mathcal{Q}(x)$. By definition of $\mathcal{Q}(x)$, for any $t \in \mathbb{R}$, $tQ \in \mathcal{Q}(x)$. We now show that for $Q \in \mathcal{Q}(x)$ there exists $\bar{\epsilon}$ such that $M_k \pm (\epsilon Q)_k > 0$ for all $k = 1, \ldots, n$. The requirement that $M_k \pm \epsilon Q_k > 0$ is equivalent to $|\epsilon (Q_k)_{ij}| < (M_k)_{ij}$ for all i, j, k where $(Q_k)_{ij}$ denotes the ijth element of the corresponding kth matrix. Thus, we show there exists $\bar{\epsilon} > 0$ such that for all $0 \le \epsilon \le \bar{\epsilon}$, $\epsilon |(Q_k)_{ij}| \le \bar{\epsilon} |(Q_k)_{ij}| < (M_k)_{ij}$.

The case $Q = 0$ is trivial. For $Q \neq 0$, let $\alpha = \max\{|(Q_k)_{ij}| \text{ for all } i, j, k\}$ and let $\beta = \min\{(M_k)_{ij} \text{ for all } i, j, k\}$. Take $\bar{\epsilon} \in \mathbb{R}^+$ such that $\frac{\beta}{\alpha} > \bar{\epsilon}$. Since for all i, j, k,

$$(M_k)_{ij} > \min\{(M_k)_{ij} \text{ for all } i, j, k\} \ge \beta$$

and

$$\epsilon\alpha > \epsilon(\max\{|(Q_k)_{ij}| \text{ for all } i, j, k\}) \ge \epsilon |(Q_k)_{ij}|$$

we have for all i, j, k and $0 \le \epsilon \le \bar{\epsilon}$ that $(M_k)_{ij} > \epsilon |(Q_k)_{ij}|$. Thus $\epsilon Q \in \mathcal{P}(x, M)$ for $0 \le \epsilon \le \bar{\epsilon}$. □

Corollary 11. *If $P \in \mathcal{P}(x, M)$, then $tP \in \mathcal{P}(x, M)$ for $0 \le t \le 1$.*

That the set $\mathcal{P}(x, M) \neq \emptyset$ follows readily.

We now show that the above constructed set $\mathcal{P}(x, M)$ defines a collection of perturbations of M with the desired fixed point preserving property for the GA map (2). For $P \in \mathcal{P}(x, M)$, let $M_P := M + P$ and $G_P = M_P \circ F$.

Lemma 12. *Let $G = M \circ F$ be a GA map (2). Assume $x \in \Lambda$ is a fixed point of G. If $P \in \mathcal{P}(x, M)$, then $M_P = M + P$ satisfies*

(1) $M_P \in \mathcal{M}$
(2) $G_P(x) = M_P \circ F(x) = x$.

That is, G_P has the same fixed point x as G.

Proof. Let $P \in \mathcal{P}(x, M)$. Consider a quadratic operator, $M_P = ([M_1 + P_1], \ldots, [M_n + P_n])$. We first show $M_P \in \mathcal{M}$. That for $k = 1, \ldots, n$, $(M_P)_k = M_k + P_k \in Mat_n(\mathbb{R})$ is symmetric and has $(M_P)_k > 0$ follows readily. To show part (3) of the definition of $\mathcal{M}$, we show $\sum_{k=1}^n (M_k)_{ij} = 1$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, n$. Since $\mathcal{P}(x, M) \subset \mathcal{Q}(x)$, by (2) of the definition of $\mathcal{Q}(x)$, and (3) of the definition of $\mathcal{M}$,

$$\begin{aligned}
\sum_{k=1}^n ((M_P)_k)_{ij} &= \sum_{k=1}^n (M_k + P_k)_{ij} \\
&= \sum_{k=1}^n (M_k)_{ij} + \sum_{k=1}^n (P_k)_{ij} \\
&= 1 + 0 \\
&= 1.
\end{aligned}$$

Thus we have shown $M_P \in \mathcal{M}$.

Now, we prove $G_P(x) = M_P \circ F(x) = x$. Clearly,

$$\begin{aligned} G_P(x) &= M_P \circ F(x) \\ &= (M + P) \circ F(x) \\ &= M \circ F(x) + P \circ F(x) \\ &= G(x) + (F^T(x)P_1F(x), \dots, F^T(x)P_nF(x))^T \\ &= G(x) \end{aligned}$$

since $P \in \mathcal{P}(x, M) \subset \mathcal{Q}(x)$. □

Corollary 13. *Let $G = M \circ F$ be a GA map (2). Assume $x \in \Lambda$ is a fixed point of G and $P^1, \dots, P^l \in \mathcal{P}(x, M)$. There exists $\epsilon > 0$ such that*

(1) $\epsilon \sum_{i=1}^n P^i \in \mathcal{P}(x, M)$;
(2) $M + \epsilon \sum_{i=1}^l P^i \in \mathcal{M}$;
(3) $G_P := (M + \epsilon \sum_{i=1}^l P^i) \circ F$ *admits the same fixed point x as the map $G = M \circ F$.*

Proof. For part (1), it suffices to show that if $P^1, \dots, P^l \in \mathcal{P}(x, M)$, then $\sum_{i=1}^l P^i \in \mathcal{Q}(x)$, since by Lemma 10, it follows that there exists $\epsilon > 0$ such that $\epsilon \sum_{i=1}^l P^i \in \mathcal{P}(x, M)$. Let $P^1, \dots, P^l \in \mathcal{P}(x, M)$ and consider $\sum_{i=1}^l P^i$. We show that the definition of $\mathcal{Q}(x)$ is satisfied. Clearly $\left(\sum_{i=1}^l P^i\right)_k \in Mat_n(\mathbb{R})$ is symmetric and $\left(\sum_{i=1}^l P^i\right)_k > 0$ for $l = 1, \dots, n$. Similarly, since for each $i = 1, \dots, l$; $P^i = (P_1^i, \dots, P_n^i)$, and $\sum_{j=1}^n P_j^i = 0$, it follows that $\sum_{j=1}^n \sum_{i=1}^l P_j^i = 0$. By definition of $\mathcal{P}(x, M)$, for $i = 1, \dots, l$, and $j = 1, \dots, n$, $[F(x)]^T P_j^i F(x) = 0$. Thus, for $i = 1, \dots, l$,

$$[F(x)]^T \left(\sum_j P^i\right)_j F(x) = \sum_j [F(x)]^T (P^i)_j F(x) = 0.$$

So we have shown $\sum_{i=1}^l P^i \in \mathcal{P}(x, M)$, which leads to the desired result.

Part (2) follows automatically from part (1). Further, to show part (3), if $\epsilon \sum_{i=1}^l P^i \in \mathcal{P}(x, M)$, by Lemma 12, $G_P := (M + \epsilon \sum_{i=1}^l P^i) \circ F$ admits the same fixed point x as the map $G = M \circ F$. □

We observe that $G_P = M_P \circ F = (M + P) \circ F = (M \circ F) + (P \circ F) = G + (P \circ F)$. Thus,

$$\begin{aligned} DG_P(x) &= D[G + (P \circ F)](x) \\ &= DG(x) + H \end{aligned}$$

where $H \in Mat_n(\mathbb{R})$. In order to trace the effects of perturbations of M on the derivative DG_P, we define

$$\mathcal{H} = \{H \in Mat_n(\mathbb{R}) | H = D(P \circ F)(x) \text{ for } P \in \mathcal{P}(x, M)\}.$$

Before we construct another admissible perturbation we need a technical lemma.

