



Gravitational and electromagnetic potentials of the stationary Einstein-Maxwell field equations  
by Thaddeus Charles Jones

A thesis submitted in partial fulfillment of the requirements for the degree of DOCTOR OF  
PHILOSOPHY in Physics  
Montana State University  
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Abstract:

Associated with the stationary Einstein-Maxwell field equations is an infinite hierarchy of potentials.

The basic characteristics of these potentials are examined in general and then in greater detail for the particular case of the Reissner-Nordstrom metric. Their essential utility in the process of solution generation is elucidated and the necessary equations for solution generation are developed.

Appropriate generating functions, which contain the complete infinite hierarchy of potentials, are developed and analyzed. Particular attention is paid to the inherent gauge freedom of these generating functions.

Two methods of solution generation, which yield asymptotically flat solutions in vacuum, are generalized to include electromagnetism. One method, using potentials consistent with the Harrison transformation and the Reissner-Nordstrom metric, is discussed in detail and its resultant difficulties explored.

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FIELD EQUATIONS

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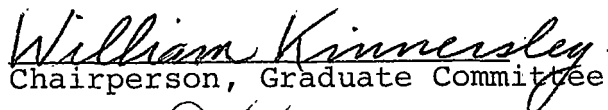
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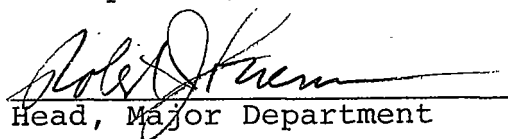
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## ABSTRACT

Associated with the stationary Einstein-Maxwell field equations is an infinite hierarchy of potentials. The basic characteristics of these potentials are examined in general and then in greater detail for the particular case of the Reissner-Nordstrom metric. Their essential utility in the process of solution generation is elucidated and the necessary equations for solution generation are developed.

Appropriate generating functions, which contain the complete infinite hierarchy of potentials, are developed and analyzed. Particular attention is paid to the inherent gauge freedom of these generating functions.

Two methods of solution generation, which yield asymptotically flat solutions in vacuum, are generalized to include electromagnetism. One method, using potentials consistent with the Harrison transformation and the Reissner-Nordstrom metric, is discussed in detail and its resultant difficulties explored.

## CHAPTER 1

### INTRODUCTION

#### Historical Background

We are fortunate to be alive at a time when, during the Einstein centennial, many of the more exotic predictions of General Relativity appear to be coming into the arena of experimental verification, e.g., black holes and gravity waves. Unfortunately, in order to fully describe such phenomena, linearized equations will not suffice, and the full Einstein equations must be solved. As approximation methods are now ruled out, we are left with numerical solutions, which certainly have their own difficulties in addition to lacking aesthetic appeal, or exact solutions.

In spite of the notorious difficulties posed by Einstein's non-linear, coupled, partial differential equations, a considerable number of exact solutions have been obtained. Unluckily, the majority of these are essentially mathematical curiosities with no relevance to the physical universe. Moreover, progress in attaining solutions of astrophysical significance has, until recently, been quite infrequent.

Initially there was impressive progress. In 1916 Einstein derived his field equations, and in that same year

Schwarzschild<sup>1</sup> discovered an exact solution for the gravitational field of a spherically symmetric mass distribution. The next year, 1917, Weyl<sup>2</sup> obtained the general form of the time independent metric with axial symmetry. He showed that the field equations could be solved to obtain the gravitational field around any static, axially symmetric, mass distribution. The next solution, the Kerr metric,<sup>3</sup> was obtained fourty-six years later, in 1963. This is the simplest case of a rotating black hole. Ten years after that, in 1973, Tomimatsu and Sato<sup>4</sup> discovered a family of solutions, of which the Kerr solution is the first member, representing the gravitational field of axially symmetric spinning masses. These solutions may be viewed as a class of Weyl solutions generalized to include rotation.

Since the method of a direct frontal attack on Einstein's equations had been so desultory, attention has shifted recently to the problem of generating solutions from those previously known. [Ehlers,<sup>5</sup> in 1957, was an early investigator in this area.] A surprising number of solutions have been obtained in this fashion, but once again the majority lack astrophysical significance. Furthermore, one had no way to predict beforehand if the nature of a new

solution might be of physical interest. Nevertheless, work went ahead on attempting to discover the largest class of transformations which allow one to generate new solutions from old. Geroch,<sup>6</sup> in 1971, was able to discover an infinite-dimensional symmetry group which acted on an infinite hierarchy of potentials and generated an infinite number of conservation laws. Although he was unable to obtain a finite form for these transformations, he postulated that one solution might prove sufficient to generate all the stationary, axially symmetric, vacuum metrics. Finally, Kinnersley et al.,<sup>7,8,9,10,11</sup> in a major tour de force, have recently developed methods which will give the finite form of new solutions, but, more importantly, these new solutions are guaranteed to possess the prime attribute of being asymptotically flat. Moreover, they conjecture that these methods, if applied to the general static metric, can be used to generate all stationary, axisymmetric, asymptotically flat metrics.

It is instructive to note that this important procedure was developed via an examination of the combined Maxwell-Einstein equations. Historically, Einstein was able to discover the real nature of spacetime by realizing that a choice had to be made between Maxwell's equations and

Newton's ideas of spacetime. He decided that Newton was incorrect and set out to discover a spacetime compatible with Maxwell's equations. It should not be too surprising then that, when Maxwell's equations are coupled to Einstein's, a wealth of additional insight is provided.

### Simplicity and Physics

One could take the viewpoint that the key to the finding of exact solutions lay in a concept that all physicists deeply believe in, i.e., simplicity. We feel that an underlying structure of simplicity is a key to much of physics. How can this be true? Nature appears to be exceedingly complex. To put the matter into perspective, it is instructive to examine the game of chess. The number of possible board positions has been estimated to be  $10^{43}$  with  $10^{125}$  ways to reach them. Yet a child of five can easily learn to play chess. This is because each piece always moves according to a simple set of rules, no matter how the situation may vary. Physicists are convinced of the existence of such underlying simplicities. Another way to state the matter would be to say we believe that all events have various unifying features such as conservation of momentum or energy. Not all the sciences share this

belief equally. Biology, for instance, seems to dwell much more on the diversity of life rather than its common features. In fact, in the 1930's and 1940's a group of physicists went into microbiology and garnered a number of Nobel prizes because of their basic beliefs.

Thus far our discussion has involved more of an aesthetic content as opposed to an idea with direct mathematical implications. Therefore, it is appropriate to try to couch this rather vague concept in the language of mathematics. It is well known that the discovery of various symmetries in nature has been instrumental in reinforcing the essential belief in the simplicity and unity of the universe. Accordingly, one might say that our requirements for simplicity could be couched in the language of mathematics by acknowledgment of the symmetries or invariance properties of nature. Thus, when it was determined that the inclusion of the Maxwell equations into the stationary Einstein equations not only maintained the original symmetry group, but actually expanded it, one had to sense that this could not be merely fortuitous.

All of this may be very fine, but how are these ideas relevant to the problem at hand, namely, solution generation? The basic message of the preceding was, "Use

simplicity to overcome complexity." In the more specific language of physicists one could reword this to say:

"Apply the invariance properties of Einstein's equations to known solutions in order to generate new solutions." It is clear that before one could initiate this process the invariance properties had to be discovered. Some of these properties were obvious, e.g., coordinate transformations. Then certain potentials were defined using the field equations and subsequently used to rewrite the field equations. At this juncture an unexpected "internal" invariance group revealed itself. The amalgamation of this surprising "internal" group with the "external" group of coordinate and gauge conditions gives the entire symmetry group discovered by Geroch. The inclusion of an electromagnetic field enlarges this group, and its representation includes an infinite hierarchy of potentials. These potentials were not only essential to the discovery of new invariance properties but, moreover, turn out to be a sine qua non of the solution generating method. In this thesis we will attempt to discuss the combined electromagnetic and gravitational potential hierarchy in general and then in greater detail for particular cases. Moreover, we will discover the particular modifications which must be put into effect when

electromagnetism is included into the scheme of solution generation.

## CHAPTER 2

### STATIONARY, AXIALLY SYMMETRIC SPACETIMES

#### The Metric

In order to motivate the particular notation that we will be using, a brief description of the general form of the metric for stationary, axially symmetric spacetimes is necessary.

What is the physical situation we wish to consider? We have in mind the spacetime external to the body actually producing the associated gravitational and electromagnetic fields. We do not possess nor do we require any information concerning the equation of state for the internal composition of the body. The mass must possess axial symmetry but also may exhibit two types of differential rotation. A point near the equator on the surface may move at a different angular velocity than a point near the poles. In addition, a point at a distance  $\rho$  from the axis of rotation may have a different angular velocity than a point at a distance  $\rho'$  from the axis of rotation. However, no pulsations or mass distributions which violate the conditions of axial symmetry and reflection symmetry will be permitted.

Given that the fields under consideration are not arbitrary but possess stationarity, axial symmetry, and motion

reversal, how may we incorporate this physical information into the explicit mathematics of the metric? Axial symmetry means that the field doesn't have any angular  $\phi$  dependence. "Stationarity" is equivalent to the existence of a time coordinate for which  $g_{\alpha\beta,t} = 0$ . Motion reversal implies that  $(t, \phi) \leftrightarrow (-t, -\phi)$ , i.e., the situation is unchanged if both the time coordinate and the  $\phi$  angular coordinate are reversed. Thus, we have that  $g_{\alpha\beta,\phi} = g_{\alpha\beta,t} = 0$  and motion reversal implies  $g_{ti} = g_{\phi i} = 0$ .  $g_{tt}$ ,  $g_{\phi\phi}$ , and  $g_{\phi t}$  are permitted to be nonzero. Another way to view this situation is to note that the metric for an axially symmetric rotating body admits two commuting, orthogonally transitive, Killing vectors.

By transformation of the remaining two spacelike coordinates among themselves, the line element can always be reduced to block diagonal form.

$$\begin{aligned} ds^2 &= ds_1^2 - ds_2^2 \\ ds_1^2 &= f_{AB} dx^A dx^B \quad A, B = 1, 2 \quad (1.1) \\ ds_2^2 &= e^{2\Gamma} \delta_{MN} dx^M dx^N = h_{MN} dx^M dx^N \quad M, N = 3, 4 \end{aligned}$$

Specifically, the canonical form introduced by Lewis<sup>12</sup> is

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2] , \quad (1.2)$$

thus

$$f_{AB} = \begin{pmatrix} f & -f\omega \\ -f\omega & f^{-1}\rho^2 - \omega^2 f \end{pmatrix}. \quad (1.3)$$

### Notation

Notation which combines clarity with ease of useage is always to be valued. Accordingly, we will discuss the somewhat nonstandard form used here that takes full advantage of the particular type of metric under consideration.

Normally the metric tensor  $g_{\mu\nu}$  is used to raise and lower indices, but, as we have our metric broken down into two two-dimensional spaces, there are more alternatives.

$$g_{\mu\nu} = \begin{pmatrix} f_{AB} & 0 \\ 0 & h_{MN} \end{pmatrix} \quad (2.1)$$

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} (f_{AB})^{-1} & 0 \\ 0 & (h_{MN})^{-1} \end{pmatrix}. \quad (2.2)$$

In the two-dimensional space  $(t, \phi)$  we may employ either the inverse metric  $(f_{AB})^{-1}$  or the alternating symbol

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.3)$$

to raise indices. We choose  $\epsilon^{AB}$  for simplicity. Contracted indices in the  $(t, \phi)$  space will be denoted by X, Y, or Z.

A few examples are in order so as to give an explicit demonstration of the action of  $\epsilon^{AB}$ .

We have

$$\epsilon^{AX} \epsilon_{XC} = -\delta^A_C \quad (2.4)$$

$$V^A = \epsilon^{AX} V_X \quad (2.5)$$

where  $V^A$  is a vector in the  $(t, \phi)$  space.

Furthermore,

$$V_C = V^X \epsilon_{XC} \quad (2.6)$$

$$-V_C = \epsilon_{CX} V^X. \quad (2.7)$$

Also

$$V_X W^X = V_X \epsilon^{XZ} W_Z = \epsilon^{XZ} V_X W_Z, \quad (2.8)$$

but

$$V_X W_X = \epsilon^{XZ} V_Z W_X = -\epsilon^{ZX} V_Z W_X = -\epsilon^{XZ} V_X W_Z \quad (2.9)$$

where dummy indices were exchanged in the last step in Equation (2.9).

Therefore,

$$V_X W^X = -V^X W_X \quad (2.10)$$

so

$$V_X V^X = 0. \quad (2.11)$$

In the  $(X^3, X^4)$  space we have  $h^{MN}$  to raise indices.

When it is necessary to use  $(f_{AB})^{-1}$  or  $\epsilon^{MN}$  instead, that

index shall be marked with a tilde.

So

$$(f_{AB})^{-1} = -\rho^{-2} \epsilon^{AX} \epsilon^{BZ} f_{XZ} = \tilde{f}^{AB} \quad (2.12)$$

where

$$-\rho^2 \equiv \det(f_{AB}). \quad (2.13)$$

Then

$$\tilde{f}^{AB} = -\rho^2 f^{AB} \quad (2.14)$$

and

$$g^{\mu\nu} = \begin{pmatrix} \tilde{f}^{AB} & 0 \\ 0 & h^{MN} \end{pmatrix}. \quad (2.15)$$

What about derivatives?

Normally,

$$\vec{\nabla} \cdot \vec{\nabla} = (-g)^{-\frac{1}{2}} ((-g)^{\frac{1}{2}} g^{\mu\nu} V_{\mu})_{,\nu} \quad (2.16)$$

where

$$g \equiv \det(g_{\mu\nu}).$$

Clearly,

$$\vec{\nabla}_2 \cdot \vec{\nabla} = h^{-\frac{1}{2}} (h^{\frac{1}{2}} h^{MN} V_M)_{,N} \quad (2.17)$$

would then be associated with  $ds_2^2$ . But Equation (2.17) is two-dimensional while Equation (2.16) is three-dimensional. How may we express a three-dimensional derivative in our notation?

Let

$$\begin{aligned} ds_3^2 &= H_{RS} dx^R dx^S & R, S &= 2, 3, 4 \\ &= ds_2^2 + \rho^2 d\phi^2 \end{aligned} \quad (2.18)$$

then

$$H_{RS} = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \left( h_{MN} \right) \\ 0 & & \end{pmatrix}. \quad (2.19)$$

Thus

$$\vec{\nabla}_3 \cdot \vec{U} = H^{-\frac{1}{2}} (H^{\frac{1}{2}} H^{RS} U_R)_{,S} \quad (2.20)$$

where

$$H \equiv \det(H_{RS}) = \rho^2 \det(h_{MN}) = \rho^2 h. \quad (2.21)$$

Let

$$\vec{U} = (U_2, V_3, V_4) \equiv (U_2, V). \quad (2.22)$$

This implies

$$\begin{aligned} \vec{\nabla}_3 \cdot \vec{U} &= \rho^{-1} h^{-\frac{1}{2}} (\rho h^{\frac{1}{2}} H^{RS} U_R)_{,S} \\ &= \rho^{-1} h^{-\frac{1}{2}} [ (\rho h^{\frac{1}{2}} H^{2S} U_2)_{,S} \\ &\quad + (\rho h^{\frac{1}{2}} H^{3S} V_3)_{,S} \\ &\quad + (\rho h^{\frac{1}{2}} H^{4S} V_4)_{,S} ]. \end{aligned} \quad (2.23)$$

So

$$\vec{\nabla}_3 \cdot \vec{U} = \rho^{-1} h^{-\frac{1}{2}} [ h^{\frac{1}{2}} \rho^{-1} U_2 ]_{,2} + (h^{\frac{1}{2}} h^{MN} \rho V_M)_{,N}. \quad (2.24)$$

However,

$$\frac{\partial}{\partial X^2} = \frac{\partial}{\partial X^\phi} = 0 \quad (2.25)$$

therefore,

$$\vec{\nabla}_3 \cdot \vec{U} = \rho^{-1} h^{-\frac{1}{2}} (h^{\frac{1}{2}} h^{MN} \rho V_M)_{,N} = \rho^{-1} \vec{\nabla}_2 \cdot (\rho \vec{V}). \quad (2.26)$$

Note that  $\vec{V}$  is the  $(X^3, X^4)$  part of the three-dimensional vector  $\vec{U}$ .

What about the action of the tilde operation?

$$\vec{V} = (V_3, V_4) = (V_M) \quad (2.27)$$

$$\tilde{\vec{V}} = (h_{MN} V^N) = (h_{MN} \epsilon^{NP} V_P) = (V_4, -V_3) \quad (2.28)$$

as a consequence,

$$\tilde{\tilde{\vec{V}}} = (-V_3, -V_4) = -\vec{V}. \quad (2.29)$$

Accordingly,

$$\begin{aligned} X^4 &= \tilde{X}^3 \\ X^3 &= -\tilde{X}^4. \end{aligned} \quad (2.30)$$

So, by the chain rule,

$$\frac{\partial}{\partial \tilde{X}^3} = \frac{\partial X^3}{\partial \tilde{X}^3} \frac{\partial}{\partial X^3} + \frac{\partial X^4}{\partial \tilde{X}^3} \frac{\partial}{\partial X^4} = \frac{\partial}{\partial X^4}. \quad (2.31)$$

Similarly

$$\frac{\partial}{\partial \tilde{X}^4} = -\frac{\partial}{\partial X^3}. \quad (2.32)$$

Therefore,

$$\tilde{\vec{\nabla}}_2 = \left( \frac{\partial}{\partial X^4}, -\frac{\partial}{\partial X^3} \right). \quad (2.33)$$

Twist Potentials

For any scalar U we have

$$\begin{aligned}
 \vec{\nabla}_3 \cdot [\rho^{-1} \vec{\nabla} U] &= \rho^{-1} \nabla_2 \cdot (\vec{\nabla} U) \\
 &= \rho^{-1} \left( \frac{\partial^2 U}{\partial X^3 \partial X^4} - \frac{\partial^2 U}{\partial X^4 \partial X^3} \right) \\
 &= 0
 \end{aligned} \tag{3.1}$$

where we have employed Equations (2.26) and (2.33).

Therefore, if we have an equation in the form of a vanishing divergence,  $\vec{\nabla}_3 \cdot \vec{\nabla} = 0$ , then this implies the local existence of a potential U, such that

$$\vec{\nabla} = \rho^{-1} \vec{\nabla} U. \tag{3.2}$$

In Paper I, for the vacuum case, it is shown that the field equations can be written as divergences, e.g.,

$$R^A_C = -\frac{1}{2} \rho \vec{\nabla}_2 \cdot (\rho^{-1} f^{AX} \vec{\nabla}_2 f_{XC}). \tag{3.3}$$

Therefore, one may glimpse their possible future usefulness, and we will capitalize upon this fact to develop the associated twist potentials.

These rather simple concepts are important because Equations (3.1) and (3.2) will be used repeatedly. These potentials will be employed to reveal information about an important internal symmetry group. Moreover, it will be demonstrated that these twist potentials generalize previous

ideas by Ernst, and they will be instrumental in our process of solving the Einstein-Maxwell equations in particular cases.

## CHAPTER III

### EINSTEIN-MAXWELL FIELD EQUATIONS

Now we wish to obtain the combined Einstein-Maxwell field equations. First we will consider the familiar Maxwell equations, apply the conditions of independence from  $t$  and  $\phi$  derivatives, and then reformulate these equations in our notation. Next, these equations will be put in the form of a total divergence. The Maxwell equations are connected to the gravitational field because covariant derivatives are involved.

It will be advantageous to use the fact that

$$A^\alpha{}_{;\alpha} = (-g)^{-\frac{1}{2}} ( (-g)^{\frac{1}{2}} A^\alpha )_{,\alpha} \quad (4.1)$$

and

$$F^{\alpha\beta}{}_{;\beta} = (-g)^{-\frac{1}{2}} ( (-g)^{\frac{1}{2}} F^{\alpha\beta} )_{,\beta} \quad (4.2)$$

since  $F^{\alpha\beta}$  is anti-symmetric.

Next we will consider the basic gravitational equations involving  $R_{\mu\nu}$  and the stress-energy tensor  $T_{\mu\nu}$ . The electromagnetic field will make its appearance in this equation as  $T_{\mu\nu}$  is expressed in terms of  $F_{\mu\nu}$ . Then, as above, our main objective will be to formulate this equation in our notation as a total divergence.

Our spacetime is stationary and axially symmetric,

and we assume that the electromagnetic field contained therein will also be independent of the  $t$  and  $\phi$  coordinates. Likewise, the same properties are accorded to the four-vector potential  $A_\mu$ . It then follows that the electromagnetic field tensor  $F_{\mu\nu}$  reduces to

$$\begin{aligned} F_{AB} &= A_{B,A} - A_{A,B} = 0 \\ F_{AM} &= A_{M,A} - A_{A,M} = -A_{A,M} \\ F_{MN} &= A_{N,M} - A_{M,N}. \end{aligned} \quad (4.3)$$

Since we are working only in the source-free arena outside of the body that is generating the gravitational and electromagnetic fields, Maxwell's equations may be written

$$F^{\mu\nu}{}_{;\nu} = (-g)^{\frac{1}{2}} F^{\mu\nu}{}_{,\nu} = 0 \quad (4.4)$$

$$F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} + F_{\alpha\beta,\gamma} = 0. \quad (4.5)$$

Written in this fashion we have a set of coupled, first order, partial differential equations relating components of the field variables. If the vector potential is introduced, then a smaller number of second order equations are obtained. In particular, Equation (4.5) will be satisfied identically.

Consider Equation (4.4). For the  $(X^3, X^4)$  space,

$$(\rho h^{\frac{1}{2}} F^{MN})_{,N} = (\rho h^{\frac{1}{2}} h^{NR} F^M_R)_{,N} . \quad (4.6)$$

This implies, using Equation (2.17),

$$\vec{\nabla}_2 \cdot (\rho \vec{V}^M) = 0 \quad (4.7)$$

where

$$\vec{V}^M \equiv (F^M_3, F^M_4) . \quad (4.8)$$

An immediate solution is

$$V^M_N = \varepsilon^M_N \rho^{-1} \quad (4.9)$$

or

$$F^{MN} = C \rho^{-1} \varepsilon^{MN} . \quad (4.10)$$

The equivalent physical situation is a magnetic field in the  $\phi$  direction, falling off as  $\rho^{-1}$ , produced by a line current along the symmetry axis. Such situations are to be avoided because they are unphysical, so set  $C = 0$ .

This implies

$$A_N = 0 . \quad (4.11)$$

The final set of equations, from Equation (4.3) are

$$\begin{aligned} 0 &= ((-g)^{\frac{1}{2}} F^{AM})_{,M} \\ &= (\rho h^{\frac{1}{2}} f^{AB} h^{MN} F_{BN})_{,N} . \end{aligned} \quad (4.12)$$

Now

$$F_{BN} = -A_{B,N} = -[\nabla_2 A_B]_N \quad (4.13)$$

[N<sup>th</sup> component].

Therefore, using Equation (2.17),

$$\vec{\nabla}_2 \cdot [\rho f^{\overset{\circ}{\circ}}{AX} \vec{\nabla}_2 A_X] = \vec{\nabla}_2 \cdot [\rho^{-1} f^{\overset{\circ}{\circ}}{AX} \vec{\nabla}_2 A_X] = 0, \quad (4.14)$$

then, using Equation (2.26)

$$\vec{\nabla}_3 \cdot [f^{\overset{\circ}{\circ}}{AX} \vec{\nabla}_3 A_X] = \vec{\nabla}_3 \cdot [\rho^{-2} f^{\overset{\circ}{\circ}}{AX} \vec{\nabla}_3 A_X] = 0. \quad (4.15)$$

Now we must connect this with the full Einstein equations. The presence of the electromagnetic field is incorporated into the geometry of spacetime via its stress-energy tensor

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \quad (4.16)$$

The electromagnetic stress-energy tensor is given by

$$4\pi T_{\mu\nu} = F_{\mu}^{\sigma} F_{\nu\sigma} - \frac{1}{2}g_{\mu\nu} F_{\sigma}^{\tau} F_{\tau}^{\sigma}. \quad (4.17)$$

So

$$\begin{aligned} 4\pi T^A_B &= F^{AX} F_{BX} + F^{AM} F_{BM} - \frac{1}{2} \delta^A_B F^X_Z F^Z_X \\ &\quad - \frac{1}{2} \delta^A_B F^M_X F^X_M - \frac{1}{2} \delta^A_B F^X_M F^M_X \\ &\quad - \frac{1}{2} \delta^A_B F^M_N F^N_M. \end{aligned} \quad (4.18)$$

Using Equations (4.3) and (4.11) we obtain

$$\begin{aligned} 4\pi T^A_B &= A^{A,M} A_{B,M} - \frac{1}{2} \delta^A_B A_X^M A_X^M \\ &\quad - \frac{1}{2} \delta^A_B A_X^M A_X^M \\ &= f^{AX} A_X^M A_{B,M} - \frac{1}{2} \delta^A_B f^{XZ} A_X^M A_X^M \\ &\quad - \frac{1}{2} \delta^A_B f^{ZX} A_X^M A_X^M \\ &= f^{AX} \vec{\nabla}_2 A_X \cdot \vec{\nabla}_2 A_B - \frac{1}{2} \delta^A_B f^{XZ} \vec{\nabla}_2 A_X \cdot \vec{\nabla}_2 A_Z. \end{aligned} \quad (4.19)$$

Previously it was noted that the field equations could be put into the form of a vanishing divergence, so with that objective in mind, we note that

$$\begin{aligned} & \vec{\nabla}_2 \cdot [\rho^{-1} (f^{AX} A_B \vec{\nabla}_X - \frac{1}{2} \delta_B^A f^{XZ} A_X \vec{\nabla}_Z)] \\ &= A_B \vec{\nabla}_2 \cdot [\rho^{-1} f^{AX} \vec{\nabla}_X] + \rho^{-1} f^{AX} \vec{\nabla}_2 A_B \cdot \vec{\nabla}_2 A_X \\ & \quad - \frac{1}{2} \delta_B^A A_X \vec{\nabla}_2 \cdot [\rho^{-1} f^{XZ} \vec{\nabla}_Z] - \frac{1}{2} \delta_B^A \rho^{-1} f^{XZ} \vec{\nabla}_2 A_X \cdot \vec{\nabla}_2 A_Z \end{aligned} \quad (4.20)$$

Employing Equation (4.14) in Equation (4.19) yields

$$\begin{aligned} 4\pi T_B^A &= \rho [\rho^{-1} f^{AX} \vec{\nabla}_X \cdot \vec{\nabla}_A - \frac{1}{2} \delta_B^A f^{XZ} \vec{\nabla}_X \cdot \vec{\nabla}_Z] \\ &= \rho \vec{\nabla}_2 \cdot [\rho^{-1} (f^{AX} A_B \vec{\nabla}_X - \frac{1}{2} \delta_B^A f^{XZ} A_X \vec{\nabla}_Z)] \end{aligned} \quad (4.21)$$

Referring back to Equation (4.16) we see that we need to know the trace of  $T_{\mu\nu}$  and, in fact,

$$T = T^\mu{}_\mu = \frac{1}{4\pi} [F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} 4 F_{\mu\nu} F^{\mu\nu}] = 0 \quad (4.22)$$

Moreover, as is clear from Equation (4.21)

$$T^X{}_X = 0 \quad (4.23)$$

Therefore, Equation (4.16) becomes

$$R_B^A = -8\pi T_B^A \quad (4.24)$$

Recalling our previous expression for  $R_B^A$ , Equation (3.3), we have

$$\vec{\nabla}_2 \cdot [\rho^{-1} (f^{AX} \vec{\nabla}_X f_{BX} - 4f^{AX} A_B \vec{\nabla}_X + 2\delta_B^A f^{XZ} A_X \vec{\nabla}_Z)] = 0 \quad (4.25)$$

Therefore, finally,

$$\vec{\nabla}_2 \cdot [\rho^{-1} (f^{AX} \vec{\nabla}_X f_{BX} - 2f^{AX} A_B \vec{\nabla}_X - 2f_B^X A^A \vec{\nabla}_X)] = 0 \quad (4.26)$$

using the identity

$$V_B^A - V_B^A = \delta_B^A V_X^X. \quad (4.27)$$

Note that Equations (4.14) and (4.26) are the Einstein-Maxwell equations written in the desired form of total divergences.

## CHAPTER IV

### REFORMULATION

#### Gravitational and Electromagnetic Potentials

If one could but solve Equations (4.14) and (4.26), then our work would be complete. However, as such is not the case, we begin to examine these key equations to see what we can discover. That is, what symmetries do they possess, what gauge freedoms, and what are the transformations that leave them invariant?

First, we recall that we went to the bother of writing the Einstein-Maxwell equations in the form of total divergences for a specific reason. Our purpose was to develop the potentials which, by reason of the discussion on Twist Potentials, must exist as a result of these vanishing divergences.

In this section a reformulation of the Einstein-Maxwell equations into a compact form possessing surprising utility will be accomplished.

As we have given much advance fanfare concerning the usefulness of the potentials associated with vanishing divergences, we will now take an initial look at them and develop some key equations.

Consider the divergence equation involving the

electromagnetic potential, i.e., Equation (4.14). As detailed in the section on twist potentials, this equation implies the existence of a potential,  $B_A$ , such that

$$\vec{\nabla} B_A = -\rho^{-1} f_A^{X\gamma} \vec{\nabla}_X A_\gamma . \quad (5.1)$$

Before we proceed any further, we should attempt to determine how the potential  $B_A$  fits into the usual scheme of electromagnetic notation. We begin with a consideration of Maxwell's equations in a source-free space. This will lead to a set of potentials which will possess the same relationship to the customary electromagnetic potentials as  $B_A$  has to  $A_A$  in flat space.

