Gravitational and electromagnetic potentials of the stationary Einstein-Maxwell field equations
by Thaddeus Charles Jones

A thesis submitted in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Physics
Montana State University

Abstract:
Associated with the stationary Einstein-Maxwell field equations is an infinite hierarchy of potentials.

The basic characteristics of these potentials are examined in general and then in greater detail for the particular case of the Reissner-Nordstrom metric. Their essential utility in the process of solution generation is elucidated and the necessary equations for solution generation are developed.

Appropriate generating functions, which contain the complete infinite hierarchy of potentials, are developed and analyzed. Particular attention is paid to the inherent gauge freedom of these generating functions.

Two methods of solution generation, which yield asymptotically flat solutions in vacuum, are generalized to include electromagnetism. One method, using potentials consistent with the Harrison transformation and the Reissner-Nordstrom metric, is discussed in detail and its resultant difficulties explored.

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ABSTRACT

Associated with the stationary Einstein-Maxwell field equations is an infinite hierarchy of potentials. The basic characteristics of these potentials are examined in general and then in greater detail for the particular case of the Reissner-Nordstrom metric. Their essential utility in the process of solution generation is elucidated and the necessary equations for solution generation are developed.

Appropriate generating functions, which contain the complete infinite hierarchy of potentials, are developed and analyzed. Particular attention is paid to the inherent gauge freedom of these generating functions.

Two methods of solution generation, which yield asymptotically flat solutions in vacuum, are generalized to include electromagnetism. One method, using potentials consistent with the Harrison transformation and the Reissner-Nordstrom metric, is discussed in detail and its resultant difficulties explored.

CHAPTER 1

INTRODUCTION

Historical Background

We are fortunate to be alive at a time when, during the Einstein centennial, many of the more exotic predictions of General Relativity appear to be coming into the arena of experimental verification, e.g., black holes and gravity waves. Unfortunately, in order to fully describe such phenomena, linearized equations will not suffice, and the full Einstein equations must be solved. As approximation methods are now ruled out, we are left with numerical solutions, which certainly have their own difficulties in addition to lacking aesthetic appeal, or exact solutions.

In spite of the notorious difficulties posed by Einstein's non-linear, coupled, partial differential equations, a considerable number of exact solutions have been obtained. Unluckily, the majority of these are essentially mathematical curiosities with no relevance to the physical universe. Moreover, progress in attaining solutions of astrophysical significance has, until recently, been quite infrequent.

Initially there was impressive progress. In 1916 Einstein derived his field equations, and in that same year

Schwarzschild[1] discovered an exact solution for the gravitational field of a spherically symmetric mass distribution. The next year, 1917, Weyl[2] obtained the general form of the time independent metric with axial symmetry. He showed that the field equations could be solved to obtain the gravitational field around any static, axially symmetric, mass distribution. The next solution, the Kerr metric,[3] was obtained fourty-six years later, in 1963. This is the simplest case of a rotating black hole. Ten years after that, in 1973, Tomimatsu and Sato[4] discovered a family of solutions, of which the Kerr solution is the first member, representing the gravitational field of axially symmetric spinning masses. These solutions may be viewed as a class of Weyl solutions generalized to include rotation.

Since the method of a direct frontal attack on Einstein's equations had been so desultory, attention has shifted recently to the problem of generating solutions from those previously known. [Ehlers,[5] in 1957, was an early investigator in this area.] A surprising number of solutions have been obtained in this fashion, but once again the majority lack astrophysical significance. Furthermore, one had no way to predict beforehand if the nature of a new solution might be of physical interest. Nevertheless, work went ahead on attempting to discover the largest class of transformations which allow one to generate new solutions from old. Geroch,[6] in 1971, was able to discover an infinite-dimensional symmetry group which acted on an infinite hierarchy of potentials and generated an infinite number of conservation laws. Although he was unable to obtain a finite form for these transformations, he postulated that one solution might prove sufficient to generate all the stationary, axially symmetric, vacuum metrics. Finally, Kinnersley et al.,[7,8,9,10,11] in a major tour de force, have recently developed methods which will give the finite form of new solutions, but, more importantly, these new solutions are guaranteed to possess the prime attribute of being asymptotically flat. Moreover, they conjecture that these methods, if applied to the general static metric, can be used to generate all stationary, axisymmetric, asymptotically flat metrics.

It is instructive to note that this important procedure was developed via an examination of the combined Maxwell-Einstein equations. Historically, Einstein was able to discover the real nature of spacetime by realizing that a choice had to be made between Maxwell's equations and

Newton's ideas of spacetime. He decided that Newton was incorrect and set out to discover a spacetime compatible with Maxwell's equations. It should not be too surprising then that, when Maxwell's equations are coupled to Einstein's, a wealth of additional insight is provided.

Simplicity and Physics

One could take the viewpoint that the key to the finding of exact solutions lay in a concept that all physicists deeply believe in, i.e., simplicity. We feel that an underlying structure of simplicity is a key to much of physics. How can this be true? Nature appears to be exceedingly complex. To put the matter into perspective, it is instructive to examine the game of chess. The number of possible board positions has been estimated to be 10^{43} with 10^{125} ways to reach them. Yet a child of five can easily learn to play chess. This is because each piece always moves according to a simple set of rules, no matter how the situation may vary. Physicists are convinced of the existence of such underlying simplicities. Another way to state the matter would be to say we believe that all events have various unifying features such as conservation of momentum or energy. Not all the sciences share this belief equally. Biology, for instance, seems to dwell much more on the diversity of life rather than its common features. In fact, in the 1930's and 1940's a group of physicists went into microbiology and garnered a number of Nobel prizes because of their basic beliefs.

Thus far our discussion has involved more of an aesthetic content as opposed to an idea with direct mathematical implications. Therefore, it is appropriate to try to couch this rather vague concept in the language of mathematics. It is well known that the discovery of various symmetries in nature has been instrumental in reinforcing the essential belief in the simplicity and unity of the universe. Accordingly, one might say that our requirements for simplicity could be couched in the language of mathematics by acknowledgment of the symmetries or invariance properties of nature. Thus, when it was determined that the inclusion of the Maxwell equations into the stationary Einstein equations not only maintained the original symmetry group, but actually expanded it, one had to sense that this could not be merely fortuitous.

All of this may be very fine, but how are these ideas relevant to the problem at hand, namely, solution generation? The basic message of the preceeding was, "Use simplicity to overcome complexity." In the more specific language of physicists one could reword this to say: "Apply the invariance properties of Einstein's equations to known solutions in order to generate new solutions." It is clear that before one could initiate this process the invariance properties had to be discovered. Some of these properties were obvious, e.g., coordinate transformations. Then certain potentials were defined using the field equations and subsequently used to rewrite the field equations. At this juncture an unexpected "internal" invariance group revealed itself. The amalgamation of this surprising "internal" group with the "external" group of coordinate and gauge conditions gives the entire symmetry group discovered by Geroch. The inclusion of an electromagnetic field enlarges this group, and its representation includes an infinite hierarchy of potentials. These potentials were not only essential to the discovery of new invariance properties but, moreover, turn out to be a sine qua non of the solution generating method. In this thesis we will attempt to discuss the combined electromagnetic and gravitational potential hierarchy in general and then in greater detail for particular cases. Moreover, we will discover the particular modifications which must be put into effect when electromagnetism is included into the scheme of solution generation.

