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ORIGINAL PAPER

Encountering ideas about teaching and learning mathematics in undergraduate mathematics courses

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Abstract
We study the ideas about teaching and learning mathematics that undergraduate students generate when they encounter tasks designed to embed approximations of teaching practice in mathematics courses taken by a general population of students. These tasks attend to the dual goals of developing an understanding of mathematics content and an understanding of how teachers provide classroom experiences that foster mathematics learning. The study employs a qualitative, multiple-case study methodology, with four cases bounded by the content areas of abstract algebra, single variable calculus, discrete mathematics, and introductory statistics. The data for the study come from undergraduate students' written work on mathematical tasks, interviews with a subset of students from each course, and interviews with each instructor throughout the term during which they implemented the tasks. Our findings indicate that students identified the broad applicability of teaching skills (discussed by 32 of the 61 interviewed students), recognized the value of examining hypothetical learners' mathematical work (discussed by 59 of the 61 interviewed students), and reported empathy for hypothetical learners (discussed by 38 of the 61 interviewed students). These findings persisted across the course content and course levels we studied, leading us to conclude that our findings can transfer to additional mathematics courses in secondary mathematics teacher preparation.

Keywords Approximations of teaching practice · Prospective teachers · Secondary mathematics · Mathematics content courses

1 Introduction

How can teacher education programs support strong mathematics preparation among prospective secondary mathematics teachers in mathematics content courses? University mathematics coursework often assumes that prospective teachers will indirectly derive relevant knowledge about mathematics for teaching from this coursework (Schmidt et al., 2017; Tatto et al., 2010). Though there is wide variation in what such coursework entails, as a whole, it overlaps substantially with mathematics coursework taken by undergraduate students studying mathematics who do not intend to pursue teaching. Yet, research studies conducted worldwide have found that prospective secondary mathematics teachers complete their university studies without having a deep understanding of secondary mathematics content (Moreira & David, 2008; Speer et al., 2015) and find their undergraduate mathematical study disconnected from the practice of teaching secondary mathematics (Moreira & David, 2008; Winsor et al., 2020; Zazkis & Leikin, 2010).

University-level mathematics courses provide a venue where knowledge about mathematics and knowledge about teaching secondary mathematics can be integrated to address both mathematics content and content about teaching and learning mathematics (Lischka et al., 2020; Martin et al., 2020; Weber et al., 2021), and recent evidence suggests that mathematical tasks that link university mathematics courses and school mathematics can counter the disconnect prospective teachers have felt between their university-level

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coursework and secondary mathematics (Rach, 2022). The present study reports findings about how undergraduate students, both prospective secondary mathematics teachers and others studying mathematics, engaged with opportunities to learn mathematics content and learn about teaching mathematics through tasks included in undergraduate mathematics content courses. Throughout, we ground this research in a human context through our stance that mathematics teachers require a deep underlying understanding of the mathematics they will teach as well as of the learners they will teach mathematics to, because teachers' fluent understanding of the mathematical content they teach is always coupled with an understanding of how to interact with learners and their mathematical work (Álvarez et al., 2020; Baldinger, 2020; Ko & Rose, 2021).

2 Background and literature

There is general agreement that the mathematical knowledge needed for teaching differs from the mathematical knowledge needed for graduate school and other careers (e.g., Ball et al., 2008; Tatto et al., 2010). Thus, a central component of secondary mathematics teacher preparation involves creating experiences for prospective teachers to develop teaching knowledge; that is, to develop an understanding of mathematics as well as an understanding of teaching mathematics. Such experiences rely on content knowledge for teaching, incorporating pedagogical practice, and making connections between mathematics content of secondary school and undergraduate study. We examine some research literature that undergirds each of these features.

2.1 Frameworks about teaching knowledge

Shulman's work in 1986 highlighted that teachers must possess more than a separate understanding of content and pedagogy, and he introduced the phrase *pedagogical content knowledge*. He examined the transition from expert student to novice teacher and identified the distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK includes understanding the structures of the subject matter as well as understanding the difference between topics central to the discipline versus those that are not. PCK includes knowledge of ways to represent and formulate the subject to make it comprehensible to others and an understanding of "what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9).