Lemma 14. *Let f be a differentiable map with range Λ. For all v, $(Df(x))v \cdot (1, \dots, 1) = 0$, where $Df(x)$ is the Jacobian of f at the point x.*

Proof. By definition of directional derivatives,

$$\begin{aligned} [Df(x)v]_j &= \frac{\partial f_j}{\partial x_1} v_1 + \frac{\partial f_j}{\partial x_2} v_2 + \dots + \frac{\partial f_j}{\partial x_n} v_n \\ &= \lim_{\alpha \mapsto 0} \frac{f_j(x + \alpha v) - f_j(x)}{\alpha}. \end{aligned}$$

We compute

$$\begin{aligned} Df(x)v \cdot (1, \dots, 1) &= \sum_j [Df(x)v]_j \\ &= \lim_{\alpha \mapsto 0} \frac{1}{\alpha} \left[\sum_j f_j(x + \alpha v) - \sum_j f_j(x)\right]. \end{aligned}$$

Since the range of f is Λ we have $\sum_j f_j(x) = 1$ for all x. Therefore the last bracket is zero. □

Lemma 15. *Let $G = M \circ F$ admit a fixed point x and* rank $(DF(x)) = n - 1$ *where F is non-negative and not identically zero. There exists an admissible perturbation P such that $H = D(P \circ F)$ has rank $n - 1$.*

Proof. Consider the $n-1$ dimensional space $(F(x))^\perp$ and select its basis $\{v_1, \ldots, v_{n-1}\}$. Take $\alpha \in (0, 1)$. Assume without loss of generality that for $i < k$, $F_i(x) = 0$, and for $i \geq k$, $F_i(x) > 0$. We define $P = (P_1, \ldots, P_n)$, where for $l < n$ we set

$$(P_l)_{rs} := \begin{cases} \dfrac{(v_l)_s - \alpha \sum_{i \neq k} F_i(x)}{F_k(x)} & r = k \text{ and } s \leq k-1, \\ \dfrac{(v_l)_r - \alpha \sum_{i \neq k} F_i(x)}{F_k(x)} & s = k \text{ and } r \leq k-1, \\ \dfrac{(v_l)_r - \alpha \sum_{i \neq r} F_i(x)}{F_r(x)} & r = s \text{ and } r > k-1, \\ \alpha & \text{elsewhere;} \end{cases}$$

and for $l = n$

$$(P_n)_{rs} := -\sum_{l=1}^{n-1} (P_l)_{rs}.$$

Straightforward computation shows that there is an ϵ sufficiently small such that ϵP is an admissible perturbation. Now we show that rank $(H) = n-1$ where $H = D[(P \circ F)](x)$. Then

$$H = 2 \begin{pmatrix} (P_1 F(x))^T \\ (P_2 F(x))^T \\ \vdots \\ (P_n F(x))^T \end{pmatrix} DF(x). \tag{4}$$

Observe for $l < n$, $P_l \circ F(x) = v_l$, and, since $P_n = -\sum_{k=1}^{n-1} P_k$,

$$P_n \circ F(x) = -\sum_{k=1}^{n-1} v_l.$$

Thus, from Eq. (4), we see that

$$H = 2 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -\sum_{k=1}^{n-1} v_k \end{pmatrix} DF(x).$$

Since $v_1, \ldots, v_{n-1}$ form a basis of $(F(x))^\perp$ rank of the first matrix is $n-1$. By construction of $v_1, \ldots, v_{n-1}$,

$$\text{null} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -\sum_{k=1}^{n-1} v_k \end{pmatrix} = \text{span}[(F(x))].$$

By the hypothesis, rank $(DF(x)) = n-1$. Therefore rank $(H) = n-1$ provided that $\text{span}(F(x)) \not\subseteq \text{Range } DF(x)$. Assume to the contrary that

$$F(x) \subset \text{Range } DF(x).$$

Then there is a vector v such that $DF(x)v = F(x)$. Taking a dot product with the vector $(1, \ldots, 1)$ we get

$$DF(x)v \cdot (1, \ldots, 1) = F(x) \cdot (1, \ldots, 1).$$

By Lemma 14 the left hand side is zero, while $F(x) \in \Lambda$ implies that the right hand side is equal to one. This contradiction shows that $\text{span}(F(x)) \not\subseteq \text{Range } DF(x)$ and hence rank $(H) = n-1$. □

Lemma 16. *Let $G = M \circ F$ be a GA map. Assume $G(x) = x$ for some $x \in \Lambda$ and that* rank $(DF(x)) = n - 1$. *Let $\mathcal{R} := \{h | h = v^T DF(x),\ \text{and } v \in F(x)^\perp\}$. Then $\mathcal{R}$ has dimension $n - 1$.*

Proof. By assumption, rank $(DF(x)) = n - 1$, so if

$$\text{null}(DF(x))^T \cap (F(x))^\perp = \{0\}$$

then $\dim(\mathcal{R}) = n - 1$. We now show

$$\text{null}(DF(x))^T \cap (F(x))^\perp = \{0\}.$$

By Lemma 14, we know that $(\text{Image}(DF(x)))^\perp = \text{span}(1, \dots, 1)$, and by the Fredholm alternative, $\text{null}(DF(x))^T = \text{Image}(DF(x))^\perp = \text{span}(1, \dots, 1)$. That is, $\text{null}(DF(x))^T = \text{span}(1, \dots, 1)$. Take $v \in \text{null}(DF(x))^T \cap (F(x))^\perp,\ v \neq 0$. Then $v = \alpha(1, \dots, 1)$ with $\alpha \neq 0$ and $v \perp F(x)$. Then

$$v \cdot F(x) = \alpha(1, \dots, 1) \cdot F(x) = \alpha \sum_i F_i(x) = \alpha \neq 0,$$

which is a contradiction, thus $\text{null}(DF(x))^T \cap F(x)^\perp = \{0\}$. □

Lemma 17. *Let $G = M \circ F$ be a GA map with a fixed point x and assume* rank $(DF(x)) = n - 1$. *Given $h \in \mathcal{R}$ with $h \neq 0$, for all $1 \le i < j \le n$ there exists $H \in \mathcal{H}$ such that*

(1) $h = h_j = -h_i \neq 0$;
(2) $h_k = 0$ *for* $k \neq i, j$.

Proof. That such an H exists can be shown by explicitly forming an operator $P \in \mathcal{P}(x, M)$ so that the corresponding $H \in \mathcal{H}$ with $H = PF(x)DF(x)$ has the desired properties.

Let $v \in (F(x))^\perp := \{u \in \mathbb{R}^n \mid u \cdot F(x) = 0\}$ with $v \neq 0$. Assume without loss of generality that $F_i(x) = 0$ for $i < k$ and $F_i(x) \neq 0$ for $i \ge k$. Select arbitrary integers $i, j, i \neq j$, with $1 \le i < j \le n$ and create a quadratic operator $Q = (Q_1, \dots, Q_n)$ as follows: for $l \neq i, j$, let $Q_l = 0$, the zero matrix and let $Q_j = -Q_i$ with entries

$$(Q_j)_{rs} := \begin{cases} \frac{v_s}{F_k(x)} & r = k \text{ and } s \le k - 1 \\ \frac{v_r}{F_k(x)} & s = k \text{ and } r \le k - 1 \\ \frac{v_r}{F_r(x)} & r = s \text{ and } r > k - 1 \\ 0 & \text{elsewhere.} \end{cases}$$

By construction $Q_j F(x) = v$ for $j = 1, \dots, n$. By Lemma 10, there exists $\epsilon > 0$ such that $P := \epsilon Q \in \mathcal{P}(x, M)$.

