An analytical approach to finite slope stability analysis
by William Arthur Vischer

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Civil Engineering
Montana State University

Abstract:
The development of an analytic approach to the finite slope stability problem is initialized in this paper. This method is similar to the "method of slices," in terms of the static analysis; however, exact integration is used for determining the actuating and resisting forces.

An equation was derived expressing the safety factor of a homogeneous, finite slope in terms of the slope geometry, geometry of a circular failure arc, and soil parameters. Safety factors obtained from the derived equation, were compared with those obtained by methods currently in use. Differences of up to five percent were noted in the comparison. The equation for the safety factor was then differentiated, with respect to the radius of the failure arc, in a futile attempt to derive an analytical expression for the radius that yields the minimum factor of safety for any given center. Results of the differentiated expression and the basic expression were compared. This comparison showed that when the differentiated expression was nearly satisfied, the center yielding the minimum safety factors was normally defined.

Further extensive studies are required before any definite conclusion can be made concerning the characteristics of the differentiated expression.

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Signature *William Arthur Visscher*

Date *December 5, 1969*

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WILLIAM ARTHUR VISCHER

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

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Approved:

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ABSTRACT

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An equation was derived expressing the safety factor of a homogeneous, finite slope in terms of the slope geometry, geometry of a circular failure arc, and soil parameters. Safety factors obtained from the derived equation, were compared with those obtained by methods currently in use. Differences of up to five percent were noted in the comparison. The equation for the safety factor was then differentiated, with respect to the radius of the failure arc, in a futile attempt to derive an analytical expression for the radius that yields the minimum factor of safety for any given center. Results of the differentiated expression and the basic expression were compared. This comparison showed that when the differentiated expression was nearly satisfied, the center yielding the minimum safety factors was normally defined.

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CHAPTER I

INTRODUCTION

Probably the most widely used method of analysis for finite slopes is the "method of slices" based on circular failure surfaces. According to Taylor (1948), K. E. Petterson is thought to be the first to use such a method in the study of a quay wall in 1915. Further investigations and studies revealed that actual failure surfaces do not deviate greatly from this assumed circular failure surface.

The method of slices basically consists of dividing the soil mass into vertical slices and performing a static analysis on the soil above the assumed failure surface. As there are many possible circular arcs for a given cross-section, a trial and error procedure must be used to locate both the location of the center of the critical arc and the radius of the critical arc. The time and labor involved in the graphical trial and error solutions is excessive. With the advent of the digital computer, the problems associated with this trial and error analysis were alleviated.

The above remarks indicate the benefit of a direct analytical analysis to the engineering profession. An analytical analysis would not only eliminate the repetitive calculations, but would also allow more efficient use of the computer in slope stability problems.

CHAPTER II

ANALYTICAL DEVELOPMENT

The initial portion of the derivation of the analytic expression is similar to the analysis used in the "method of slices;" the only difference being that exact integration is used to obtain the actuating and resisting forces. After performing the required exact integrations, the actuating and resisting forces can be expressed in terms of the radius, R, and the known geometric and soil parameters. The factor of safety can similarly be expressed as it is a function of these actuating and resisting forces. Differentiating the factor of safety with respect to the radius, R, and setting the resultant equal to zero, maximizes or minimizes the factor of safety with respect to the radius. Solving the expression for R should then yield the radius, for a given slope geometry and arc center, that produces the minimum factor of safety.

For a simple finite slope, three geometric failure possibilities, as shown in Figures 1, 2, and 3, must be investigated. The failure geometry shown in Figure 1 is probably the most common failure noted in soft, cohesive materials, whereas the failure geometry shown in Figure 2 is most common in "mixed" soils. The failure geometry of Figure 1 is the general case and degenerates to the cases shown in Figures 2 and 3 when the appropriate geometric substitutions are made.

Definition of Geometry

The initial step of the derivation consists of defining the intercepts of the slopes and failure arc.