CHAPTER 2

STATIONARY, AXIALLY SYMMETRIC SPACETIMES

The Metric

In order to motivate the particular notation that we will be using, a brief description of the general form of the metric for stationary, axially symmetric spacetimes is necessary.

What is the physical situation we wish to consider? We have in mind the spacetime external to the body actually producing the associated gravitational and electromagnetic fields. We do not possess nor do we require any information concerning the equation of state for the internal composition of the body. The mass must possess axial symmetry but also may exhibit two types of differential rotation. A point near the equator on the surface may move at a different angular velocity than a point near the poles. In addition, a point at a distance ρ from the axis of rotation may have a different angular velocity than a point at a distance ρ' from the axis of rotation. However, no pulsations or mass distributions which violate the conditions of axial symmetry and reflection symmetry will be permitted.

Given that the fields under consideration are not arbitrary but possess stationarity, axial symmetry, and motion reversal, how may we incorporate this physical information into the explicit mathematics of the metric? Axial symmetry means that the field doesn't have any angular ϕ dependence. "Stationarity" is equivalent to the existence of a time coordinate for which $g_{\alpha\beta,t}=0$. Motion reversal implies that $(t,\phi) \leftrightarrow (-t,-\phi)$, i.e., the situation is unchanged if both the time coordinate and the ϕ angular coordinate are reversed. Thus, we have that $g_{\alpha\beta,\phi}=g_{\alpha\beta,t}=0$ and motion reversal implies $g_{ti} = g_{\phi i} = 0$. g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$ are permitted to be nonzero. Another way to view this situation is to note that the metric for an axially symmetric rotating body admits two commuting, orthogonally transitive, Killing vectors.

By transformation of the remaining two spacelike coordinates among themselves, the line element can always be reduced to block diagonal form.

$$ds^2 = ds_1{}^2 - ds_2{}^2$$

$$ds_1{}^2 = f_{AB}dX^A dX^B \qquad A,B = 1,2 \qquad (1.1)$$

$$ds_2{}^2 = e^{2\Gamma}\delta_{MN}dX^M dX^N = h_{MN}dX^M dX^N \qquad M,N = 3,4$$

Specifically, the canonical form introduced by Lewis[12] is

$$ds^2 = f(dt-\omega d\phi)^2 - f^{-1}[e^{2\gamma}(d\rho^2+dZ^2)+\rho^2 d\phi^2] \,, \qquad (1.2)$$

thus

$$f_{AB} = \begin{pmatrix} f & -f\omega \\ -f\omega & f^{-1}\rho^2-\omega^2 f \end{pmatrix} . \qquad (1.3)$$

<u>Notation</u>

Notation which combines clarity with ease of useage is always to be valued. Accordingly, we will discuss the somewhat nonstandard form used here that takes full advantage of the particular type of metric under consideration.

Normally the metric tensor $g_{\mu\nu}$ is used to raise and lower indices, but, as we have our metric broken down into two two-dimensional spaces, there are more alternatives.

$$g_{\mu\nu} = \begin{pmatrix} f_{AB} & 0 \\ 0 & h_{MN} \end{pmatrix} \qquad (2.1)$$

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} (f_{AB})^{-1} & 0 \\ 0 & (h_{MN})^{-1} \end{pmatrix} . \qquad (2.2)$$

In the two-dimensional space (t,ϕ) we may employ either the inverse metric $(f_{AB})^{-1}$ or the alternating symbol

$$\varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (2.3)$$

to raise indices. We choose ε^{AB} for simplicity. Contracted indices in the (t,ϕ) space will be denoted by X, Y, or Z.

A few examples are in order so as to give an explicit demonstration of the action of ε^{AB}.

We have

$$\varepsilon^{AX}\varepsilon_{XC} = -\delta^{A}{}_{C} \tag{2.4}$$

$$V^{A} = \varepsilon^{AX}V_{X} \tag{2.5}$$

where V^A is a vector in the (t,ϕ) space.

Furthermore,

$$V_{C} = V^{X}\varepsilon_{XC} \tag{2.6}$$

$$-V_{C} = \varepsilon_{CX}V^{X}. \tag{2.7}$$

Also

$$V_{X}W^{X} = V_{X}\varepsilon^{XZ}W_{Z} = \varepsilon^{XZ}V_{X}W_{Z}, \tag{2.8}$$

but

$$V^{X}W_{X} = \varepsilon^{XZ}V_{Z}W_{X} = -\varepsilon^{ZX}V_{Z}W_{X} = -\varepsilon^{XZ}V_{X}W_{Z} \tag{2.9}$$

where dummy indices were exchanged in the last step in Equation (2.9).

Therefore,

$$V_{X}W^{X} = -V^{X}W_{X} \tag{2.10}$$

so

$$V_{X}V^{X} = 0. \tag{2.11}$$

In the (X^3,X^4) space we have h^{MN} to raise indices. When it is necessary to use $(f_{AB})^{-1}$ or ε^{MN} instead, that index shall be marked with a tilde.

So

$$(f_{AB})^{-1} = -\rho^{-2}\varepsilon^{AX}\varepsilon^{BZ}f_{XZ} = f^{\tilde{A}\tilde{B}} \tag{2.12}$$

where

$$-\rho^2 \equiv \det(f_{AB}). \tag{2.13}$$

Then

$$f^{AB} = -\rho^2 f^{\tilde{A}\tilde{B}} \tag{2.14}$$

and

$$g^{\mu\nu} = \begin{bmatrix} f^{\tilde{A}\tilde{B}} & 0 \\ 0 & h^{MN} \end{bmatrix}. \tag{2.15}$$

What about derivatives?

Normally,

$$\vec{\nabla}\cdot\vec{V} = (-g)^{-\frac{1}{2}}((-g)^{\frac{1}{2}}g^{\mu\nu}V_{\mu})_{,\nu} \tag{2.16}$$

where

$$g \equiv \det(g_{\mu\nu}).$$

Clearly,

$$\vec{\nabla}_2\cdot\vec{V} = h^{-\frac{1}{2}}(h^{\frac{1}{2}}h^{MN}V_M)_{,N} \tag{2.17}$$

would then be associated with $ds_2{}^2$. But Equation (2.17) is two-dimensional while Equation (2.16) is three-dimensional. How may we express a three-dimensional derivative in our notation?

