



A constitutive theory for high rate multiaxial deformation of snow  
by Andrew Christian Hansen

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in  
Mechanical Engineering  
Montana State University  
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**Abstract:**

The mechanical properties of snow under high strain rates and finite deformation are highly nonlinear and rate dependent. Unlike most geologic materials, snow can undergo substantial volumetric strains which are irreversible. Pressures may increase by orders of magnitude as snow is compressed from initially low densities of  $100 \text{ kg/m}^3$  to densities beyond  $500 \text{ kg/m}^3$ . At the same time, the shearing stiffness can increase dramatically.

One of the most difficult problems encountered when trying to quantify snow mathematically is to account for the microstructure of the material. Naturally occurring snow generally has a complex microstructure which varies dramatically depending on the weather conditions to which it is subjected. The microstructural properties of snow have a significant impact on its mechanical properties.

In this thesis, a constitutive theory is developed to describe the mechanical properties of snow at finite strain under high rate multiaxial deformation. The theory is based on nonequilibrium thermodynamics with internal state variables. The internal variable approach is based on the view that the state of the material at any given time in a deformation process is adequately determined by the strain, internal energy, and a finite collection of internal state variables. The state variables may be thought of as characterizing the extent of microstructural rearrangement within a sample. This approach is appealing in that it incorporates parameters reflecting the microstructure of snow, while at the same time essentially remaining a continuum theory.

Results are presented for low to medium density snow subjected to a variety of multiaxial loading conditions. Volumetric strains in excess of 100 percent are considered. The theory is shown to reflect mechanical properties of snow such as rate dependence, stress relaxation, and strain recovery.

The constitutive theory is significant in that it is consistent with the first and second laws of thermodynamics as well as satisfying the principle of material frame indifference. Equally significant is the inclusion of microstructural properties in the formulation. The internal state variable approach provides a powerful method of incorporating the microstructure of snow in a precise mathematical fashion.

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of

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APPROVAL

of a thesis submitted by

Andrew Christian Hansen

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citation, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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## ABSTRACT

The mechanical properties of snow under high strain rates and finite deformation are highly nonlinear and rate dependent. Unlike most geologic materials, snow can undergo substantial volumetric strains which are irreversible. Pressures may increase by orders of magnitude as snow is compressed from initially low densities of  $100 \text{ kg/m}^3$  to densities beyond  $500 \text{ kg/m}^3$ . At the same time, the shearing stiffness can increase dramatically.

One of the most difficult problems encountered when trying to quantify snow mathematically is to account for the microstructure of the material. Naturally occurring snow generally has a complex microstructure which varies dramatically depending on the weather conditions to which it is subjected. The microstructural properties of snow have a significant impact on its mechanical properties.

In this thesis, a constitutive theory is developed to describe the mechanical properties of snow at finite strain under high rate multiaxial deformation. The theory is based on non-equilibrium thermodynamics with internal state variables. The internal variable approach is based on the view that the state of the material at any given time in a deformation process is adequately determined by the strain, internal energy, and a finite collection of internal state variables. The state variables may be thought of as characterizing the extent of microstructural rearrangement within a sample. This approach is appealing in that it incorporates parameters reflecting the microstructure of snow, while at the same time essentially remaining a continuum theory.

Results are presented for low to medium density snow subjected to a variety of multiaxial loading conditions. Volumetric strains in excess of 100 percent are considered. The theory is shown to reflect mechanical properties of snow such as rate dependence, stress relaxation, and strain recovery.

The constitutive theory is significant in that it is consistent with the first and second laws of thermodynamics as well as satisfying the principle of material frame indifference. Equally significant is the inclusion of microstructural properties in the formulation. The internal state variable approach provides a powerful method of incorporating the microstructure of snow in a precise mathematical fashion.

## CHAPTER 1

## INTRODUCTION

Snow is a geologic material which dominates the polar climates and intermittently covers a much larger portion of the earth. Historically, it has received little attention from the rheological community. This is primarily because problems requiring knowledge of its material properties have been limited in the past. However, polar and alpine climates are receiving much more attention and use in recent years. Recreational use in the alpine back-country has increased dramatically. Polar regions are now being explored heavily for oil and natural gas reserves. Finally, recognition of the significance of snow from a military standpoint has increased.

Most of the past work in characterizing the material properties of snow has involved snow under low strain rates. However, the properties of snow are strongly rate dependent and there are a number of classical problems which require knowledge of the high rate properties of snow. Probably the most publicized is that of the snow avalanche. When an avalanche occurs in an area developed by man, the results are often disastrous. A recent example of this was the total destruction of the lift lodge and death of seven people at Alpine Meadows, California in March 1982.

Stress waves are another aspect of snow which require knowledge of its high rate properties. For example, snow has the ability to absorb a great deal of energy through material compaction resulting in tremendous attenuation rates for stress waves. Studies indicate that most of the energy produced by a classical one kilogram explosive is dissipated within a meter of the bomb sight. This becomes a significant problem when trying to initiate avalanches or clear mine fields covered by snow with explosives.

Vehicle mobility in alpine and polar terrain also requires knowledge of the high rate properties of snow. A snow cover has two degrading effects on the performance of a vehicle. First, large volumetric deformations occur from compaction by the wheels or tracks of the vehicle. This has the effect of reducing the total amount of energy available to propel the vehicle forward. Brown (1979b) has studied this problem and found that for some tracked military vehicles traveling at 20 mph in medium density snow (300-400 kg/m<sup>3</sup>), the power absorbed by the snow cover can exceed 40 percent of the available engine power.

The second degrading effect of snow on vehicle performance is the loss of traction and danger in becoming stuck. This problem is much more difficult to analyze since slippage involves both deviatoric and volumetric deformations. Little success in characterizing this type of multiaxial deformation has been obtained.

The use of snow as a protective material has recently received attention from the field of penetration mechanics. Experimental evidence indicates snow is very effective at retarding projectile penetration (Aitken, 1978). At the same time, artillery shells tend to exhibit a delayed fuse action when impacting a snow cover. This dramatically decreases the effectiveness of the explosive since most of the energy is absorbed in the snow. Knowledge of the high rate multiaxial properties of snow would yield a better understanding of both of these phenomena.

Finally, although this dissertation is directed towards an investigation of snow, a valid constitutive law would represent a significant advancement in the theory of rate dependent deformation of granular materials under finite strain. This includes many geologic materials such as cohesive and cohesionless soils. The theory is also applicable to the field of high temperature powder metallurgy.

### Some Unique Properties of Snow

Among all geologic materials, snow may be regarded as one of the most difficult to quantify mathematically. Naturally occurring snow generally has a complex stratigraphy which varies dramatically depending on the conditions it is subjected to while on the ground. These include rain, wind, extreme temperatures, melt-freeze cycles, and temperature gradient effects. The large variation in microstructure precludes modeling the snow as a simple collection of spheres or ellipses.

The reasons for the complex characteristics of a natural snow cover are two fold. First, the material is thermodynamically active because its natural state generally lies close to its melting point. Also, the material is highly porous with the ice phase typically occupying 5 to 50 percent of the material volume. These factors make the structure of snow susceptible to change resulting from changes in the weather. For instance, the ground temperature under a seasonal snow cover typically lies very close to 0°C. If this coincides with cooler ambient air conditions the snow is subjected to a vertical temperature gradient.<sup>1</sup> Under the influence of a temperature gradient, a mass flux of vapor from the zone of high vapor pressure deeper in the pack to the zone of low vapor pressure occurs. This facilitates the development of faceted crystals near the top of an air pore as water vapor is sublimated off the surfaces in the lower regions. Also, the bonds deteriorate most readily, causing an overall weakening of the snow cover. This process, known as temperature gradient metamorphism, has a dramatic effect on the properties of snow.<sup>2</sup>

Figure 1 shows the variation in compressive strength of four 0.1-m-thick layers of snow subjected to a constant temperature gradient. Notice the drastic loss of strength in the lower layers.

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<sup>1</sup> This situation may also exist on temperate glaciers or polar ice which has been warmed during the summer.

<sup>2</sup> Adams and Brown (1983) review temperature gradient metamorphism.

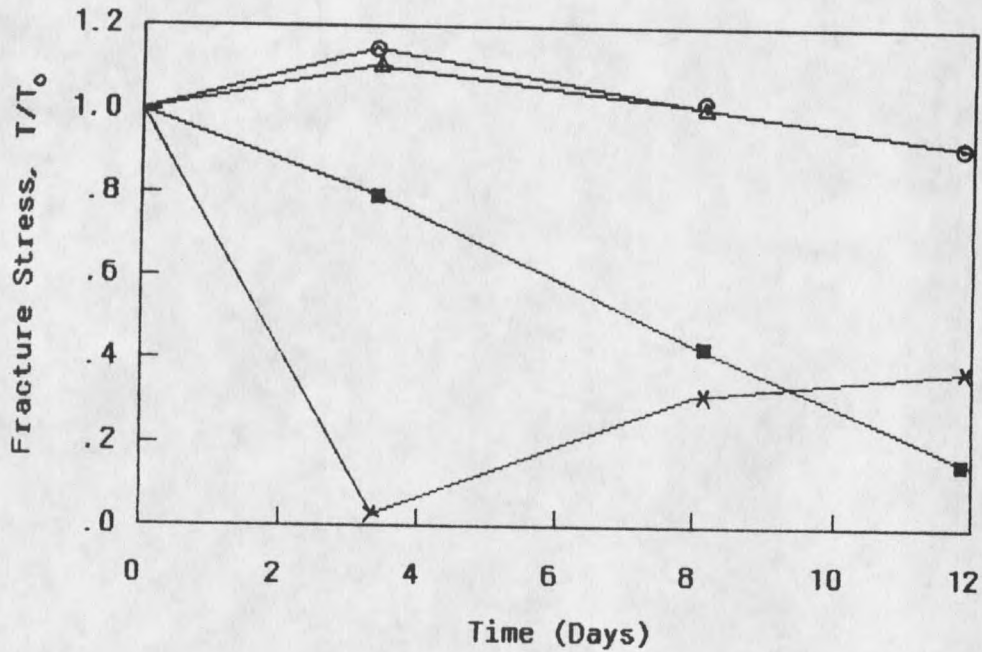


Figure 1. Characteristic reduction in strength in 0.4-m deep snow due to temperature gradient metamorphism (Bradley et al., 1977).  $T = 9500 \text{ Pa}$ ,  $d\theta/dz = -50 \text{ K/m}$ , (X) bottom layer, (■) second layer, (△) third layer, (○) top layer,  $\rho = 180 \text{ kg/m}^3$ .