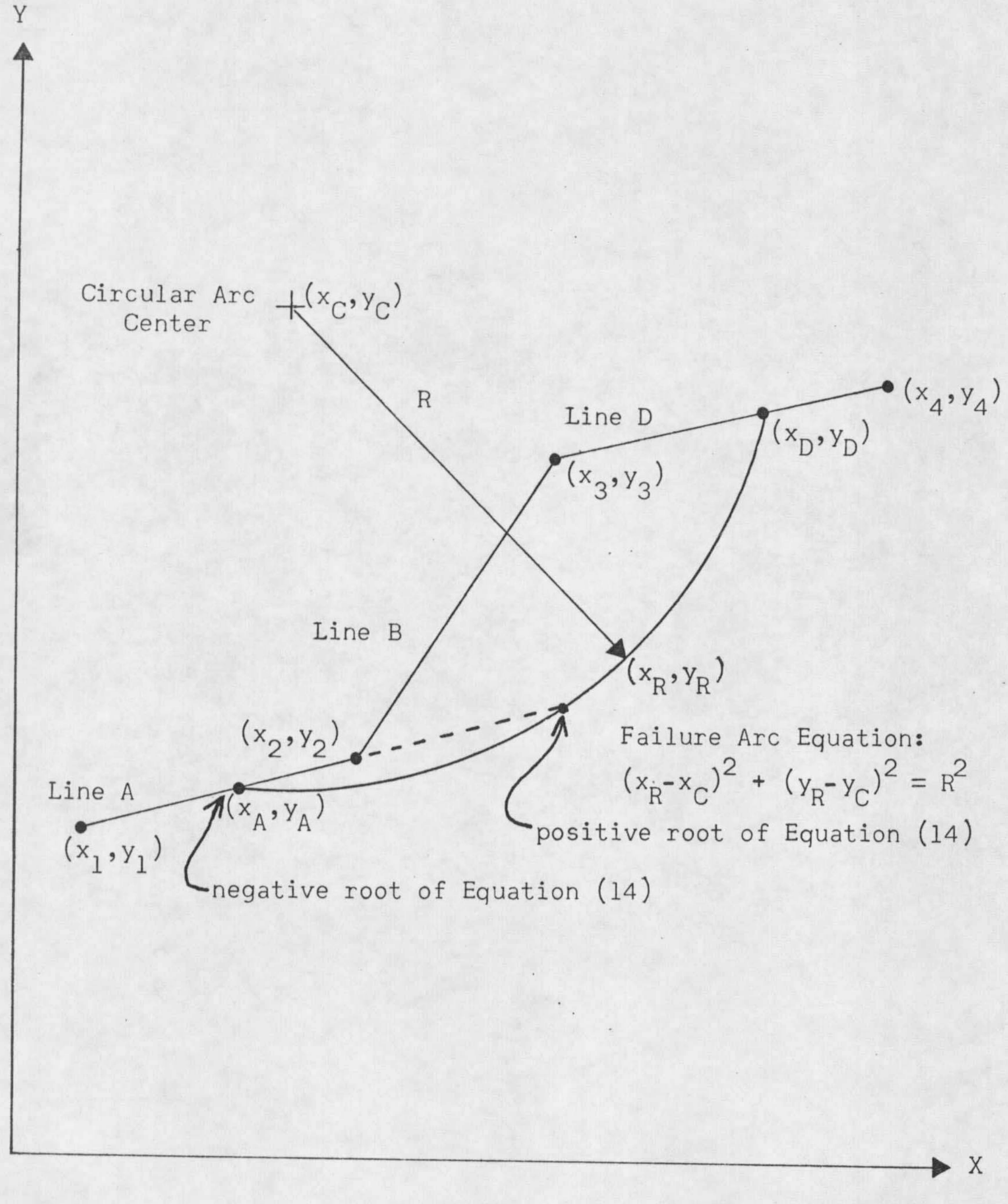


Figure 1. Possible failure geometry of a simple, finite slope, Case 1.

