Uncountable collections of unimodal continua

by Paul James Johanson

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Inverse limit spaces occur often in dynamical systems, in particular, those with unimodal bonding maps. Barge and Diamond showed uncountably many indecomposable unimodal inverse limit spaces exist. We show that uncountably many hereditarily decomposable unimodal inverse limit spaces exist by actually creating two different uncountable collections of hereditarily decomposable unimodal inverse limit spaces. Finally we consider how to expand upon these and combine them to create even more unimodal inverse limit spaces.

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Paul James Johanson

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Jim Johanson

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Carl Johanson

Mary Johanson

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ABSTRACT

Inverse limit spaces occur often in dynamical systems, in particular, those with unimodal bonding maps. Barge and Diamond showed uncountably many indecomposable unimodal inverse limit spaces exist. We show that uncountably many hereditarily decomposable unimodal inverse limit spaces exist by actually creating two different uncountable collections of hereditarily decomposable unimodal inverse limit spaces. Finally we consider how to expand upon these and combine them to create even more unimodal inverse limit spaces.

CHAPTER 1

INTRODUCTION

We begin by defining what we mean by the term *inverse limit space*. This definition is general, but for most of this paper we will make a few restrictions that will make it easier to visualize and think about the inverse limit spaces we will be dealing with. For our definition, let $i \in \{0, 1, 2, 3, \dots\}$ and let $\{X_i\}_{i=1}^{\infty}$ be a sequence compact metric spaces each with metric d_i such that for all $x, y \in X_i$, $d_i(x, y) \leq 1$. Also let $\{f_i\}_{i=0}^{\infty}$ be a sequence of continuous functions so that $f_i : X_{i+1} \to X_i$ (these functions will be referred to as *bonding maps*). We define the *inverse limit space* of $\{f_i\}$, denoted $\varprojlim\{f_i\}$, to be the subset of $\prod_{i=0}^{\infty} X_i$ given by

$$\varprojlim\{f_i\} = \{\underline{x} = (x_0, x_1, x_2, x_3, \dots) : x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i\}$$

with the metric defined to be $\underline{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$. Another common notation used in some manuscripts is (X_i, f_i). It can be shown that $\varprojlim\{f_i\}$ is a compact metric space, and connected if each X_i is connected. ([14], pg. 163). If we are repeating a single bonding map, that is $f_i = f$ for all i, we will use the notation $\varprojlim f$ rather than $\varprojlim\{f_i\}$. We notice that the topology from our metric on the inverse limit space is equivalent to the subspace topology inherited from the product topology on $\prod_{i=0}^{\infty} X_i$. We define a *continuum* to be any compact, connected, metric space.

For this thesis, we are mostly interested in the case where $X_i = [a_i, b_i]$, a compact interval in the real numbers, $\mathbb{R}$, with the metric $d_i(x, y) = \frac{|x-y|}{b_i - a_i}$ for all $i \in \{0, 1, 2, \dots\}$. We say that the continuous bonding maps $f_i : X_{i+1} \to X_i$ are *unimodal* if there is a critical value $c_i \in X_i$, dividing X_i into two subintervals, $[a_i, c_i]$ and $[c_i, b_i]$, so that f_i is monotone each subinterval. We allow $c_i = a_i$ or $c_i = b_i$, thus we will also say that f_i is unimodal if it monotone on all of X_i. In some of the proofs included in this thesis we may assume our bonding maps are strictly monotone on each subinterval, but we will not require strict monotonicity here due to a result of Morton Brown ([5], Theorem 3) stating that if we have functions f_i and g_i so that $f_i(x) - g_i(x) < \epsilon_i$ for all $x \in X_i$ and for all i, with ϵ_i converging to zero quickly enough, then the inverse limit spaces are homeomorphic. This, together with the fact that any monotone function can be approximated by a strictly monotone function, means we lose no generality in considering strictly monotone functions. We will refer to any space as a *unimodal continuum* if it can be realized as, or is homeomorphic to, the inverse limit space of unimodal bonding maps $f_i : X_{i+1} \to X_i$.

In 1954, C.E. Capel ([7]) used inverse limit spaces to prove theoerms concerning Alexander-Kolmogoroff cohomology theory. Inverse limit spaces of one-dimensional bonding maps are often attractors in dynamical systems. R.F. Williams ([16]) proved in 1967 that hyperbolic one-dimensional attractors are inverse limits of maps on branched one manifolds. The same year, W.S. Mahavier ([9]) looked at inverse limits on $[0, 1]$ with a single bonding map. Sam B. Nadler ([13],[14]) and Dorothy S. Marsh ([10]) both created inverse limit spaces that we will later refer to as generalized $\sin \frac{1}{x}$ spaces, but without using unimodal bonding maps.

Research of my advisor, Marcy Barge, along with coauthors Beverly Diamond and Karen Brucks ([1], [3]), as well as the research of others, has shown that often the inverse limit spaces that occur in dynamical systems are unimodal continua. For this reason we are interested in what these unimodal continua "look" like, what properties they have and how we can tell if a continuum is a unimodal continuum or not. Some examples of unimodal continua include the "topologist's sine curve" (Figure 1), the "double topologist's sine curve" with two limit bars, one on each end (Figure 2), as well as the Knaster continuum, also called the bucket handle continuum (Figure 3).

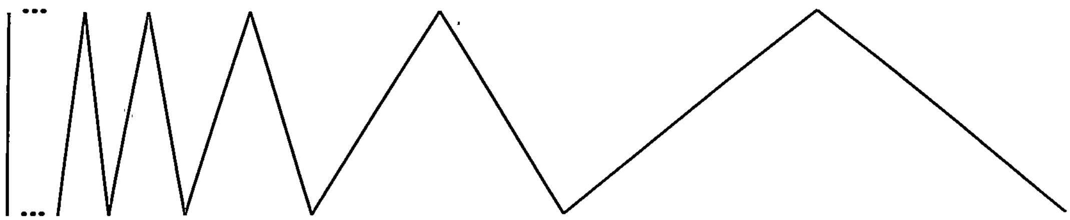


Figure 1: The Topologist's Sine Curve

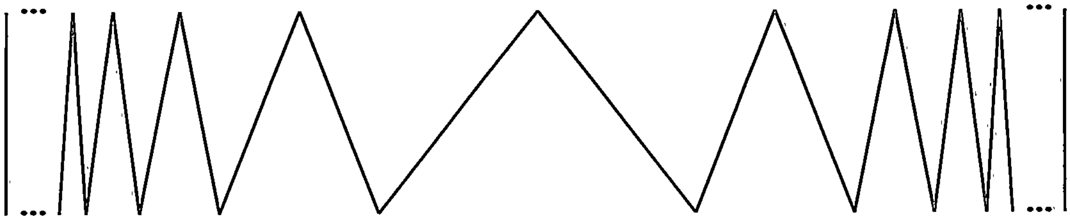


Figure 2: The Double Topologist's Sine Curve

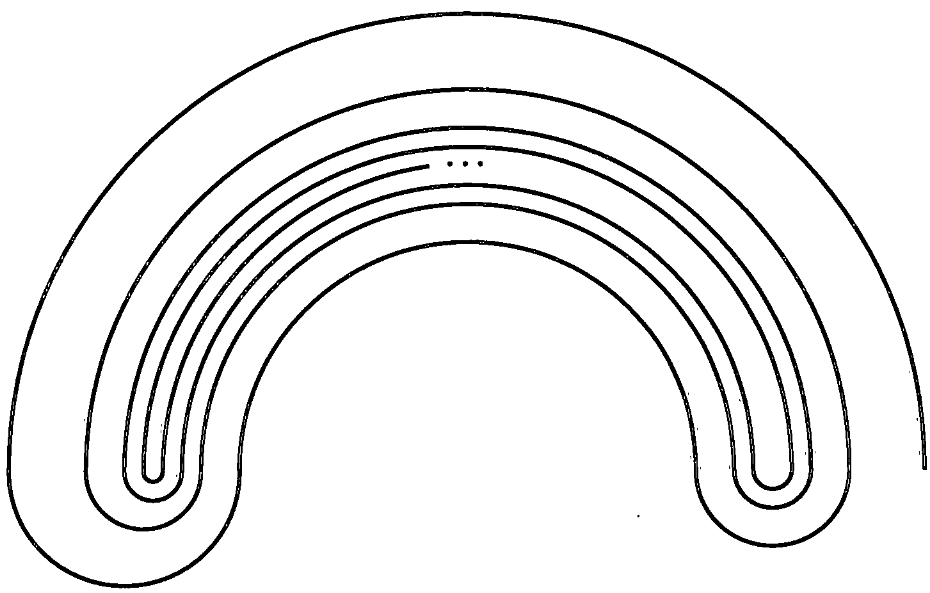


Figure 3: The Knaster Continuum

A continuum is said to be *decomposable* if it is the union of two proper subcontinua, otherwise it is an *indecomposable continuum.* We say that a continuum is *hereditarily indecomposable* provided each of its subcontinua is indecomposable and a continuum is *hereditarily decomposable* if each of its subcontinua is decomposable.

The Knaster bucket-handle continuum (Figure 3) is indecomposable (See [4]). One can see that if we pick a point along the ray path component, but not the end point, of the topologist's sine curve and include everything to one side of that point as well as the point itself in one set, while putting everything on the other side of the point including the point itself in another set, the result is two proper subcontinua whose union is the entire topologist's sine curve (Figure 1). We could repeat this process on each of these subcontinua, thus the curve is not only decomposable, but hereditarily decomposable. A similar procedure can be done on the double topologist's sine curve (Figure 2) by selecting a point on the middle (infinitely long) path component and proceeding as above. Hence the double topologist's sine curve is also hereditarily decomposable. Barge and Diamond [2] showed that there are uncountably many (non-homeomorphic) indecomposable unimodal continua. In this thesis we will show that there are uncountably many hereditarily decomposable unimodal continua by constructing a class of uncountably many non-homeomorphic hereditarily decomposable unimodal continua.

CHAPTER 2

SOME PROPERTIES OF INVERSE LIMIT SPACES AND UNIMODAL CONTINUA

Here we state a couple of elementary topology lemmas because they will be referred to in the following pages. We then state and prove some well known basic results concerning inverse limit spaces, followed by our first original theorem of this dissertation.

Lemma 2.1 *Let $h : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then h is a homeomorphism.*

Proof: See ([12] pg. 167) *Q. E. D.*

We recall here that all metric spaces are Hausdorff. Next denote the projection mapping by π_i, thus $\pi_i(\underline{x}) = \pi_i(x_0, x_1, x_2, \dots) = x_i$.

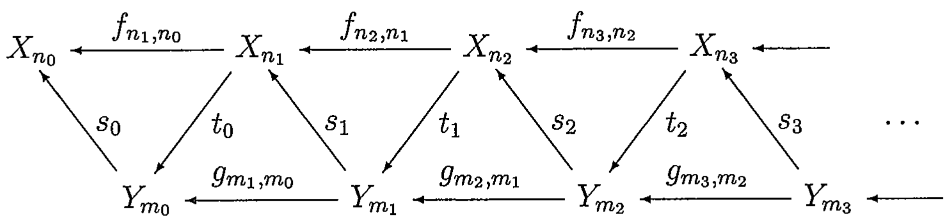
Lemma 2.2 *Let A be any space and let h be any function from A to $\underleftarrow{\lim}\{f_i\}$. Then h is continuous if and only if $\pi_i \circ h : A \to X_i$ is continuous for all $i \in \mathbb{N} \cup \{0\}$.*

Proof: The theorem in [12] (pg. 115) proves that $h : A \to \prod_{i=0}^{\infty} X_i$ is continuous if and only if $\pi_i \circ h : A \to X_i$ is continuous for all $i \in \{0, 1, 2, \dots\}$. Then by ([12] pg. 107) we know that restricting the range of a function does not affect the continuity of the function, thus the lemma holds. *Q. E. D.*

The next therem, along with it's corollaries, and the lemma that follows them are a compilation and contemporary restatement of results in [8], pages 215 through 220. The proofs are included here for completeness.

Define $f_{n,m} : X_n \to X_m$ by $f_{n,m} = f_m \circ f_{m+1} \circ \cdots \circ f_{n-2} \circ f_{n-1}$ for $n \geq m+1$ (and not defined otherwise). Let $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ be any two sequences in $\mathbb{N} \cup \{0\}$ such that $0 \leq n_0 < n_1 < n_2 < \cdots$ and $0 \leq m_0 < m_1 < m_2 < \cdots$. Also assume $f_i : X_{i+1} \to X_i$ and $g_i : Y_{i+1} \to Y_i$ are continuous bonding maps.

Theorem 2.3 *Suppose there exists the following commuting diagram with s_i and t_i continuous maps:*



Then $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$.

Proof: We warn the reader to pay close attention to the important but visually subtle difference in the subscripting between x_{n_k-1} and $x_{n_{k-1}}$ throughout this proof.

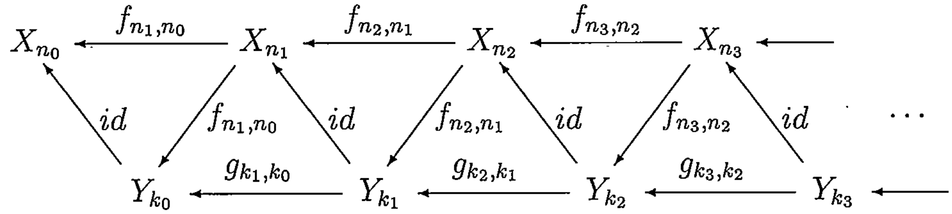
Let us define $H : \varprojlim\{f_i\} \to \varprojlim\{g_i\}$ by $H(\underline{x}) = \underline{y}$, where $\underline{x} = (x_0, x_1, x_2, \dots)$ and $\underline{y} = (y_0, y_1, y_2, \dots)$ with $y_{m_i} = t_i(x_{n_{i+1}})$. This does not specify all the coordinates of $\underline{y}$, but by specifying infinitely many of the coordinates, the point is uniquely determined, because to find any y_k if $k \neq m_i$ for some i, then pick the smallest i so that $m_i > k$ and simply define $y_k = g_{m_i,k}(y_{m_i})$. Now to show that $\underline{y} = H(\underline{x}) \in \varprojlim\{g_i\}$, if $k \neq m_i$ for any i, again choose the smallest i so that $m_i > k$. If $k = m_i - 1$ then $g_k(y_{k+1}) = g_{m_i-1}(y_{m_i}) = g_{m_i,m_i-1}(y_{m_i}) = g_{m_i,k}(y_{m_i}) = y_k$ from the definition of y_k above, otherwise we have $g_k(y_{k+1}) = g_k(g_{m_i,k+1}(y_{m_i})) = g_k \circ g_{k+1} \circ \cdots \circ g_{m_{i-1}}(y_{m_i}) = g_{m_i,k}(y_{m_i}) = y_k$ as desired. But for $k = m_i$, we use the fact that our diagram commutes to get $g_k(y_{k+1}) = g_{m_i}(y_{m_i+1}) = g_{m_i} \circ g_{m_i+1} \circ \cdots \circ g_{m_{i+1}-1}(y_{m_{i+1}}) = g_{m_{i+1},m_i} \circ t_{i+1}(x_{n_{i+2}}) = t_i \circ f_{n_{i+1},n_i}(x_{n_{i+2}}) = t_i(x_{n_{i+1}}) = y_{m_i} = y_k$, that is $g_k(y_{k+1}) = y_k$ in all cases, so $\underline{y} \in \varprojlim\{g_i\}$.

