An engineering analysis of systems containing time-varying elements
by Philip Alan Buckley

A thesis submitted to the graduate faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University

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Electrical Engineering

Approved:

Paul E. Uhlrich
Head, Major Department

Donald A. Rudberg
Chairman, Examining Committee

K. Goering
Graduate Dean

MONTANA STATE UNIVERSITY
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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (continued)

LIST OF FIGURES (continued)

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 INTRODUCTION

Most of the techniques that engineers apply to the solution of linear systems with constant parameters fail to be of much value to them in the solution to problems dealing with systems with varying parameters. If the parameters vary in such a way as to be functions of one or more of the system's dependent variables, then the system is immediately nonlinear and the method of solution is sensitive to the type and severity of the nonlinearity. If, on the other hand, the parameters vary in such a way as to be functions of the system's independent variable (most often time), then the system remains linear and is described by a linear differential equation with varying coefficients. In problems of the latter type, the engineer is equipped with several tools which he commonly applies to constant parameter systems which are also valid and powerful tools for the solution of systems with variable parameters. One of the most important of these is the principle of superposition.

It would be ideal if all systems with time-varying parameters could be characterized in complete mathematical detail by the appropriate differential equation and solved to yield all the necessary information. However, apart from simple first order differential equations with varying coefficients, little is known about such solutions. In an effort to obtain some sort of answer, the engineer usually attempts to simplify the system model or to solve the problem by numerical techniques, neither of which may yield the desired qualitative or quantitative result or insight. It would be desirable if there were other techniques available which an engineer could apply to obtain useful information from time-varying parameter systems without having to resort to doubtful approximations or numerical techniques. In the following section, works from the technical literature directed toward this end are discussed.

1.2 LITERATURE REVIEW

Recorded history of solutions to time-varying equations begins with the publication, in 1868, of the solutions to the Mathieu equation (50). In 1877, Hill (37) generalized the Mathieu equation and in 1883 Floquet (28) published a general treatment of linear differential equations with periodic coefficients. Mathieu and Hill equations are special cases in the Floquet theory (52). In 1887, by means of Hill's analysis, Lord Rayleigh studied the Melde experiment, which concerned the maintenance of vibrations in a body by forces of double frequency (47).

Variable parameter system analysis attracted scant attention among engineers and physicists prior to 1912. A possible explanation was the lack of physical application and the analytical difficulties. Whittaker (85) in 1912, and van der Pol (77) in 1928, studied the stability of solutions obtained by Mathieu, Hill, and Floquet. The phenomenon of parametric excitation was investigated in 1931 by Mandelstam and Papalexi (49) and later by Lazarev (44), Migulin (55), Barrow (8,9,10), and Gorelik (34,35).

In 1917 Nichols (60), and in 1921 Carson (18) investigated systems such as the variable capacitor microphone and the induction generator with variable mutual inductance. Carson showed that a system described by a differential equation with varying coefficients could also be written as an integral equation of the Volterra type, possessing an infinite series solution, which unfortunately and unexplainedly diverged in many cases of practical interest. Carson neglected to give any criteria relative to conditions under which his series solution would converge. Later in the same paper Carson suggested yet another series for steady-state solutions but did not develop the technique in detail. Bolle (15), in 1955, furthered this technique, commonly referred to as Harmonic Balance, by using phasor notation to obtain the first steady-state solution usable to an engineer. Later, Desoer (24) in 1959 and Sandberg (66) in 1964 contributed to Bolle's technique.

A parallel path of development has taken place since 1922, due to the increasing interest in frequency modulation. Analysis was done on constant parameter systems with variable frequency excitation and on the similarity between these and time-varying systems. Carson and Fry (19) in 1937, studied the conditions for a quasi-steady-state to apply, and their work was extended by several others including Armstrong (2), Stumpers (75), Weiner (80), and Baghdady (4,5). In 1964, Weiner and Leon (81,82) actually considered the quasi-steady-state plus distortion response of a simple resistor and time-varying capacitor network excited by an FM signal.

