Tchebycheff Approximations by General Spline Functions
by LEROY AMUNRUD

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

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This thesis presents a development of the theory of Tchebycheff approximations by polynomials with imposed boundary conditions and the theory of Tchebycheff approximations by general spline functions. Existence and characterization theorems are given along with computational procedures and examples.

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Abstract

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1. Introduction

A problem encountered repeatedly in scientific research is the following:

> Given a function $f(x)$, a norm $||\cdot||$, and a class S of admissible approximating functions, find a function $P(x) \in S$ such that $||f(x) - P(x)||$ is a minimum.

The function which is to be approximated may be given in many different ways. For example, the function may be a continuous function defined by a slowly converging power series or it may be a discrete function defined as the numerical solution of a differential equation. The characteristics of the function $f(x)$ and the intended use of the approximation influence the choice of the norm and the class of admissible approximating functions.

A desirable norm in many applications is the Tchebycheff norm (called the ℓ_∞ norm in the discrete case, the uniform norm in the continuous case) described by

$$||X^*|| = \sup_{x \in I} \{|X^*(x)|\}$$

where I is some given set of points. Tchebycheff [12] and de la Vallée Poussin [8] developed much of the early theory associated with this norm. In the past few years E. K. Blum [3], P. C. Curtis, Jr. [7], E. W. Cheney [5], A. A. Goldstein [5], C. W. Clenshaw [6], J. C. C. Nitsche [10], and many others have made contributions to the advancement of this theory. Also in a recent publication [9] C. L. Lawson presents characterization theorems and solution procedures associated with the problem of partitioning an interval such that the largest error incurred in approximating a continuous function by separate polynomials or rational forms on each subinterval is minimized. This type of approximation, i.e. one in which different approximating functions are used on different subintervals of the argument domain, is called a segmented approximation. If the end points of the subintervals are not specified, the problem of finding the "best" segmented approximating function is more complicated than the standard fixed interval approximation problem due to the added difficulty of finding the optimum set of end points for the subintervals. However, the increased precision of a segmented approximation often justifies the additional work.

A logical extension of Lawson's type of approximation is to require that the approximating function be continuous or have derivatives up through some order at the end points of the subintervals. One class of functions of this type is the class of general spline functions.

Definition 1.

Let $\nu_1, \nu_2, \ldots, \nu_k$ be non-negative integers and $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_k$ be a set of points in $[\alpha, \beta]$.

A function $s_k(x)$ which satisfies the conditions:

i) $s_k(x)$ is a polynomial in x on each subinterval $[\alpha,\delta_1]$, $[\delta_r,\delta_{r+1}]$, $r = 1,2,\ldots,k-1$, and $[\delta_k,\beta]$;

ii) $s_k(x)$ is continuous on $[\alpha,\beta]$,

iii) $s_k(x)$ has derivatives up through order ν_r at δ_r, $r = 1,2,\ldots,k$,

is called a <u>general spline function</u>. The points $\delta_1,\delta_2,\ldots,\delta_k$ are called the <u>join points</u> of $s_k(x)$.

For a definition of a spline function and other associated definitions see [1], [2] and [11].

The purpose of this thesis is to present characterization theorems and solution procedures associated with Tchebycheff approximations where the class of approximating functions is the class of general spline functions. Basic to the theory of Tchebycheff approximations by general spline functions is the theory of Tchebycheff approximations by polynomials with imposed boundary conditions. Consequently this theory is developed first. It should be noted that all polynomials used in the following discussions are real polynomials.

2. Tchebycheff Theory With Imposed Boundary Conditions

In order to simplify the notation, the zero derivative of a function shall be used to designate the function itself.

Lemma 1. (Existence Lemma)

Let f be a continuous real valued function defined on the non-degenerate interval $[\alpha,\beta]$, and let f have derivatives up through some order $\nu \geqq 0$ at some fixed point δ in $[\alpha,\beta]$. Let n be a non-negative integer such that $n \geqq \nu$. Then there exists a polynomial $P_n(x)$ which has the same values for its derivatives up through order ν as f has at δ, is of degree less than or equal to n, and is a best approximation in the sense of the Tchebycheff norm relative to the conditions imposed on the derivatives at δ.

Proof:

Let $P_n(x)$ be given by

$$P_n(x) = \sum_{j=0}^{n} \lambda_j x^j$$

and let

$$\varphi(\lambda_1,\lambda_2,\ldots,\lambda_n) = ||f - P_n|| = \max_{\alpha \leqq x \leqq \beta} \left| f(x) - P_n(x) \right|$$

The polynomial $P_n(x)$ must satisfy the conditions

$$P_n^{(m)}(\delta) = f^{(m)}(\delta), \qquad m = 0,1,2,\ldots,\nu$$

where in general $f^{(m)}(x)$ denotes the m^{th} derivative of f(x). If one sets $\delta^0 = 1$ even when δ is zero, and $f^{(m)}(\delta) = C_m$, then this system of equations can be written in the form

$$\sum_{j=\alpha}^{n} \frac{j!}{(j-\alpha)!} \lambda_j \delta^{j-\alpha} = C_\alpha, \qquad \alpha = 0,1,\ldots,\nu$$

In this system of equations, the coefficient matrix A of $\lambda_0,\lambda_1,\ldots,\lambda_\nu$ is upper triangular and has non-zero elements on its diagonal. Such a matrix is non-singular. Consequently this system of $(\nu + 1)$ equations can be used to express $\lambda_0,\lambda_1,\ldots,\lambda_\nu$ as continuous functions of the remaining λ_j's. That is

$$P_n(x) = a_0 + a_1 x + \ldots + a_\nu x^\nu + \lambda_{\nu+1}x^{\nu+1}+\ldots+\lambda_n x^n$$

where from the form of A it follows that each a_j can be written in the form

$$a_j = \sum_{\ell=0}^{\nu} k_{j\ell}c_\ell + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell, \quad j = 0,1,\ldots,\nu$$

Here each $k_{j\ell}$ is a constant. Thus each a_j is also of the form

$$a_j = K_j + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell, \quad j = 0,1,\ldots,\nu$$

where each K_j is a constant.

Consider the function ψ defined by

$$\psi(\lambda_{\nu+1},\lambda_{\nu+2},\ldots,\lambda_n)=||f(x)-\sum_{j=0}^{\nu} a_j x^j-\sum_{j=\nu+1}^{n}\lambda_j x^j|| \quad (1)$$

First it will be shown that ψ is a continuous function of the vector argument $\lambda = (\lambda_{\nu+1},\lambda_{\nu+2},\ldots,\lambda_n)$.

$$|\psi(\lambda')-\psi(\lambda)| = \Big|\, ||f(x)-\sum_{j=0}^{\nu} a_j' x^j-\sum_{j=\nu+1}^{n}\lambda_j' x^j|| - ||f(x)-\sum_{j=0}^{\nu} a_j x^j-\sum_{j=\nu+1}^{n}\lambda_j x^j|| \,\Big|$$

Hence

$$|\psi(\lambda')-\lambda(\lambda)| \leq \left|\left| \sum_{j=0}^{\nu} (a_j'-a_j)x^j + \sum_{j=\nu+1}^{n} (\lambda_j'-\lambda_j)x^j \right|\right|$$

$$\leq \max_{\substack{0 \leq j \leq \nu \\ \nu+1\leq i\leq n}} \left\{|a_j'-a_j|, |\lambda_i'-\lambda_i|\right\} \sum_{j=0}^{n} ||x^j||$$

Each a_j' is a continuous function of $\lambda_{\nu+1}', \lambda_{\nu+2}', \ldots, \lambda_n'$. Thus $a_j' \to a_j$, $j = 0,1,\ldots,\nu$, as $\lambda' \to \lambda$. Furthermore each $||x^j||$, $j = 0,1,\ldots,n$ is bounded. This completes the proof that $\psi(\lambda)$ is a continuous function of λ. Compare this proof with that on page 130 of [13].

Now either ν is equal to n or ν is less than n. Consider first the case where ν is equal to n. Then

$$P_n(x) = \sum_{j=0}^{\nu} a_j x^j$$

where each a_j is uniquely determined. Consequently there is one and only one polynomial which satisfies the given conditions, and for this case the lemma is seen to be true.