Finally, we show that $H = DP \circ F(x) = DP(F(x))DF(x)$ has the advertised properties. Since

$$h_l = (P_l F(x))^T DF(x)$$

we have $h_l = 0$ for $l \neq i, j$. For $l = i$,

$$h_i = (P_i F(x))^T DF(x) = v^T DF(x).$$

Clearly, $h_j = -h_i$. □

The following lemma describes relationship between the choice of the vector h and the change of basis matrix C that takes $DG(x)$ to its Jordan form.

Lemma 18. *Consider $\lambda_0 \in \mathbb{R}$ a simple eigenvalue of $DG(x)$. Let the first column of C, denoted C^1, be the eigenvector corresponding to λ_0. Then there is a vector v with $v \perp F(x)$ such that $h := v^T DF(x)$ satisfies $h \cdot C^1 \neq 0$.*

Proof. Assume that for all vectors v with $v \cdot F(x) = 0$ we have $h \cdot C^1 = 0$. That is,

$$v^T DF(x) C^1 = 0 \quad \text{for all } v \text{ such that } v^T F(x) = 0.$$

This happens if, and only if, $DF(x)C^1 = aF(x)$ for some $a \in \mathbb{R}$. Note that $a \neq 0$, since C^1 is an λ-eigenvector of $DG(x) = 2M(F(x))DF(x)$. Finally, by Lemma 14 applied to map F, $DF(x)C^1 \cdot (1, \dots, 1) = 0$. Since $F(x) \in \Lambda$,

$$\begin{aligned} 0 &= DF(x)C^1 \cdot (1, \dots, 1) = aF(x) \cdot (1, \dots, 1) \\ &= a \sum_{i=1}^{n} F_i(x) = a, \end{aligned}$$

which is a contradiction. This shows that there is an h such that $h \cdot C^1 \neq 0$. □

Lemma 19. *Let $\lambda_0 = \alpha + i\beta$, $\beta \neq 0$, be a simple eigenvalue of $DG(p)$. Let the first and second columns, C^1 and C^2, of C be the real and complex parts of the eigenvector corresponding to λ_0, respectively. Then there are indices i and j such that*

$$-(C^{-1})_{1i} + (C^{-1})_{1j} \neq 0.$$

Proof. Assume that $\lambda_0 = \alpha + i\beta$ is a simple complex eigenvalue. Then

$$DG(p)(C^1 + iC^2) = (\alpha + i\beta)(C^1 + iC^2).$$

Collecting real and imaginary parts and applying the Lemma 14 to map G we get

$$(\alpha C^1 - \beta C^2) \cdot (1, \dots, 1) = 0, \qquad (\beta C^1 + \alpha C^2) \cdot (1, \dots, 1) = 0.$$

Since $\alpha^2 + \beta^2 \neq 0$, a short computation shows that

$$C^1 \cdot (1, \dots, 1) = 0 \quad \text{and} \quad C^2 \cdot (1, \dots, 1) = 0. \tag{5}$$

We now prove the result by contradiction. Assume that for all $i, j \in \{1, \dots, n\}$,

$$-(C^{-1})_{1i} + (C^{-1})_{1j} = 0. \tag{6}$$

Because C is invertible, there exists an index i such that $C_{2i}^{-1} \neq 0$. Thus (6) implies that there is $a \neq 0$ such that the first row of the matrix C^{-1}, $C_1^{-1} = a(1, \dots, 1)$. Since $C^{-1}C = I$

$$1 = C_1^{-1} \cdot C^1 = a(1, \dots, 1) \cdot C^1.$$

By (5) we have $a(1, \dots, 1) \cdot C^1 = 0$ and this contradiction finishes the proof. □

5. Perturbation of the characteristic polynomial

To simplify calculations, we make use of $DG(x)$ in Jordan normal form. Let $C \in Mat_n(\mathbb{R})$ be the change of basis matrix so that $B = C^{-1}[DG(x)]C$ is in Jordan normal form, set $B(\lambda) := DG(p) - \lambda I$. We observe that

$$\begin{aligned} C^{-1}[DG(x) - \lambda I + \epsilon H]C &= C^{-1}[DG(x)]C - \lambda I + \epsilon C^{-1}HC \\ &= B(\lambda) + \epsilon K. \end{aligned}$$

Corresponding to the set $\mathcal{H}$, we define $\mathcal{K} := \{K \mid \text{there exists } H \in \mathcal{H} \text{ such that } K = C^{-1}HC\}$. Finally, for any matrix $A \in Mat_n(\mathbb{R})$, let spec A denote the spectrum of A and let $A_{ij} \in Mat_{n-1}(\mathbb{R})$, denote the matrix obtained by removing row i and column j from the matrix A.

Theorem 20. *Assume* $\text{spec}(DG(x)) = \{\lambda_1, \dots, \lambda_n\}$ *and* λ_1 *is a simple eigenvalue. Then for any perturbation* $P \in \mathcal{P}(x, M)$ *and its corresponding matrix* $K \in \mathcal{K}$, *there is* $\delta > 0$ *and polynomials* $q(\lambda)$ *and* $s(\lambda))$ *such that*

$$\det(B(\lambda) + \epsilon K) = \prod_{i=2}^{n}(\lambda_i - \lambda)\left[(\lambda_1 - \lambda) + \frac{\epsilon q(\lambda) + \epsilon(\lambda_1 - \lambda)s(\lambda) + \mathcal{O}(\epsilon^2)}{\prod_{i=2}^{n}(\lambda_i - \lambda)}\right] \tag{7}$$

is well defined for $\lambda \in N_\delta(\lambda_1)$. *Furthermore,*

(a) *for* $\lambda_1 \in \mathbb{R}$, $\quad q(\lambda_1) = k_{11} \det B_{11}(\lambda_1)$;
(b) *for* $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, $\quad q(\lambda_1) = [(k_{11} + k_{22}) + i(k_{12} - k_{21})] \ \det B_{11}(\lambda_1)$.

Proof. Note that for any two matrices V and W the determinant of $V + \epsilon W$ can be expanded as

$$\begin{aligned} \det(V + \epsilon W) &= \det(V) + \epsilon W_1 \cdot (\det V_{11}, -\det V_{12}, \dots, \pm\det V_{1n}) + \cdots \\ &\quad + \epsilon W_n \cdot (\pm\det V_{n1}, -(\pm)\det V_{n2}, \dots, \pm V_{nn}) + \mathcal{O}(\epsilon^2) \end{aligned}$$

where W_j denotes the jth row of the matrix W. We apply this expansion to $V := B(\lambda)$ and $W := K$. We then define

$$q_i(\lambda) := K_i \cdot (\det B_{i1}(\lambda), -\det B_{i2}(\lambda), \det B_{i3}(\lambda), \dots, \pm\det B_{in}(\lambda)) \tag{8}$$

for all $i > 1$ and get

$$\begin{aligned} \det(B(\lambda) + \epsilon K) &= \det(B(\lambda) + \epsilon \sum_{i=1}^{n} q_i(\lambda) + \mathcal{O}(\epsilon^2) \\ &= \prod_{i=1}^{n}(\lambda_i - \lambda) + \epsilon \sum_{i=1}^{n} q_i(\lambda) + \mathcal{O}(\epsilon^2) \\ &= \prod_{i=2}^{n}(\lambda_i - \lambda)\left[(\lambda_1 - \lambda)\frac{\epsilon \sum_{i=1}^{n} q_i(\lambda)}{\prod_{i=2}^{n}(\lambda_i - \lambda)}\right] + \mathcal{O}(\epsilon^2). \end{aligned}$$

Since λ_1 is simple, $\prod_{i=2}^{n}(\lambda_i - \lambda_1) \neq 0$. The existence of a δ advertised in the Theorem follows now from the continuity of the denominator in λ.