Researchers have extended Shulman's work to develop different frameworks that describe types of knowledge for teaching that integrate mathematical content knowledge and pedagogical knowledge. For example, Dreher et al. (2018) theorized a construct, "school-related content knowledge" (p. 322), that attends to specific content knowledge that secondary mathematics teachers need. Carrillo-Yañez et al. (2018) constructed a model of mathematics teachers' "specialised knowledge" (p. 236), which focuses on the unique nature of knowledge for teaching that primary, secondary, and university level teachers use during instruction. Turner and Rowland (2011) also focus on how beginning teachers use their mathematical content knowledge during instruction, which they analyze using their "knowledge quartet" framework (p. 195).

Ball et al. (2008) refined Shulman's domains of knowledge for mathematics teaching into three sub-domains of content knowledge (common content knowledge, specialized content knowledge, and horizon content knowledge) and three sub-domains of pedagogical content knowledge (knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum) in their Mathematical Knowledge for Teaching (MKT) framework. MKT was developed as a construct for explaining elementary mathematics teaching practice, and researchers acknowledge that it is incomplete as a construct to explain secondary mathematics teaching practice. Speer et al. (2015) propose that secondary MKT consists of both subject matter knowledge and pedagogical content knowledge, as indicated in Ball et al.'s (2008) MKT framework, but clarify that common content knowledge, as a component of content knowledge, should refer to that knowledge held about mathematics that is common to all university mathematics majors.

In an alternative framing of knowledge for teaching, Wasserman (2018) differentiates the "relative location of mathematical ideas within a broader landscape" (p. 118). He describes the "local" and "non-local" mathematical neighborhoods that demarcate mathematical ideas that are relatively close to the mathematics being taught (e.g., ideas in first-year secondary school algebra) and mathematical ideas outside the scope of what is being taught (e.g., university-level mathematics), respectively.

Evident among each of these frameworks is the importance of both content knowledge and pedagogical knowledge for teaching, and Martin et al. (2020) encourage teacher preparation programs to support prospective teachers' encounters with these types of knowledge. In the present study, we explore prospective teachers' encounters with content and pedagogical knowledge within the context of undergraduate mathematics content courses that serve a wide range of students, including those who intend to pursue secondary teaching and those who do not. Our study is informed, in particular, by Wasserman's (2018) description of local and nonlocal mathematical neighborhoods and by Speer et al.'s (2015) extension of Ball et al.'s (2008) MKT construct to secondary teaching.

2.2 Approximations of teaching practice in prospective teacher coursework

A common approach for addressing the pedagogical needs of teacher preparation is to include the use of *approximations of practice*. Grossman et al. (2009) define these as "opportunities to engage in practices that are more or less proximal to the practices of a profession" (p. 2058) and underscore the importance of providing prospective teachers exposure to approximations of practice in settings of reduced complexity before they begin teaching in their own classrooms.

Scholars have used different approximations of practice in pedagogy-focused coursework to engage prospective teachers with opportunities to focus their attention on pedagogical considerations in the context of their future students' mathematical learning. For example, Herbst et al. (2011) use comic-based representations of teaching to scaffold learning about the complexities of teaching mathematics. Heid et al. (2015) present situations from secondary classrooms as a way to deepen prospective secondary teachers' mathematical understandings. Ghousseini and Herbst (2016) use constructed dialogues that offer prospective teachers opportunities with various pedagogies of practice such as leading classroom discussions. Campbell et al. (2020) use planted errors in coached rehearsals with prospective secondary mathematics teachers to focus on the practice of responding to errors in whole-class discussion.

Other scholars have incorporated approximations of practice in the context of mathematics content courses that are designed specifically for the population of prospective teachers. For example, in script writing (e.g., Zazkis 2018; Zazkis & Zazkis, 2014), prospective teachers in mathematics courses engage in an imagined role-play through the process of writing a script, which provides them with opportunities to imagine the situation, consider the characters' conceptions, and refine the interactions while removed from a classroom environment. Wasserman et al. (2019) leverage secondary curriculum and secondary mathematics practices as a foundation for building activities for prospective teachers in real analysis, using pedagogical teaching situations to motivate the mathematics prospective teachers are to learn. Their multi-phase instructional model for teaching advanced mathematics to secondary teachers draws on prospective teachers' knowledge of secondary content, on teaching advanced content, and on engaging prospective teachers in "stepping down to (teaching) practice" (p. 387). Lischka et al. (2020) created modules to use in courses for teaching algebra, geometry, modeling, and statistics. These modules provide curriculum developed through an examination of the content of secondary mathematics with an eye toward developing knowledge about mathematics as it is used in teaching.

