The maximum deflection of a blast loaded cantilever beam by the modified Galerkin method
by Daniel Michael Koszuta

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Aerospace and Mechanical Engineering
Montana State University

Abstract:
This paper presents an analytical method for calculating the maximum deflection of a rectangular cantilever beam subjected to blast loading. The method is based on the modified Galerkin method and uses a nonlinear stress-strain relation together with the nonlinear geometry changes of large deflection motion.

The paper also presents a method whereby the approximate solution can be improved. By variation of the shape functions used the corresponding equation residuals are studied. A preferred or "better residual" is defined and then used to select the shape function which best satisfies the governing equation of motion.

Results show that a one term Galerkin solution gives satisfactory accuracy. Both the linearized and nonlinear beam equations are solved. Residual properties of both the linear and nonlinear problems are found to be identical. This suggests first solving the linearized version of a physical system by this method and using the shape functions which give good results in the linear case to solve the nonlinear version.

The investigation revealed that combining shape functions resulted in an averaging of their residuals which can be used to good advantage when attempting to reduce equation residuals.

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Signature Daniel Michael Koszuta

Date 11-21-69

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DANIEL MICHAEL KOSZUTA

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Approved:

E. R. Bishop
Head, Major Department

Dennis O. Blackketter
Chairman, Examining Committee

K. Goering
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

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ABSTRACT

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CHAPTER I: INTRODUCTION

The classical linearized theory of beam deflections is known well and has been used extensively with good results in some engineering problems encountered in the past. This theory is based on the restrictive assumptions of small deflections and a linear stress-strain relation. Both assumptions eliminate highly nonlinear terms from the governing equations of motion. Previous works [1,2][1] have included some nonlinear aspects of beam motion but still only approach the general form of the equation of motion as developed by Eringen [3]. Another popular approach used in the past [4-8] has been the Rigid-Plastic theory, but in most cases it is again restricted to small or moderately large deflections. The literature presently available to the author suggests a need to investigate more of the nonlinear terms involved in the general governing equations.

The purpose of this paper is to present an approximate method of solution for the large deflection motion of a blast loaded rectangular cantilever beam. Fewer restrictive assumptions have been made in the derivation of an equation of motion which is a better approximation to the general form. An inverse tangent function was used to approximate a true stress-strain relation. An approximate solution for the maximum deflection is obtained by the modified Galerkin method. This method involves choosing deflection shape functions which reduce the continuous system to one of a finite number of degrees of freedom. The difficulty

[1] Numbers in brackets refer to literature consulted.

of solving nonlinear equations led to a one term solution of the motion. A procedure to improve the approximate solution by mode shape variation is then described and evaluated. To aid this study the method is applied first to the classical beam equation using the corresponding exact eigenfunctions with variation. The results of mode shape variation in the linear case are studied, then the same technique is used on the nonlinear equation of motion. In this case the deflection shape of the nonlinear motion [9] is assumed to be approximately that of the linearized beam.

CHAPTER II: SOLUTION METHOD

2.1 The Galerkin Method as a Weighted Residual Method

Many physical problems encountered in engineering, such as the large deflections of a beam, lead to nonlinear mathematical models. The present methods available do not provide exact solutions in such cases. The engineer, quite practically, turns to methods which yield approximate solutions. The method of weighted residuals [10], of which the Galerkin method is one type, is such a method. The method involves assuming a particular form of the solution which is a sum of products of space and time functions. The space or shape functions are known from assumptions made through experience and intuition in most cases. Nonlinear problems make the choice of shape functions much more difficult. The time functions are unknown. The assumed solution, being inexact, will not satisfy the system model but will yield an error termed the equation residual and/or boundary residuals. These residuals are weighted in different fashions resulting in differential equations to be solved for the time functions.

The accuracy of the resulting approximate solution depends on the accuracy of the chosen shape functions. Finlayson and Scriven [10] point out the difficulties involved and the scant knowledge available on convergence, especially in a nonlinear problem. Crandl [11] tacitly assumed that convergence occurs with an increasing number of terms in the assumed solution.