Now we prove the last part of the Theorem. Since B is the Jordan form of the matrix $DG(p)$ the matrix $B(\lambda)$ has the form

$$\begin{pmatrix} \lambda_1-\lambda & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}, \qquad \begin{pmatrix} \alpha-\lambda & \beta & 0 & \dots & 0 \\ -\beta & \alpha-\lambda & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \tag{9}$$

in case $\lambda_1 \in \mathbb{R}$ and $\lambda_1 = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, respectively. Direct computation now shows that in the case $\lambda_1 \in \mathbb{R}$ all determinants in the expression for $q_i(\lambda)$ (see (8)) for all $i \geq 2$ contain the factor $\lambda_1 - \lambda$, while none of the determinants in $q_1(\lambda)$ do. However, all determinants $\det(B_{1j})$ with $j \geq 2$ contain a zero first column and hence are zero. Since λ_1 is simple eigenvalue, $q_1(\lambda_1) \neq 0$. Now the polynomials

$$q(\lambda) := q_1(\lambda), \qquad s(\lambda) = \frac{\sum_{i\geq 2} q_i(\lambda)}{\lambda_1 - \lambda}$$

satisfy the assertions of the Theorem.

In the case $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ a similar computation shows that all determinants in $q_i(\lambda)$ for all $i \geq 3$ contain the factor $\lambda_1 - \lambda$, while none of the determinants in $q_1(\lambda)$ and $q_2(\lambda)$ do. Further, determinants $\det(B_{ij})$ with $i = 1, 2$ and $j \geq 3$ are also zero. We define

$$q(\lambda) := q_1(\lambda) + q_2(\lambda), \qquad s(\lambda) = \frac{\sum_{i\geq 3} q_i(\lambda)}{\lambda_1 - \lambda}.$$

Then (8) implies $q(\lambda_1) = k_{11}\det B_{11}(\lambda_1) - k_{12}\det B_{12}(\lambda_1) - k_{21}\det B_{21}(\lambda_1) + k_{22}\det B_{22}(\lambda_1)$. Furthermore, the form of the matrix $B(\lambda)$ in (9) implies that $B(\lambda_1)$ has the form

$$\begin{pmatrix} -i\beta & \beta & 0 & \dots & 0 \\ -\beta & -i\beta & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

This implies that the matrices $B_{11}(\lambda_1) = B_{22}(\lambda_1)$ and that $B_{12}(\lambda_1) = -B_{21}(\lambda_1)$. Computing the determinant shows that $\det B_{11}(\lambda_1) = i \det B_{12}(\lambda_1)$. These equations imply the statement of the Theorem. □

Lemma 21. *Let $G = M \circ F$ be a GA map with fixed point x. Assume λ_0 is a simple, real eigenvalue of $DG(x)$. Then there is an admissible perturbation $P \in \mathcal{P}(x, M)$ with the corresponding matrix $K \in \mathcal{K}$ such that*

$$q(\lambda_0) \neq 0,$$

and $q(\lambda_0) \in \mathbb{R}$ is a real number.

Proof. By Theorem 20, $q(\lambda_0) = k_{11}\det B_{11}(\lambda_0)$. Since λ_0 is a real eigenvalue and the matrix $DG(x)$ is real valued, the minor $B_{11}(\lambda_0)$ is a real number. Now we show that there is a perturbation matrix K such that $k_{11} \neq 0$ and $k_{11} \in \mathbb{R}$.

We present a proof by contradiction. Assume for all K that correspond to an admissible P, we have $k_{11} = 0$. Recall that C is the change of basis matrix corresponding to $DG(x)$ in Jordan normal form and we can select the first column C^1 of C to be the eigenvector corresponding to λ_0. By direct computation one can show that since $K = C^{-1}HC$ and $k_{ij} = [C^{-1}HC]_{ij}$ we get

$$k_{ij} = (C^{-1})_{i1}(H_1 \cdot C^j) + (C^{-1})_{i2}(H_2 \cdot C^j) + \cdots + (C^{-1})_{in}(H_n \cdot C^j). \tag{10}$$

Here $(C^{-1})_{i1}$ is the ij element of the matrix C^{-1}. By Lemma 18 there exists a vector h such that $h \cdot C^1 \neq 0$. We construct a admissible perturbation P with a matrix H as in Lemma 17 with $H_i = h$ and $H_j = -h$ and $H_l = 0$ for $l \neq i, j$. Then the assumption $k_{11} = 0$ and formula (10) imply

$$0 = k_{11} = ((C^{-1})_{1i} - (C^{-1})_{1j})(h \cdot C^1),$$

and hence

$$(C^{-1})_{1i} = (C^{-1})_{1j},$$

since $h \cdot C^1 \neq 0$ by assumption. Since i, j were arbitrary, this implies $(C^{-1})_{11} = (C^{-1})_{12} = (C^{-1})_{13} = \cdots = (C^{-1})_{1n}$ and the first row of the matrix C^{-1} must be

$$(C^{-1})_1 = a(1, \ldots, 1)$$

for $a = (C^{-1})_{11}$. Clearly, since $(C^{-1})_1$ is a row of an invertible matrix C^{-1} we have $a \neq 0$ and $a \in \mathbb{R}$. Since $C^{-1}C = I$, we see that

$$1 = (C^{-1})_1 \cdot C^1 = a(1, \ldots, 1) \cdot C^1. \tag{11}$$

Because λ_0 is a simple eigenvalue of $DG(x)$,

$$DG(x)C^1 = \lambda_0 C^1. \tag{12}$$

Thus, by Eqs. (11) and (12),

$$(DG(x)C^1) \cdot (1, \ldots, 1) = \lambda_0 C^1 \cdot (1, \ldots, 1) = \frac{\lambda_0}{a} \neq 0.$$

This contradicts Corollary 14 applied to the map G. This contradiction finishes the proof. □

We continue with a series of lemmas, the first of which shows that, given a fixed point x, if $DG(x)$ has an eigenvalue λ_1 with multiplicity $k > 1$, then there is an admissible perturbation $P \in \mathcal{P}(x, M)$ such that the multiplicity of λ_1 for the perturbed Jacobian $DG_P(x)$ is less or equal to one. The second lemma then shows that if $DG(x)$ has an eigenvalue λ_1 with multiplicity 1, then there is an admissible perturbation P which moves this eigenvalue off of the unit circle, $\mathcal{S}^1$.

Finally, we show that given a GA map $G = M \circ F$ with fixed point x and $\text{spec}(DG(x)) \cap \mathcal{S}^1 \neq \emptyset$, that we can perturb using $P \in \mathcal{P}(x, M)$ such that the resulting map G_P has $\text{spec}(DG_P(x)) \cap \mathcal{S}^1 = \emptyset$. These three lemmas then allow us to prove denseness.