Next we notice that $t_i \circ \pi_{n_{i+1}} = \pi_{m_i} \circ H$ and since t_i and $\pi_{n_{i+1}}$ are continuous, $\pi_{m_i} \circ H$ is also. Now if $k \neq m_i$ for some i, $\pi_k \circ H(\underline{x}) = y_k = g_k \circ g_{k+1} \circ \cdots \circ g_{m_i - 1}(y_{m_i}) = g_k \circ g_{k+1} \circ \cdots \circ g_{m_i-1} \circ t_i(x_{n_{i+1}})$ which is continuous as the composite of continuous functions. In either case, $\pi_k \circ H$ is continuous for all k, therefore by Lemma 2.2, H is continuous.

Let us now define $G : \underleftarrow{\lim}\{g_i\} \to \underleftarrow{\lim}\{f_i\}$ by letting $G(\underline{y}) = \underline{x}$ where $x_{n_i} = s_i(y_{m_i})$ and $x_k = f_{n_i,k}(x_{n_i})$ for the smallest i so that $n_i > k$. By similar arguments to those given for H, we can see G is continuous also. Consider $G \circ H(\underline{x})$. For any fixed i, if $H(\underline{x}) = \underline{y}$, $y_{m_i} = t_i(x_{n_{i+1}})$. Then our G function takes y_{m_i} to $s_i(y_{m_i}) = x_{n_i}$ so $s_i \circ t_i(x_{n_{i+1}}) = x_{n_i}$. Thus $G \circ H(\underline{x})$ is specified by $x_{n_{i+1}} \mapsto x_{n_i}$. Again, even though not all coordinates have been specified, infinitely many have been, so $G \circ H(\underline{x})$ is completely determined, and since $G \circ H(\underline{x})$ has x_{n_i} as coordinates, which are the coordinates of $\underline{x}$, $G \circ H(\underline{x}) = \underline{x}$, so $G \circ H$ is the identity map. Similarly $H \circ G(\underline{y}) = \underline{y}$, hence G is the inverse of H. Therefore H is a homeomorphism. *Q. E. D.*

Corollary 2.1 *Suppose $\{f_i\}_{i=0}^{\infty}$ and $\{g_i\}_{i=0}^{\infty}$ are sequences of bonding maps so that $f_i : X_{i+1} \to X_i$ and $g_i : Y_{i+1} \to Y_i$ and there exists an $N \in \mathbb{N} \cup \{0\}$ and an $m \in \mathbb{Z}$ where $X_i = Y_{i+m}$ and $f_i = g_{i+m}$ for all $i \geq N$. Then $\underleftarrow{\lim}\{f_i\}$ is homeomorphic to $\underleftarrow{\lim}\{g_i\}$.*

Proof: Let $\{n_i\}_{i=0}^{\infty}$ be any sequence with $N \leq n_0 < n_1 < n_2 < \ldots$ and let $k_i = n_i + m$ for all $i = 0, 1, 2, \ldots$, thus $X_{n_i} = Y_{k_i}$ and $f_{n_i} = g_{k_i}$. Consider the diagram below:

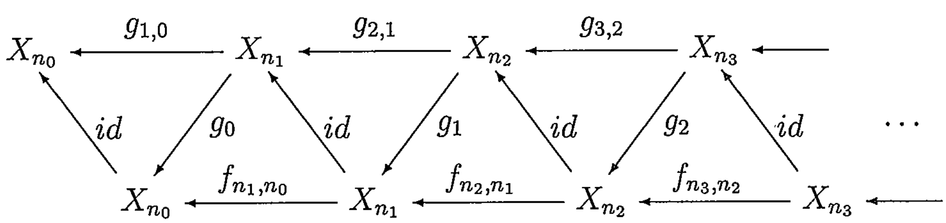


The reader can quickly check that this diagram commutes. So by Theorem 2.3, $\varprojlim\{f_i\}$ is homeomorphic to $\varprojlim\{g_i\}$. *Q. E. D.*

What this tells us is that if two sequences of bonding maps differ only in finitely many positions, they have homeomorphic inverse limit spaces. We can also ignore or remove the first finitely many bonding maps and get a homeomorphic inverse limit space to the original inverse limit space.

Corollary 2.2 *Suppose* $\{f_i\}_{i=0}^{\infty}$ *is a sequence of bonding maps so that* $f_i : X_{i+1} \to X_i$ *and* $\{n_i\}_{i=0}^{\infty}$ *is any sequence of integers with* $0 \leq n_0 < n_1 < n_2 < \cdots$. *If we let* $g_j = f_{n_j} \circ f_{n_j+1} \circ f_{n_j+2} \circ \cdots \circ f_{n_{j+1}-1}$ *then* $\varprojlim\{f_i\}$ *is homeomorphic to* $\varprojlim\{g_j\}$.

Proof: Consider the following diagram:



Since $g_j = g_{j+1,j}$, again the reader can easily see that the diagram commutes, thus by Theorem 2.3, $\varprojlim\{f_i\} \simeq \varprojlim\{g_j\}$. *Q. E. D.*

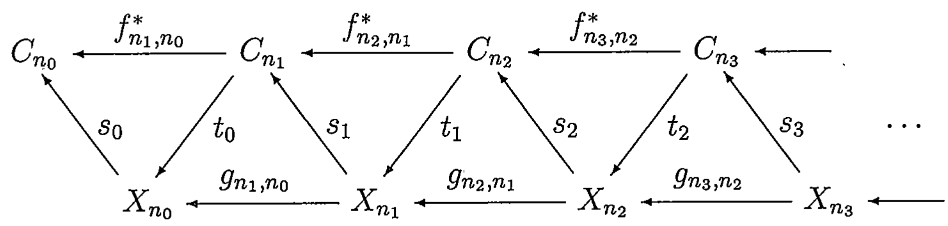
Suppose that we are using a single bonding map, that is in our sequence of bonding maps, $f_i = f$ for all i. If we use the notation that $f^3 = f \circ f \circ f$ then a common application of the above corollary is to show that $\varprojlim f^3 \simeq \varprojlim f$.

Lemma 2.4 *Suppose* $\{X_i\}_{i=0}^{\infty}$*is a sequence of spaces with* $X_i = [a_i, b_i]$ *and* $\{f_i\}_{i=0}^{\infty}$, *with* $f_i : X_{i+1} \to X_i$ *is a sequence of continuous unimodal bonding maps for* $i \in \{0, 1, 2, \ldots\}$. *Thus* $\varprojlim\{f_i\}$ *is a unimodal continuum. Further assume that* $\varprojlim\{f_i\}$ *is* *larger than a single point. Then* $\varprojlim\{f_i\}$ *is homeomorphic to* $\varprojlim\{g_i\}$ *where* $\{g_i\}_{i=0}^{\infty}$ *is a sequence of bonding maps such that* $g_i : X_{i+1} \to X_i$ *are continuous unimodal* <u>onto</u> *functions.*

Proof: Let $C_n = \bigcap_{i \geq 0} f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$. Since the composite of continuous functions is continuous and the continuous image of a connected set is itself connected, $f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$ is connected for all i. Similarly $f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1})$ is compact as the continuous image of a compact set and thus is closed. So C_n is the nested intersection of closed connected nonempty sets, thus C_n is closed, connected and nonempty ([12] pg. 170-171). Since $C_n \subset X_n$, C_n is a closed bounded interval or a single point. Define $f_n^* : C_{n+1} \to C_n$ by $f_n^* = f_n|_{C_{n+1}}$. By the definition of C_n, f_n^* is onto.

Next we show that $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$. Let $\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{f_n^*\}$, then since $C_n \subset X_n$ and $f_n^* = f_n|_{C_{n+1}}$, it is clear $\underline{x} \in \varprojlim\{f_n\}$. Thus, as sets, $\varprojlim\{f_n^*\} \subset \varprojlim\{f_n\}$. Next suppose $\underline{x} \in \varprojlim\{f_n\}$, then for any n, $x_n = f_n(x_{n+1}) = f_n \circ f_{n+1}(x_{n+2}) = \dots = f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(x_{n+i+1})$ for any $i \geq 0$ and of course $x_{n+i+1} \in X_{n+i+1}$, so $x_n \in \bigcap_{i \geq 0} f_n \circ f_{n+1} \circ \cdots \circ f_{n+i}(X_{n+i+1}) = C_n$ and because $f_{n-1}^* = f_{n-1}|_{C_n}$, $x_{n-1} = f_{n-1}^*(x_n)$ so $\underline{x} \in \varprojlim\{f_n^*\}$. Thus as sets, $\varprojlim\{f_n\} \subset \varprojlim\{f_n^*\}$ and therefore $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ as sets, and since the metrics on $\varprojlim\{f_n\}$ and $\varprojlim\{f_n^*\}$ induce the same topology on this set, $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ as desired.

Because $\varprojlim\{f_n\} = \varprojlim\{f_n^*\}$ is not just a single point, there exists an $M \in \mathbb{N} \cup \{0\}$ so that C_n is a non-degenerate interval for all $n \geq M$, that is $C_n = [p_n, q_n]$ for $p_n < q_n$ if $n \geq M$. So for $n \geq M$, define the function $\alpha_n : C_n \to X_n$ by $\alpha_n(x) = \frac{b_n - a_n}{q_n - p_n}(x - p_n) + a_n$, hence $\alpha_n(p_n) = a_n$ and $\alpha_n(q_n) = b_n$. Also, α_n is an affine function in $\mathbb{R}$ with positive slope so it is a continuous, strictly increasing, onto function, thus is a homeomorphism. Next let $g_n : X_{n+1} \to X_n$ be defined by $g_n = \alpha_n \circ f_n^* \circ \alpha_{n+1}^{-1}$, for $n \geq M$. So g_n is continuous and onto as the composition of continuous, onto functions. Let $\{n_i\}_{i=0}^{\infty}$ be any sequence in $\mathbb{N} \cup \{0\}$ with $M \leq n_0 < n_1 < n_2 < \dots$ and let $s_i = \alpha_{n_i}^{-1}$, and $t_i = \alpha_{n_i} \circ f^*_{n_{i+1},n_i}$. Consider the following diagram:



So let us quickly check that this diagram commutes. Looking at any triangle of it with $C_{n_{i+1}}$, C_{n_i} and X_{n_i} at the corners, if we go from $C_{n_{i+1}}$ to C_{n_i}, passing through X_{n_i}, we get $s_i \circ t_i = \alpha_{n_i}^{-1} \circ \alpha_{n_i} \circ f^*_{n_{i+1},n_i} = f^*_{n_{i+1},n_i}$, which is exactly what we get going directly from $C_{n_{i+1}}$ to C_{n_i}. Checking a triangle with X_{n_i}, $X_{n_{i-1}}$ and C_{n_i} at the corners, going through C_{n_i} we have $t_{i-1} \circ s_i = \alpha_{n_{i-1}} \circ f^*_{n_i,n_{i-1}} \circ \alpha_{n_i}^{-1} = \alpha_{n_{i-1}} \circ f^*_{n_{i-1}} \circ f^*_{n_{i-1}+1} \circ f^*_{n_{i-1}+2} \circ \cdots f^*_{n_i-1} \circ \alpha_{n_i}^{-1} = \alpha_{n_{i-1}} \circ f^*_{n_{i-1}} \circ \alpha^{-1}_{n_{i-1}+1} \circ \alpha_{n_{i-1}+1} \circ f^*_{n_{i-1}+1} \circ \alpha^{-1}_{n_{i-1}+2} \circ \cdots \circ \alpha_{n_i-1} \circ f^*_{n_i-1} \circ \alpha_{n_i}^{-1} = g_{n_{i-1}} \circ g_{n_{i-1}+1} \circ \cdots \circ g_{n_i-1} = g_{n_i,n_{i-1}}$, which is the function going directly from X_{n_i} to $X_{n_{i-1}}$. Since all the maps on our diagram are continuous, we use Lemma 2.3 to get that $\varprojlim\{g_i\} \simeq \varprojlim\{f_i^*\} = \varprojlim\{f_i\}$.

Notice that to create these g_i functions we restricted the domain and range of f_i to get the f_i^* functions and rescaled them with the affine homeomorphism α_i, which did nothing to change the fact that we have unimodal maps, hence g_i is unimodal for all i, as desired. *Q. E. D.*

This next theorem is one we have recently proven and, although it will not be used in the remainder of this thesis, it illustrates an important property all unimodal continua must have.

Theorem 2.5 *All unimodal continua must have a dense arc component.*

Proof: Suppose we have any unimodal continuum. If this continuum is a single point, then we have a degenerate dense arc component which is that point itself. Otherwise the unimodal continuum is more than a single point and we can assume that there exist a sequence of bonding maps $\{f_i\}_{i=0}^{\infty}$ with $f_i : X_{i+1} \to X_i$ which are continuous unimodal and, by Lemma 2.4, we can also assume they are onto, so that $\varprojlim\{f_i\}$ is homeomorphic to our chosen unimodal continuum. Since f_i is unimodal, there is a critical point of f_i, say $c_{i+1} \in X_{i+1}$. Because f_i is onto, there is a subinterval $X_{i+1}^* \subset X_{i+1}$ with $X_{i+1}^* = [a_{i+1}, c_{i+1}]$ or $X_{i+1}^* = [c_{i+1}, b_{i+1}]$ so that $f_i|_{X_{i+1}^*} : X_{i+1}^* \to X_i$ is one-to-one and onto. Define $C_N = \{\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{f_i\} : x_i \in X_i^*$ for $i \geq N\}$ where $N \in \mathbb{N} \cup \{0\}$. Let $C = \bigcup_{N \geq 0} C_N$. We will show that the set C is path connected and dense in $\varprojlim\{f_i\}$.