The high porosity of snow is also responsible for the complex deformation mechanisms observed in snow. Bond growth, bond fracture, pressure melting, and intergranular glide all contribute to the total deformation making modeling the deformation process difficult. These factors are compounded even further by the nonlinear rheological properties of the matrix material, ice. For instance, under compression ice is known to behave as an elastic-rate-sensitive plastic material. Under high rate deformation, the rate effects are pronounced.



Finally, snow has been shown to have the ability to rapidly heal itself giving it a limited memory of its past. This is primarily due to sintering of the ice grains. Sintering is the process by which a collection of loose particles form permanent bonds at their points of contact forming a solid framework. This process occurs more readily at temperatures near to, but still below the melting point. Studies indicate that for ice grains, a bond of measurable strength can form within minutes (Hobbs, 1974). Also, deformation induced cracks are thought to heal due to diffusion of vapor and sintering effects.

The driving force for the material rearrangement in sintering is supplied by a surplus of energy associated with the excess of free surface area in the material. Hence, the sintering process is intricately related to the microstructural properties of the material, specifically the mean surface area per unit volume and mean curvature (Reid, 1984). Furthermore, the sintering process significantly alters the microstructural state of the snow resulting in an overall strengthening of the snow cover.

### Approach

In light of the above discussion, it is not surprising that previous efforts at characterizing the thermomechanical properties of snow have met with limited success. This is because the microstructural properties of snow are important in characterizing the material response. These properties vary dramatically depending on the weather conditions the snow is subjected to while on the ground. Therefore, the approach taken here is to formulate a continuum theory which incorporates parameters reflecting the microstructure of the snow.

The constitutive theory developed is designed to characterize the properties of snow under high rate multiaxial deformation. The theory is based on non-equilibrium thermodynamics with internal state variables. The internal variable approach is based on the view that the state of a material at any given time in the deformation process is adequately

determined by the strain, internal energy, and a finite collection of state variables. The state variables may be thought of as characterizing the extent of microstructural rearrangement within the sample. Furthermore, at any point in the deformation process, the variables may be fixed in theory, if not in principle, through imposition of appropriate constraints. In this manner, the deformation may be viewed as a series of constrained equilibrium states. The relation between the state variables and other state properties such as stress is then determined by the fictitious state of thermodynamic equilibrium referred to as the accompanying state.

The internal state variables identified in this work are of the "averaging" type on the granular level. By the term "averaging," it is meant that the variables refer to average characteristics of the structural rearrangements taken over all operative sites within the system. With some noted exceptions, this approach follows a formulation by Rice (1971) who developed a theory to describe the constitutive relations for metal plasticity using internal state variables at the microstructural level, e.g., crystalline slip, phase transformations, diffusional transport, etc.

Finally, the theory presented represents a significant deviation from classical plasticity. First, in classical plasticity, volumetric strains are purely elastic. This is hardly the case for snow which may undergo density changes of an order of magnitude under compression. The internal variable approach readily accommodates this property. Furthermore, unlike classical plasticity, the theory is rate dependent through the internal state variables which are governed by temporal evolution equations. This is extremely important for snow which has been shown to have strong rate dependent properties.

Results

Results of the constitutive theory are presented for uniaxial confined compression tests and combined compression and shear tests for multiple strain rates and volumetric strains exceeding 100 percent. The theory is shown to be capable of modeling phenomenological properties of snow such as rate dependence, stress relaxation, and limited creep recovery.

## CHAPTER 2

## OVERVIEW

In this chapter, a general overview of the problems associated with characterizing the rheological properties of snow under high rate deformation and finite strain is given. First, a review of previous investigations on the high rate properties of snow is presented including experimental and analytical results. This is followed by a discussion of previous investigations of general thermomechanical constitutive relations. Included in this discussion is the internal state variable approach as well as other theories which may be applicable to snow. Finally, a description of the proposed problem is presented along with some of the important properties which should be incorporated into the constitutive theory.

It should be noted that the literature review presented is by no means exhaustive and is merely intended to provide a basis for subsequent work performed in this study. For an extensive review of snow research, the reader is referred to review articles by Mellor (1974, 1977). Also, an excellent foundation for studying thermomechanical constitutive relations can be found in Truesdell and Noll (1965).

#### Previous Investigations on the High Rate Properties of Snow

Much of the laboratory experimental work on high rate deformation of snow under finite strain was done by Napadenski (1964), Abele and Gow (1975, 1976), and Wakahama and Sato (1977). Napadenski's work was concerned with the response of snow subjected to high intensity shock waves. These experiments were conducted by accelerating a metal plate into a snow sample with the use of explosives. A streak camera was used to record the plastic wave speed and particle velocity at the shock front. This data was then used to

determine the shock adiabat (Hugoniot) for the snow sample using hydrodynamic theory. The shock adiabat is useful in that it defines an upper limit to the compressibility of snow for a given pressure. For an extensive review of hydrodynamic theory, the reader is referred to Duvall and Fowles (1963).

To this day the stress wave experiments performed by Napadenski provide the only Hugoniot data for snow. The snow samples for her work ranged from 390 to 530 kg/m<sup>3</sup> and are significantly higher than those normally encountered in a seasonal snow cover. However, a current set of shock tube experiments is being conducted at Los Alamos National Laboratory in conjunction with the United States Army Cold Regions Research and Engineering Laboratory (USACRREL). This should provide much needed information on the response of low density snow to shock waves.

Stress wave data for solid ice is also important to the field of snow mechanics. This has particular relevance to projectile penetration. For instance, inert steel and aluminum projectiles fired at velocities of around 10 m/s into snow with densities of 400 to 500 kg/m<sup>3</sup> are deformed plastically (Swinzow, 1972). These materials had yield stresses of  $5(10)^2$  and  $3(10)^2$  MPa respectively which are approaching the shock adiabat for ice. To this end, the works of Anderson (1968) and Gaffney (1985) on the shock adiabat of ice are cited.

Abele and Gow (1975, 1976) performed an extensive set of uniaxial compression tests on virgin and precompressed snow. Their work was primarily concerned with investigating the effects of temperature, initial density, and rate of deformation on the stresses produced. The tests were conducted with a 10,000 kg load capacity servo-controlled MTS machine with an environmental test chamber capable of maintaining temperatures down to -50°C. The cross head speeds ranged from 0.1 to 0.4 cm/s while the stresses produced ranged from 0.1 to 75 bar. To supplement this data, some microstructural analysis of the snow before and after the test was made. This was accomplished by preparing thin sections

of the snow and making photomicrographs. Full details of this technique are given in Gow (1969).

Wakahama and Sato (1977) studied the response of snow to a plastic wave by dropping a 1 kg mass 2 meters to a block of snow. The plastic wave speed was monitored using a high speed camera and the results were compared for snow whose initial density ranged from 170 to 460 kg/m<sup>3</sup> and free-water content from 0 to 17 percent. Changes in the snow structure were also studied using thin sections.

A number of field experiments have been conducted to supplement data taken in the laboratory. Stress wave studies using explosives have been conducted by numerous investigators. A review of work previous to 1965 is given by Mellor (1965). Albert (1983) provides still another review of more recent work. Brown and Hansen (1985) have conducted research on the significance of the pore pressure to the overall plastic wave. Studies indicate that the pore pressure is very significant to the overall response. Hansen and Brown (1985) have also analyzed the energy dissipation caused by ice fracture in a plastic wave. The results indicate that at least for pressures up to one MPa, energy dissipation by ice fracture is negligible.

The vast majority of analytical work on the constitutive properties of snow under high rate deformation has been done by Brown (1979a, 1980a). Brown (1979a) formulated a volumetric constitutive law for medium to high density snow based on a pore collapse model put forth by Carroll and Holt (1972). The model consists of spherical pores in a matrix of ice subjected to a hydrostatic pressure. A second volumetric constitutive law based on neck growth of grain bonds was formulated by Brown (1980a) for low density snow. This formulation is significant in that it successfully incorporates microstructural parameters for snow such as bond radius and bond length. Brown (1979b, 1980b,c,d; 1981a,b) has used these equations to study a number of significant problems including vehicle mobility and stress wave propagation.

As mentioned previously, the microstructure of snow plays an important role in the rheological characteristics of the material. Therefore, the study of the structure of snow has received a considerable amount of attention. Kry (1975a) has made an analysis of grain bonds in snow using quantitative stereology. Using this analysis, Kry (1975b) followed the evolution and subsequent effect of grain bonds on material properties under quasi-static uniaxial compression. Gubler (1978) has made significant advances in determining the mean number of bonds per snow grain using stereology. This is a very difficult problem to solve using only two-dimensional information. Finally, Good (1974) has developed a set of numerical parameters to identify the structure of snow.

In closing, there are numerous gaps in the knowledge of the material properties of snow under high rate deformation. This is particularly true for multiaxial states of stress where both experimental and theoretical results are scarce.

#### Principles of Thermomechanical Constitutive Theories

In describing the dynamic behavior of a continuum, there are three general principles which govern the material response (Truesdell and Noll, 1965). The principles of determinism and local action collectively state that the stress at a point in the body is determined by the history of an arbitrarily small neighborhood of the particle. The principle of material frame indifference states that the response of the material is independent of the observer. These principles are applicable to all classical work in continuum mechanics.