Lemma 22. *Let $G = M \circ F$ be a GA map with fixed point x. If $DG(x)$ has eigenvalue $\lambda_0 \in \mathcal{S}^1$ and multiplicity $k > 1$, then there exists $P \in \mathcal{P}(x, M)$ such that for $0 < t \leq 1$, $DG_{tP}(x)$ has eigenvalue $\lambda_0 \in \mathcal{S}^1$ with multiplicity at most 1.*

Proof. Assume $\lambda_0 \in \mathcal{S}^1$ is the eigenvalue of $DG(x)$ with multiplicity $k > 1$. Since the polynomial $g(c) = \det(DG(x) - \lambda_0 I + cH)$, $g : \mathbb{R} \to \mathbb{C}$ defines an analytic function in c, either (see [8])

(1) $g \equiv 0$; or
(2) $g(c)$ has isolated zeros.

By Lemma 15, we can choose H to have rank $n - 1$. Thus 0 is a simple eigenvalue of H. For large values of c, we have $0 \in \text{spec}(cH)$ but for $\mu \in \text{spec}(cH)\backslash\{0\}$, $|\mu| > L$ for some large $L = \mathcal{O}(c)$. If $\|DG(x)\| \ll L$, then we can view $DG(x)$ as a small perturbation of cH. Two possibilities arise:

(a) There exists $c \in \mathbb{R}$ such that $g(c) = \det(cH + DG(x) - \lambda_0 I) \neq 0$.
(b) For all $c \in \mathbb{R}$, $g(c) = \det(cH + DG(x) - \lambda_0 I) = 0$.

Case (a) implies (2), i.e. g has isolated zeros. Since $g(0) = 0$, there is δ arbitrarily close to 0 such that $g(\delta) \neq 0$. Observe that $\delta H \in \mathcal{H}$ for all $0 < \delta \leq \delta_0$ and the corresponding $\delta P \in \mathcal{P}(x, M)$ by definition of $\mathcal{H}$.

In case (b), we note that since H has a simple eigenvalue 0, λ_0 must be a simple eigenvalue of $(cH + DG(x))$ for large c. Since $g \equiv 0$, λ_0 is an eigenvalue of $cH + DG(x)$ for all c. Therefore there exists a function $h(c, \lambda)$ such that $\det(cH + DG(x) - \lambda_0 I) = (\lambda - \lambda_0)h(c, \lambda)$. Observe that $h(c, \lambda)$ is a polynomial in λ and, since $g(c)$ is analytic in c, the function

$$h(c) := h(c, \lambda_0)$$

is also analytic in c. Since λ_0 is a simple eigenvalue of $(cH + DG(x))$ for large c, $h(c) \neq 0$ for large c. Therefore $h(c)$ has isolated zeros and there is $\delta_0 > 0$ such that for all $\delta < \delta_0$ we have $h(\delta) \neq 0$. Therefore, the Jacobian of the map $G_{\delta P}$ corresponding to a perturbation $\delta P \in \mathcal{P}(x, M)$ with $\delta < \delta_0$, has eigenvalue λ_0 with multiplicity 1. Set $P = \delta P$, it follows that for $0 < t \leq 1$, $tP \in \mathcal{P}(x, M)$ and the Jacobian of the map G_P has eigenvalue λ_0 with multiplicity 1. □

Theorem 23. *Let $G = M \circ F$ be a GA map with a fixed point x and assume* rank $(DF(x)) = n - 1$. *If $DG(x)$ has eigenvalue $\lambda_0 \in \mathcal{S}^1$ with multiplicity $k = 1$, then there exists an admissible perturbation $P \in \mathcal{P}(x, M)$ and $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$, λ_0 is not an eigenvalue of $DG_{\epsilon P}(x)$.*

Proof. Assume $\lambda_0 \in \mathcal{S}^1$ is a simple eigenvalue of $DG(x)$. As before we let $B(\lambda) = DG(x) - \lambda I$. Let K be a perturbation of $B(\lambda)$ corresponding to an admissible perturbation $P \in \mathcal{P}(x, M)$. By Theorem 20

$$\det(B(\lambda) + \epsilon K) = r(\lambda)\left[(\lambda_0 - \lambda) + \frac{\epsilon q(\lambda) + \epsilon(\lambda_0 - \lambda)s(\lambda) + \mathcal{O}(\epsilon^2)}{r(\lambda)}\right], \tag{13}$$

where $r(\lambda) := \prod_{i=2}^{n}(\lambda_i - \lambda)$, $q(\lambda)$ and $s(\lambda)$ are polynomials in λ of degree less than n. Evaluating (13) at $\epsilon = 0$ we get $\det(B(\lambda) + \epsilon K) = r(\lambda)(\lambda_0 - \lambda)$. Since λ_0 is a simple eigenvalue by assumption, we must have

$$r(\lambda_0) \neq 0. \tag{14}$$

We expand the polynomials $r(\lambda)$, $q(\lambda)$ and $s(\lambda)$ in a Taylor expansion about $\lambda = \lambda_0$:

$$r(\lambda) = r(\lambda_0) + \frac{r'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: R_0 + R_1(\lambda - \lambda_0) + \ldots$$
$$s(\lambda) = s(\lambda_0) + \frac{s'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: S_0 + S_1(\lambda - \lambda_0) + \ldots$$
$$q(\lambda) = q(\lambda_0) + \frac{q'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: Q_0 + Q_1(\lambda - \lambda_0) + \ldots.$$

Let $\lambda_\epsilon := f(\epsilon)$ with $\lambda_0 = f(0)$ be the continuation in ϵ of the root λ_0. By (14), λ_ϵ satisfies

$$0 = (\lambda_0 - \lambda_\epsilon) + \frac{\epsilon q(\lambda_\epsilon) + \epsilon(\lambda_0 - \lambda_\epsilon)s(\lambda_\epsilon) + \mathcal{O}(\epsilon^2)}{r(\lambda_\epsilon)}. \tag{15}$$

Consider the Taylor series expansion of $\lambda_\epsilon = f(\epsilon)$ about the point $\epsilon = 0$ given by

$$\lambda_\epsilon = f(\epsilon) = f(0) + \epsilon f'(0) + O(\epsilon^2).$$

By definition of $\lambda_\epsilon, f(0) = \lambda_0$, thus we get

$$\lambda_\epsilon - \lambda_0 = \epsilon f'(0) + \mathcal{O}(\epsilon^2).$$

The term $f'(0)$ describes the first order direction of movement of λ_0 as we perturb by ϵK. To find this direction, we expand all factors in (15) to Taylor series

$$\begin{aligned} 0 \ &= \epsilon f'(0) + \mathcal{O}(\epsilon^2) + \epsilon \frac{Q_0 + Q_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)}{R_0 + R_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)} \\ &\quad - \epsilon(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) \frac{S_0 + S_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)}{R_0 + R_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Taking the common denominator and then equating the resulting numerator to zero we get

$$0 = \epsilon(f'(0)R_0 + Q_0) + \mathcal{O}(\epsilon)^2.$$

Equating the order ϵ term to zero we get

$$f'(0) = -\frac{Q_0}{R_0} + \mathcal{O}(\epsilon)^2. \tag{16}$$

The rest of the proof consists of showing that there is a perturbation matrix K such that $f'(0)$ is not only non-zero, but also not tangent to the unit circle. We consider the cases $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ separately.