Next let ν be less than n. It follows from the continuity proof given above that for the continuous function

$$\sum_{j=0}^{\nu} K_j x^j$$

$$\varphi = \left|\left| \sum_{j=0}^{\nu} K_j x^j - \sum_{j=0}^{\nu} a_j x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j \right|\right|$$

$$= \left|\left| - \sum_{j=0}^{\nu} \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j \right|\right|$$

is a continuous function of λ. The shell

$$\lambda_{\nu+1}^2 + \ldots + \lambda_n^2 = 1$$

is a bounded, closed (compact) set in ordinary $(n-\nu)$ dimensional space, and on it the continuous function φ must assume a minimum σ. Since a norm by definition is always greater than or equal to zero, $\sigma \geqq 0$. The functions $1, x, x^2, \ldots$ are linearly independent. Thus, if at least one of the λ_j's, $j = \nu+1, \ldots, n$ is not zero, then $\sigma \neq 0$. It follows, by the homogeneity of φ, that for any $(\lambda_{\nu+1}, \lambda_{\nu+2}, \ldots, \lambda_n)$ with at least one non-zero component,

$$\varphi \geqq \sigma \sqrt{\lambda_{\nu+1}^2 + \ldots + \lambda_n^2} > 0$$

Let ρ be the lower bound of $\psi(\lambda)$. Then it is true that $\rho \geqq 0$. Now one only needs to show that this bound is attained. That is, one only needs to show that there exists a λ^* such that

$$\psi(\lambda^*) = \rho \tag{2}$$

Assume that

$$\sqrt{\sum_{j=\nu+1}^{n} \lambda_j^2} > R = (\rho + 1 + ||f(x) - \sum_{j=0}^{\nu} K_j x^j||)/\sigma$$

Then

$$\psi(\lambda) = ||f(x) - \sum_{j=0}^{\nu} a_j x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j||$$

$$\psi(\lambda) = ||f(x) - \sum_{j=0}^{\nu} (K_j + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell)x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j||$$

$$\geqq ||-\sum_{j=0}^{\nu} \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j|| - ||f(x) - \sum_{j=0}^{\nu} K_j x^j||$$

$$\geqq \sigma(\rho + 1 + ||f(x) - \sum_{j=0}^{\nu} K_j x^j||)/\sigma - ||f(x) - \sum_{j=0}^{\nu} K_j x^j||$$

$$= \rho + 1$$

Hence the lower bound of $\psi(\lambda)$, for all λ, is the same as the lower bound when λ is restricted by $\sqrt{\sum_{j=\nu+1}^{n} \lambda_j^2} \leqq R$. Since this sphere is closed and bounded, the lower bound is attained, and the existence of λ^* is established.

This lemma can be generalized to include the case where the derivatives up through order ν_1 are given at one point $\delta_1 \in [\alpha,\beta]$ and up through order ν_2 at a second point $\delta_2 \in [\alpha,\beta]$, provided the degree n of the polynomial satisfies the relation $n \geqq \nu_1 + \nu_2 + 1$. The only modification required is that the argument associated with the matrix A must be applied twice, first at δ_1 (if zero is involved, set $\delta_1 = 0$) and secondly at δ_2, where the form of the matrix is slightly different. In fact this lemma can be generalized to include any finite number of points at which conditions are imposed. Also one could allow the constants, $C_0, C_1, \ldots, C_\nu$, to be chosen arbitrarily rather than being chosen equal to derivatives of the function f. This last generalization would require no changes in the proof of the lemma.

When the conditions on the derivatives are imposed at an end point of the interval $[\alpha,\beta]$, then the imposed conditions shall be called boundary conditions. In the case of boundary conditions, the next two lemmas give a characterization of the solution whose existence was established in lemma 1.

Definition 2.

Let $f(x)$ be a continuous function on $[\alpha,\beta]$, n be a non-negative integer, $Q_n(x)$ be a polynomial of degree less than or equal to n and let

$$\max_{\alpha \leqq x \leqq \beta} \left| f(x) - Q_n(x) \right| = \rho_{Q_n}(f)$$

Consider a set of points

$$\alpha \leqq x_1 < x_2 < \dots < x_V \leqq \beta$$

subject to the conditions that

$$\text{i)} \quad |f(x_i) - Q_n(x_i)| = \rho_{Q_n}(f), \quad i = 1,2,\dots,V$$

and

$$\text{ii)} \quad f(x_i)-Q_n(x_i)=-[f(x_{i-1})-Q_n(x_{i-1})], \quad i = 2,3,\dots,V.$$

The maximum number of points x_i which can be made to satisfy these two conditions is called the <u>oscillation number</u> of $(f(x) - Q_n(x))$ and is designated $V(f - Q_n)$.

If the number of x_i can be made large without bound, then one writes

$$V(f - Q_n) = \infty$$

Lemma 2.

Let n, ν_1 and ν_2 be non-negative integers with $n \geqq \nu_1+\nu_2+1$. Let $P_n(x)$ be a polynomial of degree n or less with prescribed values for its derivatives of order less than or equal to ν_1 at α and ν_2 at β. If no values are prescribed at $\alpha(\beta)$ set $\nu_1(\nu_2)$ equal to -1. Let $\nu = \nu_1 + \nu_2$ and let

$$\alpha < x_1 \leqq x_2 \leqq x_3 \leqq \dots \leqq x_{n-\nu} < \beta$$

be $(n-\nu)$ points in the interval (α,β). (When $\nu_1 = -1$ include α in the interval and when $\nu_2 = -1$ include β.) If the difference

$$g(x) = f(x) - P_n(x)$$

has the values

$$g(x_i) = \epsilon(-1)^{i-1}\gamma_i,\ \gamma_i > 0,\ i = 1,2,\dots,n-\nu, \text{ and}$$
$$\epsilon = 1 \text{ if } g(x_1) > 0, \text{ or } \epsilon = -1 \text{ if } g(x_1) < 0,$$

then for any polynomial $Q_n(x)$ of degree n or less with prescribed values for all derivatives up through ν_1 at α and up through ν_2 at β,

$$\rho_{Q_n}(f) \geqq \min\,(\gamma_1,\gamma_2,\dots,\gamma_{n-\nu})$$

Proof:

Assume the lemma is false. Then there exists a polynomial $r_n(x)$ of degree n or less which satisfies the given boundary conditions and such that

$$\max_{x_1 \leqq x \leqq x_{n-\nu}} \left| f(x) - r_n(x) \right| < \min(\gamma_1, \gamma_2, \ldots, \gamma_{n-\nu})$$

Consider the polynomial

$$\Delta(x) = r_n(x) - P_n(x) = (f(x) - P_n(x)) - (f(x) - r_n(x))$$

At the $(n-\nu)$ points, $x_1, x_2, \ldots, x_{n-\nu}$, the sign of $\Delta(x)$ is the same as the sign of $f(x) - P_n(x)$. Thus $\Delta(x)$ has $(n-\nu-1)$ zeros in the interval (α,β) (possibly including α and/or β). Furthermore $\Delta(x)$ has a zero of order (ν_1+1) at α and (ν_2+1) at β. Consequently $\Delta(x)$ must have $(n+1)$ zeros in the interval $[\alpha,\beta]$, which is a contradiction.

Lemma 3.

Using the notation of (1) and (2) introduced in the proof of lemma 1, let

$$\psi(\lambda^*) = \rho$$

and let n, ν_1 and ν_2 be non-negative integers such that $\nu_1 + \nu_2 = \nu \leqq n-1$. Assume there exists a polynomial $\mathbb{P}_n(x)$ of best approximation of degree less than or equal to n with prescribed values for its derivatives of each order less than or equal to ν_1 at α and less than or equal to ν_2 at β, and such that if $\rho > 0$ and $\nu_1 \geqq 0$, it is true that

$$|f(\alpha) - \mathbb{P}_n(\alpha)| < \rho \tag{3}$$

and if $\rho > 0$ and $\nu_2 \geqq 0$, it is true that

$$|f(\beta) - \mathbb{P}_n(\beta)| < \rho \tag{4}$$

Then $\mathbb{P}_n(x)$ is unique and is completely characterized by the property that the oscillation number satisfies the inequality

$$V(f(x) - \mathbb{P}_n(x)) \geqq n - \nu .$$

Proof:

Assume $V(f(x) - \mathbb{P}_n(x)) \geqq n - \nu$. If $\rho_{\mathbb{P}_n}(f)$ is zero, then $\mathbb{P}_n(x)$ is certainly a polynomial of best approximation. If $\rho_{\mathbb{P}_n}(f)$ is not zero, then in lemma 2 let

$$\gamma_1 = \rho_{\mathbb{P}_n}(f)$$

$$\gamma_2 = \rho_{\mathbb{P}_n}(f)$$

$$\vdots$$

$$\gamma_{n-\nu} = \rho_{\mathbb{P}_n}(f)$$

Then it follows that for any polynomial $Q_n(x)$

$$\rho_{Q_n}(f) \geqq \rho_{\mathbb{P}_n}(f)$$

Thus

$$\rho_{\mathbb{P}_n}(f) = \rho$$

Assume $V(f(x) - \mathbb{P}_n(x)) \leqq n-\nu-1$ and that $\mathbb{P}_n(x)$ is a polynomial of best approximation relative to the conditions imposed on the derivatives. There are only two possible cases: either $f(x)$ is a polynomial of degree less than or equal to n with the prescribed values for its derivatives or it is not.

Case I

Let $f(x)$ be a polynomial of degree less than or equal to n having the prescribed values for its derivatives. For this function $\rho = 0$. In order for $V(f(x) - \mathbb{P}_n(x))$ to be less than $n-\nu$, $\mathbb{P}_n(x)$ cannot be identically $f(x)$. Thus

$$\rho_{\mathbb{P}_n}(f) > \rho$$

and $\mathbb{P}_n(x)$ is not a polynomial of best approximation.

