Uncountable collections of unimodal continua
by Paul James Johanson

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of
Philosophy in Mathematical Sciences
Montana State University
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Abstract:
Inverse limit spaces occur often in dynamical systems, in particular, those with unimodal bonding
maps. Barge and Diamond showed uncountably many indecomposable unimodal inverse limit spaces
exist. We show that uncountably many hereditarily decomposable unimodal inverse limit spaces exist
by actually creating two different uncountable collections of hereditarily decomposable unimodal
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more unimodal inverse limit spaces.

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APPROVAL

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Date

Marcy Barge

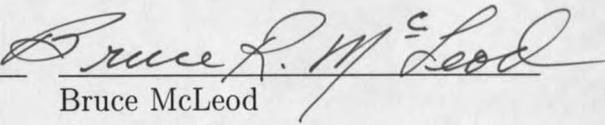Chairperson, Graduate Committee

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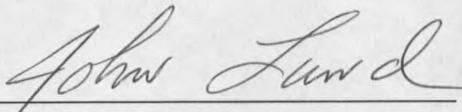
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Jim Johanson

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Kirsten Johanson

Kelsey Johanson

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Mary Johanson

</div>

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ABSTRACT

Inverse limit spaces occur often in dynamical systems, in particular, those with unimodal bonding maps. Barge and Diamond showed uncountably many indecomposable unimodal inverse limit spaces exist. We show that uncountably many hereditarily decomposable unimodal inverse limit spaces exist by actually creating two different uncountable collections of hereditarily decomposable unimodal inverse limit spaces. Finally we consider how to expand upon these and combine them to create even more unimodal inverse limit spaces.

CHAPTER 1

INTRODUCTION

We begin by defining what we mean by the term *inverse limit space*. This definition is general, but for most of this paper we will make a few restrictions that will make it easier to visualize and think about the inverse limit spaces we will be dealing with. For our definition, let $i \in \{0, 1, 2, 3, \dots\}$ and let $\{X_i\}_{i=1}^{\infty}$ be a sequence compact metric spaces each with metric d_i such that for all $x, y \in X_i$, $d_i(x, y) \leq 1$. Also let $\{f_i\}_{i=0}^{\infty}$ be a sequence of continuous functions so that $f_i : X_{i+1} \to X_i$ (these functions will be referred to as *bonding maps*). We define the *inverse limit space* of $\{f_i\}$, denoted $\varprojlim\{f_i\}$, to be the subset of $\prod_{i=0}^{\infty} X_i$ given by

$$\varprojlim\{f_i\} = \{\underline{x} = (x_0, x_1, x_2, x_3, \dots) : x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i\}$$

with the metric defined to be $\underline{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$. Another common notation used in some manuscripts is (X_i, f_i). It can be shown that $\varprojlim\{f_i\}$ is a compact metric space, and connected if each X_i is connected. ([14], pg. 163). If we are repeating a single bonding map, that is $f_i = f$ for all i, we will use the notation $\varprojlim f$ rather than $\varprojlim\{f_i\}$. We notice that the topology from our metric on the inverse limit space is equivalent to the subspace topology inherited from the product topology on $\prod_{i=0}^{\infty} X_i$. We define a *continuum* to be any compact, connected, metric space.

For this thesis, we are mostly interested in the case where $X_i = [a_i, b_i]$, a compact interval in the real numbers, $\mathbb{R}$, with the metric $d_i(x, y) = \frac{|x-y|}{b_i - a_i}$ for all $i \in \{0, 1, 2, \dots\}$. We say that the continuous bonding maps $f_i : X_{i+1} \to X_i$ are *unimodal* if there is a critical value $c_i \in X_i$, dividing X_i into two subintervals, $[a_i, c_i]$ and $[c_i, b_i]$, so that f_i is monotone each subinterval. We allow $c_i = a_i$ or $c_i = b_i$, thus

we will also say that f_i is unimodal if it monotone on all of X_i. In some of the proofs included in this thesis we may assume our bonding maps are strictly monotone on each subinterval, but we will not require strict monotonicity here due to a result of Morton Brown ([5], Theorem 3) stating that if we have functions f_i and g_i so that $f_i(x) - g_i(x) < \epsilon_i$ for all $x \in X_i$ and for all i, with ϵ_i converging to zero quickly enough, then the inverse limit spaces are homeomorphic. This, together with the fact that any monotone function can be approximated by a strictly monotone function, means we lose no generality in considering strictly monotone functions. We will refer to any space as a *unimodal continuum* if it can be realized as, or is homeomorphic to, the inverse limit space of unimodal bonding maps $f_i : X_{i+1} \to X_i$.

In 1954, C.E. Capel ([7]) used inverse limit spaces to prove theoerms concerning Alexander-Kolmogoroff cohomology theory. Inverse limit spaces of one-dimensional bonding maps are often attractors in dynamical systems. R.F. Williams ([16]) proved in 1967 that hyperbolic one-dimensional attractors are inverse limits of maps on branched one manifolds. The same year, W.S. Mahavier ([9]) looked at inverse limits on $[0, 1]$ with a single bonding map. Sam B. Nadler ([13],[14]) and Dorothy S. Marsh ([10]) both created inverse limit spaces that we will later refer to as generalized $\sin\frac{1}{x}$ spaces, but without using unimodal bonding maps.

Research of my advisor, Marcy Barge, along with coauthors Beverly Diamond and Karen Brucks ([1], [3]), as well as the research of others, has shown that often the inverse limit spaces that occur in dynamical systems are unimodal continua. For this reason we are interested in what these unimodal continua "look" like, what properties they have and how we can tell if a continuum is a unimodal continuum or not. Some examples of unimodal continua include the "topologist's sine curve" (Figure 1), the "double topologist's sine curve" with two limit bars, one on each end (Figure 2), as well as the Knaster continuum, also called the bucket handle continuum (Figure 3).
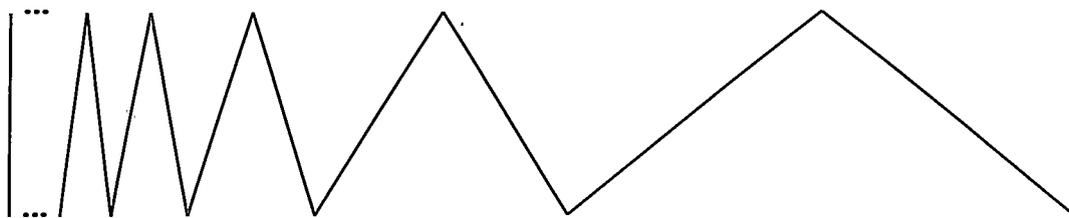
Figure 1: The Topologist's Sine Curve

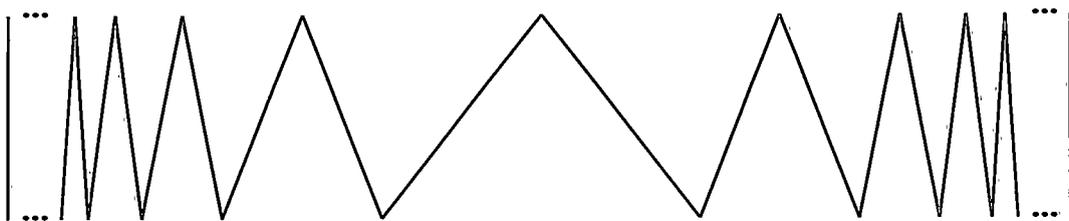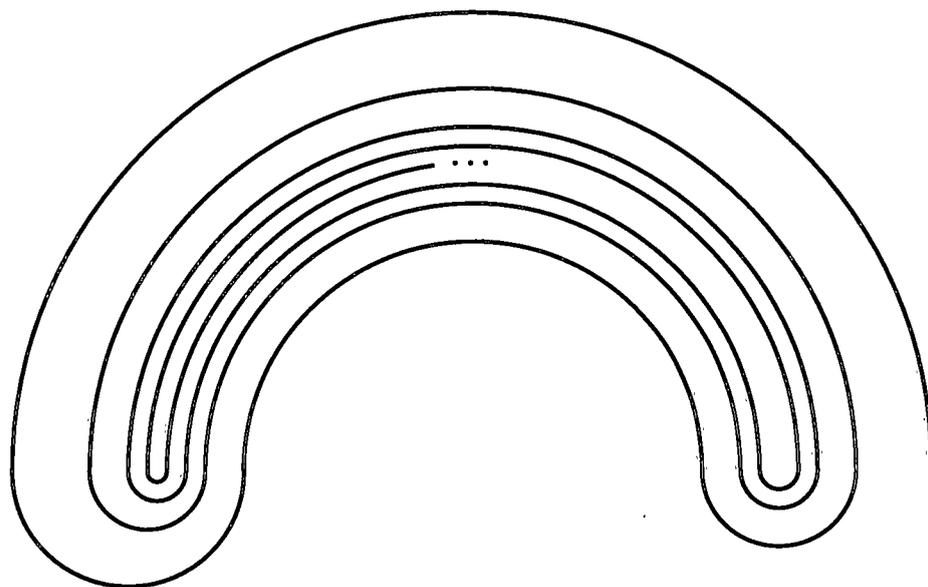Figure 2: The Double Topologist's Sine Curve

Figure 3: The Knaster Continuum

A continuum is said to be *decomposable* if it is the union of two proper sub-continua, otherwise it is an *indecomposable continuum*. We say that a continuum is *hereditarily indecomposable* provided each of its subcontinua is indecomposable and a continuum is *hereditarily decomposable* if each of its subcontinua is decomposable.

The Knaster bucket-handle continuum (Figure 3) is indecomposable (See [4]). One can see that if we pick a point along the ray path component, but not the end point, of the topologist's sine curve and include everything to one side of that point as well as the point itself in one set, while putting everything on the other side of the point including the point itself in another set, the result is two proper subcontinua whose union is the entire topologist's sine curve (Figure 1). We could repeat this process on each of these subcontinua, thus the curve is not only decomposable, but hereditarily decomposable. A similar procedure can be done on the double topologist's sine curve (Figure 2) by selecting a point on the middle (infinitely long) path component and proceeding as above. Hence the double topologist's sine curve is also hereditarily decomposable. Barge and Diamond [2] showed that there are uncountably many (non-homeomorphic) indecomposable unimodal continua. In this thesis we will show that there are uncountably many hereditarily decomposable unimodal continua by constructing a class of uncountably many non-homeomorphic hereditarily decomposable unimodal continua.

CHAPTER 2

SOME PROPERTIES OF INVERSE LIMIT SPACES AND UNIMODAL CONTINUA

Here we state a couple of elementary topology lemmas because they will be referred to in the following pages. We then state and prove some well known basic results concerning inverse limit spaces, followed by our first original theorem of this dissertation.

Lemma 2.1 *Let $h : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then h is a homeomorphism.*

Proof: See ([12] pg. 167) *Q. E. D.*

We recall here that all metric spaces are Hausdorff. Next denote the projection mapping by π_i, thus $\pi_i(\underline{x}) = \pi_i(x_0, x_1, x_2, \dots) = x_i$.

Lemma 2.2 *Let A be any space and let h be any function from A to $\varprojlim\{f_i\}$. Then h is continuous if and only if $\pi_i \circ h : A \to X_i$ is continuous for all $i \in \mathbb{N} \cup \{0\}$.*

Proof: The theorem in [12] (pg. 115) proves that $h : A \to \prod_{i=0}^{\infty} X_i$ is continuous if and only if $\pi_i \circ h : A \to X_i$ is continuous for all $i \in \{0, 1, 2, \dots\}$. Then by ([12] pg. 107) we know that restricting the range of a function does not affect the continuity of the function, thus the lemma holds. *Q. E. D.*

The next therem, along with it's corollaries, and the lemma that follows them are a compilation and contemporary restatement of results in [8], pages 215 through 220. The proofs are included here for completeness.

Define $f_{n,m} : X_n \to X_m$ by $f_{n,m} = f_m \circ f_{m+1} \circ \cdots \circ f_{n-2} \circ f_{n-1}$ for $n \geq m+1$ (and not defined otherwise). Let $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ be any two sequences in $\mathbb{N} \cup \{0\}$ such that $0 \leq n_0 < n_1 < n_2 < \cdots$ and $0 \leq m_0 < m_1 < m_2 < \cdots$. Also assume $f_i : X_{i+1} \to X_i$ and $g_i : Y_{i+1} \to Y_i$ are continuous bonding maps.

Theorem 2.3 *Suppose there exists the following commuting diagram with s_i and t_i continuous maps:*
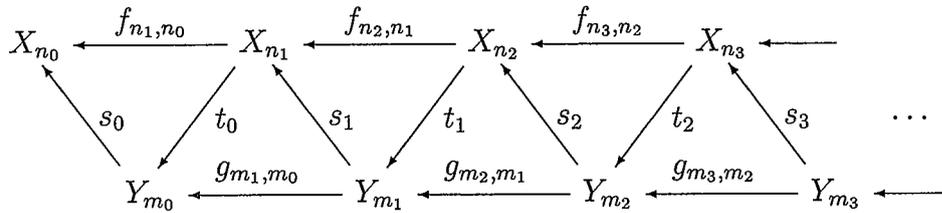
Then $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$.

Proof: We warn the reader to pay close attention to the important but visually subtle difference in the subscripting between x_{n_k-1} and $x_{n_{k-1}}$ throughout this proof.

Let us define $H : \varprojlim\{f_i\} \to \varprojlim\{g_i\}$ by $H(\underline{x}) = \underline{y}$, where $\underline{x} = (x_0, x_1, x_2, \dots)$ and $\underline{y} = (y_0, y_1, y_2, \dots)$ with $y_{m_i} = t_i(x_{n_{i+1}})$. This does not specify all the coordinates of $\underline{y}$, but by specifying infinitely many of the coordinates, the point is uniquely determined, because to find any y_k if $k \neq m_i$ for some i, then pick the smallest i so that $m_i > k$ and simply define $y_k = g_{m_i,k}(y_{m_i})$. Now to show that $\underline{y} = H(\underline{x}) \in \varprojlim\{g_i\}$, if $k \neq m_i$ for any i, again choose the smallest i so that $m_i > k$. If $k = m_i - 1$ then $g_k(y_{k+1}) = g_{m_i-1}(y_{m_i}) = g_{m_i,m_i-1}(y_{m_i}) = g_{m_i,k}(y_{m_i}) = y_k$ from the definition of y_k above, otherwise we have $g_k(y_{k+1}) = g_k(g_{m_i,k+1}(y_{m_i})) = g_k \circ g_{k+1} \circ \cdots \circ g_{m_i-1}(y_{m_i}) = g_{m_i,k}(y_{m_i}) = y_k$ as desired. But for $k = m_i$, we use the fact that our diagram commutes to get $g_k(y_{k+1}) = g_{m_i}(y_{m_i+1}) = g_{m_i} \circ g_{m_i+1} \circ \cdots \circ g_{m_{i+1}-1}(y_{m_{i+1}}) = g_{m_{i+1},m_i} \circ t_{i+1}(x_{n_{i+2}}) = t_i \circ f_{n_{i+1},n_i}(x_{n_{i+2}}) = t_i(x_{n_{i+1}}) = y_{m_i} = y_k$, that is $g_k(y_{k+1}) = y_k$ in all cases, so $\underline{y} \in \varprojlim\{g_i\}$.

Next we notice that $t_i \circ \pi_{n_{i+1}} = \pi_{m_i} \circ H$ and since t_i and $\pi_{n_{i+1}}$ are continuous, $\pi_{m_i} \circ H$ is also. Now if $k \neq m_i$ for some i, $\pi_k \circ H(\underline{x}) = y_k = g_k \circ g_{k+1} \circ \cdots \circ g_{m_i-1}(y_{m_i}) = g_k \circ g_{k+1} \circ \cdots \circ g_{m_i-1} \circ t_i(x_{n_{i+1}})$ which is continuous as the composite of continuous functions. In either case, $\pi_k \circ H$ is continuous for all k, therefore by Lemma 2.2, H is continuous.

Let us now define $G : \varprojlim\{g_i\} \to \varprojlim\{f_i\}$ by letting $G(\underline{y}) = \underline{x}$ where $x_{n_i} = s_i(y_{m_i})$ and $x_k = f_{n_i,k}(x_{n_i})$ for the smallest i so that $n_i > k$. By similar arguments to those given for H, we can see G is continuous also. Consider $G \circ H(\underline{x})$. For any fixed i, if $H(\underline{x}) = \underline{y}$, $y_{m_i} = t_i(x_{n_{i+1}})$. Then our G function takes y_{m_i} to $s_i(y_{m_i}) = x_{n_i}$ so $s_i \circ t_i(x_{n_{i+1}}) = x_{n_i}$. Thus $G \circ H(\underline{x})$ is specified by $x_{n_{i+1}} \mapsto x_{n_i}$. Again, even though not all coordinates have been specified, infinitely many have been, so $G \circ H(\underline{x})$ is completely determined, and since $G \circ H(\underline{x})$ has x_{n_i} as coordinates, which are the coordinates of $\underline{x}$, $G \circ H(\underline{x}) = \underline{x}$, so $G \circ H$ is the identity map. Similarly $H \circ G(\underline{y}) = \underline{y}$, hence G is the inverse of H. Therefore H is a homeomorphism. *Q. E. D.*

Corollary 2.1 *Suppose $\{f_i\}_{i=0}^{\infty}$ and $\{g_i\}_{i=0}^{\infty}$ are sequences of bonding maps so that $f_i : X_{i+1} \to X_i$ and $g_i : Y_{i+1} \to Y_i$ and there exists an $N \in \mathbb{N} \cup \{0\}$ and an $m \in \mathbb{Z}$ where $X_i = Y_{i+m}$ and $f_i = g_{i+m}$ for all $i \geq N$. Then $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$.*

Proof: Let $\{n_i\}_{i=0}^{\infty}$ be any sequence with $N \leq n_0 < n_1 < n_2 < \ldots$ and let $k_i = n_i + m$ for all $i = 0, 1, 2, \ldots$, thus $X_{n_i} = Y_{k_i}$ and $f_{n_i} = g_{k_i}$. Consider the diagram below:
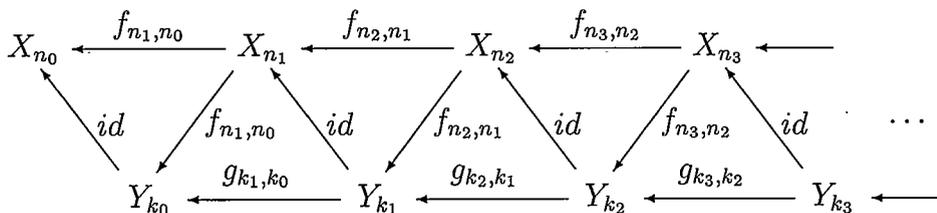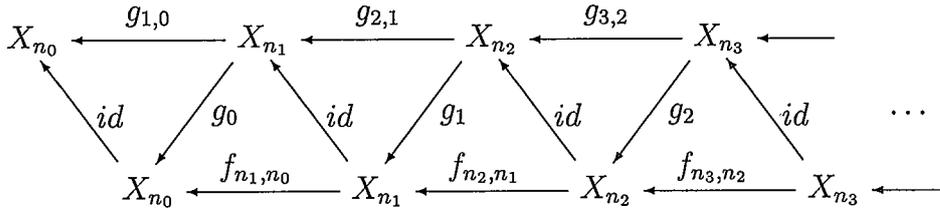
The reader can quickly check that this diagram commutes. So by Theorem 2.3, $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$. *Q. E. D.*

What this tells us is that if two sequences of bonding maps differ only in finitely many positions, they have homeomorphic inverse limit spaces. We can also ignore or remove the first finitely many bonding maps and get a homeomorphic inverse limit space to the original inverse limit space.