The difference in material response is governed by the specific constitutive assumptions made for the material. These assumptions can be mathematically stated using the principles of determinism and local action. The principle of frame indifference can then be used to place restrictions on the assumptions.

By introducing the thermodynamics of a continuum, an additional set of restrictions can be developed by application of the Second Law. This law takes the form of the

Clausius-Duhem inequality. The inequality implies there is internal entropy production in any irreversible process. Coleman and Noll (1963) present logical consequences of this law for a particular class of materials. These results have since become standard in making thermodynamic arguments.

One specific approach to describing the response of a material is put forth by Coleman and Noll (1960, 1961) in which the principle of fading memory is postulated.<sup>3</sup> This theory assumes that the entire past history of strain influences the stress. The principle is a smoothness postulate which requires the stress to be more sensitive to strains in the recent past as opposed to the distant past. Coleman (1964) developed a general theory of thermodynamics of simple materials with fading memory. Here, he assumed the stress was a function of the past history of the deformation gradient, temperature, and temperature gradient. The article is primarily concerned with restrictions placed on this type of material using the Second Law.

An alternative approach to the principle of fading memory is to postulate the existence of a set of internal state variables which characterize the state of the material at any given time and whose rate of change is governed by a set of temporal evolution equations. Coleman and Gurtin (1967) provide a detailed analysis of the thermodynamics of internal state variables. In this work no attempt is made to identify the specific nature of the variables as they are treated as abstract quantities. However, restrictions on the form of the state variables are developed based on the principle of material frame indifference.

Rice (1971) developed a general internal variable thermodynamic formalism for a class of solids at finite strain exhibiting inelasticity due to specific structural rearrangements on the microscale. In particular, he applied the theory to metals deforming plastically through dislocation motions. Specific kinetic relations for the rate of change of the

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<sup>3</sup> This principle was later developed from a set of elementary axioms by Coleman and Mizel (1966).



state variables are postulated and discussed. Rice (1975) discusses the relation between macroscopic deformation and inelastic structural rearrangements which operate on the microscale. These microstructural changes include crystalline slip, diffusion, phase changes, and Griffith cracks. The thermodynamics of internal state variables is further investigated by Bataille and Kestin (1979). Finally, Lubliner (1972) examines evolution equations for internal variables yielding descriptions of viscoelastic, viscoplastic, and plastic behavior.

The work of Valanis (1966; 1971a,b) closely paralleled that of Coleman and Gurtin. Valanis (1966) applied the theory of irreversible thermodynamics to large deformation of viscoelastic materials. In this work he referred to the internal state variables as "hidden" coordinates and shows them to be scalar functionals of the history of the deformation. Valanis (1971a,b) used an internal variable formalism to describe the theory of viscoplasticity without a yield surface. To describe materials of this type he introduced an intrinsic time scale which is related to the external or real time through scalar material properties. As a result, he coined the term, "endochronic" theory.

Numerous papers have been written on the use of endochronic theory in geologic materials. Valanis and Read (1980) provide a brief review of endochronic theory including thermodynamics while introducing a new endochronic model for soils. Read et al. (1981) introduced an inelastic constitutive model for soils based on endochronic theory which incorporates critical state soil mechanics. A review of critical state soil mechanics can be found in Schofield and Wroth (1960). Harrison and Berger (1975, 1976) discuss the possible application of critical state theory to snow mechanics.

Finally, some recent theories of granular materials have shown promise for snow. Specifically, the pioneering work of Goodman (1969) and Goodman and Cowin (1972) are noted. In this theory, the bulk density of a granular material is written as the product of two scalars consisting of the matrix density and a volume distribution function. The

latter may be thought of as characterizing the spatial and time dependence of the porosity. By breaking the bulk density into two parts, an additional degree of freedom in describing the kinematics of the deformation has been introduced. This results in an additional balance equation known as the balance of equilibrated stress. Goodman (1969) shows that the equilibrated stress can represent a center of compression or dilatation in the material.

The theory of Goodman and Cowin has been expanded by Passman (1974, 1977) and Nunziato and Walsh (1980) by treating the volume fraction of each constituent of a mixture as an independent kinematical quantity.<sup>4</sup> This gives rise to additional balance laws describing the microstructural material response.

#### Description of the Problem

Most engineering materials, including a large class of geologic materials such as sand and many soils are largely incompressible with volumetric strains being small and elastic. By contrast, snow can undergo substantial volumetric strains under compression which are irreversible. Pressures may increase by orders of magnitude as snow is compressed from initially low densities of  $100 \text{ kg/m}^3$  to densities beyond  $500 \text{ kg/m}^3$ . At the same time, the shearing stiffness may increase dramatically. The problem of characterizing the response of snow is further complicated by the material properties of ice which is the matrix material. Ice is a nonlinear, rate dependent material and these properties must be incorporated into the constitutive law.

During plastic deformation, the granular properties of the snow structure change dramatically as the microstructure adjusts to the applied loads. Under compression, the intergranular slip distances close as grains and grain bonds fracture, thereby allowing the grains to reorganize. At the same time, pressure sintering causes new bonds to be formed and "neck growth" in old bonds. Therefore, it is not sufficient to describe the state of the

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<sup>4</sup> An extensive review of mixture theory is given by Bowen (1976).

material in terms of density alone. The introduction of internal state variables then provides a means of characterizing the plastic state of the material much more accurately.

As mentioned previously, I am concerned with the material response of snow to high rate deformation. However, only states of stress which produce a mean compressive state in addition to shearing will be considered. A mean tensile stress would simply fracture the material since snow behaves in a brittle fashion under this type of loading.

Finally, the constitutive theory developed should satisfy the following guidelines which are considered important in the modern theory of constitutive relations.

1. The constitutive law should satisfy the principle of material frame indifference.<sup>5</sup>

This implies the constitutive equations are invariant under changes of frame of reference. That is, the response of a body to an applied load is independent of the observer.

Frame indifference is extremely important when dealing with large deformation theory. Many constitutive equations developed in the past do not satisfy this condition. Such a shortcoming can produce erroneous results, particularly when finite rotations are involved.

2. The constitutive law should be consistent with the first and second laws of thermodynamics. In this manner, some knowledge of energy dissipation rates and recoverable energy can be obtained. Furthermore, the Second Law provides valuable restrictions on constitutive assumptions.
3. The constitutive law should be partially defined by the structural properties of the material.
4. The constitutive law should reflect the dominant deformation mechanisms for the material. For snow, some of the mechanisms considered are bond fracture and intergranular glide, and pressure sintering.

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<sup>5</sup> Malvern (1969) provides an indepth discussion of frame indifference.

A constitutive theory based on the above criteria will by nature be very complex. However, the complex properties of snow require a formulation which is very general and requires few specific assumptions regarding material response. Some simplification to the resulting theory can be obtained by examining specific stress states.

## CHAPTER 3

## KINEMATICS, BASIC PRINCIPLES

In this chapter, the kinematics and basic thermomechanical principles of the dynamic behavior of a continuum are laid down. These include the balance of mass and momentum as well as the first and second laws of thermodynamics. The equations are developed within the context of large deformation theory.

Notation

Before proceeding with the theory, a brief discussion of the notation is presented. Unless otherwise noted, all work is referred to a general curvilinear coordinate system. Second order tensors will be represented by a subscripted tilde while first order tensors will be underlined. Dot and double dot products represent first and second order contractions of tensors. The transpose and inverse properties of a second order tensor will be denoted with the superscripts T and  $-1$  respectively. Finally, the identity tensor will be denoted by  $\underline{\underline{1}}$ .

The following examples serve to clarify the notation discussed.

$$\underline{\underline{A}} = A^{ij} (\underline{g}_i \underline{g}_j) ; \underline{\underline{B}} = B_{ij} (\underline{g}^i \underline{g}^j) ; \underline{u} = u^i \underline{g}_i$$

$$\underline{\underline{A}} : \underline{\underline{B}} = A^{ij} B_{ij} = \text{tr}(\underline{\underline{A}}^T \underline{\underline{B}}),$$

where tr denotes the trace (first invariant) of a second order tensor defined by

$$\text{tr}(\underline{\underline{T}}) = T^k_k.$$

$$\underline{\underline{A}} \cdot \cdot \underline{\underline{B}} = A^{ij} B_{ji} = \text{tr}(\underline{\underline{A}} \underline{\underline{B}})$$

$$\underline{\underline{B}} \underline{u} = B_{ij} u^j \underline{g}^i$$

$$\underline{\underline{B}} \underline{u} = \underline{u} \underline{\underline{B}}^T$$

The reader is referred to Malvern (1969) for an in-depth review of tensor algebra and tensor calculus.

### Kinematics

A "body"  $\beta$  is a three-dimensional differential manifold, whose elements are called particles. The particles are denoted simply by  $X$ . If the particles at any instant occupy the region  $B$ , then  $B$  is called a "configuration."<sup>6</sup>

Normally, some configuration is chosen as one which is used to refer back to. This is called the "reference configuration,"  $B_r$ . The body may be referred to a system of coordinates in this configuration which establishes a relationship between particles and triples of real numbers.

$$\underline{X} = \kappa (X) \tag{1}$$

where  $\underline{X} = (X^1, X^2, X^3)$

The  $X^i$  are referred to as the material coordinates of the particle  $X$ . This relationship is one-to-one so there is an inverse relationship.

$$X = \kappa^{-1} (\underline{X}) . \tag{2}$$

As the body is deformed, an entire family of configurations  $B$  are formed. Therefore, the position of a particle  $X$  changes with time. This can be expressed by the following relationship.

$$\underline{x} = \underline{\phi} (X, t) \tag{3}$$

The point  $\underline{x} = \underline{\phi} (X, t)$  is the position occupied by the particle  $X$  at time  $t$ .

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<sup>6</sup> A body  $\beta$  is invariant. The region it occupies is not.