Distinct from the work described above, the present study embeds approximations of teaching practice in the mathematical tasks used in mathematics content courses whose population consists of prospective teachers, mathematics majors not pursuing teaching, and non-mathematics majors. These tasks offer opportunities for undergraduate students to encounter ideas about mathematics and ideas about teaching mathematics. As part of our study design, we stipulate that while approximations of practice situate mathematics in the realm of prospective teachers' future careers, they are equally beneficial for students who are not obtaining a teaching credential. Moreover, including approximations of practice as an application of mathematics to teaching does not detract from the mathematics learning goals of the content course (Fulton et al., 2022) just as it is generally accepted that including applications to engineering, say, does not detract from the learning of mathematics.

2.3 Connections to teaching secondary mathematics

Emphasizing connections between undergraduate mathematics and school mathematics in content courses has been positioned as a way to help foster prospective teachers' mathematical knowledge for teaching (Baldinger, 2020; Lai & Patterson, 2017; Murray et al., 2017). Klein (1932) first identified the lack of connection between the mathematics that prospective teachers study and that which they teach. Usiskin (1974) was among those to describe mathematical connections between mathematical topics in undergraduate mathematics courses and mathematical topics in secondary mathematics. He articulated "concept to concept" connections between abstract algebra and school mathematics by identifying how mathematical concepts in undergraduate mathematics (such as groups and fields) relate to mathematical concepts in school mathematics (such as properties of the real number system). More recently, Wasserman (2016) and Patterson (2020) used this approach to make connections between school mathematics and undergraduate courses such as real analysis and abstract algebra, by identifying content taught in secondary schools and examining how teaching this content may be informed by teachers' knowledge of a given concept in an undergraduate course.

Arnold et al. (2020) formulated a set of five types of "connections to teaching," informed by Ball et al.'s (2008) MKT framework and with attention to the particular needs of prospective secondary mathematics teachers. Three of these five types of connections feature opportunities to encounter learners' interactions with mathematics: *Explaining Mathematical Content*; *School Student Thinking*; and *Guiding School Students' Understanding* (Table 1). We use mathematical tasks that embed these connections to teaching secondary mathematics, presenting each of these connections in the context of an explicit or implicit mathematical learner. The implementation of these tasks in undergraduate

Table 1 Three types of connections to teaching that align with the human context of mathematics (Arnold et al., 2020)

Type of connection	Description
Explaining Mathematical Content	Undergraduates justify mathematical procedures or theorems and use of related mathematical concepts
School Student Thinking	Undergraduates evaluate the mathematics underlying a hypothetical student's work and explain what that student may understand
Guiding School Students' Understanding	Undergraduates pose or evaluate guiding questions to help a hypothetical student understand a mathematical concept and explain how the questions may guide the student's learning

mathematics courses form the basis for our study; we use these three connections to teaching as our analytic framework for our study.

3 Mathematical tasks

Our study uses mathematical tasks (Álvarez et al., 2020, 2022; Fulton et al., 2022) that embed approximations of teaching practice in order to provide undergraduate students opportunities to encounter ideas about mathematics and ideas about teaching and learning mathematics. The tasks provide focused opportunities to engage in analyzing, anticipating, or guiding student thinking in the context of expanding or deepening undergraduate learning of mathematical ideas and concepts. These tasks present hypothetical learners and descriptions of their mathematical ideas so that mathematical content is addressed in the context of guiding learners.

Figure 1 displays an example of a mathematical task for a discrete mathematics lesson on the binomial theorem. In this task, undergraduate students are presented with the work of Henry, a hypothetical secondary school student, who has not yet correctly applied the binomial theorem to expand $(2x - y)^4$. By working through this task, undergraduate students examine the hypothetical student work and demonstrate an understanding of how to correctly apply the binomial theorem. They are prompted to explain what they think Henry both does and does not yet understand about the binomial theorem. Undergraduate students are prompted to evaluate questions one might ask Henry to help guide his mathematical understanding and revise his work. This task attends to developing undergraduate students' understanding and use of the binomial theorem and encourages them to consider the hypothetical student's conceptions, introducing them to some of the complexity that accompanies assisting others as they develop their own mathematical understanding.