Corollary 2.2 *Suppose $\{f_i\}_{i=0}^{\infty}$ is a sequence of bonding maps so that $f_i : X_{i+1} \to X_i$ and $\{n_i\}_{i=0}^{\infty}$ is any sequence of integers with $0 \leq n_0 < n_1 < n_2 < \cdots$. If we let $g_j = f_{n_j} \circ f_{n_j+1} \circ f_{n_j+2} \circ \cdots \circ f_{n_{j+1}-1}$ then $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_j\}$.*

Proof: Consider the following diagram:

$$X_{n_0} \xleftarrow{\;g_{1,0}\;} X_{n_1} \xleftarrow{\;g_{2,1}\;} X_{n_2} \xleftarrow{\;g_{3,2}\;} X_{n_3} \longleftarrow$$

with diagonal maps id, g_0, id, g_1, id, g_2, id, $\cdots$

$$X_{n_0} \xleftarrow{\;f_{n_1,n_0}\;} X_{n_1} \xleftarrow{\;f_{n_2,n_1}\;} X_{n_2} \xleftarrow{\;f_{n_3,n_2}\;} X_{n_3} \longleftarrow$$

Since $g_j = g_{j+1,j}$, again the reader can easily see that the diagram commutes, thus by Theorem 2.3, $\varprojlim\{f_i\} \simeq \varprojlim\{g_j\}$. *Q. E. D.*

Suppose that we are using a single bonding map, that is in our sequence of bonding maps, $f_i = f$ for all i. If we use the notation that $f^3 = f \circ f \circ f$ then a common application of the above corollary is to show that $\varprojlim f^3 \simeq \varprojlim f$.

Lemma 2.4 *Suppose $\{X_i\}_{i=0}^{\infty}$ is a sequence of spaces with $X_i = [a_i, b_i]$ and $\{f_i\}_{i=0}^{\infty}$, with $f_i : X_{i+1} \to X_i$ is a sequence of continuous unimodal bonding maps for $i \in \{0, 1, 2, \dots\}$. Thus $\varprojlim\{f_i\}$ is a unimodal continuum. Further assume that $\varprojlim\{f_i\}$ is*
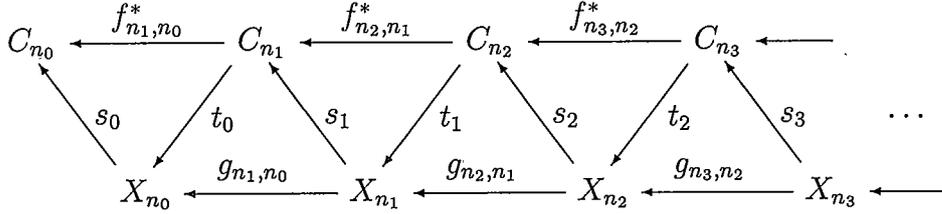
larger than a single point. Then $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$ where $\{g_i\}_{i=0}^{\infty}$ is a sequence of bonding maps such that $g_i : X_{i+1} \to X_i$ are continuous unimodal <u>onto</u> functions.

Proof: Let $C_n = \bigcap_{i \geq 0} f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$. Since the composite of continuous functions is continuous and the continuous image of a connected set is itself connected, $f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$ is connected for all i. Similarly $f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$ is compact as the continuous image of a compact set and thus is closed. So C_n is the nested intersection of closed connected nonempty sets, thus C_n is closed, connected and nonempty ([12] pg. 170-171). Since $C_n \subset X_n$, C_n is a closed bounded interval or a single point. Define $f_n^* : C_{n+1} \to C_n$ by $f_n^* = f_n|_{C_{n+1}}$. By the definition of C_n, f_n^* is onto.

Next we show that $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$. Let $\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{f_n^*\}$, then since $C_n \subset X_n$ and $f_n^* = f_n|_{C_{n+1}}$, it is clear $\underline{x} \in \varprojlim\{f_n\}$. Thus, as sets, $\varprojlim\{f_n^*\} \subset \varprojlim\{f_n\}$. Next suppose $\underline{x} \in \varprojlim\{f_n\}$, then for any n, $x_n = f_n(x_{n+1}) = f_n \circ f_{n+1}(x_{n+2}) = \dots = f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(x_{n+i+1})$ for any $i \geq 0$ and of course $x_{n+i+1} \in X_{n+i+1}$, so $x_n \in \bigcap_{i \geq 0} f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1}) = C_n$ and because $f_{n-1}^* = f_{n-1}|_{C_n}$, $x_{n-1} = f_{n-1}^*(x_n)$ so $\underline{x} \in \varprojlim\{f_n^*\}$. Thus as sets, $\varprojlim\{f_n\} \subset \varprojlim\{f_n^*\}$ and therefore $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ as sets, and since the metrics on $\varprojlim\{f_n\}$ and $\varprojlim\{f_n^*\}$ induce the same topology on this set, $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ as desired.

Because $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ is not just a single point, there exists an $M \in \mathbb{N} \cup \{0\}$ so that C_n is a non-degenerate interval for all $n \geq M$, that is $C_n = [p_n, q_n]$ for $p_n < q_n$ if $n \geq M$. So for $n \geq M$, define the function $\alpha_n : C_n \to X_n$ by $\alpha_n(x) = \frac{b_n - a_n}{q_n - p_n}(x - p_n) + a_n$, hence $\alpha_n(p_n) = a_n$ and $\alpha_n(q_n) = b_n$. Also, α_n is an affine function in $\mathbb{R}$ with positive slope so it is a continuous, strictly increasing, onto function, thus is a homeomorphism. Next let $g_n : X_{n+1} \to X_n$ be defined by $g_n = \alpha_n \circ f_n^* \circ \alpha_{n+1}^{-1}$, for $n \geq M$. So g_n is continuous and onto as the composition of continuous, onto

functions. Let $\{n_i\}_{i=0}^{\infty}$ be any sequence in $\mathbb{N} \cup \{0\}$ with $M \leq n_0 < n_1 < n_2 < \dots$ and let $s_i = \alpha_{n_i}^{-1}$, and $t_i = \alpha_{n_i} \circ f_{n_{i+1},n_i}^*$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
C_{n_0} & \xleftarrow{\ f_{n_1,n_0}^*\ } & C_{n_1} & \xleftarrow{\ f_{n_2,n_1}^*\ } & C_{n_2} & \xleftarrow{\ f_{n_3,n_2}^*\ } & C_{n_3} & \longleftarrow \\
& {\scriptstyle s_0}\ \ {\scriptstyle t_0} & & {\scriptstyle s_1}\ \ {\scriptstyle t_1} & & {\scriptstyle s_2}\ \ {\scriptstyle t_2} & & {\scriptstyle s_3} & \cdots \\
& X_{n_0} & \xleftarrow{\ g_{n_1,n_0}\ } & X_{n_1} & \xleftarrow{\ g_{n_2,n_1}\ } & X_{n_2} & \xleftarrow{\ g_{n_3,n_2}\ } & X_{n_3} & \longleftarrow
\end{array}
$$

So let us quickly check that this diagram commutes. Looking at any triangle of it with $C_{n_{i+1}}$, C_{n_i} and X_{n_i} at the corners, if we go from $C_{n_{i+1}}$ to C_{n_i}, passing through X_{n_i}, we get $s_i \circ t_i = \alpha_{n_i}^{-1} \circ \alpha_{n_i} \circ f_{n_{i+1},n_i}^* = f_{n_{i+1},n_i}^*$, which is exactly what we get going directly from $C_{n_{i+1}}$ to C_{n_i}. Checking a triangle with X_{n_i}, $X_{n_{i-1}}$ and C_{n_i} at the corners, going through C_{n_i} we have $t_{i-1} \circ s_i = \alpha_{n_{i-1}} \circ f_{n_i,n_{i-1}}^* \circ \alpha_{n_i}^{-1} = \alpha_{n_{i-1}} \circ f_{n_{i-1}}^* \circ f_{n_{i-1}+1}^* \circ f_{n_{i-1}+2}^* \circ \cdots f_{n_i-1}^* \circ \alpha_{n_i}^{-1} = \alpha_{n_{i-1}} \circ f_{n_{i-1}}^* \circ \alpha_{n_{i-1}+1}^{-1} \circ \alpha_{n_{i-1}+1} \circ f_{n_{i-1}+1}^* \circ \alpha_{n_{i-1}+2}^{-1} \circ \cdots \circ \alpha_{n_i-1} \circ f_{n_i-1}^* \circ \alpha_{n_i}^{-1} = g_{n_{i-1}} \circ g_{n_{i-1}+1} \circ \cdots \circ g_{n_i-1} = g_{n_i,n_{i-1}}$, which is the function going directly from X_{n_i} to $X_{n_{i-1}}$. Since all the maps on our diagram are continuous, we use Lemma 2.3 to get that $\varprojlim\{g_i\} \simeq \varprojlim\{f_i^*\} = \varprojlim\{f_i\}$.

Notice that to create these g_i functions we restricted the domain and range of f_i to get the f_i^* functions and rescaled them with the affine homeomorphism α_i, which did nothing to change the fact that we have unimodal maps, hence g_i is unimodal for all i, as desired. *Q. E. D.*

This next theorem is one we have recently proven and, although it will not be used in the remainder of this thesis, it illustrates an important property all unimodal continua must have.

Theorem 2.5 *All unimodal continua must have a dense arc component.*

Proof: Suppose we have any unimodal continuum. If this continuum is a single point, then we have a degenerate dense arc component which is that point itself. Otherwise the unimodal continuum is more than a single point and we can assume that there exist a sequence of bonding maps $\{f_i\}_{i=0}^{\infty}$ with $f_i : X_{i+1} \to X_i$ which are continuous unimodal and, by Lemma 2.4, we can also assume they are onto, so that $\varprojlim\{f_i\}$ is homeomorphic to our chosen unimodal continuum. Since f_i is unimodal, there is a critical point of f_i, say $c_{i+1} \in X_{i+1}$. Because f_i is onto, there is a subinterval $X_{i+1}^* \subset X_{i+1}$ with $X_{i+1}^* = [a_{i+1}, c_{i+1}]$ or $X_{i+1}^* = [c_{i+1}, b_{i+1}]$ so that $f_i|_{X_{i+1}^*} : X_{i+1}^* \to X_i$ is one-to-one and onto. Define $C_N = \{\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{f_i\} : x_i \in X_i^*$ for $i \geq N\}$ where $N \in \mathbb{N} \cup \{0\}$. Let $C = \bigcup_{N \geq 0} C_N$. We will show that the set C is path connected and dense in $\varprojlim\{f_i\}$.

First, let us show C is path connected. Let $\underline{x}, \underline{y} \in C$, then if $\underline{x} = (x_0, x_1, x_2, \dots)$ and $\underline{y} = (y_0, y_1, y_2, \dots)$, there exists an $N_1 \in \mathbb{N} \cup \{0\}$ so $x_i \in X_i^*$ for all $i \geq N_1$. Similarly, there is an $N_2 \in \mathbb{N} \cup \{0\}$ such that $y_i \in X_i^*$ for all $i \geq N_2$. Let $N = \max\{N_1, N_2\}$, thus $x_i, y_i \in X_i^*$ for all $i \geq N$ and $\underline{x}, \underline{y} \in C_N$. Now define the function $g_i : X_i \to X_{i+1}^*$ by $g_i = (f_i|_{X_{i+1}^*})^{-1}$. Also, let the functions $p_k : [0, 1] \to I$ for $k = 0, 1, 2, \dots$ be defined by $p_N(t) = t x_N + (1-t) y_N$ and $p_{N+n}(t) = g_{N+n} \circ g_{N+n-1} \circ g_{N+n-2} \circ \cdots \circ g_N(p_N(t))$ for $n = 1, 2, 3, \dots$ and $p_m(t) = f_m \circ f_{m+1} \circ f_{m+2} \circ \cdots \circ f_{N-1}(p_N(t))$ for $m = 0, 1, 2, \dots, N-1$. Next let $P : [0, 1] \to C$ be defined by $P(t) = (p_0(t), p_1(t), p_2(t), \dots)$.

To see that P is a path from $\underline{y}$ to $\underline{x}$, we notice that:

$$P(0) = (p_0(0), p_1(0), p_2(0), \dots, p_N(0), \dots) = (p_0(0), p_1(0), p_2(0), \dots, y_N, \dots)$$

$$= (f_0 \circ f_1 \circ \cdots \circ f_{N-1}(y_N), f_1 \circ f_2 \circ \cdots \circ f_{N-1}(y_N), f_2 \circ f_3 \circ \cdots \circ f_{N-1}(y_N), \dots,$$

$$f_{N-1}(y_N), y_N, g_N(y_N), g_{N+1} \circ g_N(y_N), g_{N+2} \circ g_{N+1} \circ g_N(y_N), \dots)$$

$$= (y_0, y_1, \dots, y_N, (f_N|_{X_{N+1}^*})^{-1}(y_N), (f_{N+1}|_{X_{N+2}^*})^{-1} \circ (f_N|_{X_{N+1}})^{-1}(y_N), \dots)$$

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But since $\underline{y} \in C_N$, $(f_N|_{X_{N+1}}^*)^{-1}(y_N) = y_{N+1}$ and $(f_{N+n}|_{X_{N+n+1}}^*)^{-1}(y_{N+n}) = y_{N+n+1}$ for $n \geq 0$, so $P(0) = (y_0, y_1, \ldots, y_N, y_{N+1}, y_{N+2}, \ldots) = \underline{y}$. Since $p_N(1) = x_N$, by a similar argument, $P(1) = \underline{x}$.

It is clear from the construction of $P(t)$ that $P(t) \in C_N$ for all N and thus $P(t) \in C$ for all t. All that remains in the proof that P is a path is to show P is continuous. Since $P(t) = (p_0(t), p_1(t), p_2(t), \ldots)$, $\pi_i \circ P(t) = p_i(t)$ which is continuous. Hence by Lemma 2.2, P is continuous and thus P is a path from $\underline{y}$ to $\underline{x}$.

Next we will show C is dense in $\varprojlim\{f_n\}$. Suppose that $\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{f_n\}$ and let $\epsilon > 0$ be given. Since $\sum_{i=0}^{\infty} \frac{1}{2^i}$ converges, there is an $M \in \mathbb{N} \cup \{0\}$ so that $\sum_{i=M}^{\infty} \frac{1}{2^i} < \epsilon$. Now let $\underline{y} = (y_0, y_1, y_2, \ldots)$ be defined by $y_i = x_i$ for $i \in \{0, 1, 2, \ldots, M-1\}$ and let $y_i = g_i \circ g_{i-1} \circ g_{i-2} \circ \cdots \circ g_{M+1} \circ g_M(x_{M-1})$ for $i \in \{M, M+1, M+2, \ldots\}$. Then $d(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} = \sum_{i=0}^{M-1} \frac{|x_i - y_i|}{2^i} + \sum_{i=M}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=0}^{M-1} \frac{|x_i - x_i|}{2^i} + \sum_{i=M}^{\infty} \frac{1}{2^i} < \sum_{i=0}^{M-1} \frac{0}{2^i} + \sum_{i=M}^{\infty} \frac{1}{2^i} = \sum_{i=M}^{\infty} \frac{1}{2^i} < \epsilon$. We notice that from the construction of $\underline{y}$, $\underline{y} \in C_M$ and hence in C. Thus the closure of the points of C is all of $\varprojlim\{f_n\}$, and therefore C is dense. $Q.\ E.\ D.$

Below is an example of a continuum that appears to be similar to Figures 1 and 2, but from Theorem 2.5 we know it is not a unimodal continua.
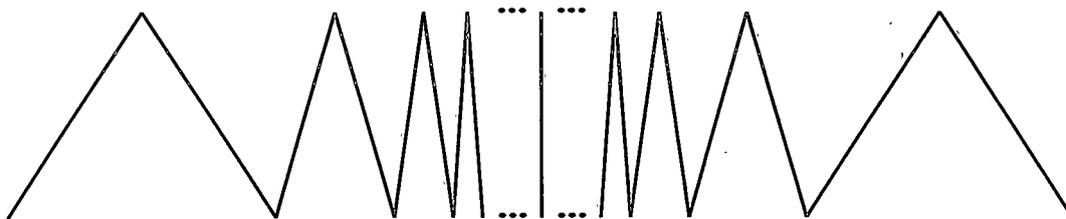

Figure 4: An Example of a Non-unimodal Continua

CHAPTER 3

GENERALIZED $\sin\frac{1}{x}$ SPACES

In this chapter we start with bonding maps (that are not unimodal) with certain general properties and show that their inverse limit spaces each are homeomorphic to a space that has many similarities to the "topologist's sine curve" space. Below are a couple of examples of what we mean by similar to the "topologist's sine curve" space.
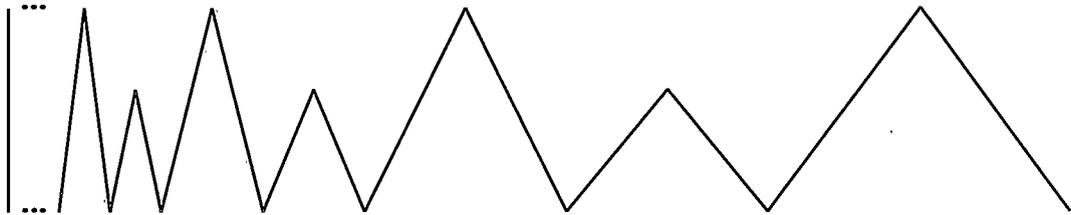
Figure 5: An M Continuum
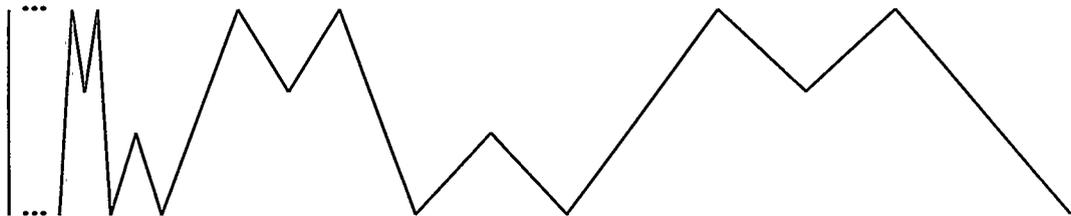
Figure 6: An MW Continuum

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a sequence of continuous functions so that each $\varphi_n : I \to I$, with $I = [0, 1]$, has the following properties:

(a) $\varphi_n(0) = 0$,

(b) $\varphi_n([0, \frac{1}{2})) = [0, 1]$,

(c) $\varphi_n(x) = x$, for all $x \in [\frac{1}{2}, 1]$,

(d) there exists a real number η, independent of φ_n, and point a_n for each φ_n so that

  (i) $0 < \eta \le a_n < \frac{1}{2}$,

  (ii) φ_n is strictly increasing on $[0, a_n]$,

  (iii) $\varphi_n(a_n) = \frac{1}{2}$

  (iv) and $\varphi_n(x) \ge \frac{1}{2}$ for all $x \in [a_n, 1]$.

We consider the inverse limit space of $\{\varphi_n\}_{n=0}^{\infty}$, denoted by $\varprojlim\{\varphi_n\}$, and will use the notation $\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{\varphi_n\}$ if $\varphi_n(x_{n+1}) = x_n$. Define $c_n = \min\{x \in [0, \frac{1}{2}] : \varphi_n(x) = 1\}$. Since $\varphi_n(\frac{1}{2}) = \frac{1}{2}$ and φ_n is strictly increasing on $[0, a_n]$ with $\varphi_n(a_n) = \frac{1}{2}$, $a_n < c_n < \frac{1}{2}$. Notice that by our selection of a_n and c_n, neither $(a_0, a_1, a_2, \dots)$ nor $(c_0, c_1, c_2, \dots)$ is in $\varprojlim\{\varphi_n\}$. The space $\varprojlim\{\varphi_n\}$ is homeomorphic to a space similar to a "topologist's sine curve" space, so we will soon label it as a generalized $\sin\frac{1}{x}$ space. But first we state a well known theorem (without proof) and a couple of lemmas that we will need later.

Consider a function $\psi : [a, b] \to [a, b]$, $a < b$ and $a, b \in \mathbb{R}$. Divide $[a, b]$ into k distinct subintervals $[a, p_1] = I_1, [p_1, p_2] = I_2, [p_2, p_3] = I_3, \dots, [p_{k-1}, b] = I_k$, that intersect at no more than their end points. Let us use a $k \times k$ transition matrix M to encode information about ψ. In matrix M, let the ij^{th} entry, call it m_{ij}, be either 0 or 1, if $I_i \subset \psi(I_j)$ then let $m_{ij} = 1$ and let $m_{ij} = 0$ otherwise.