There are four approaches to describing the motion of a continuum in common use. These are the material, referential, spatial, and relative descriptions (Truesdell, 1965). Malvern (1969) provides a brief discussion of each of these approaches. The intent here is to provide a basis for the work which follows. Therefore, only the referential and spatial descriptions will be discussed as the others do not appear in any of the discussions.

The referential description is the preferred approach in elasticity where it is assumed the body will return to its natural state when unloaded. This will be the approach followed. However, under finite deformation it is often difficult to attach physical significance to some of the variables in the referential description. In contrast, the physical significance of variables in the deformed configuration is much clearer. Furthermore, the basic principles which govern the dynamic behavior of a continuum are usually derived in the deformed configuration and then transferred back to the reference configuration when necessary. Therefore, a discussion of the spatial description is also provided.

### Referential Description

The referential description refers the motion of the body to a reference configuration in which the particle  $X$  occupies the position  $\underline{X}$ . I will define the reference position to be the position occupied at  $t=0$ . This is often referred to as the Lagrangian description.

Recall the following relationships.

$$\underline{X} = \kappa(\underline{X}) ; \underline{X} = \kappa^{-1}(\underline{X})$$

$$\underline{x} = \underline{\phi}(\underline{X}, t)$$

where  $\underline{x}$  = the position of the particle  $X$  in the deformed configuration,

and  $\underline{X}$  = the position of the particle  $X$  in the reference configuration.

The spatial position can be put in terms of the reference configuration through the inverse mapping  $\kappa^{-1}$ .

$$\underline{x} = \underline{\phi}(\kappa^{-1}(\underline{X}), t) = \underline{\chi}(\underline{X}, t) \quad (4)$$

The velocity of the particle  $X$  which occupies the position  $\underline{X}$  in  $B_r$  is then

$$\underline{V} = \dot{\underline{x}} = \frac{\partial \underline{\chi}(\underline{X}, t)}{\partial t} = \frac{d\underline{\chi}(\underline{X}, t)}{dt} \quad (5)$$

since  $\underline{X}$  is not a function of time. Similarly, the acceleration of the particle  $X$  is given by

$$\underline{A} = \ddot{\underline{x}} = \frac{\partial^2 \underline{\chi}(\underline{X}, t)}{\partial t^2} = \frac{d^2 \underline{\chi}(\underline{X}, t)}{dt^2} \quad (6)$$

More generally, if  $\Psi(\underline{X}, t)$  is any scalar, vector, or tensor valued property of the material, the material time derivatives of  $\Psi$  are just

$$\dot{\Psi} = \frac{d\Psi(\underline{X}, t)}{dt} \quad \text{and} \quad \Psi^{(n)} = \frac{d^n \Psi(\underline{X}, t)}{dt^n} \quad (7)$$

### Spatial (Eulerian) Description

The spatial description focuses attention on a given region of space as opposed to a given body of matter. Hence, the kinematics are described in terms of the instantaneous position  $\underline{x}$ . This can be accomplished since the relation

$$\underline{x} = \underline{\chi}(\underline{X}, t)$$

is assumed to be invertible, i.e., one can write

$$\psi(\underline{x}, t) = \Psi(\underline{\chi}^{-1}(\underline{x}, t), t), \quad (8)$$

where  $\psi$  may be any scalar, vector, or tensor valued property of the material.

Derivatives of  $\psi(\underline{x}, t)$  must take into account the fact that  $\underline{x}$  is now a function of time. Therefore, the material derivative of  $\psi(\underline{x}, t)$  is

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{\partial \psi(\underline{x}, t)}{\partial t} + (\psi \overleftarrow{\nabla}_{\underline{x}}) \frac{d\underline{x}}{dt} \\ &= \frac{\partial \psi(\underline{x}, t)}{\partial t} + (\psi \overleftarrow{\nabla}_{\underline{x}}) \underline{v} \end{aligned} \quad (9)$$

where  $\frac{\partial \psi}{\partial t}$  = local rate of change,



and  $(\psi \overleftarrow{\nabla}_x) \underline{v}$  = convected rate of change.

In the above,  $(\psi \overleftarrow{\nabla}_x)$  is the gradient of  $\psi$  with respect to spatial coordinates. This can be distinguished from the gradient  $(\Psi \overleftarrow{\nabla})$  with respect to material coordinates by the subscript  $x$ . Hence, I adopt the following notation.

$$(\psi \overleftarrow{\nabla}_x) = \underline{g}^i \frac{\partial \psi}{\partial x^i}$$

$$(\Psi \overleftarrow{\nabla}) = \underline{G}^I \frac{\partial \Psi}{\partial X^I}$$

The distinction between the two sets of coordinates is further emphasized by the use of capital letters for coordinates and indices referred to the reference configuration.

As an example of the material derivative in spatial coordinates, the acceleration is just

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \overleftarrow{\nabla}_x) \underline{v}, \quad (10)$$

where  $(\underline{v} \overleftarrow{\nabla}_x) = v^i \underline{i}_j (\underline{g}_i \underline{g}^j)$ ,

and  $v^i \underline{i}_j$  are the covariant derivatives of the vector  $\underline{v}$ .

### Large Deformation Theory

Due to the large deformations encountered when snow is loaded, the distinction between the deformed and undeformed states becomes significant. Therefore, it is necessary to develop the theory in terms of a finite deformation measure. This is much more difficult to use than the infinitesimal theory commonly used in engineering since additive decomposition of the displacement gradient into the sum of a pure rotation and pure strain is no longer possible.

### Deformation Gradient

There are several different approaches to characterizing finite deformation using the Lagrangian description. The most fundamental approach is the deformation gradient. During a deformation, the following function is defined as a smooth mapping.

$$\underline{x} = \underline{\chi}(\underline{X}, t),$$

where  $\underline{X} = \kappa(\underline{X})$ .

Given the above deformation, the "deformation gradient" is a second order tensor field with the form

$$\underline{F} = (\underline{x} \overleftarrow{\nabla}) = (\underline{\chi}(\underline{X}, t) \overleftarrow{\nabla}). \quad (11)$$

Hence, it is the gradient of the current position with respect to a reference configuration.

As defined, the deformation gradient is a tensor with the following component form.

$$\underline{F} = x^i |_{|J} (\underline{g}_i \underline{G}^J), \quad (12)$$

where  $x^i |_{|J}$  are the covariant derivatives of the vector components of  $\underline{x}$ .

This is a mixed tensor in the sense that the basis for  $\underline{F}$  consists of  $\underline{g}_i$  in the deformed configuration and  $\underline{G}^J$  in the reference configuration.

For a physical interpretation, the deformation gradient maps neighborhoods of a particle  $X$  in  $B_r$  into neighborhoods of  $X$  in  $B$ , Figure 2. More precisely,  $\underline{F}$  is a linear mapping which operates on an arbitrary infinitesimal vector  $d\underline{X}$  at  $\underline{X}$  to associate with it a vector  $d\underline{x}$  at  $\underline{x}$ .

$$d\underline{x} = \underline{F} d\underline{X} \quad (13)$$

This result comes directly from the definition of the gradient of a vector.

Consider any vector field  $\underline{v}(\underline{X})$ .

$$\underline{v}(\underline{X} + \underline{u}) = \underline{v}(\underline{X}) + \underline{v}'(\underline{X}, \underline{u}) + \epsilon(\underline{X}, \underline{u}) \quad (14)$$

The Frechet derivative  $\underline{v}'(\underline{X}, \underline{u})$  is given by

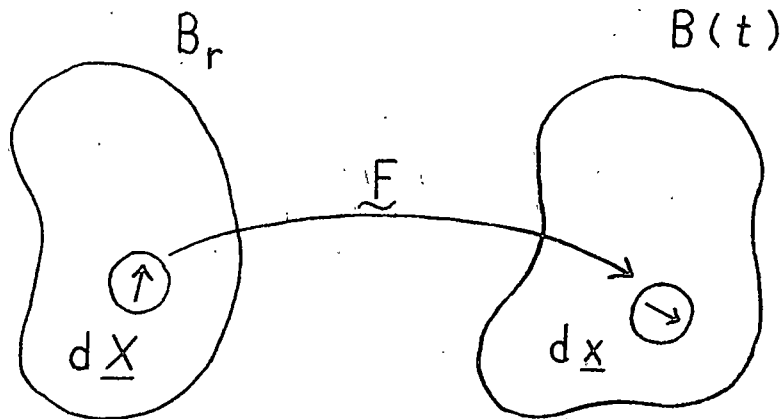


Figure 2. The mapping of the deformation gradient.

$$\underline{v}'(\underline{X}, \underline{u}) = (\underline{v} \overleftarrow{\nabla}) \underline{u}. \quad (15)$$

Therefore,

$$\underline{x}(\underline{X} + \underline{u}) = \underline{x}(\underline{X}) + (\underline{x} \overleftarrow{\nabla}) \underline{u} + \underline{\epsilon}(\underline{X}, \underline{u}). \quad (16)$$

Setting  $\underline{u} = \Delta \underline{X}$  and letting  $\Delta \underline{X} \rightarrow 0$  gives

$$d\underline{x} = (\underline{x} \overleftarrow{\nabla}) d\underline{X} = \tilde{F} d\underline{X}.$$

The deformation gradient is interesting in that it contains information about the rotation as well as the strain of the body. As a result, constitutive relations employing this tensor must be constructed so as not to predict a stress due to rigid-body rotation. Furthermore, the deformation gradient is not symmetric and can have nine independent components in general.