Case II

Assume $f(x)$ is not a polynomial of degree less than or equal to n having the prescribed values for its derivatives. In this case $\rho \neq 0$.

Subdivide the interval from α to β into the intervals

$$[u_0,u_1],\ [u_1,u_2],\ \ldots$$

so small that the oscillation (maximum minus the minimum) of $[f(x) - \mathbb{P}_n(x)]$ in each subinterval is less than or equal to $\frac{1}{2}\,\rho_{\mathbb{P}_n}(f)$. (This is possible in that a function which is continuous on a closed, bounded set is uniformly continuous there.)

If $|f(x) - \mathbb{P}_n(x)| = \rho_{\mathbb{P}_n}(f)$ in the interval $[u_k,u_{k+1}]$, the interval $[u_k,u_{k+1}]$ is called a distinguished interval. Label each such interval + or - according as $(f(x) - \mathbb{P}_n(x))$ is positive or negative respectively.

Label the distinguished intervals D_1, D_2,..... in order. Without loss of generality assume D_1 is +. Starting with D_1 proceed through the distinguished intervals until the first - distinguished interval is encountered. Call this group of + distinguished intervals group 1. Starting with the first encountered - distinguished interval proceed until the next + distinguished interval is encountered. Call this group of - distinguished intervals group 2, and continue in this fashion. Since $V(f - \mathbb{P}_n)$ is $\leq n-\nu-1$, the total number of groups, T, is less than or equal to $(n-\nu-1)$. Construct a polynomial $r(x)$ of degree n which is negative in the intervals of group 1, positive in the intervals of group 2, etc., and such that the derivatives of $r(x)$ are zero at α up to order ν_1 and at β up to order ν_2.

Consider the polynomial

$$Q_n(x) = \mathbb{P}_n(x) - \epsilon\, r(x)$$

for an arbitrary $\epsilon > 0$. If ϵ is chosen sufficiently small, $Q_n(x)$ is a better fitting polynomial than $\mathbb{P}_n(x)$, and also satisfies the conditions imposed at the boundaries. This contradicts the assumption that $\mathbb{P}_n(x)$ is a best fitting polynomial.

To prove the uniqueness of the best approximating polynomial, assume there exist two polynomials of best approximation $\mathbb{P}_n(x)$ and $Q_n^*(x)$. Consider the polynomial

$$\varphi(x) = \tfrac{1}{2}[\mathbb{P}_n(x) + Q_n^*(x)]$$

$\varphi(x)$ is also a polynomial of best approximation, and satisfies the boundary conditions. Thus from the first part of this lemma, it must have an oscillation number $\geq (n-\nu)$. From this it follows that $\mathbb{P}_n(x) = Q_n^*(x)$ at $(n-\nu)$ interior points. Furthermore they have the same $(\nu+2)$ conditions imposed upon their derivatives at the boundary points. This implies that

$$\mathbb{P}_n(x) \equiv Q_n^*(x)$$

Thus the solution is unique.

If a boundary condition is imposed at a boundary point δ such that

$$|f(\delta) - \mathbb{P}_n(\delta)| = \rho$$

then the polynomial of best approximation $\mathbb{P}_n(x)$ may not be unique. For example if

$$f(x) = -x^2 , \qquad x \in [-1,0]$$

and if a first degree approximating polynomial must have the value 10 at $x = 0$, then any polynomial of the form

$$\mathbb{P}_1(x) = cx + 10, \qquad 1.0 \leqq c \leqq 21.0$$

is a polynomial of best approximation relative to the imposed conditions. See figure 1.

It should be noted that this non-uniqueness can never occur when the boundary conditions require the approximating polynomial to have the same value as the given function at the boundary points. For in this case either $\rho = 0$ or conditions (3) and (4) of lemma 3 are satisfied.

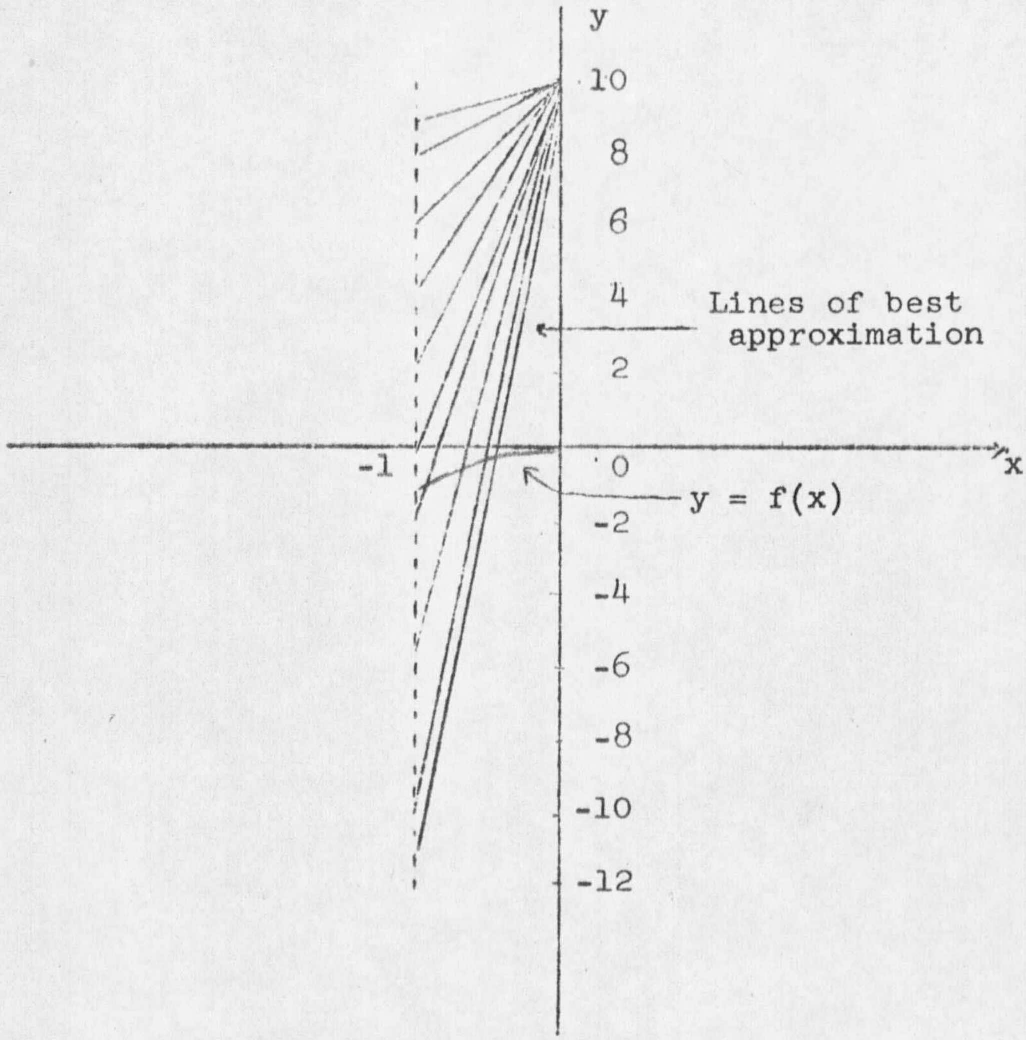


Figure 1. Family of lines of best approximation

3. Spline Function Approximation with the Join Points Given

Theorem 1.

Let $n_0, n_1, \ldots, n_k$ and $\nu_1, \nu_2, \ldots, \nu_k$ be non-negative integers with $n_0 \geqq \nu_1$, $n_r \geqq \nu_r + \nu_{r+1} + 1$, $r = 1, 2, \ldots, k-1$, and $n_k \geqq \nu_k$. Let f be a continuous real valued function on the non-degenerate interval $[\alpha, \beta]$ with (bounded) values for its derivatives up through order $\nu_1, \nu_2, \ldots, \nu_k$ at the respective fixed join points $\alpha \leqq \delta_1 < \delta_2 < \ldots < \delta_k \leqq \beta$. Let S be the class of general spline functions which have the same values for their derivatives as f at $\delta_1, \delta_2, \ldots, \delta_k$ and are of degree less than or equal to n_0 on $[\alpha, \delta_1]$, to n_r on $[\delta_r, \delta_{r+1}]$, $r = 1, 2, \ldots, k-1$, and to n_k on $[\delta_k, \beta]$. Then there exists an $s_k(x) \in S$ which makes the Tchebycheff norm $||f(x) - s_k(x)||$ a minimum. Furthermore, if it is required that $s_k(x)$ be a best approximation on each subinterval, then $s_k(x)$ is unique and is completely characterized by the property that the oscillation number $V(f(x) - s_k(x))$ is at least as large as $n_o - \nu_1 + 1$ on $[\alpha, \delta]$, $n_r - \nu_r - \nu_{r+1}$ on $[\delta_r, \delta_{r+1}]$, $r = 1, 2, \ldots, k-1$, and $n_k - \nu_k + 1$ on $[\delta_k, \beta]$.