First, let us show C is path connected. Let $\underline{x}, \underline{y} \in C$, then if $\underline{x} = (x_0, x_1, x_2, \dots)$ and $\underline{y} = (y_0, y_1, y_2, \dots)$, there exists an $N_1 \in \mathbb{N} \cup \{0\}$ so $x_i \in X_i^*$ for all $i \geq N_1$. Similarly, there is an $N_2 \in \mathbb{N} \cup \{0\}$ such that $y_i \in X_i^*$ for all $i \geq N_2$. Let $N = \max\{N_1, N_2\}$, thus $x_i, y_i \in X_i^*$ for all $i \geq N$ and $\underline{x}, \underline{y} \in C_N$. Now define the function $g_i : X_i \to X_{i+1}^*$ by $g_i = (f_i|_{X_{i+1}^*})^{-1}$. Also, let the functions $p_k : [0,1] \to I$ for $k = 0, 1, 2, \dots$ be defined by $p_N(t) = t x_N + (1-t) y_N$ and $p_{N+n}(t) = g_{N+n} \circ g_{N+n-1} \circ g_{N+n-2} \circ \dots \circ g_N(p_N(t))$ for $n = 1, 2, 3, \dots$ and $p_m(t) = f_m \circ f_{m+1} \circ f_{m+2} \circ \dots \circ f_{N-1}(p_N(t))$ for $m = 0, 1, 2, \dots, N-1$. Next let $P : [0,1] \to C$ be defined by $P(t) = (p_0(t), p_1(t), p_2(t), \dots)$.

To see that P is a path from $\underline{y}$ to $\underline{x}$, we notice that:

$$P(0) = (p_0(0), p_1(0), p_2(0), \dots, p_N(0), \dots) = (p_0(0), p_1(0), p_2(0), \dots, y_N, \dots)$$

$$= (f_0 \circ f_1 \circ \dots \circ f_{N-1}(y_N), f_1 \circ f_2 \circ \dots \circ f_{N-1}(y_N), f_2 \circ f_3 \circ \dots \circ f_{N-1}(y_N), \dots,$$

$$f_{N-1}(y_N), y_N, g_N(y_N), g_{N+1} \circ g_N(y_N), g_{N+2} \circ g_{N+1} \circ g_N(y_N), \dots)$$

$$= (y_0, y_1, \dots, y_N, (f_N|_{X_{N+1}^*})^{-1}(y_N), (f_{N+1}|_{X_{N+2}^*})^{-1} \circ (f_N|_{X_{N+1}})^{-1}(y_N), \dots)$$

But since $\underline{y} \in C_N$, $(f_N|^*_{X_{N+1}})^{-1}(y_N) = y_{N+1}$ and $(f_{N+n}|_{X^*_{N+n+1}})^{-1}(y_{N+n}) = y_{N+n+1}$ for $n \geq 0$, so $P(0) = (y_0, y_1, \ldots, y_N, y_{N+1}, y_{N+2}, \ldots) = \underline{y}$. Since $p_N(1) = x_N$, by a similar argument, $P(1) = \underline{x}$.

It is clear from the construction of $P(t)$ that $P(t) \in C_N$ for all N and thus $P(t) \in C$ for all t. All that remains in the proof that P is a path is to show P is continuous. Since $P(t) = (p_0(t), p_1(t), p_2(t), \ldots)$, $\pi_i \circ P(t) = p_i(t)$ which is continuous. Hence by Lemma 2.2, P is continuous and thus P is a path from $\underline{y}$ to $\underline{x}$.

Next we will show C is dense in $\varprojlim\{f_n\}$. Suppose that $\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{f_n\}$ and let $\epsilon > 0$ be given. Since $\displaystyle\sum_{i=0}^{\infty} \frac{1}{2^i}$ converges, there is an $M \in \mathbb{N} \cup \{0\}$ so that $\displaystyle\sum_{i=M}^{\infty} \frac{1}{2^i} < \epsilon$. Now let $\underline{y} = (y_0, y_1, y_2, \ldots)$ be defined by $y_i = x_i$ for $i \in \{0, 1, 2, \ldots, M-1\}$ and let $y_i = g_i \circ g_{i-1} \circ g_{i-2} \circ \cdots \circ g_{M+1} \circ g_M(x_{M-1})$ for $i \in \{M, M+1, M+2, \ldots\}$. Then $\displaystyle d(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} = \sum_{i=0}^{M-1} \frac{|x_i - y_i|}{2^i} + \sum_{i=M}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=0}^{M-1} \frac{|x_i - x_i|}{2^i} + \sum_{i=M}^{\infty} \frac{1}{2^i} < \sum_{i=0}^{M-1} \frac{0}{2^i} + \sum_{i=M}^{\infty} \frac{1}{2^i} = \sum_{i=M}^{\infty} \frac{1}{2^i} < \epsilon$. We notice that from the construction of $\underline{y}$, $\underline{y} \in C_M$ and hence in C. Thus the closure of the points of C is all of $\varprojlim\{f_n\}$, and therefore C is dense. *Q. E. D.*

Below is an example of a continuum that appears to be similar to Figures 1 and 2, but from Theorem 2.5 we know it is not a unimodal continua.

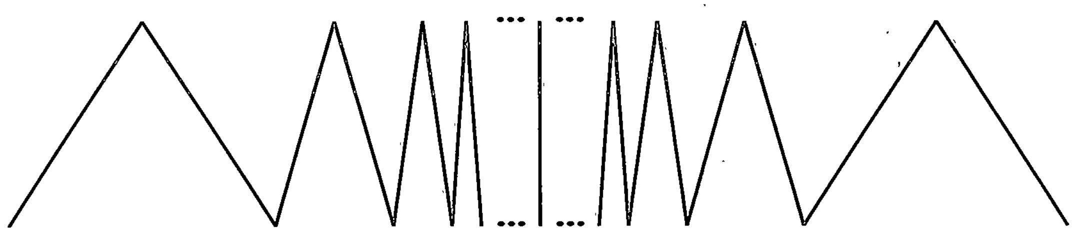


Figure 4: An Example of a Non-unimodal Continua

CHAPTER 3

GENERALIZED $\sin\frac{1}{x}$ SPACES

In this chapter we start with bonding maps (that are not unimodal) with certain general properties and show that their inverse limit spaces each are homeomorphic to a space that has many similarities to the "topologist's sine curve" space. Below are a couple of examples of what we mean by similar to the "topologist's sine curve" space.

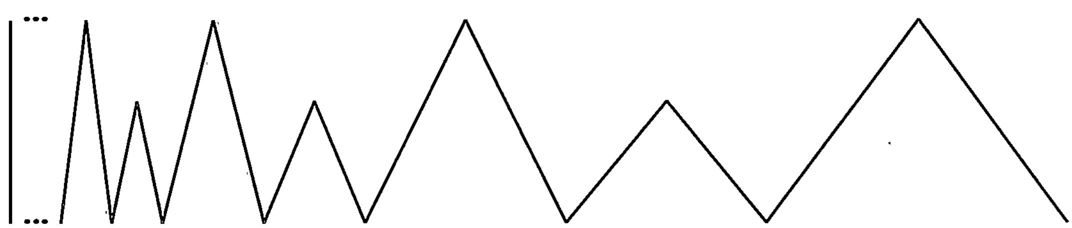


Figure 5: An M Continuum

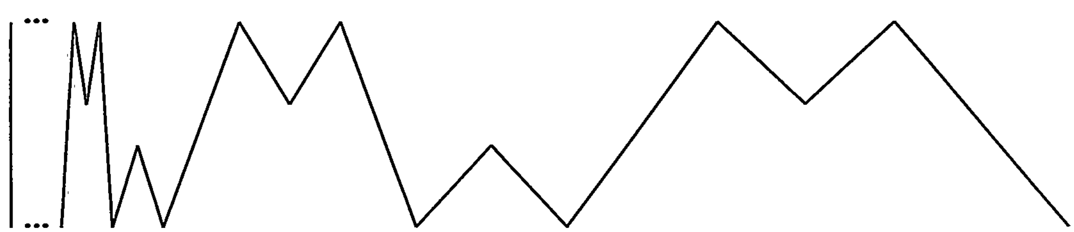


Figure 6: An MW Continuum

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a sequence of continuous functions so that each $\varphi_n : I \to I$, with $I = [0, 1]$, has the following properties:

(a) $\varphi_n(0) = 0$,

(b) $\varphi_n([0, \frac{1}{2})) = [0, 1]$,

(c) $\varphi_n(x) = x$, for all $x \in [\frac{1}{2}, 1]$,

(d) there exists a real number η, independent of φ_n, and point a_n for each φ_n so that

(i) $0 < \eta \leq a_n < \frac{1}{2}$,

(ii) φ_n is strictly increasing on $[0, a_n]$,

(iii) $\varphi_n(a_n) = \frac{1}{2}$

(iv) and $\varphi_n(x) \geq \frac{1}{2}$ for all $x \in [a_n, 1]$.

We consider the inverse limit space of $\{\varphi_n\}_{n=0}^{\infty}$, denoted by $\varprojlim\{\varphi_n\}$, and will use the notation $\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{\varphi_n\}$ if $\varphi_n(x_{n+1}) = x_n$. Define $c_n = \min\{x \in [0, \frac{1}{2}] : \varphi_n(x) = 1\}$. Since $\varphi_n(\frac{1}{2}) = \frac{1}{2}$ and φ_n is strictly increasing on $[0, a_n]$ with $\varphi_n(a_n) = \frac{1}{2}$, $a_n < c_n < \frac{1}{2}$. Notice that by our selection of a_n and c_n, neither $(a_0, a_1, a_2, \dots)$ nor $(c_0, c_1, c_2, \dots)$ is in $\varprojlim\{\varphi_n\}$. The space $\varprojlim\{\varphi_n\}$ is homeomorphic to a space similar to a "topologist's sine curve" space, so we will soon label it as a generalized $\sin\frac{1}{x}$ space. But first we state a well known theorem (without proof) and a couple of lemmas that we will need later.

Consider a function $\psi : [a, b] \to [a, b]$, $a < b$ and $a, b \in \mathbb{R}$. Divide $[a, b]$ into k distinct subintervals $[a, p_1] = I_1, [p_1, p_2] = I_2, [p_2, p_3] = I_3, \dots, [p_{k-1}, b] = I_k$, that intersect at no more than their end points. Let us use a $k \times k$ transition matrix M to encode information about ψ. In matrix M, let the ij^{th} entry, call it m_{ij}, be either 0 or 1, if $I_i \subset \psi(I_j)$ then let $m_{ij} = 1$ and let $m_{ij} = 0$ otherwise.

Theorem 3.1 *For all $n \in \mathbb{N}$, let $\psi_n : [a, b] \to [a, b]$, then let M_n be a $k \times k$ transition matrix of zeros and ones constructed as above from ψ_n. If the ij^{th} entry of the matrix product $M_0 M_1 M_2 \cdots M_n$ is λ, there are λ subintervals, $J_1, J_2, J_3, \dots, J_\lambda$, of I_i so that $I_j \subset \psi_0 \circ \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(J_m)$ for $m \in \{1, 2, 3, \dots, \lambda\}$ and $J_i \cap J_j$ is at most one point if $i \neq j$.*

The above theorem is actually a compilation of two lemmas in [15]. For a proof of the above theorem, one is referred to [15], Lemma 1.2, pg. 64 and Lemma 2.2 pg. 74 and related definitions on the pages in between these lemmas.

In the next lemma we assume the following: X is some metric space and, for all $n \in \mathbb{N} \cup \{0\}$, R_n is a subspace of X so that R_n is homeomorphic to a compact interval in $\mathbb{R}$ with positive length. Further assume that R_n is a proper subset of R_{n+1} and p is a common endpoint of each R_n. Let $R = \bigcup_{n \geq 0} R_n$. Also let $\{n_i\}_{i=1}^{\infty}$ be a sequence of integers such that $0 < n_1 < n_2 < n_3 < \dots$, and define $\{x_i\}_{i=1}^{\infty}$ to be any sequence with $x_i \in R_{n_i} \setminus R_{n_i - 1}$ for all $i \in \mathbb{N}$. We will also assume that R has the property that the set of limit points from any such sequence $\{x_i\}_{i=1}^{\infty}$ is disjoint from R.

Lemma 3.2 *If R and R_n are as given above, R is homeomorphic to $\mathbb{R}_+ \cup \{0\}$.*

Proof: Recall that p is a common endpoint of all our R_n. We know that each R_n is homeomorphic to a compact interval in $\mathbb{R}$ with positive length, so let $h_0 : [0,1] \to R_0$ be a homeomorphism so that $h_0(0) = p$. Next let $h_1 : [1,2] \to \overline{R_1 \setminus R_0}$ be a homeomorphism with $h_1(1) = h_0(1)$ and let $h_2 : [2,3] \to \overline{R_2 \setminus R_1}$ be a homeomorphism with $h_2(2) = h_1(2)$ and so on, thus in general we have $h_n : [n, n+1] \to \overline{R_n \setminus R_{n-1}}$ is a homeomorphism with $h_n(n) = h_{n-1}(n)$. Let us show that $R = R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$, by showing mutual containment. First assume $x \in R = \bigcup_{n \geq 0} R_n$ so $x \in R_n$ for some n. Since the R_n's are nested with $R_{n-1} \subset R_n$, there exists an $N \in \mathbb{N} \cup \{0\}$ so that $x \in R_N$ but $x \notin R_{N-1}$. If $N = 0$ then $x \in R_0$, otherwise $N \geq 1$ and $x \in R_N \setminus R_{N-1}$ so $x \in \overline{R_N \setminus R_{N-1}}$, thus either way $x \in R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$ and $R \subset R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{n-1}})$. Next assume $x \in R_0 \cup (\bigcup_{n \geq 1} \overline{R_n \setminus R_{N-1}})$. If $x \in R_0$ then $x \in R = \bigcup_{n \geq 0} R_n$. Otherwise $x \in \overline{R_n \setminus R_{n-1}}$ for some $n \geq 1$. Recall that from the definition of set closure, $\overline{R_n \setminus R_{n-1}}$ is the intersection of all the closed sets containing $R_n \setminus R_{n-1}$, and since R_n is a closed set containing $R_n \setminus R_{n-1}$, $x \in R_n$ so $x \in R = \bigcup_{n\geq 0} R_n$. Hence $R_0 \cup (\bigcup_{n\geq 1} \overline{R_n \setminus R_{n-1}}) \subset R$ and therefore $R = R_0 \cup (\bigcup_{n\geq 1} \overline{R_n \setminus R_{n-1}})$.