### Strain

A particular finite deformation measure which involves only the strain of the material vector  $d\underline{X}$  is given by the Lagrangian strain tensor  $\underline{E}$ . This tensor is defined so that it gives the change in the squared length of the material vector  $d\underline{X}$  as follows.

$$(ds)^2 - (dS)^2 = 2 d\underline{X} \cdot \underline{E} \cdot d\underline{X}, \quad (17)$$

where  $(ds)^2$  = the squared length of the material vector in the deformed configuration, and  $(dS)^2$  = the squared length of the material vector in the reference configuration. The Lagrangian strain tensor may be expressed in terms of the deformation gradient as follows.

$$(ds)^2 = d\underline{x} \cdot d\underline{x} = (d\underline{X} \underline{F}^T) \cdot (\underline{F} d\underline{X}) = d\underline{X} (\underline{F}^T \underline{F}) d\underline{X} \quad (18)$$

Therefore,

$$(ds)^2 - (dS)^2 = d\underline{X} (\underline{F}^T \underline{F}) d\underline{X} - d\underline{X} \underline{1} d\underline{X} = d\underline{X} (\underline{F}^T \underline{F} - \underline{1}) d\underline{X}. \quad (19)$$

Comparing the above results there follows,

$$\begin{aligned} \underline{E} &= \frac{1}{2} (\underline{F}^T \underline{F} - \underline{1}) \\ &= \frac{1}{2} (\underline{C} - \underline{1}), \end{aligned} \quad (20)$$

where  $\underline{C} = \underline{F}^T \underline{F}$  is defined as the Green deformation tensor.

For a comparison with the classical infinitesimal strain tensor one can write

$$x_i = X_i + u_i.$$

Substituting the above into Equation 20 and using rectangular Cartesian coordinates there follows,

$$E_{IJ} = \frac{1}{2} \left( \frac{\partial u_I}{\partial X^J} + \frac{\partial u_J}{\partial X^I} + \frac{\partial u_K}{\partial X^I} \frac{\partial u_K}{\partial X^J} \right). \quad (21)$$

This shows that if the partial derivatives of the displacement gradients  $U_I$  with respect to the material coordinates are small compared to unity, the second order terms may be neglected. This results in the infinitesimal strain tensor often used in engineering and classical elasticity.

It is interesting to note that the finite strain components of  $\underline{\underline{E}}$  involve only linear and quadratic terms in the components of the displacement gradients. However, this is a complete finite strain tensor and not merely a second order approximation to it.

Finally, there are numerous other finite strain measures which can be referred to the reference configuration. In particular, multiplicative decompositions of the deformation gradient into a stretch and rotation tensor are common. However, I intend to develop the theory in terms of the Lagrangian strain and alternate strain definitions will no longer be considered.

### General Mechanical Principles

In the following section, equations will be derived which give additional information on the way stress and deformation can vary in the neighborhood of a point with time. These equations express locally the conservation of mass and the balance of linear and angular momentum respectively. The equations are derived from integral forms of balance equations which express fundamental postulates of continuum mechanics and are therefore valid for any continuous medium.

The equations presented are usually expressed in terms of spatial coordinates and are well documented in this form. In contrast, the Lagrangian forms of these principles appear much less frequently. It is for this reason that the derivation of these equations is presented.

#### Conservation of Mass

Consider a body  $\beta$  and an arbitrary part  $\mathcal{P}$  of  $\beta$ , Figure 3. The body  $\beta$  occupies the regions  $B_r$  in the reference configuration and  $B(t)$  in the configuration  $\underline{x} = \underline{x}(\underline{X}, t)$ .  $\mathcal{P}$  occupies the region  $P_r$  in the reference configuration and  $P(t)$  in the spatial configuration. The boundaries of  $B$  and  $P$  are denoted respectively by  $\partial B(t)$ ,  $\partial P(t)$ .

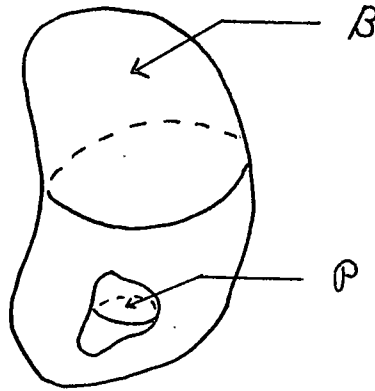


Figure 3. The body  $\beta$  and an arbitrary part  $\rho$  of  $\beta$ .

The principle of conservation of mass states that for any part  $\rho$  of  $\beta$ , there corresponds a positive number,  $m(\rho)$ , called the *mass of part  $\rho$* , which remains invariant from one configuration to another. The mass is assumed to be smooth enough to permit the existence of a *mass density function*,  $\rho_{\underline{x}}(\underline{x}, t)$ , such that

$$m(\rho) = \int_{P(t)} \rho_{\underline{x}}(\underline{x}, t) dV. \quad (22)$$

Since the mass is independent of the configuration; if  $\underline{x}(\underline{x}, t)$  and  $\underline{\psi}(\underline{x}, t)$  are two different possible configurations, then there follows,

$$m(\rho) = \int_{P_{\underline{x}}} \rho_{\underline{x}}(\underline{x}, t) dV = \int_{P_{\underline{\psi}}} \rho_{\underline{\psi}}(\underline{x}, t) dV. \quad (23)$$

$P_{\underline{x}}$  and  $P_{\underline{\psi}}$  are just the two regions occupied respectively by the configurations  $\underline{x}$  and  $\underline{\psi}$ . Unless there is a need, the subscripts for  $\rho$  will be dropped, i.e.,

$$\rho_{\underline{x}}(\underline{x}, t) = \rho(\underline{x}, t).$$

The one exception to this is the mass density in the reference configuration. In this case the notation  $\rho_0(\underline{X})$  will be used and one can write

$$m(\rho) = \int_{P_r} \rho_0(\underline{X}) dV_0 = \int_{P(t)} \rho(\underline{x}, t) dV. \quad (24)$$

The above is a global statement of the principle of conservation of mass. Now put this in local form in terms of the Lagrangian coordinates. Malvern (1969) provides the following kinematic relationship between the deformed volume and the reference volume.

$$dV = \det(\underline{\tilde{F}}) dV_0 \quad (25)$$

Using this expression, the spatial form of the mass measure may be referred back to the reference configuration.

$$\int_{P(t)} \rho(\underline{x}, t) dV = \int_{P_r} \rho(\underline{x}(\underline{X}, t), t) \det(\underline{\tilde{F}}) dV_0 \quad (26)$$

Combining the above result with Equation 24 and collecting terms gives

$$\int_{P_r} [\rho_0(\underline{X}) - \rho(\underline{x}(\underline{X}, t), t) \det(\underline{\tilde{F}})] dV_0 = 0. \quad (27)$$

This equation is valid for any part of the body  $\beta$ . This then requires

$$\rho_0(\underline{X}) - \rho(\underline{x}, t) \det(\underline{\tilde{F}}) = 0;$$

$$\text{or } \frac{\rho_0}{\rho} = \det(\underline{\tilde{F}}). \quad (28)$$

This expression is the Lagrangian form of the principle of conservation of mass.

### Balance of Momentum

For the momentum balance laws, two types of forces are considered.

- a. *Surface forces* are forces distributed over the boundary of  $\mathcal{P}$  and are caused by contact with other parts of  $\beta$  or other foreign bodies.
- b. *External body forces* are forces distributed over the interior and are caused by the effects of an external body.

In the case of surface forces, it is assumed there exists a vector valued function,  $\underline{t}_n(\underline{x}, t)$ , called the *stress traction* which gives a measure of the force intensity transmitted across the surface (force/unit area) of  $\partial P$ . The total resultant of this is

$$\underline{P} = \int_{\partial P(t)} \underline{t}_n dS. \quad (29)$$

The stress traction may be expressed in terms of the Cauchy stress tensor,  $\underline{\underline{t}}$ , by the following.

$$\underline{t}_n = \underline{n} \underline{\underline{t}} = \underline{\underline{t}}^T \underline{n}, \quad (30)$$

where  $\underline{\underline{t}} = t^{ij} (\underline{g}_i \underline{g}_j)$ ,

and the  $i$ th superscript represents the face acted on while the  $j$ th superscript represents the direction.

In a similar manner, a vector valued function  $\underline{b}(\underline{x}, t)$  called the *body force density* (force/unit mass) is assumed to exist such that the total body force is

$$\underline{P} = \int_P \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV. \quad (31)$$

The momentum principle states that the time rate of change of the total momentum of a collection of particles equals the vector sum of all external forces acting on the particle. This fundamental postulate takes the form of the following axioms in continuum mechanics.

(M<sub>1</sub>) Balance of Linear Momentum

$$\int_{\partial P(t)} \underline{t}_n dS + \int_{P(t)} \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV = \frac{d}{dt} \int_{P(t)} \underline{v}(\underline{x}, t) \rho(\underline{x}, t) dV \quad (32)$$

(M<sub>2</sub>) Balance of Angular Momentum

$$\begin{aligned} \int_{\partial P(t)} (\underline{x} - \underline{x}_0) \times \underline{t}_n(\underline{x}, t) dS + \int_{P(t)} (\underline{x} - \underline{x}_0) \times \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV \\ = \frac{d}{dt} \int_{P(t)} (\underline{x} - \underline{x}_0) \times \underline{v}(\underline{x}, t) \rho(\underline{x}, t) dV \end{aligned} \quad (33)$$

where  $\underline{x}_0$  is some fixed point of a part  $\mathcal{P}$  of  $\beta$ .

Consider the balance of linear momentum. Using the Reynold's Transport Theorem, axiom (M<sub>1</sub>) may be rewritten as



$$\begin{aligned} \int_{\partial P(t)} \underline{t}_n dS + \int_{P(t)} \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV \\ = \int_{P(t)} \underline{a}(\underline{x}, t) \rho(\underline{x}, t) dV. \end{aligned} \quad (34)$$

Now transform this expression to the reference configuration. Using conservation of mass there follows,

$$\begin{aligned} \int_{P(t)} \rho(\underline{x}, t) \underline{b}(\underline{x}, t) dV = \int_{P_r} \rho(\underline{x}(\underline{X}, t), t) \underline{b}(\underline{x}(\underline{X}, t)) dV_0 \\ = \int_{P_r} \underline{B}(\underline{X}, t) \rho_0 dV_0. \end{aligned} \quad (35)$$

Likewise, for the acceleration,

$$\int_{P(t)} \rho \underline{a}(\underline{x}, t) dV = \int_{P_r} \rho_0 \underline{A}(\underline{X}, t) dV_0. \quad (36)$$

For the surface integral, define the force across the surface  $dS$  by  $d\underline{P}$ .

$$d\underline{P} = \underline{t}_n dS = \underline{n} \underline{t} dS = (\underline{n} dS) \underline{t} \quad (37)$$

The areas  $dS_0$  in  $B_r$  and  $dS$  in  $B(t)$  are related by the following expression (Malvern, 1969).

$$\underline{n} dS = \underline{N} \underline{F}^{-1} \frac{\rho_0}{\rho} dS_0 \quad (38)$$

where  $\underline{n}$  is the unit normal in the deformed configuration,

and  $\underline{N}$  is the unit normal in the reference configuration.