Proof:

The direct application of the generalized form of lemma 1 and lemma 3 to each of the intervals $[\alpha, \delta_1]$, $[\delta_1, \delta_2], \ldots, [\delta_k, \beta]$ proves the theorem.

Corollary 1.

Let $n_0, n_1, \ldots, n_k$ and $\nu_1, \nu_2, \ldots, \nu_k$ be non-negative integers with $n_0 \geqq \nu_1$, $n_r \geqq \nu_r + \nu_{r+1} + 1$, $r = 1,2,\ldots,k-1$, and $n_k \geqq \nu_k$. Let f be a continuous real valued function on the non-degenerate interval $[\alpha,\beta]$. Let S be the class of general spline functions which have the same prescribed values for their derivatives up through order $\nu_1, \nu_2, \ldots, \nu_k$ at the respective fixed join points $\alpha \leqq \delta_1 < \delta_2 < \cdots < \delta_k \leqq \beta$, and are of degree less than or equal to n_0 on $[\alpha,\delta_1]$, to n_r on $[\delta_r,\delta_{r+1}]$, $r = 1,2,\ldots,k-1$, and to n_k on $[\delta_k,\beta]$. Then there exists an $s_k(x) \in S$ which makes the Tchebycheff norm $||f(x) - s_k(x)||$ a minimum. Furthermore, if it is required that $s_k(x)$ be a best approximation on each subinterval, and if the maximum value of $|f(x) - s_k(x)|$ on each subinterval $[\delta_r,\delta_{r+1}]$ does not occur at either of the join points δ_r or δ_{r+1}, then $s_k(x)$ is unique on the interval $[\delta_r,\delta_{r+1}]$ and completely characterized by the property that on $[\delta_r,\delta_{r+1}]$, $V(f(x) - s_k(x)) \geqq n_r - \nu_r - \nu_{r+1}$, $r = 1,2,\ldots,k-1$. If the maximum value of $|f(x) - s_k(x)|$ on $[\alpha,\delta_1]$ does not occur at δ_1, then $s_k(x)$ is unique on $[\alpha,\delta_1]$ and completely characterized by the property that on $[\alpha,\delta_1]$, $V(f(x) - s_k(x)) \geqq n_0 - \nu_1 + 1$. Also if the maximum value of $|f(x) - s_k(x)|$ on $[\delta_k,\beta]$ does not occur at δ_k, then $s_k(x)$ is unique on $[\delta_k,\beta]$ and completely characterized by the property that on $[\delta_k,\beta]$, $V(f(x) - s_k(x)) \geqq n_k - \nu_k + 1$.

4. Spline Function Approximations With Variable Join Points

Theorem 2.

Let f be a continuous real valued function on the non-degenerate interval $[\alpha,\beta]$ and let f possess bounded values for its derivatives up through some order ν on $[\alpha,\beta]$. Let $\mu_1,\mu_2,\ldots,\mu_k$ and $n_0,n_1,\ldots,n_k$ be finite sets of non-negative integers with

1) $\max_i \mu_i \leqq \nu$

2) $n_0 \geqq \max_i \mu_i$

3) $n_k \geqq \max_i \mu_i$

4) $n_r \geqq \max_{\substack{i,j \\ i\neq j}} (\mu_i + \mu_j + 1)$, $r = 1,2,\ldots,k-1$.

Let $\nu_1,\nu_2,\ldots,\nu_k$ be an arbitrary permutation of $\mu_1,\mu_2,\ldots,\mu_k$. Let $S = S(\nu_1,\nu_2,\ldots,\nu_k)$ be the class of all general spline functions which (a) have join points $\delta_1,\delta_2,\ldots,\delta_k$, arbitrarily selected from the interval $[\alpha,\beta]$ and labeled such that $\delta_1 \leqq \delta_2 \leqq \ldots \leqq \delta_k$, (b) have the same derivatives as f up through order $\nu_1,\nu_2,\ldots,\nu_k$ at the respective join points $\delta_1,\delta_2,\ldots,\delta_k$, and (c) are of degree less than or equal to n_0 on $[\alpha,\delta_1]$, to n_r on $[\delta_r,\delta_{r+1}]$, $r = 1,2,\ldots,k-1$, and to n_k on $[\delta_k,\beta]$. Then among all functions in S there is at least one function s_k such that $||f - s_k||$ is a minimum. Furthermore, among all possible permutations of $\mu_1, \mu_2, \ldots, \mu_k$ there exists at least one permutation which makes this Tchebycheff norm a minimum.

It should be noted that in this theorem the norm $||f - s_k||$ is a function of the join points $\delta_1, \delta_2, \ldots, \delta_k$, the permutation $\nu_1, \nu_2, \ldots, \nu_k$, the function f, the interval $[\alpha, \beta]$ and the coefficients of the general spline function s_k.

Proof:

Let an arbitrary general spline function s_k be described by

$$s_k(x) = \begin{cases} \sum_{j=0}^{n_0} \lambda_{0,j} x^j, & \alpha \leqq x \leqq \delta_1 \\ \vdots & \\ \sum_{j=0}^{n_r} \lambda_{r,j} x^j, & \delta_r \leqq x \leqq \delta_{r+1} \\ \vdots & \\ \sum_{j=0}^{n_k} \lambda_{k,j} x^j, & \delta_k \leqq x \leqq \beta \end{cases} \tag{5}$$

In order to simplify the notation, let $\delta_0 = \alpha$, $\delta_{k+1} = \beta$, $n = \max_{0 \leqq r \leqq k} (n_r)$ and let $\dot{\bar{\lambda}}_j$ be defined by:

$$\dot{\bar{\lambda}}_j = \begin{cases} \lambda_{r,j} \text{ when } \delta_r \leqq x \leqq \delta_{r+1}, & j = 0,1,2,\ldots,n_r \\ & r = 0,1,\ldots,k \\ 0 \text{ when } \delta_r \leqq x \leqq \delta_{r+1}, & j = n_{r+1},\ldots,n \\ & r = 0,1,\ldots,k \end{cases} \tag{6}$$

It should be noted that for each fixed value of r, $\dot{\bar{\lambda}}_j$ is unique. However, for $x = \delta_r$, $r = 1,2,\ldots,k$, there are at least two possible choices for $\dot{\bar{\lambda}}_j$ which are consistent with (5). Using this notation one has

$$s_k(x) = \sum_{j=0}^{n} \dot{\bar{\lambda}}_j x^j$$

The $s_k(x)$ must satisfy the conditions:

$$s_k^{(m_i)}(\delta_i) = f^{(m_i)}(\delta_i) \qquad m_i = 0,1,2,\ldots,\nu_i$$
$$i = 1,2,\ldots,k$$

For $i = 1$ this system of equations is of the form:

$$\sum_{j=0}^{n_0} \lambda_{0,j}\delta_1^j = f(\delta_1), \quad \sum_{j=0}^{n_1} \lambda_{1,j}\delta_1^j = f(\delta_1)$$

$$\sum_{j=1}^{n_0} j\lambda_{0,j}\delta_1^{j-1} = f'(\delta_1), \quad \sum_{j=0}^{n_1} j\lambda_{1,j}\delta_1^{j-1} = f'(\delta_1) \qquad (7)$$

$$\vdots$$

$$\sum_{j=\nu_1}^{n_0} \frac{j!}{(j-\nu_1)!} \lambda_{0,j}\delta_1^{j-\nu_1} = f^{(\nu_1)}(\delta_1),$$

$$\sum_{j=\nu_1}^{n_1} \frac{j!}{(j-\nu_1)!} \lambda_{1,j}\delta_1^{j-\nu_1} = f^{(\nu_1)}(\delta_1)$$

A similar set of equations exists for $i = 2,3,\ldots,k$.