We provided support for the implementation of these tasks by creating instructor notes (see Hiebert & Morris, 2012) that accompanied each task. These notes were written by members of the research team with experience implementing the tasks and provide detailed recommendations on how to implement the tasks. The notes help instructors decide how to structure the class and incorporate class

7. Henry, a high school student, expanded $(2x - y)^4$ using the binomial theorem and made some errors. Below is his work.

$$(2x - y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

   (a) What does Henry understand about the binomial theorem?

   (b) What does Henry not yet understand about binomial theorem?

   (c) Consider the following questions that one might ask Henry about his work.

      i. Explain how the following question could help Henry to advance in his understanding of the binomial theorem:

      How is $(2x - y)^4$ similar to $(a + b)^4$ and how is it different?

      ii. Explain how the following question can help you assess what Henry understands about the binomial theorem:

      Why doesn't $(-3y)^2 = -3y^2$?

      iii. Explain why the following question would not help Henry:

      Do the exponents and the coefficients look right?

Fig. 1 A mathematical task for a discrete mathematics lesson about the binomial theorem. (Adapted from Fulton et al., 2022, p. 705) discussions by describing specific content and pedagogical knowledge to emphasize. The notes also highlight the ways in which these tasks help prepare prospective teachers for interactions with their future students' mathematical work and conceptions.

The tasks we used for this study were intentionally designed to situate university mathematics topics in the context of teaching secondary mathematics (see Álvarez et al., 2020, for additional details about the design of these tasks). These tasks target opportunities to deepen undergraduate students' reasoning about key mathematical concepts while also planting seeds for ways in which mathematical interactions can be respectful of others' thinking. They provide examples of how teachers can encourage learning by understanding a student's developing notions.

4 Research question

Our motivation for this study originates with our stance that prospective teachers and all undergraduate students can benefit from experiencing approximations of teaching practice that provide them with encounters of how learners interact with mathematical ideas. We examine four content-based mathematics courses taken by many mathematics majors: single variable calculus and introductory statistics, where mathematical content is addressed by providing undergraduate students with foundational mathematics knowledge for a broad range of undergraduate majors, and discrete mathematics and abstract algebra, where mathematics content is addressed by providing undergraduate students mathematical knowledge for mathematics or closely related majors. We investigate the following research question: *In content-based mathematics courses, what kinds of ideas about teaching and learning mathematics do undergraduate students generate when they encounter mathematical tasks that embed approximations of teaching practice?* In investigating this question we remained open to the types of ideas that undergraduate students might describe based on their encounters with the tasks, whether these ideas were about mathematics content, about beliefs about teaching and learning mathematics, or about something else we did not anticipate.

5 Methods of Inquiry

This study employs a qualitative, multiple-case study methodology (Miles et al., 2020), and our work is situated within the context of four undergraduate mathematics content courses: abstract algebra, discrete mathematics, introductory statistics, and single variable calculus.

5.1 Participants and context of each case

Eleven mathematics and statistics instructors from universities across the United States each implemented two lessons that contained mathematical tasks that embed approximations of teaching practice in their respective courses (Table 2). Each lesson spanned approximately two days of instruction and each instructor decided when in their curriculum to implement each lesson. All instructors possess a Ph.D. in mathematics or statistics and none had been specifically trained in mathematics teacher preparation. Further, the population of students in each of these courses did not consist solely of prospective secondary mathematics teachers. These courses served prospective teachers, mathematics majors not pursuing teaching licensure, and non-mathematics majors.

The four cases are bounded by the content areas of abstract algebra, discrete mathematics, introductory statistics, and single variable calculus. For example, the "abstract algebra case" is bounded by the instructors and students within the abstract algebra courses where the two lessons that contained mathematical tasks were implemented. The use of a multiple-case study design allows us to understand the differences and similarities of any potential influence of course level or course content on the ideas about teaching and learning mathematics that students generated after encountering mathematical tasks that embed approximations of practice.

Table 2 Topic and content area of each lesson

Content area	Topic of lesson
Abstract algebra	Solving equations in alternative number systems
	Groups of transformations
	Logarithms and isomorphisms
Discrete mathematics	Binomial theorem
	Foundations of divisibility
Introductory statistics	Variability: mean absolute deviation and standard deviation
	Understanding margin of error
Single variable calculus	Newton's method
	Finding inverse functions and their derivatives

5.2 Data collection

The primary data for this study come from students' written work on the mathematical tasks during the two lessons and student interviews from each undergraduate course (see Table 3). We supplemented these data by conducting interviews with each instructor throughout the school term they implemented the lessons.