Using the above result, the force  $d\underline{P}$  may be expressed as

$$d\underline{P} = (\underline{N} dS_0 \underline{F}^{-1} \frac{\rho_0}{\rho} \underline{t}) = \underline{N} \underline{T} dS_0, \quad (39)$$

where  $\underline{T} = \frac{\rho_0}{\rho} \underline{F}^{-1} \underline{t}$  is defined as the first Piola-Kirchhoff stress tensor.

Therefore, using Gauss's theorem,

$$\int_{\partial P(t)} \underline{n} \underline{t} dS = \int_{P_r} \underline{N} \underline{T} dS_0 = \int_{P_r} \underline{\nabla} \cdot \underline{T} dV_0. \quad (40)$$

Combining Equations 32, 33 and 36 the balance of linear momentum acquires the form

$$\int_{P_r} [\underline{\nabla} \cdot \underline{T} + \rho_0 \underline{B} - \rho_0 \underline{A}] dV_0 = 0. \quad (41)$$

Since P is arbitrary, this requires

$$\underline{\nabla} \cdot \underline{T} + \rho_0 \underline{B} = \rho_0 \underline{A}. \quad (42)$$

The above is the equation of motion in the Lagrangian description. This is contrasted to the equation of motion in the Eulerian description shown below.

$$\underline{\nabla}_x \cdot \underline{t} + \rho \underline{b} = \rho \underline{a} \quad (43)$$

For non-polar materials, the Cauchy stress tensor is symmetric, i.e.,

$$\underline{t}^T = \underline{t}. \quad (44)$$

This is actually the local form of the balance of angular momentum in spatial coordinates. In contrast, the first Piola-Kirchhoff stress tensor is non-symmetric in general. Therefore, often a second Piola-Kirchhoff stress tensor is introduced which is symmetric. This is given by

$$\underline{S} = \frac{\rho_0}{\rho} \underline{F}^{-1} \underline{t} \underline{F}^{-1 T}. \quad (45)$$

The equation of motion using this tensor is given by

$$\underline{\nabla} \cdot (\underline{S} \underline{F}^T) + \rho_0 \underline{B} = \rho_0 \underline{A}. \quad (46)$$

The physical relationships between the three stress tensors is shown in Figure 4. The Cauchy stress tensor gives the force transmitted across the area  $dS$  when operated on the unit normal,  $\underline{n}$ .

$$d\underline{P} = \underline{n} \underline{t} dS \quad (47)$$

The first Piola-Kirchhoff stress tensor also gives the force transmitted across the area  $dS$ , but it does it with respect to the undeformed area.

$$d\underline{P} = \underline{N} \underline{T} dS_0 = \underline{n} \underline{t} dS \quad (48)$$

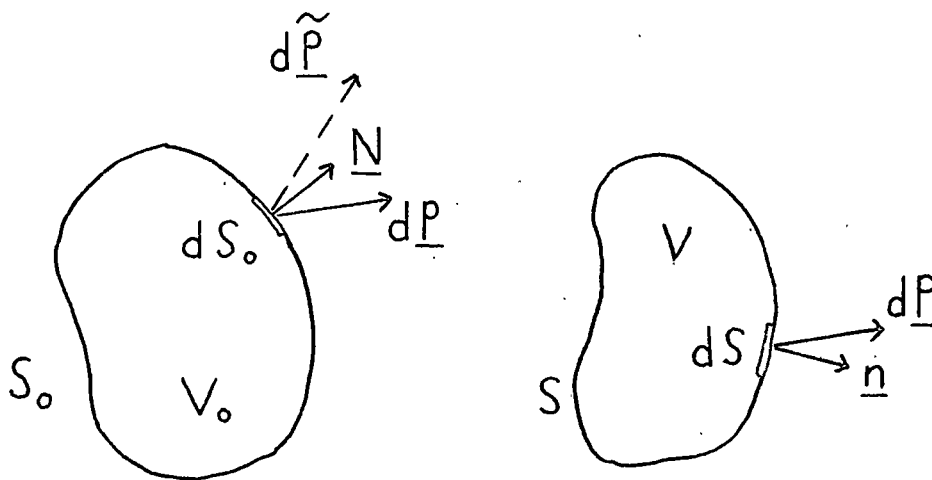


Figure 4. The relationship between the Cauchy stress tensor and the Piola-Kirchhoff stress tensors (Malvern, 1969).

In contrast, the second Piola-Kirchhoff stress tensor gives a force  $d\tilde{\underline{P}}$  which is related to the force  $d\underline{P}$  by the inverse of the deformation gradient.

$$d\tilde{\underline{P}} = \underline{N} \underline{S} dS_0 = \underline{F}^{-1} d\underline{P} \quad (49)$$

Hence, the physical significance of the second Piola-Kirchhoff stress is often difficult to interpret.

### Thermodynamics

Here the fundamental laws of thermodynamics are laid down as they apply to a continuum. The First Law provides a basis for determining the proper thermodynamic conjugate to be used with an associated choice of strain. The Second Law produces valuable mathematical restrictions on the constitutive assumptions.

#### First Law

The First Law requires that for any thermomechanical process, all energy exchanges within a body or a part of it must be accounted for. The global form of this law is given by

$$\begin{aligned} \frac{d}{dt} \int_{P(t)} \left( u + \frac{1}{2} \underline{v} \cdot \underline{v} \right) \rho \, dV &= \int_{\partial P(t)} [\underline{t}_n \cdot \underline{v} - \underline{q} \cdot \underline{v}] \, dS \\ &+ \int_{P(t)} [\rho r + \rho \underline{b} \cdot \underline{v}] \, dV \end{aligned} \quad (50)$$

In the above expression:

$u$  = the specific internal energy of the body. (This includes things such as strain energy, thermal energy, surface energy, etc.),

$\frac{1}{2} \underline{v} \cdot \underline{v}$  = the specific kinetic energy,

$\underline{q}$  = the heat flux vector,

and  $r$  = the heat supply.

The First Law states that the rate of change of energy (both internal and kinetic) is equated to the sum of:

1. The rate of work of boundary forces
2. The flux of heat across the boundary
3. Heat sources such as radiation
4. The rate of work of body forces

Using the Reynold's Transport Theorem and Gauss's Theorem, the global form of the First Law becomes,

$$\begin{aligned} \int_{P(t)} [\rho \dot{u} + \rho \underline{a} \cdot \underline{v}] \, dV \\ = \int_{P(t)} \left[ \vec{\nabla}_x \cdot (\underline{t} \underline{v}) - \vec{\nabla}_x \cdot \underline{q} + \rho r + \rho \underline{b} \cdot \underline{v} \right] \, dV. \end{aligned} \quad (51)$$

Now note the following identity.

$$\vec{\nabla}_x \cdot (\underline{t} \underline{v}) = (\vec{\nabla}_x \cdot \underline{t}) \underline{v} + \text{tr}(\underline{t} \underline{D}), \quad (52)$$

where  $\underline{D}$  is the rate of deformation tensor.

$$\underline{D} = \frac{1}{2} (\underline{v} \vec{\nabla}_x + \vec{\nabla}_x \underline{v}) \quad (53)$$

Substituting Equation 52 into the First Law and collecting terms gives

$$\int_{P(t)} [\rho \underline{a} - \rho \underline{b} - \underline{\nabla}_x \cdot \underline{t}] \cdot \underline{v} + \rho \dot{u} - \text{tr}(\underline{t} \underline{D}) - \rho r + \underline{\nabla}_x \cdot \underline{q}] dV = 0. \quad (54)$$

The first term in the above expression is simply the equation of motion in spatial coordinates and is equal to zero. Therefore, one is left with

$$\int_{P(t)} [\rho \dot{u} - \text{tr}(\underline{t} \underline{D}) - \rho r + \underline{\nabla}_x \cdot \underline{q}] dV = 0. \quad (55)$$

Now express the above result in the reference configuration. Consider the stress power defined by  $\text{tr}(\underline{t} \underline{D})$ . The rate of deformation tensor is related to the time rate of change of the Lagrangian strain tensor by the following.

$$\underline{D} = \underline{F}^{-1 T} \underline{\dot{E}} \underline{F}^{-1} \quad (56)$$

Using the above the stress power can be written as

$$\begin{aligned} \text{tr}(\underline{t} \underline{D}) &= \text{tr}(\underline{t} \underline{F}^{-1 T} \underline{\dot{E}} \underline{F}^{-1}) \\ &= \text{tr}((\underline{F}^{-1} \underline{t} \underline{F}^{-1 T}) \underline{\dot{E}}) = \frac{\rho}{\rho_0} \text{tr}(\underline{S} \underline{\dot{E}}). \end{aligned} \quad (57)$$

The divergence of the heat flux may be expressed in the reference configuration as follows.

$$\begin{aligned} \int_{P(t)} \underline{\nabla}_x \cdot \underline{q} dV &= \int_{\partial P(t)} \underline{n} \cdot \underline{q} dS \\ &= \int_{\partial P_r} \frac{\rho_0}{\rho} \underline{N} \underline{F}^{-1} \underline{q} dS = \int_{P_r} \frac{\rho_0}{\rho} \underline{\nabla} \cdot (\underline{F}^{-1} \underline{q}) dV_0 \end{aligned} \quad (58)$$

Using Equations 57 and 58 the global form of the First Law in the reference configuration becomes

$$\int_{P_r} [\rho_0 \dot{U} - \text{tr}(\underline{S} \underline{\dot{E}}) + \rho_0 R + \frac{\rho_0}{\rho} \underline{\nabla} \cdot (\underline{F}^{-1} \underline{q})] dV_0 = 0. \quad (59)$$

Since the above integral is equal to zero for all parts  $P_r$  there follows,

$$\rho_0 \dot{U} = \text{tr}(\underline{S} \underline{\dot{E}}) + \rho_0 R + \frac{\rho_0}{\rho} \underline{\nabla} \cdot (\underline{F}^{-1} \underline{q}). \quad (60)$$

Equation 60 is the local form of the First Law in the reference configuration. This shows that the second Piola-Kirchhoff stress tensor is the thermodynamic conjugate of the rate of change of the Lagrangian strain. In a similar manner, it can be shown that the first Piola-Kirchhoff stress tensor is conjugate to the rate of change of the deformation gradient.