Theorem 3.1 *For all $n \in \mathbb{N}$, let $\psi_n : [a, b] \to [a, b]$, then let M_n be a $k \times k$ transition matrix of zeros and ones constructed as above from ψ_n. If the ij^{th} entry of the matrix product $M_0 M_1 M_2 \cdots M_n$ is λ, there are λ subintervals, $J_1, J_2, J_3, \dots, J_\lambda$, of I_i so that $I_j \subset \psi_0 \circ \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(J_m)$ for $m \in \{1, 2, 3, \dots, \lambda\}$ and $J_i \cap J_j$ is at most one point if $i \ne j$.*

The above theorem is actually a compilation of two lemmas in [15]. For a proof of the above theorem, one is referred to [15], Lemma 1.2, pg. 64 and Lemma 2.2 pg. 74 and related definitions on the pages in between these lemmas.

In the next lemma we assume the following: X is some metric space and, for all $n \in \mathbb{N} \cup \{0\}$, R_n is a subspace of X so that R_n is homeomorphic to a compact interval in $\mathbb{R}$ with positive length. Further assume that R_n is a proper subset of R_{n+1} and p is a common endpoint of each R_n. Let $R = \bigcup_{n \geq 0} R_n$. Also let $\{n_i\}_{i=1}^{\infty}$ be a sequence of integers such that $0 < n_1 < n_2 < n_3 < \ldots$, and define $\{x_i\}_{i=1}^{\infty}$ to be any sequence with $x_i \in R_{n_i} \setminus R_{n_i-1}$ for all $i \in \mathbb{N}$. We will also assume that R has the property that the set of limit points from any such sequence $\{x_i\}_{i=1}^{\infty}$ is disjoint from R.

Lemma 3.2 *If R and R_n are as given above, R is homeomorphic to $\mathbb{R}_+ \cup \{0\}$.*

Proof: Recall that p is a common endpoint of all our R_n. We know that each R_n is homeomorphic to a compact interval in $\mathbb{R}$ with positive length, so let $h_0 : [0,1] \to R_0$ be a homeomorphism so that $h_0(0) = p$. Next let $h_1 : [1,2] \to \overline{R_1 \setminus R_0}$ be a homeomorphism with $h_1(1) = h_0(1)$ and let $h_2 : [2,3] \to \overline{R_2 \setminus R_1}$ be a homeomorphism with $h_2(2) = h_1(2)$ and so on, thus in general we have $h_n : [n, n+1] \to \overline{R_n \setminus R_{n-1}}$ is a homeomorphism with $h_n(n) = h_{n-1}(n)$. Let us show that $R = R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$, by showing mutual containment. First assume $x \in R = \bigcup_{n \geq 0} R_n$ so $x \in R_n$ for some n. Since the R_n's are nested with $R_{n-1} \subset R_n$, there exists an $N \in \mathbb{N} \cup \{0\}$ so that $x \in R_N$ but $x \notin R_{N-1}$. If $N = 0$ then $x \in R_0$, otherwise $N \geq 1$ and $x \in R_N \setminus R_{N-1}$ so $x \in \overline{R_N \setminus R_{N-1}}$, thus either way $x \in R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$ and $R \subset R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$. Next assume $x \in R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{N-1}})$. If $x \in R_0$ then $x \in R = \bigcup_{n \geq 0} R_n$. Otherwise $x \in \overline{R_n \setminus R_{n-1}}$ for some $n \geq 1$. Recall that from the definition of set closure, $\overline{R_n \setminus R_{n-1}}$ is the

intersection of all the closed sets containing $R_n \setminus R_{n-1}$, and since R_n is a closed set containing $R_n \setminus R_{n-1}$, $x \in R_n$ so $x \in R = \bigcup_{n \geq 0} R_n$. Hence $R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}}) \subset R$ and therefore $R = R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$.

Now let $h : \mathbb{R}_+ \cup \{0\} \to R$ be defined by $h(x) = h_n(x)$ where $x \in [n, n+1]$. By the "pasting lemma" ([12], pg. 108) h is a continuous function. We also note that h is a bijection by construction and since each h_n is bijective. To prove h is a homeomorphism, we still need to show $h^{-1} : R \to \mathbb{R}_+ \cup \{0\}$ is continuous. Consider any sequence $\{y_i\}_{i=1}^{\infty} \in R$ that converges to $y \in R$. If we can show $\{h^{-1}(y_i)\}_{i=1}^{\infty}$ converges to $h^{-1}(y)$ in $\mathbb{R}_+ \cup \{0\}$ then h^{-1} is continuous ([12] pg. 128). There must exist an $N \in \mathbb{N} \cup \{0\}$ so that y and y_i are in R_N for all i, for if this was not the case, then there exists a subsequence $\{y_{i_j}\}_{j=1}^{\infty}$ and natural numbers $n_1 < n_2 < n_3 < \dots$ so that $y_{i_j} \in R_{n_j} \setminus R_{n_j - 1}$. But $\{y_{i_j}\}_{j=1}^{\infty}$ converges to $y \in R$ which contradicts the hypothesis that such sequences have there limit points outside R. So there exists an N such that y and y_i are elements of R_N for all i. Let us look at $h|_{[0,N]} : [0, N] \to \bigcup_{n=0}^{N-1} R_n$. Because $R_n \simeq [n, n+1]$ for all n and the countable union of compact sets is compact, $h|_{[0,N]}$ is a function between compact metric spaces that is continuous and bijective since h is and we selected our range space to make it so. Thus by Lemma 2.1, $h|_{[0,N]}$ is a homeomorphism and so $\{(h|_{[0,N]})^{-1}(y_i)\}_{i=1}^{\infty} = \{h^{-1}(y_i)\}_{i=1}^{\infty}$ must converge to $(h|_{[0,N]})^{-1}(y) = h^{-1}(y)$, so h^{-1} is continuous as desired.

Therefore h is a homeomorphism from $\mathbb{R}_+ \cup \{0\}$ onto R. Q. E. D.

We define a topological space X to be a *generalized* $\sin \frac{1}{x}$ *space* if:

(a) X is compact and connected,

(b) X has two path components, one homeomorphic to a compact interval in $\mathbb{R}$, and the other homeomorphic to $\mathbb{R}_+ \cup \{0\}$, and

(c) the non-compact path component is dense in X.

Lemma 3.3 *The space $\varprojlim\{\varphi_n\}$ is a generalized $\sin\frac{1}{x}$ space.*

Proof: As mentioned in the first chapter, we know that $\varprojlim\{\varphi_n\}$ is compact and connected. Define a subset of $\varprojlim\{\varphi_n\}$ by $A = \{\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{\varphi_n\} : x_n \in [\frac{1}{2}, 1] \text{ for all } n\}$. Since $\varphi_n|_{[\frac{1}{2}, 1]}(x) = x$, elements of A have the form $(x_0, x_0, x_0, \dots)$. Let $h : A \to [\frac{1}{2}, 1]$ be equal to $\pi_0|_A$, thus h is continuous. Next let $g : [\frac{1}{2}, 1] \to A$ be defined by $g(x) = (x, x, x, \dots)$. If $x \in [\frac{1}{2}, 1]$, $\pi_n \circ g(x) = x$ for all n, so $\pi_n \circ g$ is the identity map and thus continuous. Hence by Lemma 2.2, g is continuous. Also, if $\underline{x} = (x_0, x_1, x_2, \dots) = (x_0, x_0, x_0, \dots) \in A$, $g \circ h(\underline{x}) = g(x_0) = (x_0, x_0, x_0, \dots) = \underline{x}$ and if $x \in [\frac{1}{2}, 1]$, $h \circ g(x) = h(x, x, x, \dots) = x$. Therefore $g = h^{-1}$ and h is a homeomorphism from A to the interval $[\frac{1}{2}, 1] \in \mathbb{R}$.

Next let $B = (\varprojlim\{\varphi_n\}) \setminus A$. Since $\varprojlim\{\varphi_n\}$ is compact and connected, B is not compact, otherwise A and B form a separation of $\varprojlim\{\varphi_n\}$.

We now show $\varprojlim\{\varphi_n\}$ is not path connected. Assume, by way of contradiction, that $\gamma : [0, 1] \to \varprojlim\{\varphi_n\}$ is a path from $\underline{0} = (0, 0, 0, \dots)$ to $\underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, thus we can write $\gamma(t) = (x_0(t), x_1(t), x_2(t), \dots)$ where $x_n : [0, 1] \to I$ is a continuous function such that $x_n(0) = 0$, $x_n(1) = \frac{1}{2}$ and, since γ is a path in $\varprojlim\{\varphi_n\}$, $\varphi_n(x_{n+1}(t)) = x_n(t)$, for all n and t. Thus $x_0(t) = \varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1} \circ x_n(t)$, for all $t \in [0, 1]$ and all $n \in \mathbb{N}$.

Let M_n be a transition matrix for φ_n. First let $I_0 = [0, \frac{1}{2}]$ and $I_1 = [\frac{1}{2}, 1]$. Then, from our hypotheses about φ_n, we have $\varphi_n(I_0) = I_0 \cup I_1$ and $\varphi_n(I_1) = I_1$. Notice we have the same transition matrix for all φ_n so let $M_n = M$ where:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We note that it is easy to show (by induction) that

$$M_0 M_1 M_2 \cdots M_{n-1} = M^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

So by Theorem 3.1, for any $N^* \in \mathbb{N}$ there exists subintervals $J_1, J_2, J_3, \ldots, J_{N^*}$ of I_1 so that $x_0(J_m) = \varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{N^*-1}(x_{N^*}(J_m)) \supset I_2$ for $m = 1, 2, 3, \ldots, N^*$. Hence there exists $t_n, s_n \in I_1$ such $x_0(t_n) = \frac{1}{2}$ and $x_0(s_n) = 1$, also we can order these either $t_1 < s_1 < t_2 < s_2 < \cdots < s_{N-1} < t_N$ or $s_1 < t_1 < s_2 < t_2 < \cdots < t_{N-1} < s_N$ if N^* is even and $N = \frac{N^*}{2} + 1$ or else, if N^* is odd and $N = \frac{N^*+1}{2}$, $t_1 < s_1 < t_2 < s_2 < \cdots < t_N < s_N$ or $s_1 < t_1 < s_2 < t_2 < \cdots < t_N < s_N$. The arguments are identical regardless, so assume we have the case where $t_1 < s_1 < t_2 < s_2 < \cdots < s_{N-1} < t_N$. Since N^* was arbitrary, we can choose N as large as we need.

Because x_0 is continuous on $[0,1]$, x_0 is uniformly continuous and there is a $\delta > 0$ so that if $|t - s| < \delta$, then $|x_0(t) - x_0(s)| < \frac{1}{4}$. We choose N large enough so that $\frac{1}{2N} < \delta$. Then there must be a t_n and s_m, where m is $n-1$ or n, so that $|s_m - t_n| < \delta$ but $|x_0(s_m) - x_0(t_n)| = |1 - \frac{1}{2}| = \frac{1}{2} > \frac{1}{4}$. This a contradiction, therefore $\varprojlim\{\varphi_n\}$ is not path connected.

It is clear that A, homeomorphic to $[\frac{1}{2}, 1]$, is contained in a path component, so we wish next to show that the only other path component is B, and that it is homeomorphic to a ray. Recall we defined $c_n = \min\{x \in [0, \frac{1}{2}] : \varphi_n(x) = 1\}$. Let $B_0 = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_0 \in [0, c_0]\}$ and $B_1 = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_1 \in [0, c_1]\}$ and continue like this to get the general form: $B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, c_k]\}$ then $\bigcup_{k \geq 0} B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, c_k] \text{ for some } k\}$, but $[0, c_k] \subset [0, \frac{1}{2})$ and if $x_k \in [0, \frac{1}{2})$ then $x_{k+1} \in [0, a_k) \subset [0, c_k]$, so $\bigcup_{k \geq 0} B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, \frac{1}{2}) \text{ for some } k\} = (\varprojlim\{\varphi_n\}) \setminus A = B$. Notice that since $[0, c_n] \subset [0, \frac{1}{2})$, φ_n is strictly increasing on $[0, a_n]$, $\varphi_n(a_n) = \frac{1}{2}$ and $\varphi_n(x) \geq \frac{1}{2}$ for $x \in [a_n, 1]$, it is the case that if $x_n \in [0, c_n]$ then x_n has only one preimage, that is $x_{n+1} = [\varphi_n]^{-1}(x_n) \in [0, a_{n+1}) \subset [0, c_{n+1}]$. So in general we know that if $x_n \in [0, c_n]$ then $x_{n+1} \in [0, c_{n+1}]$. Thus $B_n \subset B_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. In addition we notice $\underline{0} = (0, 0, 0, \ldots)$ is an endpoint of B_n for all n.

Next we show B_n is homeomorphic to a compact interval in $\mathbb{R}$. Let $h_n : B_n \to [0, c_n]$ be defined as $h_n = \pi_n|_{B_n}$, so h_n is continuous. From the hypotheses about φ_n on $[0, a_n]$, $\varphi_n|_{[0,a_n]}$ is a homeomorphism, so let $\gamma_n = (\varphi_n|_{[0,a_n]})^{-1}$, thus if $x_n \in [0, \frac{1}{2})$, x_n has only one preimage under φ_n and so $x_{n+1} = \gamma_n(x_n)$. Define $g_n : [0, c_n] \to B_n$ by $g_n(x) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{n-1}(x), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1}(x), \ldots, \varphi_{n-1}(x), x, \gamma_n(x), \gamma_{n+1} \circ \gamma_n(x), \gamma_{n+2} \circ \gamma_{n+1} \circ \gamma_n(x), \ldots)$. Since φ_k and γ_k are continuous for all k, $\pi_k \circ g_n$ is continuous for all k, so by Lemma 2.2, g_n is continuous. We quickly see that $h_n \circ g_n(x) = x$ and $g_n \circ h_n(\underline{x}) = \underline{x}$, so $h_n^{-1} = g_n$ and h_n is a homeomorphism as desired.

Notice that if $\underline{y} = (y_0, y_1, y_2, \ldots) \in B_n \backslash B_{n-1}$ then because $\underline{y} \notin B_{n-1}$, $y_i \in [\frac{1}{2}, 1]$ for all $i < n$ and since $\varphi_i|_{[\frac{1}{2},1]}(x) = x$, $y_0 = y_1 = y_2 = \cdots = y_{n-1}$, but $\underline{y} \in B_n$ so $y_i \in [0, \frac{1}{2})$ for all $i \geq n$. Hence $\underline{y} = (y_0, y_0, y_0, \ldots, y_0, y_n, y_{n+1}, \ldots)$. Now suppose $i = 1, 2, 3, \ldots$ and $\{n_i\}_{i=1}^{\infty}$ is a sequence of integers such that $0 < n_1 < n_2 < n_3 < \ldots$ and $\{\underline{x}^i\}_{i=1}^{\infty}$ is a sequence of elements of B so that $\underline{x}^i \in B_{n_i} \backslash B_{n_i-1}$ and let $\underline{x}$ be a limit point of $\{\underline{x}^i\}_{i=1}^{\infty}$. Assume, by way of contradiction, that $\underline{x} \in B = \bigcup_{n \geq 0} B_n$, so there exists some $N \in \mathbb{N} \cup \{0\}$ so that $\underline{x} \in B_N \backslash B_{N-1}$ (where B_{0-1} is taken to be $\{\}$), thus $\underline{x} = (x_0, x_0, x_0, \ldots, x_0, x_N, x_{N+1}, \ldots)$. There also must be an $i^* \in \mathbb{N}$ so that if $i > i^*$, $n_i > N$. Therefore if $i > i^*$, then $\underline{x}^i \in B_{n_i} \backslash B_{n_i-1}$ and $\underline{x}^i = (x_0^i, x_1^i, x_2^i, \ldots, x_N^i, \ldots, x_{n_i-1}^i, x_{n_i}^i, \ldots) = (x_0^i, x_0^i, x_0^i, \ldots, x_0^i, x_{n_i}^i, \ldots)$ where $x_0^i \in [\frac{1}{2}, 1]$. Since $x_N \in [0, \frac{1}{2})$, $|x_N - \frac{1}{2}| > 0$, let $\epsilon = \frac{|x_N - \frac{1}{2}|}{2^{N+1}}$. Hence if $i > i^*$ then $n_i > N$ and $d(\underline{x}, \underline{x}^i) = \sum_{j=0}^{\infty} \frac{|x_j - x_j^i|}{2^j} \geq \frac{|x_N - x_N^i|}{2^N} \geq \frac{|x_N - \frac{1}{2}|}{2^N} = 2\frac{|x_N - \frac{1}{2}|}{2^{N+1}} = 2\epsilon > \epsilon$. Therefore we have found an $\epsilon > 0$ such that for all $i > i^*$, $\underline{x}^i$ is not in the ϵ-ball about $\underline{x}$, so at most finitely many elements of $\{\underline{x}^i\}_{i=1}^{\infty}$ are in the ϵ-ball, which contradicts the assumption that $\underline{x}$ is a limit point of $\{\underline{x}\}_{i=1}^{\infty}$. Thus all the limits points of the sequence are outside of B. Therefore B and B_n satisfy all the hypotheses of Lemma 3.2, so B is homeomorphic to a ray.

Then since $B = (\varprojlim\{\varphi_n\}) \setminus A$ and B is path connected, not only is A in a path component, A itself is the only other path component, besides B.

Next we show that B is dense in $\varprojlim\{\varphi_n\}$. Recall that every point of A is of the form $\underline{z} = (z, z, z, \ldots)$ where $z \in [\frac{1}{2}, 1]$. Since $\varphi_n([0, \frac{1}{2}]) = [0, 1]$, if $z \in (\frac{1}{2}, 1]$, there exists an $x_0 \in [0, \frac{1}{2})$ such that $\varphi_n(x_0) = z$ and if $z = \frac{1}{2}$, then let $x_0 = a_n$, thus $\varphi_n(x_0) = z$ and $x_0 \notin [\frac{1}{2}, 1]$. Now consider the sequence $\{\underline{z}^i\}_{i=0}^{\infty}$ defined by $\underline{z}^i = (z, z, \ldots, z, z, x_0, x_1, x_2, \ldots)$ where x_0 is in the $(i+1)^{st}$ position and as before $\varphi_i(x_{i+1}) = x_i$, hence $\underline{z}^i \in \varprojlim\{\varphi_n\}$. Because $x_0 \notin [\frac{1}{2}, 1]$, $\underline{z}^i \in B$. Now to show $\{\underline{z}^i\}_{i=0}^{\infty}$ converges to $\underline{z}$, let $\epsilon > 0$ be given. Therefore there is an $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \epsilon$, so for $i > N$, $d(\underline{z}, \underline{z}^i) = \sum_{i=0}^{N-1} \frac{|z-z|}{2^i} + \sum_{i=N}^{\infty} \frac{|z - x_{i-N}|}{2^i} \le 0 + \sum_{i=N}^{\infty} \frac{1}{2^i} < 0 + \epsilon = \epsilon$. Hence $\{\underline{z}^i\}_{i=0}^{\infty}$ converges to $\underline{z}$, as desired.

Whence $\varprojlim\{\varphi_n\}$ is a generalized $\sin\frac{1}{x}$ space. *Q. E. D.*

For the next lemma we make another restriction on the sequence of φ_n functions. In addition to the properties (a)-(d) at the beginning of this chapter, assume

(e) φ_n is monotone increasing on $[0, c_n]$.

Thus we can see $\varphi_n|_{[0,c_n]} = [0, 1]$, with $\varphi_n(c_n) = 1$ and $\varphi_n|_{[0,c_n]}$ is a homeomorphism. Define $(\varphi_n|_{[0,c_n]})^{-1} = \gamma_n : [0, 1] \to [0, c_n]$ and let $\Upsilon_{\varphi_n,k} : [0, 1] \to \varprojlim\{\varphi_n\}$ be defined by $\Upsilon_{\varphi_n,k}(t) = (\varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(t), \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \cdots \circ \varphi_k(t), \ldots, \varphi_k(t), t, \gamma_{k+2}(t), \gamma_{k+2} \circ \gamma_{k+3}(t), \ldots)$. Notice that t is in the $(k+1)^{st}$ position.