This complete system of equations can be used to eliminate $\nu_1 + 1$ of the $\lambda_{0,j}$'s, $\nu_1 + \nu_2 + 2$ of the $\lambda_{1,j}$'s,..., $\nu_{k-1} + \nu_k + 2$ of the $\lambda_{k-1,j}$'s and $\nu_k + 1$ of the $\lambda_{k,j}$'s. Thus $s_k(x)$ can be written in the form

$$s_k(x) = a_{r,0} + a_{r,1}x + \dots + a_{r,n_r}x^{n_r} \quad \text{for}$$

$$\delta_r \leqq x \leqq \delta_{r+1}, \quad r = 0,1,\dots,k$$

where each $a_{r,j}$ is a rational function of the remaining $\lambda_{r,j}$ coefficients and the join points δ_r and δ_{r+1}. Because of the form of the system of equations and the coefficient matrix (See the description of matrix A in lemma 1.) it is easy to show that each $a_{r,j}$ is also homogeneous of degree one, in the $\lambda_{r,j}$'s that appear, with the exception of one term which is only a function of the join points. That is

$$a_{0,j} = \begin{cases} K_{0,j} + \sum_{\ell=\nu_1+1}^{n_0} k_{0,j,\ell}\lambda_{0,\ell}, & j = 0,1,\dots,\nu_1 \\ \lambda_{0,j}, & j = \nu_1 + 1, \nu_1 + 2,\dots,n_0 \end{cases}$$

$$a_{r,j} = \begin{cases} K_{r,j} + \sum_{\ell=\nu_r+\nu_{r+1}+2}^{n_r} k_{r,j,\ell}\lambda_{r,\ell}, & j=0,1,\dots,\nu_r+\nu_{r+1}+1 \\ \lambda_{r,j}, & j = \nu_r + \nu_{r+1} + 2,\dots,n_r \end{cases}$$

for $r = 1,2,\dots,k-1$, and

$$a_{k,j} = \begin{cases} K_{k,j} + \sum_{\ell=\nu_k+1}^{n_k} k_{k,j,\ell}\lambda_{k,\ell}, & j = 0,1,\ldots,\nu_k \\ \lambda_{k,j}, & j = \nu_k + 1, \nu_k + 2, \ldots, n_k \end{cases}$$

where each $K_{r,j}$ and each $k_{r,j,\ell}$, $r = 0,1,2,\ldots,k$, is a rational function of δ_r and δ_{r+1} only. Using the notation previously introduced in (6), $s_k(x)$ can be written

$$s_k(x) = \sum_{j=0}^{n} \dot{\bar{a}}_j x^j$$

Now let the function ψ be defined by

$$\psi(\lambda_{0,\nu_1+1},\ldots,\lambda_{0,n_0},\ldots,\lambda_{k,\nu_k+1},\ldots,\lambda_{k,n_k},\delta_1,\delta_2,\ldots,\delta_k) =$$

$$= ||f(x) - s_k(x)||$$

$$= \max_{\alpha \leqq x \leqq \beta} \left| f(x) - \sum_{j=0}^{n} \dot{\bar{a}}_j x^j \right|$$

Define ω to be the vector with $\bar{n}$ components $(\lambda_{0,\nu_1+1},\ldots,\delta_k)$ and ω' the vector with $\bar{n}$ components $(\lambda'_{0,\nu_1+1},\ldots,\delta'_k)$. Then the continuity of ψ will be established if it can be shown that $|\psi(\omega') - \psi(\omega)| \to 0$ as $||\omega' - \omega|| \to 0$ (Tchebycheff norm).

Two cases will now be examined: Case I, the case in which no two join points δ_i, δ_j are equal, and Case II, the case in which at least two join points are equal. If Case I is true, then $\alpha \leqq \delta_1 < \delta_2 < \cdots < \delta_k \leqq \beta$. In the limit as $\omega' \to \omega$, it must be true that $\delta'_r \to \delta_r$, $r = 1,2,\ldots,k$. First for an arbitrary r, assume $\delta_r' \geqq \delta_r$. Let $|\delta_{r-1}' - \delta_{r-1}|$, $|\delta_r' - \delta_r|$ and $|\delta_{r+1}' - \delta_{r+1}|$ all be so small that $\delta_r' < \delta_{r+1}'$, $\delta_r' < \delta_{r+1}$ and $\delta_{r-1}' < \delta_r$. Next designate the general spline function associated with the vectors ω and ω' by $s_k(x)$ and $s_k'(x)$ respectively. Then for any $x \in [\delta_r, \delta_r']$

$$s_k(x) = \sum_{j=0}^{n} \dot{\bar{a}}_j x^j = \sum_{j=0}^{n_r} a_{r,j} x^j$$

$$s_k'(x) = \sum_{j=0}^{n} \dot{\bar{a}}_j' x^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j}' x^j$$

For fixed ω, $\sum_{j=0}^{n_r} a_{r,j} x^j$ is a continuous (polynomial) function of x. Thus it follows that

$$\sum_{j=0}^{n_r} a_{r,j} x^j = \sum_{j=0}^{n_r} a_{r,j} (\delta_r)^j + \bar{\epsilon}_r$$

where $\bar{\epsilon}_r \to 0$ as $x \to \delta_r$. Similarly

$$\sum_{j=0}^{n_{r-1}} a_{r-1,j} x^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j} (\delta_r)^j + \epsilon_r^*$$

where $\epsilon_r^* \to 0$ as $x \to \delta_r$. But

$$\sum_{j=0}^{n_r} a_{r,j} (\delta_r)^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j} (\delta_r)^j$$

Thus for $x \in [\delta_r, \delta'_r]$

$$\sum_{j=0}^{n} \dot{\bar{a}}_j x^j = \sum_{j=0}^{n_r} a_{r,j} x^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j} x^j + \bar{\epsilon}_r - \epsilon_r^*$$

$$\sum_{j=0}^{n} \dot{\bar{a}}'_j x^j = \sum_{j=0}^{n_{r-1}} a'_{r-1,j} x^j$$

When $\delta'_r < \delta_r$ and $||\omega' - \omega||$ is sufficiently small, one obtains for $x \in [\delta'_r, \delta_r]$

$$\sum_{j=0}^{n} \dot{\bar{a}}_j x^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j} x^j = \sum_{j=0}^{n_r} a_{r,j} x^j + \bar{\bar{\epsilon}}_r - \epsilon_r^{**}$$

$$\sum_{j=0}^{n} \dot{\bar{a}}'_j x^j = \sum_{j=0}^{n_r} a'_{r,j} x^j$$

where $\bar{\bar{\epsilon}}_r \to 0$ and $\epsilon_r^{**} \to 0$ as $\delta'_r \to \delta_r$.

Now for $x \in [\max(\delta_{r-1}, \delta'_{r-1}), \min(\delta_r, \delta'_r)]$

$$\sum_{j=0}^{n} \dot{\bar{a}}_j x^j = \sum_{j=0}^{n_{r-1}} a_{r-1,j} x^j$$

$$\sum_{j=0}^{n} \dot{\bar{a}}'_j x^j = \sum_{j=0}^{n_{r-1}} a'_{r-1,j} x^j$$

and for $x \in [\max(\delta_r, \delta'_r), \min(\delta_{r+1}, \delta'_{r+1})]$

$$\sum_{j=0}^{n} \dot{\bar{a}}_j x^j = \sum_{j=0}^{n_r} a_{r,j} x^j$$

$$\sum_{j=0}^{n} \dot{\bar{a}}'_j x^j = \sum_{j=0}^{n_r} a'_{r,j} x^j$$

Furthermore, from the properties of a norm it follows that

$$|\psi(\omega')-\psi(\psi)| = \Big|\,||f-s_k'|| - ||f-s_k||\,\Big| \leqq ||s_k'-s_k||$$

$$= \Big|\Big| \sum_{j=0}^{n} (\dot{\bar{a}}_j' - \dot{\bar{a}}_j)x^j \Big|\Big|$$

Thus for all $x \in [\alpha,\beta]$,

$$\Big|\Big| \sum_{j=0}^{n} (\dot{\bar{a}}_j'-\dot{\bar{a}}_j)x^j \Big|\Big| \leqq \max_{r,j}\big|a_{r,j}'-a_{r,j}\big| \sum_{j=0}^{n} \big|\big|x^j\big|\big|$$

$$+ 2 \max_r \left(\max\{|\bar{\epsilon}_r|,|\epsilon_r^*|,|\bar{\bar{\epsilon}}_r|,|\epsilon_r^{**}|\}\right)$$

Each $a_{r,j}'$ is a continuous function of its arguments. Thus in the limit as $\omega' \to \omega$, each $a_{r,j}' \to a_{r,j}$. Also each $\bar{\epsilon}_r$, ϵ_r^*, $\bar{\bar{\epsilon}}_r$ and ϵ_r^{**} approaches zero as $\omega' \to \omega$. This shows that $\psi(\dot{\omega})$ is a continuous function of ω on a subspace $\bar{E}$ of Euclidean $\bar{n}$-space in Case I.

Now consider Case II. If any two or more of the join points $\delta_1,\delta_2,\ldots,\delta_k$ are the same point $\bar{\delta}$, then in a sufficiently small neighborhood of $\bar{\delta}$ the continuous spline functions $s_k(x)$ and $s_k'(x)$ satisfy the relation

$$f(\bar{\delta}) = s_k(x) + \bar{\epsilon}_1 = s_k'(x) + \bar{\epsilon}_2$$

where $\bar{\epsilon}_1 \to 0$ and $\bar{\epsilon}_2 \to 0$ as $x \to \bar{\delta}$. Let $\bar{\delta}_1,\bar{\delta}_2,\ldots,\bar{\delta}_s$ be the components of $\dot{\omega}$ which are equal to $\bar{\delta}$, and let $\bar{\delta}_1',\bar{\delta}_2',\ldots,\bar{\delta}_s'$ be the corresponding components of ω'. If $x \in [\min_i \bar{\delta}_i', \max_i \bar{\delta}_i']$, then $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ can be made arbitrarily small by making $\max_i|\bar{\delta}_i' - \bar{\delta}|$ sufficiently small. But this implies that as $\omega' \to \omega$,

$$\max_{x \in [\min_i \bar{\delta}_i', \max_i \bar{\delta}_i']} \left| s_k' - s_k \right| \to 0$$

Thus the continuity of $\psi(\omega)$ is not destroyed at a point where two or more join points are equal. Case I covers all the other points x in the interval $[\alpha,\beta]$. Consequently $\psi(\omega)$ is also a continuous function in Case II.