Let

$$ds_3{}^2 = H_{RS}dX^R dX^S \qquad R,S = 2,3,4$$
$$= ds_2{}^2 + \rho^2 d\phi^2 \tag{2.18}$$

then

$$H_{RS} = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\left[h_{MN}\right]$}} \\ 0 & & \end{pmatrix}. \tag{2.19}$$

Thus

$$\vec{\nabla}_3 \cdot \vec{U} = H^{-\frac{1}{2}}(H^{\frac{1}{2}}H^{RS}U_R)_{,S} \tag{2.20}$$

where

$$H \equiv \det(H_{RS}) = \rho^2 \det(h_{MN}) = \rho^2 h. \tag{2.21}$$

Let

$$\vec{U} = (U_2, V_3, V_4) \equiv (U_2, V). \tag{2.22}$$

This implies

$$\vec{\nabla}_3 \cdot \vec{U} = \rho^{-1}h^{-\frac{1}{2}}(\rho h^{\frac{1}{2}}H^{RS}U_R)_{,S}$$
$$= \rho^{-1}h^{-\frac{1}{2}}[\ (\rho h^{\frac{1}{2}}H^{2S}U_2)_{,S} + (\rho h^{\frac{1}{2}}H^{3S}V_3)_{,S} + (\rho h^{\frac{1}{2}}H^{4S}V_4)_{,S}\ ]. \tag{2.23}$$

So

$$\vec{\nabla}_3 \cdot \vec{U} = \rho^{-1}h^{-\frac{1}{2}}[h^{\frac{1}{2}}\rho^{-1}U_2)_{,2} + (h^{\frac{1}{2}}h^{MN}\rho V_M)_{,N}]. \tag{2.24}$$

However,

$$\frac{\partial}{\partial X^2} = \frac{\partial}{\partial X^\phi} = 0 \tag{2.25}$$

therefore,

$$\vec{\nabla}_3 \cdot \vec{U} = \rho^{-1} h^{-\frac{1}{2}} (h^{\frac{1}{2}} h^{MN} \rho V_M)_{,N} = \rho^{-1} \vec{\nabla}_2 \cdot (\rho \vec{V}). \tag{2.26}$$

Note that $\vec{V}$ is the (X^3, X^4) part of the three-dimensional vector $\vec{U}$.

What about the action of the tilde operation?

$$\vec{V} = (V_3, V_4) = (V_M) \tag{2.27}$$

$$\tilde{V} = (h_{MN} V^N) = (h_{MN} \varepsilon^{NP} V_P) = (V_4, -V_3) \tag{2.28}$$

as a consequence,

$$\tilde{\tilde{V}} = (-V_3, -V_4) = -\vec{V}. \tag{2.29}$$

Accordingly,

$$\begin{aligned} X^4 &= \tilde{X}^3 \\ X^3 &= -\tilde{X}^4. \end{aligned} \tag{2.30}$$

So, by the chain rule,

$$\frac{\partial}{\partial \tilde{X}^3} = \frac{\partial X^3}{\partial \tilde{X}^3} \frac{\partial}{\partial X^3} + \frac{\partial X^4}{\partial \tilde{X}^3} \frac{\partial}{\partial X^4} = \frac{\partial}{\partial X^4}. \tag{2.31}$$

Similarly

$$\frac{\partial}{\partial \tilde{X}^4} = -\frac{\partial}{\partial X^3}. \tag{2.32}$$

Therefore,

$$\tilde{\nabla}_2 = \left(\frac{\partial}{\partial X^4}, -\frac{\partial}{\partial X^3} \right). \tag{2.33}$$

Twist Potentials

For any scalar U we have

$$\vec{\nabla}_3 \cdot [\rho^{-1}\tilde{\nabla}U] = \rho^{-1}\nabla_2 \cdot (\tilde{\nabla}U)$$

$$= \rho^{-1}\left(\frac{\partial^2 U}{\partial X^3 \partial X^4} - \frac{\partial^2 U}{\partial X^4 \partial X^3}\right) \tag{3.1}$$

$$= 0$$

where we have employed Equations (2.26) and (2.33).

Therefore, if we have an equation in the form of a vanishing divergence, $\vec{\nabla}_3 \cdot \vec{V} = 0$, then this implies the local existence of a potential U, such that

$$\vec{V} = \rho^{-1}\tilde{\nabla}U. \tag{3.2}$$

In Paper I, for the vacuum case, it is shown that the field equations can be written as divergences, e.g.,

$$R^A{}_C = -\tfrac{1}{2}\rho\vec{\nabla}_2 \cdot (\rho^{-1} f^{AX}\vec{\nabla}_2 f_{XC}). \tag{3.3}$$

Therefore, one may glimpse their possible future usefulness, and we will capitalize upon this fact to develop the associated twist potentials.

These rather simple concepts are important because Equations (3.1) and (3.2) will be used repeatedly. These potentials will be employed to reveal information about an important internal symmetry group. Moreover, it will be demonstrated that these twist potentials generalize previous ideas by Ernst, and they will be instrumental in our process of solving the Einstein-Maxwell equations in particular cases.

CHAPTER III

EINSTEIN-MAXWELL FIELD EQUATIONS

Now we wish to obtain the combined Einstein-Maxwell field equations. First we will consider the familiar Maxwell equations, apply the conditions of independence from t and ϕ derivatives, and then reformulate these equations in our notation. Next, these equations will be put in the form of a total divergence. The Maxwell equations are connected to the gravitational field because covariant derivatives are involved.

It will be advantageous to use the fact that

$$A^{\alpha}{}_{;\alpha} = (-g)^{-\frac{1}{2}}(\ (-g)^{\frac{1}{2}}A^{\alpha}\ )_{,\alpha} \tag{4.1}$$

and

$$F^{\alpha\beta}{}_{;\beta} = (-g)^{-\frac{1}{2}}\ (\ (-g)^{\frac{1}{2}}F^{\alpha\beta}{}_{;\beta})_{,\beta} \tag{4.2}$$

since $F^{\alpha\beta}$ is anti-symmetric.

Next we will consider the basic gravitational equations involving $R_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$. The electromagnetic field will make its appearance in this equation as $T_{\mu\nu}$ is expressed in terms of $F_{\mu\nu}$. Then, as above, our main objective will be to formulate this equation in our notation as a total divergence.

Our spacetime is stationary and axially symmetric, and we assume that the electromagnetic field contained therein will also be independent of the t and ϕ coordinates. Likewise, the same properties are accorded to the four-vector potential A_μ. It then follows that the electromagnetic field tensor $F_{\mu\nu}$ reduces to

$$\begin{aligned} F_{AB} &= A_{B,A} - A_{A,B} = 0 \\ F_{AM} &= A_{M,A} - A_{A,M} = -A_{A,M} \\ F_{MN} &= A_{N,M} - A_{M,N}. \end{aligned} \tag{4.3}$$

Since we are working only in the source-free arena outside of the body that is generating the gravitational and electromagnetic fields, Maxwell's equations may be written

$$F^{\mu\nu}{}_{;\nu} = \left((-g)^{\frac{1}{2}} F^{\mu\nu} \right)_{,\nu} = 0 \tag{4.4}$$

$$F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} + F_{\alpha\beta,\gamma} = 0. \tag{4.5}$$

Written in this fashion we have a set of coupled, first order, partial differential equations relating components of the field variables. If the vector potential is introduced, then a smaller number of second order equations are obtained. In particular, Equation (4.5) will be satisfied identically.

Consider Equation (4.4). For the (X^3, X^4) space,

$$(\rho h^{\frac{1}{2}} F^{MN})_{,N} = (\rho h^{\frac{1}{2}} h^{NR} F^{M}{}_{R})_{,N} \ . \qquad (4.6)$$

This implies, using Equation (2.17),

$$\vec{\nabla}_2 \cdot (\rho \vec{V}^M) = 0 \qquad (4.7)$$

where

$$\vec{V}^M \equiv (F^M{}_3, F^M{}_4) \ . \qquad (4.8)$$

An immediate solution is

$$V^M{}_N = \varepsilon^M{}_N \rho^{-1} \qquad (4.9)$$

or

$$F^{MN} = C\rho^{-1}\varepsilon^{MN} \ . \qquad (4.10)$$

The equivalent physical situation is a magnetic field in the ϕ direction, falling off as ρ^{-1}, produced by a line current along the symmetry axis. Such situations are to be avoided because they are unphysical, so set C = 0.