From 1950 to 1960, Zadeh (87,88,89,90,91,92,93,94,95,96) published many papers dealing with time-varying and stochastic parameter systems. In particular, he contributed a great deal to the theory of time-varying systems in the specific area of time-varying weighting functions (impulse responses) and their transforms. The papers of Zadeh contribute much to the theory of time-varying systems but the material contained therein is little comfort in that it does not provide a usable engineering analysis unless the system parameters vary slowly with time.

In 1960, Leon (45) reduced a second order differential equation describing a network containing a single sinusoidally varying element to a second order difference equation by transform techniques. In 1962, Schweizer (71,72) using phasor methods, solved essentially the same sort of problem. Confined to the steady-state response, both authors obtained an infinite series solution. The coefficients of the series were continued fractions. Although Leon developed convergence criteria for the continued fractions and estimated the truncation error, no physical significance is attributed to them.

Analytical techniques commonly used in the analysis of nonlinear systems have often also applied to linear time-varying systems with much success (97,98). Perturbation methods, for example, when suitably applied to the problems discussed within this thesis, yield results in the time-domain very similar to the frequency domain results obtained in this thesis.

1.3 PURPOSE OF THE THESIS

The purpose of the thesis is to contribute a sound engineering method of analysis applicable to linear systems containing elements which are functions of the system's independent variable. The types of systems considered are single input, single output, lumped linear systems containing parameters which are arbitrary deterministic functions of time.

A key term in the statement of the purpose of the thesis is the adjective 'engineering'. To be of any engineering value, the method of analysis must provide a solution in such a form that the quantities of engineering value are readily accessible and the numerical results require as little human effort as possible. The method should be mathematically sound and limitations, if any, should be clearly defined and related to the system's physical parameters whenever possible.

CHAPTER 2

A SERIES SOLUTION FOR LINEAR TIME-VARYING SYSTEMS

2.1 INTRODUCTION

Equations characterizing linear time-varying systems are similar to those characterizing linear time-invariant systems with the exception that in the time-varying case, one or more of the coefficients of the equation are functions of time. Figure 2.1 illustrates a time-varying linear system with a single input $x(t)$ and a single output $y(t)$.

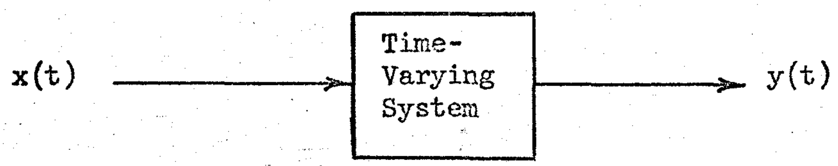

Figure 2.1 Single Input, Single Output Time-Varying System

The linear time-varying system shown in Figure 2.1 is assumed to be characterized by equations of the following form:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_o(t)y(t)$$

$$= b_m(t)x^{(m)}(t) + b_{m-1}(t)x^{(m-1)}(t) + \dots + b_o(t)x(t) \qquad (2.1)$$

Any set of n linearly independent functions $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ satisfying the homogeneous, or unforced, part of equation 2.1, but not necessarily satisfying any particular boundary conditions, is termed a fundamental set of solutions or a set of basis functions. Any solution satisfying the homogeneous or unforced part of equation 2.1 must be a linear combination of $y_1(t)$, $y_2(t)$, ..., $y_n(t)$. Therefore, once any set of basis functions is known, the particular solution satisfying a given set of n boundary conditions can be found by algebraic manipulations of the known basis functions and their derivatives (1).

The classical approach to the solution of time-varying differential equations such as equation 2.1 hinges on the determination of a set of basis functions for the unforced portion of the solution, and the use of the superposition, or convolution, integral to determine the response of the system to particular inputs. The major problem herein is that the basis functions are generally difficult to determine and no general procedure exists for determining a set of basis functions for an arbitrary linear differential equation with time-varying coefficients. If a set of basis functions for a particular linear time-varying differential equation is not known (i.e., the differential equation doesn't belong to a class of equations for which the basis functions are known) or cannot be determined, then the solution to the equation cannot be obtained by the classical method and another technique must be found.