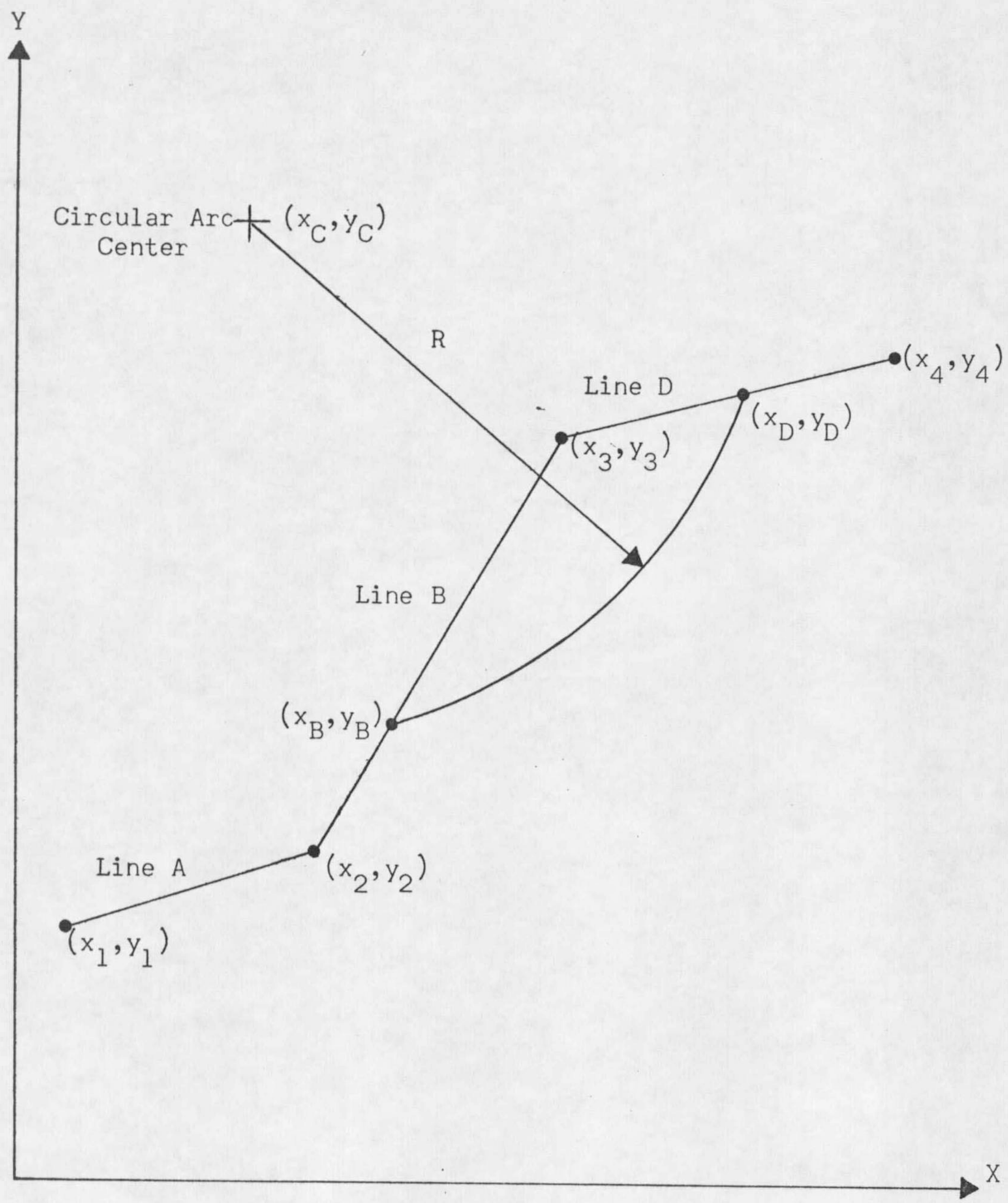


Figure 2. Possible failure geometry of a simple, finite slope, Case 2.