Recall $\overline{E}$ is the subspace of Euclidean $\overline{n}$-space such that if $\omega \in \overline{E}$, then $\alpha \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_k \leqq \beta$. Let ρ be the lower bound of $\psi(\omega)$ for $\omega \in \overline{E}$. Then $\rho \geqq 0$. Now one only needs to show that there exists an $\omega^* \in \overline{E}$ such that $\psi(\omega^*) = \rho$. The following two cases will be discussed in order:

(i) All coefficients $\lambda_{r,j}$ are determined by the join conditions.

(ii) There is at least one $\lambda_{r,j}$ coefficient which is not determined by the join conditions.

In the first case $\psi(\omega)$ is only a function of the join points $\delta_1, \delta_2, \ldots, \delta_k$. As each δ_r, $r = 1,2,\ldots,k$, satisfies $\alpha \leqq \delta_{r-1} \leqq \delta_r \leqq \delta_{r+1} \leqq \beta$, it follows from the continuity of $\psi(\omega)$ that there exists a set of join points $\delta_1^*, \delta_2^*, \ldots, \delta_k^*$ such that $\psi(\delta_1^*, \delta_2^*, \ldots, \delta_k^*) = \rho$.

In the second case define $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_k$ as follows:

$$\bar{\lambda}_0^2 = \lambda_{0,\nu_1+1}^2 + \cdots + \lambda_{0,n_0}^2 \quad \text{if } n_0 \geqq \nu_1 + 1$$

$$\bar{\lambda}_0^2 \equiv 0 \quad \text{if } n_0 < \nu_1 + 1$$

$$\bar{\lambda}_1^{\,2} = \lambda_{1,\nu_1+\nu_2+2}^2 + \cdots + \lambda_{1,n_1}^2 \qquad \text{if } n_1 \geqq \nu_1 + \nu_2 + 2$$

$$\bar{\lambda}_1^{\,2} \equiv 0 \qquad \text{if } n_1 < \nu_1 + \nu_2 + 2$$

$$\vdots$$

$$\bar{\lambda}_k^{\,2} = \lambda_{k,\nu_k+1}^2 + \cdots + \lambda_{k,n_k}^2 \qquad \text{if } n_k \geqq \nu_k + 1$$

$$\bar{\lambda}_k^{\,2} \equiv 0 \qquad \text{if } n_k < \nu_k + 1$$

Let N be the set of integers r such that $\bar{\lambda}_r \not\equiv 0$. Note N is not empty. Now $\alpha \leqq \delta_r \leqq \beta$, $r = 1,2,\ldots,k$. It follows that on the union of the closed shells

$$\bar{\lambda}_r^{\,2} = R, \quad r \in N, \qquad R \text{ a positive constant,}$$

$$\bar{\lambda}_r^{\,2} = 0, \quad r \notin N$$

the continuous function $\psi(\omega)$ attains a minimum value $\bar{\rho}(R)$ where

$$\bar{\rho}(R) \geqq \rho$$

One is now faced with the problem of showing that there exists some finite R, such that when $\bar{\lambda}_r^{\,2} > R$ for any r, $r = 0,1,2,\ldots,k$, and for any arbitrary set of join points $\delta_1, \delta_2, \ldots, \delta_k$, it is also true that $\psi(\omega) \geqq \bar{\rho}(R)$. If this can be shown, then the lower bound of $\psi(\omega)$ for all ω is the same as the lower bound when ω is restricted by $\bar{\lambda}_r^{\,2} \leqq R$, $r = 0,1,2,\ldots,k$. Let the non-eliminated $\lambda_{r,j}$ coefficients be broken into two types. Those coefficients which are associated with polynomials evaluated at only one point shall be called type I coefficients. (This happens when two join points are equal.) Those coefficients which are associated with polynomials evaluated on a non-degenerate interval shall be called type II coefficients.

At the join points, the general spline function fits the function f exactly whenever the system of equations (7) is satisfied. In this system of equations, each $\lambda_{r,j}$ coefficient, which has not been eliminated, appears in a linear equation with finite coefficients. (Note that $\delta_1, \delta_2, \ldots, \delta_k$ are arbitrary but fixed.) Such an equation always has a finite solution. Thus there exists a positive constant R_I such that the value of $(f(x) - s_k(x))$ is zero at each join point $x = \delta_r$, $r = 1,2,\ldots,k$, and

$$\bar{\lambda}_r^{\,2} \leqq R_I \;, \qquad r = 0,1,2,\ldots,k$$

In fact here R_I can be taken to be zero. As the coefficients of type I only influence $(f(x) - s_k(x))$ at a join point, it follows that nothing is to be gained by having any type I coefficients satisfy

$$\bar{\lambda}_r^{\,2} > R_I$$

for any r. Consequently if there are no coefficients of type II, then there must exist an ω^* on the closed domain described by $\bar{\lambda}_r^{\,2} \leqq R_I$, $r = 0,1,\ldots,k$, $\alpha \leqq \delta_{r-1} \leqq \delta_r \leqq \delta_{r+1} \leqq \beta$, $r = 1,2,\ldots,k$, such that $\psi(\omega^*) = \rho$.

Next assume there is at least one coefficient of type II. Let $\delta_1, \delta_2, \ldots, \delta_k$ be an arbitrary set of fixed join points, and let $\overline{s}(x)$ be the function obtained from $s_k(x)$ by setting all non-eliminated $\lambda_{r,j}$ coefficients equal to zero. $\overline{s}(x)$ is a continuous function on $[\alpha, \beta]$. It has been proved that $||f(x) - s_k(x)||$ is a continuous function of ω. Thus it follows that $||\overline{s}(x) - s_k(x)||$ is a continuous function of ω. On the closed shells

$$\overline{\lambda}_r^{\,2} = 1, \qquad r \in N$$

$$\overline{\lambda}_r^{\,2} = 0, \qquad r \notin N$$

the continuous function $||\overline{s}(x) - s_k(x)||$ attains a minimum value σ. On any closed interval $[\delta_r, \delta_{r+1}]$ which has an associated non-eliminated $\lambda_{r,j}$ coefficient of type II, the independence of the functions $1, x, x^2, \ldots$ implies that

$$\max_{x \in [\delta_r, \delta_{r+1}]} \left|\overline{s}(x) - s_k(x)\right| = \sigma_r > 0$$

Thus

$$||\overline{s}(x) - s_k(x)|| \geqq \sigma \geqq \sigma_r > 0, \qquad r \in N$$

The homogeneity in the $\lambda_{r,j}$'s of the expression $(\overline{s}(x) - s_k(x))$ implies that for any vector ω with fixed $\delta_1, \delta_2, \ldots, \delta_k$,

$$||\overline{s}(x) - s_k(x)|| \geqq \sigma_r \sqrt{\overline{\lambda}_r^{\,2}}\ , \qquad r \in N$$

Thus when

$$\sqrt{\overline{\lambda}_r^{\,2}} > R_r = (\rho + 1 + ||f(x) - \overline{s}(x)||)/\sigma_r,$$

for any $r \in N$, it follows that:

$$\psi(\omega) = ||f(x) - s_k(x)|| = ||f(x) - \overline{s}(x) + \overline{s}(x) - s_k(x)||$$
$$\geqq ||\overline{s}(x) - s_k(x)|| - ||f(x) - \overline{s}(x)||$$
$$\geqq \sigma_r[(\rho+1+||f(x)-\overline{s}(x)||)/\sigma_r] - ||f(x)-\overline{s}(x)||$$
$$= \rho + 1$$

Hence the lower bound of $\psi(\omega)$, for all ω with fixed $\delta_1, \delta_2, \ldots, \delta_k$ is the same as the lower bound when ω is restricted by

$$\sqrt{\overline{\lambda}_r^{\,2}} \leqq R_{II}, \quad r = 0,1,\ldots,k$$

where

$$R_{II} = \max_{r \in N} R_r$$

It will now be established that R_{II} is a continuous function of the join points $\delta_1, \delta_2, \ldots, \delta_k$. It has already been established that $\psi(\omega) = ||f - s_k||$ is a continuous function of the join points. This implies for any general spline function s_k, that $||s_k||$ is a continuous function of the join points. Let $\overline{\overline{s}}_r(x)$ be defined by:

$$\overline{\overline{s}}_r(x) = \begin{cases} \overline{s}(x) - s_k(x), & x \in [\delta_r, \delta_{r+1}] \\ 0, & x \notin [\delta_r, \delta_{r+1}] \end{cases}$$

It is easily shown that $\overline{\overline{s}}_r(x)$ is a general spline function. Consequently $||\overline{\overline{s}}_r(x)||$ is a continuous function of the join points. Now

$$||\overline{\overline{s}}_r(x)|| = \max_{x \in [\delta_r, \delta_{r+1}]} |\overline{s}(x) - s_k(x)| = \sigma_r$$

Thus σ_r is a continuous function of $\delta_1, \delta_2, \ldots, \delta_k$.