2.2 LAPLACE TRANSFORM METHOD

The use of Laplace transforms for analyzing linear time-invariant systems is attractive because a differential equation may be transformed into an algebraic equation and algebraic equations as a class tend to be much easier to solve than do differential equations. On the other hand, the use of Laplace transforms in this usual way for analyzing linear time-varying systems is not so attractive. The reason lies in the fact that the transformed equation is rarely an algebraic equation as it is when analyzing time-invariant systems. Most often the resulting transformed equation is a differential equation in the complex variable s, or it may be a difference equation, or it may be a combination of both -- a differential-difference equation. In addition, even the resulting transformed equations generally contain varying coefficients and may or may not be easier to solve than the original equation. Fortunately, these difficulties in the use of the Laplace transform for analyzing time-varying systems may be circumvented by the method to be introduced in Section 2.3.

There may be, however, other difficulties associated with the use of Laplace transforms as an equation solving technique whether applied to linear time-varying or time-invariant systems. First of all, the Laplace transform must exist for each term $a_i(t)x^{(i)}(t)$ of the differential equation, which means that the integral

$$\int_0^t a_i(t)x^{(i)}(t)e^{-st}dt \tag{2.2}$$

must converge for some positive, real value of s (46). It is quite possible that equations could be encountered for which the integral 2.2

would diverge for one or more terms of the equation, with the result that the Laplace transform method would fail to yield a solution.

Another difficulty in the use of the Laplace transform method is that after the differential equation is Laplace transformed and the resulting equation solved, the inverse transform of the result must be obtained. This inversion process may, depending upon the complexity of the transformed expression, involve lengthy calculations although the procedure in most cases is straightforward (7,46).

Despite the several difficulties mentioned above, there are still some advantages of the Laplace transform method over and above its sometime ability to yield a simpler-to-solve transformed equation. The first of these is that the transformed variables contain valuable information as to their frequency characteristics that is lost when the differential equations are solved by the classical method of Section 2.1. It is this information about frequency characteristics that is necessary so that the designer may determine such important information as the frequencies present in both the input and output of the system, the bandwidth of the system, the frequency dependent system gain, and the figure of merit Q of the system.

The second additional advantage is that when the inverse of the transformed solution is obtained, the solution is complete. Both the transient solution and the steady-state solution are obtained simultaneously. There is no need, as in the classical method of Section 2.1, to find a set of basis functions to determine the unforced solution and then to evaluate the particular convolution integral to obtain the steady-state solution. The Laplace transform method obtains both parts of the solution simultaneously.

2.3 A SERIES SOLUTION FOR TIME-VARYING NETWORKS

Often, not all the coefficients of a time-varying linear differential equation vary with time. In some cases, only a few or perhaps just a single coefficient actually is time dependent and even then the coefficient may vary about a fixed value; in which case it may be characterized by the sum of a constant part and a time-varying part, such as

$$a(t) = a_o + a_1(t) \tag{2.3}$$

where $a(t)$ is the total coefficient, a_o is the constant part, and $a_1(t)$ is the time-varying part of the coefficient.

As a specific instance of the sort of time-varying differential equation described above, consider the time-varying electrical network shown in Figure 2.2.

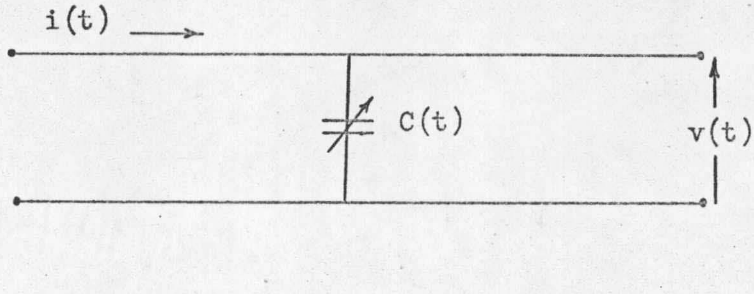

Figure 2.2 Time-Varying Electrical Network

The network contains only a single capacitor varying at a sinusoidal rate of w_c radians per second according to the relation

$$C(t) = C_o + C_1 \cos(w_c t) \qquad (2.4)$$

In a physically realizable network the total capacitance would never be negative so the additional requirement that $C_o > C_1$ would guarantee that the equation 2.4 represents a realizable network. Equation 2.4 could also be visualized as describing the time variations of the capacitor network shown in Figure 2.3.