This implies

$$A_N = 0 \ . \qquad (4.11)$$

The final set of equations, from Equation (4.3) are

$$0 = ((-g)^{\frac{1}{2}} F^{AM})_{,M} \qquad (4.12)$$
$$= (\rho h^{\frac{1}{2}} f^{\hat{A}\hat{B}} h^{MN} F_{BN})_{,N} \ .$$

Now

$$F_{BN} = -A_{B,N} = -[\nabla_2 A_B]_N \qquad (4.13)$$

[N^{th} component].

Therefore, using Equation (2.17),

$$\vec{\nabla}_2\cdot[\rho \tilde{\tilde{f}}^{AX}\vec{\nabla}_2 A_X] = \vec{\nabla}_2\cdot[\rho^{-1}f^{AX}\vec{\nabla}_2 A_X] = 0, \tag{4.14}$$

then, using Equation (2.26)

$$\vec{\nabla}_3\cdot[\tilde{\tilde{f}}^{AX}\vec{\nabla}_3 A_X] = \vec{\nabla}_3\cdot[\rho^{-2}f^{AX}\vec{\nabla}_3 A_X] = 0. \tag{4.15}$$

Now we must connect this with the full Einstein equations. The presence of the electromagnetic field is incorporated into the geometry of spacetime via its stress-energy tensor

$$R_{\mu\nu} = -8\pi(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T) \ . \tag{4.16}$$

The electromagnetic stress-energy tensor is given by

$$4\pi T_{\mu\nu} = F_{\mu}{}^{\sigma}F_{\nu\sigma}-\tfrac{1}{4}g_{\mu\nu}F_{\sigma}{}^{\tau}F^{\sigma}{}_{\tau} \ . \tag{4.17}$$

So

$$\begin{aligned} 4\pi T^{A}{}_{B} &= F^{AX}F_{BX} + F^{AM}F_{BM}-\tfrac{1}{4}\delta^{A}{}_{B}F_{Z}{}^{X}F^{Z}{}_{X} \\ &\quad -\tfrac{1}{4}\delta^{A}{}_{B}F_{X}{}^{M}F^{X}{}_{M}-\tfrac{1}{4}\delta^{A}{}_{B}F_{M}{}^{X}F^{M}{}_{X} \\ &\quad -\tfrac{1}{4}\delta^{A}{}_{B}F_{M}{}^{N}F^{M}{}_{N} \ . \end{aligned} \tag{4.18}$$

Using Equations (4.3) and (4.11) we obtain

$$\begin{aligned} 4\pi T^{A}{}_{B} &= A^{A,M}A_{B,M}-\tfrac{1}{4}\delta^{A}{}_{B}A_{X}{}^{,M}A^{X}{}_{,M} \\ &\quad -\tfrac{1}{4}\delta^{A}{}_{B}A^{X}{}_{,M}A_{X}{}^{,M} \\ &= f^{AX}A_{X}{}^{,M}A_{B,M}-\tfrac{1}{4}\delta^{A}{}_{B}f^{XZ}A_{X}{}^{,M}A_{Z,M} \\ &\quad -\tfrac{1}{4}\delta^{A}{}_{B}f^{ZX}A_{X,M}A_{Z}{}^{,M} \\ &= f^{AX}\vec{\nabla}_2 A_X\cdot\vec{\nabla}_2 A_B-\tfrac{1}{2}\delta^{A}{}_{B}f^{XZ}\vec{\nabla}_2 A_X\cdot\vec{\nabla}_2 A_Z \ . \end{aligned} \tag{4.19}$$

Previously it was noted that the field equations could be put into the form of a vanishing divergence, so with that objective in mind, we note that

$$\vec{\nabla}_2\cdot[\rho^{-1}(f^{AX}A_B\vec{\nabla}A_X-\tfrac{1}{2}\delta^A{}_B f^{XZ}A_X\vec{\nabla}A_Z)]$$
$$= A_B\vec{\nabla}_2\cdot[\rho^{-1}f^{AX}\vec{\nabla}_2A_X] + \rho^{-1}f^{AX}\vec{\nabla}_2A_B\cdot\vec{\nabla}_2A_X \tag{4.20}$$
$$-\tfrac{1}{2}\delta^A{}_BA_X\vec{\nabla}_2\cdot[\rho^{-1}f^{XZ}\vec{\nabla}_2A_Z]-\tfrac{1}{2}\delta^A{}_B\rho^{-1}f^{XZ}\vec{\nabla}_2A_X\cdot\vec{\nabla}_2A_Z \;.$$

Employing Equation (4.14) in Equation (4.19) yields

$$4\pi T^A{}_B = \rho[\rho^{-1}f^{AX}\vec{\nabla}A_X\cdot\vec{\nabla}A_B-\tfrac{1}{2}\delta^A{}_Bf^{XZ}\vec{\nabla}A_X\cdot\vec{\nabla}A_Z]$$
$$= \rho\vec{\nabla}_2\cdot[\rho^{-1}(f^{AX}A_B\vec{\nabla}A_X-\tfrac{1}{2}\delta^A{}_Bf^{XZ}A_X\vec{\nabla}A_Z] \;. \tag{4.21}$$

Referring back to Equation (4.16) we see that we need to know the trace of $T_{\mu\nu}$ and, in fact,

$$T = T^\mu{}_\mu = \frac{1}{4\pi}[F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{4}4F_{\mu\nu}F^{\mu\nu}] = 0 \;. \tag{4.22}$$

Moreover, as is clear from Equation (4.21)

$$T^X{}_X = 0 \;. \tag{4.23}$$

Therefore, Equation (4.16) becomes

$$R^A{}_B = -8\pi T^A{}_B \;. \tag{4.24}$$

Recalling our previous expression for $R^A{}_B$, Equation (3.3), we have

$$\vec{\nabla}_2\cdot[\rho^{-1}(f^{AX}\vec{\nabla}f_{BX}-4f^{AX}A_B\vec{\nabla}A_X + 2\delta^A{}_Bf^{XZ}A_X\vec{\nabla}A_Z)] = 0. \tag{4.25}$$

Therefore, finally,

$$\vec{\nabla}_2\cdot[\rho^{-1}(f^{AX}\vec{\nabla}f_{BX}-2f^{AX}A_B\vec{\nabla}A_X-2f_B{}^XA^A\vec{\nabla}A_X)] = 0 \tag{4.26}$$

using the identity

$$V^A{}_B - V_B{}^A = \delta^A{}_B V^X{}_X \ . \tag{4.27}$$

Note that Equations (4.14) and (4.26) are the Einstein-Maxwell equations written in the desired form of total divergences.

CHAPTER IV

REFORMULATION

Gravitational and Electromagnetic Potentials

If one could but solve Equations (4.14) and (4.26), then our work would be complete. However, as such is not the case, we begin to examine these key equations to see what we can discover. That is, what symmetries do they possess, what gauge freedoms, and what are the transformations that leave them invariant?