Consider the mathematical model in two independent variables for y(x,t):

$$N\left[y(x,t)\right] - \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad \text{for } x \text{ in } V,\ t > 0 \qquad \text{(Eq. 1)}$$

where $N\left[y\right]$ denotes a general differential operator involving spacial derivatives of y in the three dimensional domain V with boundary S, and t represents time.

The inital and boundary conditions are:

$$y(x,0) = y_0(x) \qquad \text{for } x \text{ in } V$$

$$y(x,t) = f_s(x,t) \qquad \text{for } x \text{ on } S \qquad \text{(Eq. 2)}$$

A trial solution is assumed of the form:

$$y^*(x,t) = y_s(x,t) + \sum_{i=1}^{n} q_i(t)\ \theta_i(x) \qquad \text{(Eq. 3)}$$

where $y_s(x,t)$ satisfies all the nonhomogeneous boundary conditions and $\theta_i(x)$ satisfies all homogeneous boundary conditions; i.e.,

$$y_s = f_s$$

$$\text{for } x \text{ on } S$$

$$\theta_i = 0 \qquad \text{(Eq. 4)}$$

Substitution of the assumed solution Equation (3) into the governing Equation (1) yields the equation residual:

$$R\left[y^*(x,t)\right] = N(y^*) - \frac{\partial^2 y^*}{\partial t^2} \qquad \text{(Eq. 5)}$$

The residual is a measure of the exactness of the assumed solution. When the assumed solution is the exact solution, the residual will be identically zero throughout the domain V. It is a premise of this paper that the magnitude and distribution of the residual can be used to judge the merit of the assumed solution.

The next step in the procedure is an approximation to the ideal case of a zero residual. The weighted integrals of the residual are set equal to zero over the domain of interest:

$$\int_V W_j \, R(y^*) \, dV = 0 \qquad j = 1,2,3,\ldots,n \qquad \text{(Eq. 6)}$$

where the W_j are prescribed weighting functions and can be chosen in several different ways [10]. Each criterion for choosing the weighting functions corresponds to a particular weighted residual method. The Galerkin method uses the chosen space modes as weighting functions;

$$\int_V \theta_j \, R(y^*) \, dV = 0 \qquad j = 1,2,3,\ldots,n \qquad \text{(Eq. 7)}$$

Equations (7) represent a system of n ordinary second order differential equations in time for the time modes $q_i(t)$. They may be linear or nonlinear, coupled or uncoupled, depending on the spacial operator N(y) and the orthogonality of the space modes with respect to the residual. The original initial conditions are imposed on the $q_i(t)$, term by term, thus completing the approximate solution, Equation 3.

2.2 The Modified Galerkin Method

The modified Galerkin method is nearly identical to the Galerkin method. As discussed by Anderson [12], it is used for problems which present great difficulty in choosing shape functions which satisfy all boundary conditions. The modified Galerkin method permits the use of functions which satisfy only the displacement boundary conditions. As an example, let y(x,t) represent the nonlinear large deflection motion of a cantilever beam fixed at x = 0 and free at x = L. The displacement boundary conditions at the wall are:

$$y(0,t) = 0 \qquad \text{(zero displacement)}$$

$$\frac{\partial y(0,t)}{\partial x} = 0 \qquad \text{(zero slope)} \qquad \text{(Eq. 8)}$$

which can be easily satisfied. The force boundary conditions of zero moment and shear at the free end are very difficult to satisfy since these expressions are highly nonlinear. Using space modes $\Theta_i(x)$ which satisfy only the displacement boundary conditions will result in errors in force at the free end termed boundary residuals. The modified Galerkin method includes the boundary errors in the weighted residual condition, Equation (7):

$$\int_V \Theta_j \, R(y^*) \, dV + E_M \, \Phi_j + E_V \, \Theta_j = 0 \qquad j = 1,2,3,\ldots,n \qquad \text{(Eq. 9)}$$

where E_M and E_V are the moment and shear force errors at the free end.

Φ_j and Θ_j are weighting terms usually chosen as a rotation and displacement of the free end in the j_{th} space mode. Equations (9) similarly result in n differential equations in the time modes q_i.