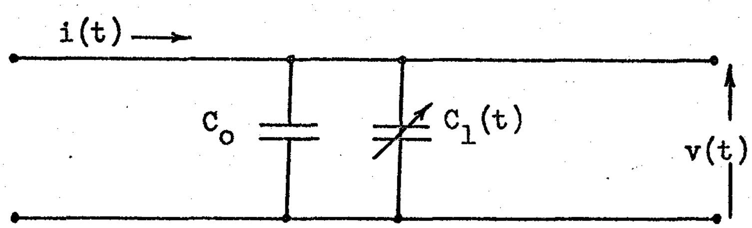

Figure 2.3 Capacitor Network Described by Equation 2.4

The network of Figure 2.3 consists of a fixed capacitor C_o and a time-varying capacitor $C_1(t)$, where $C_1(t)$ is as defined in equation 2.5.

$$C_1(t) = C_1 \cos(w_c t) \qquad (2.5)$$

Therefore, it can be seen that the networks of Figures 2.2 and 2.3 are mathematically identical. As shown in both figures, the networks are

are excited by a current i(t), and the problem is to determine the output voltage response v(t). Both networks are described by the time-varying differential equation 2.6.

$$i(t) = \frac{d}{dt}\Big[\, C(t)v(t) \,\Big] \qquad v(t=0) = 0$$

$$= C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\Big[\cos(w_c t) \cdot v(t)\Big] \tag{2.6}$$

The first term on the right side of the above equation represents the current flowing through capacitor C_o in Figure 2.3, and the second term represents the current flowing through capacitor $C_1(t)$ and is designated $i_c(t)$. Equation 2.6 may then be rewritten as equation 2.7.

$$i(t) = C_o \frac{dv(t)}{dt} + i_c(t) \tag{2.7}$$

Equation 2.7 may then be Laplace transformed and written as

$$I(s) - I_c(s) = C_o sV(s) \tag{2.8}$$

or

$$V(s) = \frac{1}{sC_o}\Big[I(s) - I_c(s)\Big] = Z(s)\Big[I(s) - I_c(s)\Big] \tag{2.9}$$

Z(s), as it appears in equation 2.9, is the impedance of the fixed part of the network of Figure 2.3. In this case, it is the impedance of the capacitor C_o. Furthermore, equation 2.9 suggests the following very interesting interpretation. The voltage response V(s) of a time-varying network to a current I(s) is exactly the same as the voltage response V(s) of the fixed part of the same network to the new forcing current $I_T(s)$ shown in equation 2.10.

$$I_T(s) = I(s) - I_c(s) \tag{2.10}$$

This fact becomes apparent after examining equation 2.9. Figure 2.4 portrays the meaning of equation 2.9 in block diagram form.

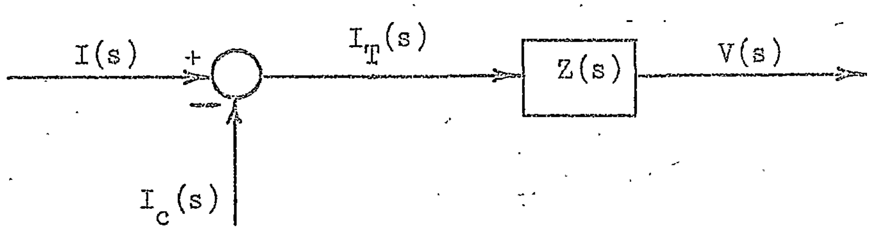

Figure 2.4 Block Diagram Model of Equation 2.9

The current $i_c(t)$ which flows through the time-varying capacitor is a function of the voltage v(t) across the capacitor. This means that $I_c(s)$ must necessarily be some specific function of V(s), and let this functional relationship be represented by the symbol F.