Now let $h : \mathbb{R}_+ \cup \{0\} \to R$ be defined by $h(x) = h_n(x)$ where $x \in [n, n+1]$. By the "pasting lemma" ([12], pg. 108) h is a continuous function. We also note that h is a bijection by construction and since each h_n is bijective. To prove h is a homeomorphism, we still need to show $h^{-1} : R \to \mathbb{R}_+ \cup \{0\}$ is continuous. Consider any sequence $\{y_i\}_{i=1}^{\infty} \in R$ that converges to $y \in R$. If we can show $\{h^{-1}(y_i)\}_{i=1}^{\infty}$ converges to $h^{-1}(y)$ in $\mathbb{R}_+ \cup \{0\}$ then h^{-1} is continuous ([12] pg. 128). There must exist an $N \in \mathbb{N} \cup \{0\}$ so that y and y_i are in R_N for all i, for if this was not the case, then there exists a subsequence $\{y_{i_j}\}_{j=1}^{\infty}$ and natural numbers $n_1 < n_2 < n_3 < \ldots$ so that $y_{i_j} \in R_{n_j} \setminus R_{n_j-1}$. But $\{y_{i_j}\}_{j=1}^{\infty}$ converges to $y \in R$ which contradicts the hypothesis that such sequences have there limit points outside R. So there exists an N such that y and y_i are elements of R_N for all i. Let us look at $h|_{[0,N]} : [0, N] \to \bigcup_{n=0}^{N-1} R_n$. Because $R_n \simeq [n, n+1]$ for all n and the countable union of compact sets is compact, $h|_{[0,N]}$ is a function between compact metric spaces that is continuous and bijective since h is and we selected our range space to make it so. Thus by Lemma 2.1, $h|_{[0,N]}$ is a homeomorphism and so $\{(h|_{[0,N]})^{-1}(y_i)\}_{i=1}^{\infty} = \{h^{-1}(y_i)\}_{i=1}^{\infty}$ must converge to $(h|_{[0,N]})^{-1}(y) = h^{-1}(y)$, so h^{-1} is continuous as desired.

Therefore h is a homeomorphism from $\mathbb{R}_+ \cup \{0\}$ onto R. *Q. E. D.*

We define a topological space X to be a *generalized* $\sin \frac{1}{x}$ *space* if:

(a) X is compact and connected,

(b) X has two path components, one homeomorphic to a compact interval in $\mathbb{R}$, and the other homeomorphic to $\mathbb{R}_+ \cup \{0\}$, and

(c) the non-compact path component is dense in X.

Lemma 3.3 *The space* $\varprojlim\{\varphi_n\}$ *is a generalized* $\sin\frac{1}{x}$ *space.*

Proof: As mentioned in the first chapter, we know that $\varprojlim\{\varphi_n\}$ is compact and connected. Define a subset of $\varprojlim\{\varphi_n\}$ by $A = \{\underline{x} = (x_0, x_1, x_2, \dots) \in \varprojlim\{\varphi_n\} : x_n \in [\frac{1}{2}, 1] \text{ for all } n\}$. Since $\varphi_n|_{[\frac{1}{2},1]}(x) = x$, elements of A have the form $(x_0, x_0, x_0, \dots)$. Let $h : A \to [\frac{1}{2}, 1]$ be equal to $\pi_0|_A$, thus h is continuous. Next let $g : [\frac{1}{2}, 1] \to A$ be defined by $g(x) = (x, x, x, \dots)$. If $x \in [\frac{1}{2}, 1]$, $\pi_n \circ g(x) = x$ for all n, so $\pi_n \circ g$ is the identity map and thus continuous. Hence by Lemma 2.2, g is continuous. Also, if $\underline{x} = (x_0, x_1, x_2, \dots) = (x_0, x_0, x_0, \dots) \in A$, $g \circ h(\underline{x}) = g(x_0) = (x_0, x_0, x_0, \dots) = \underline{x}$ and if $x \in [\frac{1}{2}, 1]$, $h \circ g(x) = h(x, x, x, \dots) = x$. Therefore $g = h^{-1}$ and h is a homeomorphism from A to the interval $[\frac{1}{2}, 1] \in \mathbb{R}$.

Next let $B = (\varprojlim\{\varphi_n\}) \setminus A$. Since $\varprojlim\{\varphi_n\}$ is compact and connected, B is not compact, otherwise A and B form a separation of $\varprojlim\{\varphi_n\}$.

We now show $\varprojlim\{\varphi_n\}$ is not path connected. Assume, by way of contradiction, that $\gamma : [0, 1] \to \varprojlim\{\varphi_n\}$ is a path from $\underline{0} = (0, 0, 0, \dots)$ to $\underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, thus we can write $\gamma(t) = (x_0(t), x_1(t), x_2(t), \dots)$ where $x_n : [0, 1] \to I$ is a continuous function such that $x_n(0) = 0$, $x_n(1) = \frac{1}{2}$ and, since γ is a path in $\varprojlim\{\varphi_n\}$, $\varphi_n(x_{n+1}(t)) = x_n(t)$, for all n and t. Thus $x_0(t) = \varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{n-1} \circ x_n(t)$, for all $t \in [0, 1]$ and all $n \in \mathbb{N}$.

Let M_n be a transition matrix for φ_n. First let $I_0 = [0, \frac{1}{2}]$ and $I_1 = [\frac{1}{2}, 1]$. Then, from our hypotheses about φ_n, we have $\varphi_n(I_0) = I_0 \cup I_1$ and $\varphi_n(I_1) = I_1$. Notice we have the same transition matrix for all φ_n so let $M_n = M$ where:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We note that it is easy to show (by induction) that

$$M_0 M_1 M_2 \cdots M_{n-1} = M^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

So by Theorem 3.1, for any $N^* \in \mathbb{N}$ there exists subintervals $J_1, J_2, J_3, \ldots, J_{N^*}$ of I_1 so that $x_0(J_m) = \varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{N^*-1}(x_{N^*}(J_m)) \supset I_2$ for $m = 1, 2, 3, \ldots, N^*$. Hence there exists $t_n, s_n \in I_1$ such $x_0(t_n) = \frac{1}{2}$ and $x_0(s_n) = 1$, also we can order these either $t_1 < s_1 < t_2 < s_2 < \cdots < s_{N-1} < t_N$ or $s_1 < t_1 < s_2 < t_2 < \cdots < t_{N-1} < s_N$ if N^* is even and $N = \frac{N^*}{2} + 1$ or else, if N^* is odd and $N = \frac{N^*+1}{2}$, $t_1 < s_1 < t_2 < s_2 < \cdots < t_N < s_N$ or $s_1 < t_1 < s_2 < t_2 < \cdots < t_N < s_N$. The arguments are identical regardless, so assume we have the case where $t_1 < s_1 < t_2 < s_2 < \cdots < s_{N-1} < t_N$. Since N^* was arbitrary, we can choose N as large as we need.

Because x_0 is continuous on $[0,1]$, x_0 is uniformly continuous and there is a $\delta > 0$ so that if $|t - s| < \delta$, then $|x_0(t) - x_0(s)| < \frac{1}{4}$. We choose N large enough so that $\frac{1}{2N} < \delta$. Then there must be a t_n and s_m, where m is $n-1$ or n, so that $|s_m - t_n| < \delta$ but $|x_0(s_m) - x_0(t_n)| = |1 - \frac{1}{2}| = \frac{1}{2} > \frac{1}{4}$. This a contradiction, therefore $\varprojlim\{\varphi_n\}$ is not path connected.

It is clear that A, homeomorphic to $[\frac{1}{2}, 1]$, is contained in a path component, so we wish next to show that the only other path component is B, and that it is homeomorphic to a ray. Recall we defined $c_n = \min\{x \in [0, \frac{1}{2}] : \varphi_n(x) = 1\}$. Let $B_0 = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_0 \in [0, c_0]\}$ and $B_1 = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_1 \in [0, c_1]\}$ and continue like this to get the general form: $B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, c_k]\}$ then $\bigcup_{k \geq 0} B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, c_k]$ for some $k\}$, but $[0, c_k] \subset [0, \frac{1}{2})$ and if $x_k \in [0, \frac{1}{2})$ then $x_{k+1} \in [0, a_k) \subset [0, c_k]$, so $\bigcup_{k \geq 0} B_k = \{\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim\{\varphi_n\} : x_k \in [0, \frac{1}{2})$ for some $k\} = (\varprojlim\{\varphi_n\}) \setminus A = B$. Notice that since $[0, c_n] \subset [0, \frac{1}{2})$, φ_n is strictly increasing on $[0, a_n]$, $\varphi_n(a_n) = \frac{1}{2}$ and $\varphi_n(x) \geq \frac{1}{2}$ for $x \in [a_n, 1]$, it is the case that if $x_n \in [0, c_n]$ then x_n has only one preimage, that is $x_{n+1} = [\varphi_n]^{-1}(x_n) \in [0, a_{n+1}) \subset [0, c_{n+1}]$. So in general we know that if $x_n \in [0, c_n]$ then $x_{n+1} \in [0, c_{n+1}]$. Thus $B_n \subset B_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. In addition we notice $\underline{0} = (0, 0, 0, \ldots)$ is an endpoint of B_n for all n.

Next we show B_n is homeomorphic to a compact interval in $\mathbb{R}$. Let $h_n : B_n \to [0, c_n]$ be defined as $h_n = \pi_n|_{B_n}$, so h_n is continuous. From the hypotheses about φ_n on $[0, a_n]$, $\varphi_n|_{[0,a_n]}$ is a homeomorphism, so let $\gamma_n = (\varphi_n|_{[0,a_n]})^{-1}$, thus if $x_n \in [0, \frac{1}{2})$, x_n has only one preimage under φ_n and so $x_{n+1} = \gamma_n(x_n)$. Define $g_n : [0, c_n] \to B_n$ by $g_n(x) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{n-1}(x), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1}(x), \ldots, \varphi_{n-1}(x), x, \gamma_n(x), \gamma_{n+1} \circ \gamma_n(x), \gamma_{n+2} \circ \gamma_{n+1} \circ \gamma_n(x), \ldots)$. Since φ_k and γ_k are continuous for all k, $\pi_k \circ g_n$ is continuous for all k, so by Lemma 2.2, g_n is continuous. We quickly see that $h_n \circ g_n(x) = x$ and $g_n \circ h_n(\underline{x}) = \underline{x}$, so $h_n^{-1} = g_n$ and h_n is a homeomorphism as desired.

Notice that if $\underline{y} = (y_0, y_1, y_2, \ldots) \in B_n \backslash B_{n-1}$ then because $\underline{y} \notin B_{n-1}$, $y_i \in [\frac{1}{2}, 1]$ for all $i < n$ and since $\varphi_i|_{[\frac{1}{2},1]}(x) = x$, $y_0 = y_1 = y_2 = \cdots = y_{n-1}$, but $\underline{y} \in B_n$ so $y_i \in [0, \frac{1}{2})$ for all $i \geq n$. Hence $\underline{y} = (y_0, y_0, y_0, \ldots, y_0, y_n, y_{n+1}, \ldots)$. Now suppose $i = 1, 2, 3, \ldots$ and $\{n_i\}_{i=1}^{\infty}$ is a sequence of integers such that $0 < n_1 < n_2 < n_3 < \ldots$ and $\{\underline{x}^i\}_{i=1}^{\infty}$ is a sequence of elements of B so that $\underline{x}^i \in B_{n_i} \setminus B_{n_i - 1}$ and let $\underline{x}$ be a limit point of $\{\underline{x}^i\}_{i=1}^{\infty}$. Assume, by way of contradiction, that $\underline{x} \in B = \bigcup_{n \geq 0} B_n$, so there exists some $N \in \mathbb{N} \cup \{0\}$ so that $\underline{x} \in B_N \setminus B_{N-1}$ (where B_{0-1} is taken to be $\{\}$), thus $\underline{x} = (x_0, x_0, x_0, \ldots, x_0, x_N, x_{N+1}, \ldots)$. There also must be an $i^* \in \mathbb{N}$ so that if $i > i^*$, $n_i > N$. Therefore if $i > i^*$, then $\underline{x}^i \in B_{n_i} \setminus B_{n_i - 1}$ and $\underline{x}^i = (x_0^i, x_1^i, x_2^i, \ldots, x_N^i, \ldots, x_{n_i - 1}^i, x_{n_i}^i, \ldots) = (x_0^i, x_0^i, x_0^i, \ldots, x_0^i, x_{n_i}^i, \ldots)$ where $x_0^i \in [\frac{1}{2}, 1]$. Since $x_N \in [0, \frac{1}{2})$, $|x_N - \frac{1}{2}| > 0$, let $\epsilon = \frac{|x_N - \frac{1}{2}|}{2^{N+1}}$. Hence if $i > i^*$ then $n_i > N$ and $d(\underline{x}, \underline{x}^i) = \sum_{j=0}^{\infty} \frac{|x_j - x_j^i|}{2^j} \geq \frac{|x_N - x_N^i|}{2^N} \geq \frac{|x_N - \frac{1}{2}|}{2^N} = 2\frac{|x_N - \frac{1}{2}|}{2^{N+1}} = 2\epsilon > \epsilon$. Therefore we have found an $\epsilon > 0$ such that for all $i > i^*$, $\underline{x}^i$ is not in the ϵ-ball about $\underline{x}$, so at most finitely many elements of $\{\underline{x}^i\}_{i=1}^{\infty}$ are in the ϵ-ball, which contradicts the assumption that $\underline{x}$ is a limit point of $\{\underline{x}\}_{i=1}^{\infty}$. Thus all the limits points of the sequence are outside of B. Therefore B and B_n satisfy all the hypotheses of Lemma 3.2, so B is homeomorphic to a ray.

Then since $B = (\varprojlim\{\varphi_n\}) \setminus A$ and B is path connected, not only is A in a path component, A itself is the only other path component, besides B.