Case I: $\lambda_0 \in \mathbb{R}$.
By Lemma 21 there exists an admissible perturbation $P \in \mathcal{P}(x, M)$ with the corresponding matrix $K \in \mathcal{K}$ such that

$$Q_0 = q(\lambda_0) = k_{11} \det B_{11}(\lambda_0) \neq 0$$

is a real, non-zero number.

Comparing formulas (7) and (13) we see that

$$r(\lambda) = \prod_{i=2}^{n}(\lambda_i - \lambda). \tag{17}$$

Since $R_0 = r(\lambda_0)$ and λ_0 is a simple real eigenvalue, R_0 must be a real number. It follows from (14) that $R_0 \neq 0$.

Since both Q_0 and R_0 are non-zero real numbers, $f'(0) \neq 0$ and real. Therefore the direction at which λ_0 is moving off of the unit circle is perpendicular to the unit circle.

Case II: $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.
By Theorem 20 in this case

$$Q_0 = q(\lambda_0) = [(k_{11} + k_{22}) + \mathrm{i}(k_{12} - k_{21})] \det B_{11}(\lambda_0). \tag{18}$$

Further, the direct computation using second matrix in (9) shows

$$\det B_{11}(\lambda_0) = -\mathrm{Im}(\lambda_0)\mathrm{i}u(\lambda)$$

where $u(\lambda) := \prod_{i=3}^{n}(\lambda_0 - \lambda_i)$ and λ_i for $i = 3, \ldots, n$ are eigenvalues off of the unit circle. The constant R_0 can be computed comparing (13) and (17)

$$R_0 = r(\lambda_0) = (\overline{\lambda_0} - \lambda_0)\prod_{i=3}^{n}(\lambda_i - \lambda_0) = 2\mathrm{i}(\mathrm{Im}(\lambda_0))\prod_{i=3}^{n}(\lambda_i - \lambda_0).$$

Finally, combining (16), (18) and (19) we get

$$\begin{aligned} f'(0) &= -\frac{Q_0}{R_0} + \mathcal{O}(\epsilon)^2 \\ &= -\frac{[(k_{11}+k_{22})(-\mathrm{Im}(\lambda_0))\mathrm{i} + (k_{12}-k_{21})\mathrm{Im}(\lambda_0)]\prod_{i=3}^{n}(\lambda_i - \lambda_0)}{2\mathrm{i}(\mathrm{Im}(\lambda_0))\prod_{i=3}^{n}(\lambda_0 - \lambda_i)} + \mathcal{O}(\epsilon)^2 \\ &= \frac{(k_{11}+k_{22})}{2} + \frac{(k_{12}-k_{21})\mathrm{i}}{2} + \mathcal{O}(\epsilon)^2. \end{aligned}$$

This represents the first order approximation of the motion of the perturbed eigenvalue λ_0. We need to show that we can find a perturbation matrix $K \in \mathcal{K}$ such that this direction is not tangent to the unit circle. To do that we express the numbers k_{ij} in terms of perturbation matrix H and ultimately, the perturbation vector h. Recall that by (10) ,

$$k_{ij} = (C^{-1})_{i1}(H_1 \cdot C^j) + (C^{-1})_{i2}(H_2 \cdot C^j) + \cdots + (C^{-1})_{in}(H_n \cdot C^j), \tag{19}$$

where C^i is the ith column of the change of basis matrix C. We can choose the first two columns C^1 and C^2 in such a way that the complex eigenvector v_{λ_0} corresponding to λ_0 has the form $v_{\lambda_0} = C^1 + \mathrm{i}C^2$. We set $q : \mathbb{R}^n \mapsto \mathbb{C}$, where

$$q(h) := f'(0) = \frac{(k_{11}+k_{22})}{2} + \frac{(k_{12}-k_{21})\mathrm{i}}{2}.$$

By Lemma 19 there are coordinates i and j such that $-(C^{-1})_{1i} + (C^{-1})_{1j} \neq 0$. Fixing this i and j, by Lemma 17 we define a perturbation matrix with $H_j = -H_i = h$ and $H_l = 0$ for $l \neq i, j$. Then using Eq. (19) we see that

$$\begin{aligned} k_{11} + k_{22} &= (-(C^{-1})_{1i} + (C^{-1})_{1j})(h \cdot C^1) + (-(C^{-1})_{2i} + (C^{-1})_{2j})(h \cdot C^2) \\ k_{12} - k_{21} &= (-(C^{-1})_{1i} + (C^{-1})_{1j})(h \cdot C^2) - (-(C^{-1})_{2i} + (C^{-1})_{2j})(h \cdot C^1). \end{aligned}$$

Now let $A := \frac{1}{2}(-(C^{-1})_{1i} + (C^{-1})_{1j})$ and $B := \frac{1}{2}(-(C^{-1})_{2i} + (C^{-1})_{2j}) \neq 0$. Then

$$q(h) = A(h \cdot C^1) + B(h \cdot C^2)) + \mathrm{i}(A(h \cdot C^2)) - B(h \cdot C^1) = (v \cdot h) + \mathrm{i}(w \cdot h)$$

where $v := (AC^1 + BC^2)$ and let $w := (AC^2 - BC^1)$. Note also that since C^1 and C^2 are the real and complex parts an eigenvector and hence linearly independent and $A \neq 0$ by Lemma 19,

$$v \neq 0 \quad \text{and} \quad w \neq 0. \tag{20}$$

Now that we have shown $v \neq 0$ and $w \neq 0$, to complete the proof of the theorem it suffices to show that there is a perturbation vector $h \in \mathcal{R}$ such that,

$$q(h) \neq 0 \quad \text{and} \quad q(h) \neq a(\beta - \mathrm{i}\alpha),$$

$a \in \mathbb{R}$. In other words, there is a perturbation vector h such that $f'(0)$ is not tangent to the unit circle at λ_0.

Let $\theta := \beta - \mathrm{i}\alpha$ be the complex number representing the tangent direction at λ_0. Assume, by the way of contradiction, that for all $h \in \mathcal{R}$, the function $q(h) = a\theta$, $a \in \mathbb{R}$. That is, we assume

$$q(h) = (h \cdot v) + \mathrm{i}(h \cdot w) = a\theta.$$

Since $C^1 + \mathrm{i}C^2$ is a λ_0 eigenvector of $DG(x)$, Lemma 14 applied to the map G implies $C^i \cdot (1, \ldots, 1) = 0$ for $i = 1, 2$. Since $h \in \mathcal{R}$, $h = v^T DF(x)$ and by Lemma 14 applied to F this implies $h \cdot (1, \ldots, 1) = 0$. Furthermore, by Corollary 16 the space $\mathcal{R}$ of available vectors h has a dimension $n - 1$. More precisely, by (20) $v \neq 0$, $w \neq 0$, and when $n \geq 3$ there exists a vector h_1 such that $h_1 \cdot v = 0$ and $h_1 \cdot w \neq 0$. Similarly, there exists h_2 such that $h_2 \cdot v \neq 0$ and $h_2 \cdot w = 0$. Then $q(h_1) = \mathrm{i}(h_1 \cdot w)$ and $q(h_2) = h_2 \cdot v$ which is a contradiction, since $q(h_1) \neq aq(h_2)$ for $a \in \mathbb{R}$. Therefore for $n \geq 3$ there exists an an h such that $q(h) \neq 0$ and $q(h) \neq a(\beta - \mathrm{i}\alpha)$.