$$I_c(s) = F\left[V(s)\right] \tag{2.11}$$

The transformed current $I_c(s)$ may be obtained by performing upon $V(s)$ the operation signified by F. The symbol F, then, is not a transfer function in the usual sense of the word; rather, it represents an operation to be performed.

With the concept of the operator F understood, a feedback path containing the element marked F may be inserted in the block diagram describing the network.

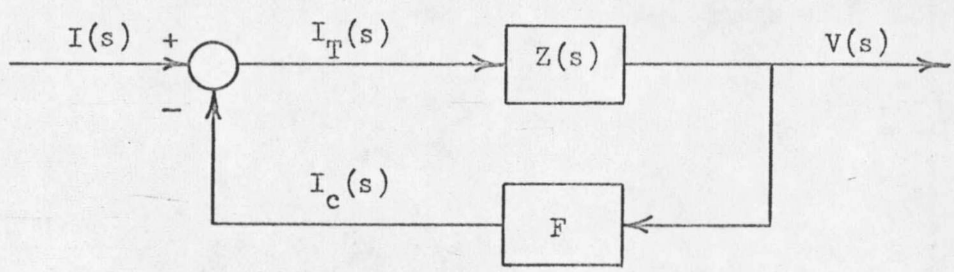

Figure 2.5 Block Diagram Model With Feedback Path

For this specific network, the operation designated by F may be determined by Laplace transforming equation 2.12, which relates $i_c(t)$ to $v(t)$

$$i_c(t) = C_1 \frac{d}{dt}\left[v(t)\cos(w_c t)\right] , \quad v(t=0) = 0$$

$$= \tfrac{1}{2}C_1 \frac{d}{dt}\left[v(t)e^{jw_c t} + v(t)e^{-jw_c t} \right] , \qquad (2.12)$$

to obtain equation 2.13 which relates $I_c(s)$ to $V(s)$.

$$I_c(s) = \tfrac{1}{2} s C_1 \left[V(s+jw_c) + V(s-jw_c) \right]$$

$$= Y(s) \left[V(s+jw_c) + V(s-jw_c) \right] \qquad (2.13)$$

Here the operation indicated by F means take the sum of V(s) shifted up in frequency by jw_c and V(s) shifted down in frequency by jw_c and multiply this sum by the admittance function $\tfrac{1}{2}sC_1$.

Suppose, in Figure 2.5 above, a current I(s) is applied to the system, and also suppose that the feedback path is, for the moment, open circuited. At the output will appear a voltage, call it $V_o(s)$.

$$V(s) = V_o(s) = I(s)Z(s) \qquad (2.14)$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = F\left[V_o(s)\right] \qquad (2.15)$$

$$V(s) = V_o(s) + V_1(s) \qquad (2.16)$$

$$V_1(s) = -I_{co}(s)Z(s) \qquad (2.17)$$

This new voltage $V_1(s)$ will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = F\left[V_1(s)\right] \tag{2.18}$$

$$V(s) = V_o(s) + V_1(s) + V_2(s) \tag{2.19}$$

$$V_2(s) = -I_{c1}(s)Z(s) \tag{2.20}$$

As this process is repeated again and again, the total transformed output voltage V(s) is arrived at by an infinite number of terms, as indicated in equation 2.21, and illustrated in Figure 2.6.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \dots + V_n(s) + \dots \tag{2.21}$$

The above series expression for V(s) enables one to gain special insight into the solution because each term of the series has the following significance. $V_o(s)$ is the response of the fixed part of the system to an external current stimulus I(s).

$$V_o(s) = I(s)Z(s) \tag{2.22}$$

$V_1(s)$ is the response of the fixed part of the system to a current $I_{co}(s)$.

$$V_1(s) = -I_{co}(s)Z(s) \tag{2.23}$$

$V_2(s)$ is the response of the fixed part of the system to a current $I_{c1}(s)$.