First, we recall that we went to the bother of writing the Einstein-Maxwell equations in the form of total divergences for a specific reason. Our purpose was to develop the potentials which, by reason of the discussion on Twist Potentials, must exist as a result of these vanishing divergences.

In this section a reformulation of the Einstein-Maxwell equations into a compact form possessing surprising utility will be accomplished.

As we have given much advance fanfare concerning the usefulness of the potentials associated with vanishing divergences, we will now take an initial look at them and develop some key equations.

Consider the divergence equation involving the electromagnetic potential, i.e., Equation (4.14). As detailed in the section on twist potentials, this equation implies the existence of a potential, B_A, such that

$$\vec{\nabla} B_A = -\rho^{-1} f_A{}^X \vec{\nabla} A_X \ . \qquad (5.1)$$

Before we proceed any further, we should attempt to determine how the potential B_A fits into the usual scheme of electromagnetic notation. We begin with a consideration of Maxwell's equations in a source-free space. This will lead to a set of potentials which will possess the same relationship to the customary electromagnetic potentials as B_A has to A_A in flat space.

The basic equations are:

$$\vec{\nabla} X \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad (5.2)$$

$$\vec{\nabla} X \vec{D} = -\frac{\partial \vec{H}}{\partial t} \qquad (5.3)$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad (5.4)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \ . \qquad (5.5)$$

Equation (5.5) implies the familiar relation

$$\vec{H} = \vec{\nabla} X \vec{A} \ . \qquad (5.6)$$

Likewise, Equation (5.4) implies the less common identity

$$\vec{D} = \vec{\nabla} X \vec{\zeta} \ . \qquad (5.7)$$

Equations (5.3) and (5.6) imply the existence of a scalar potential

$$-\vec{\nabla}\phi = \vec{D} + \frac{\partial\vec{A}}{\partial t} \; . \tag{5.8}$$

Likewise, Equations (5.2) and (5.7) imply the existence of a magnetic scalar potential[13]

$$-\vec{\nabla}\chi = \vec{H} - \frac{\partial\vec{\zeta}}{\partial t} \; . \tag{5.9}$$

Combining Equations (5.6), (5.7), (5.8), and (5.9) we obtain

$$\vec{\nabla}X\vec{A} = -\vec{\nabla}\chi + \frac{\partial\vec{\zeta}}{\partial t} \tag{5.10}$$

$$\vec{\nabla}X\vec{\zeta} = -\vec{\nabla}\phi + \frac{\partial\vec{A}}{\partial t} \; . \tag{5.11}$$

Now apply $\frac{\partial}{\partial t} = \frac{\partial}{\partial\phi} = 0$ and Equation (4.11). Then in component form, we have [using physical coordinates[14]]

$$\begin{aligned} \bar{\chi}_{,\rho} &= \rho^{-1}(\rho\bar{A}_{\phi})_{,Z} \\ \bar{\chi}_{,Z} &= -\rho^{-1}(\rho\bar{A}_{\phi})_{,\rho} \\ \bar{A}_{t,\rho} &= \rho^{-1}(\rho\bar{\zeta}_{\phi})_{,Z} \\ \bar{A}_{t,Z} &= -\rho^{-1}(\rho\bar{\zeta}_{\phi})_{,\rho} \; . \end{aligned} \tag{5.12}$$

At this point it is informative to put things in a covariant form. For simplicity we will work in flat space. The line element is, in cylindrical coordinates,

$$ds^2 = dt^2 - \rho^2 d\phi^2 - d\rho^2 - dZ^2 \; . \tag{5.13}$$

Since

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{5.14}$$

it seems appropriate, in the spirit of this section, to define a "dual" four-vector potential

$$\overline{F}_{\mu\nu} \equiv a_{\nu,\mu} - a_{\mu,\nu} \quad . \tag{5.15}$$

How are a^{μ} and A^{μ} related?

$$\overline{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = -\tfrac{1}{2}(-g)^{-\frac{1}{2}}[\mu\nu\alpha\beta]F_{\alpha\beta} \tag{5.16}$$

where we have employed the fact that the Levi-Civita tensor $\varepsilon^{\mu\nu\alpha\beta}$ is related to the completely antisymmetric symbol $[\mu\nu\alpha\beta]$ by

$$\varepsilon^{\mu\nu\alpha\beta} = -(-g)^{-\frac{1}{2}}[\mu\nu\alpha\beta] \quad . \tag{5.17}$$

Accordingly,

$$a^{\nu,\mu} - a^{\mu,\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}[A_{\beta,\alpha} - A_{\alpha,\beta}] \quad . \tag{5.18}$$

Then we find that, with our symmetries,

$$a^{\tilde{A},M} = -\rho^{-1}[MAND]A_{D,N} \tag{5.19}$$

where

$$\begin{aligned} A,D &\rightarrow t,\phi \rightarrow 1,3 \\ M,N &\rightarrow \rho,Z \rightarrow 2,4 \quad . \end{aligned} \tag{5.20}$$

For example,

$$a^{\tilde{t},\rho} = -\rho^{-1}[2143]A_{\phi,Z} \tag{5.21}$$

or

$$\overline{a}_{t,\rho} = \rho^{-1}(\rho\overline{A}_{\phi})_{,Z} \tag{5.22}$$

using physical coordinates. $[\overline{A}_{\phi} = \rho A^{\tilde{\phi}} = -\rho^{-1}A_{\phi}]$

Comparison with Equation (5.12) reveals that

$$A_{A} \leftrightarrow (\phi, A_{\phi}) \tag{5.23}$$

$a_A \leftrightarrow (-\chi, \zeta_\phi)$.

In order to relate this to B_A we look at Equation (5.1) in flat space and find

$$B_{t,\rho} = \rho^{-1} A_{\phi,z} \ . \tag{5.24}$$

So, in more common electromagnetic formalism,

$$\begin{aligned} A_A &\leftrightarrow (\phi, A_\phi) \\ B_A &\leftrightarrow (-\chi, -\zeta_\phi) \ . \end{aligned} \tag{5.25}$$

We now define the complex potential ϕ_A

$$\phi_A \equiv A_A + iB_A \ . \tag{5.26}$$

Equation (5.1) and the inverse relation

$$\vec{\nabla} A_A = -\rho^{-1} f_A{}^X \tilde{\nabla} B_X \tag{5.27}$$

imply

$$\vec{\nabla}\phi_A = -i\rho^{-1} f_A{}^X \tilde{\nabla}\phi_X \ . \tag{5.28}$$

It so happens that ϕ_A is only the first of a remarkable series of potentials all of which will obey the same key equation!

The Maxwell relations were used in developing Equation (5.28), so now let us try to use our other field relation, Equation (4.26), to define a complex tensor quantity which, mirabile dictu, will obey a relation identical with Equation (5.28).