The basic equations are:

$$\vec{\nabla}_X \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad (5.2)$$

$$\vec{\nabla}_X \vec{D} = -\frac{\partial \vec{H}}{\partial t} \quad (5.3)$$

$$\vec{\nabla} \cdot \vec{D} = 0 \quad (5.4)$$

$$\vec{\nabla} \cdot \vec{H} = 0 . \quad (5.5)$$

Equation (5.5) implies the familiar relation

$$\vec{H} = \vec{\nabla}_X \vec{A} . \quad (5.6)$$

Likewise, Equation (5.4) implies the less common identity

$$\vec{D} = \vec{\nabla}_X \vec{\zeta} . \quad (5.7)$$

Equations (5.3) and (5.6) imply the existence of a scalar potential

$$-\vec{\nabla}\phi = \vec{D} + \frac{\partial \vec{A}}{\partial t} . \quad (5.8)$$

Likewise, Equations (5.2) and (5.7) imply the existence of a magnetic scalar potential<sup>13</sup>

$$-\vec{\nabla}\chi = \vec{H} - \frac{\partial \vec{\zeta}}{\partial t} . \quad (5.9)$$

Combining Equations (5.6), (5.7), (5.8), and (5.9) we obtain

$$\vec{\nabla}_X \vec{A} = -\vec{\nabla}\chi + \frac{\partial \vec{\zeta}}{\partial t} \quad (5.10)$$

$$\vec{\nabla}_X \vec{\zeta} = -\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t} . \quad (5.11)$$

Now apply  $\frac{\partial}{\partial t} = \frac{\partial}{\partial \phi} = 0$  and Equation (4.11). Then in component form, we have [using physical coordinates<sup>14</sup>]

$$\begin{aligned} \bar{\chi}_{,\rho} &= \rho^{-1} (\rho \bar{A}_\phi)_{,Z} \\ \bar{\chi}_{,Z} &= -\rho^{-1} (\rho \bar{A}_\phi)_{,\rho} \\ \bar{A}_{t,\rho} &= \rho^{-1} (\rho \bar{\zeta}_\phi)_{,Z} \\ \bar{A}_{t,Z} &= -\rho^{-1} (\rho \bar{\zeta}_\phi)_{,\rho} . \end{aligned} \quad (5.12)$$

At this point it is informative to put things in a covariant form. For simplicity we will work in flat space. The line element is, in cylindrical coordinates,

$$ds^2 = dt^2 - \rho^2 d\phi^2 - d\rho^2 - dz^2 . \quad (5.13)$$

Since

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (5.14)$$

it seems appropriate, in the spirit of this section, to

define a "dual" four-vector potential

$$\bar{F}_{\mu\nu} \equiv a_{\nu,\mu} - a_{\mu,\nu} \quad (5.15)$$

How are  $a^\mu$  and  $A^\mu$  related?

$$\bar{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = -\frac{1}{2}(-g)^{-\frac{1}{2}}[\mu\nu\alpha\beta]F_{\alpha\beta} \quad (5.16)$$

where we have employed the fact that the Levi-Civita tensor  $\epsilon^{\mu\nu\alpha\beta}$  is related to the completely antisymmetric symbol  $[\mu\nu\alpha\beta]$  by

$$\epsilon^{\mu\nu\alpha\beta} = -(-g)^{-\frac{1}{2}}[\mu\nu\alpha\beta] \quad (5.17)$$

Accordingly,

$$a^{\nu,\mu} - a^{\mu,\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}[A_{\beta,\alpha} - A_{\alpha,\beta}] \quad (5.18)$$

Then we find that, with our symmetries,

$$a^{\hat{A},M} = -\rho^{-1}[MAND]A_{D,N} \quad (5.19)$$

where

$$\begin{aligned} A,D &\rightarrow t,\phi \rightarrow 1,3 \\ M,N &\rightarrow \rho,Z \rightarrow 2,4 \end{aligned} \quad (5.20)$$

For example,

$$a^{\hat{t},\rho} = -\rho^{-1}[2143]A_{\phi,Z} \quad (5.21)$$

or

$$\bar{a}_{t,\rho} = \rho^{-1}(\rho\bar{A}_\phi)_{,Z} \quad (5.22)$$

using physical coordinates.  $[\bar{A}_\phi = \rho A^\phi = -\rho^{-1}A_\phi]$

Comparison with Equation (5.12) reveals that

$$A_A \leftrightarrow (\phi, A_\phi) \quad (5.23)$$

$$a_A \leftrightarrow (-\chi, \zeta_\phi) .$$

In order to relate this to  $B_A$  we look at Equation (5.1) in flat space and find

$$B_{t,\rho} = \rho^{-1} A_{\phi,Z} . \quad (5.24)$$

So, in more common electromagnetic formalism,

$$A_A \leftrightarrow (\phi, A_\phi) \quad (5.25)$$

$$B_A \leftrightarrow (-\chi, -\zeta_\phi) .$$

We now define the complex potential  $\phi_A$

$$\phi_A \equiv A_A + iB_A . \quad (5.26)$$

Equation (5.1) and the inverse relation

$$\vec{\nabla}_A A_A = -\rho^{-1} f_A^{XY} \vec{\nabla}_B B_X \quad (5.27)$$

imply

$$\vec{\nabla}_A \phi_A = -i\rho^{-1} f_A^{XY} \vec{\nabla}_X \phi_X . \quad (5.28)$$

It so happens that  $\phi_A$  is only the first of a remarkable series of potentials all of which will obey the same key equation!

The Maxwell relations were used in developing Equation (5.28), so now let us try to use our other field relation, Equation (4.26), to define a complex tensor quantity which, mirabile dictu, will obey a relation identical with Equation (5.28).

Using Equation (5.1) as a guide, we define the

twist potential related to Equation (4.26)

$$\vec{\nabla}\psi_{AC} = -\rho^{-1} (f_A^X \vec{\nabla} f_{XC} - 2f_A^X \vec{\nabla} A_C - 2f_C^X \vec{\nabla} A_X) . \quad (5.29)$$

Note that although  $f_{AC}$  is symmetric,  $\psi_{AC}$  is not. The inverse relation is

$$\vec{\nabla} f_{AC} = \rho^{-1} f_A^X (\vec{\nabla}\psi_{XC} + 2A_C \vec{\nabla} B_X + 2A_X \vec{\nabla} B_C) . \quad (5.30)$$

At this stage one might follow Equation (5.26) and consider the quantity  $f_{AC} + i\psi_{AC}$ . However, it turns out that we are still a stage or two from our final objective and it is, in fact, more fruitful to look at the quantity

$$\bar{\varepsilon}_{AC} = f_{AC} + i(\psi_{AC} + 2A_A B_C) . \quad (5.31)$$

Then combining Equations (5.29) and (5.30) we obtain

$$\vec{\nabla} \bar{\varepsilon}_{AC} - \phi_C \vec{\nabla} \phi_A^* = -i\rho^{-1} f_A^X (\vec{\nabla} \bar{\varepsilon}_{XC} - \phi_C \vec{\nabla} \phi_X^*) . \quad (5.32)$$

Then defining

$$\bar{G}_{AC} \equiv \bar{\varepsilon}_{AC} - \phi_A^* \phi_C , \quad (5.33)$$

Equation (5.32) may be written as

$$\vec{\nabla} \bar{G}_{AC} + \phi_A^* \vec{\nabla} \phi_C = -i\rho^{-1} f_A^X (\vec{\nabla} \bar{G}_{XC} + \phi_X^* \vec{\nabla} \phi_C) . \quad (5.34)$$

Noting that both Equation (5.32) and (5.34) are very nearly in the required form, we finally see that if one defines

$$H_{AB} \equiv \bar{G}_{AB} + \varepsilon_{AB}^K \quad (5.35)$$

where

$$\vec{\nabla}K = \phi_X^* \vec{\nabla}\phi^X, \quad (5.36)$$

then we have

$$\vec{\nabla}H_{AB} = -i\rho^{-1}f_A^X \vec{\nabla}H_{XB}. \quad (5.37)$$

The  $H_{AB}$  and  $\phi_A$  are the basic gravitational and electromagnetic potentials which we have been seeking. They will be the key by which the door to the generation of new solutions is opened. We should notice that at this moment and in the future, the close correlation between the electromagnetic and the gravitational potentials. This similarity occurs frequently, and the only difference results from the vector nature of electromagnetism and the tensor nature of gravitation. Thus, the electromagnetic field need not be thought of as having been included as an afterthought but participates from a position of equality. Furthermore, it is appropriate to note that Campbell and Morgan<sup>15</sup> have shown that the linear theory of gravity can be cast into a form which is very similar to the usual Maxwell form of electromagnetic theory, i.e., in terms of E and B. The gravitational E and B-like quantities obey a set of dyadic equations which are identical in appearance to the usual Maxwell equations in empty space.

Generalized, Covariant Ernst Equations

Now that we have in hand  $\phi_A$  and  $H_{AB}$ , is there anything which might presage their future promise? A review of Ernst's<sup>16</sup> work immediately reveals that we have obtained a covariant generalization of his potentials. His complex potential  $\Phi$  is equivalent to  $\phi_1$ , and his twist potential  $\phi$  is equivalent to our  $\psi_{11}$ . Additionally, he reformulates the Einstein-Maxwell equations in terms of a complex function

$$\varepsilon = \frac{\xi-1}{\xi+1} = f - \phi\phi^* + i\phi = f_{11} - \phi_1\phi_1^* + i\psi_{11}, \quad (6.1)$$

which is the  $_{1,1}$  component of our  $\bar{\varepsilon}_{AC}$  [see Equation (5.31)]. This reformulation is quite important as Ernst was able to show that various well-known solutions are particularly simple when expressed in terms of  $\xi$ , using prolate spheroidal coordinates. For example, the Schwarzschild solution, remarkably, is  $\xi = X$ . The Tomimatsu-Sato solutions<sup>4</sup> were also obtained via simplifications induced by Ernst's reformulation.

His version of the coupled Einstein-Maxwell equations may be written

$$\begin{aligned} (\text{Re}\varepsilon + \phi\phi^*) \nabla^2 \varepsilon &= (\vec{\nabla}\varepsilon + 2\phi^* \vec{\nabla}\phi) \cdot \vec{\nabla}\varepsilon \\ (\text{Re}\varepsilon + \phi\phi^*) \nabla^2 \phi &= (\vec{\nabla}\varepsilon + 2\phi^* \vec{\nabla}\phi) \cdot \vec{\nabla}\phi \end{aligned} \quad (6.2)$$

Referring back to Equation (6.1) for the definition of  $\xi$ , we note that the vacuum version of Equation (6.2) can be written

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\vec{\nabla}\xi \cdot \vec{\nabla}\xi . \quad (6.3)$$

If we suppose

$$\xi = R + iM , \quad (6.4)$$

then in the weak field, vacuum version of Equation (6.1) [where the  $\epsilon = f$ ], we have

$$f \rightarrow 1 - 2R, \quad (6.5)$$

and one also discovers that both  $R$  and  $M$  obey Laplace's equation.

It is well known that in the Newtonian limit of General Relativity one may write  $f \rightarrow 1 - 2\eta$  where  $\eta$  is the Newtonian gravitational potential. Therefore,  $R$  plays the role of a Newtonian potential, and, as Kinnersley and Kelley<sup>17</sup> have pointed out, the imaginary part of  $\xi$  can be viewed as a gravitational analogue of Equation (5.9), i.e., a magnetic scalar potential.

Let us now derive Equation (6.2) and put it into a slightly more useful form.

We begin by rewriting Equation (5.37)

$$\rho^{-1}\vec{\nabla}H_{AB} = i\rho^{-2}f_A^X \vec{\nabla}H_{XB} . \quad (6.6)$$

Then by Equation (3.1)

$$\vec{\nabla} \cdot (\rho^{-2} f_A^X \vec{\nabla} H_{XB}) = 0 . \quad (6.7)$$

Breaking the covariance let  $A = 2, B = 1$

$$\vec{\nabla} \cdot (\rho^2 f_{22} \vec{\nabla} H_{11} - \rho^{-2} f_{21} \vec{\nabla} H_{21}) = 0 . \quad (6.8)$$

In Equation (5.37) let  $A = 1, B = 1$  in order to eliminate the  $H_{21}$  term

$$\vec{\nabla} H_{11} + i\rho^{-1} f_{12} \hat{\nabla} H_{11} = i\rho^{-1} f_{11} \hat{\nabla} H_{21} . \quad (6.9)$$

Combining Equations (6.8), (6.9) and using Equation (1.3)

[the Lewis canonical form] we obtain

$$\vec{\nabla} \cdot (\rho^{-2} f_{22} \vec{\nabla} H_{11} + i\rho^{-1} \omega \hat{\nabla} H_{11} - \rho^{-2} f \omega^2 \vec{\nabla} H_{11}) = 0 . \quad (6.10)$$

Using Equation (1.3) again, then

$$\vec{\nabla} \cdot (-f^{-1} \vec{\nabla} H_{11} + i\rho^{-1} \omega \hat{\nabla} H_{11}) = 0 \quad (6.11)$$

or

$$f \nabla^2 H_{11} = (\vec{\nabla} f - i\rho^{-1} f^2 \hat{\nabla} \omega) \cdot \vec{\nabla} H_{11} . \quad (6.12)$$

Now we need to eliminate the  $\hat{\nabla} \omega$  term. Returning

to Equations (5.27) and (5.29) we note that

$$\vec{\nabla} B_1 = \rho^{-1} f (\hat{\nabla} A_2 + \omega \hat{\nabla} A_1) \quad (6.13)$$

and

$$\vec{\nabla} \psi_{11} = -\rho^{-1} f^2 \hat{\nabla} \omega - 4\rho^{-1} f A_1 (\hat{\nabla} A_2 + \omega \hat{\nabla} A_1) . \quad (6.14)$$

Solving for  $\hat{\nabla} \omega$ , we obtain

$$\hat{\nabla} \omega = -\rho f^{-2} (\vec{\nabla} \psi_{11} + 4A_1 \vec{\nabla} B_1) , \quad (6.15)$$

therefore,

$$f \nabla^2 H_{11} = [\vec{\nabla} f + i(\vec{\nabla} \psi_{11} + 4A_1 \vec{\nabla} B_1)] \cdot \vec{\nabla} H_{11} . \quad (6.16)$$

Using Equation (5.35) we find

$$H_{11} = f - A_1^2 - B_1^2 + i(\psi_{11} + 2A_1B_1) . \quad (6.17)$$

Taking the gradient and rearranging terms, we have, referring to the terms in Equation (6.16),

$$\vec{\nabla}f + i(\vec{\nabla}\psi_{11} + 4A_1\vec{\nabla}B_1) = \vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1 . \quad (6.18)$$

So, finally,

$$f\nabla^2 H_{11} = [\vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1] \cdot \vec{\nabla}H_{11} . \quad (6.19)$$

Comparing Equations (5.28) and (5.37) we see that the above derivation will go through in the same manner if we replace  $H_{11}$  by  $\phi_1$  down through Equation (6.16) and then continue as before.

Doing so, we obtain

$$f\nabla^2 \phi_1 = (\vec{\nabla}H_{11} + 2\phi_1^*\vec{\nabla}\phi_1) \cdot \vec{\nabla}\phi_1 . \quad (6.20)$$

In order to demonstrate that Equations (6.19) and (6.20) are the same as Equation (6.2), we need only note that  $\text{Re}\varepsilon + \phi\phi^* = f$  via Equation (6.1), and that  $(\varepsilon, \phi) \leftrightarrow (H_{11}, \phi_1)$  .

In summation, we see that not only does our formalism contain Ernst's work, but it generalizes it and places it on a covariant basis.

## CHAPTER 5

### THE GROUPS $G$ AND $H$

In general, whenever an object is left unchanged (invariant) by some operation, one says that the operation is a symmetry of the object. With this idea in mind, let us explore some of the symmetries exhibited by axially symmetric stationary field equations. Basically then, we are seeking transformations that leave the line element [see Equation (1.2)], the field equations [see Equations (4.14) and (4.26)], or various reformulations of the field equations invariant.

An examination of the field equations reveals that they are manifestly covariant with respect to linear transformations of the coordinates  $(t, \phi)$ . These transformations must be linear if  $t$  and  $\phi$  derivatives are to be avoided. Thus we have a three parameter group  $G$  with a particular representation of its generators given by

$$t \rightarrow t + a\phi \tag{7.1}$$

$$\phi \rightarrow \phi$$

$$t \rightarrow t \tag{7.2}$$

$$\phi \rightarrow \phi + bt$$

$$t \rightarrow ct \tag{7.3}$$

$$\phi \rightarrow c^{-1}\phi$$

It should be noted at this juncture that our attention is being directed only to the local properties of the metric, and we are not interested in global complications at the moment.

Another idea that should spring to mind, especially as we have incorporated Maxwell's equations, is the consideration of the gauge freedom involved. We will focus our attention on the basic potentials  $\phi_A$  and  $H_{AB}$ . Since the potentials are initially defined by means of differential equations, we are looking for quantities that can be added to these potentials and leave the defining equations invariant. Furthermore, the  $f_{AB}$  must remain unchanged.

$$A_A \rightarrow A_A + \gamma_A \quad (7.4)$$

$$B_A \rightarrow B_A + \sigma_A \quad (7.5)$$

So the gauge freedom for  $\phi_A$  is

$$\phi_A \rightarrow A_A + \gamma_A + i(B_A + \sigma_A) = \phi_A + a_A \quad \text{[two gauge freedoms]} \quad (7.6)$$

where  $a_A$  is an arbitrary complex constant.

Equations (5.26), (5.31), and (5.33) imply

$$\bar{G}_{AB} = f_{AB} - A_A A_B - B_A B_B + i(\psi_{AB} + A_A B_B + B_A A_B) \quad (7.7)$$

Equations (5.27) and (5.29) may be combined to give

$$\vec{\nabla} \psi_{AB} = -\rho^{-1} f_A^{X\gamma} \vec{\nabla} f_{XB} - 2A_B \vec{\nabla} B_A - 2A_A \vec{\nabla} B_B, \quad (7.8)$$

so

$$\psi_{AB} \rightarrow \psi_{AB} - 2\gamma_B^A - 2\gamma_A^B + \alpha_{AB} \quad (7.9)$$

where  $\alpha_{AB}$  is a real constant.

Combining Equations (7.4)-(7.7), and (7.9) we have

$$\bar{G}_{AB} \rightarrow \bar{G}_{AB} + i\alpha_{AB} - a_A^* \phi_B - a_B^* \phi_A - a_A^* a_B + 2i\gamma_A^B \sigma_B. \quad (7.10)$$

By redefining  $\alpha_{AB}$  this equation may be placed into conformity with Equation (7.26) of Paper I

$$\alpha_{AB} \rightarrow \alpha_{AB} + 2\gamma_A^B \sigma_B. \quad (7.11)$$

Then

$$\bar{G}_{AB} \rightarrow \bar{G}_{AB} + i\alpha_{AB} - a_A^* \phi_B - \phi_A a_B^* - a_A^* a_B. \quad (7.12)$$

In order to obtain the gauge condition for  $H_{AB}$  we refer back to Equation (5.35) and determined that only the gauge freedom of  $K$  remains to be discovered.

$$\bar{\nabla} K = \phi_X^* \bar{\nabla} \phi^X = \phi_1^* \bar{\nabla} \phi_2 - \phi_2^* \bar{\nabla} \phi_1 \quad (7.13)$$

thus,

$$K \rightarrow K + a_1^* \phi_2 - a_2^* \phi_1 + \frac{1}{2} a_X^* a^X \quad (7.14)$$

so

$$H_{AB} \rightarrow H_{AB} + i\alpha_{AB} - a_A^* \phi_B - \phi_A a_B^* - a_A^* a_B + \epsilon_{AB} (a_X^* \phi^X + \frac{1}{2} a_X^* a^X) \quad (7.15)$$

Thus  $H_{AB}$  has four gauge freedoms.

Now we have the group  $G'$ . It is composed of the coordinate transformations which make up the group  $G$  in addition to the gauge conditions detailed above. These

are the linear transformations of  $\phi_A$  and  $H_{AB}$  that preserve  $f_{AB}$ .

Various less obvious transformations are also known: the generalized Ehlers transformation [see Equation (7.16)], a "duality rotation" for gravitation [see Equation (7.24)], or the Harrison transformation<sup>18</sup> [see Equation (7.17)] which maps vacuum fields into charged fields. This leads us to a consideration of the "internal" group  $H'$ .

First, consider the actual form of the Ehlers and Harrison transformations

$$\phi_1 \rightarrow \frac{\phi_1}{1+i\gamma H_{11}} \quad H_{11} \rightarrow \frac{H_{11}}{1+i\gamma H_{11}} \quad (7.16)$$

and

$$\phi_1 \rightarrow \frac{\phi_1 + c H_{11}}{1 - 2c^* \phi_1 - c c^* H_{11}} \quad H_{11} \rightarrow \frac{H_{11}}{1 - 2c^* \phi_1 - c c^* H_{11}} \quad (7.17)$$

The field equations are, in fact, invariant under these transformations. Naturally one would be hard pressed to recognize this fact if only Equations (4.14) and (4.26) were considered. Only when one rewrites these equations, using some of the "internal" potentials, e.g., Equations (6.19) and (6.20), can one hope to discover such transformations. Now that we have the field equations in the appropriate form, we may more readily determine such mappings.

For instance, using Equations (7.6) and (7.15),

$$\phi_1 \rightarrow \phi_1 \quad H_{11} \rightarrow H_{11} + i\alpha \quad (7.18)$$

or

$$\phi_1 \rightarrow \phi_1 + a \quad H_{11} \rightarrow H_{11} - 2a^* \phi_1 - aa^* \quad (7.19)$$

where  $\alpha$  is real and  $a$  is complex. These gauge transformations will retain the invariance of the field equations.

One might also consider

$$\phi_1 \rightarrow H_{11}^{-1} \phi_1 \quad H_{11} \rightarrow H_{11}^{-1} \quad (7.20)$$

or

$$\phi_1 \rightarrow \beta e^{i\alpha} \phi_1 \quad H_{11} \rightarrow \beta^2 H_{11} \quad (7.21)$$

where  $\alpha$  and  $\beta$  are real. This latter operation combines a rescaling with an electromagnetic duality rotation. This duality rotation sends Schwarzschild mass into magnetic mass. In order to have a feel for this operation, consider what a duality rotation does for the case of electromagnetism.

In order to make our equations look symmetric, suppose we have both magnetic and electric currents and charge densities present. Then Maxwell's equations are

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_e & \vec{\nabla} \times \vec{H} &= \vec{J}_e + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= \rho_m & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \end{aligned} \quad (7.22)$$

Now consider the following "duality" transformation<sup>19</sup>

$$\begin{aligned}
 \begin{pmatrix} E \\ H \end{pmatrix} &= A \begin{pmatrix} E' \\ H' \end{pmatrix} & \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix} &= A \begin{pmatrix} \rho_e' \\ \rho_m' \end{pmatrix} \\
 \begin{pmatrix} D \\ B \end{pmatrix} &= A \begin{pmatrix} D' \\ H' \end{pmatrix} & \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix} &= A \begin{pmatrix} \vec{J}_e' \\ \vec{J}_m' \end{pmatrix}
 \end{aligned} \tag{7.23}$$

where

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} .$$

It is easy to show that Maxwell's equations are invariant under such a transformation. Thus, it is only a matter of convention whether we consider a particle to possess only electric charge and no magnetic charge. The key point is that if all particles have the property that the ratio of magnetic to electric charge is the same, then duality rotations are possible, such that the angle  $\theta$  may be chosen to make  $\vec{J}_m$  and  $\rho_m$  zero.

Now, consider performing the transformation in Equation (7.20), followed by the operation described by Equation (7.18).

Then

$$\phi_1 \rightarrow \frac{\phi_1 H_{11}^{-1}}{H_{11}^{-1} + i\gamma} \quad H_{11} \rightarrow \frac{1}{H_{11}^{-1} + i\gamma} . \tag{7.24}$$

But this is the same as Equation (7.16).

So, we are now producing the typical group action whereby one member of the group is transformed into another.

It should be noted that the gauge transformations do not commute with the other operations.

We find that if one applies the above six transformations to each other in various orders, no new members result. Thus we have the  $H'$  group. It has eight real parameters [Ehlers (1), Harrison (2), electromagnetic gauge (2), gravitational gauge (1), scaling (1), and duality (1)]. Three of the eight parameters in  $H'$  are only related to gauge transformations, but the remaining five parameters provide an automatic procedure for generating five-parameter "families" of stationary Einstein-Maxwell solutions from each known solution.  $G'$  integrated with  $H'$  yields the full symmetry group  $K'$ .

## CHAPTER 6

### POTENTIAL CHARACTERISTICS

#### Reissner-Nordstrom Potentials

Before we examine more completely the ramifications of the  $\phi_A$  and  $H_{AB}$ , it might be appropriate to examine their appearance for a specific case. A natural simple example is the charged Schwarzschild black hole [the Reissner-Nordstrom solution].

To obtain the Reissner-Nordstrom geometry one solves the Einstein field equations for a spherically symmetric, static gravitational field with no matter present but a radial electric field in the static orthonormal frame.

The line element is determined to be

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2d\Omega^2, \quad (8.1)$$

where

$$f = 1 - \frac{2m}{r} + \frac{e^2}{r^2} = \frac{r^2 - 2mr + e^2}{r^2}. \quad (8.2)$$

Since the metric is static,  $f_{12} = 0$  [see Equation (1.3)]. The electric field is given by

$$\vec{E} = \frac{e}{r^2} \hat{r}. \quad (8.3)$$

For  $e > m$  the coordinate system is well behaved down to  $r = 0$ . It is of interest to note that for an electron  $e/m = 2 \times 10^{21}$  in dimensionless units.

More pertinent to our situation, Ernst<sup>16</sup> shows how a vacuum solution of the Einstein equations may be easily transformed to a charged solution, provided that  $\epsilon$  and  $\phi$  are functionally related, i.e., via Equation (6.1) and

$$\phi = \frac{q}{\xi+1} \quad q \equiv e/m . \quad (8.4)$$

Then, if  $\xi_0$  is a vacuum solution,  $\xi = \xi_0 (1-q^2)^{\frac{1}{2}}$  is a charged version of the solution. That is, given the Kerr solution, we may immediately obtain the Kerr-Newman solution. This functional relationship also holds for the Reissner-Nordstrom and the charged Tomimatsu-Sato metrics.

As was previously forecast in dealing with the Ernst formalism, simplifications occur in many instances when prolate spheroidal coordinates are employed. [see Appendix A for a discussion of these coordinates].

Therefore, using Equations (6.1) and (8.4), we obtain [remember from the section on Generalized, Covariant Ernst Equations that  $\xi_0 = X$  is equivalent to the Schwarzschild solution]

$$f = \text{Re}\epsilon + |\phi|^2 = \frac{\xi_0 \xi_0^* - 1}{|\xi_0 + \beta|^2} = \frac{X^2 - 1}{(X + \beta)^2} , \quad (8.5)$$

where

$$\beta \equiv (1-q^2)^{-\frac{1}{2}} \quad (8.6)$$

and

$$\psi = \psi_{11} = \frac{\beta(\xi_0 - \xi_0^*)}{|\xi_0 + \beta|^2} = 0 \quad (8.7)$$

and

$$\phi = \phi_1 = \frac{q\beta}{x+\beta} \quad (8.8)$$

or

$$A_A = \begin{pmatrix} \frac{q\beta}{x+\beta} \\ 0 \end{pmatrix} \quad (8.9)$$

Using Equation (5.1) to solve for  $B_A$  then, up to a constant of integration,

$$B_A = \begin{pmatrix} 0 \\ q\beta Yk \end{pmatrix} \quad (8.10)$$

Since

$$B_1 = A_2 = f_{12} = f_{21} = 0 \quad (8.11)$$

then,

$$\begin{aligned} \vec{\nabla}\psi_{11} &= \vec{\nabla}\psi_{22} = 0 \\ \vec{\nabla}\psi_{12} &= -\rho^{-1}(-f_{11}\hat{\nabla}f_{22} - 2f_{22}A_1\hat{\nabla}A_1) \\ \vec{\nabla}\psi_{21} &= -\rho^{-1}(f_{22}\hat{\nabla}f_{11} - 2f_{22}A_1\hat{\nabla}A_1) \end{aligned} \quad (8.12)$$

Accordingly, up to a constant of integration,

$$\psi_{AB} = \begin{pmatrix} 0 & -2\beta Yk \\ 2Y(x-\beta)k & 0 \end{pmatrix} \quad (8.13)$$

Equation (5.36) implies

$$\vec{\nabla}K = i\vec{\nabla}(A_1B_2) \quad (8.14)$$

so

$$K = \frac{ikq^2\beta^2Y}{X+\beta} \quad (8.15)$$

$H_{AB}$  may be written out in full as

$$H_{AB} = (f_{AB}^{-A} A_B^{-B} B_A + \epsilon_{AB} K) + i(\epsilon_{AB}^{+A} B_B^{+B} A_A) \quad (8.16)$$

therefore,

$$H_{AB} = \begin{pmatrix} \frac{X-\beta}{X+\beta} & \frac{2ikY(X^2-1)}{X+\beta} \\ -2i\beta kY & [(X+\beta)^2(Y^2-1) - Y^2q^2\beta^2]k^2 \end{pmatrix} \quad (8.17)$$

Although prolate spheroidal coordinates are useful in that they greatly simplify calculations, they do not give most of us any immediate picture of what is involved. Therefore, let us use Equations (A.5) and (A.6) from Appendix A to convert  $(X, Y)$  to  $(r, \theta)$  Schwarzschild coordinates.

We then find

$$f = \frac{\left(\frac{r-m}{k}\right)^2 - 1}{\left(\frac{r-m}{k} + \frac{m}{k}\right)^2} = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \quad (8.18)$$

This is exactly what we expected from Equation (8.2).

Furthermore,

$$H_{11} = 1 - \frac{2m}{r} \quad (8.19)$$

$$\phi_1 = \frac{e}{r} \quad (8.20)$$

This, of course, is the expected form of the electromagnetic

potential. The others have less apparent physical meaning, e.g.,

$$\phi_2 = ie \cos\theta \quad (8.21)$$

$$K = \frac{ie^2 \cos\theta}{r} \quad (8.22)$$

$$H_{AB} = \begin{pmatrix} \frac{1-2m}{r} & 2ir \cos\theta \left[1 - \frac{2m}{r} + \frac{e^2}{r^2}\right] \\ -2im \cos\theta & -r^2 \sin^2\theta - e^2 \cos^2\theta \end{pmatrix} \quad (8.23)$$

### The Potential Hierarchy

Those of a curious turn of mind may well be attracted by Equation (5.36). Seeing that an essential part of our key gravitational potential  $H_{AB}$  is a "quadratic" potential  $K$ , one may well believe that other "quadratic" combinations may be useful. Let's form the possible groups with our basic potentials  $\phi_A$  and  $H_{AB}$ .

$$\vec{\nabla}K = \phi_X^* \vec{\nabla}\phi^X \quad (9.1)$$

$$\vec{\nabla}L_B = \phi_X^* \vec{\nabla}H^X_B \quad (9.2)$$

$$\vec{\nabla}M_A = H_{XA}^* \vec{\nabla}\phi^X \quad (9.3)$$

$$\vec{\nabla}N_{AB} = H_{XA}^* \vec{\nabla}H^X_B \quad (9.4)$$

Astonishingly, one can form quantities  $R_A$  and  $P_{AB}$ .

$$R_A = M_A + 2K\phi_A + H_{AX}\phi^X \quad (9.5)$$

$$P_{AB} = N_{AB} + 2\phi_A L_B + H_{AX}H^X_B \quad (9.6)$$

that obey the same equation as  $\phi_A$  and  $H_{AB}$ . [see Appendix B] We should be able to see the pattern forming. Not only do  $\phi_A$  and  $H_{AB}$  obey the same basic equation, but an apparently endless series of combinations of these basic potentials can be found which obey this same remarkable equation.