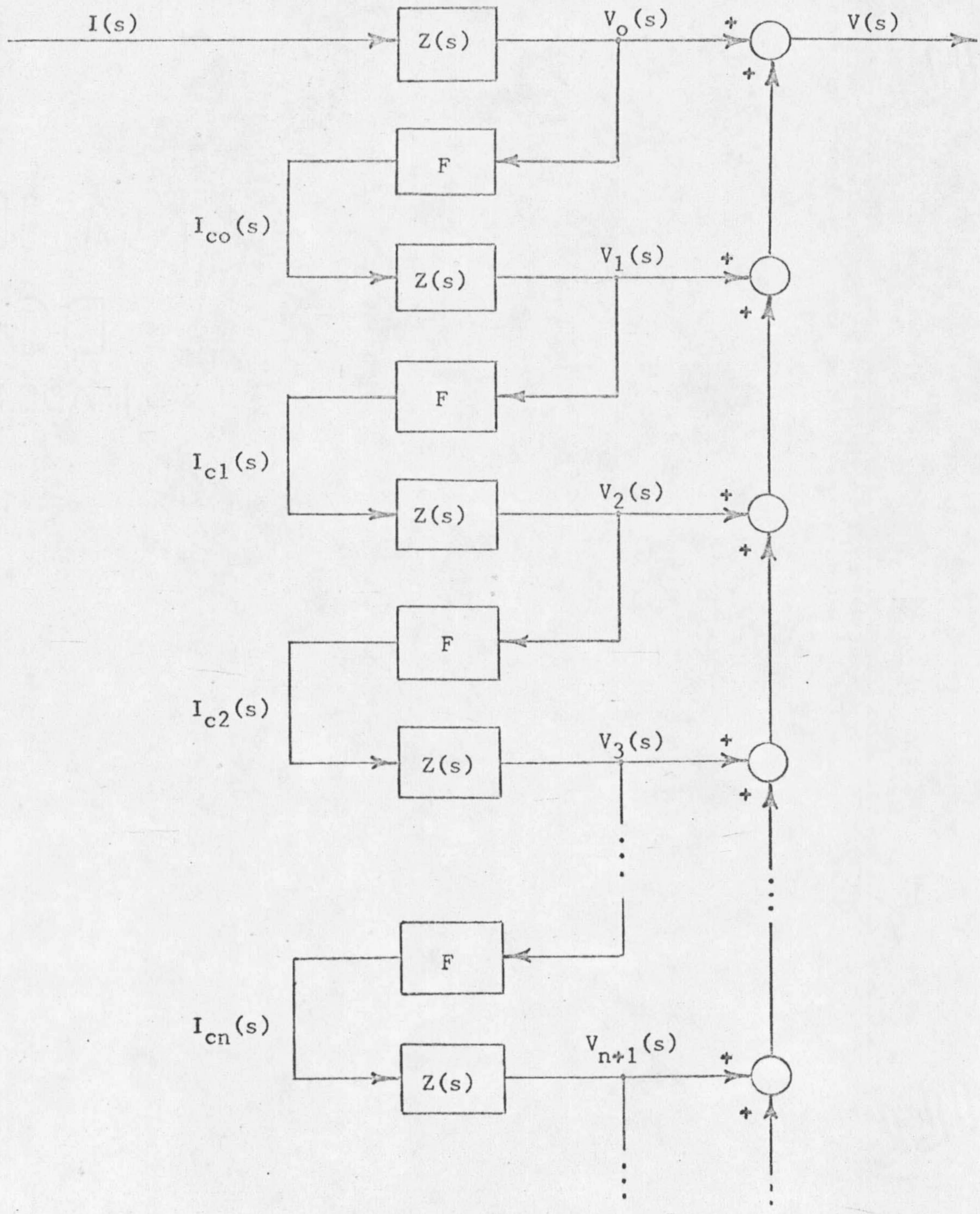

Figure 2.6 An Illustration of the Process By Which the Series Solution is Generated.