Lemma 3.4 *For any fixed* $k \in \mathbb{N} \cup \{0\}$, $\Upsilon_{\varphi_n,k}$ *is an embedding.*

Proof: This amount to showing that $\Upsilon_{\varphi_n,k} : [0, 1] \to \text{Range}(\Upsilon_{\varphi_n,k})$ is a homeomorphism. First let us show $\Upsilon_{\varphi_n,k}$ is one-to one. Let $x, y \in [0, 1]$ so that $\Upsilon_{\varphi_n,k}(x) =$

$\Upsilon_{\varphi_n,k}(y)$, that is $\Upsilon_{\varphi_n,k}(x) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k(x), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(x), \ldots, \varphi_k(x), x,$

$\gamma_{k+2}(x), \ldots) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k(y), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(y), \ldots, \varphi_k(y), y, \gamma_{k+2}(y), \ldots) =$

$\Upsilon_{\varphi_n,k}(y)$. Therefore $x = y$ since they both occupy the $(k+1)^{st}$ position.

Now because φ_n and γ_n are continuous for all n and the compositions of continuous functions are continuous, $\pi_k \circ \Upsilon_{\varphi_n,k}$ is continuous so by Lemma 2.2, $\Upsilon_{\varphi_n,k}$ is continuous. Thus $\Upsilon_{\varphi_n,k}$ is a bijective continuous function from compact space $[0,1]$ onto its range, which is a subset of the metric space $\varprojlim\{\varphi_n\}$, so by Lemma 2.1, $\Upsilon_{\varphi_n,k}$ is a homeomorphism onto its range and is thus $\Upsilon_{\varphi_n,k} : [0,1] \to \varprojlim\{\varphi_n\}$ is an embedding. $Q.\ E.\ D.$

CHAPTER 4

AN EXAMPLE: TWO DIFFERENT UNIMODAL GENERALIZED $\sin\frac{1}{x}$ CONTINUA

This chapter contains two examples of unimodal continua that are also generalized $\sin\frac{1}{x}$ spaces, referred to from here on as unimodal generalized $\sin\frac{1}{x}$ continua. We will see that these two examples of continua are not homeomorphic to each other.

Let us first look at some motivational material. Define $f : I = [0,1] \to I$ to be the continuous, piecewise linear, unimodal function:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

We next consider $f \circ f = f^2$ which is:

$$f^2(x) = \begin{cases} 4x & \text{if } 0 \le x \le \frac{1}{4} \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} \le x \le \frac{1}{2} \\ x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

A quick look at the sketch of the graph of f^2 below (Figure 8) indicates that f^2 satisfies the assumptions (a)-(e) of our φ_n functions in Chapter 2, so by Lemma 3.3, $\varprojlim f^2$ is a generalized $\sin\frac{1}{x}$ space. The same thing goes for f^4 sketched below (Figure 9). In fact, $\varprojlim f \simeq \varprojlim f^2 \simeq \varprojlim f^4$ by Corollary 2.2. But f is a unimodal function, so $\varprojlim f^2$ is a unimodal generalized $\sin\frac{1}{x}$ continuum. It turns out that $\varprojlim f^2$ is homeomorphic to the space illustrated in Figure 10 ([11]).

It occurred to us that perhaps a function similar to f^4, but with one of the two "spikes" shortened (see Figure 11), repeated as the only bonding map would yield a different inverse limit space than that of $\varprojlim f^2$. The bonding map pictured in Figure 11 has an inverse limit space homeomorphic to the space illustrated in Figure 12 (This will be clear from a later proof).
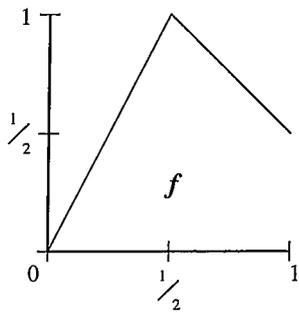
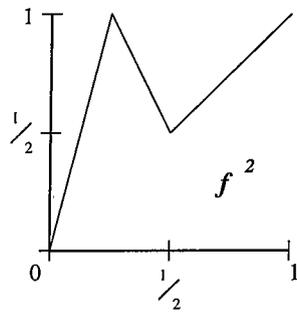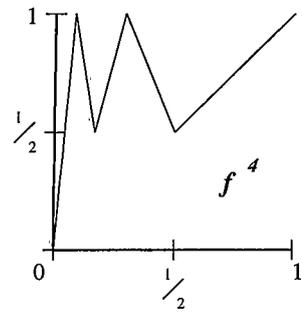Figure 7: Function f Figure 8: Function f^2 Figure 9: Function f^4
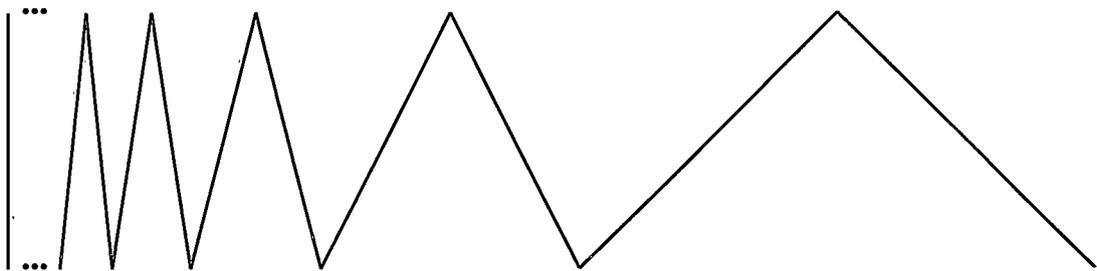
Figure 10: A Visualization of $\varprojlim f$
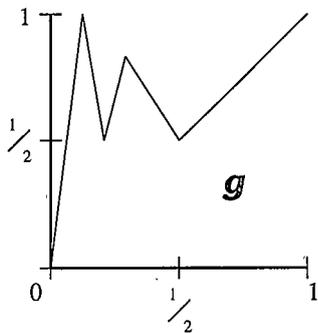
Figure 11: Function g Figure 12: A Visualization of $\varprojlim g$

But because we are interested in unimodal continua, we wish to find a few unimodal functions from I to I so that the composition of these unimodal functions will have a graph whose sketch looks like Figure 11. After many hours composing piecewise linear unimodal functions, we have succeeded.

Let $u, v,$ and z be functions from I onto I defined by:

$$u(x) = \begin{cases} \frac{4}{3}x & \text{if } 0 \le x \le \frac{3}{4} \\ -x + \frac{7}{4} & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$
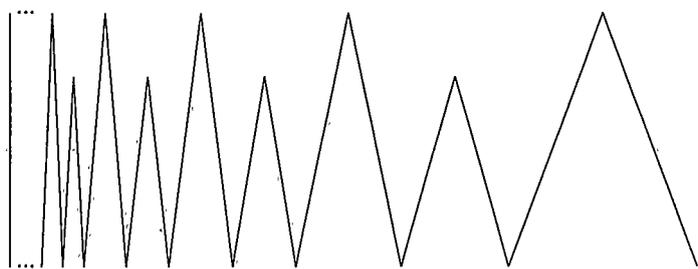
$$v(x) = \begin{cases} \frac{8}{3}x & \text{if } 0 \le x \le \frac{3}{8} \\ -x + \frac{11}{8} & \text{if } \frac{3}{8} \le x \le 1 \end{cases}$$

$$z(x) = \begin{cases} \frac{5}{4}x & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{3}{4}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

We pause for a look at the sketches of the graphs of these functions.
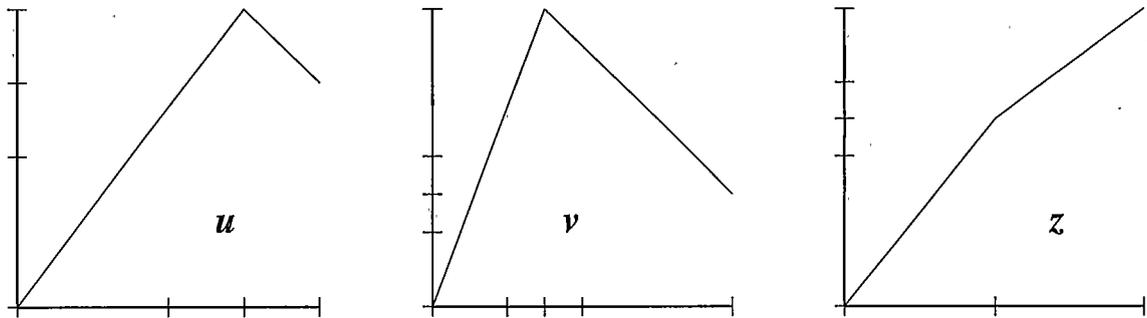
Figure 13: Functions u, v and z

Now let:

$$g(x) = f \circ u \circ v \circ z(x) = \begin{cases} \frac{80}{9}x & \text{if } 0 \le x \le \frac{9}{80} \\ -\frac{40}{9}x + \frac{3}{2} & \text{if } \frac{9}{80} \le x \le \frac{9}{40} \\ \frac{10}{3}x - \frac{1}{4} & \text{if } \frac{9}{40} \le x \le \frac{3}{10} \\ -\frac{5}{4}x + \frac{9}{8} & \text{if } \frac{3}{10} \le x \le \frac{1}{2} \\ x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Then Figure 11 is the sketch of the graph of g. If we let $\tau_{4n} = f, \tau_{4n+1} = u, \tau_{4n+2} = v$ and $\tau_{4n+3} = z$ for $n = 0, 1, 2, \ldots$, then $\varprojlim\{\tau_i\}$ is a unimodal continuum, but

$\varprojlim\{\tau_i\} \simeq \varprojlim g$ by Corollary 2.2. Since g also satisfies the assumptions (a)-(e) in Chapter 3, $\varprojlim g$ is a unimodal generalized $\sin\frac{1}{x}$ continuum. We will show that $\varprojlim f^2$ is not homeomorphic to $\varprojlim g$.

For ease of notation, let $X = \varprojlim g$ and $Y = \varprojlim f^2$. Let A_X be the "limit bar" of X which is formally defined as $A_X = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in X : x_i \in [\frac{1}{2}, 1]$ for $i = 0, 1, 2, \ldots\}$. We define A_Y to be the "limit bar" of Y, so $A_Y = \{\underline{y} = (y_0, y_1, y_2, \ldots) \in Y : y_i \in [\frac{1}{2}, 1]$ for $i = 0, 1, 2, \ldots\}$. As mentioned in the previous chapter, points in A_X and A_Y have the form $(x_0, x_0, x_0, \ldots)$. Let $B_X = X \setminus A_X$ and $B_Y = Y \setminus A_Y$, so that A_X and A_Y are each compact while B_X and B_Y are non-compact rays.

Recall that we defined $\Upsilon_{\varphi_n,k} : [0,1] \to \varprojlim\{\varphi_n\}$ in the last chapter by $\Upsilon_{\varphi_n,k}(t) = (\varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(t), \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \cdots \circ \varphi_k(t), \ldots, \varphi_k(t), t, \gamma_{k+2}(t), \gamma_{k+2} \circ \gamma_{k+3}(t), \ldots)$, where $\gamma_n = (\varphi_n|_{[0,c_n]})^{-1}$. Next we define $\underline{y}_k = \Upsilon_{g,k}(\frac{1}{2})$, for all $k \in \mathbb{N}$. Let us also adopt the notation $\frac{1}{2} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$.

Let us take a closer look at the sketch of the graph of the functions g and f^2 in Figures 14 and 15 below.
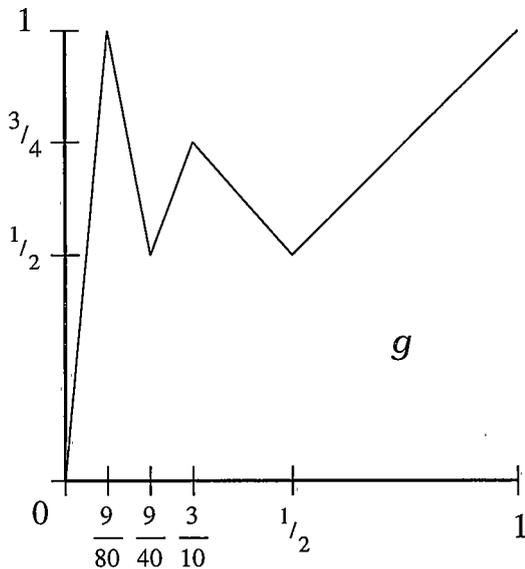
Figure 14: Details of g
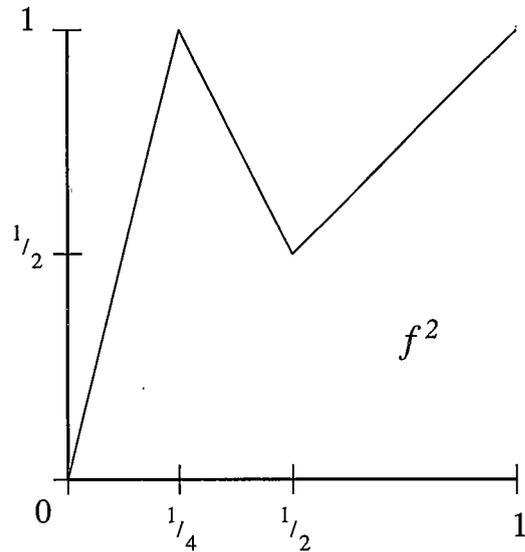
Figure 15: Details of f^2

Lemma 4.1 *The sequence $\{\underline{y}_k\}_{k=0}^{\infty}$ converges to $\frac{1}{2}$.*

Proof: Let $\epsilon > 0$ be given. There exists an $N \in \mathbb{N}$ so that $(\sum_{i=N+2}^{\infty} \frac{1}{2^i}) < \epsilon$. Note that $\underline{y}_k = \Upsilon_{g,k}(\frac{1}{2}) = (g^k(\frac{1}{2}), g^{k-1}(\frac{1}{2}), \ldots, g(\frac{1}{2}), \frac{1}{2}, g_1^{-1}(\frac{1}{2}), g_1^{-2}(\frac{1}{2}), \ldots)$ where $g_1 = g|_{[0,\frac{9}{80}]}$. But $g(\frac{1}{2}) = \frac{1}{2}$ so $\underline{y}_k = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, g_1^{-1}(\frac{1}{2}), g_1^{-2}(\frac{1}{2}), \ldots)$. Hence for $k \geq N$, if we let $y_{k,i}$ be the i_{th} coordinate of $\underline{y}_k$, then $\underline{d}(\underline{y}_k, \frac{1}{2}) = \sum_{i=0}^{\infty} \frac{|y_{k,i}-\frac{1}{2}|}{2^i} = (\sum_{i=0}^{N} \frac{\frac{1}{2}-\frac{1}{2}}{2^i}) + (\sum_{i=N+1}^{\infty} \frac{|y_{k,i}-\frac{1}{2}|}{2^i}) \leq 0 + \sum_{i=N+1}^{\infty} \frac{1}{2^i} < 0 + \epsilon = \epsilon$. Therefore $\underline{y}_k$ converges to $\frac{1}{2}$. Q. E. D.

Next let $\underline{w}_k = \Upsilon_{g,k}(\frac{9}{40})$, for all $k \in \mathbb{N}$.

Lemma 4.2 *The sequence $\{\underline{w}_k\}_{k=0}^{\infty}$ converges to $\frac{1}{2}$.*

Proof: Notice that $g(\frac{9}{40}) = \frac{1}{2}$ and $g(\frac{1}{2}) = \frac{1}{2}$, so $\underline{w}_k = \Upsilon_{g,k}(\frac{9}{40}) = (g^k(\frac{9}{40}), g^{k-1}(\frac{9}{40}), \ldots, g(\frac{9}{40}), \frac{9}{40}, g_1^{-1}(\frac{9}{40}), g_1^{-2}(\frac{9}{40}), \ldots) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{9}{40}, g_1^{-1}(\frac{9}{40}), g_1^{-2}(\frac{9}{40}), \ldots)$. The rest of this proof is similar to the last proof. Q. E. D.

Let $\underline{x}_k$ be defined by $\underline{x}_k = \Upsilon_{g,k}(\frac{3}{10})$, for all $k \in \mathbb{N}$.

Lemma 4.3 *The sequence $\{\underline{x}_k\}_{k=0}^{\infty}$ converges to $\frac{3}{4} = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots)$.*

Proof: Similar to the proof above, we quickly notice that $g(\frac{9}{34}) = \frac{3}{4}$ and $g(\frac{3}{4}) = \frac{3}{4}$. Thus $\underline{x}_k = \Upsilon_{g,k}(\frac{9}{34}) = (g^k(\frac{9}{34}), g^{k-1}(\frac{9}{34}), \ldots, g(\frac{9}{34}), \frac{9}{34}, g_1^{-1}(\frac{9}{34}), g_1^{-2}(\frac{9}{34}), \ldots) = (\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{9}{34}, g_1^{-1}(\frac{9}{34}), g_1^{-2}(\frac{9}{34}), \ldots)$ Then by an analogous proof to that of Lemma 4.1, $\underline{x}_k$ converges to $\frac{3}{4} \in A_X$. Q. E. D.

Next we state the pivotal theorem for this chapter.

Theorem 4.4 *The space $\varprojlim g = X$ is not homeomorphic to the space $\varprojlim f^2 = Y$.*

Proof: Assume, by way of contradiction, that X is homeomorphic to Y and that $h : X \to Y$ is a homeomorphism. Recall that from the previous chapter and the definition of A_X and B_X, the sets A_X and B_X are the arc components of $X = \varprojlim g$ and similarly A_Y and B_Y are the arc components of $Y = \varprojlim f^2$. Since h is a homeomorphism, compact A_X maps to compact A_Y and thus $h(B_X) = B_Y$. Also the end points of A_X must map to the end points A_Y, so without loss of generality, assume $h(\frac{1}{2}) = \frac{1}{2} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) \in A_Y$ and $h(\underline{1}) = \underline{1} \in A_Y$. Hence $h(\underline{w}_k)_{k=1}^{\infty}$ and $h(\underline{y}_k)_{k=1}^{\infty}$ must converge to $\frac{1}{2} \in A_Y$. Since $\frac{3}{10} \in (\frac{9}{40}, \frac{1}{2})$ and $\Upsilon_{g,k}$ is an embedding, $\Upsilon_{g,k}(\frac{3}{10}) = \underline{x}_k$ lies on the ray B_X between $\underline{w}_k$ and $\underline{y}_k$. Now then $\{h(\underline{w}_k)\}_{k=1}^{\infty}$ and $\{h(\underline{y}_k)\}_{k=1}^{\infty}$ converge to $\frac{1}{2}$, since $\{\underline{w}_k\}_{k=1}^{\infty}$ and $\{\underline{y}_k\}_{k=1}^{\infty}$ converge to $\frac{1}{2}$ and $h(\frac{1}{2}) = \frac{1}{2}$. Also $\{h(\underline{x}_k)\}_{k=1}^{\infty}$ converges to $h(\frac{3}{4})$ with $\frac{1}{2} \neq h(\frac{3}{4}) \neq \underline{1}$. For notation, let $h(\underline{w}_k) = (w_{k,0}, w_{k,1}, w_{k,2}, \dots)$ with analogous definitions for $h(\underline{y}_k)$ and $h(\underline{x}_k)$ and since $\{h(\underline{x}_k)\}_{k=1}^{\infty}$ converges to $h(\frac{3}{4})$, define $h(\frac{3}{4}) = (x_{0,0}, x_{0,1}, x_{0,2}, \dots)$, but since $h(\frac{3}{4}) \in A_Y$, $h(\frac{3}{4}) = (x_{0,0}, x_{0,1}, x_{0,2}, \dots) = (x_{0,0}, x_{0,0}, x_{0,0}, \dots)$. Because $h(\frac{3}{4}) \neq \frac{1}{2}$, $x_{0,0} \neq \frac{1}{2}$.