Clearly, we now need to develop a compact notation. Therefore, supposing that we are dealing with an infinite set of fields, let us initialize the series by setting

$$\overset{1}{\phi}_A = \phi_A \quad \overset{1}{H}_{AB} = H_{AB} \quad (9.7)$$

and

$$\overset{2}{\phi}_A = iR_A \quad \overset{2}{H}_{AB} = iP_{AB} \quad (9.8)$$

In general

$$\overset{n}{\nabla} \phi_A = -i\rho^{-1} f_A^X \overset{n}{\nabla} \phi_X \quad (9.9)$$

$$\overset{n}{\nabla} H_{AB} = -i\rho^{-1} f_A^X \overset{n}{\nabla} H_{XB} \quad (9.10)$$

In a similar fashion we generalize Equations (9.1) - (9.4)

$$\overset{mn}{\nabla} K = \phi_X^* \overset{n}{\nabla} \phi^X \quad (9.11)$$

$$\overset{mn}{\nabla} L_B = \phi_X^* \overset{n}{\nabla} H^X_B \quad (9.12)$$

$$\frac{m_n}{\sqrt{V}} M_A = H_{XA}^* \frac{n_X}{\sqrt{V}} \phi^X \quad (9.13)$$

$$\frac{m_n}{\sqrt{V}} N_{AB} = H_{XA}^* \frac{n_X}{\sqrt{V}} H_B^X \quad (9.14)$$

We notice that  $R_A$  and  $P_{AB}$  give us recursive relations for  $\phi_A$  and  $H_{AB}$ , i.e.,

$$\phi_A^{n+1} = i (M_A^{n+1} + 2\phi_A^{n+1} K + H_{AX}^n \phi^X) \quad (9.15)$$

$$H_{AB}^{n+1} = i (N_{AB}^{n+1} + 2\phi_A^{n+1} L_B + H_{AX}^n H_B^X) \quad (9.16)$$

Given this rather unexpected profusion of potentials, an analysis of their properties should follow. One might, however, question if we are not somehow going astray. After all, the metric is essentially governed by  $H_{11}^1$  and  $\phi_1^1$ , and this is all that is required. Nevertheless, our concern is not with known metrics, so much as with the attempt to generate new metrics. Unfortunately, it so happens that  $H_{AB}^1$  and  $\phi_A^1$  for the new metric cannot be ascertained until one has, in essence, all the potentials, both for the old and the new metric. This being the case, it seems appropriate to begin a consideration of the properties of these potentials.

For the specific example of the Schwarzschild metric, one of the properties first noted was that with an

appropriate choice of integration constants, one could always have the relation

$${}_{i}^{n+1} H_{A1} = {}_{A2}^n H_{A2} \quad (9.17)$$

In fact, Chitre<sup>20</sup> was able to show that if Equation (9.17) holds for some value  $n$ , for a particular stationary axially symmetric vacuum metric, then it holds for all higher values also. In particular, if  $\delta$  is the integer distortion parameter for the Zipoy-Voorhees metrics<sup>24,25</sup> [ $H_{11} = \left(\frac{x-1}{x+1}\right)^\delta$ ], then Equation (9.17) first holds for  $n = \delta$  and then for all higher orders. It is not difficult to prove that this result continues to hold even if an electromagnetic field is included. [See Appendix C for proof.]

We might also comment on the interdependence of  $\phi_A$  and  $H_{AB}$ . Noting that

$$\vec{\nabla}^n L_B = \phi_X^{1*} \vec{\nabla}^n H_B^X \quad (9.18)$$

$$\vec{\nabla}^n M_A = H_{XA}^{1*} \nabla^n \phi^X \quad (9.19)$$

then a glance at Equations (9.15) and (9.16) reveals that

the  ${}_{AB}^{n+1} H_{AB}(\phi_A)$  are only coupled to the  $\phi_A({}_{AB}^n H_{AB})$  by  $\phi_A({}_{AB}^1 H_{AB})$ . Thus, in  ${}_{AB}^{n+1} H_{AB}$  only  ${}_{AB}^n H_{AB}$  terms and  $\phi_A$  contribute, and conversely for  $\phi_A$ .

Separation of Variables

Return for the moment to Equation (6.7). Using flat space, i.e.,

$$f_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -\rho^2 \end{pmatrix} \quad (10.1)$$

and setting  $A = 1$ ,  $B = 1$ , we find

$$\nabla^2 H_{11} = 0. \quad (10.2)$$

This is, of course, Laplace's equation.

One supposes that even if the space is not flat, it may prove advantageous to apply some of the usual methods for the solution of Laplace's equation. For example, what about separation of variables? For simplicity, we will use the Schwarzschild metric as an example.

Remembering that  $f_{12} = f_{21} = 0$ , then Equation (6.7) yields [using our new notation - see Equation (9.10)] for  $A = 2$

$$\vec{\nabla} \cdot [f_{11}^{-1} \vec{\nabla}_{H_{1B}}^n] = 0, \quad (10.3)$$

and for  $A = 1$

$$\vec{\nabla} \cdot [f_{22}^{-1} \vec{\nabla}_{H_{2B}}^n] = 0. \quad (10.4)$$

Consider Equation (10.3). Using the results from Appendix A on prolate spheroidal coordinates, then Equation (10.3) may be rewritten as [ $f = f_{11}$ ]

$$-f^{-2} \vec{\nabla} f \cdot \vec{\nabla}_{H_{1B}}^n + f^{-1} \vec{\nabla} \cdot (\vec{\nabla}_{H_{1B}}^n) = 0 \quad (10.5)$$

or

$$\begin{aligned}
 -f^{-1} [f, {}_X^n H_{1B,X} (X^2-1)] + \frac{\partial}{\partial X} [(X^2-1) {}_X^n H_{1B,X}] \\
 + \frac{\partial}{\partial Y} [(1-Y^2) {}_Y^n H_{1B,Y}] = 0 .
 \end{aligned}
 \tag{10.6}$$

Setting  $\beta = 1$ , to convert Reissner-Nordstrom to Schwarzschild in Equation (8.5), then Equation (10.6) may be written

$$2(X-1) {}_X^n H_{1B,X} + (X^2-1) {}_X^n H_{1B,XX} = 2Y {}_Y^n H_{1B,Y} - (1-Y^2) {}_Y^n H_{1B,YY} .
 \tag{10.7}$$

Following a similar procedure for Equation (10.4) yields

$$(1-Y^2) {}_Y^n H_{2B,YY} = -2 {}_Y^n H_{2B,Y} - (X^2-1) {}_X^n H_{2B,XX} .
 \tag{10.8}$$

Equations (10.7) and (10.8) show that we have indeed been able to achieve our objective, and, furthermore, a comparison of Equations (9.9) and (9.10) shows that we may replace  ${}_X^n H_{1B}$  with  $\phi_1$  and  ${}_Y^n H_{2B}$  with  $\phi_2$  in the above analysis, and the same equations will apply.

Let us deal first with Equation (10.7). Assume the ansatz

$${}_X^n H_{1B} = F(X)G(Y) ,
 \tag{10.9}$$

then, using a separation constant of  $n(n+1)$ , we obtain two differential equations

$$(1-Y^2)G'' - 2YG' + n(n+1)G = 0
 \tag{10.10}$$

$$(1-X^2)F'' - 2(X-1)F' + n(n+1)F = 0 .
 \tag{10.11}$$

Referring to Appendix D we see that the solution of Equation (10.10) is immediate:

$$G(Y) = P_n(Y) , \quad (10.12)$$

where the  $P_n(Y)$  are the familiar Legendre polynomials. Equation (10.10) has two constants of integration to be determined, and this gives us the following options:

$$F(X) = P_n^{(-1,1)}(X) \quad (10.13)$$

or

$$F(X) = \frac{1-X}{1+X} P_n^{(1,-1)}(X) , \quad (10.14)$$

where the  $P_n^{(\alpha,\beta)}(X)$  are the Jacobi polynomials.

We know that we should initialize the series by

$$\frac{1}{H_{11}} = \frac{X-1}{X+1} \quad (10.15)$$

$$\frac{1}{H_{12}} = 2iY(X-1) .$$

This allows us to make a definitive choice between the options, and decide that

$$\frac{n+1}{H_{11}} = \frac{X-1}{X+1} P_n^{(1,-1)}(X) P_n(Y) \quad n = 0, 1, 2, \dots \quad (10.16)$$

$$\frac{n}{H_{12}} = iP_n^{(-1,1)}(X) P_n(Y) \quad n = 1, 2, \dots \quad (10.17)$$

Returning to Equation (10.8) and using the ansatz

$$H_{2B}^n = R(X)K(Y) , \quad (10.18)$$

with  $-n(n+1)$  for the separation constant, we obtain

$$(1-Y^2)K'' + n(n+1)K = 0 \quad (10.19)$$

$$(1-X^2)R'' - 2R' + n(n+1)R = 0 \quad (10.20)$$

Referring to Appendix D we find that Equation

(10.20) has a unique solution

$$R(X) = P_{n+1}^{(0, -2)}(X) \quad (10.21)$$

Equation (10.19) affords us the choice between the following two options:

$$K(Y) = P_{n+1}^{(-1, -1)}(Y) \quad (10.22)$$

or

$$K(Y) = C_{n+1}^{(-\frac{1}{2})}(Y) \quad [\text{Gegenbauer polynomials}]. \quad (10.23)$$

However, the possibilities of Equations (10.19) and (10.20) have not yet been exhausted. If we had chosen a different separation constant, say  $-(n+1)(n+2)$ , then we would have obtained

$$K(Y) = (1-Y^2)P_n^{(1, 1)}(Y) \quad (10.24)$$

or

$$K(Y) = (1-Y^2)C_n^{(\frac{3}{2})}(Y) \quad (10.25)$$

and

$$R(X) = (1+X)^2 P_n^{(0, 2)}(X) \quad (10.26)$$

Again the issue may be decided by a matching with

the initial terms

$$\begin{aligned} H_{21}^1 &= -2iY \\ H_{22}^1 &= (X+1)^2 (Y^2-1) \end{aligned} \quad (10.27)$$

The corresponding unique solutions are

$$H_{21}^{n+1} = 2i P_{n+1}^{(0,-2)}(X) C_{n+1}^{(-\frac{1}{2})}(Y) \quad n = 0, 1, \dots \quad (10.28)$$

$$H_{22}^n = (1+X)^2 P_n^{(0,2)}(X) (1-Y^2) P_n^{(1,1)}(Y) \quad n = 0, 1, \dots \quad (10.29)$$

It is of interest to note that earlier we noticed it was possible to select integration constants such that Equation (9.17) held. A glance at Equations (10.7) and (10.8), derived for the  $H_{AB}^n$  supposing separation of variables, reveals that the two methods are compatible. In particular, by Appendix D

$$P_n^{(-1,1)}(X) = \frac{X-1}{X+1} P_n^{(1,-1)}(X) \quad (10.30)$$

or

$$H_{12}^n = i \frac{X-1}{X+1} P_n^{(1,-1)}(X) P_n(Y) \quad (10.31)$$

so

$$i H_{11}^{n+1} = H_{12}^n \quad (10.32)$$

Unfortunately, these conditions will not be consistent

with a series of relations to be derived in the section on the Reissner-Nordstrom Generating Functions. These relations will be intimately connected with the action of the group  $K'$  on the potentials.

A comparison of the equations for flat space as opposed to those for the Schwarzschild metric reveals that, for separation of variables, while the radial coordinate equations differ, the angular equations are exactly the same. Therefore, as listed in Paper III, we find that the angular part of  $H_{21}^n$  and  $H_{22}^n$  may be placed in more familiar form by using the associated Legendre functions. Accordingly,

$$H_{21}^n \propto \sin \theta P_{n-1}^1(\cos \theta)$$

$$H_{22}^n \propto \sin \theta P_n^1(\cos \theta) .$$
(10.33)

To approach the problem from another viewpoint, we will consider the Reissner-Nordstrom solution. Although the physical implications of  $\phi_A^1$  and  $H_{AB}^1$  are understood, it is difficult to see what is the basic physical importance of the higher order potentials. One might ask, therefore, if it is ever possible to make the  $\phi_A^n$ , for  $n > 1$ , vanish. An examination of the recursive relation, Equation (9.15), plus Equations (9.11) and (9.13) shows that, if for any  $n$ ,

$\phi_A^n$  vanishes, then, with the appropriate integration constants,  $\phi_A^{n+1}$  may also be made to vanish. Specifically, in the case of Reissner-Nordstrom if we take

$$M_1^{11} = \frac{iq\beta Y(X-\beta)}{X+\beta}$$

$$M_2^{11} = \frac{q\beta}{X+\beta} [(X^2-1)Y^2 - (X+\beta)^2] \quad (10.34)$$

$$K^{11} = \frac{iY(\beta^2-1)}{X+\beta}$$

Then using the  $H_{AB}^1$  and  $\phi_A^1$  given in the section on the Reissner-Nordstrom Potentials, we see that

$$\phi_A^n = 0 \quad n = 2, 3, \dots \quad (10.35)$$

Thus at some point the higher order electromagnetic potentials may be made to vanish, at least for some static metrics.

## CHAPTER 7

### GENERATING FUNCTIONS

#### Differential Equations

Having developed this infinite hierarchy of potentials, we feel an immediate need for a compact way to express all the information contained therein. The generating function method fits the bill as it contains the total information of the potentials in a rather simple expression and also allows an expansion from which each potential may be subsequently determined.

At this point we will be concerned only with the process of developing generating functions for  $H_{AB}^n$  and  $\phi_A^n$ . We will, therefore, define the appropriate generating functions and determine the differential equations they must fulfill. Afterwards, the general solution will be found using the method of characteristics. Of course, the entire situation is not quite that simple. It might have been possible to stop at this point before we discovered the "quadratic" potentials  $N_{AB}^{mn}$ ,  $M_A^{mn}$ ,  $L_A^{mn}$ , and  $K$ . A glance back at the defining equations for these quantities reveals that knowledge of  $H_{AB}^n$  and  $\phi_A^n$  is not entirely sufficient as the defining equations are differential, and we will require the integrated forms. Surprisingly enough, it so happens

that, once we have finished the somewhat difficult process of finding the generating functions for  $H_{AB}^n$  and  $\phi_A^n$ , the generating functions for  $N_{AB}^{mn}$ ,  $M_A^{mn}$ ,  $L_A^{mn}$ , and  $K$  can be obtained merely as algebraic consequences of these earlier generating functions.

Upon examination, the recursion relations, Equations (9.15) and (9.16), reveal that not only do we require generating functions for the basic potentials  $H_{AB}^n$  and  $\phi_A^n$ , but also for  $K$  and  $L_A^{ln}$ .

Accordingly, let us define

$$F_{AB}(t) = \sum_{n=0}^{\infty} t^n H_{AB}^n \tag{11.1}$$

$$D_A(t) = \sum_{n=0}^{\infty} t^n \phi_A^n \tag{11.2}$$

$$S_A(t) = \sum_{n=0}^{\infty} t^n L_A^{ln} \tag{11.3}$$

$$Q(t) = \sum_{n=0}^{\infty} t^n K \tag{11.4}$$

where  $t$  is not the time coordinate but only an expansion parameter.

$Q$  and  $D_A$  are related as are  $S_A$  and  $F_{AB}$ , i.e.,

$$\vec{\nabla} Q = \sum t^n \vec{\nabla} K = \sum t^n \phi_X^* \vec{\nabla} \phi^X = \phi_X^* \vec{\nabla} D^X, \tag{11.5}$$

likewise,

$$\vec{\nabla} S_A = \phi_X^* \vec{\nabla} F_A^X. \tag{11.6}$$

Consider the basic relation for the  $H_{AB}^n$ , Equation

(9.10). Multiply both sides by  $t^n$  and sum

$$\sum_{n=0}^{\infty} t^n \vec{\nabla} H_{AB}^n = -i\rho^{-1} f_A^X \sum_{n=0}^{\infty} t^n \vec{\nabla} H_{XB}^n \quad (11.7)$$

or

$$\vec{\nabla} F_{AB} = -i\rho^{-1} f_A^X \vec{\nabla} F_{XB} . \quad (11.8)$$

Similarly, using Equation (9.9)

$$\vec{\nabla} D_A = -i\rho^{-1} f_A^X \vec{\nabla} D_X . \quad (11.9)$$

Once again the basic equations reassert themselves.

If we reconsider the derivation of the Ernst equations given in the section on Generalized, Covariant Ernst Equations, we note immediately that, instead of starting with Equation (5.37), we could have begun with Equation (9.10) or now, even better, Equation (11.8). Then, we would have the generalized Ernst equation at all levels. The results are, following the same steps as in the Generalized, Covariant Ernst Equations section,

$$f \nabla^2 F_{1B} = [\vec{\nabla} H_{11} + 2\phi_1^* \vec{\nabla} \phi_1] \cdot \vec{\nabla} F_{1B} \quad (11.10)$$

$$f \nabla^2 D_1 = [\vec{\nabla} H_{11} + 2\phi_1^* \vec{\nabla} \phi_1] \cdot \vec{\nabla} D_1 , \quad (11.11)$$

and, if there is no rotation, i.e.,  $\omega = 0$ , then the equations for  $F_{2B}$  and  $D_2$  take the form

$$\vec{\nabla} \cdot (\rho^{-2} f \vec{\nabla} F_{2B}) = 0 \quad (11.12)$$

$$\vec{\nabla} \cdot (\rho^{-2} f \vec{\nabla} D_2) = 0 . \quad (11.13)$$

Although the generalized Ernst equations have a simple looking appearance, they are not at all easy to solve, and, therefore, another approach must be found. We find that we are able to write first-order, coupled equations for the generating functions. These equations can be decoupled in general<sup>28</sup> [see Appendix E]. For particular situations, however, the coupling term drops out naturally, and the equation assumes a simple form. In this form one is able to use the method of characteristics to solve the equations. This gives a general solution in terms of an arbitrary function  $\gamma$ . One then discovers that Equation (11.10) enables us to determine a second order, ordinary, differential equation which sets the exact form of  $\gamma$ .

We begin with the recursion relations, Equations (9.15) and (9.16). Take the gradient of both sides of the equations, then multiply both sides of each equation by  $t^{n+1}$  and sum.

Then,

$$\vec{\nabla} F_{AB} = it \{ [H_{XA}^* + H_{AX} + 2\phi_A \phi_X^*] \vec{\nabla} F_B^X + 2S_B \vec{\nabla} \phi_A + F_B^X \vec{\nabla} H_{AX} \} \quad (11.14)$$

$$\vec{\nabla} D_A = it \{ [H_{XA}^* + H_{AX} + 2\phi_A \phi_X^*] \vec{\nabla} D^X + 2Q \vec{\nabla} \phi_A + D^X \vec{\nabla} H_{AX} \} \quad (11.15)$$

Defining

$$G_X^A \equiv \delta_X^A - it [H_X^{*A} + H_X^A + 2\phi_X^A \phi_X^*] \quad (11.16)$$

then

$$G^A_X = \delta^A_X [1 - 2tZ] - 2itf^A_X \quad (11.17)$$

where Equation (B.17) from Appendix B was used. Regrouping terms, we have

$$G^A_X \vec{\nabla}_F^X - itF^X_B \vec{\nabla}_H^A_X = 2itS_B \vec{\nabla}\phi^A \quad (11.18)$$

$$G^A_X \vec{\nabla}_D^X - itD^X \vec{\nabla}_H^A_X = 2itQ \vec{\nabla}\phi^A \quad (11.19)$$

Note that once we focus on a particular metric, the  $G_{AB}$ ,  $H_{AB}$ , and  $\phi_A$  are readily determined. What we require are the  $H_{AB}$  and  $\phi_A$ , i.e.,  $F_{AB}$  and  $D_A$ . Unfortunately, Equations (11.18) and (11.19) involve  $S_B$  and  $Q$  and not in just differential form either. Unluckily, we have  $F_{AB}$  ( $D_A$ ) involved in Equation (11.18) [(11.19)] in really three forms; differentiated, undifferentiated, and as part of an integral [see Equation (11.6)]. Other than the method outlined in Appendix E, we have two ways in which we might obtain decoupled differential equations for the generating functions. We already know of one method, namely the generalized Ernst equations we derived [Equations (11.10) - (11.13)]. The other alternative is to use metrics in which the terms naturally decouple. Surprisingly, this occurs in many of the cases of greatest physical interest. To see this latter result, we need to rewrite the equations so as to obtain, in so far as is possible, an equation involving

only one component of the generating function.

First we abandon covariance and look at particular indices.

Let  $A = 2$

$$G^2_1 \vec{\nabla} F^1_B + G^2_2 \vec{\nabla} F^2_B - it F^1_B \vec{\nabla} H^2_1 - it F^2_B \vec{\nabla} H^2_2 = 2it S_B \vec{\nabla} \phi^2 \quad (11.20)$$

$$G^2_1 \vec{\nabla} D^1 + G^2_2 \vec{\nabla} D^2 - it D^1 \vec{\nabla} H^2_1 - it D^2 \vec{\nabla} H^2_2 = 2it Q \vec{\nabla} \phi^2 \quad (11.21)$$

From Equations (11.8) and (11.9) we have, with  $A = 2$

$$\vec{\nabla} F^2_B - i\rho^{-1} f_{12} \vec{\nabla} F^2_B = i\rho^{-1} f_{11} \vec{\nabla} F^1_B \quad (11.22)$$

$$\vec{\nabla} D^2 - i\rho^{-1} f_{12} \vec{\nabla} D^2 = i\rho^{-1} f_{11} \vec{\nabla} D^1, \quad (11.23)$$

thereby obtaining

$$\begin{aligned} & \frac{G^2_1 [\vec{\nabla} F^2_B - i\rho^{-1} f_{12} \vec{\nabla} F^2_B] + G^2_2 \vec{\nabla} F^2_B + it F^1_B \vec{\nabla} H_{11} + it F^2_B \vec{\nabla} H_{12}}{i\rho^{-1} f_{11}} \\ & = -2it S_B \vec{\nabla} \phi_1 \end{aligned} \quad (11.24)$$

$$\begin{aligned} & \frac{G^2_1 [\vec{\nabla} D^2 - i\rho^{-1} f_{12} \vec{\nabla} D^2] + G^2_2 \vec{\nabla} D^2 + it D^1 \vec{\nabla} H_{11} + it D^2 \vec{\nabla} H_{12}}{i\rho^{-1} f_{11}} \\ & = -2it Q \vec{\nabla} \phi_1. \end{aligned} \quad (11.25)$$

Using Equation (11.17) and combining terms, we have

$$(1-2tZ) \vec{\nabla} F^2_B + 2t\rho \vec{\nabla} F^2_B + it F^1_B \vec{\nabla} H_{11} + it F^2_B \vec{\nabla} H_{12} = -2it S_B \vec{\nabla} \phi_1 \quad (11.26)$$

$$(1-2tZ) \vec{\nabla} D^2 + 2t\rho \vec{\nabla} D^2 + it D^1 \vec{\nabla} H_{11} + it D^2 \vec{\nabla} H_{12} = -2it Q \vec{\nabla} \phi_1. \quad (11.27)$$

In order to focus on  $F_{11}[D_1]$  choose  $B = 1$

$$-(1-2tZ) \vec{\nabla} F_{11} - 2t\rho \vec{\nabla} F_{11} + it F_{21} \vec{\nabla} H_{11} - it F_{11} \vec{\nabla} H_{12} = -2it S_1 \vec{\nabla} \phi_1 \quad (11.28)$$

$$-(1-2tZ) \vec{\nabla} D_1 - 2t\rho \vec{\nabla} D_1 + it D_2 \vec{\nabla} H_{11} - it D_1 \vec{\nabla} H_{12} = -2it Q \vec{\nabla} \phi_1. \quad (11.29)$$

We are now faced with two choices: eliminate  $F_{21}[D_2]$  algebraically and obtain

$$\begin{aligned}
& F_{11,X}[(1-2tZ)H_{11,Y}+2t(X^2-1)H_{11,X}] \\
& + F_{11,Y}[2t(1-Y^2)H_{11,Y}-(1-2tZ)H_{11,X}] \\
& = itF_{11}[H_{11,X}H_{12,Y}^{-1}H_{11,Y}H_{12,X}] \\
& + 2itS_1[\phi_{1,X}H_{11,Y}^{-1}\phi_{1,Y}H_{11,X}]
\end{aligned} \tag{11.30}$$

$$\begin{aligned}
& D_{1,X}[(1-2tZ)H_{11,Y}+2t(X^2-1)H_{11,X}] \\
& + D_{1,Y}[2t(1-Y^2)H_{11,Y}-(1-2tZ)H_{11,X}] \\
& = itD_1[H_{11,X}H_{12,Y}^{-1}H_{11,Y}H_{12,X}] \\
& + 2itQ[\phi_{1,X}H_{11,Y}^{-1}\phi_{1,Y}H_{11,X}]
\end{aligned} \tag{11.31}$$

or eliminate  $S_1[Q]$  and see that

$$\begin{aligned}
& F_{11,X}[2t(X^2-1)\phi_{1,X}+(1-2tZ)\phi_{1,Y}] \\
& + F_{11,Y}[2t(1-Y^2)\phi_{1,Y}-(1-2tZ)\phi_{1,X}] \\
& = itF_{11}[\phi_{1,X}H_{12,Y}^{-1}\phi_{1,Y}H_{12,X}] \\
& + itF_{21}[\phi_{1,Y}H_{11,X}^{-1}\phi_{1,X}H_{11,Y}]
\end{aligned} \tag{11.32}$$

$$\begin{aligned}
& D_{1,X}[2t(X^2-1)\phi_{1,X}+(1-2tZ)\phi_{1,Y}] \\
& + D_{1,Y}[2t(1-Y^2)\phi_{1,Y}-(1-2tZ)\phi_{1,X}]
\end{aligned} \tag{11.33}$$

$$= itD_1[\phi_{1,X}^{H_{12,Y}} - \phi_{1,Y}^{H_{12,X}}] \\ + itD_2[\phi_{1,Y}^{H_{11,X}} - \phi_{1,X}^{H_{11,Y}}] .$$

Similar equations follow the other  $F_{AB}$  and  $D_A$ .

An examination of Equations (11.30) and (11.31) reveals that we will have a first-order, differential equation for  $F_{11}$  or  $D_1$  alone if the quantity  $[\phi_{1,X}^{H_{11,Y}} - \phi_{1,Y}^{H_{11,X}}]$  happens to vanish. Is this at all likely? Recalling Equation (8.4) we note that  $H_{11}$  and  $\phi_1$  are functionally related for a number of physically interesting cases.

In particular, Equations (6.1) and (8.4) rewritten in our notation are

$$H_{11} = \frac{\xi-1}{\xi+1} \quad (11.34)$$

and

$$\phi_1 = \frac{q}{\xi+1} \quad (11.35)$$

However,

$$[\phi_{1,X}^{H_{11,Y}} - \phi_{1,Y}^{H_{11,X}}] \propto \nabla_{H_{11}} \cdot \vec{\nabla} \phi_1 \quad (11.36)$$

thus,

$$\nabla_{H_{11}} \cdot \vec{\nabla} \phi_1 \propto \nabla_{\xi} \cdot \vec{\nabla} \xi \quad (11.37)$$

by Equations (11.34) and (11.35).

Fortunately,

$$\vec{\nabla}\xi \cdot \nabla\xi = 0 \quad (11.38)$$

since

$$\vec{\nabla}U \cdot \nabla U \equiv 0 \quad (11.39)$$

for any potential U.

This simplifying result allows one to put Equations (11.30) - (11.33) into a rather compact form, i.e., a first-order, differential equation for  $F_{11}[D_1]$  involving only  $H_{11}$  or  $\phi_1$ . [See Appendix F for proof.] For example,

$$\frac{dF_{11}}{F_{11}} = \left( \frac{\frac{1}{2}\nabla^2 H_{11} - \rho^{-1} \vec{\nabla}H_{11} \cdot \vec{\nabla}\rho}{\vec{\nabla}H_{11} \cdot \vec{\nabla}H_{11}} \right) dH_{11} \quad (11.40)$$

Now let us take a particular case for which the equations for the generating functions do in fact decouple in the above fashion, e.g., the Reissner-Nordstrom solution, and actually solve the differential equation in full.

#### Reissner-Nordstrom Generating Functions

Keeping in mind the results of the section on the Reissner-Nordstrom Potentials, we rewrite Equation (11.30) in characteristic form since along each characteristic curve a partial differential equation may be reduced to an ordinary differential equation

$$\frac{F_{11}^{-1} dF_{11}}{it[H_{12,Y}H_{11,X} - H_{12,X}H_{11,Y}]} = \frac{dx}{(1-2tz)H_{11,Y} + 2t(X^2-1)H_{11,X}} \quad (12.1)$$

$$= \frac{dY}{2t(1-Y^2)H_{11,Y} - (1-2tZ)H_{11,X}} .$$

We then find

$$\frac{dY}{dX} = \frac{2tXY-1}{2t(X^2-1)} \quad (12.2)$$

$$\frac{dF}{F_{11}} = \frac{-dX}{X+\beta} \quad (12.3)$$

The solutions are

$$a_1 = (2tY-X)(X^2-1)^{-\frac{1}{2}} \quad (12.4)$$

$$F_{11} = \frac{a_2}{X+\beta} \quad (12.5)$$

where  $a_1$  and  $a_2$  are constant along each characteristic curve.

Therefore, the general solution is given by

$$F_{11} = (X+\beta)^{-1} \gamma \quad (12.6)$$

where  $\gamma$  is an arbitrary function of  $(2tY-X)(X^2-1)^{-\frac{1}{2}}$ . In order to determine the precise form of  $\gamma$ , we require another equation involving only  $F_{11}$ . Equation (11.10) fulfills this requirement.

Using the result of Appendix A on prolate spheroidal coordinate, we rewrite Equation (11.10) as

$$\frac{2(X^2-1)}{X+\beta} F_{11,X} + (X^2-1) F_{11,XX} - 2Y F_{11,Y} + (1-Y^2) F_{11,YY} = 0 \quad (12.7)$$

Now

$$F_{11} = (X+\beta)^{-1}\gamma(R) \quad \text{where } R = (2tY-X)(X^2-1)^{-\frac{1}{2}} \quad (12.8)$$

After a bit of algebra we obtain a second-order, regular differential equation for  $\gamma(R)$ .  $[\gamma' \equiv \frac{d\gamma}{dR}]$

$$\gamma''[R^2+4t^2-1] + 3R\gamma' = 0 \quad (12.9)$$

Its solution is

$$\gamma(R) = -K(t)R[R^2+4t^2-1]^{-\frac{1}{2}} + K_1(t) \quad (12.10)$$

Initially, it was not realized that  $K$  and  $K_1$  could be functions of the expansion parameter and so they were treated merely as constants. Acting under that incorrect assumption, it was a simple matter to determine  $K$  and  $K_1$ . We note that  $F_{11}$  is expanded  $H_{11}^0 t^0 + H_{11}^1 t^1 + \dots$ . Furthermore, it is clear that, if  $\gamma$  is a solution of Equation (12.9), so is  $\eta(t)\gamma$  where  $\eta(t)$  is an arbitrary function of the expansion parameter. We select  $\eta(t) = t$  in order to pick out  $K$  and  $K_1$  more readily.