Within each instructor's course, students who consented to participate granted us permission to collect and analyze their written responses to the mathematical tasks from the two lessons. Of the 202 students who agreed to participate in the study, we conducted semi-structured interviews with 61 of them, obtaining at least five students from each of the 11 courses to interview. We were intentional to invite all prospective secondary mathematics teachers to an interview and then selected remaining interview participants by examining their written responses to the mathematical tasks and asking instructors for recommendations. This allowed us to identify students who we believed would provide a variety of ideas about teaching and learning mathematics. During each interview, we gave each student their original written responses to the mathematical tasks and asked them to explain their responses, and to adjust them if they desired. Then, we asked students a series of semi-structured questions to learn about their experience working on these tasks and the types of ideas about teaching and learning mathematics that emerged from this work.

Our secondary source of data was instructor interviews. Each instructor participated in a sequence of three, hour-long semi-structured interviews, where each of the first two interviews occurred after they implemented a lesson, and the third interview occurred at the end of their school term. Collectively, these interviews provided instructors an opportunity to discuss whether and how the mathematical tasks engaged their students in learning mathematical content alongside mathematics as it is used in teaching. Instructors also reflected on their overall experience implementing the tasks, and by discussing their implementations of the task, we were able to ensure that the tasks were implemented as intended across universities. The instructor interviews served as a way to triangulate findings from student data.

Table 3 The number of participating instructors and students

Content area	Number of instructors	Number of undergraduate students	
		Written work	Interviewed
Abstract algebra	4	51	20
Single variable calculus	2	63	12
Discrete mathematics	3	47	16
Introductory statistics	2	41	13

5.3 Data analysis

We employed two stages of data analysis in this multiple-case study: within-case analysis and cross-case analysis (Miles et al., 2020). We first transcribed the interviews and written responses and then, within each case, used a descriptive coding scheme to code the data. The three types of connections to teaching defined in Sect. 2.3 (*Explaining Mathematical Content*; *School Student Thinking*; and *Guiding School Students' Understanding*) served as our initial descriptive codes. After coding interview transcripts and students' written work with these three codes, we performed a secondary round of coding. That is, within each of these three primary codes, we found emergent codes related to other ideas about the teaching and learning of mathematics, some of which involved beliefs undergraduates generated about learners of mathematics and others that involved pedagogical strategies, for example. Next, we employed thematic analysis techniques (Braun & Clarke, 2006) based on our coding scheme, within each case; this process resulted in identifying themes within data from a single case. Then, we conducted a cross-case analysis by comparing the themes across each case, with special attention to the course level and content. This process resulted in the three themes we highlight in our findings. Finally, we examined instructor interviews to triangulate the student data by looking for any corroboration or discrepancies between the instructors and their students.

As a team, the authors ensured intercoder reliability by first individually coding a subset of interview transcripts and written responses and then comparing our coding process. When discrepancies arose, we clarified the boundaries of the codes in question until we arrived at full consensus. This process served as one source of trustworthiness (Miles et al., 2020, p. 79).

6 Findings

Our findings reflect the themes we identified regarding the ideas about teaching and learning mathematics that students displayed or reported after encountering the approximations of practice embedded in the tasks. We found that students recognized the broad applicability of teaching skills, recognized the value of examining hypothetical learners' mathematical work, and reported empathy for hypothetical learners.

6.1 Teaching skills have broad applicability

Data consistently show that students explicitly acknowledge that teaching skills are broadly relevant, whether or not one intends to teach secondary school. Of the 61 students we interviewed across the four content areas, 32 (52.5%) discussed some aspect of the broad applicability of teaching skills during the interview. This finding held among those who were prospective teachers and those with other majors. In all four cases, the majority of students in the courses were not intending to become formal teachers of secondary mathematics. As such, we wondered whether the students would resist engaging with tasks in mathematics content courses that attend to hypothetical learners' interactions with mathematics. On the contrary, students across the four cases repeatedly expressed finding value in these tasks. We do not see a difference between the introductory level courses and the more advanced courses, nor differences between the types of mathematical content. Illustrative quotations from student interviews for each of the four cases are displayed in Table 4.