For the case $n = 2$ we notice that by Lemma 14 applied to map G the rank of $DG(x) = 1$. Hence in this case $DG(x)$ has only real eigenvalues. This finishes the proof of the theorem. □

We have dealt with the simple eigenvalue case and the repeated eigenvalue case separately. We now show that given a GA map $G = M \circ F$ with fixed point x and $\text{spec}(DG(x)) \cap \mathcal{S}^1 \neq \emptyset$, that we can perturb using $P \in \mathcal{P}(x, M)$ such that the resulting map G_P has $\text{spec}(DG_P(x)) \cap \mathcal{S}^1 = \emptyset$.

Lemma 24. *Let $G = M \circ F$ with fixed point x and F with* rank $(DF(x)) = n - 1$. *If x is non-hyperbolic, there exists $P \in \mathcal{P}(x, M)$ such that $G_P = (M + cP) \circ F$ has hyperbolic fixed point x for all $0 < c \leq 1$.*

Proof. Let $\text{spec}(DG(x)) \cap \mathcal{S}^1 = \{\lambda_1, \ldots, \lambda_k\}$ with multiplicities $m_1, \ldots, m_k$, respectively and let $\text{spec}(DG(x)) \backslash \mathcal{S}^1 = \{\lambda_{k+1}, \ldots, \lambda_n\}$. We define

$$\epsilon := \min_{i \in \{k+1, \ldots, n\}} (\mathrm{d}(\mathcal{S}^1, \lambda_i)),$$

where d denotes Euclidean distance in the complex plane. If $m_1 > 1$, by Lemma 22, there exists $P^r \in \mathcal{P}(x, M)$ such that $DG_{P^r}(x)$ has eigenvalue λ_1 with multiplicity at most 1. If, or once, this eigenvalue does have multiplicity 1, then by Theorem 23, there exists $P \in \mathcal{P}(x, M)$ such that the perturbed map G_{P^r} has $\lambda_0 \notin \text{spec}(DG_x(x))$. By Corollary 13, there exists $\delta > 0$ such that $\delta(P^r + P) \in \mathcal{P}(x, M)$. By Corollary 11, since $\delta(P^r + P) \in \mathcal{P}(x, M)$, for any $t \in [0, 1)$, $t(\delta(P^r + P)) \in \mathcal{P}(x, M)$. We choose t small enough so that the perturbed eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are still outside the unit circle.

Set $P^1 = t(\delta(P^r + P))$. Note that for $0 < c \leq 1$, $ct < t$, thus for $cP^1 = ct(\delta(P^r + P)) \in \mathcal{P}(x, M)$, perturbed eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are still outside the unit circle. Clearly G_{cP^1} has $\lambda_1 \notin \text{spec}(DG_{cP^1}(x))$ for all c with $0 < c \leq 1$ and $\text{spec}(DG_{cP^1}(x)) \cap \mathcal{S}^1 \subset \{\lambda_2, \ldots, \lambda_k\} = \{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_l}\}$. We repeat the argument and in every step the number of eigenvalues on the unit circle is strictly smaller. Since we started with a finite number of eigenvalues this terminates in a finite number of steps. Therefore there is a perturbation $P = P^1 + P^2 + \cdots + P^s$ such that $G_P = (M + cP) \circ F$ has a hyperbolic fixed point x for all $0 < c \leq 1$. □

6. Hyperbolic fixed points are dense: Proof of Proposition 8

We will use the following notation. For $M \in \mathcal{M}$, define $G^M := M \circ F$ and recall that $\mathcal{M}_{\{0\}} = \{M \in \mathcal{M} | [M(F(x)) - x] \pitchfork 0\}$ denotes the set of parameter values in $\mathcal{M}$ for which $M(F(x)) - x$ has finitely many solutions in Λ. Let $\text{Fix}(f)$ denote the set of fixed points of f, while $\text{NonHyp}(f) \subset \text{Fix}(f)$ and $\text{Hyp}(f) \subset \text{Fix}(f)$ denote the sets of non-hyperbolic and hyperbolic fixed points of f, respectively. For $x \in \text{Fix}(f)$, with eigenvalues $\lambda_i \in \text{spec}\, Df(x)$,

$$d(\text{spec}(DG^M), \mathcal{S}^1) = \min_i d(\lambda_i, \mathcal{S}^1).$$

Lemma 25. *Let $M \in \mathcal{M}_{\{0\}}$, then there exists $\epsilon > 0$ such that if $||M' - M|| < \epsilon$, then $\#\{\text{Fix}(M \circ F)\} = \#\{\text{Fix}(M' \circ F)\} < \infty$.*

Proof. Since $M \in \mathcal{M}_{\{0\}}$, there exists $j < \infty$ such that $\#\{\text{Fix}(M \circ F)\} = j$. Thus it suffices to show that $\#\{\text{Fix}(M \circ F)\} = \#\{\text{Fix}(M' \circ F)\}$. By Proposition 6, $M_{\{0\}}$ is open and dense in $\mathcal{M}$, thus there exists $\delta > 0$ such that if $||M - M'|| < \delta$, then $M' \in M_{\{0\}}$. Also by Proposition 6, since $M' \in M_{\{0\}}$ there exists $k < \infty$ such that $\#\{\text{Fix}(M' \circ F)\} = k$. We now show that for δ above chosen small enough, $k = j$.

By Proposition 1.1.4 of [7], because x_i is non-degenerate, for all $i = 1, \ldots, j$, there exist $\eta_i > 0$, such that for all $\gamma_i < \eta_i$, there exist $\epsilon_i > 0$ such that for all $\epsilon < \epsilon_i$, if $||(M \circ F) - (M' \circ F)||_1 < \epsilon$, then there is a unique fixed point $x' \in \text{Fix}(M' \circ F)$ with $||x_i - x_i'|| < \gamma_i$. Pick $\gamma = \min_i\{\gamma_i/2\}$. Now choose $\epsilon = \min_i\{\epsilon_i(\gamma)\}$. By this choice, if $||M \circ F - M' \circ F|| < \epsilon$, then for $i = 1, \ldots, j$ there exist unique fixed points x_i' with $||x_i - x_i'|| < \gamma$.

Note that for all $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that if $||M - M'|| < \delta_1$ then $||(M \circ F) - (M' \circ F)|| < \epsilon$ on Λ in the $\mathcal{C}^1$ topology.

Let U_i be a neighborhood of radius γ of x_i, and let $U = \bigcup_{i=1,\ldots,j} U_i$. Having fixed neighborhood U, the previous argument shows that there exists $\delta_1 > 0$ such that if $||M - M'|| < \delta_1$ then

$$\#\{\text{Fix}(M \circ F|_U)\} = \#\{\text{Fix}(M' \circ F)|_U\}.$$

We now show that on the compact set $K := \Lambda \backslash U$,

$$\#\{\text{Fix}(M \circ F|_K\} = \#\{\text{Fix}(M' \circ F)|_K\} = 0,$$

provided M' is close enough to M. Since $d(x, M \circ F(x)) : K \to \mathbb{R}$ is continuous, and K is compact, by the Minimum Value Theorem [9], there exists $c > 0$ such that for all $x \in K$, $d(x, M \circ F(x)) > c$. Thus, there exists $\delta_2 > 0$ such that if $||M' - M|| < \delta_2$, then $d(x, M' \circ F(x)) > c/2$ for all $x \in K$. This implies that if $||M - M'|| < \delta_2$, then $M' \circ F$ has no fixed points in K.