Using Equation (5.1) as a guide, we define the twist potential related to Equation (4.26)

$$\vec{\nabla}\psi_{AC} = -\rho^{-1}(f_A{}^X\tilde{\nabla}f_{XC}-2f_A{}^X A_C\tilde{\nabla}A_X-2f_C{}^X A_A\tilde{\nabla}A_X) \quad . \tag{5.29}$$

Note that although f_{AC} is symmetric, ψ_{AC} is not. The inverse relation is

$$\vec{\nabla}f_{AC} = \rho^{-1}f_A{}^X(\tilde{\nabla}\psi_{XC}+2A_C\tilde{\nabla}B_X+2A_X\tilde{\nabla}B_C) \quad . \tag{5.30}$$

At this stage one might follow Equation (5.26) and consider the quantity $f_{AC} + i\psi_{AC}$. However, it turns out that we are still a stage or two from our final objective and it is, in fact, more fruitful to look at the quantity

$$\bar{\varepsilon}_{AC} = f_{AC} + i(\psi_{AC} + 2A_A B_C) \quad . \tag{5.31}$$

Then combining Equations (5.29) and (5.30) we obtain

$$\vec{\nabla}\bar{\varepsilon}_{AC}-\phi_C\vec{\nabla}\overset{*}{\phi}_A = -i\rho^{-1}f_A{}^X(\tilde{\nabla}\bar{\varepsilon}_{XC}-\phi_C\tilde{\nabla}\overset{*}{\phi}_X) \quad . \tag{5.32}$$

Then defining

$$\bar{G}_{AC} \equiv \bar{\varepsilon}_{AC}-\overset{*}{\phi}_A\phi_C \quad , \tag{5.33}$$

Equation (5.32) may be written as

$$\vec{\nabla}\bar{G}_{AC}+\overset{*}{\phi}_A\vec{\nabla}\phi_C = -i\rho^{-1}f_A{}^X(\tilde{\nabla}\bar{G}_{XC}+\overset{*}{\phi}_X\tilde{\nabla}\phi_C) \quad . \tag{5.34}$$

Noting that both Equation (5.32) and (5.34) are very nearly in the required form, we finally see that if one defines

$$H_{AB} \equiv \bar{G}_{AB} + \varepsilon_{AB}K \tag{5.35}$$

where

$$\vec{\nabla} K = \phi_X^* \vec{\nabla} \phi^X ,\tag{5.36}$$

then we have

$$\vec{\nabla} H_{AB} = -i\rho^{-1} f_A{}^X \tilde{\nabla} H_{XB} .\tag{5.37}$$

The H_{AB} and ϕ_A are the basic gravitational and electromagnetic potentials which we have been seeking. They will be the key by which the door to the generation of new solutions is opened. We should notice that at this moment and in the future, the close correlation between the electromagnetic and the gravitational potentials. This similarity occurs frequently, and the only difference results from the vector nature of electromagnetism and the tensor nature of gravitation. Thus, the electromagnetic field need not be thought of as having been included as an afterthought but participates from a position of equality. Furthermore, it is appropriate to note that Campbell and Morgan[15] have shown that the linear theory of gravity can be cast into a form which is very similar to the usual Maxwell form of electromagnetic theory, i.e., in terms of E and B. The gravitational E and B-like quantities obey a set of dyadic equations which are identical in appearance to the usual Maxwell equations in empty space.

Generalized, Covariant Ernst Equations

Now that we have in hand ϕ_A and H_{AB}, is there anything which might presage their future promise? A review of Ernst's[16] work immediately reveals that we have obtained a covariant generalization of his potentials. His complex potential Φ is equivalent to ϕ_1, and his twist potential ϕ is equivalent to our ψ_{11}. Additionally, he reformulates the Einstein-Maxwell equations in terms of a complex function

$$\varepsilon = \frac{\xi-1}{\xi+1} = f-\Phi\Phi^* + i\phi = f_{11}-\phi_1\phi_1{}^* + i\psi_{11} \ , \qquad (6.1)$$

which is the $_{1,1}$ component of our $\bar{\varepsilon}_{AC}$ [see Equation (5.31)]. This reformulation is quite important as Ernst was able to show that various well-known solutions are particularly simple when expressed in terms of ξ, using prolate spheroidal coordinates. For example, the Schwarzschild solution, remarkably, is $\xi = X$. The Tomimatsu-Sato solutions[4] were also obtained via simplifications induced by Ernst's reformulation.

His version of the coupled Einstein-Maxwell equations may be written

$$\begin{aligned} (\mathrm{Re}\,\varepsilon+\Phi\Phi^*)\nabla^2\varepsilon &= (\vec{\nabla}\varepsilon+2\Phi^*\vec{\nabla}\Phi)\cdot\vec{\nabla}\varepsilon \\ (\mathrm{Re}\,\varepsilon+\Phi\Phi^*)\nabla^2\Phi &= (\vec{\nabla}\varepsilon+2\Phi^*\vec{\nabla}\Phi)\cdot\vec{\nabla}\Phi \ . \end{aligned} \qquad (6.2)$$

Referring back to Equation (6.1) for the definition of ξ, we note that the vacuum version of Equation (6.2) can be written

$$(\xi\xi^{*}-1)\nabla^2\xi = 2\xi^{*}\vec{\nabla}\xi\cdot\vec{\nabla}\xi \ . \qquad (6.3)$$

If we suppose

$$\xi = R + iM \ , \qquad (6.4)$$

then in the weak field, vacuum version of Equation (6.1) [where the $\varepsilon = f$], we have

$$f \rightarrow 1 - 2R, \qquad (6.5)$$

and one also discovers that both R and M obey Laplace's equation.

It is well known that in the Newtonian limit of General Relativity one may write $f \rightarrow 1 - 2\eta$ where η is the Newtonian gravitational potential. Therefore, R plays the role of a Newtonian potential, and, as Kinnersley and Kelley[17] have pointed out, the imaginary part of ξ can be viewed as a gravitational analogue of Equation (5.9), i.e., a magnetic scalar potential.

Let us now derive Equation (6.2) and put it into a slightly more useful form.

We begin by rewriting Equation (5.37)

$$\rho^{-1}\tilde{\nabla}H_{AB} = i\rho^{-2}f_{A}{}^{X}\vec{\nabla}H_{XB} \ . \qquad (6.6)$$

Then by Equation (3.1)

$$\vec{\nabla}\cdot(\rho^{-2} f_A{}^X \vec{\nabla} H_{XB}) = 0 \, . \qquad (6.7)$$

Breaking the covariance let A = 2, B = 1

$$\vec{\nabla}\cdot(\rho^{2} f_{22}\vec{\nabla}H_{11} - \rho^{-2} f_{21}\vec{\nabla}H_{21}) = 0 \, . \qquad (6.8)$$

In Equation (5.37) let A = 1, B = 1 in order to eliminate the H_{21} term

$$\vec{\nabla}H_{11} + i\rho^{-1} f_{12}\tilde{\nabla}H_{11} = i\rho^{-1} f_{11}\tilde{\nabla}H_{21} \, . \qquad (6.9)$$

Combining Equations (6.8), (6.9) and using Equation (1.3) [the Lewis canonical form] we obtain

$$\vec{\nabla}\cdot(\rho^{-2} f_{22}\vec{\nabla}H_{11} + i\rho^{-1}\omega\tilde{\nabla}H_{11} - \rho^{-2} f\omega^2\vec{\nabla}H_{11}) = 0 \, . \qquad (6.10)$$

Using Equation (1.3) again, then

$$\vec{\nabla}\cdot(-f^{-1}\vec{\nabla}H_{11} + i\rho^{-1}\omega\tilde{\nabla}H_{11}) = 0 \qquad (6.11)$$

or

$$f\nabla^2 H_{11} = (\vec{\nabla}f - i\rho^{-1} f^2\tilde{\nabla}\omega)\cdot\vec{\nabla}H_{11} \, . \qquad (6.12)$$