Therefore, we know

$$H_{11} = \frac{X-\beta}{X+\beta} = t\gamma(R(t=0)) \quad (12.11)$$

using  $R$  at  $t = 0$  in order to exclude higher order terms.

Then we find, supposing for simplicity that  $K$  and  $K_1$  are constants,

$$K = 1 \quad K_1 = -\beta \quad (12.12)$$

or

$$F_{11} = \frac{t}{X+\beta} \left( \frac{X-2tY}{[1-4tXY+4t^2(X^2+Y^2-1)]^{\frac{1}{2}}} - \beta \right) \quad (12.13)$$

Unfortunately, the situation is not that simple, and we cannot ignore the full possibilities of  $K$  and  $K_1$ . This surprising freedom is a result of the gauge freedom of the potentials, and we will not be able to determine a more specific form for them until we examine a particular set of relations among the potentials of the entire infinite hierarchy. Even then, the gauge freedom is so large that we will have only half the number of equations necessary to determine the particular form of the generating functions and will be forced to impose, ad hoc, six additional constraints.

Therefore, thus far, we have

$$F_{11} = \frac{t}{(X+\beta)S} [K(t)[X-2tY] + K_1(t)S] \quad (12.14)$$

where

$$S^2 \equiv 1-4tXY + 4t^2(X^2+Y^2-1) \quad (12.15)$$

where  $K(t)$  and  $K_1(t)$  are functions of the expansion parameter.

A complete list of the generating functions with their full gauge freedom is detailed in Appendix G. We also note that  $F_{12}$  and  $D_1$  obey the same characteristic

equations as  $F_{11}$ . Equation (11.8) [(11.9)] allows us to relate  $F_{1B}[D_1]$  and  $F_{2B}[D_2]$ . Upon examination, Appendix G reveals that  $F_{11}$  has an initial multiplier of  $t$  whereas  $F_{12}$  does not. This is to place them into conformity with the notation to be used in the section on Recursion Relations, i.e.,

$${}^0 H_{AB} \equiv i\epsilon_{AB} . \quad (12.16)$$

So

$$F_{11} = 0 + tH_{11} + \dots \quad \text{while} \quad F_{12} = i + tH_{12} + \dots .$$

The coupling generating functions  $Q$  and  $S_A$  may be determined from Equations (11.5) and (11.6) now that we know  $F_{AB}$  and  $D_A$ . The process is lengthy but straight forward.

### Recursion Relations

To further investigate the properties of the potentials, we will now list equations which relate various potentials and their complex conjugates. We will also derive relations among adjacent potentials. These equations were stated in Paper II and are an essential ingredient in explicating the action of the group  $K'$  on the individual potentials. In the following, it should be kept in mind that, as we are dealing with the actual potentials and not

merely differential relations, various choices of integration constants have to be made at each step of the way. In doing so, we are implicitly imposing certain constraints, and, if we are later going to use the results of the action of the group  $K$  on the potentials, we must make sure that the generating functions we are using are in full conformity with these recursion relations.

First, consider the relation between the potentials and their complex conjugates.

Equation (9.11) allows us to write

$$\begin{aligned}
 \vec{\nabla}_K^{mn} - \vec{\nabla}_K^{nm*} &= \phi_X^{m*} \vec{\nabla} \phi_X^n - \phi_X^n \vec{\nabla} \phi_X^{m*} \\
 &= \phi_X^{m*} \vec{\nabla} \phi_X^n + \phi_X^{m*} X \vec{\nabla} \phi_X^n \\
 &= \vec{\nabla} (\phi_X^{m*} \phi_X^n) ,
 \end{aligned} \tag{13.1}$$

therefore,

$$\vec{\nabla}_K^{mn} - \vec{\nabla}_K^{nm*} = \phi_X^{m*} \phi_X^n . \tag{13.2}$$

In a similar fashion we obtain

$$\vec{L}_B^{mn} - \vec{M}_B^{nm*} = \phi_X^{m*} H_B^n . \tag{13.3}$$

$$\vec{N}_{AB}^{mn} - \vec{N}_{BA}^{nm*} = H_{XA}^{m*} H_B^n + \epsilon_{AB} \delta_0^m \delta_0^n . \tag{13.4}$$

The last term in Equation (13.4) is needed for conformity

with Equation (12.16).

What about adjacent potentials? By Equation (9.11)

$$\vec{\nabla}^{m,n+1} K - \vec{\nabla}^{m+1,n} K = \phi_X^{m*} \vec{\nabla}^{n+1} \phi_X - \phi_X^{m+1} \vec{\nabla}^n \phi_X, \quad (13.5)$$

the recursion relation, Equation (9.15), allows one to write

$$\vec{\nabla}^{n+1} \phi_X = i M_X^{ln} + 2i \phi_X^{ln} K + i H_Z^{nZ} \phi_X. \quad (13.6)$$

So

$$\begin{aligned} \vec{\nabla}^{m,n+1} K - \vec{\nabla}^{m+1,n} K &= \phi_X^{m*} \vec{\nabla} [i M_X^{ln} + 2i \phi_X^{ln} K + i H_Z^{nZ} \phi_X] \\ &+ \vec{\nabla}^n \phi_X [i M_X^{lm*} + 2i \phi_X^{*lm*} K + i H_{XZ}^{*m*Z} \phi_X]. \end{aligned} \quad (13.7)$$

Using Equation (9.13) and rearranging terms, then

$$\begin{aligned} \vec{\nabla}^{m,n+1} K - \vec{\nabla}^{m+1,n} K &= -i \phi_X^{m*} H_{ZX}^{*nZ} \vec{\nabla} \phi_X + i H_{XZ}^{*m*Z} \phi_X \vec{\nabla} \phi_X \\ &+ 2i [\phi_Z^{m*} \phi_X^{*n} \vec{\nabla} \phi_X + K \phi_X^{lm*} \vec{\nabla} \phi_X + \phi_X^{*lm*} K \vec{\nabla} \phi_X] \\ &+ i [\phi_X^{m*nZ} \vec{\nabla} H_Z^X + i H_Z^{Xm*} \phi_X \vec{\nabla} \phi_X + i M_X^{lm*} \vec{\nabla} \phi_X]. \end{aligned} \quad (13.8)$$

Now

$$\vec{\nabla}^{m1} (K \quad K) = K \phi_X^{*nX} \vec{\nabla} \phi_X + K \phi_X^{m*} \vec{\nabla} \phi_X \quad (13.9)$$

$$\vec{\nabla}^{m1} (L_X \phi_X) = \phi_X^{m*} \vec{\nabla} H_Z^X + L_X \vec{\nabla} \phi_X. \quad (13.10)$$

Then, making use of Equation (13.2) and (13.3), we have

$$L_X^{m1} = M_X^{lm*} + \phi_Z^{m*} H_X^Z \quad (13.11)$$

$$K = K^{m1} + \phi_X^{m*} \phi^X \quad (13.12)$$

Accordingly,

$$\vec{\nabla} (L_X^{m1} \phi^X) = \phi_Z^{n_Z m_Z^*} \vec{\nabla} H_X^Z + M_X^{lm*} \vec{\nabla} \phi^X + H_X^Z \phi_Z^{m*} \vec{\nabla} \phi^X \quad (13.13)$$

$$\vec{\nabla} (K \phi^X) = K \phi_X^{*} \vec{\nabla} \phi^X + \phi_Z^{m*} \phi_Z^{n_Z} \vec{\nabla} \phi^X + K \phi_X^{m*} \vec{\nabla} \phi^X \quad (13.14)$$

Comparison with Equation (13.8) reveals that we may write

$$\vec{\nabla}^{m,n+1} K - \vec{\nabla}^{m+1,n} K = 2i \vec{\nabla} (K \phi^X) + i \vec{\nabla} (L_X^{m1} \phi^X) \quad (13.15)$$

so

$$\vec{\nabla}^{m,n+1} K - \vec{\nabla}^{m+1,n} K = 2i K \phi^X + i L_X^{m1} \phi^X \quad (13.16)$$

However, as we have previously mentioned, these relations have a considerable interdependence, and it turns out that if we require Equations (9.15) and (9.16) to hold for  $n = 0$  then it follows that we must define

$$K^0 = -\frac{1}{2}i \quad (13.17)$$

and  $H_{AB}^0 = i\epsilon_{AB}$  which we already used in Equation (12.16).

It is useful to note that with this notation we may incorporate the  $\phi_A^n$  and  $H_{AB}^n$  within the framework of the quadratic

potentials. That is, from Equations (9.13) and (9.14),

$$\text{on } M_A = -i\phi_A \quad (13.18)$$

$$\text{on } N_{AB} = -iH_{AB} \quad (13.19)$$

Thus, the integration constant for Equation (13.16) must be modified. Therefore,

$${}^{m,n+1}K - {}^{m+1,n}K = 2i {}^{m1}K {}^{ln}L_X + \frac{i}{2} \delta_0^m \delta_0^n \quad (13.20)$$

Now there are no inconsistencies.

In a similar manner, we obtain<sup>22</sup>

$${}^{m,n+1}L_B - {}^{m+1,n}L_B = 2i {}^{m1}K {}^{ln}L_B + i {}^{m1}L_X {}^{nX}H_B \quad (13.21)$$

$${}^{m,n+1}M_A - {}^{m+1,n}M_A = 2i {}^{m1}M_A {}^{ln}K + i {}^{m1}N_{AX} {}^{nX}\phi \quad (13.22)$$

$${}^{m,n+1}N_{AB} - {}^{m+1,n}N_{AB} = 2i {}^{m1}M_A {}^{ln}L_B + i {}^{m1}N_{AX} {}^{nX}H_B - 3\epsilon_{AB} \delta_0^m \delta_0^n \quad (13.23)$$

To insure that Equations (13.2) - (13.4) and Equations (13.20) - (13.23) hold for both positive and negative values of  $m$  and  $n$ , we have used the notation established in Paper II that

$${}^{1-P,P}K = - {}^{P,1-P}K \equiv \frac{1}{2}i \quad (13.24)$$

$${}^{P,-P}N_{AB} = - {}^{-P,P}N_{AB} \equiv \epsilon_{AB} \quad (13.25)$$

for  $P \geq 0$ .

It is possible to continue to iterate Equations (13.20) - (13.23) and obtain more general relations, i.e.,

$$K_{m,n+k} - K_{m+k,n} = \sum_{s=1}^k (2i K_{ms, k-s+1, n} - L_X^{ms, k-s, n} M_X^{k-s, n}) \quad (13.26)$$

$$L_A^{m,n+k} - L_A^{m+k,n} = \sum_{s=1}^k (2i K_{ms, k-s+1, n} L_A^{ms, k-s, n} - L_X^{ms, k-s, n} N_A^{k-s, n}) \quad (13.27)$$

$$M_A^{m,n+k} - M_A^{m+k,n} = \sum_{s=1}^k (2i M_A^{ms, k-s+1, n} K_{ms, k-s, n} - N_{AX}^{ms, k-s, n} M_X^{k-s, n}) \quad (13.28)$$

$$N_{AB}^{m,n+k} - N_{AB}^{m+k,n} = \sum_{s=1}^k (2i M_A^{ms, k-s+1, n} L_B^{ms, k-s, n} - N_{AX}^{ms, k-s, n} N_B^{k-s, n}) \quad (13.29)$$

In Appendix H we give the details for the derivation of Equation (13.26).

Although the series of relations we have derived in this section may not appear very restrictive, it turns out that neither of the previously discussed relations, i.e.,  $iH_{A1}^{n+1} = H_{A2}^n$  and  $H_{AB}^n = F(X)G(Y)$  are compatible with the above.

And as Paper III points out, not all the potentials  $N_{AB}^{mn}$  are algebraically independent. Consider all the  $N_{AB}^{mn}$ ,  $M_A^{mn}$ ,  $L_A^{mn}$ , and  $K$  such that  $m+n = q$ . Using Equations (13.20) - (13.23) and considering all potentials whose indices  $m, n$  are such that  $m+n = q+1$  we find that, for the  $q+1$  potentials to be completely determined, we require the

potentials such that  $m+n \geq q$ , and only one potential of each type (say  $K, L_B, M_A, N_{AB}$ ) to determine all the potentials, such that  $m+n = q+1$ . In addition, we must keep in mind Equations (13.2) - (13.4). In the final analysis one may add a real constant to  $K$ , a complex constant to  $L_A$  and  $M_A$ , and a Hermitian constant to  $N_{AB}$ .

Therefore, we cannot afford to be cavalier in our approach but must derive an efficient procedure to ensure compliance with our derived relations. Once again, in the next section, we will turn to the powerful generating function approach.

#### Double Generating Functions

The situation is now slightly different from our former experience with generating functions. Previously, we dealt with only one superscript and, thus, required only one expansion parameter. Now we are using two indices, and, therefore, two expansion parameters will be needed.

We define the following:

$$\begin{aligned}
 K &\equiv \sum_{m,n=0}^{mn} K r^m t^n & 'K &\equiv \sum_{n=0}^{ln} K t^n = Q & K' &\equiv \sum_{m=0}^{ml} K r^m \\
 L_A &\equiv \sum_{m,n=0}^{mn} L_A r^m t^n & 'L_A &\equiv \sum_{n=0}^{ln} L_A t^n & L'_A &\equiv \sum_{m=0}^{ml} L_A r^m
 \end{aligned}
 \tag{14.1}$$

$$M_A \equiv \sum_{m,n=0}^{mn} M_A r^m t^n \quad \dot{M}_A \equiv \sum_{n=0}^{ln} M_A t^n \quad M'_A \equiv \sum_{m=0}^{ml} M_A r^m$$

$$N_{AB} \equiv \sum_{m,n=0}^{mn} N_{AB} r^m t^n \quad \dot{N}_{AB} \equiv \sum_{n=0}^{ln} N_{AB} t^n \quad N'_{AB} \equiv \sum_{m=0}^{ml} N_{AB} r^m .$$

The four generating functions  $K$ ,  $L_A$ ,  $M_A$ , and  $N_{AB}$  contain all the information that the primed quantities possess, and, thus, we will eliminate as many of the latter as possible in the following.

Multiply Equations (13.2) - (13.4) by  $r^m t^n$  and sum.

We acquire the following:

$$K - K^* = D_X^*(r) D^X(t) \quad (14.2)$$

$$L_A - M_A^* = D_X^*(r) F_A^X(t) \quad (14.3)$$

$$N_{AB} - N_{BA}^* = F_{XA}^*(r) F_B^X(t) + \epsilon_{AB} . \quad (14.4)$$

In like manner Equations (13.20) - (13.23) yield,

$$t^{-1} \left( K + \frac{ir}{2} \right) - r^{-1} \left( K - \frac{it}{2} \right) = 2iK'(r)Q(t) + iL_X'(r)D^X(t) + \frac{1}{2}i \quad (14.5)$$

$$t^{-1} L_A - r^{-1} L_A = 2iK'(r)S_A(t) + iL_X'(r)F_A^X(t) \quad (14.6)$$

$$t^{-1} M_A - r^{-1} [M_A + iD_A(t)] = 2iM_A'(r)Q(t) + iN_{AX}'(r)D^X(t) \quad (14.7)$$

$$t^{-1} (N_{AB} - \epsilon_{AB}) - r^{-1} (N_{AB} + iF_{AB}(t)) = 2M_A'(r)S_B(t) + iN_{AX}'(r)F_B^X(t) . \quad (14.8)$$

The recursive definitions for  $\phi_A^n$  and  $H_{AB}^n$  give

$$D_A(t) = it(\dot{M}_A(t) + 2\phi_A Q(t) + H_{AX}^n D^X(t)) \quad (14.9)$$

$$F_{AB}(t) - i\epsilon_{AB} = it[\hat{N}_{AB}(t) + 2\phi_A S_B(t) + H_{AX} F_B^X(t)] . \quad (14.10)$$

In order to eliminate the primed quantities, we need to develop a few more equations.

In Equation (13.2), set  $m = 1$  multiply by  $\lambda$  and sum.

$$\sum_K \lambda^n - \sum_K \lambda^{n*} = \sum_X \phi_X^* \phi_X \lambda^n \quad (14.11)$$

or

$$Q - \sum_K \lambda^{n*} r^m = \phi_X^* D^X , \quad (14.12)$$

thus,

$$Q(t) - K^*(t) = \phi_X^* D^X(t) . \quad (14.13)$$

In a like manner Equations (13.3) and (13.4) produce

$$S_A(t) - M_A^*(t) = \phi_X^* F_A^X(t) \quad (14.14)$$

$$\hat{N}_{AB}(t) - N_{BA}^*(t) = H_{XA}^* F_B^X(t) . \quad (14.15)$$

Setting  $n = 1$ , multiplying by  $\lambda^m$ , and summing over  $m$ , Equation (13.3) yields

$$L_A^*(r) - M_A^*(r) = D_X^*(r) H_A^X . \quad (14.16)$$

This is the final bit of new information we are able to exact from Equations (13.2) - (13.4).

Using Equations (14.8), (14.10), (14.14) and (14.15) we obtain

$$\left(\frac{r-t}{rt}\right) N_{AB} = t^{-1} \epsilon_{AB} + 2i S_B(t) S_A^*(r) - 2i S_B(t) \phi_Z^* F_A^{*Z}(r)$$

$$\begin{aligned}
& -r^{-1}F_B^X(t)F_{XA}^*(r) - 2i\phi_X^*S_A^*(r)F_B^X(t) \quad (14.17) \\
& -i(H_{ZX}^* + H_{XZ}^*)F_B^X(t)F_A^{*Z}(r).
\end{aligned}$$

Equations (14.7), (14.9), (14.14) and (14.15) yield

$$\begin{aligned}
\left(\frac{r-t}{rt}\right)M_A &= 2iQ(t)S_A^*(r) - 2iQ(t)\phi_ZF_A^{*Z}(r) \\
& - 2iS_A^*(r)\phi_ZD^Z(t) - r^{-1}D^X(t)F_{XA}^*(r) \quad (14.18) \\
& - i(H_{ZX}^* + H_{XZ}^*)F_A^{*Z}(r)D^X(t).
\end{aligned}$$

Equations (14.6), (14.9), (14.11) and (14.16) give

$$\begin{aligned}
\left(\frac{r-t}{rt}\right)L_A &= 2iS_A(t)Q^*(r) - 2iS_A(t)\phi_ZD^{*Z}(r) \\
& - r^{-1}F_A^X(t)D_X^*(r) - 2i\phi_X^*F_A^X(t)Q^*(r) \quad (14.19) \\
& - i(H_{XZ}^* + H_{ZX}^*)D^{*Z}(r)F_A^X(t).
\end{aligned}$$

Equations (14.5), (14.9), (14.13) and (14.16) produce

$$\begin{aligned}
\left(\frac{r-t}{rt}\right)K &= \frac{i}{2}\left(1 - \frac{t}{r} - \frac{r}{t}\right) + 2iQ(t)Q^*(r) \\
& - 2iQ(t)\phi_ZD^{*Z}(r) - 2iQ^*(r)\phi_Z^*D^Z(t) \quad (14.20) \\
& - r^{-1}D^Z(t)D_Z^*(r) - i(H_{XZ}^* + H_{ZX}^*)D^{*Z}(r)D^X(t).
\end{aligned}$$

Notice that our results are expressed entirely in terms of quantities which were previously calculated.

It is clear that if  $r$  is set equal to  $t$  on both sides of Equations (14.17) - (14.20), there remain only

algebraic combinations of the known generating functions. Perforce, these results must have the same gauge conditions as the initial relations, i.e., Equations (13.2) - (13.4) and Equations (13.20) - (13.23).

When one considers the struggle entailed in the production of the earlier generating functions, when only one expansion parameter was involved, it is only with much thankfulness that it is realized that the generating functions involving two expansion parameters were obtained through relatively straightforward, algebraic manipulations.

We now wish to determine the actual number of independent equations we are dealing with. Equations (14.18) and (14.19) are complex conjugates. If we drop the covariant notation in Equation (14.17), we find that the choices of  $A = 1, B = 2$  and  $A = 2, B = 1$  yield another complex conjugate pair. Accordingly, Equation (14.17) produces three distinct equations, Equation (14.18) [or (14.19)] gives two distinct equations, and Equation (14.20) yields one unique equation. Thus, we have a total of six distinct equations which involve the quantities  $F_{AB}$ ,  $D_A$ ,  $Q$ , and  $S_A$ . A glance at Appendix G reveals that these nine functions have a total of twelve functions of the expansion parameter  $t$  which are yet to be determined, i.e.,

$K_i, C_i, E_i$ .

We remember that Equations (7.6) and (7.15) predicted a total of six gauge freedoms for the electromagnetic and gravitational potentials. Thus, these are exactly the correct number of freedoms remaining as our six equations will fix six out of twelve gauge freedoms leaving six freedoms to be specified.

#### Particular Reissner-Nordstrom Generating Functions

Now we wish to examine the generating functions for a specific metric, where the equations constructed in the previous section are employed in conjunction with six additional constraints of our own selection, in order to finally determine a specific form for the generating functions. If we don't have this specific form, then it would be impossible to use the generating function for the purpose for which it was originally proposed. That is, until we know the exact forms of the functions of the expansion parameter, we are unable to expand the generating function in powers of the expansion parameter and, thereby, select the various potentials.

An examination of the basic potentials for the Reissner-Nordstrom metric reveals that  $S_2, D_1, \phi_1, H_{11}, H_{22},$

$F_{11}$ , and  $F_{22}$  are real while  $S_1$ ,  $D_2$ ,  $\phi_2$ ,  $H_{12}$ ,  $H_{21}$ ,  $F_{12}$ ,  $F_{21}$ , and  $Q$  are imaginary. Accordingly, the six distinct relations from Equations (14.17) - (14.20) reduce to

$$\begin{aligned} & -F_{11}F_{21}t^{-1} - iS_1^2 + 2iS_1\phi_1F_{21} + 2iS_1\phi_2F_{11} \\ & = -iH_{11}F_{21}^2 + iH_{22}F_{11}^2 + i(H_{12} - H_{21})F_{11}F_{21} \end{aligned} \quad (15.1)$$

$$\begin{aligned} & \frac{1}{t} -F_{11}F_{22}t^{-1} - F_{21}F_{12}t^{-1} - 2iS_1S_2 + 2iS_2\phi_1F_{21} \\ & + 2iS_2\phi_2F_{11} + 2i\phi_1S_1F_{22} + 2i\phi_2S_1F_{12} = -i(H_{21} - H_{12})F_{22}F_{11} \\ & + i(H_{12} - H_{21})F_{12}F_{21} - 2iH_{11}F_{22}F_{21} + 2iH_{22}F_{12}F_{11} \end{aligned} \quad (15.2)$$

$$\begin{aligned} & F_{12}F_{22}t^{-1} + iS_2^2 - 2iS_2\phi_1F_{22} - 2iS_2\phi_2F_{12} \\ & = iH_{11}F_{22}^2 - iH_{22}F_{12}^2 - i(H_{12} - H_{21})F_{12}F_{22} \end{aligned} \quad (15.3)$$

$$\begin{aligned} & F_{12}D_2t^{-1} + F_{22}D_1t^{-1} + 2iQS_2 - 2iQ\phi_1F_{22} - 2iQ\phi_2F_{12} \\ & - 2i\phi_1S_2D_2 - 2i\phi_2S_2D_1 = 2iH_{11}F_{22}D_2 - 2iH_{22}F_{12}D_1 \\ & - i(H_{12} - H_{21})F_{22}D_1 + i(H_{21} - H_{12})F_{12}D_2 \end{aligned} \quad (15.4)$$

$$\begin{aligned} & -F_{11}D_2t^{-1} - F_{21}D_1t^{-1} - 2iQS_1 + 2iQ\phi_1F_{21} + 2iQ\phi_2F_{11} \\ & + 2i\phi_1S_1D_2 + 2i\phi_2S_1D_1 = -2iH_{11}F_{21}D_2 + 2iH_{22}F_{11}D_1 \\ & + i(H_{12} - H_{21})F_{21}D_1 - i(H_{21} - H_{12})F_{11}D_2 \end{aligned} \quad (15.5)$$

$$\begin{aligned} & -\frac{1}{4} + iD_1D_2t^{-1} - Q^2 + 2Q\phi_1D_2 + 2Q\phi_2D_1 = -H_{11}D_2^2 + H_{22}D_1^2 \\ & + (H_{12} - H_{21})D_1D_2 \end{aligned} \quad (15.6)$$

A glance at Appendix G for the full form of the generating functions reveals that we have a formidable algebraic task ahead. Then we notice that if the equations are correct for all  $X, Y$ , they must hold in particular for  $X = Y = 0$ . Luckily, this reduction still retains all the functions of the expansion parameter we wish to determine.

In particular, for  $X = Y = 0$

$$H_{AB} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -\beta^2 \end{pmatrix} \quad \phi_A \rightarrow \begin{pmatrix} q \\ 0 \end{pmatrix}$$

$$Q \rightarrow iE_3 \quad S_A \rightarrow \begin{pmatrix} iK_3 \\ C_3 \end{pmatrix}$$

$$F_{AB} \rightarrow \begin{pmatrix} -tK_1 & \frac{iC_1}{\beta} \\ iK_2 - \frac{iK(1-4t^2)^{\frac{1}{2}}}{2} & \frac{-C_2}{2t} + \frac{C(1-4t^2)^{\frac{1}{2}}}{2} \end{pmatrix} \quad (15.7)$$

$$D_A \rightarrow \begin{pmatrix} \frac{tE_1}{\beta} \\ iE_2 - \frac{iE(1-4t^2)^{\frac{1}{2}}}{2} \end{pmatrix} .$$

Defining  $\eta = (1-4t^2)^{\frac{1}{2}}$  then Equation (15.1) reduces

to

$$\left[ K_2 - \frac{K\eta}{2} \right]^2 - \left[ K_2 - \frac{K\eta}{2} \right] \left[ \frac{K_1}{\beta} + 2K_3q \right] + t^2 K_1^2 + K_3^2 = 0 \quad (15.8)$$

Equation (15.2) may now be written

$$\begin{aligned} & \left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right] \left[ K_2 - \frac{K\eta}{2} \right] - \frac{1}{2} \left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right] \left[ \frac{K_1}{\beta} + 2qK_3 \right] \\ & - \frac{1}{2} \left[ K_2 - \frac{K\eta}{2} \right] \left[ \frac{C_1}{t\beta} - 2C_3q \right] + tK_1C_1 - K_3C_3 - \frac{1}{2}t = 0 . \end{aligned} \quad (15.9)$$

Equation (15.3) becomes

$$\left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right]^2 - \left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right] \left[ \frac{C_1}{t\beta} - 2C_3q \right] + C_1^2 + C_3^2 = 0 . \quad (15.10)$$

Equation (15.4) is, then,

$$\begin{aligned} & \left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right] \left[ E_2 - \frac{E\eta}{2} \right] - \frac{1}{2} \left[ \frac{C_2}{2t} - \frac{C\eta}{2t} \right] \left[ \frac{E_1}{\beta} + 2qE_3 \right] \\ & - \frac{1}{2} \left[ E_2 - \frac{E\eta}{2} \right] \left[ \frac{C_1}{t\beta} - 2C_3q \right] + tC_1E_1 - C_3E_3 = 0 . \end{aligned} \quad (15.11)$$

Equation (15.5) reduces to

$$\begin{aligned} & \left[ K_2 - \frac{K\eta}{2} \right] \left[ E_2 - \frac{E\eta}{2} \right] - \frac{1}{2} \left[ K_2 - \frac{K\eta}{2} \right] \left[ \frac{E_1}{\beta} + 2qE_3 \right] \\ & - \frac{1}{2} \left[ E_2 - \frac{E\eta}{2} \right] \left[ \frac{K_1}{\beta} + 2K_3q \right] + t^2K_1E_1 + K_3E_3 = 0 . \end{aligned} \quad (15.12)$$

And finally, Equation (15.6) becomes

$$\left[ E_2 - \frac{E\eta}{2} \right]^2 - \left[ E_2 - \frac{E\eta}{2} \right] \left[ \frac{E_1}{\beta} + 2qE_3 \right] + t^2E_1^2 + E_3^2 - \frac{1}{4} = 0 . \quad (15.13)$$

Of course, we still have only six equations but twelve unknowns. Therefore, we will now impose six ad hoc conditions.

Consider Equation (15.13). We choose to make  $D_A$  as simple as possible and to require asymptotic flatness for the electromagnetic potentials  $\phi_1$ , i.e.,  $\phi_1 \rightarrow 0$  as  $X \rightarrow \infty$ . Therefore, set  $E(t) = E_2(t) = 0$ . Equation (15.13) then

implies that

$$E_3(t) = \frac{1}{2}[1+4t^2(\beta^2-1)]^{-\frac{1}{2}} \quad (15.14)$$

$$E_1(t) = -q\beta[1+4t^2(\beta^2-1)]^{-\frac{1}{2}} \quad (15.15)$$

This completely fixes  $Q$  and  $D_A$ . The particular selection of signs was determined by the fact that  $D_A = \phi_A + t^2 \phi_A + \dots$ .

In order to exploit the rest of the gauge freedom, a combination of the  $F_{AB}$ , found advantageous in Paper IV for the generation of new solutions, i.e.,  $F_{A1} + itF_{A2}$ , will be made as simple as possible. Using Appendix G we see that

$$F_{11} + itF_{12} = (K-C) \left\{ \frac{t(X-2tY)}{(X+\beta)S} \right\} + (K_1 - C_1) \left\{ \frac{t}{X+\beta} \right\} \quad (15.16)$$

$$F_{21} + itF_{22} = i(K_2 - \frac{C_2}{2}) + itY(K_1 - C_1) + \frac{i(C-K)}{2S} \{1 - 2tY(X-\beta) - 4t^2(X\beta + 1 - Y^2)\} \quad (15.17)$$

Setting  $K = C$  will greatly reduce the complexity of the above. Furthermore, requiring  $C_1 = \beta C$  and  $K_1 = -\beta K$  will ensure a close similarity to the vacuum solutions given in Paper IV.

Having now selected values for six of the functions of the expansion parameters, we have used up our ad hoc possibilities, and the remaining functions must be determined by the constraint equations.