### Second Law

The Second Law is based on the observation that energy is dissipated in an irreversible process. The law postulates the existence of entropy as a state function and that internal entropy production is the result of dissipative irreversible processes. Further, the internal entropy production is zero for reversible processes. Hence, the Second Law requires

$$\text{Rate of Entropy Increase} \geq \text{Entropy Input Rate} .$$

This may be expressed in a continuum form as follows.

$$\frac{d}{dt} \int_{P(t)} \rho \eta \, dV \geq \int_{P(t)} \frac{\rho r}{\theta} \, dV + \int_{\partial P(t)} \frac{-\mathbf{q} \cdot \mathbf{n}}{\theta} \, dS \quad (61)$$

where  $\eta$  is the specific entropy,

and  $\theta$  is the temperature.

Again, using the transport theorem and Gauss's Theorem,

$$\int_{P(t)} \left[ \rho \dot{\eta} - \frac{\rho r}{\theta} + \vec{\nabla}_x \cdot \left( \frac{\mathbf{q}}{\theta} \right) \right] \, dV \geq 0 \quad (62)$$

In terms of the reference configuration, Equation 62 becomes

$$\int_{P_r} \left[ \rho_0 \dot{H} - \frac{\rho_0 R}{\Theta} + \frac{\rho_0}{\rho} \vec{\nabla} \cdot \left( \frac{\mathbf{F}^{-1} \mathbf{q}}{\Theta} \right) \right] \, dV_0 \geq 0 \quad (63)$$

This requires

$$\rho_0 \dot{H} - \frac{\rho_0 R}{\Theta} + \frac{\rho_0}{\rho} \vec{\nabla} \cdot \left( \frac{\mathbf{F}^{-1} \mathbf{q}}{\Theta} \right) \geq 0 \quad (64)$$

The above is known as the Clausius-Duhem inequality and is the local form of the Second Law. The divergence term can be expanded to give

$$\rho_0 \dot{H} - \frac{\rho_0 R}{\Theta} + \frac{\rho_0}{\rho} \frac{1}{\Theta} \vec{\nabla} \cdot (\mathbf{F}^{-1} \underline{q}) - \frac{1}{\Theta^2} \mathbf{F}^{-1} \underline{q} \cdot \vec{\nabla} \Theta \geq 0. \quad (65)$$

Now substitute the energy equation into the above to eliminate R and  $\vec{\nabla} \cdot \mathbf{F}^{-1} \underline{q}$ .

$$\rho_0 \Theta \dot{H} - \rho_0 \dot{U} + \text{tr}(\underline{\underline{S}} \underline{\underline{E}}) - \frac{1}{\Theta} \mathbf{F}^{-1} \underline{q} \cdot \vec{\nabla} \Theta \geq 0 \quad (66)$$

This expression is an alternate form of the Second Law. Finally, introduce the Helmholtz free energy,  $\Phi$ , defined by

$$\Phi = U - \Theta H \quad (67)$$

This quantity represents the recoverable strain energy in the body. The material time derivative of the free energy is given by

$$\dot{\Phi} = \dot{U} - \dot{\Theta} H - \Theta \dot{H} \quad (68)$$

Substituting the above expression into Equation 66 provides still another form of the Second Law.

$$-\rho_0 \dot{\Phi} - \rho_0 H \dot{\Theta} + \text{tr}(\underline{\underline{S}} \underline{\underline{E}}) - \frac{1}{\Theta} \mathbf{F}^{-1} \underline{q} \cdot \vec{\nabla} \Theta \geq 0 \quad (69)$$

## CHAPTER 4

## GENERAL CONSTITUTIVE THEORY

A general constitutive theory for a class of inelastic materials at finite strain is presented within the framework of non-equilibrium thermodynamics. Specific constitutive relations are postulated and restrictions on their form are developed from the Second Law. General restrictions on the form of the evolution equations for the internal state variables are also explored. Finally, a brief description of the inverted form of the constitutive law is presented.

Constitutive Assumptions

I begin by assuming the existence of a caloric equation of state such that,

$$U = U(\underline{E}, H, \underline{\xi}), \quad (70)$$

where  $U$  = the internal energy per unit *reference mass*,

$H$  = the entropy per unit reference mass,

$\underline{E}$  = the Lagrangian strain tensor,

and  $\underline{\xi}$  = a finite collection of internal state variables characterizing the current pattern of structural arrangement on the granular level.

Hence, the internal state vector may be thought of as characterizing the plastic state of the material.

Restrictions on the form of  $\underline{\xi}$  have been found by Coleman and Gurtin (1967) based on the principle of objectivity. They show that  $\underline{\xi}$  cannot represent a vector of dimension three transforming as a spatial position vector under a change of frame, i.e.,



$$\underline{\xi}^* = \underline{Q}(t) \underline{\xi}, \quad (71)$$

where  $\underline{\xi}^*$  = the internal state vector in the starred reference frame,

$\underline{\xi}$  = the internal state vector in the unstarred reference frame,

and  $\underline{Q}(t)$  = any time dependent orthogonal tensor relating the motion of the two frames.

$\underline{\xi}$  can however represent a set of N-tuple scalars, each of which remains invariant under a change of frame. The state vector then transforms similar to an objective scalar, i.e.,

$$\underline{\xi}^* = \underline{\xi}. \quad (72)$$

Thus the only restriction on the internal state variables is that they be objective scalars.

It is often convenient to transform Equation 70 into a form where the independent variable of temperature ( $\Theta$ ) replaces the entropy. This is accomplished by introducing the Helmholtz free energy  $\Phi(\underline{E}, \Theta, \underline{\xi})$  defined earlier as

$$\Phi = U - H \Theta. \quad (67)$$

Assuming equipresence of the constitutive variables, there follows,

$$\Phi = \Phi(\underline{E}, \Theta, \underline{\xi}),$$

$$H = H(\underline{E}, \Theta, \underline{\xi}),$$

$$S = S(\underline{E}, \Theta, \underline{\xi}),$$

$$\underline{F}^{-1} \underline{q} = \underline{q}_r = \underline{q}_r(\underline{E}, \Theta, \underline{\xi}),$$

$$\text{and } \dot{\underline{\xi}} = \dot{\underline{\xi}}(\underline{E}, \Theta, \underline{\xi}) \quad (73)$$

#### Restrictions Based on the Second Law

The second law of thermodynamics was given a precise mathematical meaning by the postulate of internal entropy production. This law can now be used to place valuable mathematical restrictions on the constitutive assumptions. For convenience, the law is shown below in terms of the free energy.

$$-\rho_0 \dot{\Phi} - \rho_0 H \dot{\Theta} + \text{tr}(\underline{\underline{S}} \dot{\underline{\underline{E}}}) - \frac{1}{\Theta} \underline{\underline{F}}^{-1} \underline{\underline{q}} \cdot \underline{\underline{\nabla}} \Theta \geq 0 \quad (69)$$

From Equation 73, the time derivative of the free energy is given by

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \underline{\underline{E}}} : \dot{\underline{\underline{E}}} + \frac{\partial \Phi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Phi}{\partial \underline{\underline{\xi}}} \cdot \dot{\underline{\underline{\xi}}} \quad (74)$$

Substituting the above equation into Equation 69 and collecting terms gives

$$\begin{aligned} \text{tr}((\rho_0 \frac{\partial \Phi}{\partial \underline{\underline{E}}} - \underline{\underline{S}}) \dot{\underline{\underline{E}}}) - \rho_0 (H + \frac{\partial \Phi}{\partial \Theta}) \dot{\Theta} \\ - \rho_0 \frac{\partial \Phi}{\partial \underline{\underline{\xi}}} \cdot \dot{\underline{\underline{\xi}}} - \frac{1}{\Theta} \underline{\underline{F}}^{-1} \underline{\underline{q}} \cdot \underline{\underline{\nabla}} \Theta \geq 0. \end{aligned} \quad (75)$$

By standard thermodynamic arguments, the entropy inequality above could be violated for a particular jump in strain or temperature unless,

$$\frac{\partial \Phi}{\partial \underline{\underline{E}}} = \frac{1}{\rho_0} \underline{\underline{S}} \quad \text{and} \quad H = - \frac{\partial \Phi}{\partial \Theta} \quad (76)$$

The Second Law then reduces to

$$-\rho_0 \frac{\partial \Phi}{\partial \underline{\underline{\xi}}} \cdot \dot{\underline{\underline{\xi}}} - \frac{1}{\Theta} \underline{\underline{F}}^{-1} \underline{\underline{q}} \cdot \underline{\underline{\nabla}} \Theta \geq 0. \quad (77)$$

Under isothermal conditions, the entropy inequality may be reduced even further to

$$\frac{\partial \Phi}{\partial \underline{\underline{\xi}}} \cdot \dot{\underline{\underline{\xi}}} \leq 0. \quad (78)$$

The above result provides a restriction on the relation between the time rate of change of the internal state vector and its representative thermodynamic conjugate.