Let $\epsilon = \min\{\frac{1}{16}, \frac{1}{3}(|\frac{1}{2} - x_{0,0}|)\}$, then there is an $N \in \mathbb{N}$ so that $\underline{d}(\frac{1}{2}, h(\underline{w}_k)) < \epsilon$, $\underline{d}(\frac{1}{2}, h(\underline{y}_k)) < \epsilon$ and $\underline{d}(h(\frac{3}{4}), h(\underline{x}_k)) < \epsilon$ for all $k \geq N$.

Next we will show that for any fixed $k \geq N$, there exists an $m \in \mathbb{N}$ so that $|w_{k,m} - y_{k,m}| > \frac{5}{16}$. Suppose, by way of contradiction, that $|w_{k,m} - y_{k,m}| \leq \frac{5}{16}$ for all $m \in \mathbb{N}$. First we will show that $w_{k,0}$ and $y_{k,0}$ are elements of $[\frac{1}{2}, 1]$. If $w_{k,0} \in [0, \frac{1}{2})$ then since the only preimage (under f^2) of $[0, \frac{1}{2})$ is $[0, \frac{1}{8})$, $w_{k,1} \in [0, \frac{1}{8})$, thus $|\frac{1}{2} - w_{k,1}| > |\frac{1}{2} - \frac{1}{8}| = \frac{3}{8}$ and so $\underline{d}(\frac{1}{2}, h(\underline{w}_k)) = \sum_{i=0}^{\infty} \frac{|\frac{1}{2} - w_{k,i}|}{2^i} \geq \frac{|\frac{1}{2} - w_{k,1}|}{2} \geq \frac{\frac{3}{8}}{2} = \frac{3}{16} > \frac{1}{16} \geq \epsilon$, that is to say $\underline{d}(\frac{1}{2}, h(\underline{w}_k)) > \epsilon$. But we selected N so that $\underline{d}(\frac{1}{2}, h(\underline{w}_k)) < \epsilon$ for all $n \geq N$, therefore $w_{k,0} \notin [0, \frac{1}{2})$ and it must be that $w_{k,0} \in [\frac{1}{2}, 1]$. By a similar argument, we know $y_{k,0} \in [\frac{1}{2}, 1]$ also. However, since $h(\underline{w}_k), h(\underline{y}_k) \in B_Y$, there exists an $i \in \mathbb{N}$ such that $w_{k,i} \in [0, \frac{1}{2})$ and a $j \in \mathbb{N}$ so that $y_{k,j} \in [0, \frac{1}{2})$. Again since the only preimage of $[0, \frac{1}{2})$ is $[0, \frac{1}{8})$, $w_{k,i+n} \in [0, \frac{1}{2})$ and $y_{k,i+n} \in [0, \frac{1}{2})$ for all $n = 0, 1, 2, \dots$. Now let i^* be

the smallest integer so that $w_{k,i^*} \in [0, \frac{1}{2})$ and let j^* be the smallest integer so that $y_{k,j^*} \in [0, \frac{1}{2})$. Thus w_{k,i^*-1} and y_{k,j^*-1} are in $[\frac{1}{2}, 1]$.

Now because $f^2(x) = x$ for $x \in [\frac{1}{2}, 1]$, $w_{k,i} = w_{k,i-1}$ for $i < i^*$, we get $|\frac{1}{2} - w_{k,i^*-1}| = |\frac{1}{2} - w_{k,0}| \le \sum_{i=0}^{\infty} \frac{|\frac{1}{2} - w_{k,i}|}{2^i} = \underline{d}(\frac{1}{2}, h(\underline{w}_k)) < \epsilon$. Therefore $w_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2} + \epsilon)$. The preimage of $[\frac{1}{2}, \frac{1}{2} + \epsilon)$ under f^2 is $[\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4}) \cup (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \epsilon)$ and $w_{k,i^*} \in [0, \frac{1}{2})$ with $f^2(w_{k,i^*}) = w_{k,i^*-1}$, so $w_{k,i^*} \in [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4}) \cup (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$. Similarly $y_{k,j^*} \in [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4}) \cup (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$. First, let us consider the case where $i^* < j^*$. Then $y_{k,i^*} \in [\frac{1}{2}, \frac{1}{2} + \epsilon)$ and we assumed $|w_{k,i^*} - y_{k,i^*}| \le \frac{5}{16}$ and $\epsilon \le \frac{1}{16}$, so $w_{k,i^*} \notin [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4}) \subset [\frac{1}{8}, \frac{1}{8} + \frac{1}{64})$, thus $w_{k,i^*} \in (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$ and then $f^2(w_{k,i^*}) = w_{k,i^*+1} \in [0, \frac{1}{8})$ which is the only preimage under f^2 of $[0, \frac{1}{2}) \supset (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$. Therefore, using the assumption that $|w_{k,i^*+1} - y_{k,i^*+1}| \le \frac{5}{16}$, $y_{k,i^*+1} \notin [\frac{1}{2}, \frac{1}{2} + \epsilon)$ and $y_{k,i^*+1} \notin (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$ so $y_{k,i^*+1} \in [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4})$, hence $j^* = i^* + 1$. By a analogous argument, if $j^* < i^*$, then $i^* = j^* + 1$ and $y_{k,j^*+1} \in [0, \frac{1}{8})$ with $w_{k,j^*+1} \in [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4})$. If it is the case that $i^* = j^*$, then y_{k,i^*}, $w_{k,i^*} \in [\frac{1}{8}, \frac{1}{8} + \frac{\epsilon}{4}) \cup (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2})$ and $y_{k,i^*+1}, w_{k,i^*+1} \in [0, \frac{1}{8})$. In any case, $w_{k,i^*+1}, y_{k,i^*+1} \in [0, \frac{1}{8} + \frac{\epsilon}{4})$. Now assume, without loss of generality, that $w_{k,i^*+1} < y_{k,i^*+1}$ and consider $\Upsilon_{f^2, i^*+1}([w_{k,i^*+1}, y_{k,i^*+1}])$. Since Υ_{f^2, i^*+1} is an embedding, $\Upsilon_{f^2, i^*+1}([w_{k,i^*+1}, y_{k,i^*+1}])$ is an arc in B_Y with endpoints $h(\underline{w}_k)$ and $h(\underline{y}_k)$. But $h(\underline{x}_k)$ is on that arc between $h(\underline{w}_k)$ and $h(\underline{y}_k)$, so $x_{k,i^*+1} \in [w_{k,i^*+1}, y_{k,i^*+1}] \subset [0, \frac{1}{8} + \frac{\epsilon}{4})$ and f^2 is strictly increasing on $[0, \frac{1}{8} + \frac{\epsilon}{4})$ thus $x_{k,i^*} \in f^2([w_{k,i^*+1}, y_{k,i^*+1}]) = [w_{k,i^*}, y_{k,i^*}] \subset (\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \epsilon)$ Then $f^2(x_{k,i^*}) = x_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2} + \epsilon)$ and since $f^2(x) = x$ for $x \in [\frac{1}{2}, 1]$, we see that $x_{k,0} = x_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2} + \epsilon)$. Also $|x_{0,0} - x_{k,0}| \le \sum_{i=0}^{\infty} \frac{|x_{0,i} - x_{k,i}|}{2^i} = \underline{d}(h(\frac{3}{4}), h(\underline{x}_k)) < \epsilon$. Therefore $|x_{0,0} - \frac{1}{2}| \le |x_{0,0} - x_{k,0}| + |x_{k,0} - \frac{1}{2}| < \epsilon + \epsilon = 2\epsilon$. But by the selection of ϵ, $\epsilon \le \frac{1}{3}|\frac{1}{2} - x_{0,0}|$, so $3\epsilon \le |\frac{1}{2} - x_{0,0}| < 2\epsilon$, a clear contradiction. Hence must there exist an $m \in \mathbb{N} \cup 0$ so that $|w_{k,m} - y_{k,m}| > \frac{5}{16}$ as desired.

Since $w_{k,i} \in [\frac{1}{2}, \frac{1}{2} - \epsilon)$ for $i < i^*$ and $y_{k,i} \in [\frac{1}{2}, \frac{1}{2} - \epsilon)$ for $i < j^*$, it must be that $|w_{k,i} - y_{k,i}| < \epsilon$ for $i < \min\{i^*, j^*\}$ so $m \ge \min\{i^*, j^*\}$. Hence either $w_{k,m} \in [0, \frac{1}{2})$ or

$y_{k,m} \in [0, \frac{1}{2})$ and $|w_{k,m} - y_{k,m}| > \frac{5}{16}$. Without loss of generality, assume $w_{k,m} < y_{k,m}$, so the diameter of the interval from $w_{k,m}$ to $y_{k,m}$, denoted $\mathrm{diam}([w_{k,m}, y_{k,m}])$, is greater than $\frac{5}{16}$ and $y_{k,m} < \frac{1}{2} + \epsilon \le \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$, therefore $\frac{1}{4} \in [w_{k,m}, y_{k,m}]$. Define $\underline{b}_k = \Upsilon_{f^2,k}(\frac{1}{4}) = ((f^2)^k(\frac{1}{4}), (f^2)^{k-1}(\frac{1}{4}), \ldots, f^2(\frac{1}{4}), \frac{1}{4}, \frac{1}{16}, \ldots) = (1, 1, 1, \ldots, \frac{1}{4}, \frac{1}{16}, \ldots)$ where $\frac{1}{4}$ is in the $(k+1)^{st}$ position. By a similar proof to that we have seen before, it can be shown $\{\underline{b}_k\}_{k=1}^{\infty}$ converges to $\underline{1} = (1, 1, 1, \ldots) \in A_Y$. Since h is a homeomorphism and we assumed $h(\underline{1}) = \underline{1}$, $\{h^{-1}(\underline{b}_k)\}_{k=1}^{\infty}$ converges to $\underline{1} \in A_X$. Now for any fixed k, because $\frac{1}{4} \in [w_{k,m}, y_{k,m}]$, $\underline{b}_k$ is between $h(\underline{w}_k)$ and $h(\underline{y}_k)$ on the ray B_Y, so $h^{-1}(\underline{b}_k)$ is between $\underline{w}_k$ and $\underline{y}_k$ on the ray B_X. If we let $h^{-1}(\underline{b}_k) = (b_{k,0}, b_{k,1}, b_{k,2}, \ldots)$, it must be that $b_{k,k} \in [\frac{9}{40}, \frac{1}{2}]$ since $\Upsilon_{g,k}$ is an embedding and $\Upsilon_{g,k}[\frac{9}{40}, \frac{1}{2}]$ is an arc from $\underline{w}_k$ to $\underline{y}_k$, thus containing $h^{-1}(\underline{b}_k)$. Because $\{h^{-1}(\underline{b}_k)\}_{k=1}^{\infty}$ converges to $\underline{1}$, there exists an $l \in \mathbb{N}$ so that $\underline{d}(h^{-1}(\underline{b}_k), \underline{1}) < \frac{1}{16}$ for all $k \ge l$. So $|b_{k,0} - 1| \le \sum_{i=0}^{\infty} \frac{|b_{k,i} - 1|}{2^i} = \underline{d}(h^{-1}(\underline{b}_k), \underline{1}) < \frac{1}{16}$. But the largest $g(x)$ gets for $x \in [\frac{9}{40}, \frac{1}{2}]$ is $\frac{3}{4}$, so $b_{k,k-1} \le \frac{3}{4}$ and since $g(x) = x$ for $x \in [\frac{1}{2}, 1]$, $b_{k,k-1} = b_{k,0} \le \frac{3}{4}$. That contradicts $|b_{k,0} - 1| < \frac{1}{16}$.

Therefore h cannot be a homeomorphism and thus $X = \varprojlim g$ is not homeomorphic to $Y = \varprojlim f^2$. $Q.\ E.\ D.$

Thus we have seen that X and Y are two distinct examples of unimodal generalized $\sin \frac{1}{x}$ continua.

We note here that $X = \varprojlim g$ is homeomorphic to the M-continuum discussed by Sam Nadler in [13], however Nadler does not create it as a unimodal continuum.

CHAPTER 5

AN UNCOUNTABLE COLLECTION

In this chapter we will construct a collection of unimodal generalized $\sin \frac{1}{x}$ continua for which uncountably many are non-homeomorphic with each other.

Let $\underline{a} = (a_0 a_1 a_2 \dots)$ be an element of the uncountable set $\{0,1\}^{\mathbb{N}}$ (see [12], pg. 50), that is $\underline{a}$ is an infinitely long string of 0's and 1's. Suppose we divide $\{0,1\}^{\mathbb{N}}$ into equivalence classes. Let $(a_0 a_1 a_2 \dots)$ and $(b_0 b_1 b_2 \dots)$ be in the same equivalence class if and only if there exists a $k \in \mathbb{Z}$ and an $N \in \mathbb{Z}^+$ so that $b_{i+k} = a_i$ for all $i \geq N$. Define C to be set of these equivalence classes.

Lemma 5.1 *The set C is uncountable.*

Proof: First we recognize that for any fixed k, N can take on only countably many values. Since k must be an integer, for any given $(a_0 a_1 a_2 \dots)$, the equivalence class of $(a_0 a_1 a_2 \dots)$ is a countable collection of countable sets, and thus is countable. If we assume, by way of contradiction, that C is countable, then $\{0,1\}^{\mathbb{N}}$ is the union of countably many equivalence class, each with countably many elements, and hence is countable. But we know $\{0,1\}^{\mathbb{N}}$ is uncountable. This contradiction proves that C is an uncountable set. $Q.\ E.\ D.$

Use the functions g and f^2 that were defined in the last chapter and let $\eta_0 = g$ and $\eta_1 = f^2$, then let $h_i = \eta_{a_i}$. Thus $h_i : I \to I$ is equal to g if $a_i = 0$ and equal to f^2 if $a_i = 1$. So, by definition, $\varprojlim\{h_i\}$ is a unimodal generalized $\sin \frac{1}{x}$ continuum. Hence for each member of $\{0,1\}^{\mathbb{N}}$ there is a corresponding inverse limit space.

Now let $\underline{a} = (a_0 a_1 a_2 \dots)$ and $\underline{b} = (b_0 b_1 b_2 \dots)$ be elements of $\{0,1\}^{\mathbb{N}}$ and let $\varprojlim\{h_i\}$ correspond to $\underline{a}$ and $\varprojlim\{h_i^*\}$ correspond to $\underline{b}$.

Theorem 5.2 *The continuum* $\varprojlim\{h_i\}$ *is homeomorphic to the continuum* $\varprojlim\{h_i^*\}$ *if and only if $\underline{a}$ and $\underline{b}$ are in the same equivalence class of C.*

Proof: Let us start by assuming that there exists a $k \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$ so that $b_{i+k} = a_i$ for all $i \geq N$. This could also be written as $b_{N+k+i} = a_{N+i}$ for $i = 0, 1, 2, \ldots$. Therefore $\eta_{b_{N+k+i}} = \eta_{a_{N+i}}$ so $h_{N+k+i}^* = h_{N+i}$. Now if we apply Corollary 2.1 we get that $\varprojlim\{h_i^*\} = \varprojlim\{h_i\}$, as desired.

Next assume that $\varprojlim\{h_i\} \simeq \varprojlim\{h_i^*\}$. Let $H : \varprojlim\{h_i\} \to \varprojlim\{h_i^*\}$ be a homeomorphism. If we define the "limit bars" of $\varprojlim\{h_i\}$ and $\varprojlim\{h_i^*\}$ as in the previous chapter, that is $\underline{x} = (x_0, x_0, x_0, \ldots)$ is in the limit bar for $x_0 \in [\frac{1}{2}, 1]$, it must be that H takes the limit bar of $\varprojlim\{h_i\}$ to the limit bar of $\varprojlim\{h_i^*\}$ and similarly the ray maps to the ray. Consider the sequence of points in $\varprojlim\{h_i\}$ defined by $\underline{p}_1 = (1, p_{1,1}, p_{1,2}, p_{1,3}, \ldots)$, $\underline{p}_2 = (1, 1, p_{2,2}, p_{2,3}, p_{2,4}, \ldots)$, $\underline{p}_3 = (1, 1, 1, p_{3,3}, p_{3,4}, p_{3,5}, \ldots)$ and so on, with the general term being $\underline{p}_i = (1, 1, 1, \ldots, 1, p_{i,i}, p_{i,i+1}, p_{i,i+2}, \ldots)$ where $p_{i,j} \leq \frac{1}{2}$ for $j \geq i$. Clearly $\{\underline{p}_i\}_{i=1}^{\infty}$ converges to $\underline{1} = (1, 1, 1, \ldots)$ which is an endpoint of the limit bar of $\varprojlim\{h_i\}$. Similarly define $\underline{q}_i = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, q_{i,i}, q_{i,i+1}, q_{i,i+2}, \ldots)$ so $\{q_i\}_{i=1}^{\infty}$ converges to $\underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, the other endpoint of the limit bar of $\varprojlim\{h_i\}$. So let $\{\underline{c}_i\}$ be a sequence of points of the form $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{3}{10}, c_{i,j}, c_{i,j+1}, c_{i,j+2}, \ldots)$. Note that since $\frac{3}{10} < \frac{1}{2}$, $c_{i,j+m}$ is completely determined for $m = 0, 1, 2, \ldots$. There may be none of these points $\underline{c}_i$, a finite number of these points, or they may constitute an infinite sequence. If these points form an infinite sequence, $\{\underline{c}_i\}_{i=1}^{\infty}$ converges to $\underline{\frac{3}{4}}$. Notice that since $g(\frac{3}{10}) = \frac{3}{4} \neq f^2(\frac{3}{10})$, each point of $\{\underline{c}_i\}$ corresponds a different function of h_i being equal to the function g, so $a_n = 0$ where n is different for each different i.

We similarly define $\{\underline{x}_i\}$, $\{\underline{y}_i\}$ and $\{\underline{z}_i\}$ to be sequences in $\varprojlim\{h_i^*\}$ with $\underline{x}_i = (1, 1, 1, \ldots, 1, x_{i,i}, x_{i,i+1}, x_{i,i+2}, \ldots)$, $\underline{y}_i = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, y_{i,i}, y_{i,i+1}, y_{i,i+2}, \ldots)$ and $\underline{z}_i = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{3}{10}, z_{i,j}, z_{i,i+1}, z_{i,i+2}, \ldots)$ and further assume we number the points $\underline{c}_i$

and z_i so the larger the i, the closer we are to the limit bar. As before, it can be shown that $\{x_i\}_{i=1}^{\infty} \longrightarrow \underline{1} \in \varprojlim\{h_i^*\}$, $\{y_i\}_{i=1}^{\infty} \longrightarrow \frac{1}{2} \in \varprojlim\{h_i^*\}$ and, if $\{z_i\}$ is an infinite sequence, $\{z_i\}_{i=1}^{\infty} \longrightarrow \frac{3}{4} \in \varprojlim\{h_i^*\}$.
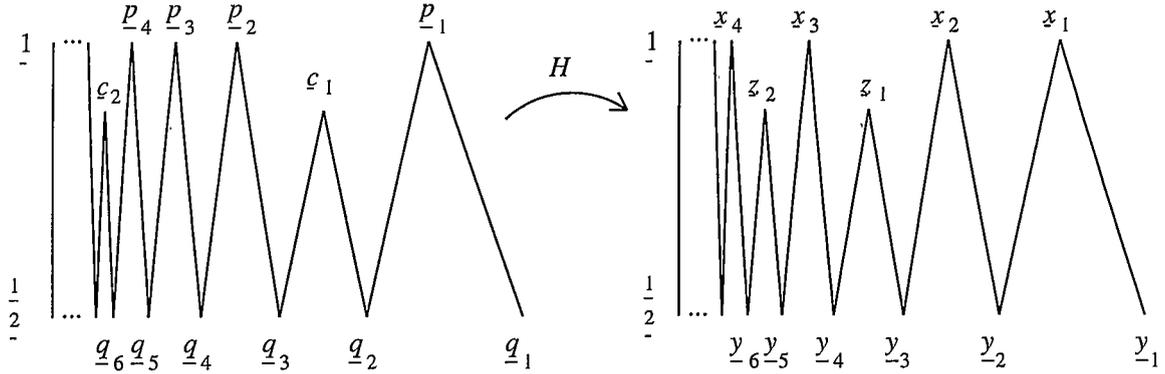

Figure 16: The Spaces $\varprojlim\{h_i\}$ and $\varprojlim\{h_i^*\}$

It is possible that neither $\{c_i\}$ nor $\{z_i\}$ is an infinite sequence, in which case for some $N_1 \in \mathbb{N}$, $h_i = f^2$ for $i \geq N_1$, for if $h_i = g$, then we would get another c_i. Similarly there exists an $N_2 \in \mathbb{N}$ so that $h_i^* = f^2$ for $i \geq N_2$. Now let $N = \max\{N_1, N_2\}$ and let $k = N_2 - N_1$, so that $a_i = 1$ for $i \geq N$ since $\eta_{a_i} = h_i = f^2 = \eta_1$ and also $b_{i+k} = 1$ for $i \geq N$ because $\eta_{b_{i+k}} = h_i^* = f^2 = \eta_1$. Thus $b_{i+k} = a_i$ for $i \geq N$.