Although we found evidence that students across each of the four cases expressed value for teaching as a broadly applicable skill after working on our tasks, this perspective was not present among instructors across the four content areas. In fact, we found evidence that instructors differed in their recognition of the motivation of this idea about the applicability of teaching, based on the level of their course. Instructors in abstract algebra and discrete mathematics, the more advanced courses for mathematics majors, expressed views similar to their students. For example, one discrete mathematics instructor described how she did not have many prospective secondary mathematics teachers in her course, so she motivated the value behind these kinds of mathematical tasks beyond teaching secondary mathematics:

> I don't think a lot of them identify as future teachers, but I gave them a little speech that was like "look if you go to grad school, you're probably going to have to teach something at some point and beyond that, sometimes you end up in a profession like getting a PhD where you wind up teaching and that wasn't necessarily [something you planned]." So I tried to motivate it beyond "yeah you might not be planning right now to be a high school teacher."

We looked for expressions of this instructor perspective in the introductory level courses, single variable calculus and introductory statistics, but did not find them. We wonder whether the diversity of undergraduate majors in the population of the introductory courses diffuses the instructors' attention to the particular aspects of teaching and learning mathematics that the tasks include, or whether the absence of this perspective is simply a reflection of the views of the particular instructors of the introductory level courses in our study.

6.2 Stance that examining work from hypothetical learners has value

Students and instructors demonstrated that they found value in the mathematical tasks that involved analyzing and guiding the work of hypothetical learners, such as that displayed in Fig. 1. All of the instructors discussed the value of these tasks in their courses, and 59 of the 61 students we interviewed (96.7%) indicated that these tasks were valuable to their learning. Instructors often explained that these tasks aligned with their curriculum, featured mathematical rigor, promoted a high level of engagement, and fostered classroom discussions in positive ways.

The interview data reflect that many instructors and students, in this setting, perceived these types of tasks to be novel. Some students described that this encounter was their first time examining work from a hypothetical learner, and in doing so, they were learning from the learner's work. For example, an undergraduate student in calculus reflected

> It helped—[the task] definitely made me look at [the mathematical content] in a different way. Because I had never seen work done like this before. And this

Table 4 Illustrative quotations from students, none of whom are prospective teachers

Case	Illustrative quotation
Abstract algebra	You might know a lot of things, like math and complex stuff, but if you're not able to share that information, to communicate it well, going from the simple and going to the complex … That might not only be applicable to the teachers but just if you want to explain anything, right?
Discrete mathematics	Teaching is a universal skill. Everyone should have some teaching skills. There's no better way to understand something than to try to teach someone those things. In my career, that's very true. I didn't understand or deeply understand a lot of things I was doing every day until I started to teach people and then they asked me probing questions.
Single variable calculus	I think everybody will end up teaching someone else something at some point …You'll always have to be able to explain something to somebody else. And it's good to have different ways to explain something because I think this activity helped by showing the different ways that students did something could show you how to explain different things to different people.
Introductory statistics	In some aspect, everyone's kind of a teacher in their own way. In pretty much every job, you're going to be teaching someone about something. In any aspect, in any field you go into, you're going to be teaching something. And so I think to be able to connect it with something that most people know is really helpful.

> helped a lot more than … if the professor would've just been writing like, this is the notation for it on the board.

During interviews, students often commented that analyzing the hypothetical learner's work deepened their own mathematical or statistical understanding by requiring them to think more critically about a problem, giving them opportunities to consider multiple approaches to a mathematical question, or helping them recognize and understand common mistakes learners make when doing mathematics. An undergraduate student in abstract algebra stated that these kinds of tasks "force you to justify why [the work] is wrong. So, it's not only, yes, I know how to get the right answer, but I know why this is a wrong answer which requires a deeper level of thinking." An undergraduate student in statistics reported a similar effect about their own critical thinking process:

> Sometimes I can explain why I think something that way but seeing somebody else's thinking and having to make a judgment of whether that's right or wrong was something that was different. That's something I haven't seen tested on before and I thought that was very interesting. So, it … did force me to think about why I didn't agree with these [choices]. And I think that was good for me.