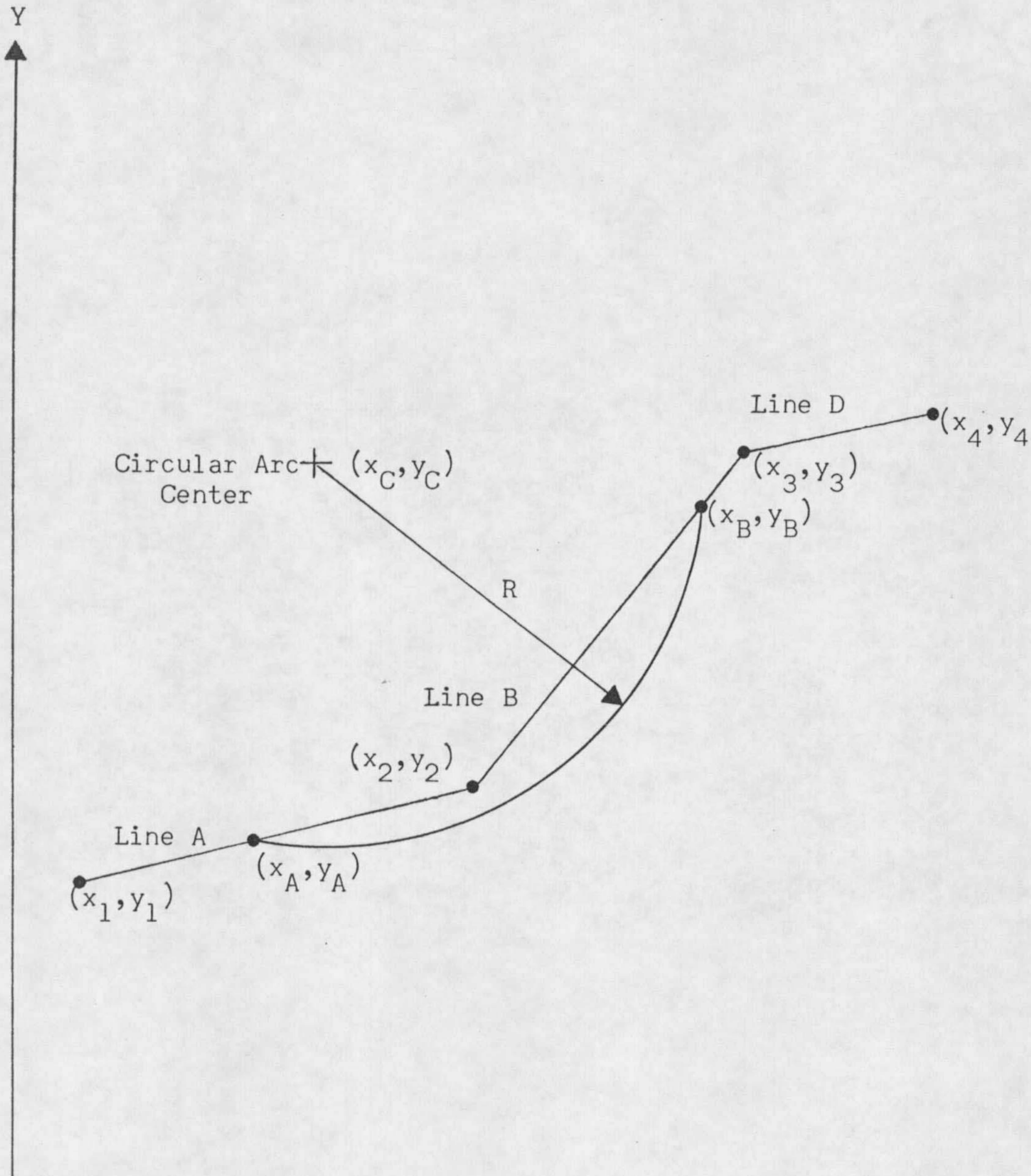


Figure 3. Possible failure geometry of a simple, finite slope, Case 3.

The equation of lines A, B, and D can be expressed in the general form:

$$y = mx + b \tag{1}$$

where: m is the slope of the line, and
b is the y intercept.

Letting:

$$e = \frac{y_2 - y_1}{x_2 - x_1} \tag{2}$$

$$g = \frac{y_3 - y_2}{x_3 - x_2} \tag{3}$$

$$i = \frac{y_4 - y_3}{x_4 - x_3} \tag{4}$$

$$f = \frac{y_1 x_2 - x_1 y_2}{x_2 - x_1} \tag{5}$$

$$h = \frac{y_2 x_3 - y_3 x_2}{x_3 - x_2} \text{ , and} \tag{6}$$

$$j = \frac{y_3 x_4 - y_4 x_3}{x_4 - x_3} \tag{7}$$

and substituting yields:

$$y_A = ex_A + f \tag{8}$$

$$y_B = gx_B + h \tag{9}$$

$$y_D = ix_D + j \tag{10}$$

The general equation for the assumed circular failure arc is:

$$r^2 = (x - h)^2 + (y - k)^2 \tag{11}$$

where: h and k are the respective x and y values denoting the center of the circle, and r is the radius.

Substituting the appropriate variables, as shown in Figure 1, into Equation (11) yields:

$$R^2 = (x_R - x_C)^2 + (y_R - y_C)^2 \tag{12}$$

Solving for y_R in Equation (12) gives:

$$y_R = y_C \pm \sqrt{R^2 - (x_R - x_C)^2} \tag{13}$$

Respectively setting Equations (8), (9), and (10) equal to Equation (13) the expressions for x_A, x_B, and x_D are:

$$x_A = \frac{ey_C - ef + x_C - \sqrt{-2efx_C + 2ey_Cx_C + e^2R^2 - e^2x_C^2 - f^2 + 2fy_C - y_C^2 + R^2}}{e^2 + 1} \tag{14}$$

$$x_B = \frac{gy_C - gh + x_C \pm \sqrt{-2ghx_C + 2gy_Cx_C + g^2R^2 - g^2x_C^2 - h^2 + 2hy_C - y_C^2 + R^2}}{g^2 + 1} \quad (15)$$

$$x_D = \frac{iy_C - ij + x_C + \sqrt{-2ijx_C + 2iy_Cx_C + iR^2 - i^2x_C^2 - j^2 + 2jy_C - y_C^2 + R^2}}{i^2 + 1} \quad (16)$$

The solution of Equation (14) involves a quadratic with the negative root being rational (see Figure 1). The solution of Equation (15) can involve either the negative or positive root depending on the failure geometry, while for Equation (16), the positive root is rational.