Next we consider the case where either $\{c_i\}$ or $\{z_i\}$ is finite and the other is infinite. Without loss of generality, assume $\{c_i\}$ is finite and $\{z_i\}_{i=1}^{\infty}$ is infinite. From the form of the coordinates of c_i versus p_i, we know only finitely many of the bonding maps h_i are equal to g, the rest are equal to f^2. Then $\varprojlim\{h_i\}$ is homeomorphic to $\varprojlim f^2$ by Lemma 2.1, thus we have $\varprojlim\{h_i^*\}$ homeomorphic to $\varprojlim f^2$. However, since $\{z_i\}_{i=1}^{\infty}$ is infinite, $h_i^* = g$ for infinitely many i's, we construct a proof very similar to the proof of Theorem 4.4 to show that $\varprojlim\{h_i^*\}$ is not homeomorphic to $\varprojlim f^2$. So to do this, assume G is a homeomorphism from $\varprojlim\{h_i^*\}$ onto $\varprojlim f^2$. Notate any arc in $\varprojlim\{h_i^*\}$ from y_n through z_m ending at y_{n+1} by $\langle y_n, z_m, y_{n+1} \rangle$ and look at $G(\langle y_n, z_m, y_{n+1} \rangle)$. Since $\{y_i\}_{i=1}^{\infty} \longrightarrow \frac{1}{2}$ and $\{z_i\}_{i=1}^{\infty} \longrightarrow \frac{3}{4}$, $\{G(y_i)\}_{i=1}^{\infty}$ converges

to an endpoint of the limit bar of $\varprojlim f^2$ and $\{G(\underline{z}_i)\}_{i=1}^{\infty}$ converges to something other than an endpoint. Thus for increasingly large m, $G(\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1} \rangle)$ contains a point $\underline{b}_m$ with the first several coordinates being the number 1, so that $\{\underline{b}_i\}_{i=1}^{\infty}$ converges to the opposite endpoint of the limit bar of $\varprojlim f^2$ from which $\{G(\underline{y}_i)\}_{i=1}^{\infty}$ converges. Then $\{G^{-1}(\underline{b}_i)\}_{i=1}^{\infty} \longrightarrow \underline{1}$ and $G^{-1}(\underline{b}_m) \in \langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1} \rangle$, but from the construction of $\underline{z}_i$ and the function g, we know that the first coordinate of every element of $\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1} \rangle$ must be smaller than $\frac{3}{4}$, so $\{\underline{b}_i\}_{i=1}^{\infty}$ cannot converge to $\underline{1}$ hence $\varprojlim f^2$ is not homeomorphic to $\varprojlim \{h_i^*\}$. This is a clear contradiction, so it must not be the case that either $\{\underline{c}_i\}$ or $\{\underline{z}_i\}$ is finite and the other one is infinite.

Now let us assume both $\{\underline{c}_i\}_{i=1}^{\infty}$ and $\{\underline{z}_i\}_{i=1}^{\infty}$ are infinite sequences, that is they have infinitely many distinct points. Since H is a homeomorphism, $\{H^{-1}(\underline{x}_i)\}_{i=1}^{\infty}$, $\{H^{-1}(\underline{y}_i)\}_{i=1}^{\infty}$, and $\{H^{-1}(\underline{z}_i)\}_{i=1}^{\infty}$ all must converge to point on the limit bar of $\varprojlim \{h_i\}$. Define $\underline{z}_0$ to be the limit point of $\{H^{-1}(\underline{z}_i)\}_{i=1}^{\infty}$ on the limit bar of $\varprojlim \{h_i\}$, and since endpoints must map to endpoints, $\underline{z}_0 \neq \underline{1}$ and $\underline{z}_0 \neq \frac{1}{2}$. Next consider $\{H(\underline{p}_i)\}_{i=1}^{\infty}$, $\{H(\underline{q}_i)\}_{i=1}^{\infty}$, and $\{H(\underline{c}_i)\}_{i=1}^{\infty}$ which must all converge in the limit bar of $\varprojlim \{h_i^*\}$. Again if we let $\{H(\underline{c}_i)\}_{i=1}^{\infty} \longrightarrow \underline{c}_0$ in the limit bar of $\varprojlim \{h_i^*\}$, $\underline{c}_0 \neq \underline{1}$ and $\underline{c}_0 \neq \frac{1}{2}$. For now let $\underline{p}_0$ be the limit point of the sequence $\{H(\underline{p}_i)\}_{i=1}^{\infty}$ and similarly let $\underline{q}_0$, $\underline{x}_0$ and $\underline{y}_0$ be limit points of $\{H(\underline{q}_i)\}_{i=1}^{\infty}$, $\{H^{-1}(\underline{x}_i)\}_{i=1}^{\infty}$ and $\{H^{-1}(\underline{y}_i)\}_{i=1}^{\infty}$, respectively. Since H is a homeomorphism, each of these limits on an endpoint of the limit bar of the space the limit point is in. Let $\epsilon > 0$ be defined as follows:

$$\epsilon < \tfrac{1}{2}\min\{\underline{d}(\underline{1}, \tfrac{3}{4}), \underline{d}(\tfrac{3}{4}, \tfrac{1}{2}), \underline{d}(\underline{1}, \underline{z}_0), \underline{d}(\tfrac{1}{2}, \underline{z}_0), \underline{d}(\underline{1}, \underline{c}_0), \underline{d}(\tfrac{1}{2}, \underline{c}_0)\}$$

We let $N^* \in \mathbb{N}$ be large enough so that all points of the sequences we have labeled are within an ϵ-ball of their limit point for all subscripts $i \geq N^*$. Choose $n, m \geq N^*$ so that for a given $\underline{c}_m$, $\underline{q}_n$ is the closest point of $\{\underline{q}_i\}_{i=1}^{\infty}$ to $\underline{c}_m$ along the ray from $\underline{c}_m$ to the endpoint of the ray part of $\varprojlim \{h_i\}$, then $\underline{q}_{n+1}$ is the closest point of $\{\underline{q}_i\}_{i=1}^{\infty}$

to $\underline{c}_m$ along the open end of the ray. Notate the arc from $\underline{q}_n$, through $\underline{c}_m$ and ending at $\underline{q}_{n+1}$ by $\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. If we look at $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, we know that $H(\underline{q}_n)$ and $H(\underline{q}_{n+1})$ are within ϵ of $\underline{q}_0$, but $H(\underline{c}_m)$ is not, it is within ϵ of $\underline{c}_0$. Thus if $\underline{q}_0 = 1$, $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ must pass through $\underline{y}_i$ for some i, but then $H^{-1}(\underline{y}_i) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. Now if $\underline{q}_0 = 1$ then $\underline{p}_0 = \frac{1}{2}$ and it must be that $\underline{y}_0 = 1$, so $H^{-1}(\underline{y}_i) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$ and $H^{-1}(y_i)$ is within ϵ of $\underline{1}$, which contradicts the selection of $\underline{q}_n$, $\underline{c}_m$ and $\underline{q}_{n+1}$, thus it must be the case that $\underline{q}_0 = \frac{1}{2} \in \varprojlim\{h_i^*\}$, $\underline{p}_0 = \underline{1} \in \varprojlim\{h_i^*\}$, $\underline{y}_0 = \frac{1}{2} \in \varprojlim\{h_i\}$ and $\underline{x}_0 = \varprojlim\{h_i\}$.

Now then, still considering $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, we know that $\underline{q}_n$ and $\underline{q}_{n+1}$ are within ϵ of $\frac{1}{2}$ and that $\underline{c}_m$ is not within ϵ of $\frac{1}{2}$, so $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ passes through either a point of $\{\underline{x}_i\}_{i=1}^{\infty}$, call it $\underline{x}_M$ or a point of $\{z_i\}_{i=1}^{\infty}$, labeled $\underline{z}_{M^*}$. If M (or M^*) is less than N^*, choose a larger n and m until we get $\underline{x}_M$ (or $\underline{z}_{M^*}$) so that $M \geq N$ ($M^* \geq N$). Thus either we have an $\underline{x}_M$ within ϵ of $\underline{1}$ or there is a $\underline{z}_{M^*}$ within ϵ of $\frac{3}{4}$. Since $\underline{x}_i$ converges to $\underline{1}$, $H^{-1}(\underline{x}_i)$ converges to $\underline{1}$ and $M \geq N^*$, so $H^{-1}(\underline{x}_M)$ is within ϵ of $\underline{1}$, but since $\underline{x}_M \in H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, $H^{-1}(\underline{x}_M) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. However $\underline{c}_m$ is the relative maximum of this arc, in other words $\underline{c}_m$ is the point with the largest first coordinate, which is $\frac{3}{4}$, so $\underline{x}_M$ cannot have a first coordinate larger than $\frac{3}{4}$, hence $H^{-1}(\underline{x}_M)$ cannot be arbitrarily close to $\underline{1}$. Therefore it is $\underline{z}_{M^*}$ in $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ and both $\underline{c}_m$ and $\underline{z}_{M^*}$, by virtue of the first coordinate being $\frac{3}{4}$ and the first coordinate less than $\frac{1}{2}$ being $\frac{3}{10}$, relate to the bonding map g in the inverse limit space. Hence $h_{a_m} = h_0 = g = h_0^* = h_{b_{M^*}}^*$ and $b_{M^*} = 0 = a_m$. Let $m = N$ and $k = M^* - m$ so $b_{i+k} = a_i$ for $i = N$.

This argument can then be repeated for $\underline{c}_{m+1}$. We know that the same k will work since $\underline{c}_{m+1}$ maps to $\underline{z}_{M^*+1}$, for if it did not, we could consider an arc $\langle \underline{y}_M, \underline{z}_{M^*+1}, \underline{y}_{M+1} \rangle$ and the arc $H^{-1}(\langle \underline{y}_M, \underline{z}_{M^*+1}, \underline{y}_{M+1} \rangle)$ and with an argument similar to above, arrive at a contradiction. Again repeat the argument for $\underline{c}_{m+2}$ and $\underline{z}_{M^*+2}$,

and so on. Therefore we have $b_{i+k} = a_i$ for $i \geq N$ as desired. $Q.\ E.\ D.$

Define D to be the collection of all inverse limit spaces $\varprojlim\{h_i\}$ of this form, that is, spaces using f^2 and g as the only bonding maps, applied in different combinations.

Corollary 5.3 *The set D contains uncountably many pairwise nonhomeomorphic spaces.*

Proof: From the theorem above, there is a one to one correspondence between our set D and the set C from Lemma 5.1. $Q.\ E.\ D.$

So we have created an uncountable set of unimodal generalized $\sin\frac{1}{x}$ continua.

The continua considered here are often called $arc + ray$ continua for obvious reasons. We wish to point out that Karen Brucks and Henk Bruin ([6]) have proven that there are uncountably many nonhomeomorphic unimodal inverse limit spaces that are non arc+ray continua.

CHAPTER 6

A DIFFERENT UNCOUNTABLE COLLECTION

In previous chapters we created an uncountable collection of unimodal gener-
alized $\sin \frac{1}{x}$ continua. If we look at the visualizations of these continua, like Figure
16, we could characterize the way we created this collection by saying we varied prox-
imity of the critical points of height $\frac{3}{4}$ relative to the limit bar. Here we define height
to be the first coordinate of our point. In all these cases, the critical points all have
heights of $\frac{1}{2}$, $\frac{3}{4}$ or 1. In this chapter we develop an uncountable collection of unimodal
generalized $\sin \frac{1}{x}$ continua by creating infinitely many different height critical points.

If all of the local extreme points that do not have a first coordinate of $\frac{1}{2}$ or
1 originate from the same side, say for example they start at the bottom and reach
part way up, we will call the inverse limit space an M-continua. (We realize our
figures show them looking more like a W shape than an M, but our Figure 5 is
homeomorphic to what Sam Nadler, [13], refers to as M-continua, so for consistency,
we will call the ones we created in the last two chapters M-continua as well.) In
the next chapter we consider how to create what Nadler refers to as MW-continua,
where the critical points of heights between $\frac{1}{2}$ and 1 are originating from both sides
(top and bottom). Refer back to Figure 6.

Let us start by letting β be any number in $[\frac{1}{4}, \frac{1}{2}]$ and define the following
functions parameterized by β.

Let $u_\beta : [0,1] \to [0,1]$ be defined by:

$$u_\beta(x) = \begin{cases} \frac{1}{2\beta}x & \text{if } 0 \leq x \leq 2\beta \\ -x + 2\beta + 1 & \text{if } 2\beta \leq x \leq 1 \end{cases}$$

and let $v_\beta : [0,1] \to [0,1]$ be defined by:

$$v_\beta(x) = \begin{cases} \frac{1}{\beta}x & \text{if } 0 \le x \le \beta \\ -x + \beta + 1 & \text{if } \beta \le x \le 1 \end{cases}$$

and define $z_\beta : [0,1] \to [0,1]$ by:

$$z_\beta(x) = \begin{cases} (2 - 2\beta)x & \text{if } 0 \le x \le \frac{1}{2} \\ 2\beta x - 2\beta + 1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

In addition recall that we defined $f : [0,1] \to [0,1]$ by:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Now let $g_\beta : [0,1] \to [0,1]$ be defined by $g_\beta = f \circ u_\beta \circ v_\beta \circ z_\beta$. One can see that the functions u, v, z and g from the previous chapters are exactly $u_{\frac{3}{8}}$, $v_{\frac{3}{8}}$, $z_{\frac{3}{8}}$ and $g_{\frac{3}{8}}$ respectively, so refer to Figure 13 and 14 for sketches of the graphs of these functions for $\beta = \frac{3}{8}$. Figure 17 shows g_β for $\beta = \frac{1}{4}$, $\frac{1}{3}$, $\frac{7}{16}$ and $\frac{1}{2}$.

A little investigation shows that $x = \frac{\beta}{2-2\beta}$ maps to β under z_β, then β is a critical point of v_β and maps to 1. The function u_β maps 1 to 2β and then $f(2\beta) = \frac{3}{2} - 2\beta$ for $\beta \in [\frac{1}{4}, \frac{1}{2}]$. Thus for $\beta \in (\frac{1}{4}, \frac{1}{2})$ we get $x = \frac{\beta}{2-2\beta}$ is a critical point of g_β leading to the local maximum of $g(\frac{\beta}{2-2\beta}) = \frac{3}{2} - 2\beta$, which is an element of $(\frac{1}{2}, 1)$. Examples of this are $g_{\frac{3}{8}}(\frac{3}{10}) = g(\frac{3}{10}) = \frac{3}{4}$ and $g_{\frac{1}{3}}(\frac{1}{4}) = \frac{5}{6}$. We also notice that the relative size of β is inversely proportional to the height of this local maximum, that is, as β gets larger, $g_\beta(\frac{\beta}{2-2\beta}) = \frac{3}{2} - 2\beta$ becomes smaller.
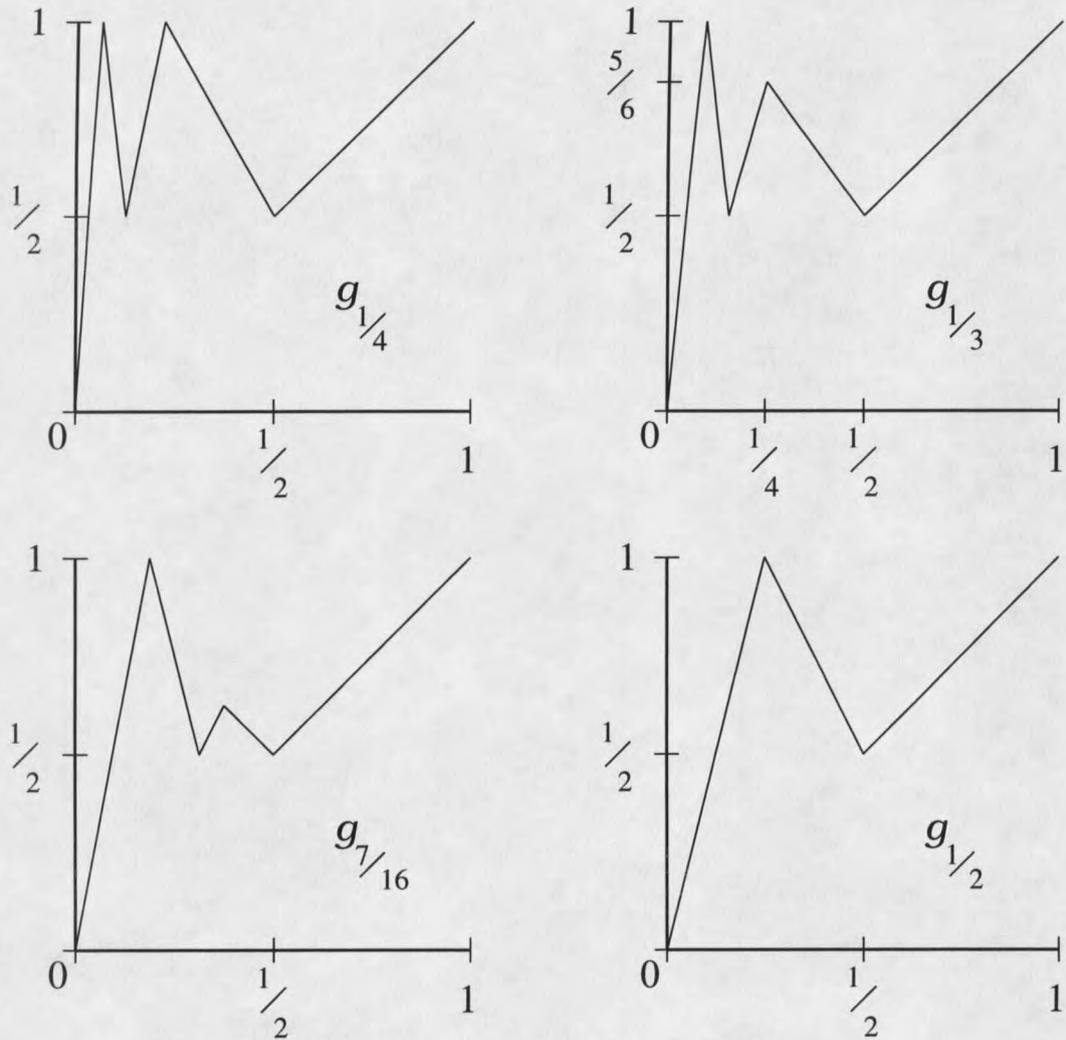
Figure 17. Function g_β for Various Values of β

Notice that $u_{\frac{1}{2}}$ and $z_{\frac{1}{2}}$ are both the identity function and $v_{\frac{1}{2}} = f$, thus $g_{\frac{1}{2}} = f^2$. Also, it can be shown that $\varprojlim g_{\frac{1}{4}}$ is homeomorphic to $\varprojlim f^2$, but for any value of $\beta \in (\frac{1}{4}, \frac{1}{2})$ we see that $\varprojlim g_\beta$ is homeomorphic to $\varprojlim g = \varprojlim g_{\frac{3}{8}}$. However if we let $h = g_{\frac{1}{3}} \circ g_{\frac{3}{8}}$ and consider $\varprojlim h$, we see the resulting inverse limit space is apparently different than (that is, not homeomorphic to) $\varprojlim g$ or $\varprojlim f^2$. See Figure 18 for a visualization of $\varprojlim h$.