CHAPTER III: FORMULATION OF SYSTEM MODELS

3.1 The Blast Loaded Linear Beam

The derivation of the linearized equation of motion of a cantilever beam is well known and is presented here only for easy comparison with that of the nonlinear large deflection equation of motion.

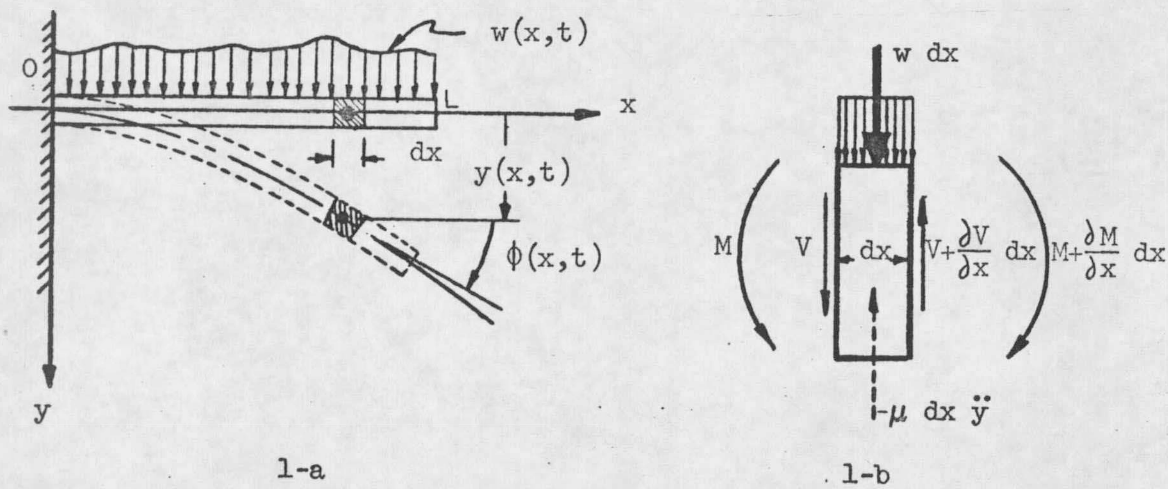

Figure 1. Linear Beam Deflection and Elemental Forces

Referring to Figure 1, it is assumed that there exists a neutral plane at the beam center, initially on the x-axis, which does not deflect longitudinally. Neglecting shear deformation and assuming that cross-sectional planes remain plane, let $y(x,t)$ represent the motion of the neutral plane. The beam is fixed at $x = 0$ and free at $x = L$. The ex-ternal loading is represented by w(lbs/in). Referring to the free body diagram (Figure 1-b) of a beam element of length dx, we can write the equation of motion from a Newtonian force balance in the y direction:

$$\overset{\downarrow}{+} \sum \text{External Forces} - \text{Inertia Force} = 0$$

$$w\,dx + V - \left(V + \frac{\partial V}{\partial x}\,dx\right) - \mu\,dx\,\frac{\partial^2 y}{\partial t^2} = 0$$

or: $$w - \frac{\partial V}{\partial x} - \mu\,\frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{(Eq. 10)}$$

where V = Shear Force (psi)

μ = Linear Beam Density (lbsm/in)

t = Time (sec)

Summing moments and neglecting rotational inertia yields:

$$\frac{\partial M}{\partial x} = V \qquad \text{(Eq. 11)}$$

where M = Internal Moment (in-lbs).

Substituting Equation (11) into Equation (10):

$$\overset{\downarrow}{+} \qquad w - \frac{\partial^2 M}{\partial x^2} - \mu\,\frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{(Eq. 12)}$$

Note that it was assumed that geometry changes do not effect the direction of forces. Assuming a linear stress-strain relation and small deflection, from elementary beam theory [13] we have:

$$\frac{\partial y}{\partial x} = \tan\phi \approx \phi$$

and $$M = EI\,\frac{\partial^2 y}{\partial x^2} \qquad \text{(Eq. 13)}$$