Next we show that B is dense in $\varprojlim\{\varphi_n\}$. Recall that every point of A is of the form $\underline{z} = (z, z, z, \ldots)$ where $z \in [\frac{1}{2}, 1]$. Since $\varphi_n([0, \frac{1}{2}]) = [0, 1]$, if $z \in (\frac{1}{2}, 1]$, there exists an $x_0 \in [0, \frac{1}{2})$ such that $\varphi_n(x_0) = z$ and if $z = \frac{1}{2}$, then let $x_0 = a_n$, thus $\varphi_n(x_0) = z$ and $x_0 \notin [\frac{1}{2}, 1]$. Now consider the sequence $\{\underline{z}^i\}_{i=0}^{\infty}$ defined by $\underline{z}^i = (z, z, \ldots, z, z, x_0, x_1, x_2, \ldots)$ where x_0 is in the $(i+1)^{st}$ position and as before $\varphi_i(x_{i+1}) = x_i$, hence $\underline{z}^i \in \varprojlim\{\varphi_n\}$. Because $x_0 \notin [\frac{1}{2}, 1]$, $\underline{z}^i \in B$. Now to show $\{\underline{z}^i\}_{i=0}^{\infty}$ converges to $\underline{z}$, let $\epsilon > 0$ be given. Therefore there is an $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \epsilon$, so for $i > N$, $d(\underline{z}, \underline{z}^i) = \sum_{i=0}^{N-1} \frac{|z-z|}{2^i} + \sum_{i=N}^{\infty} \frac{|z-x_{i-N}|}{2^i} \leq 0 + \sum_{i=N}^{\infty} \frac{1}{2^i} < 0 + \epsilon = \epsilon$. Hence $\{\underline{z}^i\}_{i=0}^{\infty}$ converges to $\underline{z}$, as desired.

Whence $\varprojlim\{\varphi_n\}$ is a generalized $\sin \frac{1}{x}$ space. *Q. E. D.*

For the next lemma we make another restriction on the sequence of φ_n functions. In addition to the properties (a)-(d) at the beginning of this chapter, assume

(e) φ_n is monotone increasing on $[0, c_n]$.

Thus we can see $\varphi_n|_{[0,c_n]} = [0, 1]$, with $\varphi_n(c_n) = 1$ and $\varphi_n|_{[0,c_n]}$ is a homeomorphism. Define $(\varphi_n|_{[0,c_n]})^{-1} = \gamma_n : [0, 1] \to [0, c_n]$ and let $\Upsilon_{\varphi_n,k} : [0, 1] \to \varprojlim\{\varphi_n\}$ be defined by $\Upsilon_{\varphi_n,k}(t) = (\varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(t), \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \cdots \circ \varphi_k(t), \ldots, \varphi_k(t), t, \gamma_{k+2}(t), \gamma_{k+2} \circ \gamma_{k+3}(t), \ldots)$. Notice that t is in the $(k+1)^{st}$ position.

Lemma 3.4 *For any fixed $k \in \mathbb{N} \cup \{0\}$, $\Upsilon_{\varphi_n,k}$ is an embedding.*

Proof: This amount to showing that $\Upsilon_{\varphi_n,k} : [0, 1] \to \text{Range}(\Upsilon_{\varphi_n,k})$ is a homeomorphism. First let us show $\Upsilon_{\varphi_n,k}$ is one-to one. Let $x, y \in [0, 1]$ so that $\Upsilon_{\varphi_n,k}(x) =$

$\Upsilon_{\varphi_n,k}(y)$, that is $\Upsilon_{\varphi_n,k}(x) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k(x), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(x), \ldots, \varphi_k(x), x, \gamma_{k+2}(x), \ldots) = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k(y), \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(y), \ldots, \varphi_k(y), y, \gamma_{k+2}(y), \ldots) = \Upsilon_{\varphi_n,k}(y)$. Therefore $x = y$ since they both occupy the $(k+1)^{st}$ position.

Now because φ_n and γ_n are continuous for all n and the compositions of continuous functions are continuous, $\pi_k \circ \Upsilon_{\varphi_n,k}$ is continuous so by Lemma 2.2, $\Upsilon_{\varphi_n,k}$ is continuous. Thus $\Upsilon_{\varphi_n,k}$ is a bijective continuous function from compact space $[0,1]$ onto its range, which is a subset of the metric space $\varprojlim\{\varphi_n\}$, so by Lemma 2.1, $\Upsilon_{\varphi_n,k}$ is a homeomorphism onto its range and is thus $\Upsilon_{\varphi_n,k} : [0,1] \to \varprojlim\{\varphi_n\}$ is an embedding. *Q. E. D.*

CHAPTER 4

AN EXAMPLE: TWO DIFFERENT UNIMODAL GENERALIZED $\sin\frac{1}{x}$ CONTINUA

This chapter contains two examples of unimodal continua that are also generalized $\sin\frac{1}{x}$ spaces, referred to from here on as unimodal generalized $\sin\frac{1}{x}$ continua. We will see that these two examples of continua are not homeomorphic to each other.

Let us first look at some motivational material. Define $f : I = [0,1] \to I$ to be the continuous, piecewise linear, unimodal function:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

We next consider $f \circ f = f^2$ which is:

$$f^2(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq \frac{1}{4} \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

A quick look at the sketch of the graph of f^2 below (Figure 8) indicates that f^2 satisfies the assumptions (a)-(e) of our φ_n functions in Chapter 2, so by Lemma 3.3, $\varprojlim f^2$ is a generalized $\sin\frac{1}{x}$ space. The same thing goes for f^4 sketched below (Figure 9). In fact, $\varprojlim f \simeq \varprojlim f^2 \simeq \varprojlim f^4$ by Corollary 2.2. But f is a unimodal function, so $\varprojlim f^2$ is a unimodal generalized $\sin\frac{1}{x}$ continuum. It turns out that $\varprojlim f^2$ is homeomorphic to the space illustrated in Figure 10 ([11]).

It occurred to us that perhaps a function similar to f^4, but with one of the two "spikes" shortened (see Figure 11), repeated as the only bonding map would yield a different inverse limit space than that of $\varprojlim f^2$. The bonding map pictured in Figure 11 has an inverse limit space homeomorphic to the space illustrated in Figure 12 (This will be clear from a later proof).

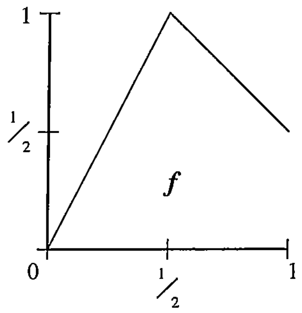


Figure 7: Function f

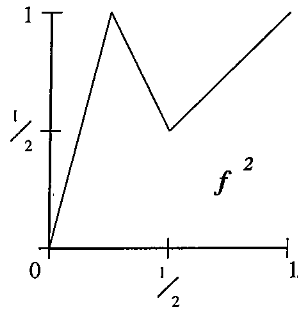


Figure 8: Function f^2

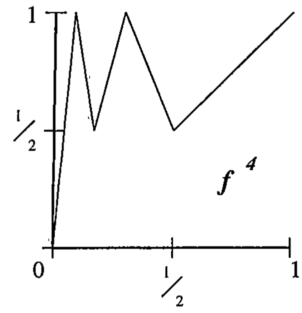


Figure 9: Function f^4

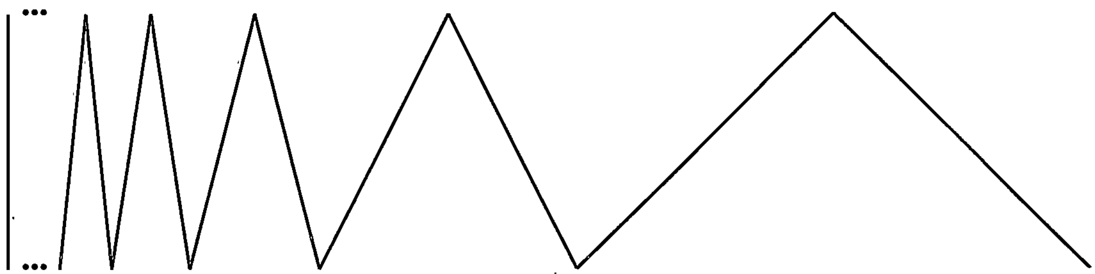


Figure 10: A Visualization of $\varprojlim f$

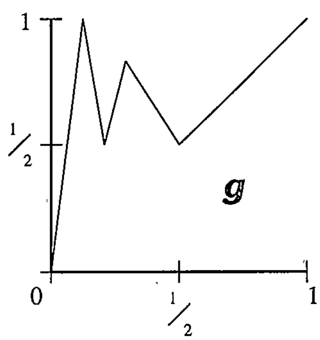


Figure 11: Function g

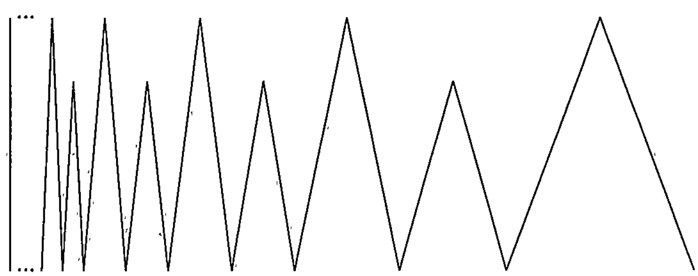


Figure 12: A Visualization of $\varprojlim g$

But because we are interested in unimodal continua, we wish to find a few unimodal functions from I to I so that the composition of these unimodal functions will have a graph whose sketch looks like Figure 11. After many hours composing piecewise linear unimodal functions, we have succeeded.

Let u, v, and z be functions from I onto I defined by:

$$u(x) = \begin{cases} \frac{4}{3}x & \text{if } 0 \leq x \leq \frac{3}{4} \\ -x + \frac{7}{4} & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$v(x) = \begin{cases} \frac{8}{3}x & \text{if } 0 \leq x \leq \frac{3}{8} \\ -x + \frac{11}{8} & \text{if } \frac{3}{8} \leq x \leq 1 \end{cases}$$

$$z(x) = \begin{cases} \frac{5}{4}x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4}x + \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

We pause for a look at the sketches of the graphs of these functions.

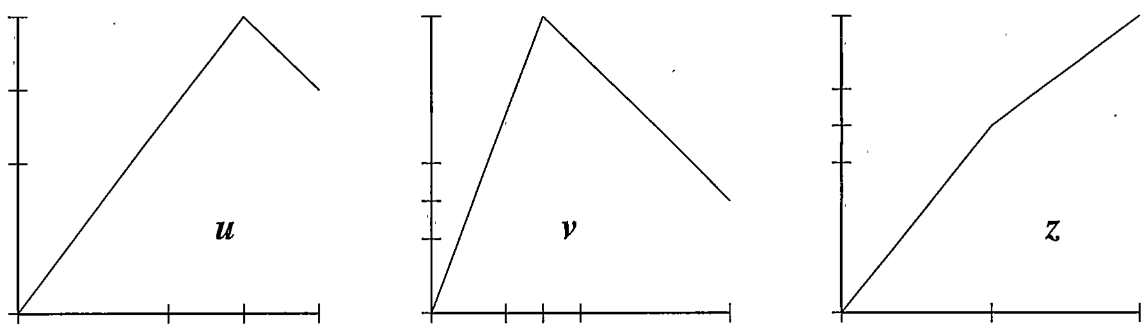


Figure 13: Functions u, v and z

Now let:

$$g(x) = f \circ u \circ v \circ z(x) = \begin{cases} \frac{80}{9}x & \text{if } 0 \leq x \leq \frac{9}{80} \\ -\frac{40}{9}x + \frac{3}{2} & \text{if } \frac{9}{80} \leq x \leq \frac{9}{40} \\ \frac{10}{3}x - \frac{1}{4} & \text{if } \frac{9}{40} \leq x \leq \frac{3}{10} \\ -\frac{5}{4}x + \frac{9}{8} & \text{if } \frac{3}{10} \leq x \leq \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then Figure 11 is the sketch of the graph of g. If we let $\tau_{4n} = f, \tau_{4n+1} = u, \tau_{4n+2} = v$ and $\tau_{4n+3} = z$ for $n = 0, 1, 2, \ldots$, then $\varprojlim\{\tau_i\}$ is a unimodal continuum, but

$\varprojlim\{\tau_i\} \simeq \varprojlim g$ by Corollary 2.2. Since g also satisfies the assumptions (a)-(e) in Chapter 3, $\varprojlim g$ is a unimodal generalized $\sin \frac{1}{x}$ continuum. We will show that $\varprojlim f^2$ is not homeomorphic to $\varprojlim g$.

For ease of notation, let $X = \varprojlim g$ and $Y = \varprojlim f^2$. Let A_X be the "limit bar" of X which is formally defined as $A_X = \{\underline{x} = (x_0, x_1, x_2, \dots) \in X : x_i \in [\frac{1}{2}, 1]$ for $i = 0, 1, 2, \dots\}$. We define A_Y to be the "limit bar" of Y, so $A_Y = \{\underline{y} = (y_0, y_1, y_2, \dots) \in Y : y_i \in [\frac{1}{2}, 1]$ for $i = 0, 1, 2, \dots\}$. As mentioned in the previous chapter, points in A_X and A_Y have the form $(x_0, x_0, x_0, \dots)$. Let $B_X = X \setminus A_X$ and $B_Y = Y \setminus A_Y$, so that A_X and A_Y are each compact while B_X and B_Y are non-compact rays.

Recall that we defined $\Upsilon_{\varphi_n,k} : [0,1] \to \varprojlim\{\varphi_n\}$ in the last chapter by $\Upsilon_{\varphi_n,k}(t) = (\varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_k(t), \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \dots \circ \varphi_k(t), \dots, \varphi_k(t), t, \gamma_{k+2}(t), \gamma_{k+2} \circ \gamma_{k+3}(t), \dots)$, where $\gamma_n = (\varphi_n|_{[0,c_n]})^{-1}$. Next we define $\underline{y}_k = \Upsilon_{g,k}(\frac{1}{2})$, for all $k \in \mathbb{N}$. Let us also adopt the notation $\underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$.

Let us take a closer look at the sketch of the graph of the functions g and f^2 in Figures 14 and 15 below.