We wish to generalize this idea, but before we can prove such a result, we need the following definitions and lemmas. We define an *undistinguished point*

$\underline{x}$ on the limit bar of a generalized $\sin \frac{1}{x}$ continua to be a point $\underline{x}$ that has an open neighborhood that is homeomorphic, with homeomorphism H, to the following subset of $\mathbb{R}^2$: $(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}) \times (-1, 1)$. See Figure 19.
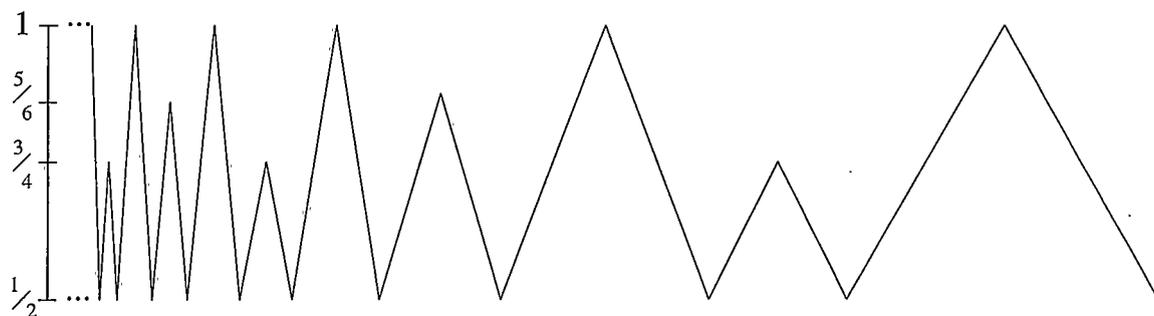

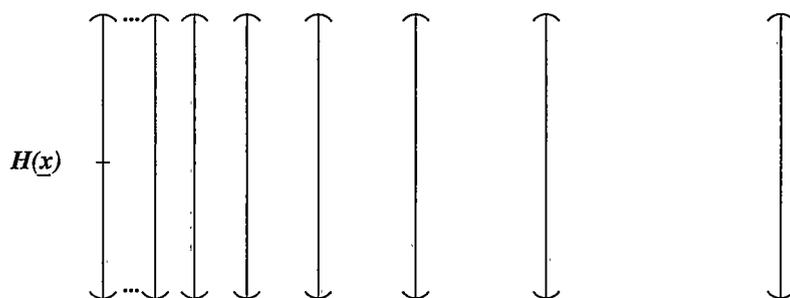Figure 18. A Visualization of $\varprojlim h$

Figure 19. A Neighborhood of H of Undistinguished Point $\underline{x}$

If a point on the limit bar is not undistinguished, we will then naturally call it a *distinguished point*. Figures 20 and 21 show visualizations to which neighborhoods of a couple of types of distinguished points would be homeomorphic.
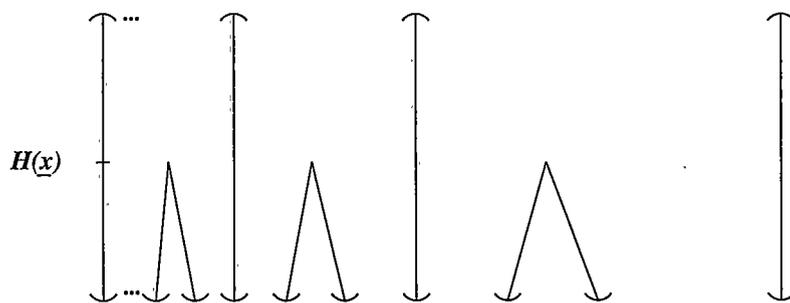

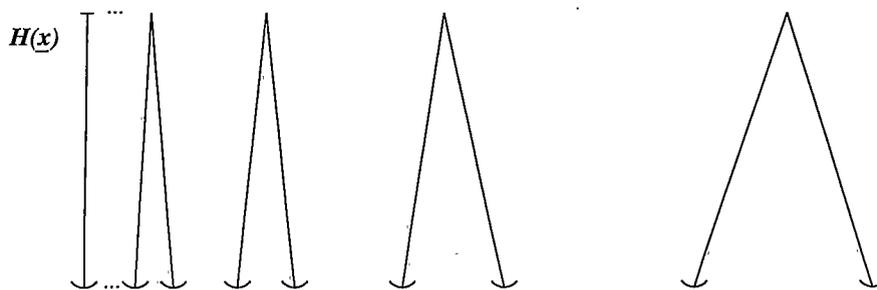Figure 20. A Neighborhood of a Distinguished Point

Figure 21. A Neighborhood of an End Distinguished Point

Now let X and Y be generalized $\sin \frac{1}{x}$ continua and $G : X \to Y$ be a homeomorphism.

Lemma 6.1 *The homeomorphism G must take distinguished points of X to distinguished points of Y.*

Proof: Since G would naturally take open neighborhoods of undistinguished points to open neighborhoods of undistinguished points, distinguished points must map to distinguished points. *Q. E. D.*

Next suppose that γ is a particular number chosen from $(\frac{1}{4}, \frac{1}{2})$ and suppose that countably many more numbers are chosen from $[\frac{1}{4}, \frac{1}{2}]$ (we allow repeating of the same number). Let us label the number γ and the other numbers as $\beta_0, \beta_1, \beta_2, \ldots$, in no particular order, but require that γ be repeated infinitely many times in the sequence, that is $\gamma = \beta_i$ for infinitely many i.

Now g_γ has a critical point $c = \frac{\gamma}{2-2\gamma}$ so $g_\gamma(c) = \frac{3}{2} - 2\gamma$. For ease of notation, let $k = g_\gamma(c) = (\frac{3}{2} - 2\gamma) \in (\frac{1}{2}, 1)$. Then, similar to what we have seen before, $\underline{k} = (k, k, k, \ldots)$ is the point on the limit bar of $\varprojlim\{g_{\beta_i}\}$ that is the limit of points of the form $(k, k, k, \ldots, k, c, \ldots)$ where of course the number of k's before the c varies. The points of the form $(k, k, k, \ldots, k, c, \ldots)$ are on the ray part of $\varprojlim\{g_{\beta_i}\}$.

Lemma 6.2 *The point $\underline{k}$ is a distinguished point of $\varprojlim\{g_{\beta_i}\}$.*

Proof: For any $\epsilon > 0$, consider the (open) ϵ-ball about $\underline{k}$. Since $\epsilon > \frac{1}{2^n}$ for some $n \in \mathbb{N}$, consider the point $\underline{k}^* = (k, k, k, \ldots, k, c, \ldots)$ where c is in the mth position, and $m \geq 2^{n+1}$. Then the distance from $\underline{k}^*$ to $\underline{k}$ is less than

$$\sum_{i=0}^{n} \frac{k-k}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = 0 + \frac{\frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1}{2^{n+1}}\left(\frac{2}{1}\right) = \frac{1}{2^n} < \epsilon$$

So $\underline{k}^*$ is in the ϵ-ball about $\underline{k}$. But c is a critical point of g_γ, so for x in a small enough open neighborhood of $c \in [0,1]$, $g_\gamma \leq k$, thus points along the line segment $\underline{k}^*$ is on, inside this ϵ-ball are of the form $(p, p, p, \ldots, p, q, \ldots)$ where $g_\gamma(q) = p$, q is in the mth position and $p \leq k$. Therefore no point near k^* on the line segment $\underline{k}^*$ is on goes above $\underline{k}$ (that is, has a first coordinate bigger than k), and so the ϵ-ball about $\underline{k}$ cannot be homeomorphic to $(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}) \times (-1, 1)$ which means $\underline{k}$ is a distinguished point. *Q. E. D.*

We wish to mention here that if we only have finitely many other numbers, say $m-1$ of them so that with γ there is a total of m numbers, labeled $\beta_0, \beta_1, \beta_2, \ldots, \beta_{m-1}$, again in no particular order, and we define $h = g_{\beta_0} \circ g_{\beta_1} \circ g_{\beta_2} \circ \cdots \circ g_{\beta_{m-1}}$ and also define $\beta_{jm} = \beta_0$, $\beta_{jm+1} = \beta_1$, $\beta_{jm+2} = \beta_2, \ldots, \beta_{(j+1)m-1} = \beta_{m-1}$ for $j = 1, 2, 3, \ldots$, then of course $\varprojlim h \simeq \varprojlim\{g_{\beta_i}\}$ by Corollary 2.2. Hence $\underline{k}$, defined similarly to the one above, is a distinguished point of $\varprojlim h$.

Now we will generalize our earlier idea. Let $m, n \in \mathbb{N}$ so that $m \neq n$ and choose m distinct numbers $\beta_0, \beta_1, \beta_2, \ldots, \beta_{m-1}$ from $(\frac{1}{4}, \frac{1}{2})$. Also, choose n numbers $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ from $(\frac{1}{4}, \frac{1}{2})$, where $\gamma_i \neq \gamma_j$ if $i \neq j$ but we will allow some of the γ's to be equal to some of the β's. Suppose $h = g_{\beta_0} \circ g_{\beta_1} \circ g_{\beta_2} \circ \cdots \circ g_{\beta_{m-1}}$ and $h^* = g_{\gamma_0} \circ g_{\gamma_1} \circ g_{\gamma_2} \circ \cdots \circ g_{\gamma_{n-1}}$.

Lemma 6.3 *The space $\varprojlim h$ is not homeomorphic to $\varprojlim h^*$.*

Proof: First, we notice that both inverse limit spaces satisfy the conditions of being unimodal generalized $\sin\frac{1}{x}$ continua. Assume, without loss of generality, that $m > n$ and further suppose, by way of contradiction, that $\varprojlim h$ is homeomorphic to $\varprojlim h^*$ and that G is such a homeomorphism between these two spaces. For each i, g_{β_i} has it's own critical point, $c_{\beta_i} = \frac{\beta_i}{2-2\beta_i}$, so let $k_{\beta_i} = g_{\beta_i}(c_{\beta_i})$. Let $\underline{k}_i$ be the point on the limit bar of $\varprojlim h$ that is the limit of points of the form $(k_{\beta_i}, k_{\beta_i}, \ldots, k_{\beta_i}, c_{\beta_i}, \ldots)$. Similarly, for each j, g_{γ_j} has a critical point $c_{\gamma_j} = \frac{\gamma_j}{2-2\gamma_j}$ with $g_{\gamma_j}(c_{\gamma_j}) = k^*_{\gamma_j}$ and let $\underline{k}^*_j$ be the point on the limit bar to which the points $(k^*_{\gamma_j}, k^*_{\gamma_j}, \ldots, k^*_{\gamma_j}, c_{\gamma_j}, \ldots)$ converge. Now since G is a homeomorphism, it must take end points of the limit bar of $\varprojlim h$ to end points of the limit bar of $\varprojlim h^*$ as well as the other distinguished points $\underline{k}_i$ to the other distinguished points $\underline{k}^*_j$, but since $m > n$ and G is one to one, there is at least one $\underline{k}_i$ that cannot map onto a $\underline{k}^*_j$, thus there is no homeomorphism from $\varprojlim h$ to $\varprojlim h^*$. *Q. E. D.*

Next consider if we used m numbers from $(\frac{1}{4}, \frac{1}{2})$ for both h and h^*, for instance, consider the distinct numbers $\beta_0, \beta_1, \beta_2, \ldots, \beta_{m-1}$ and let $\rho_0, \rho_1, \rho_2, \ldots, \rho_{m-1}$ be another m numbers from $(\frac{1}{4}, \frac{1}{2})$ so that $\rho_i \neq \rho_j$ for $i \neq j$. Suppose, as before, that $h = g_{\beta_0} \circ g_{\beta_1} \circ g_{\beta_2} \circ \cdots \circ g_{\beta_{m-1}}$ but now let $h^* = g_{\rho_0} \circ g_{\rho_1} \circ g_{\rho_2} \circ \cdots \circ g_{\rho_{m-1}}$. We are interested in what conditions are needed for $\varprojlim h$ to be homeomorphic to $\varprojlim h^*$. It turns out that if the order of the relative sizes of the β's and the ρ's are the same up to a cycle of size m they are homeomorphic. The next lemma clarifies this.

First define $\Pi|_{\{0,1,2,\ldots,m-1\}} : \{0, 1, 2, \ldots, m-1\} \to \{0, 1, 2, \ldots, m-1\}$ to be a reordering of the numbers zero through $m-1$ so that $\beta_{\Pi(0)} < \beta_{\Pi(1)} < \beta_{\Pi(2)} < \cdots < \beta_{\Pi(m-1)}$. Then let $\sigma : \mathbb{N} \to \{0, 1, 2, \ldots, m-1\}$ be defined by $\sigma(i) = (i+1)(\bmod m)$.

Lemma 6.4 *If there exists a* $k \in \mathbb{N} \cup \{0\}$ *such that* $\rho_{\Pi(\sigma^k(0))} < \rho_{\Pi(\sigma^k(1))} < \rho_{\Pi(\sigma^k(2))} < \cdots < \rho_{\Pi(\sigma^k(m-1))}$ *then* $\varprojlim h$ *is homeomorphic to* $\varprojlim h^*$.

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Proof: First let us create a sequence of points $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ so that $\alpha_{\Pi(i)} = \rho_{\Pi(\sigma^k(i))}$ for $i = 0, 1, 2, \ldots, m-1$. Thus it is then also true that $\alpha_{\Pi(0)} < \alpha_{\Pi(1)} < \alpha_{\Pi(2)} < \cdots < \alpha_{\Pi(m-1)}$. Define $h^1 : [0,1] \to [0,1]$ by $h^1 = g_{\alpha_0} \circ g_{\alpha_1} \circ g_{\alpha_2} \circ \cdots g_{\alpha_{m-1}}$ Label the points of interest as in the diagram below, for $i = 0, 1, 2, \ldots, m-1$. Thus for $g_{\beta_{\Pi(i)}}$ the smallest nonzero critical number is a_i, so $g_{\beta_{\Pi(i)}}(a_i) = 1$, followed by critical number b_i so that $g_{\beta_{\Pi(i)}}(b_i) = \frac{1}{2}$, then c_i where $g_{\beta_{\Pi(i)}}(c_i) \in (\frac{1}{2}, 1)$ and then critical numbers $\frac{1}{2}$ and 1. The critical numbers of $g_{\alpha_{\Pi(i)}}$ are labeled the same, except a superscript $*$ being added to the lettered critical numbers to differentiate them.
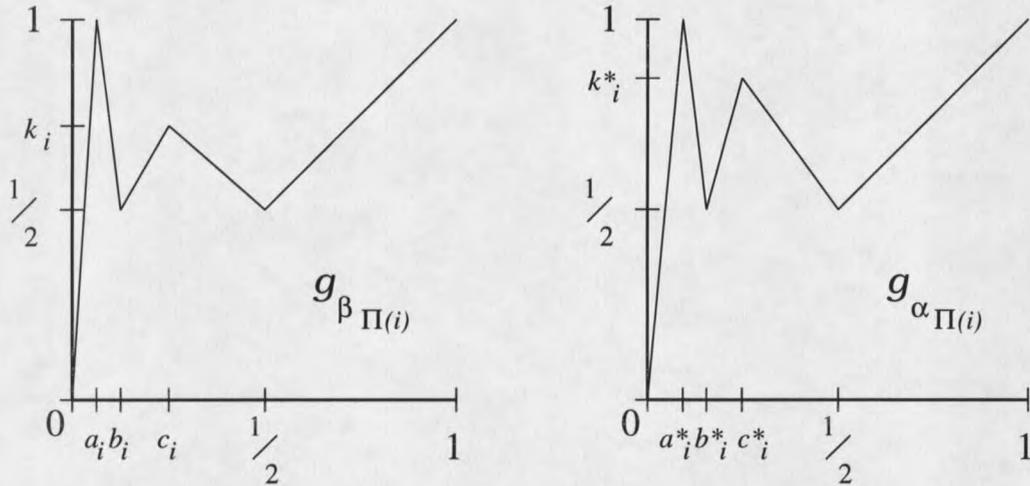

Figure 22. The Functions $g_{\beta_{\Pi(i)}}$ and $g_{\alpha_{\Pi(i)}}$

Since the relative size of the parameter drawn from $(\frac{1}{4}, \frac{1}{2})$ is inversely proportional to the size of k_i or k_i^*, it is the case that $k_{m-1} < k_{m-2} < \cdots < k_1 < k_0$ and $k_{m-1}^* < k_{m-2}^* < \cdots < k_1^* < k_0^*$.

Let $I = [0,1]$ and define $\varphi_0 : I \to I$ so that $\varphi_0(0) = 0$, $\varphi_0(\frac{1}{2}) = \frac{1}{2}$, $\varphi_0(k_0) = k_0^*$, $\varphi_0(1) = 1$ and φ_0 is a straight line connecting these points everywhere else in I. Since k_0, $k_0^* \in (\frac{1}{2}, 1)$ for all these cases, φ_0 is strictly increasing and therefore a homeomorphism. For ease of notation, define subintervals of I in the following way: For any fixed i and thus considering $g_{\beta_{\Pi(i)}}$, let $I_{i,0} = [0, a_i]$, $I_{i,1} = [a_i, b_i]$, $I_{i,2} = [b_i, c_i]$,

$I_{i,3} = [c_i, \frac{1}{2}]$, $I_{i,4} = [\frac{1}{2}, k_i]$ and $I_{i,5} = [k_i, 1]$. Similarly, considering $g_{\alpha_{\Pi(i)}}$ define $J_{i,0} = [0, a_i^*]$, $J_{i,1} = [a_i^*, b_i^*]$, $J_{i,2} = [b_i^*, c_i^*]$, $J_{i,3} = [c_i^*, \frac{1}{2}]$, $J_{i,4} = [\frac{1}{2}, k_i^*]$ and $J_{i,5} = [k_i^*, 1]$. Now for any $i \in \mathbb{N} \cup \{0\}$, we define $\varphi_{i+1} : I \to I$ by letting $\varphi_{i+1}|_{I_{i,0}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,0}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$ and $\varphi_{i+1}|_{I_{i,1}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,1}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$, then $\varphi_{i+1}|_{I_{i,2}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,2}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$ and $\varphi_{i+1}|_{I_{i,3}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,3}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$ with $\varphi_{i+1}|_{I_{i,4}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,4}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$ and lastly $\varphi_{i+1}|_{I_{i,5}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,5}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}$.

Next consider the diagram below:

We will soon show that φ_i is a homeomorphism for all i, and thus, from the way φ_i was created, we see the diagram commutes.

We will show φ_i is a homeomorphism for all i by induction. We already have seen that φ_0 is a homeomorphism. So now assume that for some fixed $i \in \mathbb{N} \cup \{0\}$, φ_i is a homeomorphism, we wish to show φ_{i+1} is therefore a homeomorphism. Since I is compact and Hausdorff, we need only show $\varphi_{i+1} : I \to I$ is a continuous bijective function (See Lemma 2.1). For any $n = 0, 1, 2, 3, 4, 5$, we see $\varphi_{i+1}|_{I_{i,n}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}} = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}|_{I_{i,n}}$ is the composite of homeomorphisms and hence is continuous. Also we notice φ_{i+1} is well defined for each of the (overlapping) endpoints of these subintervals and therefore φ_{i+1} must be continuous.

Choose a point y in the range of φ_{i+1}. Then y must be in at least one subinterval of I, say $J_{i,n}$. Let $x = [g_{\beta_{\Pi(i)}}|_{I_{i,n}}]^{-1} \circ \varphi_i^{-1} \circ g_{\alpha_{\Pi(i)}}(y)$ thus $\varphi_{i+1}(x) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}(x) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}} \circ [g_{\beta_{\Pi(i)}}|_{I_{i,n}}]^{-1} \circ \varphi_i^{-1} \circ g_{\alpha_{\Pi(i)}}(y) = y$ as desired, so φ_{i+1} is onto.