Acting on this we obtain from Equations (15.8) -

(15.12)

$$\begin{aligned}
K &= C = C_1 \beta^{-1} = -K_1 \beta^{-1} = (1-4t^2)^{-\frac{1}{2}} (1+4t^2 (\beta^2-1))^{-\frac{1}{2}} \\
K_2 &= \frac{1}{2} [1+4t^2 (\beta^2-1)]^{-\frac{1}{2}} - \frac{1}{2} [1+4t^2 (\beta^2-1)]^{\frac{1}{2}} [1-4t^2]^{-\frac{1}{2}} - \frac{1}{2} \\
K_3 &= 2t^2 K_1 q \beta \quad (15.18) \\
C_2 &= [1+4t^2 (\beta^2-1)]^{-\frac{1}{2}} + [1+4t^2 (\beta^2-1)]^{\frac{1}{2}} [1-4t^2]^{-\frac{1}{2}} - 1 \\
C_3 &= -2t C_1 q \beta .
\end{aligned}$$

Various signs have been determined by noting the first term in the series  $F_{AB} = i\epsilon_{AB} + tF_{AB} + \dots$

The above results do not represent the only possible specific representation of the generating functions, or even the most interesting one. In fact, we could well wish to examine the results when agreement with the Harrison transformation is enforced. This situation is of interest because the Harrison transformation and the procedures used in Paper IV for solution generation do not commute. If one generates a rotating solution from the Schwarzschild metric and then uses the Harrison transformation to charge it, this result should be distinct from a procedure that first charges the Schwarzschild metric and then uses the results of Paper IV to generate a solution with rotation. And yet the only natural candidate for the resulting product of both procedures is the well-known Kerr-Newman solution!

## CHAPTER 8

### THE HARRISON TRANSFORMATION

#### Action of the Harrison Transformation

In this section we wish to examine what happens to the electromagnetic and gravitational potentials when they undergo the action of the Harrison transformation. In particular, we wish to charge the Schwarzschild metric and then determine what remains to be done in order to bring it into conformity with the familiar results for the Reissner-Nordstrom metric. We will discover that appropriate changes of scale and gauge transformations are necessitated. The specific gauge transformation will be found to be related to the constant  $c$  in the Harrison transformation. The results will be specified by demanding that the form of the metric be invariant, and also making sure that the new potential  $\phi_1'$  has the familiar form  $\frac{e'}{r}$ .

The explicit action of the Harrison transformation on  $\phi_1$  and  $H_{11}$  has already been given by Equation (7.17). The Reissner-Nordstrom metric was given in Equation (8.1). Using Equation (8.16) we can determine the action on  $f$ .

For  $c$  real, using Equations (8.18) - (8.20), we find

$$f \rightarrow f' = \frac{f}{[1-2c\phi_1-c^2H_{11}]^2} \quad (16.1)$$

$$= \frac{r^2 - 2mr + e^2}{[r(1-c^2) - 2ce + 2mc^2]^2}$$

The line element may be written as

$$ds^2 = \left[ \frac{r^2 - 2mr + e^2}{r^2} \right] dt^2 - \left[ \frac{r^2}{r^2 - 2mr + e^2} \right] [dr^2 + (r^2 - 2mr + e^2) d\Omega^2] \quad (16.2)$$

Defining the following

$$\begin{aligned} m' &\equiv [m(1+c^2) - 2ce] [1-c^2]^{-1} \\ e' &\equiv [e(1+c^2) - 2mc] [1-c^2]^{-1} \end{aligned} \quad (16.3)$$

$$r' \equiv [r(1-c^2) - 2ce + 2mc^2] [1-c^2]^{-1},$$

then it turns out that

$$r'^2 - 2m'r' + e'^2 = r^2 - 2mr + e^2 \quad (16.4)$$

and

$$m'^2 - e'^2 = m^2 - e^2 \quad (16.5)$$

Applying these definitions in Equations (7.17) and (16.1)

we obtain

$$f \rightarrow f' = \left[ \frac{r'^2 - 2m'r' + e'^2}{r'^2} \right] [1-c^2]^{-2} \quad (16.6)$$

$$\phi_1 \rightarrow \phi'_1 = \left[ c + \frac{e'}{r'} \right] [1-c^2]^{-1} \quad (16.7)$$

$$H_{11} \rightarrow H'_{11} = 1 - c^2 - \frac{2m'}{r'} \left[ \frac{1-c^2}{1+c^2} \right] \quad (16.8)$$

A comparison of these last three equations with Equations (8.18) - (8.20) reveals that various changes of scale and gauge transformations are required to achieve

conformity with the desired final forms. We must also keep in mind that the Harrison transformation does not commute with the gauge transformation.

We will now elucidate the situation via an example.

$$H_{11} = 1 - \frac{2m}{r} = \frac{Xk-m}{Xk+m} \quad (16.9)$$

$$\phi_1 = \frac{e}{r} = \frac{e}{Xk+m} \quad (16.10)$$

Using Equation (16.3) we find

$$\begin{aligned} r-m &= r'-m' \\ X &= X' \\ Y &= Y' \\ S &= S' \end{aligned} \quad (16.11)$$

So under the Harrison transformation, as given by Equation (7.17), we find

$$H_{11} \rightarrow H'_{11} = [X'k' - \frac{1-c^2}{1+c^2}m'] [(1-c^2)(X'k'+m')]^{-1} \quad (16.12)$$

$$\phi_1 \rightarrow \phi'_1 = \left[ c + \frac{e'}{X'k'+m'} \right] [1-c^2]^{-1} \quad (16.13)$$

Applying the appropriate changes of scale

$$H_{11} \rightarrow (1-c^2) [X'k' - \frac{1-c^2}{1+c^2}m'] [X'k'+m']^{-1} \quad (16.14)$$

$$\phi_1 \rightarrow c + \frac{e'}{X'k'+m'} \quad (16.15)$$

Next, in seeking the appropriate gauge transformation, we refer back to Equation (7.6) and note that a choice of

$$a_1 = -c \quad (16.16)$$

will yield

$$\phi_1 \rightarrow \frac{e'}{X'k'+m'} = \frac{e'}{r'} \quad (16.17)$$

The gauge condition for  $H_{11}$ , Equation (7.15), may then be written

$$H_{11} \rightarrow H_{11} + 2c\phi_1 - c^2, \quad (16.18)$$

so,

$$\begin{aligned} H_{11} &\rightarrow [1-c^2] \left[ X'k' - \left[ \frac{1-c^2}{1+c^2} \right] m' \right] [X'k'+m']^{-1} \\ &\quad + 2c \left[ c + \frac{e'}{X'k'+m'} \right] - c^2 \\ &= [X'k' - m'] [X'k'+m']^{-1} \\ &= \frac{X' - \beta'}{X' + \beta'} \quad (16.19) \end{aligned}$$

This is the result obtained in Equation (8.17).

#### Application of the Harrison Transformation

In Appendix I we have listed the finite form of the Harrison transformation on all the potentials, in addition to the finite forms of the gauge action on the  $H_{AB}^n$  and  $\phi_A^n$ . Therefore, we are now in a position to apply the Harrison transformation to the Schwarzschild potentials, and acquire the Reissner-Nordstrom potentials.

To obtain the Schwarzschild potentials, we need only to set  $\beta = 1$  [or equivalently by Equation (8.6)  $e = 0$ ]

in the generating functions obtained in the section on the Reissner-Nordstrom Generating Functions.

Accordingly,

$$F_{11} = \frac{t[(X-2tY)-S][1-4t^2]^{-\frac{1}{2}}}{(X+1)S} \quad (17.1)$$

$$F_{12} = \frac{i[X-2tY+S][1-4t^2]^{-\frac{1}{2}}}{(X+1)S} \quad (17.2)$$

Now how do the  $F_{AB}$  transform under the Harrison and gauge transformations? As we are starting from an uncharged metric, then by Equations (I.5) and (I.6) from Appendix I and Equations (13.18) and (13.19), we have, for the Harrison transformation

$$H_{11}^n \rightarrow \frac{H_{11}^n}{1-c^2 H_{11}^n} \quad (17.3)$$

$$\phi_1^n \rightarrow \frac{c H_{11}^n}{1-c^2 H_{11}^n} \quad (17.4)$$

Therefore,

$$F_{11} \rightarrow \frac{F_{11}}{1-c^2 H_{11}} \quad (17.5)$$

$$D_1 \rightarrow \frac{c F_{11}}{1-c^2 H_{11}} \quad (17.6)$$

Also, using Equation (I.7) from Appendix I

$$H_{12}^n \rightarrow \frac{H_{12}^n - i c^2 H_{11}^{n+1}}{1-c^2 H_{11}^n} \quad (17.7)$$

so

$$\Sigma t^n H_{12} \rightarrow \frac{\Sigma t^n H_{12} - \frac{ic^2}{t} \Sigma H_{11} t^{n+1}}{1-c^2 H_{11}} \quad (17.8)$$

Therefore,

$$F_{12} \rightarrow \frac{F_{12} - ic^2 t^{-1} F_{11}}{1-c^2 H_{11}} \quad (17.9)$$

One may apply the same procedure to the gauge transformations. In fact, by Equations (I.11) and (I.14) of Appendix I

$$H_{11} \rightarrow H_{11} + 2c \phi_1 + ic^2 H_{12}, \quad (17.10)$$

thus,

$$F_{11} \rightarrow F_{11} + 2c D_1 + ic^2 t F_{12}. \quad (17.11)$$

Returning to Equation (16.3) one can determine the physical significance of the constant  $c$  in the Harrison transformation. Setting  $e = 0$ , it is easy to show that

$$\frac{e'}{m'} = \frac{-2c}{1+c^2}, \quad (17.12)$$

or in terms of  $\beta'$ ,

$$\left[1 - \left(\frac{e'}{m'}\right)^2\right]^{-\frac{1}{2}} \equiv \beta' = \frac{1+c^2}{1-c^2}. \quad (17.13)$$

Using this result we find

$$1-c^2 H_{11} = 1-c^2 \left(\frac{X-1}{X+1}\right) = (1-c^2) \left(\frac{X'+\beta'}{X'+1}\right). \quad (17.14)$$

Then, applying Equations (17.1) and (17.5) we have

$$F_{11} \rightarrow \frac{t}{(X'+1)S'} [X'-2tY'-S'] [1-4t^2]^{-\frac{1}{2}} \quad (17.15)$$

$$\frac{(1-c^2)(X'+\beta')}{(X'+1)}$$

After the appropriate scale change we have

$$F_{11} \rightarrow \frac{t(1-c^2)[X'-2tY'-S'] [1-4t^2]^{-\frac{1}{2}}}{(X'+\beta')S'} \quad (17.16)$$

In order to use the gauge transformation of  $F_{11}$  as detailed by Equation (17.11), we require the appropriate forms of  $D_1$  and  $F_{12}$ . Therefore, we must obtain the Harrison transformation of the Schwarzschild  $D_1$  and  $F_{12}$  and then apply the necessary scale changes.

Equations (17.6) and (17.9) result in

$$D_1 \rightarrow \frac{tc[X'-2tY'-S'] [1-4t^2]^{-\frac{1}{2}}}{(X'+\beta')S'} \quad (17.17)$$

$$F_{12} \rightarrow \frac{i(X'-2tY') [1-4t^2]^{-\frac{1}{2}}}{(X'+\beta')S} + \frac{i(1+c^2) [1-4t^2]^{-\frac{1}{2}}}{(1-c^2)(X'+\beta')} \quad (17.18)$$

Accordingly, Equation (17.11) gives

$$F_{11} \rightarrow \frac{t[X'-2tY'-\beta'S'] [1-4t^2]^{-\frac{1}{2}}}{(X'+\beta')S'} \quad (17.19)$$

Following the same procedure for  $D_1$  we have

$$D_1 \rightarrow \frac{tq'\beta' [1-4t^2]^{-\frac{1}{2}}}{X'+\beta'} \quad (17.20)$$

At this point we will, for convenience, drop the primes. Using Equations (11.5) and (11.9) we may derive

$$D_2 = itq\beta Y(1-4t^2)^{-\frac{1}{2}} + \alpha_2(t) \quad (17.21)$$

$$Q = \frac{it(\beta^2-1)Y(1-4t^2)^{-\frac{1}{2}}}{(X+\beta)} + \alpha_3(t) \quad (17.22)$$

Using these results in Equation (15.6) we have,  
collecting terms with like powers of Y,  
for  $Y^0$

$$\begin{aligned} -\frac{1}{4} - \frac{q\beta\alpha_2[1-4t^2]^{-\frac{1}{2}}}{X+\beta} + \alpha_3^2 - \frac{2q\beta\alpha_3\alpha_2}{X+\beta} \\ = \left(\frac{X-\beta}{X+\beta}\right)\alpha_2^2 - \frac{t^2(\beta^2-1)}{1+4t^2} \end{aligned} \quad (17.23)$$

and for  $Y^1$

$$\frac{-q\beta}{[1-4t^2]^{\frac{1}{2}}} - 2q\beta\alpha_3 = -2\beta\alpha_2 \quad (17.24)$$

thus,

$$\alpha_2 = \frac{q}{2}[1-4t^2]^{-\frac{1}{2}} + q\alpha_3 \quad (17.25)$$

Inserting this result in Equation (17.23) we find

$$\alpha_3 = \frac{\beta^2}{2}[(1-4t^2)^{-\frac{1}{2}} - 1] - \frac{1}{2}[1-4t^2]^{-\frac{1}{2}} \quad (17.26)$$

thus,

$$\alpha_2 = \frac{q\beta^2}{2}[(1-4t^2)^{-\frac{1}{2}} - 1] \quad (17.27)$$

In order to complete the production of the generating functions, return to Equations (15.8) - (15.13). We again select  $K = C$ ,  $C_1 = \beta C$ , and  $K_1 = -\beta K$ . The complete results are listed in Appendix J. Note that both the

results of this section and the section on Particular Reissner-Nordstrom Generating Functions are in total agreement with Equations (13.2) - (13.4) and Equations (13.20) - (13.23) as required.

At this juncture we find that we are ready to begin to consider the generalization of two methods of solution generation,  $\beta^k$  and  $A^{(p)}$ , which have worked with potentials  $\gamma_{AB}^k$  in the vacuum case. Both processes deal with the  $\gamma_{AB}^k$  transformations of the group K. They differ in that the  $\beta^k$  method uses finite linear combinations of the  $\gamma_{AB}^k$  while the other employs infinite linear combinations. Also, the  $\beta^k$  process leaves flat space invariant, while the other maps it into an asymptotically flat, non-flat spacetime. Additionally, the  $A^{(p)}$  transformation requires only algebraic manipulations, while  $\beta^k$  is not straightforward in its application and requires the solution of a set of differential equations.

## CHAPTER 9

### SOLUTION GENERATION

#### $\beta$ Transformations

In Paper IV it was demonstrated that the  $\gamma_{11}^k$  and the  $\gamma_{22}^{k+2}$  infinitesimal transformations both generate the same multipole structure, albeit with opposite sign, when applied to  $H_{11}$  for flat space. It turns out that, at least for the infinitesimal case, the combinations

$$\beta \equiv \gamma_{11}^k + \gamma_{22}^{k+1} \quad k = 0, 1, \dots \quad (18.1)$$

will leave  $H_{11}$  for flat space invariant. To determine the efficacy of  $\beta$  we need to determine two main features, i.e., (i) what occurs for finite transformations and (ii) how are the other potentials affected. In order to discuss these problems, a sketch of the procedure used in Paper IV for the vacuum case will be given.

As a first step in investigating the  $\beta$  transformation, we would naturally investigate its action on the  $N_{AB}^{mn}$  potentials [N.B. as we are now considering only a vacuum spacetime, all the other potentials are zero]. When this is done, by applying Equation (3.1) of Paper II, one notices that certain linear combinations of the  $N_{AB}^{mn}$  reoccur, and a concision of notation would be gained by focusing attention on these combinations. Therefore, attention is now

paid to the action of the  $\beta^k$  transformation on these new combinations. It is noted that for flat space all of these combinations vanish and what is more, after the transformation is used, they will continue to vanish and furthermore, repeated iterations of this process to achieve a finite form will not alter this result. Thus, the particular relation  $H_{11} = 1$ , which holds for flat space, will continue to hold in the transformed space, and no gauge or coordinate changes will be required to maintain this property thereafter. As a consequence of the above, if we begin with a spacetime which is asymptotically flat, then, since we may choose a gauge in which our particular linear combinations of potentials go to zero or, at worst, a constant at spatial infinity, it follows that the  $\beta^k$  transformation will preserve this property for the transformed space. In essence, the  $\beta^k$  transformation will preserve asymptotic flatness.

An examination of the above combinations of potentials reveals that the higher order [in  $m, n$ ] groups are fixed linear combinations of a small set of these potentials. Surprisingly enough, for the Zipoy-Voorhees metric,<sup>24,25</sup> of which the Schwarzschild metric is a particular case, one discovers that these relations are unchanged

by the infinitesimal action of  $\beta^k$ . Therefore, the entire set of  $\beta^k$  transformations may be reduced to a small subset. In particular, for Schwarzschild,  $\beta^0$  has only two distinct equations,  $\beta^1$  one, and higher order  $\beta^k$ 's give no new information. These equations may then be converted to differential equations and integrated to obtain the finite forms. At the conclusion of this rather simple process, one has the transformed potentials for the new spacetime, which is guaranteed to be asymptotically flat. From these potentials we may reconstruct the metric, i.e., using the relation between the Ernst potential and  $H_{11}$ .

Given this overview, we will now consider this process as applied to spacetimes possessing Maxwell fields, in particular, the by now familiar, Reissner-Nordstrom solution. A major difficulty we must watch out for is the possibility that the necessary simplifying relations among the potentials, and also among the higher order  $\beta^k$ 's, may not be applicable. In that case the resulting set of equations would be too large to manage. Furthermore, will the relations that hold before the  $\beta^k$  transformation still be valid after the transformation?

In order to incorporate an electromagnetic field we must develop the explicit action of  $\beta^k$ , not only on  $N_{AB}^{mn}$ ,

but also on all the other potentials. Additionally, we enforce two conditions upon these potentials. First, we must, as previously discussed, make sure that the potentials are not inconsistent with the relations used to derive the action of the group  $K'$ , i.e., Equations (13.2) - (13.4) and Equations (13.20) - (13.23). This was accomplished in the section on Double Generating Functions. Secondly, we must require that the appropriate potentials vanish as spatial infinity is approached. As an initial step in this direction, we used the Harrison transformation to achieve the appropriate form for the electromagnetic potentials.

$k$   
Action of the  $\beta$  Transformation

At this point let us begin to develop the equations for the explication of the action of  $\beta$ . From Equation (18.1) we see that the actions of  $\gamma_{11}$  and  $\gamma_{22}$  are required. Using Equation (3.1) of Paper II, then we obtain

$$\gamma_{11}: N_{mn} \rightarrow N_{mn} + \gamma_{11} \left( N_{21} + N_{12} + \sum_{s=1}^k N_{12} N_{21} \right) \quad (19.1)$$

$$\gamma_{22}: N_{mn} \rightarrow N_{mn} + \gamma_{22} \left( N_{11} N_{11} + N_{11} N_{11} + \sum_{s=1}^{k+2} N_{11} N_{11} \right), \quad (19.2)$$

so,

$$\beta: N_{11} \rightarrow N_{11} + \beta \left\{ \begin{array}{l} m+k, n \quad m, n+k \quad m, k+2 \quad 0n \quad m1 \quad k+1, n \\ N_{21} + N_{12} + N_{11} \quad N_{11} + N_{11} \quad N_{11} \end{array} \right. \quad (19.3)$$

$$+ \sum_{s=1}^k N_{12} \quad N_{21} + \sum_{s=1}^k N_{11} \quad N_{11} \quad \left. \begin{array}{l} k \quad ms \quad k-s, n \quad k \quad m, s+1 \quad k+1-s, n \\ N_{12} \quad N_{21} + \sum_{s=1}^k N_{11} \quad N_{11} \end{array} \right\}$$

Taking advantage of Equation (13.29) we may write

$$N_{11}^{m+k+1, n} - N_{11}^{m, n+k+1} + 2i \sum_{s=1}^{k+1} M_1 \quad L_1 \quad + \sum_{s=1}^{k+1} N_{12} \quad N_{11}^{k+1-s, n} \quad (19.4)$$

$$- \sum_{s=1}^{k+1} N_{11} \quad N_{21}^{k+1-s, n} = 0$$

If we incorporate Equation (19.4) into Equation (19.3), then

$$\beta: N_{11} \rightarrow N_{11} + \beta \left\{ i \left[ \begin{array}{l} m+k+1, n \quad m+k, n \quad m, n+k+1 \quad m, n+k \\ N_{11} \quad -i \quad N_{21} \end{array} \right] - i \left[ \begin{array}{l} N_{11} \quad +i \quad N_{12} \end{array} \right] \right. \quad (19.5)$$

$$+ N_{11} \left[ \begin{array}{l} 0n \quad m, k+2 \quad m, k+1 \quad m1 \quad k+1, n \quad kn \\ N_{11} + i \quad N_{12} \end{array} \right] + N_{11} \left[ \begin{array}{l} N_{11} - i \quad N_{21} \end{array} \right]$$

$$+ \sum_{s=1}^k \left[ \begin{array}{l} m, s+1 \quad ms \quad k+1-s, n \quad k-s, n \\ N_{11} + i \quad N_{12} \end{array} \right] \left[ \begin{array}{l} N_{11} \quad -i \quad N_{21} \end{array} \right]$$

$$- 2 \sum_{s=1}^k \left[ \begin{array}{l} m, s+1 \quad k-s+1, n \quad m1 \quad k+1, n \\ M_1 \quad L_1 \quad -2 M_1 \quad L_1 \end{array} \right]$$

Similar results for  $N_{12}$ ,  $N_{21}$ , and  $N_{22}$  are given in

Appendix K.

A comparison of the results for the  $\beta: N_{AB}$  shows that a simplification may well occur if one looks at the combination  $N_{11} + i N_{12}$ .

In fact, using Equation (19.5) and Equation (K.1) from Appendix K, we find



$$\begin{aligned}
& + \sum_{s=1}^K N_{0,s+1} N_{K+1-s,n}^{-2} M_{01} L_{K+1,n} \\
& - 2 \sum_{s=1}^k M_{01} L_{k-s+1,n}
\end{aligned}$$

Now

$$\sum_{s=1}^k N_{0,s+1} N_{k+1-s,n} = \sum_{s=1}^{k+2} N_{0s} N_{k+2-s,n}^{-N_{01}} N_{k+1,n}^{-N_{0,k+2}} N_{0n} \quad (19.13)$$

and

$$N_{01} = N_{11+i}^{01} \quad (19.14)$$

by Equation (13.25). Then, finally, using the fact that

$$M_A^{mn} = 0 \text{ for } n \leq 0 \quad (19.15)$$

$$L_A^{mn} = 0 \text{ for } m, n = 0, \quad (19.16)$$

we have

$$\beta : N_{0n} \rightarrow N_{0n} + \beta \left\{ -2i N_{0,n+k+1} + \sum_{s=1}^{k+2} N_{0s} N_{k+2-s,n}^{-2} \sum_{s=1}^{k+1} M_{0s} L_{k-s+2,n} \right\} \quad (19.17)$$

The rest of the results for the action of  $\beta$  on the quantities defined in Equations (19.8) - (19.11) are listed in Equations (K.7) - (K.13) of Appendix K.

#### Calculation of Potentials

It should, by now, be clear that while previously it was sufficient to focus attention on the explicit

representations of  $H_{AB}$  and  $\phi_A$ , we are no longer in such a position and are required to find the actual representations of  $N_{AB}$ ,  $M_A$ , etc., if we expect to carry out the  $\beta$  transformations in a specific case. If we return to Equations (14.17) - (14.20) and insert the generating functions listed in Appendix J, we may then expand the appropriate double generating functions and select out the various terms. Although the process is straightforward, there is a considerable amount of algebra involved, and even after much cancellation, the expressions retain considerable length, e.g.,

$$\begin{aligned}
 N_{11} = & \frac{irt(X-2tY)(2rX-Y)}{S(r)S(t)\eta(r)\eta(t)} - \frac{2i\beta^2 rt(r-t)}{\eta(r)\eta(t)[1-4tr-\eta(r)\eta(t)]} \\
 & + \frac{irt[2XY+8trXY-(2X^2+2Y^2)(t+r)]}{S(r)\eta(t)[S(t)\eta(r)+S(r)\eta(t)]} \\
 & + \frac{itS(r)[\beta S(t)-X+2tY]}{2(X+\beta)S(t)\eta(r)\eta(t)} \\
 & - \frac{i(\beta^2-1)t[\beta S(t)-X+2tY]}{2(X+\beta)S(t)\eta(t)} \\
 & + \frac{i\beta t(2rY+\beta)[\beta S(t)-X+2tY]}{2(X+\beta)S(t)\eta(t)\eta(r)} \\
 & - \frac{irt(2tX-Y)(\beta S(r)-X+2rY)}{S(r)S(t)\eta(r)\eta(t)}
 \end{aligned} \tag{20.1}$$

where

$$\eta(t) \equiv [1-4t^2]^{\frac{1}{2}} \tag{20.2}$$

We note that  $M_A$  and  $L_A$ , say, may be expressed in terms of generating functions in a differential form

$$\vec{\nabla} L_A = D_1^*(r) \vec{\nabla} F_{2A}(t) - D_2^*(r) \vec{\nabla} F_{1A}(t) \quad (20.3)$$

$$\vec{\nabla} M_A = F_{1A}^*(r) \vec{\nabla} D_2(t) - F_{2A}^*(r) \vec{\nabla} D_1(t) \quad (20.4)$$

These forms may then be integrated to obtain particular expressions for Equations (14.18) and (14.19), but only up to an integration constant, of course. This constant will be a function of  $r, t$  and must be obtained if one is going to use  $L_1$ , say, to obtain  $L_1$ . At first glance it may appear that one is forced to use the full expressions as given by Equations (14.18) and (14.19), but this may entail a considerable amount of algebra. Fortunately, one may combine both approaches. If one sets  $X = Y = 0$ , then the amount of algebra is considerably reduced. After performing the integration in Equations (20.3) and (20.4), also set  $X = Y = 0$ . Equations (20.3) and (14.19) and Equations (20.4) and (14.18) must be the same, and so, the appropriate constants of integration can be immediately determined. The results for  $M_A$ ,  $L_A$ , and  $K$  are given in Appendix L.

However, the situation may not be quite so complicated after all. Equations (19.7) - (19.11) were useful in simplifying the notation for the  $\beta$  transformations, and in

fact, their employment in conjunction with a specific metric will engender a considerable reduction in the amount of algebra necessitated.

Let us relate the quantities in Equations (19.7) - (19.11) to a set of generating functions:

$$\begin{aligned}
 N(r,t) &\equiv \sum N_{mn} r^m t^n \\
 &= \sum N_{11} r^m t^n - i \sum N_{21} r^{m-1} t^n + i \sum N_{12} r^m t^{n-1} + \sum N_{22} r^{m-1} t^{n-1} \\
 &= N_{11}(r,t) - irN_{21}(r,t) + itN_{12}(r,t) + rtN_{22}(r,t)
 \end{aligned} \tag{20.5}$$

$$L(r,t) \equiv \sum L_{mn} r^m t^n = L_1(r,t) + itL_2(r,t) \tag{20.6}$$

$$M(r,t) \equiv \sum M_{mn} r^m t^n = M_1(r,t) - irM_2(r,t) \tag{20.7}$$

$$\bar{K}(r,t) \equiv \sum K_{mn} r^m t^n \tag{20.8}$$

Begin with Equation (14.17). This equation, in conjunction with Equation (20.5) may be written

$$\begin{aligned}
 N(r,t) &= \frac{1}{r-t} \{ -tF_{X1}^*(r) [F_1^X(t) + itF_2^X(t)] \\
 &\quad + 2irtS_1^*(r) [S_1(t) + itS_2(t)] \\
 &\quad - 2irt\phi_Z F_1^{*Z}(r) [S_1(t) + itS_2(t)] \\
 &\quad - 2irt\phi_X S_1^{*X}(r) [F_1^X(t) + itF_2^X(t)] \}
 \end{aligned} \tag{20.9}$$

$$\begin{aligned}
& -irt(H_{ZX}+H_{XZ}^*)F^{*Z}_1(r)[F^X_1(t)+itF^X_2(t)] \\
& +ir(t+r)+2r^2tS_2^*(r)[S_1(t)+itS_2(t)] \\
& +itrF_{X2}^*(r)[F^X_1(t)+itF^X_2(t)] \\
& -2r^2t\phi_ZF^{*Z}_2(r)[S_1(t)+itS_2(t)] \\
& -2r^2t\phi_X^*S_2^*(r)[F^X_1(t)+itF^X_2(t)] \\
& -r^2t(H_{ZX}+H_{XZ}^*)F^{*Z}_2(r)[F^X_1(t)+itF^X_2(t)]
\end{aligned}$$

Making some natural definitions

$$R_A(t) \equiv F_{A1}(t)+itF_{A2}(t) \quad (20.10)$$

$$\Omega(t) \equiv S_1+itS_2 \quad (20.11)$$

then, after some algebra, we have

$$\begin{aligned}
N(r,t) = \frac{1}{r-t} \{ & 2irt-tR^X(t)R_X^*(r)+2irt\Omega(t)\Omega^*(r) \\
& -2irt\phi_Z\Omega(t)R^{*Z}(r)-2irt\phi_X^*R^X(t)\Omega^*(r) \\
& -irt(H_{ZX}+H_{XZ}^*)R^X(t)R^{*Z}(r) \} \quad (20.12)
\end{aligned}$$

The constant term has been changed from  $ir(t+r)$  to  $2irt$  to facilitate division by  $r-t$ . Furthermore,

$$\begin{aligned}
L(r,t) \equiv L_1+itL_2 = \left( \frac{rt}{r-t} \right) \{ & 2iQ(r)[\phi_1R_2(t)+\phi_2R_1(t)-\Omega(t)] \\
& +[i(H_{21}-H_{12})-\frac{1}{r}][D_1(r)R_2(t)+D_2(r)R_1(t)] \\
& +2iH_{11}D_2(r)R_2(t)-2iH_{22}D_1(r)R_1(t) \} \quad (20.13)
\end{aligned}$$

$$+2i\Omega(t) [\phi_1 D_2(r) + \phi_2 D_1(r)]$$

$$\begin{aligned} M(r,t) \equiv M_1 - irM_2 = & \frac{rt}{r-t} \{2iQ(t) [\phi_1 R_2(r) + \phi_2 R_1(r) - \Omega(r)] \\ & + [i(H_{21} - H_{12}) - \frac{1}{r}] [D_1(t) R_2(r) + D_2(t) R_1(r)] \\ & + 2iH_{11} D_2(t) R_2(r) - 2iH_{22} D_1(t) R_1(r) \\ & + 2i\Omega(r) [\phi_1 D_2(t) + \phi_2 D_1(t)] \} \end{aligned} \quad (20.14)$$

$$\begin{aligned} \bar{K}(r,t) \equiv & \left( \frac{rt}{r-t} \right) \left\{ -\frac{ir}{2t} - \frac{it}{2r} + \frac{1}{2}i - 2iQ(t)Q(r) \right. \\ & + 2iQ(t) [\phi_1 D_2(r) + \phi_2 D_1(r)] \\ & + 2iQ(r) [\phi_1 D_2(t) + \phi_2 D_1(t)] \\ & + [i(H_{21} - H_{12}) - \frac{1}{r}] [D_1(r) D_2(t) + D_1(t) D_2(r)] \\ & \left. + 2iH_{11} D_2(r) D_2(t) - 2iH_{22} D_1(r) D_1(t) \right\} , \end{aligned} \quad (20.15)$$

where we have used the fact that for the Reissner-Nordstrom metric  $R_1$  is real and  $R_2$  and  $\Omega$  are imaginary.