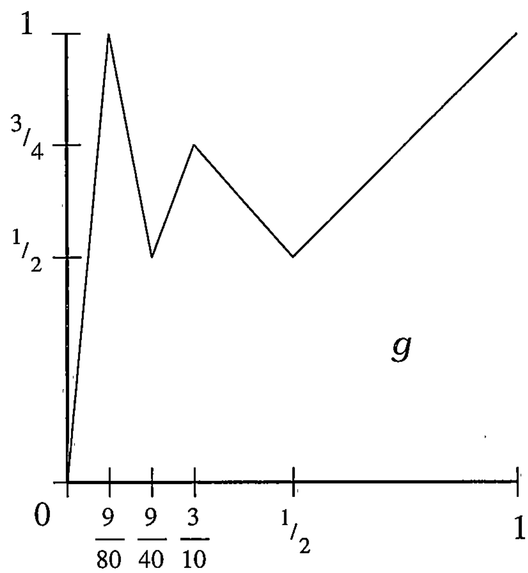


Figure 14: Details of g

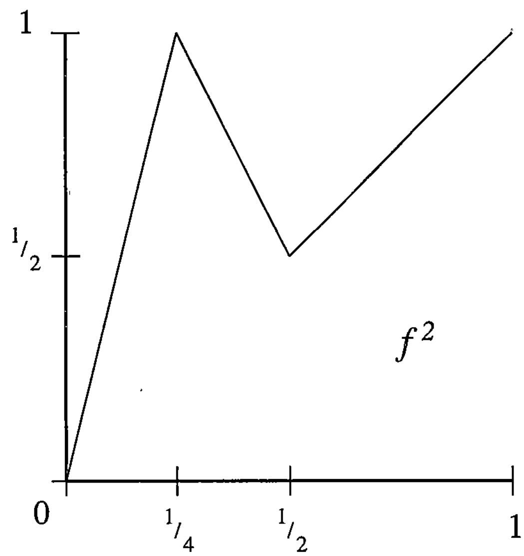


Figure 15: Details of f^2

Lemma 4.1 *The sequence* $\{\underline{y}_k\}_{k=0}^{\infty}$ *converges to* $\underline{\frac{1}{2}}$.

Proof: Let $\epsilon > 0$ be given. There exists an $N \in \mathbb{N}$ so that $(\sum_{i=N+2}^{\infty} \frac{1}{2^i}) < \epsilon$. Note that $\underline{y}_k = \Upsilon_{g,k}(\frac{1}{2}) = (g^k(\frac{1}{2}), g^{k-1}(\frac{1}{2}), \ldots, g(\frac{1}{2}), \frac{1}{2}, g_1^{-1}(\frac{1}{2}), g_1^{-2}(\frac{1}{2}), \ldots)$ where $g_1 = g|_{[0,\frac{9}{80}]}$. But $g(\frac{1}{2}) = \frac{1}{2}$ so $\underline{y}_k = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, g_1^{-1}(\frac{1}{2}), g_1^{-2}(\frac{1}{2}), \ldots)$. Hence for $k \geq N$, if we let $y_{k,i}$ be the i_{th} coordinate of $\underline{y}_k$, then $\underline{d}(\underline{y}_k, \underline{\frac{1}{2}}) = \sum_{i=0}^{\infty} \frac{|y_{k,i} - \frac{1}{2}|}{2^i} = (\sum_{i=0}^{N} \frac{\frac{1}{2}-\frac{1}{2}}{2^i}) + (\sum_{i=N+1}^{\infty} \frac{|y_{k,i}-\frac{1}{2}|}{2^i}) \leq 0 + \sum_{i=N+1}^{\infty} \frac{1}{2^i} < 0 + \epsilon = \epsilon$. Therefore $\underline{y}_k$ converges to $\underline{\frac{1}{2}}$. *Q. E. D.*

Next let $\underline{w}_k = \Upsilon_{g,k}(\frac{9}{40})$, for all $k \in \mathbb{N}$.

Lemma 4.2 *The sequence* $\{\underline{w}_k\}_{k=0}^{\infty}$ *converges to* $\underline{\frac{1}{2}}$.

Proof: Notice that $g(\frac{9}{40}) = \frac{1}{2}$ and $g(\frac{1}{2}) = \frac{1}{2}$, so $\underline{w}_k = \Upsilon_{g,k}(\frac{9}{40}) = (g^k(\frac{9}{40}), g^{k-1}(\frac{9}{40}), \ldots, g(\frac{9}{40}), \frac{9}{40}, g_1^{-1}(\frac{9}{40}), g_1^{-2}(\frac{9}{40}), \ldots) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{9}{40}, g_1^{-1}(\frac{9}{40}), g_1^{-2}(\frac{9}{40}), \ldots)$. The rest of this proof is similar to the last proof. *Q. E. D.*

Let $\underline{x}_k$ be defined by $\underline{x}_k = \Upsilon_{g,k}(\frac{3}{10})$, for all $k \in \mathbb{N}$.

Lemma 4.3 *The sequence* $\{\underline{x}_k\}_{k=0}^{\infty}$ *converges to* $\underline{\frac{3}{4}} = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots)$.

Proof: Similar to the proof above, we quickly notice that $g(\frac{9}{34}) = \frac{3}{4}$ and $g(\frac{3}{4}) = \frac{3}{4}$. Thus $\underline{x}_k = \Upsilon_{g,k}(\frac{9}{34}) = (g^k(\frac{9}{34}), g^{k-1}(\frac{9}{34}), \ldots, g(\frac{9}{34}), \frac{9}{34}, g_1^{-1}(\frac{9}{34}), g_1^{-2}(\frac{9}{34}), \ldots) = (\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{9}{34}, g_1^{-1}(\frac{9}{34}), g_1^{-2}(\frac{9}{34}), \ldots)$ Then by an analogous proof to that of Lemma 4.1, $\underline{x}_k$ converges to $\underline{\frac{3}{4}} \in A_X$. *Q. E. D.*

Next we state the pivotal theorem for this chapter.

Theorem 4.4 *The space* $\varprojlim g = X$ *is not homeomorphic to the space* $\varprojlim f^2 = Y$.

Proof: Assume, by way of contradiction, that X is homeomorphic to Y and that $h : X \to Y$ is a homeomorphism. Recall that from the previous chapter and the definition of A_X and B_X, the sets A_X and B_X are the arc components of $X = \varprojlim g$ and similarly A_Y and B_Y are the arc components of $Y = \varprojlim f^2$. Since h is a homeomorphism, compact A_X maps to compact A_Y and thus $h(B_X) = B_Y$. Also the end points of A_X must map to the end points A_Y, so without loss of generality, assume $h(\underline{\frac{1}{2}}) = \underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) \in A_Y$ and $h(\underline{1}) = \underline{1} \in A_Y$. Hence $h(\underline{w}_k)_{k=1}^{\infty}$ and $h(\underline{y}_k)_{k=1}^{\infty}$ must converge to $\underline{\frac{1}{2}} \in A_Y$. Since $\frac{3}{10} \in (\frac{9}{40}, \frac{1}{2})$ and $\Upsilon_{g,k}$ is an embedding, $\Upsilon_{g,k}(\frac{3}{10}) = \underline{x}_k$ lies on the ray B_X between $\underline{w}_k$ and $\underline{y}_k$. Now then $\{h(\underline{w}_k)\}_{k=1}^{\infty}$ and $\{h(\underline{y}_k)\}_{k=1}^{\infty}$ converge to $\underline{\frac{1}{2}}$, since $\{\underline{w}_k\}_{k=1}^{\infty}$ and $\{\underline{y}_k\}_{k=1}^{\infty}$ converge to $\underline{\frac{1}{2}}$ and $h(\underline{\frac{1}{2}}) = \underline{\frac{1}{2}}$. Also $\{h(\underline{x}_k)\}_{k=1}^{\infty}$ converges to $h(\underline{\frac{3}{4}})$ with $\underline{\frac{1}{2}} \neq h(\underline{\frac{3}{4}}) \neq \underline{1}$. For notation, let $h(\underline{w}_k) = (w_{k,0}, w_{k,1}, w_{k,2}, \dots)$ with analogous definitions for $h(\underline{y}_k)$ and $h(\underline{x}_k)$ and since $\{h(\underline{x}_k)\}_{k=1}^{\infty}$ converges to $h(\underline{\frac{3}{4}})$, define $h(\underline{\frac{3}{4}}) = (x_{0,0}, x_{0,1}, x_{0,2}, \dots)$, but since $h(\underline{\frac{3}{4}}) \in A_Y$, $h(\underline{\frac{3}{4}}) = (x_{0,0}, x_{0,1}, x_{0,2}, \dots) = (x_{0,0}, x_{0,0}, x_{0,0}, \dots)$. Because $h(\underline{\frac{3}{4}}) \neq \underline{\frac{1}{2}}$, $x_{0,0} \neq \frac{1}{2}$.

Let $\epsilon = \min\{\frac{1}{16}, \frac{1}{3}(|\frac{1}{2} - x_{0,0}|)$, then there is an $N \in \mathbb{N}$ so that $\underline{d}(\underline{\frac{1}{2}}, h(\underline{w}_k)) < \epsilon$, $\underline{d}(\underline{\frac{1}{2}}, h(\underline{y}_k)) < \epsilon$ and $\underline{d}(h(\underline{\frac{3}{4}}), h(\underline{x}_k)) < \epsilon$ for all $k \geq N$.

Next we will show that for any fixed $k \geq N$, there exists an $m \in \mathbb{N}$ so that $|w_{k,m} - y_{k,m}| > \frac{5}{16}$. Suppose, by way of contradiction, that $|w_{k,m} - y_{k,m}| \leq \frac{5}{16}$ for all $m \in \mathbb{N}$. First we will show that $w_{k,0}$ and $y_{k,0}$ are elements of $[\frac{1}{2}, 1]$. If $w_{k,0} \in [0, \frac{1}{2})$ then since the only preimage (under f^2) of $[0, \frac{1}{2})$ is $[0, \frac{1}{8})$, $w_{k,1} \in [0, \frac{1}{8})$, thus $|\frac{1}{2} - w_{k,1}| > |\frac{1}{2} - \frac{1}{8}| = \frac{3}{8}$ and so $\underline{d}(\underline{\frac{1}{2}}, h(\underline{w}_k)) = \sum_{i=0}^{\infty} \frac{|\frac{1}{2} - w_{k,i}|}{2^i} \geq \frac{|\frac{1}{2} - w_{k,1}|}{2} \geq \frac{\frac{3}{8}}{2} = \frac{3}{16} > \frac{1}{16} \geq \epsilon$, that is to say $\underline{d}(\underline{\frac{1}{2}}, h(\underline{w}_k)) > \epsilon$. But we selected N so that $\underline{d}(\underline{\frac{1}{2}}, h(\underline{w}_k)) < \epsilon$ for all $n \geq N$, therefore $w_{k,0} \notin [0, \frac{1}{2})$ and it must be that $w_{k,0} \in [\frac{1}{2}, 1]$. By a similar argument, we know $y_{k,0} \in [\frac{1}{2}, 1]$ also. However, since $h(\underline{w}_k), h(\underline{y}_k) \in B_Y$, there exists an $i \in \mathbb{N}$ such that $w_{k,i} \in [0, \frac{1}{2})$ and a $j \in \mathbb{N}$ so that $y_{k,j} \in [0, \frac{1}{2})$. Again since the only preimage of $[0, \frac{1}{2})$ is $[0, \frac{1}{8})$, $w_{k,i+n} \in [0, \frac{1}{2})$ and $y_{k,i+n} \in [0, \frac{1}{2})$ for all $n = 0, 1, 2, \dots$. Now let i^* be the smallest integer so that $w_{k,i^*} \in [0, \frac{1}{2})$ and let j^* be the smallest integer so that $y_{k,j^*} \in [0, \frac{1}{2})$. Thus w_{k,i^*-1} and y_{k,j^*-1} are in $[\frac{1}{2}, 1]$.

Now because $f^2(x) = x$ for $x \in [\frac{1}{2}, 1]$, $w_{k,i} = w_{k,i-1}$ for $i < i^*$, we get $|\frac{1}{2} - w_{k,i^*-1}| = |\frac{1}{2} - w_{k,0}| \leq \sum_{i=0}^{\infty} \frac{|\frac{1}{2}-w_{k,i}|}{2^i} = \underline{d}(\underline{\frac{1}{2}}, h(\underline{w}_k)) < \epsilon$. Therefore $w_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2}+\epsilon)$. The preimage of $[\frac{1}{2}, \frac{1}{2}+\epsilon)$ under f^2 is $[\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4}) \cup (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2}+\epsilon)$ and $w_{k,i^*} \in [0, \frac{1}{2})$ with $f^2(w_{k,i^*}) = w_{k,i^*-1}$, so $w_{k,i^*} \in [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4}) \cup (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$. Similarly $y_{k,j^*} \in [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4}) \cup (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$. First, let us consider the case where $i^* < j^*$. Then $y_{k,i^*} \in [\frac{1}{2}, \frac{1}{2}+\epsilon)$ and we assumed $|w_{k,i^*} - y_{k,i^*}| \leq \frac{5}{16}$ and $\epsilon \leq \frac{1}{16}$, so $w_{k,i^*} \notin [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4}) \subset [\frac{1}{8}, \frac{1}{8}+\frac{1}{64})$, thus $w_{k,i^*} \in (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$ and then $f^2(w_{k,i^*}) = w_{k,i^*+1} \in [0, \frac{1}{8})$ which is the only preimage under f^2 of $[0, \frac{1}{2}) \supset (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$. Therefore, using the assumption that $|w_{k,i^*+1} - y_{k,i^*+1}| \leq \frac{5}{16}$, $y_{k,i^*+1} \notin [\frac{1}{2}, \frac{1}{2}+\epsilon)$ and $y_{k,i^*+1} \notin (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$ so $y_{k,i^*+1} \in [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4})$, hence $j^* = i^*+1$. By a analogous argument, if $j^* < i^*$, then $i^* = j^*+1$ and $y_{k,j^*+1} \in [0, \frac{1}{8})$ with $w_{k,j^*+1} \in [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4})$. If it is the case that $i^* = j^*$, then y_{k,i^*}, $w_{k,i^*} \in [\frac{1}{8}, \frac{1}{8}+\frac{\epsilon}{4}) \cup (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2})$ and $y_{k,i^*+1}, w_{k,i^*+1} \in [0, \frac{1}{8})$. In any case, $w_{k,i^*+1}, y_{k,i^*+1} \in [0, \frac{1}{8}+\frac{\epsilon}{4})$. Now assume, without loss of generality, that $w_{k,i^*+1} < y_{k,i^*+1}$ and consider $\Upsilon_{f^2,i^*+1}([w_{k,i^*+1}, y_{k,i^*+1}])$. Since Υ_{f^2,i^*+1} is an embedding, $\Upsilon_{f^2,i^*+1}([w_{k,i^*+1}, y_{k,i^*+1}])$ is an arc in B_Y with endpoints $h(\underline{w}_k)$ and $h(\underline{y}_k)$. But $h(\underline{x}_k)$ is on that arc between $h(\underline{w}_k)$ and $h(\underline{y}_k)$, so $x_{k,i^*+1} \in [w_{k,i^*+1}, y_{k,i^*+1}] \subset [0, \frac{1}{8}+\frac{\epsilon}{4})$ and f^2 is strictly increasing on $[0, \frac{1}{8}+\frac{\epsilon}{4})$ thus $x_{k,i^*} \in f^2([w_{k,i^*+1}, y_{k,i^*+1}]) = [w_{k,i*}, y_{k,i^*}] \subset (\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2}+\epsilon)$ Then $f^2(x_{k,i^*}) = x_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2}+\epsilon)$ and since $f^2(x) = x$ for $x \in [\frac{1}{2}, 1]$, we see that $x_{k,0} = x_{k,i^*-1} \in [\frac{1}{2}, \frac{1}{2}+\epsilon)$. Also $|x_{0,0} - x_{k,0}| \leq \sum_{i=0}^{\infty} \frac{|x_{0,i}-x_{k,i}|}{2^i} = \underline{d}(h(\underline{\frac{3}{4}}), h(\underline{x}_k)) < \epsilon$. Therefore $|x_{0,0} - \frac{1}{2}| \leq |x_{0,0} - x_{k,0}| + |x_{k,0} - \frac{1}{2}| < \epsilon + \epsilon = 2\epsilon$. But by the selection of ϵ, $\epsilon \leq \frac{1}{3}|\frac{1}{2} - x_{0,0}|$, so $3\epsilon \leq |\frac{1}{2} - x_{0,0}| < 2\epsilon$, a clear contradiction. Hence must there exist an $m \in \mathbb{N} \cup 0$ so that $|w_{k,m} - y_{k,m}| > \frac{5}{16}$ as desired.