Now we need to eliminate the $\tilde{\nabla}\omega$ term. Returning to Equations (5.27) and (5.29) we note that

$$\vec{\nabla}B_1 = \rho^{-1} f(\tilde{\nabla}A_2 + \omega\tilde{\nabla}A_1) \qquad (6.13)$$

and

$$\vec{\nabla}\psi_{11} = -\rho^{-1} f^2\tilde{\nabla}\omega - 4\rho^{-1} f A_1(\tilde{\nabla}A_2 + \omega\tilde{\nabla}A_1) \, . \qquad (6.14)$$

Solving for $\tilde{\nabla}\omega$, we obtain

$$\tilde{\nabla}\omega = -\rho f^{-2}(\vec{\nabla}\psi_{11} + 4A_1\vec{\nabla}B_1) \, , \qquad (6.15)$$

therefore,

$$f\nabla^2 H_{11} = [\vec{\nabla}f + i(\vec{\nabla}\psi_{11} + 4A_1\vec{\nabla}B_1)]\cdot\vec{\nabla}H_{11} \, . \qquad (6.16)$$

Using Equation (5.35) we find

$$H_{11} = f - A_1^2 - B_1^2 + i(\psi_{11} + 2A_1B_1) \ . \qquad (6.17)$$

Taking the gradient and rearranging terms, we have, referring to the terms in Equation (6.16),

$$\vec{\nabla} f + i(\vec{\nabla}\psi_{11} + 4A_1\vec{\nabla}B_1) = \vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1 \ . \qquad (6.18)$$

So, finally,

$$f\nabla^2 H_{11} = [\vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1]\cdot\vec{\nabla}H_{11} \ . \qquad (6.19)$$

Comparing Equations (5.28) and (5.37) we see that the above derivation will go through in the same manner if we replace H_{11} by ϕ_1 down through Equation (6.16) and then continue as before.

Doing so, we obtain

$$f\nabla^2\phi_1 = (\vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1)\cdot\vec{\nabla}\phi_1 \ . \qquad (6.20)$$

In order to demonstrate that Equations (6.19) and (6.20) are the same as Equation (6.2), we need only note that $\mathrm{Re}\,\varepsilon + \Phi\Phi^* = f$ via Equation (6.1), and that $(\varepsilon, \Phi) \leftrightarrow (H_{11}, \phi_1)$.

In summation, we see that not only does our formalism contain Ernst's work, but it generalizes it and places it on a covariant basis.

CHAPTER 5

THE GROUPS G′ AND H′

In general, whenever an object is left unchanged (invariant) by some operation, one says that the operation is a symmetry of the object. With this idea in mind, let us explore some of the symmetries exhibited by axially symmetric stationary field equations. Basically then, we are seeking transformations that leave the line element [see Equation (1.2)], the field equations [see Equations (4.14) and (4.26)], or various reformulations of the field equations invariant.

An examination of the field equations reveals that they are manifestly covariant with respect to linear transformations of the coordinates (t,ϕ). These transformations must be linear if t and ϕ derivatives are to be avoided. Thus we have a three parameter group G with a particular representation of its generators given by

$$\begin{aligned} t &\to t + a\phi \\ \phi &\to \phi \end{aligned} \tag{7.1}$$

$$\begin{aligned} t &\to t \\ \phi &\to \phi + bt \end{aligned} \tag{7.2}$$

$$\begin{aligned} t &\to ct \\ \phi &\to c^{-1}\phi \ . \end{aligned} \tag{7.3}$$

It should be noted at this juncture that our attention is being directed only to the local properties of the metric, and we are not interested in global complications at the moment.

Another idea that should spring to mind, especially as we have incorporated Maxwell's equations, is the consideration of the gauge freedom involved. We will focus our attention on the basic potentials ϕ_A and H_{AB}. Since the potentials are initially defined by means of differential equations, we are looking for quantities that can be added to these potentials and leave the defining equations invariant. Furthermore, the f_{AB} must remain unchanged.

$$A_A \rightarrow A_A + \gamma_A \tag{7.4}$$

$$B_A \rightarrow B_A + \sigma_A \ . \tag{7.5}$$

So the gauge freedom for ϕ_A is

$$\phi_A \rightarrow A_A+\gamma_A+i(B_A+\sigma_A) = \phi_A+a_A \quad \text{[two gauge freedoms]} \tag{7.6}$$

where a_A is an arbitrary complex constant.

Equations (5.26), (5.31), and (5.33) imply

$$\overline{G}_{AB} = f_{AB}-A_A A_B-B_A B_B+i(\psi_{AB}+A_A B_B+B_A A_B) \ . \tag{7.7}$$

Equations (5.27) and (5.29) may be combined to give

$$\vec{\nabla}\psi_{AB} = -\rho^{-1} f_A{}^X \tilde{\nabla} f_{XB} - 2A_B\vec{\nabla}B_A - 2A_A\vec{\nabla}B_B \ , \tag{7.8}$$

so

$$\psi_{AB} \rightarrow \psi_{AB}-2\gamma_B B_A-2\gamma_A B_B+\alpha_{AB} \tag{7.9}$$

where α_{AB} is a real constant.

Combining Equations (7.4)-(7.7), and (7.9) we have

$$\overline{G}_{AB} \rightarrow \overline{G}_{AB}+i\alpha_{AB}-\overset{*}{a}_A\phi_B-\overset{*}{a}_B\phi_A-\overset{*}{a}_A a_B+2i\gamma_A\sigma_B \ . \tag{7.10}$$

By redefining α_{AB} this equation may be placed into conformity with Equation (7.26) of Paper I

$$\alpha_{AB} \rightarrow \alpha_{AB}+2\gamma_A\sigma_B \ . \tag{7.11}$$

Then

$$\overline{G}_{AB} \rightarrow \overline{G}_{AB}+i\alpha_{AB}-\overset{*}{a}_A\phi_B-\phi_A\overset{*}{a}_B-\overset{*}{a}_A a_B \ . \tag{7.12}$$

In order to obtain the gauge condition for H_{AB} we refer back to Equation (5.35) and determined that only the gauge freedom of K remains to be discovered.

$$\vec{\nabla}K = \overset{*}{\phi}_X\vec{\nabla}\phi^X = \overset{*}{\phi}_1\vec{\nabla}\phi_2-\overset{*}{\phi}_2\vec{\nabla}\phi_1 \tag{7.13}$$

thus,

$$K \rightarrow K+\overset{*}{a}_1\phi_2-\overset{*}{a}_2\phi_1+\tfrac{1}{2}\overset{*}{a}_X a^X \tag{7.14}$$

so

$$H_{AB} \rightarrow H_{AB}+i\alpha_{AB}-\overset{*}{a}_A\phi_B-\phi_A\overset{*}{a}_B-\overset{*}{a}_A a_B+\varepsilon_{AB}(\overset{*}{a}_X\phi^X+\tfrac{1}{2}\overset{*}{a}_X A^X) \tag{7.15}$$

Thus H_{AB} has four gauge freedoms.