In fact, the data across the four cases consistently show that students recognized that some tasks about hypothetical learners' work help them to understand mistakes learners, including themselves, might make. For example, an undergraduate student in abstract algebra stated,

> I think it helps you catch any misconceptions because obviously this is probably a common misconception when learning in this class, and so [the hypothetical learner's work] kind of puts that out there for you, like learning from other people's mistakes.

Another undergraduate student in discrete mathematics reported similarly, stating

> Giving us an example like this where we're meant to pick out a mistake sort of lets us think about it without being … concerned about making a mistake on our own behalf. And also, it primes me to look for mistakes I might make in the future trying to do something similar.

Instructors corroborated that these tasks provide a way for their students to avoid the same mistakes made by the hypothetical learners. For example, discrete mathematics instructors discussed how their students made or would make similar mistakes as Henry (the hypothetical student applying the binomial theorem in Fig. 1). As one reported,

> It's funny because I think some of them were either making the same mistake as the student or [a] worse mistake as the student. … there was sort of a key thing about how the students were interacting with [Henry's mistake]. There were specific errors in applying the binomial theorem which interestingly were turned into potential student mistakes that they had to analyze basically before [they] even had a chance to make those errors.

Not all of the hypothetical work in these types of tasks contained mistakes. Some, like the one in Fig. 2, presented different or new ways to approach a mathematical problem. This type of hypothetical work prompted students in all four content areas to consider different perspectives and see mathematical solutions in a different light. An undergraduate student in calculus discussed the task presented in Fig. 2, saying, "you have to be able to perceive how other people are going to think." Undergraduate students across the cases responded similarly to comparative tasks; for example, an undergraduate student in abstract algebra reported, especially with math there's … so many different ways to solve one problem. I always think it's nice to know how other people think because they don't think like I do." Instructors also stated that some hypothetical learner work "opened [their students'] eyes" to different approaches and that "having those different approaches can help students to understand the concept better.

6.3 Empathy elicited by the presence of hypothetical learners in mathematical tasks

Tasks embedding approximations of practice incorporated the use of hypothetical learners as part of their design (see Álvarez et al., 2020). We found that these hypothetical learners provided a crucial bridge between mathematical ideas and ideas about teaching and learning mathematics. Beyond our finding that students and instructors across the four cases found value in these activities, we also identified ways that these hypothetical learners elicited an empathetic perspective from students.

Students' interview responses indicated respect for the learners' mathematical work and what they understood. We found this perspective in the responses of 38 of the 61 students we interviewed (62.3%). For example, in abstract algebra, one student recognized that being able to empathize with a learner would allow her to appreciate their reasoning and identify their mathematical conceptions:

> It does kind of make you step back and think, okay, how is the student thinking, and just kind of get back onto that level field. And so it just kind of gives me a different view and really just reiterates that I do need to take a step back and put myself in their shoes and

Fig. 2 A task on inverse functions for single variable calculus that highlights multiple student approaches

1. Consider how Alex, Jordan, and Kelly found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Alex's Work	Jordan's Work	Kelly's Work
$y = 2/3\, x + 1$ $x = 2/3\, y + 1$ $x - 1 = 2/3\, y$ $\frac{x-1}{2/3} = y$ $\boxed{\frac{3}{2}(x-1) = y}$	$f \circ f^{-1}(y) = y$ $f(x) = \frac{2}{3}x + 1$ So: $f(f^{-1}(y)) = \frac{2}{3} f^{-1}(y) + 1 = y$ $\frac{2}{3} f^{-1}(y) = y - 1$ $\frac{3}{2} \cdot \frac{2}{3} f^{-1}(y) = \frac{3}{2}(y-1)$ $\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$	$y = \frac{2x}{3} + 1$ $y - 1 = \frac{2x}{3}$ $3(y-1) = 2x$ $\frac{3(y-1)}{2} = x$ But $f(x) = y \Rightarrow$ $f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$ $x = f^{-1}(y)$ So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

Compare and contrast the key mathematical ideas used by Alex, Jordan, and Kelly to find the inverse function of $f(x) = \frac{2}{3}x + 1$. Make sure to identify which properties of inverse functions each student uses, if any.

> see what is going on, either right or wrong, so I can [build] on their strengths and fix their weaknesses.

Another student in abstract algebra described that an empathetic perspective helps with guiding a learner's understanding, saying,

> "the first thing that you have to do to be able to ask that question to guide them in the correct direction is to understand: I know that it's not correct, but why are they thinking the way that they are thinking?"