<u>Computation of Forces</u>

The stability of the slope is dependent on the resisting moment (made up of the resisting forces acting about the assumed center of rotation) being greater than the actuating moment.

From elementary soil mechanics, the actuating force, F_{act}, can be expressed in terms of that component of the weight force acting tangentially to the assumed failure arc, as shown in Figure 4:

$$F_{act} = \sum \text{Tangential Components of Weight Force} = \sum T \quad (17)$$

Figure 4 is a profile of a lineal slope having an infinite length normal to the cross-section shown. Assuming a unit length of slope, the volume of the differential parallelepiped shown in Figure 4 is:

$$dV = (y - y_R)dx \quad (18)$$

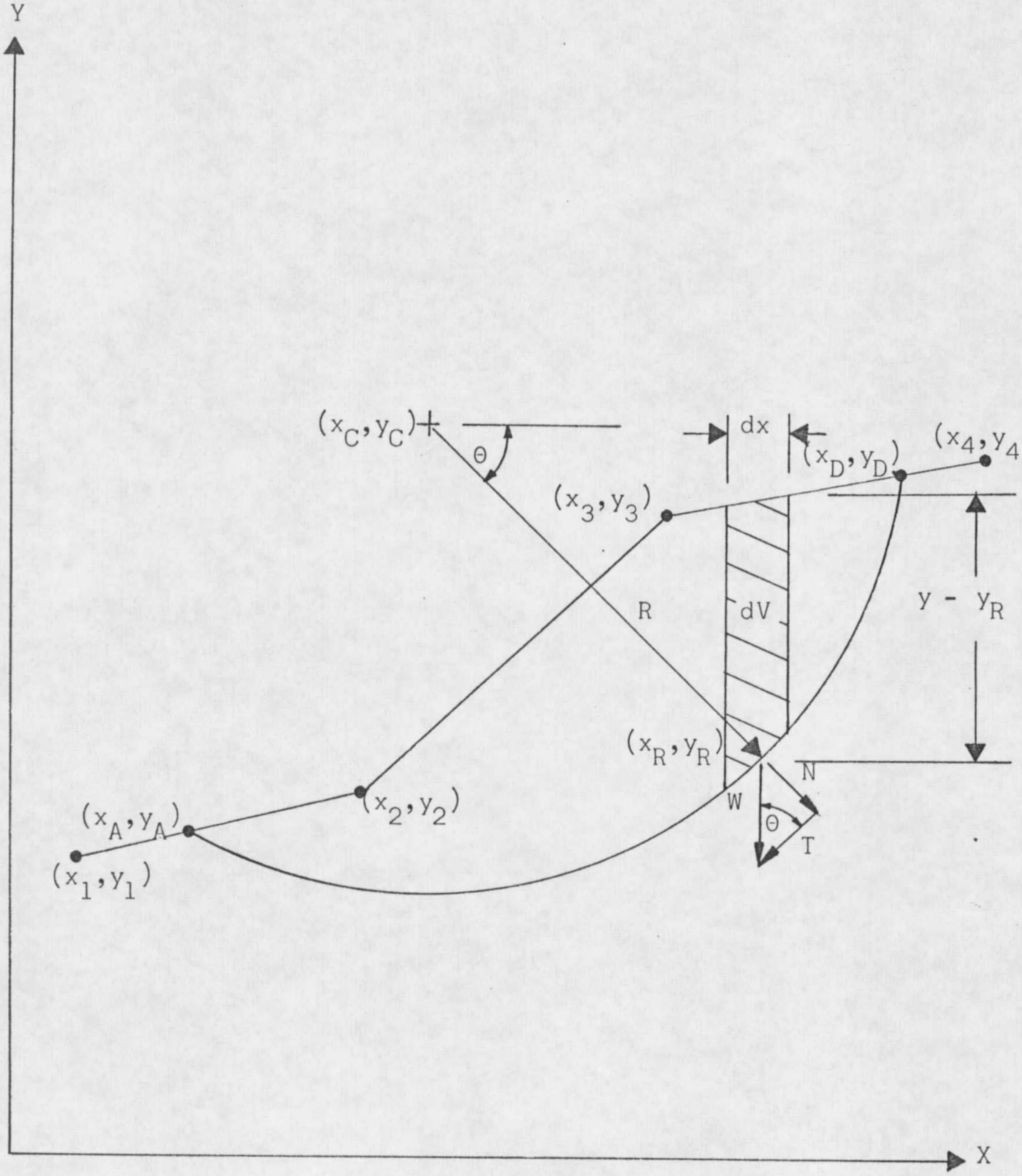


Figure 4. Diagram of forces and geometry.