Referring to Appendix J we find

$$R_1(t) = F_{11} + itF_{12} = \frac{-2\beta t}{(X+\beta)\eta(t)} \quad (20.16)$$

$$R_2(t) = F_{21} + itF_{22} = i \left[ \beta^2 - 1 - \frac{\beta^2}{\eta(t)} \right] - \frac{2i\beta t Y}{\eta(t)} \quad (20.17)$$

$$\Omega(t) = S_1 + itS_2 = iq\beta^2 \left[ 1 - \frac{1}{\eta(t)} \right] - \frac{2iq\beta^2 t Y}{(X+\beta)\eta(t)} . \quad (20.18)$$

Remember that some of the gauge conditions were earlier chosen to make these quantities as simple as possible. Therefore, one finally obtains

$$\begin{aligned}
N(r,t) = & \frac{-2irt\beta^2}{(r-t)} \left[ \frac{1-4rt-\eta(r)\eta(t)}{\eta(r)\eta(t)} \right] \\
& + \frac{2it\beta^3}{(X+\beta)\eta(r)\eta(t)} - \frac{2it\beta(\beta^2-1)}{(X+\beta)\eta(t)} \quad (20.19) \\
& + \frac{4irt\beta^2 Y}{(X+\beta)\eta(r)\eta(t)}.
\end{aligned}$$

Now we are finally in a position to look at the various powers of  $r$  and  $t$  and pick off the appropriate  $N_{mn}$ ,  $M_{mn}$ ,  $L_{mn}$ , and  $K_{mn}$ . From Equation (20.19) we find, for example,

$$N_{01} = \frac{-2i\beta(\beta^2-1)}{X+\beta} + \frac{2i\beta^3}{X+\beta} = \frac{2i\beta}{X+\beta} \quad (20.20)$$

Once having obtained these specific quantities, we need to discover, if possible, a basic set in terms of which the others are linear combinations. Next, the action of the  $k$   $\beta$  on these basic relations must be determined to see if we can find a small enough set of unique equations that can be solved. If so, we may then generate the new solution.

It is interesting to refer back to the vacuum case. Paper IV gave the results for the  $F_{AB}$  of the general Zipoy-Voorhees metric.<sup>24,25</sup> In that case we have

$$R_1(t) = \frac{t}{S(t)(X+1)^\delta} [C(t)A_-^\delta - D(t)A_+^\delta] \quad (20.21)$$

$$R_2(t) = \frac{i}{2S(t)(X-1)^\delta} [B_-D(t)A_2^\delta - B_+C(t)A_1^\delta] \quad (20.22)$$

where C and D contain the gauge freedom and

$$A_{\pm} \equiv X - 2tY \pm S(t) \quad (20.23)$$

$$B_{\pm} \equiv 1 - 2tZ \pm S(t) \quad (20.24)$$

Then,

$$N(r, t) = \frac{1}{(r-t)} \left[ 2irt \left[ 1 + \left( \frac{X-1}{X+1} \right)^{\delta} R_2(r) R_2(t) \right. \right. \\ \left. \left. + \rho^2 \left( \frac{X+1}{X-1} \right)^{\delta} R_1(r) R_1(t) \right] \right. \\ \left. - t(1-2rZ) [R_1(r) R_2(t) + R_1(t) R_2(r)] \right] \quad (20.25)$$

We may select  $C(t) = D(t)$ . In this case for flat space [ $\delta = 0$ ]

$$R_1(t) \rightarrow 0 \quad R_2(t) \rightarrow -i \quad (20.26)$$

This implies

$$N(r, t) \rightarrow 0 \quad (20.27)$$

In other words all the  $N_{mn}$  would be zero. Furthermore, for  $\delta = 0$ ,

$$F_{11} = \frac{t}{S(t)} \quad (20.28)$$

$$F_{12} = \frac{i}{S(t)} \quad (20.29)$$

Therefore, for flat space, but only for flat space, the condition  $i H_{n+1, 11} = H_{12, n}$ , which we previously wished to use, is still consistent with Equations (13.2) - (13.4) and Equations (13.20) - (13.23). However, for  $\delta > 0$ , even in a vacuum, the above condition is not viable.

$k$   
 $\beta$  and Reissner-Nordstrom

Now we need to consider the results of the expansions of  $N(r,t)$ ,  $M(r,t)$ ,  $L(r,t)$ , and  $K(r,t)$ . We will be looking for a basic set of relationships from which the others may be determined. In fact, we find

$$N_{0,2j+1} = \alpha_{0j} N_{01} \quad (21.1)$$

$$N_{2k+1,2j} = \beta_{kj} N_{12} \quad (21.2)$$

$$N_{2k+1,2j+1} = \alpha_{kj} N_{11} \quad (21.3)$$

$$N_{2k,2j+1} = \frac{\alpha_{kj}}{2} N_{21} + \beta_{kj} N_{12} \quad (21.4)$$

$$N_{0,2j} = 0 \quad (21.5)$$

$$N_{2k,2j} = 0, \quad (21.6)$$

where

$$\alpha_{kj} \equiv \begin{pmatrix} 2k \\ k \end{pmatrix} \begin{pmatrix} 2j \\ j \end{pmatrix} \quad (21.7)$$

and

$$\begin{pmatrix} 2k \\ k \end{pmatrix} \equiv \frac{(2k)!}{[k!]^2}. \quad (21.8)$$

We note that

$$\sum_k \begin{pmatrix} 2k \\ k \end{pmatrix} t^{2k} = [1-4t^2]^{-\frac{1}{2}} = \eta(t)^{-1}. \quad (21.9)$$

$\beta_{kj}$  is the coefficient of the  $r^{2k+1}t^{2j}$  term in the expansion of

$$-\frac{1}{2} \left( \frac{rt}{r-t} \right) \left[ \frac{1-4rt-\eta(r)\eta(t)}{\eta(t)\eta(r)} \right].$$

A table of values of  $\beta_{kj}$  is given in Equation (M.4) in Appendix M. The relations for M, L, and K are given in Appendix M also.

The actual values of the basic set are

$$\begin{aligned} N_{01} &= \frac{2i\beta}{X+\beta} & N_{12} &= 4i\beta^2 \\ N_{11} &= \frac{4i\beta^2 Y}{X+\beta} & N_{21} &= -4i\beta^2 + \frac{4i\beta^3}{X+\beta} \\ N_{22} &= N_{02} = 0. \end{aligned} \quad (21.10)$$

If we use the gauge selection of the section on Particular Reissner-Nordstrom Generating Functions, the situation is much more complicated, e.g.,

$$\begin{aligned} N_{12} &= 4i\beta^2 + 8i(\beta^2 - 1) \\ N_{21} &= \frac{2i\beta}{X+\beta} [2 - 2(\beta^2 - 1)] - \frac{8iX(\beta^2 - 1)}{X+\beta} - 4i\beta^2 \\ N_{03} &= \frac{2i\beta [2 - 2(\beta^2 - 1)]}{X+\beta}. \end{aligned} \quad (21.11)$$

The critical issue is what occurs when we operate on the basic relations, e.g., Equations (21.1) - (21.6), with the  $\beta$  transformation. Using Equation (19.17) and Equation (K.7) from Appendix K we find, for example,

$$\beta : N_{12} \rightarrow N_{12} + \beta \{ 2i(N_{22} - N_{13}) + N_{11}N_{12} + N_{12}N_{02} - 2M_{11}L_{12} \} \quad (21.12)$$

$$\beta : N_{02} \rightarrow N_{02} + \beta \{ -2iN_{03} + N_{01}N_{12} + N_{02}N_{02} - 2M_{01}L_{12} \}. \quad (21.13)$$

Using Equation (21.10) and Appendix M, we determine

that the quantity in the brackets of Equation (21.12) is equal to  $\frac{-8\beta^2(\beta^2-1)Y}{X+\beta}$ , while the quantity in the brackets of Equation (21.13) is equal to  $\frac{-4\beta(\beta^2-1)}{X+\beta}$ . Notice that if we go to a vacuum field, i.e.,  $\beta \rightarrow 1$ , then both of these quantities vanish. Also, we immediately see that the fortunate situation that held for the vacuum situation doesn't apply if electromagnetism is added. That is, the relations which held for the vacuum case before the  $\beta$  transformation are exactly maintained even after the transformation. If electromagnetism is included, we see that the basic relations are broken by the  $\beta$  transformations and that, in particular, constants no longer remain constant.

Additionally, we notice that

$$\begin{aligned} \beta:N_{34} &\rightarrow N_{34} + \beta\{2i(N_{44}-N_{35}) + N_{31}N_{14} + N_{32}N_{04} - 2M_{31}L_{14}\} \\ &\rightarrow N_{34} + 6\beta\{2i(N_{22}-N_{13}) + N_{11}N_{12} + N_{12}N_{02} - 2M_{11}L_{12}\} \\ &\quad + \beta\{3iL_{22} - 3iL_{12}N_{02}\}. \end{aligned} \quad (21.14)$$

So, using Equations (21.1) - (21.6), (21.10) and Appendix M, we obtain

$$\beta:N_{34} = 6\beta:N_{12}. \quad (21.15)$$

But,

$$N_{34} = 4N_{12}. \quad (21.16)$$

Therefore, the  $\beta$  transformation also affects our basic set of relations. Furthermore, although  $N_{02}$  and  $L_{22}$  are

initially zero, they do not retain this characteristic after a  $\beta^0$  transformation. Therefore, even Equation (21.15) holds only to first order.

We see that our worst fears are being realized. The basic set of relations is not maintained, constants do not remain constants, and thus, we are unable to reduce the equation to a simple set of manageable units.

Although in the vacuum situation the  $\beta^k$  repeat, e.g.,  $\beta^2 = 4\beta^0$ , perhaps we might still obtain a useable set of relations if we consider  $\beta^1$  or  $\beta^2$ , say. A systematic approach seems advisable at this point.

For definiteness consider  $\beta^{2p} : N_{0,2j+1}$ . By Equation (19.17)

$$\begin{aligned} \beta^{2p} : N_{0,2j+1} \rightarrow N_{0,2j+1}^{2p} + \beta \{ -2i N_{0,2(j+p+1)}^{2p} + \sum_{r=1}^{2p+2} N_{0r}^{2p+2-r,2j+1} \\ - 2 \sum_{r=1}^{2p+1} M_{0r}^{L_{2p+2-r,2j+1}} \} . \end{aligned} \quad (21.17)$$

Regrouping the terms on the left side, based on the notation used in Equations (21.1) - (21.6),

$$\begin{aligned} \beta^{2p} : N_{0,2j+1} \rightarrow N_{0,2j+1}^{2p} + \beta \{ -2i N_{0,2(j+p+1)}^{2p} \\ + \sum_{r=1}^{p+1} N_{0,2r}^{N_{2(p+1-r),2j+1}} \} \end{aligned} \quad (21.18)$$

$$\begin{aligned}
& + \sum_{r=0}^p N_{0,2r+1} N_{2(p-r)+1,2j+1} \\
& - 2 \sum_{r=0}^p M_{0,2r} L_{2(p-r+1),2j+1} \\
& - 2 \sum_{r=0}^p M_{0,2r+1} L_{2(p-r)+1,2j+1}
\end{aligned}$$

Using Equations (21.1) - (21.6) we have

$$\begin{aligned}
\beta : N_{0,2j+1} & \rightarrow N_{0,2j+1} + \beta \{-i\alpha_{0,j+p+1} N_{0,2} + \alpha_{j,p+1} N_{0,2} N_{0,1} \\
& + \frac{1}{2} N_{0,2} N_{2,1} \sum_{r=1}^p \alpha_0 r^{\alpha_{p+1-r,j}} + \frac{1}{2} N_{1,2} N_{0,2} \sum_{r=1}^p \alpha_0 r^{\beta_{p+1-r,j}} \\
& + N_{0,1} N_{1,1} \sum_{r=0}^p \alpha_0 r^{\alpha_{p-r,j}} - \frac{1}{2} M_{0,2} L_{2,1} \sum_{r=1}^p \alpha_0 r^{\alpha_{p-r+1,j}} \\
& - L_{1,2} M_{0,2} \sum_{r=1}^p \alpha_0 r^{\beta_{p-r+1,j}} - 2M_{0,1} L_{1,1} \sum_{r=0}^p \alpha_0 r^{\alpha_{p-r,j}}\}
\end{aligned} \tag{21.19}$$

Applying Equations (M.11) - (M.14) of Appendix M, we obtain

$$\begin{aligned}
\beta : N_{0,2j+1} & \rightarrow N_{0,2j+1} + \beta \{-i\alpha_{0,j+p+1} N_{0,2} + N_{0,2} N_{0,1} \frac{\alpha_{p+1,j}}{2} \\
& + \frac{1}{2} N_{0,2} N_{2,1} [2^{2(p+1)} \alpha_{0j} - 2\alpha_{p+1,j}] + N_{0,1} N_{1,1} [2^{2p} \alpha_{0j}] \\
& + \frac{1}{2} N_{0,2} N_{1,2} [\frac{1}{2} \alpha_{0,j+p+1} - \beta_{p+1,j} - \frac{1}{2} \alpha_{p+1,j}] \\
& - \frac{1}{2} M_{0,2} L_{2,1} [2^{2(p+1)} \alpha_{0j} - 2\alpha_{p+1,j}] - 2M_{0,1} L_{1,1} [2^{2p} \alpha_{0j}] \\
& - M_{0,2} L_{1,2} [\frac{1}{2} \alpha_{0,j+p+1} - \beta_{p+1,j} - \frac{1}{2} \alpha_{p+1,j}]\}
\end{aligned} \tag{21.20}$$

Setting  $j = 0$  we then have  $\beta^{2p} : N_{01}$ . Looking back at Equation (21.1) we see that  $\beta^{2p} : N_{0,2j+1}$  should be compared with  $\alpha_{0j} \beta^{2p} : N_{01}$ . And in fact,

$$\begin{aligned} \beta^{2p} : N_{0,2j+1} &= \alpha_{0j} \beta^{2p} : N_{01} \\ &- \beta^{2p} \left\{ \frac{1}{2} (\alpha_{j,p+1} - \alpha_{0,j+p+1}) [-4iN_{02} + N_{02}N_{12} - 2M_{02}L_{12}] \right. \\ &\quad \left. - \frac{1}{2} (\beta_{p+1,j}) [N_{02}N_{12} - 2M_{02}L_{12}] \right\}. \end{aligned} \quad (21.21)$$

For odd  $k$  we may determine, in a similar manner,

$$\begin{aligned} \beta^{2p+1} : N_{0,2j+1} &= \alpha_{0j} \beta^{2p+1} : N_{01} \\ &+ \beta^{2p+1} \left\{ \frac{1}{2} (\alpha_{j,p+1} - \alpha_{0,j+p+1}) \right. \\ &\quad \left. [-4iN_{01} + N_{01}N_{12} - 2M_{01}L_{12}] \right\}. \end{aligned} \quad (21.22)$$

The other relations of this kind are given explicitly in Appendix N.

An examination of the  $\alpha_{kj}$  reveals that, except for the trivial case, there are no values of  $j$  and  $p$  which can make the quantities in the bracket vanish. Note that if the electromagnetic field is removed, then  $N_{02}$ ,  $M_{02}$ , and  $M_{01}$  remain zero and also

$$-4iN_{01} + N_{01}N_{12} = 0, \quad (21.23)$$

so

$$\beta^k : N_{0,2j+1} = \alpha_{0j}^k \beta^k : N_{01} \quad (21.24)$$

for all values of  $k$ .

Unfortunately, a considerable effort was expended in a futile attempt to surmount these various difficulties. The action of  $\beta^0$  and  $\beta^1$  on the potentials for  $(m,n) \leq 10$  were examined in detail. There were a few rays of hope. In particular, under  $\beta^1$  both  $N_{02}$  and  $N_{22}$  remain zero. Nevertheless, the other basic relations are not maintained, e.g.,

$$\beta^1 : N_{03} = 2\beta^1 : N_{01} + \beta^1 (-4iN_{01} + N_{01}N_{12} - 2M_{01}L_{12}) \quad (21.25)$$

We also considered the possibility that our basic set of potentials should actually be broken into two separate parts; one part being a constant that the  $\beta^k$  wouldn't affect and the other part transforming under  $\beta^k$ . For example, under Equation (21.2) we have  $N_{2k+1,2j} = \beta_{kj}^k N_{12}$ , but

$$\beta^0 : N_{2k+1,2j} = \frac{\alpha_{kj}^0}{2} \beta^0 : N_{12}, \quad (21.26)$$

dropping terms that are zero before a  $\beta^k$  transformation.

Therefore, one might try

$$N_{2k+1,2} = \frac{\alpha_{kj}^k}{2} N_{12} - \beta_{jk}^k (4i\beta^2) \quad (21.27)$$

as the basic relation. From Equation (21.10) we have

$$N_{12} = 4i\beta^2 \text{ and for } k = 0, j = 1$$

$$N_{12} = \frac{\alpha_{01}}{2} N_{12} - \beta_{10} (4i\beta^2) = 4i\beta^2 \quad (21.28)$$

as required. But also, by Equation (M.5) of Appendix M

$$N_{2k+1,2j} = \frac{\alpha_{kj}}{2} (4i\beta^2) - \beta_{jk} (4i\beta^2) = \beta_{kj} (4i\beta^2) \quad (21.29)$$

This is our original equation. In addition,

$${}^0\beta : N_{2k+1,2j} = \beta : \left[ \frac{\alpha_{kj}}{2} N_{12} \right] - \beta : (\beta_{jk} 4i\beta^2) = \frac{\alpha_{kj}}{2} \beta : N_{12} \quad (21.30)$$

and this is, of course, Equation (21.26). This process failed because Equation (21.30) contains the fatal flaw, that it holds only if quantities which are zero before the transformation remain zero after its application. This is not the case, unfortunately.

However, what about the  $\beta^1$  transformation for which we noted that  $N_{02}$  and  $N_{22}$  remained zero before and after the transformation? An examination of, say, Equation (21.25) reveals that the basic relations also break down under  $\beta^1$ , i.e.,

$$-4iN_{01} + N_{01}N_{12} - 2M_{01}L_{12} = -\frac{8\beta^2(\beta^2-1)Y}{X+\beta} \neq 0 \quad (21.31)$$

Similar ideas, as writing

$$N_{2k+1,2j} = \beta_{kj} A + \alpha_{kj} B \quad (21.32)$$

and finding A and B, such that the equation holds before and after a  $\beta^k$  transformation also failed.

In Paper VI another method of solution generation

is discussed. In Appendix O we outline the necessary modifications for the inclusion of electromagnetism. Unlike the  $\beta$  transformations, this new method is not essentially more difficult if an electromagnetic field is included.

## CHAPTER 10

### SUMMARY

"Use simplicity to surmount complexity." This is the basic message which forms the bedrock of solution generation. In language a physicist would employ, we might say, more specifically: "New solutions may be generated from old solutions via invariance properties."

At the heart of both the generation of new solutions and the discovery of the invariance properties of the combined Einstein-Maxwell equations lies an infinite hierarchy of potentials. In this work we have attempted to elucidate the essential mathematical and physical characteristics of the combined gravitational and electromagnetic potential hierarchy, in general, and then in greater detail for specific cases. Moreover, we have seen that, although a method of solution generation may be eminently viable for vacuum solutions, the inclusion of an electromagnetic field may render the method unworkable.

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APPENDIX A

Prolate spheroidal coordinates are defined by

$$\rho = k(X^2 - 1)^{\frac{1}{2}}(1 - Y^2)^{\frac{1}{2}} \quad (\text{A.1})$$

$$Z = kXY \quad (\text{A.2})$$

or

$$X = \frac{1}{2k} [ ((Z+k)^2 + \rho^2)^{\frac{1}{2}} + ((Z-k)^2 + \rho^2)^{\frac{1}{2}} ] \quad (\text{A.3})$$

$$Y = \frac{1}{2k} [ ((Z+k)^2 + \rho^2)^{\frac{1}{2}} - ((Z-k)^2 + \rho^2)^{\frac{1}{2}} ] \quad (\text{A.4})$$

where  $k$  is an arbitrary constant which may be defined to convert the result to a more familiar form.

In fact, it is convenient to let

$$k \equiv (m^2 - e^2)^{\frac{1}{2}} \quad (\text{A.5})$$

In terms of radial and angular Schwarzschild coordinates, we may identify

$$X = \frac{r-m}{k} \quad Y = \cos \theta \quad (\text{A.6})$$

The gradient and Laplacian operators are

$$\vec{\nabla} = \frac{k}{(X^2 - Y^2)^{\frac{1}{2}}} [ \hat{X}(X^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial X} + \hat{Y}(1 - Y^2)^{\frac{1}{2}} \frac{\partial}{\partial Y} ] \quad (\text{A.7})$$

$$\nabla^2 = \frac{k^2}{(X^2 - Y^2)} [ \frac{\partial}{\partial X} (X^2 - 1) \frac{\partial}{\partial X} + \frac{\partial}{\partial Y} (1 - Y^2) \frac{\partial}{\partial Y} ] \quad (\text{A.8})$$

We also need to know how the twist derivative will look.

If

$$\vec{\nabla} \rightarrow \left( \frac{1}{h_X} \frac{\partial}{\partial X}, \frac{1}{h_Y} \frac{\partial}{\partial Y} \right) \quad (\text{A.9})$$

then

$$\hat{\nabla} \rightarrow \left( \frac{1}{h_Y} \frac{\partial}{\partial Y}, -\frac{1}{h_X} \frac{\partial}{\partial X} \right), \quad (\text{A.10})$$

so

$$\hat{\nabla} = \frac{k}{(X^2 - Y^2)^{\frac{1}{2}}} \left[ \hat{X}(1 - Y^2)^{\frac{1}{2}} \frac{\partial}{\partial Y} - \hat{Y}(X^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial X} \right]. \quad (\text{A.11})$$

Therefore,

$$\hat{\nabla}_X = - \left[ \frac{X^2 - 1}{1 - Y^2} \right]^{\frac{1}{2}} \hat{\nabla}_Y \quad \hat{\nabla}_X = \left[ \frac{X^2 - 1}{1 - Y^2} \right]^{\frac{1}{2}} \hat{\nabla}_Y \quad (\text{A.12})$$

and

$$\hat{\nabla}_\rho = -\hat{\nabla}_Z. \quad (\text{A.13})$$

APPENDIX B

$$R_A = M_A + 2K\phi_A + H_{AX}\phi^X \quad (B.1)$$

What differential equation does  $R_A$  obey?

Applying Equations (9.1) - (9.4) we obtain

$$\vec{\nabla} R_A = (H_{AX} + H_{XA}^* + 2\phi_A \phi_X^*) \vec{\nabla} \phi^X + 2K \vec{\nabla} \phi_A + \phi^X \vec{\nabla} H_{AX} \quad (B.2)$$

Examine the term in brackets.

Equations (5.31), (5.33), and (5.35) may be combined to give

$$H_{AB} = f_{AB} - \phi_A^* \phi_B + \epsilon_{AB} K + i(\psi_{AB} + 2A_{AB}) \quad (B.3)$$

Thus,

$$\begin{aligned} H_{AX} + H_{XA}^* + 2\phi_A \phi_X^* \\ = 2f_{AX} + \epsilon_{AX} (K - K^*) + i(\psi_{AX} + 2A_{AX} - \psi_{XA} - 2A_{XA}) \\ = 2f_{AX} + \epsilon_{AX} (K - K^*) + i\epsilon_{XA} (\psi_Z^Z + 2A_Z^Z) \end{aligned} \quad (B.4)$$

so

$$\vec{\nabla} R_A = 2f_{AX} \vec{\nabla} \phi^X + (3K - K^* + i(\psi_Z^Z + 2A_Z^Z)) \vec{\nabla} \phi_A + \phi^X \vec{\nabla} H_{AX} \quad (B.5)$$

Consider  $\psi_Z^Z$ :

$$\vec{\nabla} \psi_Z^Z = \vec{\nabla} \psi_{21} - \vec{\nabla} \psi_{12} \quad (B.6)$$

Application of Equation (5.29) gives

$$\vec{\nabla} \psi_Z^Z = -\rho^{-1} \hat{\nabla} (f_{22} f_{11} - f_{12}^2) \quad (B.7)$$

or using Equation (1.3)

$$\vec{\nabla} \psi_Z^Z = 2\hat{\nabla} \rho = -2\vec{\nabla} Z \quad (B.8)$$

where we have made use of Equation (A.13) from Appendix A.

So

$$\psi^Z_Z = -2Z . \quad (\text{B.9})$$

Now

$$K-K^* = \phi^*_X \phi^X = 2i(A_1 B_2 - A_2 B_1) , \quad (\text{B.10})$$

so

$$-2A^Z B_Z + i\phi^*_X \phi^X = 0 . \quad (\text{B.11})$$

Using Equation (B.9) and (B.11) on the term in brackets in Equation (B.5) gives

$$3K-K^* + i(\psi^Z_Z + 2A^Z B_Z) = 2(K-iZ) . \quad (\text{B.12})$$

Therefore, we may rewrite Equation (B.5)

$$\vec{\nabla} R_A = 2f_{AX} \vec{\nabla} \phi^X + 2(K-iZ) \vec{\nabla} \phi_A + \phi^X \vec{\nabla} H_{AX} . \quad (\text{B.13})$$

Using the fact that

$$f^X_A f_{XB} = -\rho^2 \epsilon_{AB} \quad (\text{B.14})$$

and applying the tilde operation, we have

$$f^X_A \vec{\nabla} R_X = -2\rho^2 \epsilon_{AZ} \vec{\nabla} \phi^Z + 2(K-iZ) f^X_A \vec{\nabla} \phi_X + \phi^Z f^X_A \vec{\nabla} H_{XZ} . \quad (\text{B.15})$$

Then, if we take advantage of our basic relations for  $\phi_A$  and  $H_{AB}$ , we obtain

$$-i\rho^{-1} f^X_A \vec{\nabla} R_X = 2f_{AX} \vec{\nabla} \phi^X + 2(K-iZ) \vec{\nabla} \phi_A + \phi^Z \vec{\nabla} H_{AZ} = \vec{\nabla} R_A \quad (\text{B.16})$$

by Equation (B.13).

Therefore,  $R_A$  obeys the same equation as  $\phi_A$  and  $H_{AB}$ ! We also note for future reference that

$$H_{AX} + H_{XA}^* + 2\phi_A \phi_X^* = 2f_{AX} - 2i\epsilon_{XA} Z, \quad (\text{B.17})$$

using Equations (B.4) and (B.11).

APPENDIX C

Prove that

$$i H_{A1}^{n+1} = H_{A2}^n \quad (C.1)$$

with an appropriate choice of integration constants. We begin with Equation (9.10)

$$\vec{\nabla}^n H_{AB} = -i\rho^{-1} f_A^X \vec{\nabla}^n H_{XB} = -i\rho^{-1} f_{A2} \vec{\nabla}^n H_{1B} + i\rho^{-1} f_{A1} \vec{\nabla}^n H_{2B} \quad (C.2)$$

Now,

$$\begin{aligned} \vec{\nabla}^{n+1} (i H_{11} - H_{12}) &= -i\rho^{-1} [f_{12} \vec{\nabla}^{n+1} (i H_{11} - H_{12})] \\ &\quad + i\rho^{-1} [f_{11} \vec{\nabla}^{n+1} (i H_{21} - H_{22})] \end{aligned} \quad (C.3)$$

and

$$\begin{aligned} \vec{\nabla}^{n+1} (i H_{21} - H_{22}) &= -i\rho^{-1} [f_{22} \vec{\nabla}^{n+1} (i H_{11} - H_{12})] \\ &\quad + i\rho^{-1} [f_{21} \vec{\nabla}^{n+1} (i H_{21} - H_{22})] . \end{aligned} \quad (C.4)$$

This implies that

$$i H_{11}^{n+1} = H_{12}^n \quad (C.5)$$

if, and only if,

$$i H_{21}^{n+1} = H_{22}^n \quad (C.6)$$

Using Equation (9.16) we have

$$\vec{\nabla}^{n+1} H_{AB} = i [H_{1A}^* + H_{A1} + 2\phi_A \phi_1^*] \vec{\nabla}^n H_{2B}$$

$$\begin{aligned}
& -i[H_{2A}^* + H_{A2} + 2\phi_A \phi^*] \vec{\nabla}^n H_{1B} \\
& + 2i L_B \vec{\nabla}^n \phi_A + i H_{2B}^* \vec{\nabla}^n H_{A1} \\
& - i H_{1B}^* \vec{\nabla}^n H_{A2} .
\end{aligned} \tag{C.7}$$

Therefore, after some algebra,

$$\begin{aligned}
\vec{\nabla}^{n+2} (i H_{11} - H_{12}) &= i [(H_{11}^* + H_{11} + 2\phi_1 \phi_1^*) \vec{\nabla}^{n+1} (i H_{21} - H_{22})] \\
& - i [(H_{21}^* + H_{12} + 2\phi_1 \phi_2^*) \vec{\nabla}^{n+1} (i H_{11} - H_{12})] \\
& + i [(i H_{21} - H_{22}) \vec{\nabla}^n H_{11}] \\
& - i [(i H_{11} - H_{12}) \vec{\nabla}^n H_{12}] \\
& + 2i [(i L_1 - L_2) \vec{\nabla}^n \phi_1]
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\vec{\nabla}^{n+2} (i H_{21} - H_{22}) &= i [(H_{12}^* + H_{21} + 2\phi_2 \phi_1^*) \vec{\nabla}^{n+1} (i H_{21} - H_{22})] \\
& - i [(H_{22}^* + H_{22} + 2\phi_2 \phi_2^*) \vec{\nabla}^{n+1} (i H_{11} - H_{12})] \\
& + i [(i H_{21} - H_{22}) \vec{\nabla}^n H_{21}] \\
& - i [(i H_{11} - H_{12}) \vec{\nabla}^n H_{22}] \\
& + 2i [(i L_1 - L_2) \vec{\nabla}^n \phi_2] .
\end{aligned} \tag{C.9}$$

Now,

$$\vec{\nabla} L_B^{ln} = \phi_1^* \vec{\nabla} H_{2B}^n - \phi_2^* \vec{\nabla} H_{1B}^n \quad (C.10)$$

so

$$i \vec{\nabla} L_1^{ln+1} - \vec{\nabla} L_2^{ln} = \phi_1^* [\vec{\nabla} (H_{21}^{n+1} - H_{22}^n)] - \phi_2^* [\vec{\nabla} (i H_{11}^{n+1} - H_{12}^n)] \quad (C.11)$$

Thus it follows from Equations (C.3), (C.4), (C.8), and (C.9) that if  $i H_{A1}^{n+1} = H_{A2}^n$  for some value of  $n$ , then it holds good for all higher values also. In fact, it is easy to chose integration constants so that  $i H_{A1}^2 = H_{A2}^1$  for Reissner-Nordstrom while for the charged  $\delta = 2$  Weyl solution, the series begins at  $i H_{A1}^3 = H_{A2}^2$ .