To show φ_{i+1} is one-to-one, suppose $\varphi_{i+1}(x) = \varphi_{i+1}(z)$. So $\varphi_{i+1}(x) = \varphi_{i+1}(z) \in J_{i,n}$ for some particular n, but by the definition of φ_{i+1}, it must be that $x, z \in I_{i,n}$ thus equivalently we get $\varphi_{i+1}(x) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}(x) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}|_{I_{i,n}}(x) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}|_{I_{i,n}}(z) = [g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1} \circ \varphi_i \circ g_{\beta_{\Pi(i)}}(z) = \varphi_{i+1}(z)$, but $[g_{\alpha_{\Pi(i)}}|_{J_{i,n}}]^{-1}$, φ_i and $g_{\beta_{\Pi(i)}}|_{I_{i,n}}$ are homeomorphisms, hence $x = z$

Therefore φ_{i+1} is a homeomorphism and by mathematical induction, this proves φ_i is a homeomorphism for all i.

Define $H : \varprojlim h \to \varprojlim h^1$ by $H(x_0, x_1, x_2, \dots) = (\varphi_0(x_0), \varphi_1(x_1), \varphi_2(x_2), \dots)$. Then H inherits the properties of being a homeomorphism from the coordinate functions, hence $\varprojlim h$ is homeomorphic to $\varprojlim h^1$.

Now to show that $\varprojlim h^1$ is homeomorphic to $\varprojlim h^*$, we look at both as inverse limit spaces of the g_β functions and realize they are the same except one is missing the first finite number of terms of the other, so we apply Corollary 2.1. Therefore $\varprojlim h$ is homeomorphic to $\varprojlim h^*$. Q. E. D.

Let L be the subset of points of product space $\Pi_{i=0}^\infty \mathbb{R}$ of the form $(x, x, x, \dots)$ where $x \in [\frac{1}{2}, 1]$. One can see that L is homeomorphic to the interval $[\frac{1}{2}, 1]$ by defining the homeomorphism $H : [\frac{1}{2}, 1] \to L$ by $H(x) = (x, x, x, \dots)$. The proof that H is a homeomorphism is similar to a portion of the proof of Lemma 3.3, and is left to the interested reader. Next let C be any countable subset of L. We will now create an inverse limit space, using the g_β functions, that has every point of C and every accumulation point of C along with $\frac{1}{2}$ and $\underline{1}$ as a distinguished point.

Now for $i = 0, 1, 2, \dots$, let $\underline{x}_i = (x_i, x_i, x_i, \dots) \in C$, then there exists a $\beta_i \in [\frac{1}{4}, \frac{1}{2}]$ so that g_{β_i} has a critical number $c_i = \frac{\beta_i}{2 - 2\beta_i}$ and $g_{\beta_i}(c_i) = x_i$. If C is of finite cardinality n, let $\gamma_i = \beta_{i(\mathrm{mod} n)}$ for $i \in \mathbb{N}$. If C is infinitely countable, consider the sequence $\{a_j\}_{j=0}^\infty = (0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, \dots)$. Define γ_j by $\gamma_j = \beta_{a_j}$ for $j \in \mathbb{N} \cup \{0\}$.

Lemma 6.5 *If the point $\underline{p} = (p, p, p, \dots) \in L$ is an accumulation point of distinguished points, $\underline{p}$ is also a distinguished point.*

Proof: If $\underline{p}$ were an undistinguished point, then every point in an open neighborhood of $\underline{p}$ would also have to be an undistinguished point, but $\underline{p}$ is an accumulation point of distinguished points so that cannot happen. *Q. E. D.*

Theorem 6.6 *The points of C along with the set of accumulation points of C and $\{\frac{1}{2}, \underline{1}\}$ are the only distinguished points of $\varprojlim\{g_{\gamma_j}\}$.*

Proof: Clearly since $\frac{1}{2}$ and $\underline{1}$ are end points of the limit bar, they cannot be mapped by a homeomorphism to a point in an open segment, so they are distinguished points.

Next choose any $\underline{k}_i \in C$. Then there is critical number $c_i \in I$ of g_{γ_j} so that $g_{\gamma_j}(c_i) = k_i$. Hence by Lemma 6.2, $\underline{k}_i$ is a distinguished point of $\varprojlim\{g_{\gamma_j}\}$. Then from Lemma 6.5, every accumulation point of distinguished points is also a distinguished point.

Now to show they are the only distinguished points of $\varprojlim\{g_{\gamma_j}\}$. Suppose that $\underline{p} = (p, p, p, \dots) \in L \setminus \overline{C} \cup \{\frac{1}{2}, \underline{1}\})$. Recall also that $\varprojlim\{g_{\gamma_j}\}$ is a unimodal generalized $\sin\frac{1}{x}$ continua, so it consists of a ray winding down onto a limit bar. There must be an (open) ϵ-ball about $\underline{p}$ so that the ϵ-ball does not intersect C. But this means that no element in that ϵ-ball about $\underline{p}$ has a first coordinate that is a local extreme value of some g_{γ_j}. So every open line segment in ϵ-ball about $\underline{p}$ has points with first element both above and below p, so they can contribute elements to a sequence that converges to points on L in the ϵ-ball about $\underline{p}$ that are above or below $\underline{p}$. Hence $\underline{p}$ is an undistinguished point of $\varprojlim\{g_{\gamma_j}\}$. Therefore the only distinguished points of $\varprojlim\{g_{\gamma_j}\}$ are the points of the closure of C along with $\frac{1}{2}$ and $\underline{1}$. *Q. E. D.*

Now let K be any closed subset of $\left[\frac{1}{2}, 1\right]$ that contains both $\frac{1}{2}$ and 1. Let $\underline{K} = \{(x, x, x, \dots) : x \in K\} \subset L$. Then there exists a finite or countably infinite set $C \subset L$ so that the closure of C is $\underline{K}$ and also, by the above theorem, a corresponding unimodal generalized $\sin\frac{1}{x}$ continua, call it X_K, with the distinguished points of X_K on L being precisely the points of $\underline{K}$.

We will now use the notation of a homeomorphic pair. If A and A' are sets in a topological space with B and B' subsets of A and A' respectively, then (A, B) is homeomorphic to (A', B') if and only if there is a homeomorphism from A to A' that takes B homeomorphically onto B'.

Theorem 6.7 *Let K and K' be two closed subsets of $\left[\frac{1}{2}, 1\right]$, each containing $\frac{1}{2}$ and 1. If X_K and $X_{K'}$ are homeomorphic then $(\left[\frac{1}{2}, 1\right], K)$ and $(\left[\frac{1}{2}, 1\right], K')$ are homeomorphic pairs.*

Proof: Suppose the contradiction, that X_K is homeomorphic to $X_{K'}$ and $F : X_K \to X_{K'}$ is such a homeomorphism. Then F maps the limit bar of X_K to the limit bar of $X_{K'}$ and maps $\underline{K}$ to $\underline{K'}$. But since K is homeomorphic to $\underline{K}$, K' to $\underline{K'}$ and $\left[\frac{1}{2}, 1\right]$ to both the limit bars of X_K and $X_{K'}$, we would get $(\left[\frac{1}{2}, 1\right], K)$ homeomorphic to $(\left[\frac{1}{2}, 1\right], K')$, a clear contradiction. $Q.\ E.\ D.$

Since there are uncountably many pairwise nonhomeomorphic closed subsets, K, of $\left[\frac{1}{2}, 1\right]$ containing both $\frac{1}{2}$ and 1, using the contrapositive of the above theorem, we get that the set of unimodal generalized $\sin\frac{1}{x}$ continua, X_K, is an uncountable collection.

CHAPTER 7

SOME *MW* CONTINUA AND CONJECTURES

In the previous chapter we looked at a parameterized function g_β, which was the composition of parameterized functions u_β, v_β, z_β and the function f. After looking at the sketches of the graphs of g_β for several values of parameter β chosen from $[\frac{1}{4}, \frac{1}{2}]$, one may ask "What does g_β look like for other values of β?" So next we will consider g_β for $\beta \in [\frac{1}{8}, \frac{1}{4})$. Figure 23 has some examples of sketches of the graphs of g_β with $\beta = \frac{1}{5}, \frac{3}{16}, \frac{1}{7}, \frac{1}{8}$. We see that for $\beta \in (\frac{1}{8}, \frac{1}{4})$, the critical point that has height in $(\frac{1}{2}, 1)$ is a local minimum, not a local maximum as was the case for $\beta \in (\frac{1}{4}, \frac{1}{2})$. Also, we notice that g_β satisfies the conditions to be used as a bonding map to form a unimodal generalized $\sin \frac{1}{x}$ continuum.

In this chapter we state several conjectures, which we believe to be true, with some of the proofs being very similar to those we have already established, but we leave them here as conjectures, to be proven in future work.

If we let $\beta \in (\frac{1}{4}, \frac{1}{2})$ and $\gamma \in (\frac{1}{8}, \frac{1}{4})$ and let $h = g_\beta \circ g_\gamma$, which has a local maximum and a local minimum with heights in $(\frac{1}{2}, 1)$ as well as local maximums and a local minimums with heights of $\frac{1}{2}$ and 1 and is the identity function on $[\frac{1}{2}, 1]$, then the resulting inverse limit space $\varprojlim h$ is a unimodal generalized $\sin \frac{1}{x}$ continuum.

Conjecture 7.1 *The inverse limit space* $\varprojlim h$ *is not homeomorphic to any element of either of the uncountable collections in Chapter 5 or Chapter 6.*

Next, let us further suppose that if k_β is the local maximum of g_β which is in $(\frac{1}{2}, 1)$ and k_γ is the local minimum of g_γ that is in $(\frac{1}{2}, 1)$, then β and γ were chosen

so that $k_\beta < k_\gamma$. Now choose $\alpha \in (\frac{1}{4}, \frac{1}{2})$ and $\rho \in (\frac{1}{8}, \frac{1}{4})$ so that $k_\alpha > k_\rho$, where k_α is the local maximum of g_α which is in $(\frac{1}{2}, 1)$ and k_ρ is the local minimum of g_ρ which is in $(\frac{1}{2}, 1)$. Let $h^* = g_\alpha \circ g_\rho$. Also choose $\eta \in (\frac{1}{4}, \frac{1}{2})$ and $\tau \in (\frac{1}{8}, \frac{1}{4})$ such that $k_\eta = k_\tau$, with k_η and k_τ defined in conjunction with g_η and g_τ, similar to k_β and k_γ above. Define $h^1 = g_\eta \circ g_\tau$.
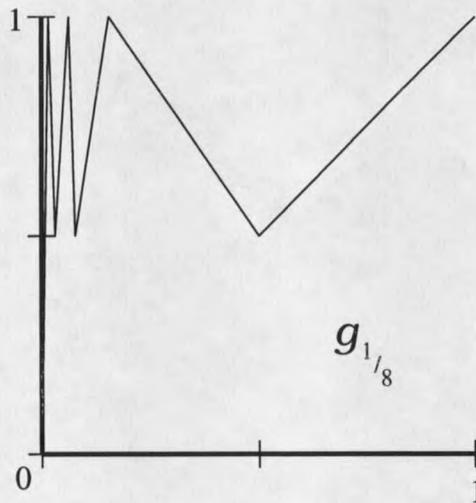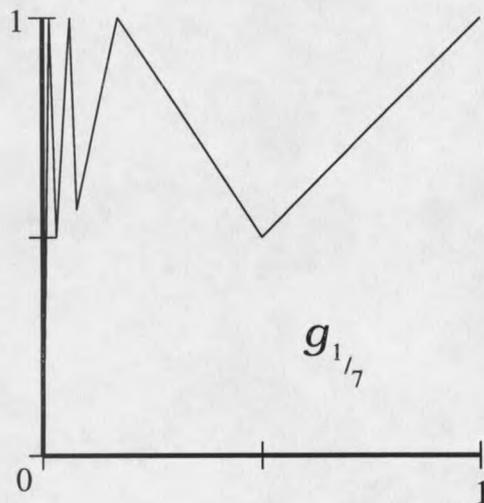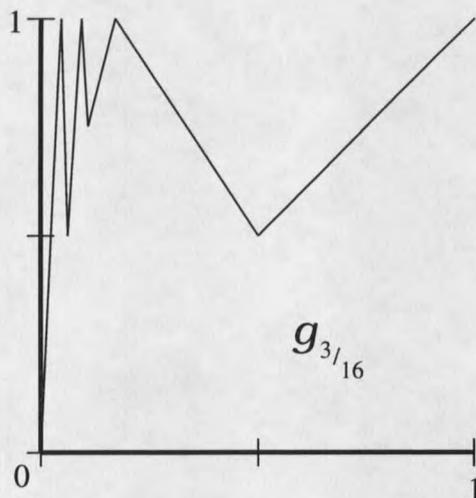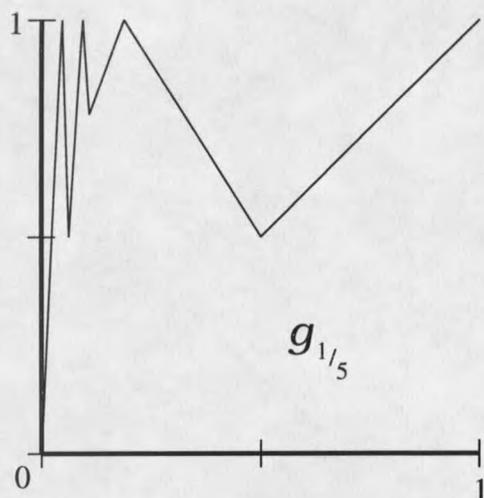

Figure 23. The Function g_β for $\beta \in [\frac{1}{8}, \frac{1}{4})$

Conjecture 7.2 *The continuum* $\varprojlim h$ *is not homeomorphic to* $\varprojlim h^*$ *nor* $\varprojlim h^1$. *Similarly,* $\varprojlim h^*$ *is not homeomorphic to* $\varprojlim h^1$.

See Figure 24 to compare visualizations of these three continua. The sketches in $\mathbb{R}^2$ represented by Figure 24 are each homeomorphic to the indicated continuum.
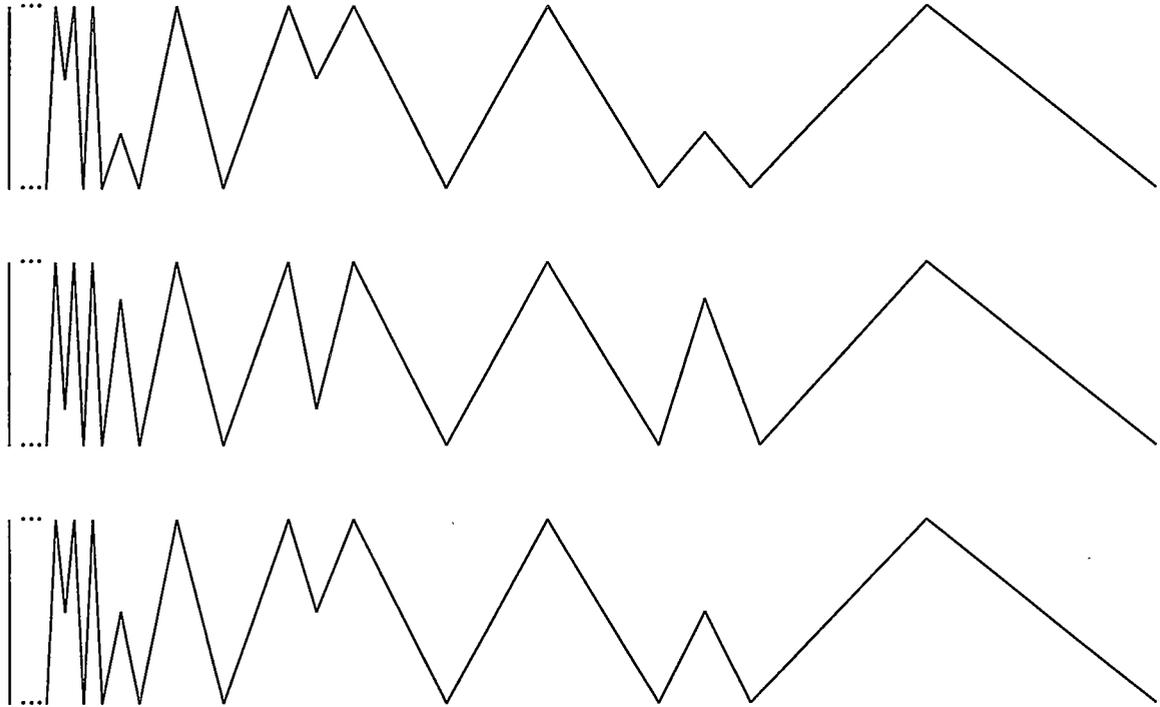

Figure 24. Top: $\varprojlim h$, center: $\varprojlim h^*$, bottom: $\varprojlim h^1$

Using ideas from the last chapter, classify distinguished points of inverse limit spaces constructed from bonding maps g_β where $\beta \in [\frac{1}{8}, \frac{1}{2}]$ into two classes: Class A, which is the set of distinguished points that are the limit points of the points $(k, k, k, \ldots, k, c, \ldots)$ where $(c, g_\alpha(c)) = (c, k)$ is a local maximum of g_α with $\alpha \in [\frac{1}{4}, \frac{1}{2}]$, with the bonding map g_α being repeated infinitely many times and the limit points of these distinguished points unioned with $\{\frac{1}{2}, 1\}$, and; Class B, which is the set of distinguished points that are the limit points of the points $(k, k, k, \ldots, k, c, \ldots)$

where $(c, g_\gamma(c)) = (c, k)$ with $\gamma \in [\frac{1}{8}, \frac{1}{4}]$ with the bonding map g_γ being repeated infinitely many times and the limit points of these distinguished points unioned with $\{\frac{1}{2}, 1\}$. Notice that a point may have more than one classification. Let K and K^* be any two (not necessarily distinct) closed subsets of $[\frac{1}{2}, 1]$ containing $\frac{1}{2}$ and 1. Let $\underline{K} = \{(k, k, k, \dots) \in L : k \in K\}$ with a similar definition for $\underline{K}^*$. Then, using techniques like those in Chapter 5, we can create a unimodal generalized $\sin \frac{1}{x}$ continuum, call it X_{K,K^*}, using the g_β function so that $\underline{K}$ is the set of Class A distinguished points of X_{K,K^*} and $\underline{K}^*$ is the set of Class B distinguished points of X_{K,K^*}.

Let us also extend our definition of a homeomorphic pair. Let A, B and C be sets so that $B \subset A$ and $C \subset A$. Similarly define A', B' and C' as sets with B' and C' subsets of A'. We will say $(A, \{B, C\})$ is homeomorphic to $(A'\{B', C'\})$ if and only if there is a homeomorphism from A onto A' that takes B homeomorphically onto B' and C homeomorphically onto C'.

Now using K_1, K_1^*, K_2, K_2^* to be sets of the form K and K^* above, and so on, we present our final theorem.

Theorem 7.1 *If X_{K_1, K_1^*} is homeomorphic to X_{K_2, K_2^*} then $([\frac{1}{2}, 1], \{K_1, K_1^*\})$ is homeomorphic with $([\frac{1}{2}, 1], \{K_2, K_2^*\})$ or $([\frac{1}{2}, 1], \{K_2^*, K_2\})$.*

Proof: This theorem is a generalization of Theorem 6.7, and the proof is analogous to that proof. *Q. E. D.*

Thus the collection of all possible nonhomeomorphic unimodal continua X_{K,K^*} is uncountable, and contains the collection X_K we created in Chapter 6.

In conclusion, in Chapter 5 we created an uncountable collection of nonhomeomorphic unimodal generalized $\sin \frac{1}{x}$ continua by varying the proximity of critical

points of height $\frac{3}{4}$ relative to the limit bar, and in Chapter 6 we developed an uncountable collection of nonhomeomorphic unimodal generalized $\sin\frac{1}{x}$ continua by varying the heights of the distinguished points, with Chapter 7 just adding the the possibility of distinguished points coming from local maximums and local minimums on the ray. Future work will involve combining these techniques to create a collection that would include each of those collections as subcollections.

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