The differential weight, dW, of the differential volume is equal to the volume of soil multiplied by the unit weight of soil, γ , or:

$$dW = \gamma(y - y_R)dx \tag{19}$$

Multiplying dW by cos θ (see Figure 4) yields the tangential component of the weight, or:

$$dT = \gamma(y - y_R)\cos\theta dx \tag{20}$$

and from Figure 4:

$$\cos\theta = \frac{x_R - x_C}{R} \tag{21}$$

Converting Equation (17) to its integral form and substituting Equation (20) yields:

$$F_{act} = \int_{x_A}^{x_2} \gamma(y_A - y_R)\cos\theta dx + \int_{x_2}^{x_3} \gamma(y_B - y_R)\cos\theta dx + \int_{x_3}^{x_D} \gamma(y_D - y_R)\cos\theta dx \tag{22}$$

Substituting Equations (8), (9), (10), (13), and (21) into Equation (22), and simplifying yields:

$$F_{act} = A'\left[\int_{x_A}^{x_2} (B' + C' + D' + E')dx + \int_{x_2}^{x_3} (F' + G' + H' - I')dx + \int_{x_3}^{x_D} (J' + K' + L' - M')dx\right] \quad (23)$$

where: $A' = \frac{\gamma}{R}$;

$B' = ex^2 + fx - y_C x$;

$C' = G' = K' = x(-x^2 + 2x_C x + R^2 - x_C{}^2)^{\frac{1}{2}}$;

$D' = -ex_C x - fx_C + y_C x_C$;

$E' = I' = M' = x_C(-x^2 + 2x_C x + R^2 - x_C{}^2)^{\frac{1}{2}}$;

$F' = gx^2 + hx - y_C x$;

$H' = -gx_C x - hx_C + y_C x_C$;

$J' = ix^2 + jx - y_C x$; and

$L' = ix_C x - jx_C + y_C x_C$

Note that the smallest root of Equation (13) was used as it is applicable as shown in Figure 1. Integrating Equation (23) produces:

$$F_{act} = A'(B'' + C'' + D'' + E'' + F'' + G'' + H'' - I'' + J'' + K'' + L'' - M'') \quad (24)$$

where:
$$B'' = \int_{x_A}^{x_2} B'dx = (\tfrac{1}{3})e(x_2^{\,3} - x_A^{\,3})$$

$$+ (\tfrac{1}{2})(f - y_C)(x_2^{\,2} - x_A^{\,2}) \quad ; \quad (25)$$

$$C'' = \int_{x_A}^{x_2} C'dx = (-\tfrac{1}{3})(-x_2^{\,2} + 2x_C x_2 + R^2 - x_C)^{\frac{3}{2}}$$

$$+(\tfrac{1}{3})(-x_A^{\,2} + 2x_C x_A + R^2 - x_C^{\,2})^{\frac{3}{2}}$$

$$- (\tfrac{1}{2})(x_C)(x_C - x_2)(-x_2^{\,2} + 2x_C x_2 + R^2 - x_C^{\,2})^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(x_C)(x_C - x_A)(-x_A^{\,2} + 2x_C x_A + R^2 - x_C^{\,2})^{\frac{1}{2}}$$

$$- (\tfrac{1}{2})(x_C R^2)(\sin^{-1} \frac{x_C - x_2}{R}) + (\tfrac{1}{2})(x_C R^2)(\sin^{-1} \frac{x_C - x_A}{R}) \; ; \quad (26)$$

$$D'' = \int_{x_A}^{x_2} D'dx = (-\tfrac{1}{2})(ex_C)(x_2^{\,2} - x_A^{\,2}) + (y_C x_C - f x_C)(x_2 - x_A); \quad (27)$$

$$E'' = \int_{x_A}^{x_2} E'dx = (\tfrac{1}{2})(x_C)(x_2 - x_C)(-x_2{}^2 + 2x_C x_2 + R^2 - x_C{}^2)^{\frac{1}{2}}$$

$$-(\tfrac{1}{2})(x_C)(x_A - x_C)(-x_A{}^2 + 2x_C x_A + R^2 - x_C{}^2)^{\frac{1}{2}}$$

$$-(\tfrac{1}{2})(x_C R^2)(\sin^{-1} \frac{x_C - x_2}{R}) + (\tfrac{1}{2})(x_C R^2)(\sin^{-1} \frac{x_C - x_A}{R}) \quad ; \qquad (28)$$