APPENDIX D

Consider the general second-order differential equation

$$g_2(X)Z'' + g_1(X)Z' + g_0(X)Z = 0. \quad (D.1)$$

Consulting a handbook on orthogonal functions,<sup>21</sup> we compile the following table of solutions:

Table 1

(1.) $P_n^{(\alpha, \beta)}(X)$	$\frac{Z}{X}$	(1.) $1 - X^2$	$S_2(X)$
(2.) $(1-X)^\alpha (1+X)^\beta P_n^{(\alpha, \beta)}(X)$		(2.) $1 - X^2$	
(3.) $C_n^{(\alpha)}(X)$		(3.) $1 - X^2$	
(4.) $(1-X^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(X)$		(4.) $1 - X^2$	
(5.) $P_N(X)$		(5.) $1 - X^2$	
	$S_1(X)$		$S_0(X)$
(1.) $\beta - \alpha - (\alpha + \beta + 2)X$		(1.) $n(n + \alpha + \beta + 1)$	
(2.) $\alpha - \beta + (\alpha + \beta - 2)X$		(2.) $(n+1)(n + \alpha + \beta)$	
(3.) $-(2\alpha + 1)X$		(3.) $n(n + 2\alpha)$	
(4.) $(2\alpha - 3)X$		(4.) $(n+1)(n + 2\alpha - 1)$	
(5.) $-2X$		(5.) $n(n+1)$	

$P_n^{(\alpha, \beta)}(X)$  are the Jacobi polynomials,  $C_n^{(\alpha)}(X)$  are Gegenbauer polynomials, and  $P_n(X)$  are the familiar Legendre polynomials. Both the Jacobi and the Gegenbauer polynomials can be expressed in terms of Legendre polynomials.

The relevant expressions for the  $H_{AB}^n$  are

$$P_n^{(1,-1)}(X) = \left( \frac{n+1}{2n+1} \right) \frac{1}{(X-1)} [P_{n+1}(X) - P_{n-1}(X)] \quad (D.2)$$

$$P_n^{(-1,1)}(X) = \left( \frac{n+1}{2n+1} \right) \frac{1}{(X+1)} [P_{n+1}(X) - P_{n-1}(X)] \quad (D.3)$$

$$P_{n-1}^{(1,1)}(X) = \frac{2n}{2n+1} \frac{1}{(X^2-1)} [P_{n+1}(X) - P_{n-1}(X)] \quad (D.4)$$

$$P_n^{(-1,-1)}(X) = \frac{n}{2(2n+1)} [P_{n+1}(X) - P_{n-1}(X)] \quad (D.5)$$

$$P_{n-1}^{(0,2)}(X) = \frac{2}{(2n+1)} \frac{1}{(X+1)^2} [(n+1)P_{n-1}(X) + nP_{n+1}(X)] \\ + \frac{2P_n(X)}{(X+1)^2} \quad (D.6)$$

$$P_n^{(0,-2)}(X) = \frac{n}{2(2n+1)} [P_{n+1}(X) + P_{n-1}(X)] \\ + \frac{P_{n-1}(X)}{2(2n+1)} + \frac{P_n(X)}{2} \quad (D.7)$$

$$C_n^{(\frac{1}{2})}(X) = P_n(X) \quad (D.8)$$

$$C_{n+1}^{(-\frac{1}{2})}(X) = \frac{1}{(2n+1)} [P_{n-1}(X) - P_{n+1}(X)] \quad (D.9)$$

$$C_n^{(\frac{3}{2})}(X) = \frac{(n+2)(n+1)}{(2n+3)} \frac{1}{(X^2-1)} [P_{n+2}(X) - P_n(X)] \quad (D.10)$$

APPENDIX E

We will now outline a general method whereby Equations (11.6) and (11.18) may be decoupled. We have

$$G^A_X \vec{\nabla} F^X_B = (it \vec{\nabla} H^A_X) F^X_B + 2it (\vec{\nabla} \phi^A) S_B \quad (E.1)$$

$$\vec{\nabla} S_B = \phi^*_X \vec{\nabla} F^X_B \quad (E.2)$$

Rewriting Equation (E.1) by means of the inverse matrices  $F^{-1A}_B$  and  $G^{-1A}_B$ , we obtain

$$(\vec{\nabla} F^C_X) F^{-1X}_D = it G^{-1C}_X [\vec{\nabla} H^X_D + 2 (\vec{\nabla} \phi^X) S_Z F^{-1Z}_D] \quad (E.3)$$

Let

$$T_D \equiv S_Z F^{-1Z}_D, \quad (E.4)$$

then

$$(\vec{\nabla} F^C_X) F^{-1X}_D = it G^{-1C}_X [\vec{\nabla} H^X_D + 2 (\vec{\nabla} \phi^X) T_D] \quad (E.5)$$

Now

$$\begin{aligned} \vec{\nabla} T_D &= (\vec{\nabla} S_Z) F^{-1Z}_D + S_Z \vec{\nabla} (F^{-1Z}_D) \\ &= (\vec{\nabla} S_Z) F^{-1Z}_D - S_Z F^{-1Z}_X (\vec{\nabla} F^X_Y) F^{-1Y}_D. \end{aligned} \quad (E.6)$$

Using Equations (E.2) and (E.5), we find

$$\vec{\nabla} T_D = it G^{-1X}_Z (\vec{\nabla} H^Z_D + 2 (\vec{\nabla} \phi^Z) T_D) (\phi^*_X - T_X) \quad (E.7)$$

Setting  $D = 1$ , we obtain

$$\begin{aligned} \vec{\nabla} T_1 &= it G^{-1^1}_Z (\vec{\nabla} H^Z_1 + 2 T_1 \vec{\nabla} \phi^Z) (\phi^*_1 - T_1) \\ &\quad + it G^{-1^2}_Z (\vec{\nabla} H^Z_1 + 2 T_1 \vec{\nabla} \phi^Z) (\phi^*_2 - T_2). \end{aligned} \quad (E.8)$$

It is clear that if we now look at the  $\hat{X}$  and  $\hat{Y}$  component parts of Equation (E.8), we will have two independent equations and thus may eliminate  $T_2$ . This will leave an equation involving only known quantities and  $T_1$ .

For convenience let us define the following:

$$\begin{aligned}
 A^1 &\equiv G^{-1}{}^1{}_1 \phi_1^* + G^{-1}{}^2{}_1 \phi_2^* \\
 A^2 &\equiv G^{-1}{}^1{}_1 \phi_1^* + G^{-1}{}^2{}_1 \phi_2^* \\
 \vec{B}^1 &\equiv G^{-1}{}^1{}_1 \vec{\nabla} \phi_2 - G^{-1}{}^1{}_2 \vec{\nabla} \phi_1 \\
 \vec{B}^2 &\equiv 2A^2 \vec{\nabla} \phi_1 - 2A^1 \vec{\nabla} \phi_2 + G^{-1}{}^1{}_1 \vec{\nabla} H_{21} - G^{-1}{}^1{}_2 \vec{\nabla} H_{11} \\
 \vec{B}^3 &\equiv A^2 \vec{\nabla} H_{11} - A^1 \vec{\nabla} H_{21} \\
 \vec{B}^4 &\equiv G^{-1}{}^2{}_1 \vec{\nabla} \phi_2 - G^{-1}{}^2{}_2 \vec{\nabla} \phi_1 \\
 \vec{B}^5 &\equiv G^{-1}{}^2{}_1 \vec{\nabla} H_{21} - G^{-1}{}^2{}_2 \vec{\nabla} H_{11} .
 \end{aligned} \tag{E.9}$$

Then, having eliminated  $T_2$  from Equation (E.8), we have

$$\begin{aligned}
 &T_{1,X} [2T_1 B^4_Y + B^5_Y] - T_{1,Y} [2T_1 B^4_X + B^5_X] \\
 &+ it \{ T_1^3 [4B^1_X B^4_Y - 4B^1_Y B^4_X] \\
 &\quad + T_1^2 [2B^1_X B^5_Y - 2B^1_Y B^5_X + 4B^2_X B^4_Y - 4B^2_Y B^4_X] \\
 &\quad + T_1 [B^2_X B^5_Y - B^2_Y B^5_X + 2B^3_X B^4_Y - 2B^3_Y B^4_X] \\
 &\quad + (B^3_X B^5_Y - B^3_Y B^5_X) \} \\
 &= 0 .
 \end{aligned} \tag{E.10}$$

Once we have  $T_D$  then we return to Equation (E.5) and discover that we now know everything on the right-hand side of the equation.

Define

$$\gamma \equiv \det(F_{B}^A) , \quad (E.11)$$

then

$$F^{-1}A_B = \frac{1}{\gamma} \begin{pmatrix} F^2_2 & -F^1_2 \\ -F^2_1 & F^1_1 \end{pmatrix} . \quad (E.12)$$

So, taking the trace of Equation (E.5), we have on the right-hand side

$$\begin{aligned} & \frac{1}{\gamma} [ (\vec{\nabla} F^1_1) F^2_2 - (\vec{\nabla} F^1_2) F^2_1 - (\vec{\nabla} F^2_1) F^1_2 + (\vec{\nabla} F^2_2) F^1_1 ] \\ & = \frac{1}{\gamma} [ \vec{\nabla} [ F^1_1 F^2_2 - F^1_2 F^2_1 ] ] = \frac{1}{\gamma} \vec{\nabla} \gamma . \end{aligned} \quad (E.13)$$

The right-hand side only contains known quantities. Therefore, once having taken the trace of Equation (E.5), we may integrate the result and obtain  $\gamma$ . This is the generalization of Equation (5.9) in Paper III.

For convenience we will define

$$\vec{\eta}^C_D \equiv itG^{-1C}_X [ \vec{\nabla} H^X_D + 2T_D \vec{\nabla} \phi^X ] . \quad (E.14)$$

Then setting  $C = 1, D = 1$  and  $C = 1, D = 2$  in Equation (E.5), we obtain

$$\vec{\eta}^1_1 = (\vec{\nabla} F^1_1) F^2_2 - (\vec{\nabla} F^1_2) F^2_1 \quad (E.15)$$

$$\gamma \vec{\eta}^1_2 = (\vec{\nabla} F^1_2) F^1_1 - (\vec{\nabla} F^1_1) F^1_2 . \quad (\text{E.16})$$

Using Equations (E.16) and (E.11), we find

$$\begin{aligned} \gamma \vec{\eta}^1_1 &= \frac{\vec{\nabla} F^1_1}{F^1_1} [\gamma + F^2_1 F^1_2] - \frac{F^2_1}{F^1_1} [\gamma \vec{\eta}^1_2 + (\vec{\nabla} F^1_1) F^1_2] \\ &= \frac{1}{F^1_1} \vec{\nabla} F^1_1 - F^2_1 \vec{\eta}^1_2 . \end{aligned} \quad (\text{E.17})$$

If we consider the  $\hat{X}$  and  $\hat{Y}$  components of Equation (E.17), we have two independent equations and so  $F^2_1$  may be eliminated, leaving us with an equation for  $F^1_1$ .

APPENDIX F

In this appendix we wish to derive an equation for  $F_{11}[D_1]$  involving only  $H_{11}$  or  $\phi_1$ .

Rewriting Equation (11.33) in characteristic form, we have

$$\begin{aligned} \frac{dx}{2t(X^2-1)\phi_{1,X} + (1-2tZ)\phi_{1,Y}} &= \frac{dy}{2t(1-Y^2)\phi_{1,Y} - (1-2tZ)\phi_{1,X}} \\ &= \frac{dD_1}{itD_1[\phi_{1,X}H_{12,Y} - \phi_{1,Y}H_{12,X}] + itD_2[\phi_{1,Y}H_{11,X} - \phi_{1,X}H_{11,Y}]} \end{aligned} \quad (F.1)$$

Now, combining terms

$$\begin{aligned} &\frac{\phi_{1,X}dx + \phi_{1,Y}dy}{2t(X^2-1)\phi_{1,X}^2 + (1-2tZ)\phi_{1,X}\phi_{1,Y} + 2t(1-Y^2)\phi_{1,Y}^2 - (1-2tZ)\phi_{1,X}\phi_{1,Y}} \\ &= \frac{d\phi_1}{2t[(X^2-1)\phi_{1,X}^2 + (1-Y^2)\phi_{1,Y}^2]} = \frac{d\phi_1}{2t(X^2-Y^2)\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1}, \end{aligned} \quad (F.2)$$

using results from Appendix A. However,

$$\vec{\nabla}A \cdot \vec{\nabla}B = \frac{\rho}{(X^2-Y^2)} (A_{,Y}B_{,X} - A_{,X}B_{,Y}) \quad (F.3)$$

So, using Equations (F.2) and (F.3) we obtain

$$\begin{aligned} &\frac{\rho dD_1}{it(X^2-Y^2)[D_1\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12} + D_2\vec{\nabla}H_{11} \cdot \vec{\nabla}\phi_1]} \\ &= \frac{d\phi_1}{2t(X^2-Y^2)\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1}. \end{aligned} \quad (F.4)$$

Equations (11.37) and (11.38) show that

$$\vec{\nabla}H_{11} \cdot \vec{\nabla}\phi_1 = 0, \quad (F.5)$$

so,

$$\frac{-idD_1}{[D_1\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12}]} = \frac{d\phi_1}{2\rho\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1}. \quad (F.6)$$

Now we need to rewrite the  $\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12}$  term. From Equation (6.9), using the tilde operation, we have

$$\vec{\nabla}H_{11} = -i\rho^{-1}f_{12}\vec{\nabla}H_{11} + i\rho^{-1}f_{11}\vec{\nabla}H_{21} . \quad (\text{F.7})$$

What is the relation between  $H_{12}$  and  $H_{21}$ ? Using Equation (8.16) we quickly see that

$$H_{21} = H_{12} - 2K + i[\psi_{21} - \psi_{12}] \quad (\text{F.8})$$

or

$$H_{21} = H_{12} - 2K + i\psi_X^X . \quad (\text{F.9})$$

However, by Equation (B.9) of Appendix B, we have

$$H_{21} = H_{12} - 2(K + iZ) . \quad (\text{F.10})$$

Therefore, rearranging terms,

$$\vec{\nabla}H_{12} = -i\rho f^{-1}\vec{\nabla}H_{11} - \omega\vec{\nabla}H_{11} + 2\vec{\nabla}(K + iZ) . \quad (\text{F.11})$$

Then, using Equation (F.5)

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12} = -i\rho f^{-1}\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{11} + 2\vec{\nabla}\phi_1 \cdot \vec{\nabla}K + 2i\vec{\nabla}\phi_1 \cdot \vec{\nabla}Z . \quad (\text{F.12})$$

Using Equation (11.39),

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}K = \phi_1^* \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 . \quad (\text{F.13})$$

Rewriting Equation (5.28) and using Equation (1.3), we see that

$$\vec{\nabla}\phi_2 = -i\rho f^{-1}\vec{\nabla}\phi_1 - \omega\vec{\nabla}\phi_1 , \quad (\text{F.14})$$

thus, using Equation (11.39) again,

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 = -i\rho f^{-1}\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1 . \quad (\text{F.15})$$

Combining the results of Equation (F.12), (F.13), and (F.15) with Equation (A.13) from Appendix A, we may write

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12} = -i\rho f^{-1} [\vec{\nabla}H_{11} + 2\phi_1^* \vec{\nabla}\phi_1] \cdot \vec{\nabla}\phi_1 + 2i\vec{\nabla}\phi_1 \cdot \vec{\nabla}\rho \quad (F.16)$$

Part of this equation looks familiar, and a quick check with the Ernst Equation shows why. So, using Equation (6.20) we obtain the simple result that

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}H_{12} = -i\rho \nabla^2 \phi_1 + 2i\vec{\nabla}\phi_1 \cdot \vec{\nabla}\rho \quad (F.17)$$

Combining Equations (F.6) and (F.16) we finally achieve our goal

$$\frac{dD_1}{D_1} = \left( \frac{\frac{1}{2}\nabla^2 \phi_1 - \rho^{-1} \vec{\nabla}\phi_1 \cdot \vec{\nabla}\rho}{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1} \right) d\phi_1 \quad (F.18)$$

A comparison of Equations (11.32) and (11.33) reveals that if we had begun with Equations (11.32), we would have obtained

$$\frac{dF_{11}}{F_{11}} = \left( \frac{\frac{1}{2}\nabla^2 \phi_1 - \rho^{-1} \vec{\nabla}\phi_1 \cdot \vec{\nabla}\rho}{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_1} \right) d\phi_1 \quad (F.19)$$

Because of the duality of the equations, it is easy to show, with Equations (11.30) and (11.31), using the same procedure as above, that we could also write

$$\frac{dF_{11}}{F_{11}} = \left( \frac{\frac{1}{2}\nabla^2 H_{11} - \rho^{-1} \vec{\nabla}H_{11} \cdot \vec{\nabla}\rho}{\vec{\nabla}H_{11} \cdot \vec{\nabla}H_{11}} \right) dH_{11} \quad (F.20)$$

and

$$\frac{dD_1}{D_1} = \left( \frac{\frac{1}{2}\nabla^2 H_{11} - \rho^{-1} \vec{\nabla} H_{11} \cdot \vec{\nabla} \rho}{\vec{\nabla} H_{11} \cdot \vec{\nabla} H_{11}} \right) dH_{11} \quad (\text{F.21})$$

We are not, of course, implying that  $F_{11}$  and  $D_1$  are identical. They are not since their initial terms,  $H_{11}$  and  $\phi_1$ , are different. The above results only reflect the fact that  $H_{AB}^n$  and  $\phi_A^n$  obey the same differential equation.

APPENDIX G

The generating functions, with full-gauge freedom, for the Reissner-Nordstrom metric are

$$F_{11} = \frac{t}{(X+\beta)S} [K(t) [X-2tY] + K_1(t) S] \quad (G.1)$$

$$F_{12} = \frac{i}{(X+\beta)S} [C(t) [X-2tY] + C_1(t) S] \quad (G.2)$$

$$F_{21} = -\frac{i}{2S} [K(t) \sigma - (2K_2(t) + 2K_1(t)tY) S] \quad (G.3)$$

$$F_{22} = \frac{1}{2tS} [C(t) \sigma - (C_2(t) + 2C_1(t)t\beta Y) S] \quad (G.4)$$

$$D_1 = \frac{t}{(X+\beta)S} [E(t) [X-2tY] + E_1(t) S] \quad (G.5)$$

$$D_2 = -\frac{i}{2S} [E(t) \sigma - (2E_2(t) + 2E_1(t)Y) S] \quad (G.6)$$

$$Q = iE_3(t) + \frac{iE_1(t)tq\beta Y}{X+\beta} - \frac{itE(t)q\beta Y[2tY-X]}{(X+\beta)S} + \frac{itE(t)q\beta[2tX-Y]}{S} \quad (G.7)$$

$$S_1 = iK_3(t) + \frac{iK_1(t)tq\beta Y}{X+\beta} + \frac{2it^2K(t)q\beta X}{S} - \frac{itK(t)q\beta Y(2tY+\beta)}{(X+\beta)S} \quad (G.8)$$

$$S_2 = C_3(t) - \frac{C_1(t)q\beta Y}{X+\beta} - \frac{2C(t)tq\beta X}{S} + \frac{C(t)q\beta Y(2tY+\beta)}{(X+\beta)S} \quad (G.9)$$

where

$$S^2 \equiv 1 - 4tXY + 4t^2(X^2 + Y^2 - 1) \quad (G.10)$$

$$\sigma \equiv 1 - 2tY(X - \beta) - 4t^2(X\beta + 1 - Y^2) \quad (G.11)$$

APPENDIX H

We wish to derive the relation given by Equation (13.26). It seems clear that we should begin with Equation (13.20), so

$$\begin{matrix} m,n+2 & m+1,n+1 \\ K & - K \end{matrix} = 2i \begin{matrix} m11,n+1 & m1 & 0,n+1 \\ K & K & -L_X \end{matrix} M^X \quad (H.1)$$

and

$$\begin{matrix} m+1,n+1 & m+2,n \\ K & - K \end{matrix} = 2i \begin{matrix} m+1,1 & 1n & m+1,1 & 0n \\ K & K & -L_X & M^X \end{matrix} . \quad (H.2)$$

Now from Equation (13.22) we have

$$\begin{matrix} 0,n+1 & 1n & 01 & 1n & 01 & n \\ M_X & = M_X + 2i M_X & K + i N & X Z \phi^Z \end{matrix} , \quad (H.3)$$

and Equation (13.20) gives

$$\begin{matrix} 1n+1 & 2n & 11 & 1n & 11 & 0n \\ K & = K + 2i K & K - L_X & M^X \end{matrix} \quad (H.4)$$

$$\begin{matrix} m+1,1 & m2 & m1 & 11 & m1 & 01 \\ K & = K - 2i K & K + L_X & M^X \end{matrix} . \quad (H.5)$$

Equation (13.21) gives

$$\begin{matrix} m+1,1 & m2 & m1 & 11 & m1 & 1 \\ L_X & = L_X - 2i K & L_X & -i L_X & Z^H & X \end{matrix} . \quad (H.6)$$

Combining Equations (H.1) and (H.2) and inserting Equations (H.3) - (H.4), we find, after some algebra, that

$$\begin{matrix} m,n+2 & m+2,n \\ K & - K \end{matrix} = 2i \left( \begin{matrix} m1 & 2n & 1n & m2 \\ K & K + K & K & K \end{matrix} \right) - \left( L_X M^X + L_X M^X \right) . \quad (H.7)$$

In the above we once used Equation (13.18).

APPENDIX I

If we refer back to Section 3 of Paper II, we find the infinitesimal transformations of the potentials for which the Harrison transformation is but a specific case. It is possible to exponentiate some of these, as outlined in Paper III and, in particular, to acquire the finite form of the action of the Harrison transformation on all the electromagnetic and gravitational potentials. As a caveat, we note that the  $c_A^k$  infinitesimal transformations should be summed  $s = 1, \dots, k$  on the first sum and  $s = 1, \dots, k-1$  on the second sum. This avoids producing any extra linear terms.

As an example, we will now derive the finite action of the Harrison transformation on  $M_1$ . As noted in Paper II, we use  $c_2^1 \equiv c$  and set  $c_1^1 = 0$ . Therefore, using Equation (3.2) from Paper II, we obtain

$$M_1 \rightarrow M_1 + 2ic^* M_1 + c N_{11} \quad (I.1)$$

From this we see that we also require the action of  $c_2^1$  on  $N_{11}$ :

$$N_{11} \rightarrow N_{11} + 2ic^* M_1 N_{11} \quad (I.2)$$

Then, building up the infinitesimal elements

$$M_1 \rightarrow M_1 + 2ic^* M_1 + c N_{11}$$

$$\begin{aligned}
& + \frac{2ic^{*01}}{2!} [M_1 [2ic^{*01} M_1 M_1 + c N_{11}]] \\
& + \frac{2ic^{*01}}{2!} [M_1 [2ic^{*01} (M_1)^2 + c N_{11}]] \\
& + \frac{c}{2!} [2ic^{*01} M_1 N_{11}] \\
& + \dots
\end{aligned} \tag{I.3}$$

In fact,

$$\begin{aligned}
M_1 & \rightarrow M_1 + 2ic^{*01} M_1 M_1 + c N_{11} \\
& + \frac{1}{2!} [-8(c^{*01})^2 (M_1)^2 M_1 + 4ic^{*01} c N_{11} M_1 \\
& \quad + 2ic^{*01} c N_{11} M_1] \\
& + \frac{1}{3!} [-48i(c^{*01})^3 (M_1)^3 M_1 - 24(c^{*01})^2 c N_{11} (M_1)^2 \\
& \quad - 24(c^{*01})^2 c M_1 N_{11} + 6c^{*01} c^2 N_{11} N_{11}] \\
& + \dots
\end{aligned} \tag{I.4}$$

One can soon realize that this is a series expansion of

$$M_1 \rightarrow \frac{M_1 + c N_{11}}{1 - 2ic^{*01} M_1 - icc^{*01} N_{11}} \tag{I.5}$$

In addition, we find

$$Q_{11} \rightarrow \frac{Q_n}{N_{11}} \frac{1}{1-2ic^* M_1 - icc^* N_{11}} \quad (I.6)$$

$$Q_{12} \rightarrow \frac{Q_n}{N_{12} - ic^* (2M_1 + c N_{11})} \frac{Q_{n+1}}{1-2ic^* M_1 - icc^* N_{11}} \quad (I.7)$$

and, in general, for all the potentials,<sup>23</sup> under the Harrison transformation

$$Q_{mn} \rightarrow A Q_{mn} B + \frac{ic A Q_{m1} D Q_{0n} B}{\Delta} \quad (I.8)$$

where

$$Q_{mn} \equiv \begin{pmatrix} mn & mn & m,n-1 \\ K & L_1 & L_2 \\ mn & mn & m,n-1 \\ M_1 & N_{11} & N_{12} \\ m-1,n & m-1,n & m-1,n-1 \\ M_2 & N_{21} & N_{22} \end{pmatrix} \quad (I.9)$$

$$A \equiv \begin{pmatrix} 1 & c^* & 0 \\ 0 & 1 & 0 \\ 2ic & icc^* & 1 \end{pmatrix} \quad B \equiv A^\dagger$$

$$D \equiv \begin{pmatrix} 0 & 2 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (I.10)$$

$$\Delta \equiv 1-2ic^* M_1 - icc^* N_{11} .$$

In addition we may also procure finite forms of the

gauge action. From Paper II we find that  $c_1^0$  produces the electromagnetic gauge transformations. Therefore, setting  $c_2^0 \equiv 0$  and using Equation (3.2) of Paper II, we obtain

$$N_{AB}^{0n} \rightarrow N_{AB}^{0n} - 2ic_B^{0*} M_A^{0n} \quad (I.11)$$

$$M_A^{0n} \rightarrow M_A^{0n} - c_1^0 N_{A2}^{0, n-1}, \quad (I.12)$$

where we have employed the fact that  $k = 0$  and  $L_B^{0n} = 0$ .

Iterating further, we have [remember that  $c_2^0 \equiv 0$ ]

$$N_{AB}^{0n} \rightarrow N_{AB}^{0n} - 2ic_B^{0*} M_A^{0n} - \frac{2ic_B^{0*}}{2!} [-c_1^0 N_{A2}^{0, n-1}]. \quad (I.13)$$

Therefore, the finite form of the electromagnetic gauge transformation is

$$N_{AB}^{0n} \rightarrow N_{AB}^{0n} - 2ic_B^{0*} M_A^{0n} + ic_B^{0*} c_1^0 N_{A2}^{0, n-1} \quad (I.14)$$

$$M_A^{0n} \rightarrow M_A^{0n} - c_1^0 N_{A2}^{0, n-1}. \quad (I.15)$$

We remark that using Equations (12.16), (13.18), and (13.19) one sees that Equation (I.11) with  $A = B = 1$  and Equation (I.12) with  $A = 1$  are identical to the gauge relations contained in Equations (7.6) and (7.15).