Finally, let $\epsilon = \min\{\delta, \delta_1, \delta_2\}$. Then, if $||M' - M|| < \epsilon$,

$$\#\{\text{Fix}(M \circ F|_K\} = \#\{\text{Fix}(M' \circ F)|_K\} = j. \quad \square$$

Lemma 26. *Assume $M \in \mathcal{M}_{\{0\}}$ and $\text{Hyp}(G^M) = \{x_1, \ldots, x_m\}$ with $m < \infty$. There exists $\delta < 0$ such that if $||M - M'|| < \delta$, then for $\{x_1', \ldots, x_m'\}$ perturbed fixed points of $G^{M'}$,*

$$\min_{i=1,\ldots,m} d(\text{spec}(DG^{M'}(x_i')), \mathcal{S}^1)) > 0.$$

Proof. First create a compact set of M's by choosing $||M - M'|| \le \epsilon/2$. Given $\gamma > 0$ by uniform continuity of $DG^{M'}(x)$ in M' and x, there are ϵ' and η' so that if $||M - M'|| < \epsilon'$ and $||x - x'|| < \eta'$, then

$$||DG^{M'}(x') - DG^{M}(x)|| < \gamma.$$

Next, if $\mathrm{Hyp}(G^M) = \{x_1, \dots, x_m\}$, then there is a $\gamma > 0$ such that if $||DG^{M'}(x_i') - DG^M(x_i)|| < \gamma$ then

$$d(\mathrm{spec}(DG^{M'}(x_i')), \mathcal{S}^1) > 0$$

for $i = 1, \dots, m$. Given this γ, choose ϵ' and η' as above. Since $x_i' = x_i'(M')$ is continuous in M', for M' near M, there exists ϵ'' such that if $||M-M'|| < \epsilon''$ then $||x_i' - x_i|| < \eta'$ for all i. Finally, let $\epsilon''' = \min(\epsilon, \epsilon', \epsilon'')$. It follows that if $||M-M'|| < \epsilon'''$, then

$$\min_{i=1,\dots,m} d(\mathrm{spec}(DG^{M'}(x_i'), \mathcal{S}^1)) > 0. \quad \square$$

Proof of Proposition 8. Let $G^M = M \circ F$ be a GA map which is not hyperbolic. We claim that for any $\epsilon > 0$, we can find $\overline{M} \in N_\epsilon(M) \subset \mathcal{M}$ such that $G^{\overline{M}} = \overline{M} \circ F$ is hyperbolic.

By Proposition 6, for any $\epsilon > 0$, there exists $M' \in \mathcal{M}_F \subset \mathcal{M}_{\{0\}}$ such that if $||M' - M|| < \epsilon$, then $G^{M'}$ has finitely many fixed points in an open and dense set B_F (see (3)), each of which has a Jacobian with no eigenvalue 1.

Then there exists $m < \infty$ such that

$$\mathrm{Fix}(G^{M'}) = \{x_1, \dots, x_m\}$$

and for some $l \le m$,

$$\mathrm{Hyp}(G^{M'}) = \{x_1, \dots, x_l\}$$
$$\mathrm{NonHyp}(G^{M'}) = \{x_{l+1}, \dots, x_m\}.$$

We now construct a finite sequence of perturbations, indexed by j, which will perturb non-hyperbolic fixed points in such a way that they will become hyperbolic.

Assume that after the jth step we have the map $G^{M_j} := M_j \circ F$ with

$$\mathrm{Fix}(G^{M^j}) = \{y_1, \dots, y_m\}$$

and

$$\mathrm{Hyp}(G^{M_j}) = \{y_1, \dots, y_k\}$$
$$\mathrm{NonHyp}(G^{M_j}) = \{y_{k+1}, \dots, y_m\},$$

where all fixed points $\{y_1, y_2, \dots, y_m\} \subset B_F$. We construct the $j+1$th perturbation M_{j+1} and define $G^{M_{j+1}} = M_{j+1} \circ F$. By Lemma 25 there exists $\epsilon_1 > 0$ such that if $||M - M_j|| < \epsilon_1$, then

$$\#\{\mathrm{Fix}(M \circ F)\} = \#\{\mathrm{Fix}(M_j \circ F)\}$$

and $M \in \mathcal{M}_{\{0\}}$. Furthermore, since $\mathcal{M}_F$ is a generic subset of $\mathcal{M}_{\{0\}}$, making ϵ_1 smaller, if necessary, we can assure that all M with $||M - M_j|| < \epsilon_1$ are in $\mathcal{M}_F$.

By Lemma 26, there exists $0 < \epsilon_2 < \epsilon_1$ such that for $M \in \mathcal{M}$ with $||M - M_j|| < \epsilon_2$

$$\tau_j = \min_{i=1,\dots,k} d(\mathrm{spec}(DG^{M_j}(y_i')), \mathcal{S}^1) > 0,$$

where $G^{M'}(y_i') = y_i'$ and this y_i' is a perturbed fixed point y_i. Since $M_j \in B_F$ by assumption, by Lemma 24, there exists $P \in \mathcal{P}(y_{k+1}, M_j)$ such that for $(M_j + cP) \circ F$, y_{k+1} is a hyperbolic fixed point for all $0 < c \le 1$.

Pick $M_{j+1} = M_j + \eta P$ where $\eta = \min\{1, \epsilon_2/2\}$. By construction, $\eta < \epsilon_1$, thus

$$\#\{\mathrm{Fix}(M_{j+1} \circ F)\} = \#\{\mathrm{Fix}(M_j \circ F)\}.$$

Additionally, because $\eta < 1$, by Corollary 11, $\eta P \in \mathcal{P}(y_{k+1}, M)$ which by definition implies $G^{M_{j+1}}(y_{k+1}) = y_{k+1}$. That is, for the perturbation ηP, y_{k+1} remains a fixed point of $G^{M_{j+1}}$, and y_{k+1} is hyperbolic.

Finally, because $\eta < \epsilon_2$, for $G^{M_{j+1}}$,

$$\tau_j = \min_{i=1,\dots,k} d(\mathrm{spec}(DG^{M_{j+1}}(y_i')), \mathcal{S}^1) > 0. \tag{21}$$

Thus, by Lemma 26,

$$\mathrm{Hyp}(G^{M_{j+1}}) \supseteq \{y_1', \dots, y_k', y_{k+1}\}$$

where $y'_1, \ldots, y'_k$ are perturbed fixed points $y_1, \ldots, y_k$ which by (21) are hyperbolic. Therefore,

$$|\text{NonHyp}(G^{M_{j+1}})| < |\text{NonHyp}(G^{M_j})|.$$

This process terminates in a finite number of steps when for j large enough $\text{NonHyp}(G^{M_j}) = \emptyset$. □

7. Conclusion

In this paper we have studied genericity of hyperbolic fixed points for the infinite population model of genetic algorithm. We have shown that given a $\mathcal{C}^1$ selection function F satisfying certain genericity criteria there is an open and dense set of mixing functions such that their composition, the GA map, has only hyperbolic fixed points. The GA maps form a small subset of the set of all maps, since it is severely restricted in its form and its domain and range. It is therefore nontrivial that such a restricted set of maps is nevertheless large enough to admit the set of perturbations that perturb arbitrary non-hyperbolic fixed point to a hyperbolic one. We interpret this fact as a signal that the set of GA maps is rich enough for regular intuition about the behavior of dynamical systems to be valid. Even though the correspondence between the infinite population model of a GA and the finite population models that are used by practitioners is not straightforward and likely depends on the details of that implementation, our result adds to the increasing body of evidence that the infinite population model can give qualitative insights into the functioning of the GA.

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