$F'' = \int_{x_2}^{x_3} F'dx =$ (same form as Equation (25) with g, h, x_3 and x_2 replacing e, f, x_2, and x_A respectively); (29)

$G'' = \int_{x_2}^{x_3} G'dx =$ (same form as Equation (26) with x_2 and x_3 replacing x_A and x_2 respectively); (30)

$H'' = \int_{x_2}^{x_3} H'dx =$ (same form as Equation (27) with g, h, x_2 and x_3 replacing e, f, x_A, and x_2 respectively); (31)

$I'' = \int_{x_2}^{x_3} I'dx =$ (same form as Equation (28) with x_2 and x_3 replacing x_A and x_2 respectively); (32)

$$J'' = \int_{x_3}^{x_D} J'dx = \text{(same form as Equation (25) with } i,\ j,\ x_3 \text{ and } x_D \text{ replacing } e,\ f,\ x_A \text{ and } x_2 \text{ respectively);} \qquad (33)$$

$$K'' = \int_{x_3}^{x_D} K'dx = \text{(same form as Equation (26) with } x_3 \text{ and } x_D \text{ replacing } x_A \text{ and } x_2 \text{ respectively);} \qquad (34)$$

$$L'' = \int_{x_3}^{x_D} L'dx = \text{(same form as Equation (27) with } i,\ j,\ x_3 \text{ and } x_D \text{ replacing } e,\ f,\ x_A \text{ and } x_2 \text{ respectively);} \qquad (35)$$

$$M'' = \int_{x_3}^{x_D} M'dx = \text{(same form as Equation (28) with } x_3 \text{ and } x_D \text{ replacing } x_A \text{ and } x_2 \text{ respectively).} \qquad (36)$$

Substituting the above values into Equation (24) and reducing, yields the following expression for the activating force:

$$F_{act} = \frac{\gamma}{R}\Big[\left(\frac{-e}{3}\right)x_A^{3} + \left(\frac{-f+y_C+ex_C}{2}\right)x_A^{2} +$$

$$(f-y_C)x_Cx_A + \left(\frac{1}{3}\right)\left(-x_A^{2}+2x_Cx_A+R^2-x_C^{2}\right)^{\frac{3}{2}} +$$

$$\left(\frac{e-g}{3}\right)x_2^{3} + \left(\frac{f-ex_C-h+gx_C}{2}\right)x_2^{2} + (h-f)x_Cx_2$$

$$+\left(\frac{g-i}{3}\right)x_3^{\,3} + \left(\frac{h-gx_C-j+ix_C}{2}\right)x_3^{\,2} + (j-h)x_Cx_3 + \left(\frac{i}{3}\right)x_D^{\,3}$$

$$+\left(\frac{j-y_C-ix_C}{2}\right)x_D^{\,2} + (y_C-j)x_Cx_D - \left(\frac{1}{3}\right)(-x_D^{\,2}+2x_Cx_D+R^2-x_C^{\,2})^{\frac{3}{2}}\Big] \quad (37)$$

The strength of the soil material, which resists any impending rotation or failure, can be expressed by Coulomb's Law:

$$S = \sigma \tan\phi + c \quad (38)$$

where: S = shear strength,
σ = normal effective stress,
ϕ = angle of internal friction, and
c = cohesion.