From this it follows that R_r is a continuous function of $\delta_1, \delta_2, \ldots, \delta_k$ and consequently R_{II} is also. The join points satisfy the relation:

$$\alpha \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_k \leqq \beta$$

On this closed subspace the continuous function R_{II} attains a maximum value. Call this value R. It now follows that the lower bound of $\psi(\omega)$, for all $\omega \in \overline{E}$ is the same as the lower bound when $\omega \in \overline{E}$ is restricted by

$$\sqrt{\overline{\lambda}_r^{\,2}} \leqq R, \quad r = 0,1,\ldots,k$$

This completes the proof of the existence of a spline function of best approximation in the sense of the Tchebycheff norm for the permutation $\nu_1, \nu_2, \ldots, \nu_k$. There are only a finite number of permutations of $\mu_1, \mu_2, \ldots, \mu_k$. Thus out of all possible permutations, there must exist at least one which makes the norm a minimum.

Theorem 3:

Let f be a continuous real valued function on the non-degenerate interval $[\alpha,\beta]$. Let $\mu_1, \mu_2, \ldots, \mu_k$ and $n_0, n_1, \ldots, n_k$ be finite sets of non-negative integers with

$$n_r \geqq \max_i \mu_i \;, \; r = 0,1,\ldots,k$$

Let $\nu_1, \nu_2, \ldots, \nu_k$ be an arbitrary permutation of $\mu_1, \mu_2, \ldots, \mu_k$.

Let $S = S(\nu_1,\nu_2,\ldots,\nu_k)$ be the class of all general spline functions which, (a) have join points $\delta_1,\delta_2,\ldots,\delta_k$, arbitrarily selected from the interval $[\alpha,\beta]$ and labeled such that $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_k$, (b) have derivatives up through order $\nu_1,\nu_2,\ldots,\nu_k$ at the respective join points $\delta_1,\delta_2,\ldots,\delta_k$, and (c) are of degree less than or equal to n_0 on $[\alpha,\delta_1]$, to n_r on $[\delta_r,\delta_{r+1}]$, $r = 1,2,\ldots,k-1$ and to n_k on $[\delta_k,\beta]$. Then among all functions in S there is at least one function s_k such that $||f - s_k||$ is a minimum. Furthermore, among all possible permutations of $\mu_1,\mu_2,\ldots,\mu_k$ there exists at least one permutation which makes this Tchebycheff norm a minimum.

The proof is similar to the proof of theorem 2.

5. Characterization of Best Spline Approximations

If the join points are not fixed, the best fitting general spline function may not be unique. For example consider the function $f(x)$ defined by

$$f(x) = \begin{cases} g(x) - h(x), & x < -\frac{1}{2} \\ 0, & -\frac{1}{2} \leqq x \leqq \frac{1}{2} \\ g(x) + h(x), & \frac{1}{2} < x \end{cases}$$

where

$$g(x) = \begin{cases} \frac{1}{2}|x - [x]| - \frac{1}{4} & \text{when } [x] \text{ is even} \\ \frac{1}{2}|[x+1] - x| - \frac{1}{4} & \text{when } [x] \text{ is odd} \end{cases}$$

and

$$h(x) = x^2 - \frac{1}{4}$$

Here [x] is the greatest integer function. The graph of this function is given in figure 2.

If one wants to fit f(x) on the interval [-4,4] using a general spline function which agrees with f(x) at one join point δ and is of degree 2 on $[-4,\delta]$ and also on $[\delta,4]$, then a best fit is given by

$$s_1(x) = \begin{cases} -(x^2 - \frac{1}{4}) , & -4 \leqq x < \delta \\ (x^2 - \frac{1}{4}) , & \delta \leqq x \leqq 4 \end{cases}$$

with

$$\delta = \pm \frac{1}{2}$$

For each of these two values of δ, the maximum error is $\frac{1}{4}$. It will now be shown that this fit cannot be improved.

On the interval [-4,-1],

$$V(f(x) + (x^2 - \frac{1}{4})) = 4$$

From lemma 3, $-(x^2 - \frac{1}{4})$ is a best fitting quadratic of f(x) on [-4,-1]. Similarly $(x^2 - \frac{1}{4})$ is a best fitting quadratic of f(x) on [1,4]. Furthermore,

$$\max_{x \in [-4,-1]} |f(x) + (x^2 - \frac{1}{4})| = \frac{1}{4}$$

$$\max_{x \in [1,4]} |f(x) - (x^2 - \frac{1}{4})| = \frac{1}{4}$$

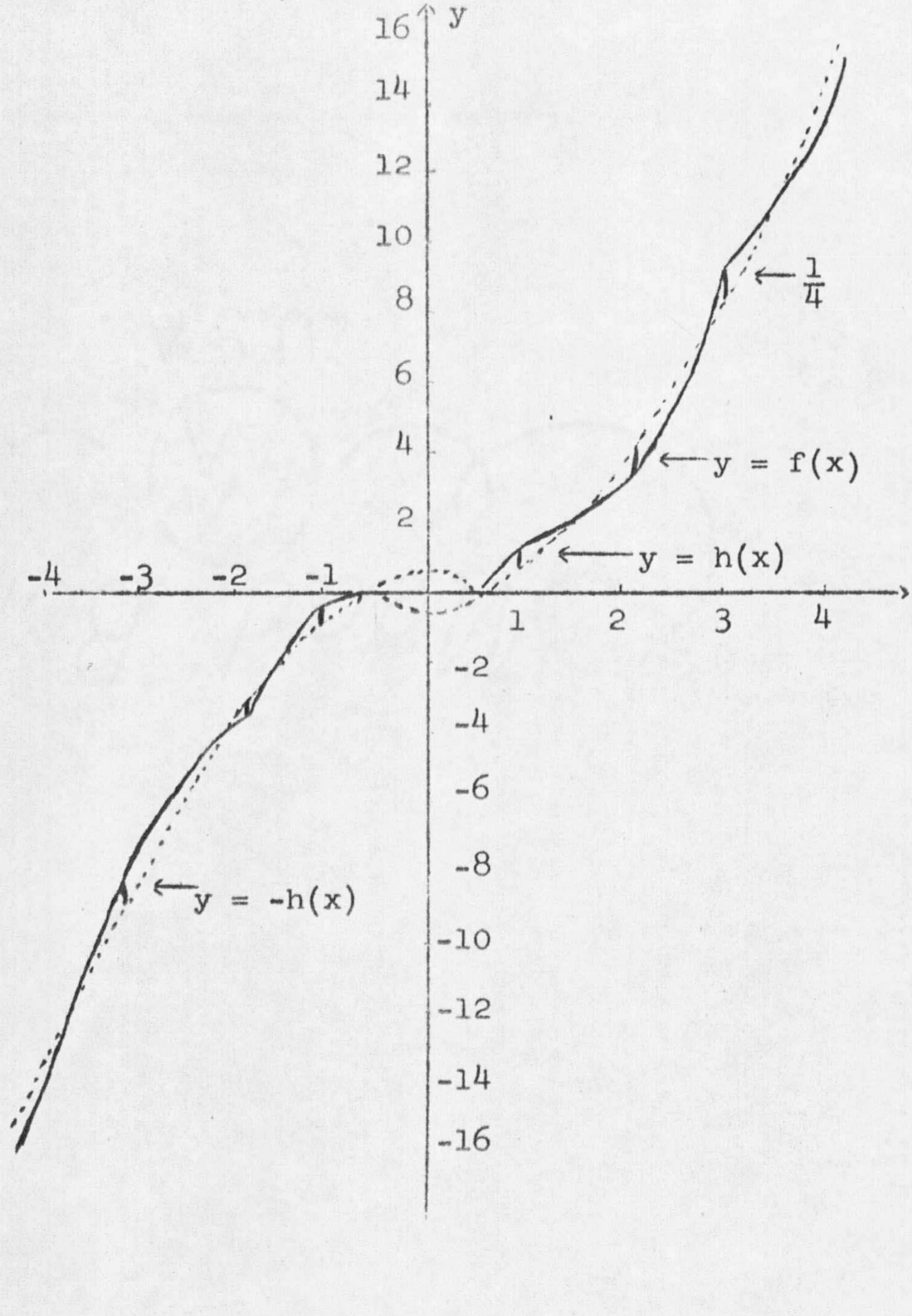


Figure 2. Illustration of a non-unique solution

This implies that for any join point $\delta \in [-4,4]$,

$$||f(x) - s_1(x)|| = \max_{x \in [-4,4]} |f(x) - s_1(x)| \geq \frac{1}{4}$$

Thus there are at least two optimum join points, and the best general spline approximation is not unique. Nevertheless, it is still true that once δ is determined, lemma 3 characterizes the best approximation on each of the subintervals.