Using Equations (I.11), (13.18), and (13.19) we may write

$$H_{11} \rightarrow H_{11} - 2ic_1^{0*} \phi_1 - c_1^0 c_1^0. \quad (I.16)$$

Comparing this with Equation (16.18), we see that the real  $c$  of the Harrison transformation is equivalent to  $ic_1^0$  in the gauge transformation, thus,

$$c = ic_1^0 \quad (I.17)$$

APPENDIX J

Defining

$$\eta(t) \equiv (1-4t^2)^{\frac{1}{2}} \quad (\text{J.1})$$

$$S(t)^2 = 1-4tXY+4t^2(X^2+Y^2-1) \quad , \quad (\text{J.2})$$

then the generating functions, in a form consistent with the Harrison transformation, are:

$$F_{11} = \frac{t[X-2tY-\beta S]}{(X+\beta)S\eta} \quad (\text{J.3})$$

$$F_{12} = \frac{i[X-2tY+\beta S]}{(X+\beta)S\eta} \quad (\text{J.4})$$

$$F_{21} = \frac{i}{2} \left( \beta^2 - 1 - \frac{\beta^2}{\eta} \right) - \frac{i\beta t Y}{\eta} \\ - \frac{i}{2S\eta} [1-2tY(X-\beta) - 4t^2(X\beta+1-Y^2)] \quad (\text{J.5})$$

$$F_{22} = \frac{1}{2t} \left( \beta^2 - 1 - \frac{\beta^2}{\eta} \right) - \frac{\beta Y}{\eta} \\ + \frac{1}{2tS\eta} [1-2tY(X-\beta) - 4t^2(X\beta+1-Y^2)] \quad (\text{J.6})$$

$$S_1 = \frac{iq\beta^2}{2} \left( 1 - \frac{1}{\eta} \right) - \frac{itq\beta^2 Y}{(X+\beta)\eta} + \frac{2it^2 q\beta X}{S\eta} - \frac{itq\beta Y(2tY+\beta)}{(X+\beta)S\eta} \quad (\text{J.7})$$

$$S_2 = \frac{q\beta^2}{2t} \left( 1 - \frac{1}{\eta} \right) - \frac{q\beta^2 Y}{(X+\beta)\eta} - \frac{2tq\beta X}{S\eta} + \frac{q\beta Y(2tY+\beta)}{(X+\beta)S\eta} \quad (\text{J.8})$$

$$D_1 = \frac{tq\beta}{(X+\beta)\eta} \quad (\text{J.9})$$

$$D_2 = \frac{itq\beta Y}{\eta} + \frac{iq\beta^2}{2} \left( \frac{1}{\eta} - 1 \right) \quad (\text{J.10})$$

$$Q = \frac{itY(\beta^2-1)}{(X+\beta)\eta} + \frac{i}{2} \left( \beta^2 \left( \frac{1}{\eta} - 1 \right) - \frac{1}{\eta} \right) \quad (\text{J.11})$$

APPENDIX K

$$\begin{aligned}
& \begin{matrix} k & mn & mn & m,n+k+2 & m,n+k+1 & m+k,n & m+k+1,n \\ \beta: N_{12} & \rightarrow & N_{12} + \beta \{ - ( & N_{11} + i & N_{12} ) + ( & N_{22} + i & N_{12} ) \end{matrix} \\
& \begin{matrix} m_1 & k+1,n & kn & 0n & m,k+2 & m,k+1 \\ + N_{11} [ & N_{12} - i N_{22} ] + N_{12} [ & N_{11} + i & N_{12} ] \end{matrix} \\
& \begin{matrix} k & m,s+1 & ms & k+1-s,n & k-s,n \\ + \sum_{s=1} [ & N_{11} + i N_{12} ] [ & N_{12} - i & N_{22} ] \end{matrix} \\
& \begin{matrix} m_1 & k+1,n & k & m,s+1 & k-s+1,n \\ - 2 M_1 & L_2 - 2 \sum_{s=1} M_1 & L_2 \end{matrix} \} \\
& \hspace{15em} (K.1)
\end{aligned}$$

$$\begin{aligned}
& \begin{matrix} k & mn & mn & m+k+2,n & m+k+1,n & m,n+k & m,n+k+1 \\ \beta: N_{21} & \rightarrow & N_{21} + \beta \{ - ( & N_{11} - i & N_{21} ) + ( & N_{22} - i & N_{21} ) \end{matrix} \\
& \begin{matrix} 0n & m,k+2 & m,k+1 & m_1 & k+1,n & kn \\ + i N_{11} [ - i & N_{21} + & N_{22} ] + N_{21} [ & N_{11} - i N_{21} ] \end{matrix} \\
& \begin{matrix} k & k+1-s,n & k-s,n & m,s+1 & ms \\ + \sum_{s=1} [ & N_{11} - i & N_{21} ] [ & N_{21} + i N_{22} ] \end{matrix} \\
& \begin{matrix} k & m,s+1 & k-s+1,n & m_1 & k+1,n \\ - 2 \sum_{s=1} ( & M_2 & L_1 - 2 M_2 & L_1 ) \} \end{matrix} \\
& \hspace{15em} (K.2)
\end{aligned}$$

$$\begin{aligned}
& \begin{matrix} k & mn & mn & m+k+2,n & m+k+1,n & m,n+k+2 & m,n+k+1 \\ \beta: N_{22} & \rightarrow & N_{22} + \beta \{ i ( i & N_{12} + & N_{22} ) - i ( - i & N_{21} + & N_{22} ) \end{matrix} \\
& \begin{matrix} 0n & m,k+2 & m,k+1 & m_1 & k+1,n & kn \\ + i N_{12} [ - i & N_{21} + & N_{22} ] - i N_{21} [ i & N_{12} + N_{22} ] \end{matrix} \\
& \begin{matrix} k & m,s+1 & ms & k-s,n & k+1-s,n \\ + \sum_{s=1} [ - i & N_{21} + N_{22} ] [ & N_{22} + i & N_{12} ] \end{matrix} \\
& \begin{matrix} k & m,s+1 & k-s+1,n & m_1 & k+1,n \\ - 2 \sum_{s=1} ( & M_2 & L_2 - 2 M_2 & L_2 ) \} \end{matrix} \\
& \hspace{15em} (K.3)
\end{aligned}$$



$$+ \sum_{s=1}^{k+2} (L_{1+i} L_2) \begin{pmatrix} m, s-1 & k+2-s, n & k+1-s, n \\ M_1 & -i & M_2 \end{pmatrix}$$

$$\begin{aligned} \beta: N_{mn} &\rightarrow N_{mn} + \beta \{ 2i (N_{m+k+1, n} - N_{m, n+k+1}) \\ &+ \sum_{s=1}^{k+2} N_{ms} N_{k+2-s, n} - 2 \sum_{s=1}^{k+1} M_{ms} L_{k-s+2, n} \} \end{aligned} \quad (K.7)$$

$$\begin{aligned} \beta: M_{0n} &\rightarrow M_{0n} + \beta \{ -i M_{0, n+k+1} - 2 \sum_{s=1}^{k+1} M_{0s} K_{k-s+2, n} \\ &+ \sum_{s=1}^{k+2} N_{0s} M_{k+2-s, n} \} \end{aligned} \quad (K.8)$$

$$\begin{aligned} \beta: M_{mn} &\rightarrow M_{mn} + \beta \{ -i M_{m, n+k+1} + 2i M_{m+k+1, n} \\ &- 2 \sum_{s=1}^{k+1} M_{ms} K_{k-s+2, n} + \sum_{s=1}^{k+2} N_{ms} M_{k+2-s, n} \} \end{aligned} \quad (K.9)$$

$$\begin{aligned} \beta: L_{0n} &\rightarrow L_{0n} + \beta \{ -2i L_{0, n+k+1} - 2 \sum_{s=1}^{k+1} K_{0s} L_{k-s+2, n} \\ &+ \sum_{s=1}^{k+2} L_{0s} N_{k+2-s, n} \} \end{aligned} \quad (K.10)$$

$$\begin{aligned}
& \beta: L_{mn}^{(k)} \rightarrow L_{mn} + \beta \{ -2iL_{m,n+k+1} + iL_{m+k+1,n} \\
& \quad - 2 \sum_{s=1}^{k+1} K_{ms} L_{k-s+2,n} + \sum_{s=1}^{k+2} L_{ms} N_{k+2-s,n} \}
\end{aligned} \tag{K.11}$$

$$\begin{aligned}
& \beta: K_{0n}^{(k)} \rightarrow K_{0n} + \beta \{ -iK_{0,n+k+1} - 2 \sum_{s=1}^{k+1} K_{0s} K_{k-s+2,n} \\
& \quad + \sum_{s=1}^{k+2} L_{0s} M_{k+2-s,n} \}
\end{aligned} \tag{K.12}$$

$$\begin{aligned}
& \beta: K_{mn}^{(k)} \rightarrow K_{mn} + \beta \{ i(K_{m+k+1,n} - K_{m,n+k+1}) \\
& \quad - 2 \sum_{s=1}^{k+1} K_{ms} K_{k-s+2,n} + \sum_{s=1}^{k+2} L_{ms} M_{k+2-s,n} \}
\end{aligned} \tag{K.13}$$

APPENDIX L

Defining

$$\eta(t) = [1-4t^2]^{\frac{1}{2}}, \quad (\text{L.1})$$

then, the explicit forms for the double generating functions

$M_A$ ,  $L_A$ , and  $K$  are:

$$M_1 = \frac{itq\beta \left( \beta^2 - 1 - \frac{\beta^2}{\eta(r)} - \frac{2\beta rY}{\eta(r)} \right)}{2(X+\beta)\eta(t)} - \frac{itq\beta S(r)}{2(X+\beta)\eta(r)\eta(t)} + \frac{iq\beta^2 tr[1-4tr-\eta(r)\eta(t)]}{2\eta(r)\eta(t)(r-t)} \quad (\text{L.2})$$

$$M_2 = \frac{tq\beta \left( 1 - \beta^2 + \frac{\beta^2}{\eta(r)} + \frac{2rY\beta}{\eta(r)} \right)}{2r(X+\beta)\eta(t)} - \frac{tq\beta S(r)}{2r(X+\beta)\eta(t)\eta(r)} - \frac{q\beta^2 t[1-4rt-\eta(r)\eta(t)]}{2\eta(r)\eta(t)(r-t)} \quad (\text{L.3})$$

$$M_1 - irM_2 = \frac{itq\beta \left( \beta^2 - 1 - \frac{\beta^2}{\eta(r)} - \frac{2\beta rY}{\eta(r)} \right)}{(X+\beta)\eta(r)} + \frac{iq\beta^2 tr[1-4tr-\eta(r)\eta(t)]}{(r-t)\eta(r)\eta(t)} \quad (\text{L.4})$$

$$L_1 = - \frac{iq\beta^2 trY}{(X+\beta)\eta(r)\eta(t)} - \frac{itrq\beta Y[2tY-X]}{(X+\beta)\eta(t)\eta(r)S(t)} + \frac{itrq\beta[2tX-Y]}{\eta(r)\eta(t)S(t)} - \frac{itq\beta^3}{2(X+\beta)\eta(r)\eta(t)} + \frac{itq\beta^3}{2(X+\beta)\eta(t)}$$

(L.5)

$$\begin{aligned}
& + \frac{itq\beta^2[X-2tY]}{2(X+\beta)\eta(r)\eta(t)S(t)} - \frac{itq\beta^2[X-2tY]}{2(X+\beta)\eta(t)S(t)} \\
& + \frac{iq\beta^2rt(1-4rt-\eta(r)\eta(t))}{2\eta(r)\eta(t)(r-t)}
\end{aligned}$$

$$\begin{aligned}
L_2 = & - \frac{q\beta^2rY}{(X+\beta)\eta(t)\eta(r)} + \frac{q\beta rY[2tY-X]}{(X+\beta)\eta(t)\eta(r)S(t)} \\
& - \frac{q\beta r[2tX-Y]}{\eta(r)\eta(t)S(t)} - \frac{q\beta^3}{2(X+\beta)\eta(t)\eta(r)} + \frac{q\beta^3}{2(X+\beta)\eta(t)} \\
& - \frac{q\beta^2(X-2tY)}{2(X+\beta)\eta(t)\eta(r)S(t)} + \frac{q\beta^2(X-2tY)}{2(X+\beta)\eta(t)S(t)} \\
& + \frac{q\beta^2r[1-4rt-\eta(r)\eta(t)]}{2\eta(r)\eta(t)(r-t)}
\end{aligned} \tag{L.6}$$

$$\begin{aligned}
L_1 + itL_2 = & - \frac{2iq\beta^2Yrt}{(X+\beta)\eta(r)\eta(t)} + \frac{iq\beta^3t}{(X+\beta)\eta(t)} \\
& - \frac{iq\beta^3t}{(X+\beta)\eta(r)\eta(t)} \\
& + iq\beta^2 \left[ \left[ \frac{rt}{r-t} \right] \left[ \frac{(1-4rt-\eta(r)\eta(t))}{\eta(r)\eta(t)} \right] \right]
\end{aligned} \tag{L.7}$$

$$\begin{aligned}
K = & - \frac{i(\beta^2-1)}{2} \left[ \frac{rt}{r-t} \right] \left[ \frac{1-4rt-\eta(r)\eta(t)}{\eta(r)\eta(t)} \right] - \frac{i}{2}(r-t) \\
& + \frac{i\beta(\beta^2-1)t}{2(X+\beta)\eta(r)\eta(t)} - \frac{i\beta(\beta^2-1)t}{2(X+\beta)\eta(t)} + \frac{i(\beta^2-1)Yrt}{(X+\beta)\eta(r)\eta(t)}
\end{aligned} \tag{L.8}$$

APPENDIX M

$$M_{0,2j+1} = \alpha_{0j} M_{01}$$

$$M_{2k,2j} = 0$$

$$M_{2k+1,2j} = \beta_{kj} M_{12}$$

(M.1)

$$M_{2k+1,2j+1} = \alpha_{kj} M_{11}$$

$$M_{2k,2j+1} = \frac{\alpha_{kj}}{2} M_{21} + \beta_{kj} M_{12}$$

$$L_{0j} = L_{k0} = 0$$

$$L_{2k,2j} = 0$$

$$L_{2k+1,2j+1} = \alpha_{kj} L_{11}$$

(M.2)

$$L_{2k+1,2j} = \beta_{kj} L_{12}$$

$$L_{2k,2j+1} = \frac{\alpha_{kj}}{2} L_{21} + \beta_{kj} L_{12}$$

$$K_{01} = -K_{10} = \frac{i}{2}$$

$$K_{0j} = K_{k0} = 0 \quad j, k > 1$$

$$K_{2k+1,2j+1} = \alpha_{kj} K_{11}$$

(M.3)

$$K_{2k+1,2j} = \beta_{kj} K_{12}$$

$$K_{2k,2j} = 0$$

$$K_{2k,2j+1} = \frac{\alpha_{kj}}{2} K_{2,1} + \beta_{kj} K_{1,2}$$

$$\beta_{kj} = \begin{pmatrix} \beta_{00} & \beta_{01} & \dots \\ \beta_{10} & & \\ \vdots & & \\ \vdots & & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 & 10 & 35 & 126 & \dots \\ 0 & 1 & 4 & 15 & 56 & 210 & \dots \\ 0 & 2 & 9 & 36 & 140 & 540 & \dots \\ 0 & 5 & 24 & 100 & 400 & 1575 & \dots \\ 0 & 14 & 70 & 300 & 1225 & 4900 & \dots \\ 0 & 42 & 216 & 945 & 3920 & 15876 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (\text{M. 4})$$

A series of useful relationships among the  $\beta_{kj}$  and  $\alpha_{kj}$  may be derived.<sup>26</sup>

$$2(\beta_{kj} + \beta_{jk}) = \alpha_{kj} \quad (\text{M. 5})$$

$$\beta_{k,j+1} + \beta_{j,k+1} = 2\alpha_{kj} \quad (\text{M. 6})$$

$$\beta_{k,k+1} = \alpha_{kk} \quad (\text{M. 7})$$

$$\beta_{0j} = \frac{1}{2}\alpha_{0j} \quad (\text{M. 8})$$

$$\beta_{j1} = 2\alpha_{0j} - \frac{1}{2}\alpha_{0,j+1} \quad (\text{M. 9})$$

$$\beta_{1j} = -\alpha_{0j} + \frac{1}{2}\alpha_{0,j+1} \quad (\text{M. 10})$$

$$\sum_{r=0}^p \alpha_{kr} \alpha_{p-r,j} = 2^{2p} \alpha_{kj} \quad (\text{M. 11})$$

$$\sum_{r=0}^p \alpha_{kr} \beta_{p-r,j} = \frac{1}{2} \alpha_{k,j+p} \quad (\text{M.12})$$

$$\sum_{r=0}^p \beta_{kr} \alpha_{p-r,j} = -\frac{1}{2} \alpha_{k+p,j} + 2^{2p-1} \alpha_{kj} \quad (\text{M.13})$$

$$\sum_{r=0}^p \beta_{kr} \beta_{p-r,j} = -\frac{1}{2} \beta_{k+p,j} + \frac{1}{2} \beta_{k,j+p} \quad (\text{M.14})$$

APPENDIX N

$$\begin{aligned}
\beta : N_{2k, 2j+1}^{2p} &= \frac{\alpha_{kj}}{2} \beta : N_{21}^{2p} + \beta \{ N_{11} [ 2i\alpha_{k+p, j}^{-i\alpha_{p+1, 0}} \alpha_{kj} ] \\
&+ \frac{1}{2} N_{22} [ i\alpha_{0, p+1} \alpha_{kj}^{-i\alpha_{k, j+p+1}} ] \\
&+ \frac{1}{4} N_{22} N_{12} [ \frac{1}{2} \alpha_{k, j+p+1}^{-\frac{1}{2} \alpha_0} j^{\alpha_{k, p+1}} k^{\alpha_{p+1, j}} ] \\
&+ \frac{1}{2} N_{11} N_{12} [ -\alpha_{k+p, j} + \frac{1}{2} \alpha_{p+1, 0} \alpha_{kj} ] \\
&- M_{12} L_{11} [ -\alpha_{k+p, j} + \frac{1}{2} \alpha_{kj} \alpha_{p+1, 0} ] \\
&- \frac{1}{2} M_{22} L_{12} [ \frac{1}{2} \alpha_{k, j+p+1}^{-\frac{1}{2} \alpha_0} j^{\alpha_{k, p+1}} k^{\alpha_{p+1, j}} ] \}
\end{aligned} \tag{N.1}$$

$$\begin{aligned}
\beta : N_{2k+1, 2j}^{2p} &= \frac{\alpha_{kj}}{2} \beta : N_{12}^{2p} + \beta \{ N_{22} [ \frac{i}{2} \alpha_{k+p+1, j}^{-\frac{i}{2} \alpha_{kj}} \alpha_{p+1, 0} ] \\
&+ N_{11} [ -2i\alpha_{k, j+p} + i\alpha_{kj} \alpha_{0, p+1} ] \\
&+ N_{12} N_{02} [ \frac{1}{2} \alpha_0 j^{\beta_{k, p+1}} k^{\alpha_{kj} \alpha_{0, p+1}} ] \\
&+ N_{12} N_{22} [ -\frac{1}{8} \alpha_{k+p+1, j}^{-\frac{1}{4} \beta_{k, p+1}} \alpha_0 j^{\frac{1}{4} \alpha_{0, p+1}} \alpha_{kj} ] \\
&+ N_{11} N_{12} [ \frac{1}{2} \alpha_{k, j+p}^{-\frac{1}{4} \alpha_{kj}} \alpha_{0, p+1} ] \\
&- 2M_{12} L_{22} [ -\frac{1}{8} \alpha_{k+p+1, j}^{-\frac{1}{4} \beta_{k, p+1}} \alpha_0 j^{\frac{1}{4} \alpha_{0, p+1}} \alpha_{kj} ] \\
&- 2M_{11} L_{12} [ \frac{1}{2} \alpha_{k, j+p}^{-\frac{1}{4} \alpha_{0, p+1}} \alpha_{kj} ] \}
\end{aligned} \tag{N.2}$$

$$\beta : N_{2k+1, 2j+1}^{2p} = \alpha_{kj} \beta : N_{11}^{2p} + \beta \{ N_{21} [ i\alpha_{k+p+1, j}^{-i\alpha_{p+1, 0}} \alpha_{kj} ]$$

$$\begin{aligned}
& +N_{12} [2i\beta_{k+p+1,j} - 2i\beta_{k,j+p+1} + i\alpha_{p+1,0} \alpha_{kj}] \\
& +N_{12}N_{01} [\alpha_{0j} \beta_{k,p+1}^{-\frac{1}{2}\alpha_{0,p+1}} \alpha_{kj}] \\
& + \frac{N_{12}N_{21}}{2} [-\frac{1}{2}\alpha_{k+p+1,j}^{-\beta_{k,p+1}} \alpha_{0j} + \alpha_{p+1,0} \alpha_{kj}] \\
& +N_{12}^2 [-\frac{1}{2}\beta_{k+p+1,j} + \frac{1}{2}\beta_{k,j+p+1} - \beta_{k,p+1} \frac{\alpha_{0j}}{2}] \\
& -M_{12}L_{21} [-\frac{1}{2}\alpha_{k+p+1,j}^{-\beta_{k,p+1}} \alpha_{0j} + \alpha_{p+1,0} \alpha_{kj}] \\
& -2M_{12}L_{12} [-\frac{1}{2}\beta_{k+p+1,j} + \frac{1}{2}\beta_{k,j+p+1} - \beta_{k,p+1} \frac{\alpha_{0j}}{2}]
\end{aligned} \tag{N.3}$$

$$\begin{aligned}
\beta : N_{0,2j}^{2p} &= \frac{\alpha_{0j}}{2} \beta : N_{0,2}^{2p} + \beta \{ [\frac{1}{2}\alpha_{0,j+p} - \frac{1}{2}\alpha_{j,p+1}] \\
& \times [-4iN_{01} + N_{01}N_{12} - 2M_{01}L_{12}] \}
\end{aligned} \tag{N.4}$$

$$\begin{aligned}
\beta : N_{2k,2j}^{2p} &= \frac{\alpha_{kj}}{4} \beta : N_{2,2}^{2p} + \beta \{ \sigma_1 [2iN_{12} - \frac{1}{2}N_{12}^2 + M_{12}L_{12}] \\
& + \sigma_2 [-iN_{21} + \frac{1}{2}N_{21}N_{12} - \frac{1}{2}M_{21}L_{12}] \}
\end{aligned} \tag{N.5}$$

where

$$\sigma_1 = \beta_{k+p,j}^{-\beta_{k,j+p}} - \frac{3}{4}\alpha_{kj} \alpha_{0,p+1} + \frac{1}{4}\alpha_{kj} \alpha_{0,p+2} \tag{N.6}$$

$$\sigma_2 = \alpha_{k,j+p}^{-\frac{1}{2}\alpha_{kj}} \alpha_{0,p+1} \tag{N.7}$$

$$\beta^{2p+1} : N_{2k, 2j+1} = \frac{\alpha_{kj}}{2} \beta^{2p+1} : N_{21}$$

$$+ \beta^{2p} \{ i N_{21} (\alpha_{k+p+1, j}^{-\alpha} \alpha_{k, j+p+1}^{-\alpha} - \frac{\alpha_{p+2, 0} \alpha_{kj}^{+\alpha} \alpha_{0, p+1} \alpha_{kj}}{2})$$

$$+ 2i N_{12} (\beta_{k+p+1, j}^{-\beta} \alpha_{k, j+p+1}^{-\beta} - \frac{\alpha_{0, p+1} \alpha_{kj}^{+\frac{1}{2}\alpha} \alpha_{0, p+2} \alpha_{kj}}{2})$$

$$+ N_{12} N_{01} [\alpha_{0j} \beta_{k, p+1}^{+\alpha} \alpha_{0, p+1} \frac{\alpha_{kj}}{2} - \frac{1}{4} \alpha_{0, p+2} \alpha_{kj}]$$

$$+ N_{21} N_{12} [\frac{1}{4} \alpha_{k, j+p+1}^{-\frac{1}{2}\alpha} \alpha_{k+p+1, j}^{-\frac{1}{2}\alpha} \beta_{k, p+1} \alpha_{0j}$$

$$- \frac{1}{2} \alpha_{kj} \alpha_{0, p+1}^{+\frac{1}{2}\alpha} \alpha_{p+2, 0} \alpha_{kj}]$$

(N. 8)

$$+ N_{12}^2 [-\frac{1}{2} \beta_{k+p+1, j}^{+\frac{1}{2}\beta} \beta_{k, j+p+1}^{+\frac{1}{2}\beta} \beta_{k, p+1} \alpha_{0j}]$$

$$- 2M_{21} L_{12} [\frac{1}{4} \alpha_{k, j+p+1}^{-\frac{1}{2}\alpha} \alpha_{k+p+1, j}^{-\frac{1}{2}\alpha} \beta_{k, p+1} \alpha_{0j}$$

$$- \frac{1}{2} \alpha_{0, p+1} \alpha_{kj}^{+\frac{1}{2}\alpha} \alpha_{p+2, 0} \alpha_{kj}]$$

$$- 2M_{12} L_{12} [-\frac{1}{2} \beta_{k+p+1, j}^{+\frac{1}{2}\beta} \beta_{k, j+p+1}^{+\frac{1}{2}\beta} \beta_{k, p+1} \alpha_{0j}]$$

$$\beta^{2p+1} : N_{2k+1, 2j} = \frac{\alpha_{kj}}{2} \beta^{2p+1} : N_{12}$$

$$+ \beta^{2p+1} \{ [2i N_{12} - \frac{1}{2} N_{12}^2 + M_{12} L_{12}]$$

(N. 9)

$$\times [\beta_{k+p+1, j}^{-\beta} \alpha_{k, j+p+1}^{-\alpha} \alpha_{0, p+1} \alpha_{kj}$$

$$+ \frac{1}{2} \alpha_{0, p+2} \alpha_{kj}] \}$$

$$\begin{aligned}
\beta : N_{2k+1, 2j+1}^{2p+1} &= \alpha_{kj} \beta : N_{11}^{2p+1} \\
&+ \beta \{ (\alpha_{k+p+1, j}^{-\alpha_{k, j+p+1}})^{2p+1} \\
&\quad \times (2iN_{11} - \frac{1}{2}N_{12}N_{11} + M_{12}L_{11}) \}
\end{aligned} \tag{N.10}$$

$$\begin{aligned}
\beta : N_{0, 2j}^{2p+1} &= \frac{\alpha_{0j}}{2} \beta : N_{02}^{2p+1} \\
&+ \beta \{ (N_{02}N_{12} - 2M_{02}L_{12}) \\
&\quad \times (\frac{1}{2}\alpha_{0, j+p+1}^{-\frac{1}{4}\alpha_{j, p+2} + \frac{1}{2}\alpha_{j, p+1} - \frac{1}{2}\beta_{p+1, j}})^{2p+1} \\
&\quad + iN_{02} [-\alpha_{0, j+p+1} + \frac{1}{2}\alpha_{j, p+2}] \}
\end{aligned} \tag{N.11}$$

$$\begin{aligned}
\beta : N_{2k, 2j}^{2p+1} &= \frac{\alpha_{kj}}{4} \beta : N_{22}^{2p+1} \\
&+ \beta \{ (N_{22}N_{12} - 2M_{12}L_{22}) \\
&\quad \times (\frac{1}{8}\alpha_{k, j+p+1}^{-\frac{1}{8}\alpha_{k+p+1, j} - \frac{1}{4}\alpha_{k0} \beta_{p+1, j}})^{2p+1} \\
&\quad - \frac{1}{4}\alpha_{0j} \beta_{k, p+1} + \frac{1}{8}\alpha_{kj} \alpha_{0, p+1} \} \\
&\quad + iN_{22} [\frac{1}{2}(\alpha_{k+p+1, j}^{-\alpha_{k, j+p+1}})] \\
&\quad + N_{12}N_{02} [\frac{1}{2}\beta_{k, p+1} \alpha_{0j} + \frac{\alpha_{kj}}{4} \alpha_{0, p+1}]
\end{aligned} \tag{N.12}$$

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$$- \frac{1}{8} \alpha_{kj} \alpha_{0,p+1} \}}}$$

APPENDIX O

In this appendix we will sketch a method which, unlike the  $\beta$  transformation, can be applied when electro-magnetism is included.

A particular problem with the  $\beta$  transformation was that its action on one potential coupled the original potential to many others. Is it possible to avoid this difficulty? Returning to Equation (19.2), we notice something promising.<sup>27</sup> The action of  $\gamma_{22}^k$  on  $N_{11}^{mn}$  only involves combinations of the  $N_{11}^k$ . Furthermore, the action of  $\gamma_{22}^k$  on  $M_1^{mn}$  is

$$\gamma_{22}^k: M_1^{mn} \rightarrow M_1^{mn} + \gamma_{22}^k \sum_{s=1}^k N_{11}^{ms} M_1^{k-s,n} \quad (0.1)$$

As  $M_1^{0n}$  and  $N_{11}^{0n}$  give us the necessary information to construct the metric, we see that we are dealing with the appropriate potentials.

Another difference from the  $\beta$  method is that an infinite number of the  $\gamma_{22}^k$ 's will be used, i.e., consider a transformation of the form  $\sum_{k=0}^{\infty} a_k \gamma_{22}^k$  where the  $a_k$  are constants.

So,

$$N_{11}^{mn} \rightarrow N_{11}^{mn} + \sum_{k=0}^{\infty} a_k \sum_{s=1}^k N_{11}^{ms} N_{11}^{k-s,n} \quad (0.2)$$

$$M_{11} \rightarrow M_{11} + \sum_{k=0}^{\infty} a_k \sum_{s=1}^{k+1} N_{11}^{k-s,n} M_{11} \quad (0.3)$$

Now consider the infinite dimensional matrices:

$$N \equiv \begin{pmatrix} 0 & N_{11}^{01} & N_{11}^{02} & \dots \\ 0 & N_{11}^{11} & N_{11}^{12} & \dots \\ 0 & N_{11}^{21} & N_{11}^{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (0.4)$$

$$M \equiv \begin{pmatrix} 0 & M_{11}^{01} & M_{11}^{02} & \dots \\ 0 & M_{11}^{11} & M_{11}^{12} & \dots \\ 0 & M_{11}^{21} & M_{11}^{22} & \dots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad (0.5)$$

$$A \equiv \begin{pmatrix} a_0 & a_1 & \dots \\ a_1 & a_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad (0.6)$$

In matrix form we may write Equations (0.2) and (0.3) as

$$N \rightarrow N + NAN \quad (0.7)$$

$$M \rightarrow M + MAN \quad (0.8)$$

Converting the above two equations to differential form we obtain

$$\frac{dN(\lambda)}{d\lambda} = N(\lambda)AN(\lambda) \quad (0.9)$$

$$\frac{dM(\lambda)}{d\lambda} = M(\lambda)AN(\lambda) \quad (0.10)$$

Upon integration, we have, where I is the identity matrix,

$$N(\lambda) = \frac{N}{I - \lambda AN} \quad (0.11)$$

$$M(\lambda) = \frac{M}{I - \lambda AN} \quad (0.12)$$

So, for  $N'$  and  $M'$  being the final transformed potential matrices, then

$$N' = \frac{N}{I - AN} \quad (0.13)$$

$$M' = \frac{M}{I - AN} \quad (0.14)$$

Then

$$N' - N = N'AN \quad (0.15)$$

$$M' - M = M'AN \quad (0.16)$$

In terms of components,

$$N_{11} - N_{11} = \sum_{s,r} N_{11} A_{sr} N_{11} \quad (0.17)$$

$$M_1 - M_1 = \sum_{s,r} N_{11} A_{sr} M_1 \quad (0.18)$$

At this juncture we apply the generating function concept again. To facilitate this, define

$$a^{rs} \equiv y^r u^s, \quad (0.19)$$

then

$$\sum_{mn} N_{11r}^{mn} t^{m,n} - \sum_{mn} N_{11r}^{mn} t^{m,n} = \sum_{\substack{ms \\ mn \\ rs}} N_{11r}^{ms} \gamma u^s u^r N_{11t}^{rn} \quad (0.20)$$

$$\sum_{mn} M_{1r}^{mn} t^{m,n} - \sum_{mn} M_{1r}^{mn} t^{m,n} = \sum_{\substack{ms \\ mn \\ rs}} N_{11r}^{ms} \gamma u^s u^r M_{1t}^{rn} \quad (0.21)$$

Therefore,

$$\begin{aligned} N_{11}(r,t) - N_{11}(r,t) &= \gamma \sum_{\substack{ms \\ mn \\ rs}} (N_{11r}^{ms} u^s) (N_{11t}^{rn} u^r) \\ &= \gamma N_{11}(r,u) N_{11}(u,t) \end{aligned} \quad (0.22)$$

Likewise,

$$M_{11}(r,t) - M_{11}(r,t) = \gamma N_{11}(r,u) M_{11}(u,t) \quad (0.23)$$

so,

$$N_{11}(r,t) = N_{11}(r,t) + \frac{\gamma N_{11}(r,u) N_{11}(u,t)}{1 - \gamma N_{11}(u,u)} \quad (0.24)$$

$$M_{11}(r,t) = M_{11}(r,t) + \frac{\gamma N_{11}(r,u) M_{11}(u,t)}{1 - \gamma N_{11}(u,u)} \quad (0.25)$$

Only algebraic manipulations remain as, fortuitously, we have previously calculated the appropriate generating functions  $N_{11}$  [see Equation (20.1)] and  $M_{11}$  [see Equation (L.2) of Appendix L].

The amount of algebra remaining is considerable, but as it is only algebra, the generation of new solutions

could be turned over to an algebraic computer program.