### General Stress-Strain Law

In the previous discussion, the constitutive theory has been formulated with strain as the independent variable. However, a significant advantage can be obtained by formulating

the strain as a function of stress. This is due to the fact that the strain can be separated into an elastic part and a plastic part respectively.

The change in independent variables can be accomplished by introducing the complementary energy,  $\Psi$ , defined by the following Legendre transformation.

$$\Psi(\underline{S}, \Theta, \underline{\xi}) = \frac{1}{\rho_0} \underline{S} : \underline{E} - \Phi. \quad (79)$$

Differentiating the above expression there follows,

$$d\Psi = \frac{1}{\rho_0} d\underline{S} : \underline{E} + \frac{1}{\rho_0} \underline{S} : d\underline{E} - \frac{\partial \Phi}{\partial \underline{E}} : d\underline{E} - \frac{\partial \Phi}{\partial \Theta} d\Theta - \frac{\partial \Phi}{\partial \underline{\xi}} \cdot d\underline{\xi}. \quad (80)$$

Recall from the Second Law,

$$\frac{\partial \Phi}{\partial \underline{E}} = \frac{1}{\rho_0} \underline{S}.$$

Therefore,

$$d\Psi = \frac{1}{\rho_0} \underline{E} : d\underline{S} - \frac{\partial \Phi}{\partial \Theta} d\Theta - \frac{\partial \Phi}{\partial \underline{\xi}} \cdot d\underline{\xi}. \quad (81)$$

Clearly, if  $\Psi$  is to be a function of  $\underline{S}$ ,  $\Theta$ , and  $\underline{\xi}$ , then

$$\frac{1}{\rho_0} \underline{E} = \frac{\partial \Psi}{\partial \underline{S}}, \quad \frac{\partial \Psi}{\partial \Theta} = - \frac{\partial \Phi}{\partial \Theta}, \quad \frac{\partial \Psi}{\partial \underline{\xi}} = - \frac{\partial \Phi}{\partial \underline{\xi}}. \quad (82)$$

From Equation 82 it follows that the strain has a constitutive relation of the form why?

$$\underline{E} = \underline{E}(\underline{S}, \Theta, \underline{\xi}). \quad (83)$$

A slight perturbation in the strain from the current thermodynamic state leads to the differential expression

$$d\underline{E} = \frac{\partial \underline{E}}{\partial \underline{S}} : d\underline{S} + \frac{\partial \underline{E}}{\partial \Theta} d\Theta + \frac{\partial \underline{E}}{\partial \underline{\xi}} \cdot d\underline{\xi}. \quad (84)$$

I require that changes in  $\underline{E}$  at fixed  $\underline{\xi}$  induce a purely elastic response (Rice, 1971).

This is physically reasonable since by fixing  $\underline{\xi}$ , the plastic state of the material remains

unchanged. Similarly, changes in  $\underline{\underline{E}}$  holding  $\underline{\underline{S}}$  and  $\Theta$  fixed results in inelastic deformation.

Therefore, the total strain increment may be written as

$$d\underline{\underline{E}} = d^e\underline{\underline{E}} + d^p\underline{\underline{E}}, \quad (85)$$

where  $d^e\underline{\underline{E}}$  = the elastic strain increment,

and  $d^p\underline{\underline{E}}$  = the inelastic strain increment.

The inelastic portion of the strain increment is given by

$$d^p\underline{\underline{E}} = \frac{\partial \underline{\underline{E}}}{\partial \underline{\underline{\xi}}} \cdot d\underline{\underline{\xi}}. \quad (86)$$

Now consider the complementary energy,  $\Psi$ . Define the thermodynamic conjugate to the internal state vector as

$$\underline{\underline{f}} = \rho_0 \frac{\partial \Psi}{\partial \underline{\underline{\xi}}}. \quad (87)$$

Taking cross derivatives of  $\Psi$  with respect to  $\underline{\underline{\xi}}$  and  $\underline{\underline{S}}$  there follows,

$$\frac{\partial^2 \Psi}{\partial \underline{\underline{\xi}} \partial \underline{\underline{S}}} = \frac{\partial}{\partial \underline{\underline{\xi}}} \left( \frac{\partial \Psi}{\partial \underline{\underline{S}}} \right) = \frac{1}{\rho_0} \frac{\partial \underline{\underline{E}}}{\partial \underline{\underline{\xi}}},$$

$$\text{and } \frac{\partial^2 \Psi}{\partial \underline{\underline{S}} \partial \underline{\underline{\xi}}} = \frac{\partial}{\partial \underline{\underline{S}}} \left( \frac{\partial \Psi}{\partial \underline{\underline{\xi}}} \right) = \frac{1}{\rho_0} \frac{\partial \underline{\underline{f}}}{\partial \underline{\underline{S}}}. \quad (88)$$

Hence, the following Maxwell relation holds.

$$\frac{\partial \underline{\underline{f}}}{\partial \underline{\underline{S}}} = \frac{\partial \underline{\underline{E}}}{\partial \underline{\underline{\xi}}} \quad (89)$$

Noting the above and expressing  $\underline{\underline{E}}$  in terms of the potential  $\Psi$  gives

$$d\underline{\underline{E}} = \underline{\underline{M}} : d\underline{\underline{S}} + \underline{\underline{A}} d\Theta + \frac{\partial \underline{\underline{f}}}{\partial \underline{\underline{S}}} \cdot d\underline{\underline{\xi}}, \quad (90)$$

$$\text{where } \underline{\underline{M}} = \frac{1}{\rho_0} \frac{\partial^2 \Psi}{\partial \underline{\underline{S}} \partial \underline{\underline{S}}},$$

and 
$$\underline{\underline{A}} = \frac{1}{\rho_0} \frac{\partial^2 \Psi}{\partial \underline{\underline{S}} \partial \Theta}$$

The rate of change of the internal state vector must now be formulated as a function of stress rather than strain. This may be accomplished in principle since,

$$\dot{\underline{\underline{\xi}}} = \dot{\underline{\underline{\xi}}}(\underline{\underline{E}}(\underline{\underline{S}}, \Theta, \underline{\underline{\xi}}), \Theta, \underline{\underline{\xi}}) = \dot{\underline{\underline{\xi}}}(\underline{\underline{S}}, \Theta, \underline{\underline{\xi}}). \quad (91)$$

Equation 90 has been shown to satisfy the principle of material frame indifference.<sup>7</sup>

This is of particular significance since I am dealing with a large deformation theory.

In dealing with the constitutive theory shown in Equation 90, one must determine how  $\underline{\underline{M}}$ ,  $\underline{\underline{A}}$ ,  $\underline{\underline{f}}$ , and  $\dot{\underline{\underline{\xi}}}$  depend on  $\underline{\underline{S}}$ ,  $\Theta$ , and  $\underline{\underline{\xi}}$ . This equation may be simplified by considering isothermal processes only, i.e.,  $\dot{\Theta} = 0$ ,  $\underline{\underline{\nabla}} \Theta = \underline{\underline{0}}$ . Hence, I am dealing with a purely mechanical theory of the form

$$\dot{\underline{\underline{E}}} = \underline{\underline{M}} : \dot{\underline{\underline{S}}} + \frac{\partial \underline{\underline{f}}}{\partial \underline{\underline{S}}} \cdot \dot{\underline{\underline{\xi}}}. \quad (92)$$

It should be noted that isothermal processes do not preclude accounting for temperature effects in a mechanical theory. For instance, the temperature may appear as a parameter in the evolution equations of the internal state vector. This can have significant effect on the resulting constitutive law.

#### Complementary Energy Expansion

In order to determine the compliance tensor,  $\underline{\underline{M}}$ , it is necessary to generate an expansion for the complementary energy. This can be used in conjunction with the Second Law to determine the functional form of  $\underline{\underline{M}}$ .

Recall  $\Psi$  is a state function of the material. Hence, under isothermal conditions one can write

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<sup>7</sup> Appendix A.

$$\Psi = \Psi(\underline{S}, \underline{\xi}). \quad (93)$$

Furthermore, the constitutive theory is restricted by requiring the material response to be elastic if no microstructural rearrangements take place while changes in  $\underline{E}$  holding  $\underline{S}$  fixed results in inelastic deformation.

Under finite strain, the elastic contribution to the strain is small compared to the plastic deformation and therefore may be assumed to be linear in stress. This implies  $\underline{M} = \underline{M}(\underline{\xi})$  only. Hence,  $\underline{M}$  is determined by the current microstructural makeup of the material.

Assuming the compliance tensor is a function of the internal state vector only, Equation 90 allows one to write

$$\underline{E} = \underline{M} : \underline{S} + \underline{E}^P(\underline{S}, \underline{\xi}). \quad (94)$$

Therefore, in terms of the potential  $\Psi$  there follows,

$$\rho_0 \frac{\partial \Psi}{\partial \underline{S}} = \underline{M} : \underline{S} + \underline{E}^P(\underline{S}, \underline{\xi}). \quad (95)$$

Integrating the above expression gives

$$\rho_0 \Psi = -\Phi^0(\underline{\xi}) + \int \underline{E}^P(\underline{S}, \underline{\xi}) : d\underline{S} + \frac{1}{2} \underline{S} : (\underline{M}(\underline{\xi}) : \underline{S}). \quad (96)$$

Rice (1975) refers to  $-\Phi^0(\underline{\xi})$  as the locked in free energy. Physically this term is related to the excess surface area in a granular material.

Equation 96 can be used in conjunction with the Second Law to determine the functional form of the compliance tensor. The details of this are left to Chapter 6.

### Evolution Equations

The development of the evolution equations is of primary importance in the formulation of this constitutive theory. Brown (1981a) has shown that kinetic relations can be developed for internal state variables at the granular level. These equations were developed



















































































































































































