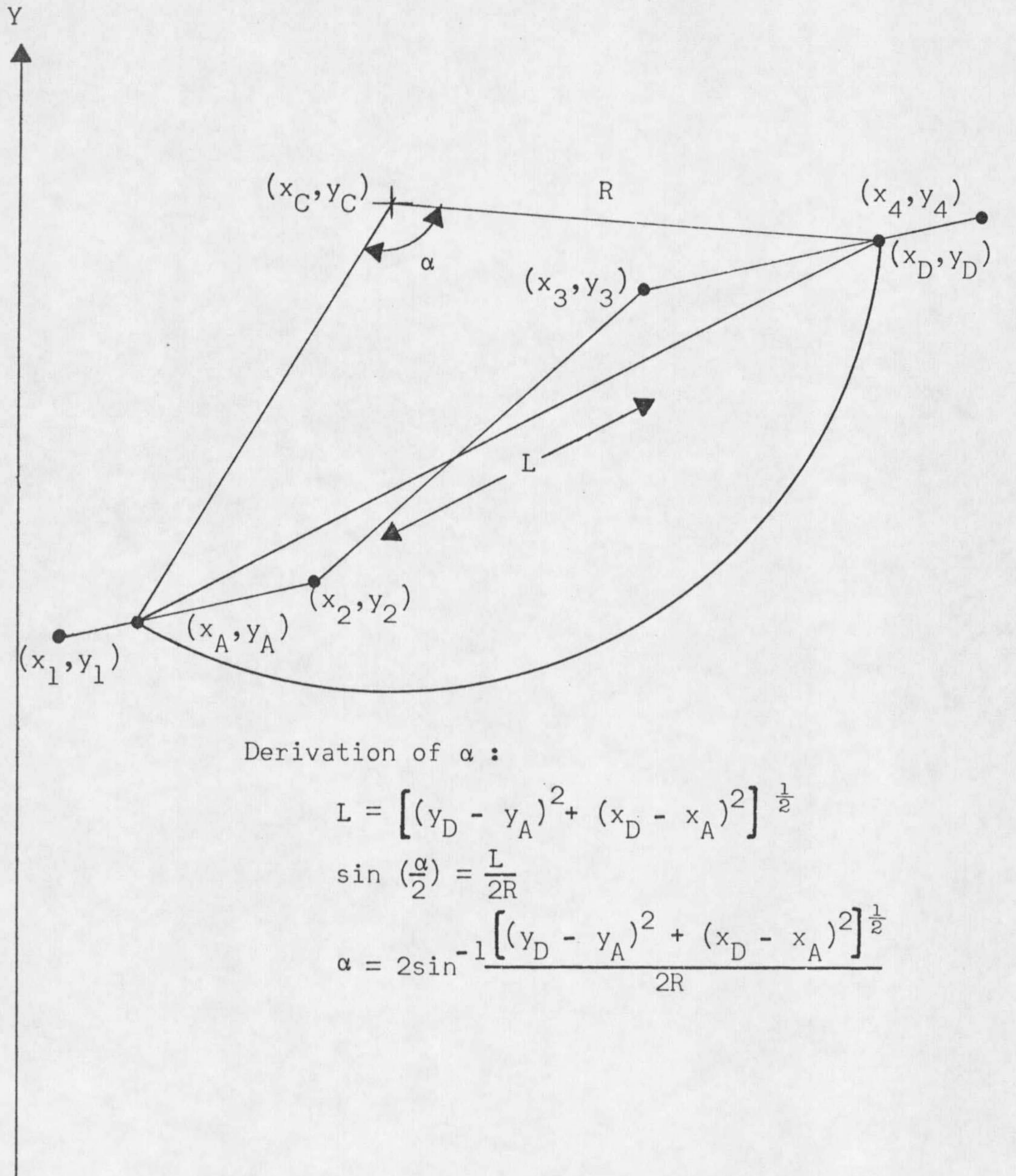
The resisting force, F_{res}, per unit length of the lineal slope (Taylor, 1948, p. 437), is expressed as:

$$F_{res} = \sum N \tan\phi + R\alpha c \quad (39)$$

where: N = normal component of differential weight (see Figure 4),
R = arc radius, and
α = central angle (see Figure 5).

Once again using the differential expression for the slice weight:

$$dW = \gamma(y - y_R)dx \quad (40)$$



Derivation of α :

$$L = \left[(y_D - y_A)^2 + (x_D - x_A)^2\right]^{\frac{1}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \frac{L}{2R}$$

$$\alpha = 2\sin^{-1}\frac{\left[(y_D - y_A)^2 + (x_D - x_A)^2\right]^{\frac{1}{2}}}{2R}$$

Figure 5. Derivation of alpha angle.

and multiplying by $\sin\theta$, yields the normal component of the differential weight:

$$dN = \gamma(y - y_R)\sin\theta dx \tag{41}$$

From Figure 4:

$$\sin\theta = \frac{y_C - y_R}{R} \tag{42}$$

Expressing Equation (39) in its integral form and substituting the value of dN from Equation (41) yields:

$$F_{res} = \tan \emptyset \Big[\int_{x_A}^{x_2} \gamma(y_A - y_R)\sin\theta \, dx + \int_{x_2}^{x_3} \gamma(y_B - y_R)\sin\theta \, dx$$

$$+ \int_{x_3}^{x_D} \gamma(y_D - y_R)\sin\theta \, dx \Big] + R\alpha c \tag{43}$$

Substituting Equations (8), (9), (10), (13) and (42) into Equation (43), integrating, and simplifying yields:

$$F_{res} = \frac{\gamma\tan\emptyset}{R}\Big[\frac{x_A^{\,3}}{3} - x_C x_A^{\,2} + \frac{e}{3} x_A^{\frac{3}{2}} + (x_C^{\,2} - R^2)x_A$$