Since $w_{k,i} \in [\frac{1}{2}, \frac{1}{2}-\epsilon)$ for $i < i^*$ and $y_{k,i} \in [\frac{1}{2}, \frac{1}{2}-\epsilon)$ for $i < j^*$, it must be that $|w_{k,i} - y_{k,i}| < \epsilon$ for $i < \min\{i^*, j^*\}$ so $m \geq \min\{i^*, j^*\}$. Hence either $w_{k,m} \in [0, \frac{1}{2})$ or

$y_{k,m} \in [0, \frac{1}{2})$ and $|w_{k,m} - y_{k,m}| > \frac{5}{16}$. Without loss of generality, assume $w_{k,m} < y_{k,m}$, so the diameter of the interval from $w_{k,m}$ to $y_{k,m}$, denoted $\mathrm{diam}([w_{k,m}, y_{k,m}])$, is greater than $\frac{5}{16}$ and $y_{k,m} < \frac{1}{2} + \epsilon \leq \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$, therefore $\frac{1}{4} \in [w_{k,m}, y_{k,m}]$. Define $\underline{b}_k = \Upsilon_{f^2,k}(\frac{1}{4}) = ((f^2)^k(\frac{1}{4}), (f^2)^{k-1}(\frac{1}{4}), \ldots, f^2(\frac{1}{4}), \frac{1}{4}, \frac{1}{16}, \ldots) = (1, 1, 1, \ldots, \frac{1}{4}, \frac{1}{16}, \ldots)$ where $\frac{1}{4}$ is in the $(k+1)^{st}$ position. By a similar proof to that we have seen before, it can be shown $\{\underline{b}_k\}_{k=1}^{\infty}$ converges to $\underline{1} = (1, 1, 1, \ldots) \in A_Y$. Since h is a homeomorphism and we assumed $h(\underline{1}) = \underline{1}$, $\{h^{-1}(\underline{b}_k)\}_{k=1}^{\infty}$ converges to $\underline{1} \in A_X$. Now for any fixed k, because $\frac{1}{4} \in [w_{k,m}, y_{k,m}]$, $\underline{b}_k$ is between $h(\underline{w}_k)$ and $h(\underline{y}_k)$ on the ray B_Y, so $h^{-1}(\underline{b}_k)$ is between $\underline{w}_k$ and $\underline{y}_k$ on the ray B_X. If we let $h^{-1}(\underline{b}_k) = (b_{k,0}, b_{k,1}, b_{k,2}, \ldots)$, it must be that $b_{k,k} \in [\frac{9}{40}, \frac{1}{2}]$ since $\Upsilon_{g,k}$ is an embedding and $\Upsilon_{g,k}[\frac{9}{40}, \frac{1}{2}]$ is an arc from $\underline{w}_k$ to $\underline{y}_k$, thus containing $h^{-1}(\underline{b}_k)$. Because $\{h^{-1}(\underline{b}_k)\}_{k=1}^{\infty}$ converges to $\underline{1}$, there exists an $l \in \mathbb{N}$ so that $\underline{d}(h^{-1}(\underline{b}_k), \underline{1}) < \frac{1}{16}$ for all $k \geq l$. So $|b_{k,0} - 1| \leq \sum_{i=0}^{\infty} \frac{|b_{k,i} - 1|}{2^i} = \underline{d}(h^{-1}(\underline{b}_k), \underline{1}) < \frac{1}{16}$. But the largest $g(x)$ gets for $x \in [\frac{9}{40}, \frac{1}{2}]$ is $\frac{3}{4}$, so $b_{k,k-1} \leq \frac{3}{4}$ and since $g(x) = x$ for $x \in [\frac{1}{2}, 1]$, $b_{k,k-1} = b_{k,0} \leq \frac{3}{4}$. That contradicts $|b_{k,0} - 1| < \frac{1}{16}$.

Therefore h cannot be a homeomorphism and thus $X = \varprojlim g$ is not homeomorphic to $Y = \varprojlim f^2$. *Q. E. D.*

Thus we have seen that X and Y are two distinct examples of unimodal generalized $\sin \frac{1}{x}$ continua.

We note here that $X = \varprojlim g$ is homeomorphic to the M-continuum discussed by Sam Nadler in [13], however Nadler does not create it as a unimodal continuum.

CHAPTER 5

AN UNCOUNTABLE COLLECTION

In this chapter we will construct a collection of unimodal generalized $\sin\frac{1}{x}$ continua for which uncountably many are non-homeomorphic with each other.

Let $\underline{a} = (a_0a_1a_2\ldots)$ be an element of the uncountable set $\{0,1\}^{\mathbb{N}}$ (see [12], pg. 50), that is $\underline{a}$ is an infinitely long string of 0's and 1's. Suppose we divide $\{0,1\}^{\mathbb{N}}$ into equivalence classes. Let $(a_0a_1a_2\ldots)$ and $(b_0b_1b_2\ldots)$ be in the same equivalence class if and only if there exists a $k \in \mathbb{Z}$ and an $N \in \mathbb{Z}^+$ so that $b_{i+k} = a_i$ for all $i \geq N$. Define C to be set of these equivalence classes.

Lemma 5.1 *The set C is uncountable.*

Proof: First we recognize that for any fixed k, N can take on only countably many values. Since k must be an integer, for any given $(a_0a_1a_2\ldots)$, the equivalence class of $(a_0a_1a_2\ldots)$ is a countable collection of countable sets, and thus is countable. If we assume, by way of contradiction, that C is countable, then $\{0,1\}^{\mathbb{N}}$ is the union of countably many equivalence class, each with countably many elements, and hence is countable. But we know $\{0,1\}^{\mathbb{N}}$ is uncountable. This contradiction proves that C is an uncountable set. *Q. E. D.*

Use the functions g and f^2 that were defined in the last chapter and let $\eta_0 = g$ and $\eta_1 = f^2$, then let $h_i = \eta_{a_i}$. Thus $h_i : I \to I$ is equal to g if $a_i = 0$ and equal to f^2 if $a_i = 1$. So, by definition, $\varprojlim\{h_i\}$ is a unimodal generalized $\sin\frac{1}{x}$ continuum. Hence for each member of $\{0,1\}^{\mathbb{N}}$ there is a corresponding inverse limit space.

Now let $\underline{a} = (a_0a_1a_2\ldots)$ and $\underline{b} = (b_0b_1b_2\ldots)$ be elements of $\{0,1\}^{\mathbb{N}}$ and let $\varprojlim\{h_i\}$ correspond to $\underline{a}$ and $\varprojlim\{h_i^*\}$ correspond to $\underline{b}$.

Theorem 5.2 *The continuum* $\varprojlim\{h_i\}$ *is homeomorphic to the continuum* $\varprojlim\{h_i^*\}$ *if and only if* $\underline{a}$ *and* $\underline{b}$ *are in the same equivalence class of* C.

Proof: Let us start by assuming that there exists a $k \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$ so that $b_{i+k} = a_i$ for all $i \geq N$. This could also be written as $b_{N+k+i} = a_{N+i}$ for $i = 0, 1, 2, \ldots$. Therefore $\eta_{b_{N+k+i}} = \eta_{a_{N+i}}$ so $h^*_{N+k+i} = h_{N+i}$. Now if we apply Corollary 2.1 we get that $\varprojlim\{h_i^*\} = \varprojlim\{h_i\}$, as desired.

Next assume that $\varprojlim\{h_i\} \simeq \varprojlim\{h_i^*\}$. Let $H : \varprojlim\{h_i\} \to \varprojlim\{h_i^*\}$ be a homeomorphism. If we define the "limit bars" of $\varprojlim\{h_i\}$ and $\varprojlim\{h_i^*\}$ as in the previous chapter, that is $\underline{x} = (x_0, x_0, x_0, \ldots)$ is in the limit bar for $x_0 \in [\frac{1}{2}, 1]$, it must be that H takes the limit bar of $\varprojlim\{h_i\}$ to the limit bar of $\varprojlim\{h_i^*\}$ and similarly the ray maps to the ray. Consider the sequence of points in $\varprojlim\{h_i\}$ defined by $\underline{p}_1 = (1, p_{1,1}, p_{1,2}, p_{1,3}, \ldots)$, $\underline{p}_2 = (1, 1, p_{2,2}, p_{2,3}, p_{2,4}, \ldots)$, $\underline{p}_3 = (1, 1, 1, p_{3,3}, p_{3,4}, p_{3,5}, \ldots)$ and so on, with the general term being $\underline{p}_i = (1, 1, 1, \ldots, 1, p_{i,i}, p_{i,i+1}, p_{i,i+2}, \ldots)$ where $p_{i,j} \leq \frac{1}{2}$ for $j \geq i$. Clearly $\{\underline{p}_i\}_{i=1}^{\infty}$ converges to $\underline{1} = (1, 1, 1, \ldots)$ which is an endpoint of the limit bar of $\varprojlim\{h_i\}$. Similarly define $\underline{q}_i = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, q_{i,i}, q_{i,i+1}, q_{i,i+2}, , \ldots)$ so $\{q_i\}_{i=1}^{\infty}$ converges to $\underline{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, the other endpoint of the limit bar of $\varprojlim\{h_i\}$. So let $\{\underline{c}_i\}$ be a sequence of points of the form $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{3}{10}, c_{i,j}, c_{i,j+1}, c_{i,j+2}, \ldots)$. Note that since $\frac{3}{10} < \frac{1}{2}$, $c_{i,j+m}$ is completely determined for $m = 0, 1, 2, \ldots$. There may be none of these points $\underline{c}_i$, a finite number of these points, or they may constitute an infinite sequence. If these points form an infinite sequence, $\{\underline{c}_i\}_{i=1}^{\infty}$ converges to $\underline{\frac{3}{4}}$. Notice that since $g(\frac{3}{10}) = \frac{3}{4} \neq f^2(\frac{3}{10})$, each point of $\{\underline{c}_i\}$ corresponds a different function of h_i being equal to the function g, so $a_n = 0$ where n is different for each different i.

We similarly define $\{\underline{x}_i\}$, $\{\underline{y}_i\}$ and $\{\underline{z}_i\}$ to be sequences in $\varprojlim\{h_i^*\}$ with $\underline{x}_i = (1, 1, 1, \ldots, 1, x_{i,i}, x_{i,i+1}, x_{i,i+2}, \ldots)$, $\underline{y}_i = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, y_{i,i}, y_{i,i+1}, y_{i,i+2}, \ldots)$ and $\underline{z}_i = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, \frac{3}{10}, z_{i,j}, z_{i,i+1}, z_{i,i+2}, \ldots)$ and further assume we number the points $\underline{c}_i$

and $\underline{z}_i$ so the larger the i, the closer we are to the limit bar. As before, it can be shown that $\{\underline{x}_i\}_{i=1}^{\infty} \longrightarrow \underline{1} \in \underleftarrow{\lim}\{h_i^*\}$, $\{\underline{y}_i\}_{i=1}^{\infty} \longrightarrow \underline{\frac{1}{2}} \in \underleftarrow{\lim}\{h_i^*\}$ and, if $\{\underline{z}_i\}$ is an infinite sequence, $\{\underline{z}_i\}_{i=1}^{\infty} \longrightarrow \underline{\frac{3}{4}} \in \underleftarrow{\lim}\{h_i^*\}$.