6. Computational Procedure

When computing a best approximation on an interval, it is often desirable to replace that interval by a finite set of points and to seek an approximation which is optimum on that set. Because of the continuity of the function being approximated, it seems reasonable to expect that if this finite set of points leaves no wide gaps, then an approximation obtained in this way will be acceptable. The process of replacing the continuum by a discrete set is called discretization. It is shown in [4] that when the approximation family satisfies the Haar condition, then the approximation obtained by discretization does indeed approach the best approximation as the discrete set "fills up" the interval, i.e. as the length of each subinterval of the partition approaches zero. The family of functions $1,x,x^2,x^3,\ldots$ satisfies this Haar condition. Consequently, best general spline function approximations can be obtained approximately using discretization whenever the join points are known. Furthermore, the fact that $||f(x) - s_k(x)||$ is a continuous function of the join points insures that discretization can be used to find a near optimum solution even when the join points are arbitrary.

The number of possible choices for the join points becomes extremely large as the discrete set "fills up" the interval in such a way that no large gaps are left. For this reason an improved method for obtaining a near optimum set of join points would be very desirable. When computing general spline function approximations, it is helpful to notice that sometimes the norm is an increasing function of the lengths of the associated intervals. Unfortunately, however, this is not always the case. For example if the function being approximated is a general spline function, then the norm may decrease as the length of a subinterval is increased.

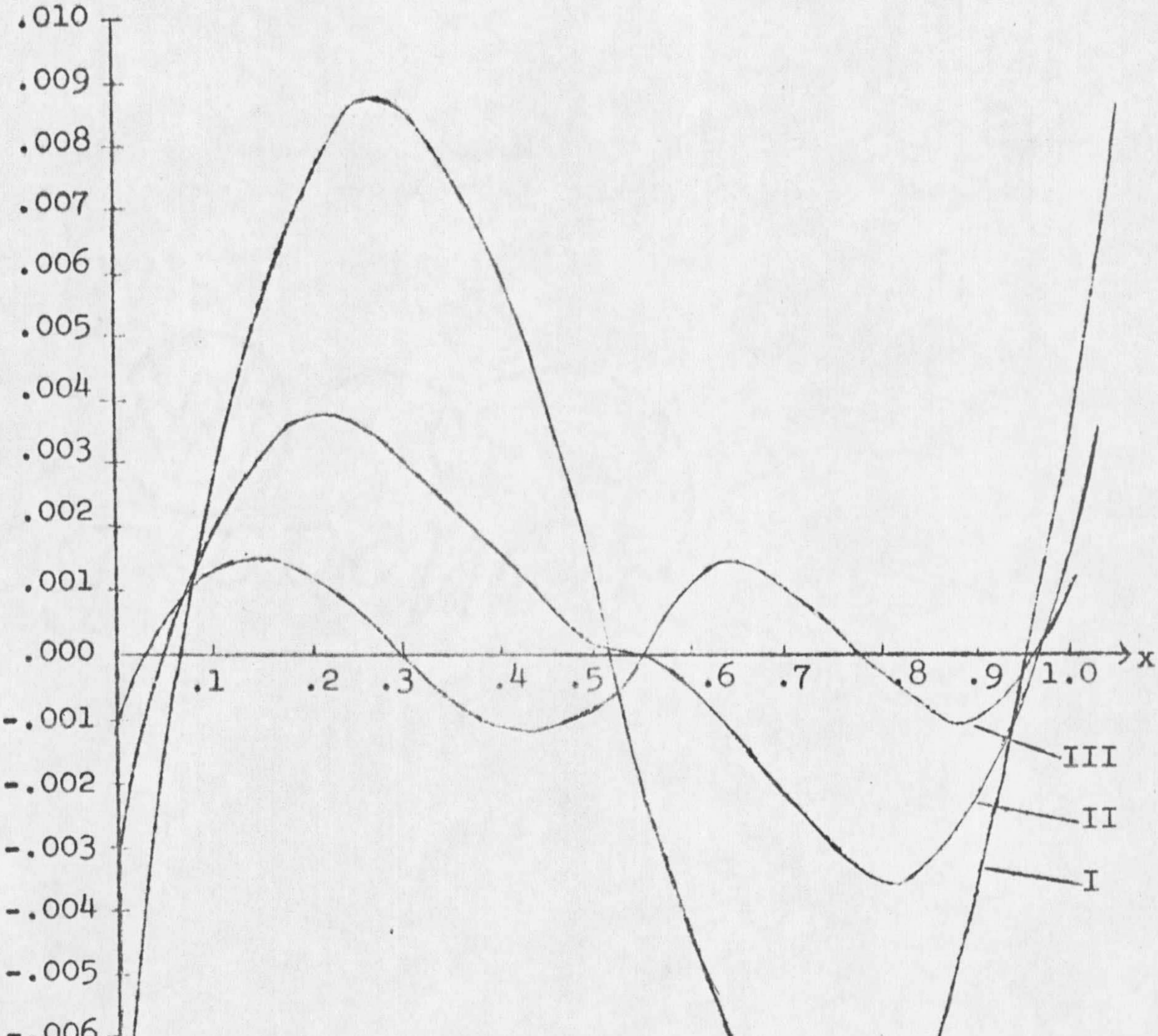
7. Examples

It is obvious from the definition of a general spline function that a general spline function approximation with no imposed boundary conditions is at least as good as a polynomial approximation if all of the polynomials used are of the same degree. The following examples show that even for the "smooth" function e^x, $x \in [0,1]$, the maximum error of the best fitting quadratic is about six times as large as the maximum error for the best fitting general spline function composed of polynomials of the same degree and having one join point $\delta = .53$. (See Table I.) Admittedly, it is true that a digital computer must test to determine whether x is larger or smaller than the join point in order to properly evaluate the general spline function, but this is a small price to pay for an answer with an error bound which is $\frac{1}{6}$ as large as that of the single quadratic. Furthermore, the second degree general spline function which is forced to fit e^x and its first derivative at $x = .53$ has an error bound which is less than half as large as the unrestricted single quadratic polynomial approximation.

The oscillation numbers associated with the examples in Table I indicate that the approximations obtained are nearly optimum for the given join points. However, the only join point which is nearly optimum is the join point $\delta = .53$. Figure 3 shows the graphs of three error curves. The graph labeled II shows a good fit of both e^x and its derivative for x near .53. This of course is expected in that the spline function associated with this error curve was made to agree exactly with both e^x and its derivative at the point $x = .53$.

Table I. General Spline Function Approximations of e^x on [0,1]

Description of the general spline function $s(x)$	Join Point(s) δ	Maximum error in absolute value by interval	Oscillation number(s) by interval	Coefficients of the general spline function		
				λ_{i0}	λ_{i1}	λ_{i2}
I. One quadratic.	none	.0089	4	1.008616,	0.855103,	0.845964
II. Two quadratics with $s(\delta) = e^{\delta}$ and $s(\delta) = e^{\delta}$.	.5	.0031	2	1.002900,	0.934583,	0.714117
		.0045	2	1.065232,	0.685171,	0.963529
"	.52	.0035	2	1.003400,	0.928001,	0.724999
		.0040	2	1.071351,	0.666653,	0.976295
"	.53	.0037	2	1.003612,	0.924829,	0.730256
		.0038	2	1.074616,	0.656889,	0.983029
"	.55	.0041	2	1.004021,	0.918690,	0.740564
		.0034	2	1.081293,	0.637689,	0.996010
III. Two quadratics with $s(x)$ continuous.	.55	.00145	3	1.001225,	0.960166,	0.673333
		.00145	3	1.120575,	0.523166,	1.073333
IV. Three quadratics with $s(.3) = e^{.3}$ and $s(.7) = e^{.7}$.	.3,.7	.0003	3	1.000185,	0.989571,	0.587142
		.0009	2	1.026275,	0.829751,	0.829999
		.0004	3	1.187701,	0.363142,	1.167142



I. $e^x - 1.008616 - 0.855103x - 0.845964x^2$, $0 \leqq x \leqq 1.0$

II. $e^x - 1.003612 - 0.924829x - 0.730256x^2$, $0 \leqq x \leqq .53$

$e^x - 1.074616 - 0.656889x - 0.983029x^2$, $.53 \leqq x \leqq 1.0$

III. $e^x - 1.001225 - 0.960166x - 0.673333x^2$, $0 \leqq x \leqq .55$

$e^x - 1.120575 - 0.523166x - 1.073333x^2$, $.55 \leqq x \leqq 1.0$

Figure 3. Error curves

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