Overall, students expressed a preference towards building on what the learners already understand rather than simply identifying where they are incorrect and providing instructions for what to do.

Other students described that the tasks with hypothetical learners underscored for them that "everyone understands things differently" and that prospective teachers ought to be aware of that. This theme was present in student interviews, regardless of the content and level of the course. For example, in statistics, a student who is not a prospective teacher indicated in their interview that the tasks "help [teachers] recognize, 'oh the students make certain mistakes, but they also do things correctly, too.'" Another undergraduate student in discrete mathematics who is a prospective secondary mathematics teacher empathized by associating herself with the learner and said, "but again, we're human, so sometimes we make mistakes. If a student has wrong work, we can also use that to our advantage."

In their interviews, instructors noted that the tasks with hypothetical learners offered students a unique way to engage in the mathematical content compared to tasks without learners as characters. In calculus, one instructor described that his students talked about the hypothetical learners as if they were peers, which he found helpful in engaging undergraduate students in the task. A discrete mathematics instructor indicated that the tasks enabled students to see things from others' perspectives, stating "there's more merit in trying to get inside somebody else's head than … you would expect. … These activities give them a chance to kind of try to figure out what other people see."

7 Discussion and conclusion

In mathematics courses offered to a general population of undergraduate students, what kinds of ideas about teaching and learning mathematics do undergraduate students generate when they encounter tasks designed to embed approximations of teaching practice in mathematics? This empirical study gives evidence that students, whether they intend to pursue teaching or not, exhibit ideas about the applicability of teaching skills, about the value of examining hypothetical learners' work, and about empathizing with hypothetical learners. These findings persisted across the course content and course levels we studied, leading us to conclude that our findings can transfer to additional mathematics courses in secondary mathematics teacher preparation.

The students we interviewed displayed dispositions that inclined them towards embracing the focus of the tasks on hypothetical learners' work, regardless of any stated intention to become secondary mathematics teachers. This finding provides a propitious counterpoint to the disconnect that has been reported in the literature between what prospective teachers study in their university-level coursework and what they expect as secondary teachers. Furthermore, the tasks themselves support students' recognition of teaching as a broadly applicable skill, and this disposition provides an opportunity for undergraduate mathematics instructors to intentionally incorporate approximations of practice into a broad variety of undergraduate mathematics courses. This finding has implications for other research that has identified ways to embed approximations of practice in teacher preparation coursework and the tools of, for example, script-writing (Zazkis, 2018), comic-based representation of teaching (Herbst et al., 2011), teaching situations (Heid et al., 2015), constructed dialogues (Ghousseini & Herbst, 2016), or planted errors (Campbell et al., 2020). Each provides strategies that enable students to acquire ideas about teaching mathematics, and the potential use of approximations of teaching practice in a variety of mathematics courses provide a ripe field for future study.

The incorporation of hypothetical learners in tasks allows us to work at the intersection of mathematical and mathematics educational learning. The hypothetical learners, as an essential component of a task, transport the students to a "local mathematics" situation (Wasserman, 2018). They provided opportunities for students to discuss the understandings of others, form conjectures about possible explanations for learners' work, empathize with another human being, and consider ways to build on correct reasoning from the perspective of the hypothetical learner, all of which encompass experiences that impact students' ideas about teaching mathematics. This finding complements the work of other researchers who find explicit ways to engage prospective teachers in developing an understanding of interacting with others about mathematics (e.g., Baldinger, 2020; Zazkis, 2018). This provides a promising avenue of research. In particular, the human quality of empathy was elicited among the participants in our study. What other qualities do undergraduate students exhibit when they work on tasks that include hypothetical learners?

The question of how mathematics content courses can attend to the dual goals of supporting prospective secondary mathematics teachers' mathematical knowledge and mathematics educational knowledge is essential for teacher preparation programs. The results from this study suggest that including tasks that embed approximations of teaching practice is a promising approach to support secondary mathematics teachers' content knowledge and pedagogical content knowledge. Those engaged in the preparation of mathematics teachers are obliged to prepare teachers for human interaction with mathematics, and we join the field in expanding research about teaching mathematics to prospective teachers in a way that recognizes mathematics as a human endeavor.

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Data availability The datasets generated and analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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