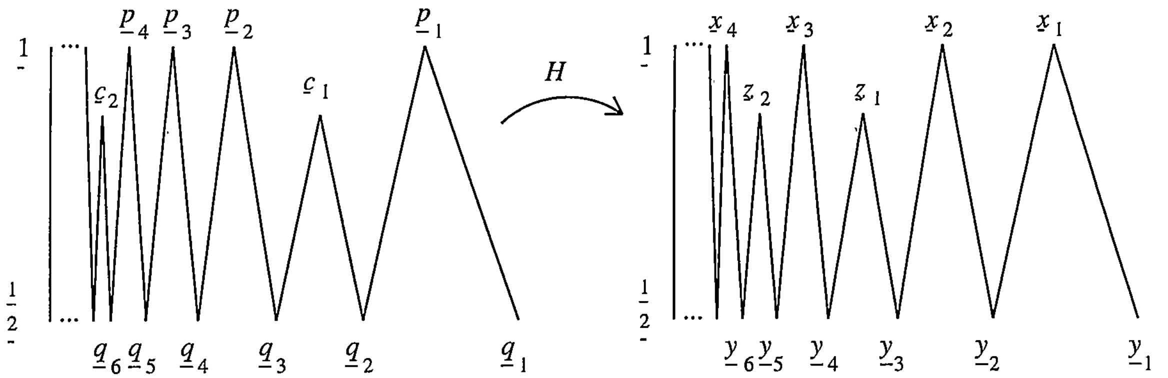


Figure 16: The Spaces $\underleftarrow{\lim}\{h_i\}$ and $\underleftarrow{\lim}\{h_i^*\}$

It is possible that neither $\{\underline{c}_i\}$ nor $\{\underline{z}_i\}$ is an infinite sequence, in which case for some $N_1 \in \mathbb{N}$, $h_i = f^2$ for $i \geq N_1$, for if $h_i = g$, then we would get another $\underline{c}_i$. Similarly there exists an $N_2 \in \mathbb{N}$ so that $h_i^* = f^2$ for $i \geq N_2$. Now let $N = \max\{N_1, N_2\}$ and let $k = N_2 - N_1$, so that $a_i = 1$ for $i \geq N$ since $\eta_{a_i} = h_i = f^2 = \eta_1$ and also $b_{i+k} = 1$ for $i \geq N$ because $\eta_{b_{i+k}} = h_i^* = f^2 = \eta_1$. Thus $b_{i+k} = a_i$ for $i \geq N$.

Next we consider the case where either $\{\underline{c}_i\}$ or $\{\underline{z}_i\}$ is finite and the other is infinite. Without loss of generality, assume $\{\underline{c}_i\}$ is finite and $\{\underline{z}_i\}_{i=1}^{\infty}$ is infinite. From the form of the coordinates of $\underline{c}_i$ versus $\underline{p}_i$, we know only finitely many of the bonding maps h_i are equal to g, the rest are equal to f^2. Then $\underleftarrow{\lim}\{h_i\}$ is homeomorphic to $\underleftarrow{\lim}f^2$ by Lemma 2.1, thus we have $\underleftarrow{\lim}\{h_i^*\}$ homeomorphic to $\underleftarrow{\lim}f^2$. However, since $\{\underline{z}_i\}_{i=1}^{\infty}$ is infinite, $h_i^* = g$ for infinitely many i's, we construct a proof very similar to the proof of Theorem 4.4 to show that $\underleftarrow{\lim}\{h_i^*\}$ is not homeomorphic to $\underleftarrow{\lim}f^2$. So to do this, assume G is a homeomorphism from $\underleftarrow{\lim}\{h_i^*\}$ onto $\underleftarrow{\lim}f^2$. Notate any arc in $\underleftarrow{\lim}\{h_i^*\}$ from $\underline{y}_n$ through $\underline{z}_m$ ending at $\underline{y}_{n+1}$ by $\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1} \rangle$ and look at $G(\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1} \rangle)$. Since $\{\underline{y}_i\}_{i=1}^{\infty} \longrightarrow \underline{\frac{1}{2}}$ and $\{\underline{z}_i\}_{i=1}^{\infty} \longrightarrow \underline{\frac{3}{4}}$, $\{G(\underline{y}_i)\}_{i=1}^{\infty}$ converges to an endpoint of the limit bar of $\varprojlim f^2$ and $\{G(\underline{z}_i)\}_{i=1}^{\infty}$ converges to something other than an endpoint. Thus for increasingly large m, $G(\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1}\rangle)$ contains a point $\underline{b}_m$ with the first several coordinates being the number 1, so that $\{\underline{b}_i\}_{i=1}^{\infty}$ converges to the opposite endpoint of the limit bar of $\varprojlim f^2$ from which $\{G(\underline{y}_i)\}_{i=1}^{\infty}$ converges. Then $\{G^{-1}(\underline{b}_i)\}_{i=1}^{\infty} \longrightarrow \underline{1}$ and $G^{-1}(\underline{b}_m) \in \langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1}\rangle$, but from the construction of $\underline{z}_i$ and the function g, we know that the first coordinate of every element of $\langle \underline{y}_n, \underline{z}_m, \underline{y}_{n+1}\rangle$ must be smaller than $\frac{3}{4}$, so $\{\underline{b}_i\}_{i=1}^{\infty}$ cannot converge to $\underline{1}$ hence $\varprojlim f^2$ is not homeomorphic to $\varprojlim\{h_i^*\}$. This is a clear contradiction, so it must not be the case that either $\{\underline{c}_i\}$ or $\{\underline{z}_i\}$ is finite and the other one is infinite.

Now let us assume both $\{\underline{c}_i\}_{i=1}^{\infty}$ and $\{\underline{z}_i\}_{i=1}^{\infty}$ are infinite sequences, that is they have infinitely many distinct points. Since H is a homeomorphism, $\{H^{-1}(\underline{x}_i)\}_{i=1}^{\infty}$, $\{H^{-1}(\underline{y}_i)\}_{i=1}^{\infty}$, and $\{H^{-1}(\underline{z}_i)\}_{i=1}^{\infty}$ all must converge to point on the limit bar of $\varprojlim\{h_i\}$. Define $\underline{z}_0$ to be the limit point of $\{H^{-1}(\underline{z}_i)\}_{i=1}^{\infty}$ on the limit bar of $\varprojlim\{h_i\}$, and since endpoints must map to endpoints, $\underline{z}_0 \neq \underline{1}$ and $\underline{z}_0 \neq \underline{\frac{1}{2}}$. Next consider $\{H(\underline{p}_i)\}_{i=1}^{\infty}$, $\{H(\underline{q}_i)\}_{i=1}^{\infty}$, and $\{H(\underline{c}_i)\}_{i=1}^{\infty}$ which must all converge in the limit bar of $\varprojlim\{h_i^*\}$. Again if we let $\{H(\underline{c}_i)\}_{i=1}^{\infty} \longrightarrow \underline{c}_0$ in the limit bar of $\varprojlim\{h_i^*\}$, $\underline{c}_0 \neq \underline{1}$ and $\underline{c}_0 \neq \underline{\frac{1}{2}}$. For now let $\underline{p}_0$ be the limit point of the sequence $\{H(\underline{p}_i)\}_{i=1}^{\infty}$ and similarly let $\underline{q}_0$, $\underline{x}_0$ and $\underline{y}_0$ be limit points of $\{H(\underline{q}_i)\}_{i=1}^{\infty}$, $\{H^{-1}(\underline{x}_i)\}_{i=1}^{\infty}$ and $\{H^{-1}(\underline{y}_i)\}_{i=1}^{\infty}$, respectively. Since H is a homeomorphism, each of these limits on an endpoint of the limit bar of the space the limit point is in. Let $\epsilon > 0$ be defined as follows:

$$\epsilon < \frac{1}{2}\min\{\underline{d}(\underline{1}, \underline{\tfrac{3}{4}}), \underline{d}(\underline{\tfrac{3}{4}}, \underline{\tfrac{1}{2}}), \underline{d}(\underline{1}, \underline{z}_0), \underline{d}(\underline{\tfrac{1}{2}}, \underline{z}_0), \underline{d}(\underline{1}, \underline{c}_0), \underline{d}(\underline{\tfrac{1}{2}}, \underline{c}_0)\}$$

We let $N^* \in \mathbb{N}$ be large enough so that all points of the sequences we have labeled are within an ϵ-ball of their limit point for all subscripts $i \geq N^*$. Choose $n, m \geq N^*$ so that for a given $\underline{c}_m$, $\underline{q}_n$ is the closest point of $\{\underline{q}_i\}_{i=1}^{\infty}$ to $\underline{c}_m$ along the ray from $\underline{c}_m$ to the endpoint of the ray part of $\varprojlim\{h_i\}$, then $\underline{q}_{n+1}$ is the closest point of $\{\underline{q}_i\}_{i=1}^{\infty}$

to $\underline{c}_m$ along the open end of the ray. Notate the arc from $\underline{q}_n$, through $\underline{c}_m$ and ending at $\underline{q}_{n+1}$ by $\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. If we look at $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, we know that $H(\underline{q}_n)$ and $H(\underline{q}_{n+1})$ are within ϵ of $\underline{q}_0$, but $H(\underline{c}_m)$ is not, it is within ϵ of $\underline{c}_0$. Thus if $\underline{q}_0 = 1$, $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ must pass through $\underline{y}_i$ for some i, but then $H^{-1}(\underline{y}_i) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. Now if $\underline{q}_0 = 1$ then $\underline{p}_0 = \frac{1}{2}$ and it must be that $\underline{y}_0 = 1$, so $H^{-1}(\underline{y}_i) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$ and $H^{-1}(y_i)$ is within ϵ of $\underline{1}$, which contradicts the selection of $\underline{q}_n$, $\underline{c}_m$ and $\underline{q}_{n+1}$, thus it must be the case that $\underline{q}_0 = \underline{\frac{1}{2}} \in \varprojlim\{h_i^*\}$, $\underline{p}_0 = \underline{1} \in \varprojlim\{h_i^*\}$, $\underline{y}_0 = \underline{\frac{1}{2}} \in \varprojlim\{h_i\}$ and $\underline{x}_0 = \varprojlim\{h_i\}$.

Now then, still considering $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, we know that $\underline{q}_n$ and $\underline{q}_{n+1}$ are within ϵ of $\underline{\frac{1}{2}}$ and that $\underline{c}_m$ is not within ϵ of $\underline{\frac{1}{2}}$, so $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ passes through either a point of $\{\underline{x}_i\}_{i=1}^{\infty}$, call it $\underline{x}_M$ or a point of $\{z_i\}_{i=1}^{\infty}$, labeled $\underline{z}_{M^*}$. If M (or M^*) is less than N^*, choose a larger n and m until we get $\underline{x}_M$ (or $\underline{z}_{M^*}$) so that $M \geq N$ ($M^* \geq N$). Thus either we have an $\underline{x}_M$ within ϵ of $\underline{1}$ or there is a $\underline{z}_{M^*}$ within ϵ of $\underline{\frac{3}{4}}$. Since $\underline{x}_i$ converges to $\underline{1}$, $H^{-1}(\underline{x}_i)$ converges to $\underline{1}$ and $M \geq N^*$, so $H^{-1}(\underline{x}_M)$ is within ϵ of $\underline{1}$, but since $\underline{x}_M \in H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$, $H^{-1}(\underline{x}_M) \in \langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle$. However $\underline{c}_m$ is the relative maximum of this arc, in other words $\underline{c}_m$ is the point with the largest first coordinate, which is $\frac{3}{4}$, so $\underline{x}_M$ cannot have a first coordinate larger than $\frac{3}{4}$, hence $H^{-1}(\underline{x}_M)$ cannot be arbitrarily close to $\underline{1}$. Therefore it is $\underline{z}_{M^*}$ in $H(\langle \underline{q}_n, \underline{c}_m, \underline{q}_{n+1} \rangle)$ and both $\underline{c}_m$ and $\underline{z}_{M^*}$, by virtue of the first coordinate being $\frac{3}{4}$ and the first coordinate less than $\frac{1}{2}$ being $\frac{3}{10}$, relate to the bonding map g in the inverse limit space. Hence $h_{a_m} = h_0 = g = h_0^* = h_{b_{M^*}}^*$ and $b_{M^*} = 0 = a_m$. Let $m = N$ and $k = M^* - m$ so $b_{i+k} = a_i$ for $i = N$.

This argument can then be repeated for $\underline{c}_{m+1}$. We know that the same k will work since $\underline{c}_{m+1}$ maps to $\underline{z}_{M^*+1}$, for if it did not, we could consider an arc $\langle \underline{y}_M, \underline{z}_{M^*+1}, \underline{y}_{M+1} \rangle$ and the arc $H^{-1}(\langle \underline{y}_M, \underline{z}_{M^*+1}, \underline{y}_{M+1} \rangle)$ and with an argument similar to above, arrive at a contradiction. Again repeat the argument for $\underline{c}_{m+2}$ and $\underline{z}_{M^*+2}$, and so on. Therefore we have $b_{i+k} = a_i$ for $i \geq N$ as desired. *Q. E. D.*

Define D to be the collection of all inverse limit spaces $\varprojlim\{h_i\}$ of this form, that is, spaces using f^2 and g as the only bonding maps, applied in different combinations.

Corollary 5.3 *The set D contains uncountably many pairwise nonhomeomorphic spaces.*

Proof: From the theorem above, there is a one to one correspondence between our set D and the set C from Lemma 5.1. *Q. E. D.*

So we have created an uncountable set of unimodal generalized $\sin \frac{1}{x}$ continua. The continua considered here are often called *arc + ray* continua for obvious reasons. We wish to point out that Karen Brucks and Henk Bruin ([6]) have proven that there are uncountably many nonhomeomorphic unimodal inverse limit spaces that are non arc+ray continua.

CHAPTER 6

A DIFFERENT UNCOUNTABLE COLLECTION

In previous chapters we created an uncountable collection of unimodal generalized $\sin\frac{1}{x}$ continua. If we look at the visualizations of these continua, like Figure 16, we could characterize the way we created this collection by saying we varied proximity of the critical points of height $\frac{3}{4}$ relative to the limit bar. Here we define height to be the first coordinate of our point. In all these cases, the critical points all have heights of $\frac{1}{2}$, $\frac{3}{4}$ or 1. In this chapter we develop an uncountable collection of unimodal generalized $\sin\frac{1}{x}$ continua by creating infinitely many different height critical points.

If all of the local extreme points that do not have a first coordinate of $\frac{1}{2}$ or 1 originate from the same side, say for example they start at the bottom and reach part way up, we will call the inverse limit space an M-continua. (We realize our figures show them looking more like a W shape than an M, but our Figure 5 is homeomorphic to what Sam Nadler, [13], refers to as M-continua, so for consistency, we will call the ones we created in the last two chapters M-continua as well.) In the next chapter we consider how to create what Nadler refers to as MW-continua, where the critical points of heights between $\frac{1}{2}$ and 1 are originating from both sides (top and bottom). Refer back to Figure 6.

Let us start by letting β be any number in $[\frac{1}{4}, \frac{1}{2}]$ and define the following functions parameterized by β.

Let $u_\beta : [0,1] \to [0,1]$ be defined by:

$$u_\beta(x) = \begin{cases} \frac{1}{2\beta}x & \text{if } 0 \le x \le 2\beta \\ -x + 2\beta + 1 & \text{if } 2\beta \le x \le 1 \end{cases}$$