Now we have the group G′. It is composed of the coordinate transformations which make up the group G in addition to the gauge conditions detailed above. These are the linear transformations of ϕ_A and H_{AB} that preserve f_{AB}.

Various less obvious transformations are also known: the generalized Ehlers transformation [see Equation (7.16)], a "duality rotation" for gravitation [see Equation (7.24)], or the Harrison transformation[18] [see Equation (7.17)] which maps vacuum fields into charged fields. This leads us to a consideration of the "internal" group H′.

First, consider the actual form of the Ehlers and Harrison transformations

$$\phi_1 \to \frac{\phi_1}{1+i\gamma H_{11}} \qquad H_{11} \to \frac{H_{11}}{1+i\gamma H_{11}} \tag{7.16}$$

and

$$\phi_1 \to \frac{\phi_1+cH_{11}}{1-2c^*\phi_1-cc^*H_{11}} \qquad H_{11} \to \frac{H_{11}}{1-2c^*\phi_1-cc^*H_{11}} \ . \tag{7.17}$$

The field equations are, in fact, invariant under these transformations. Naturally one would be hard pressed to recognize this fact if only Equations (4.14) and (4.26) were considered. Only when one rewrites these equations, using some of the "internal" potentials, e.g., Equations (6.19) and (6.20), can one hope to discover such transformations. Now that we have the field equations in the appropriate form, we may more readily determine such mappings.

For instance, using Equations (7.6) and (7.15),

$$\phi_1 \to \phi_1 \qquad H_{11} \to H_{11}+i\alpha \tag{7.18}$$

or

$$\phi_1 \to \phi_1+a \qquad H_{11} \to H_{11}-2a^*\phi_1-aa^* \tag{7.19}$$

where α is real and a is complex. These gauge transformations will retain the invariance of the field equations.

One might also consider

$$\phi_1 \to H_{11}{}^{-1}\phi_1 \qquad H_{11} \to H_{11}{}^{-1} \tag{7.20}$$

or

$$\phi_1 \to \beta e^{i\alpha}\phi_1 \qquad H_{11} \to \beta^2 H_{11} \tag{7.21}$$

where α and β are real. This latter operation combines a rescaling with an electromagnetic duality rotation. This duality rotation sends Schwarzschild mass into magnetic mass. In order to have a feel for this operation, consider what a duality rotation does for the case of electromagnetism.

In order to make our equations look symmetric, suppose we have both magnetic and electric currents and charge densities present. Then Maxwell's equations are

$$\vec{\nabla}\cdot\vec{D} = \rho_e \qquad \vec{\nabla}\times\vec{H} = \vec{J}_e + \frac{\partial\vec{D}}{\partial t} \tag{7.22}$$
$$\vec{\nabla}\cdot\vec{B} = \rho_m \qquad \vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} - \vec{J}_m .$$

Now consider the following "duality" transformation[19]

$$\begin{pmatrix} E \\ H \end{pmatrix} = A \begin{pmatrix} E' \\ H' \end{pmatrix} \qquad \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix} = A \begin{pmatrix} \rho_e' \\ \rho_m' \end{pmatrix}$$

$$\begin{pmatrix} D \\ B \end{pmatrix} = A \begin{pmatrix} D' \\ H' \end{pmatrix} \qquad \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix} = A \begin{pmatrix} \vec{J}_e' \\ \vec{J}_m' \end{pmatrix} \tag{7.23}$$

where

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

It is easy to show that Maxwell's equations are invariant under such a transformation. Thus, it is only a matter of convention whether we consider a particle to possess only electric charge and no magnetic charge. The key point is that if all particles have the property that the ratio of magnetic to electric charge is the same, then duality rotations are possible, such that the angle θ may be chosen to make $\vec{J}_m$ and ρ_m zero.

Now, consider performing the transformation in Equation (7.20), followed by the operation described by Equation (7.18).

Then

$$\phi_1 \rightarrow \frac{\phi_1 H_{11}^{-1}}{H_{11}^{-1}+i\gamma} \qquad H_{11} \rightarrow \frac{1}{H_{11}^{-1}+i\gamma} . \tag{7.24}$$

But this is the same as Equation (7.16).

So, we are now producing the typical group action whereby one member of the group is transformed into another.

It should be noted that the gauge transformations do not commute with the other operations.

We find that if one applies the above six transformations to each other in various orders, no new members result. Thus we have the H′ group. It has eight real parameters [Ehlers (1), Harrison (2), electromagnetic gauge (2), gravitational gauge (1), scaling (1), and duality (1)]. Three of the eight parameters in H′ are only related to gauge transformations, but the remaining five parameters provide an automatic procedure for generating five-parameter "families" of stationary Einstein-Maxwell solutions from each known solution. G′ integrated with H′ yields the full symmetry group K′.

CHAPTER 6

POTENTIAL CHARACTERISTICS

Reissner-Nordstrom Potentials

Before we examine more completely the ramifications of the ϕ_A and H_{AB}, it might be appropriate to examine their appearance for a specific case. A natural simple example is the charged Schwarzschild black hole [the Reissner-Nordstrom solution].

To obtain the Reissner-Nordstrom geometry one solves the Einstein field equations for a spherically symmetric, static gravitational field with no matter present but a radial electric field in the static orthonormal frame.

The line element is determined to be

$$ds^2 = -fdt^2+f^{-1}dr^2+r^2d\Omega^2 \quad , \tag{8.1}$$

where

$$f = 1- \frac{2m}{r} + \frac{e^2}{r^2} = \frac{r^2-2mr+e^2}{r^2} \quad . \tag{8.2}$$

Since the metric is static, $f_{12} = 0$ [see Equation (1.3)]. The electric field is given by

$$\vec{E} = \frac{e}{r^2}\,\hat{r} \quad . \tag{8.3}$$

For e>m the coordinate system is well behaved down to r = 0. It is of interest to note that for an electron $e/m = 2 \times 10^{21}$ in dimensionless units.

More pertinent to our situation, Ernst[16] shows how a vacuum solution of the Einstein equations may be easily transformed to a charged solution, provided that ε and Φ are functionally related, i.e., via Equation (6.1) and

$$\Phi = \frac{q}{\xi+1} \qquad q \equiv e/m \; . \tag{8.4}$$

Then, if ξ_0 is a vacuum solution, $\xi=\xi_0(1-q^2)^{\frac{1}{2}}$ is a charged version of the solution. That is, given the Kerr solution, we may immediately obtain the Kerr-Newman solution. This functional relationship also holds for the Reissner-Nordstrom and the charged Tomimatsu-Sato metrics.

As was previously forecast in dealing with the Ernst formalism, simplications occur in many instances when prolate spheroidal coordinates are employed. [see Appendix A for a discussion of these coordinates].

Therefore, using Equations (6.1) and (8.4), we obtain [remember from the section on Generalized, Covariant Ernst Equations that $\xi_0 = X$ is equivalent to the Schwarzschild solution]

$$f = \mathrm{Re}\,\varepsilon + |\Phi|^2 = \frac{\xi_0\overset{*}{\xi_0}-1}{|\xi_0+\beta|^2} = \frac{X^2-1}{(X+\beta)^2} \; , \tag{8.5}$$

where

$$\beta \equiv (1-q^2)^{-\frac{1}{2}} \tag{8.6}$$