$$V_2(s) = -I_{c1}(s)Z(s) \tag{2.24}$$

And, in general, $V_{n+1}(s)$ is the response of the fixed part of the system to a current $I_{cn}(s)$.

$$V_{n+1}(s) = -I_{cn}(s)Z(s) \tag{2.25}$$

Using the procedure described above, it is now possible to obtain a series solution for linear, time-varying, differential equations, provided that each term of the equation is Laplace transformable. In addition, this series solution not only provides the steady-state solution but the total solution, transient included, and each term of the series has the useful physical significance explained above.

2.4 EXAMPLES OF THE APPLICATION OF THE SERIES SOLUTION

For the first of three examples, consider the network in Figure 2.7, introduced in Section 2.3.

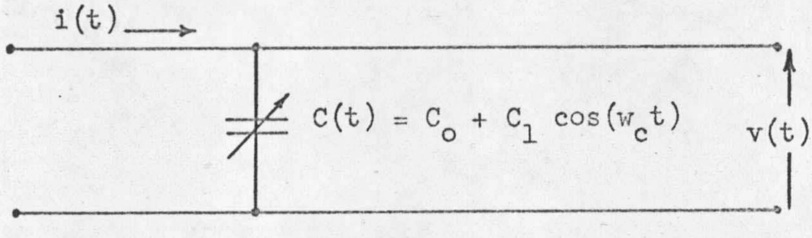

Figure 2.7 A Time-Varying Electrical Network

This network is described by the time-varying differential equation 2.26.

$$i(t) = C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\left[\cos(w_c t) \cdot v(t) \right], \qquad v(t=0) = 0$$

$$= C_o \frac{dv(t)}{dt} + \tfrac{1}{2} C_1 \frac{d}{dt}\left[v(t)e^{jw_c t} + v(t)e^{-jw_c t} \right] \qquad (2.26)$$

Equation 2.27 is obtained by Laplace transforming equation 2.26.

$$I(s) = sC_o V(s) + \tfrac{1}{2} sC_1 \left[V(s+jw_c) + V(s-jw_c) \right] \qquad (2.27)$$

The above equation suggests the use of the following complete system block diagram.

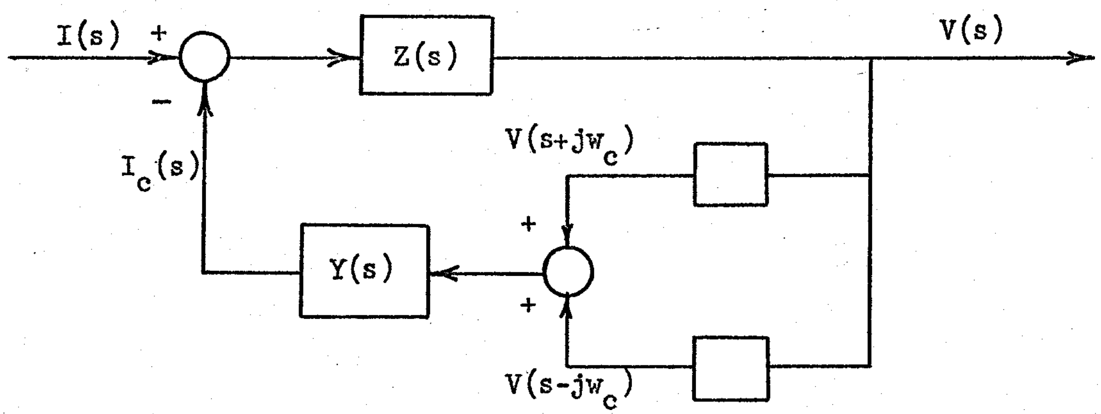

Figure 2.8 Complete Block Diagram of the System Described by Equation 2.27

The impedance Z(s) shown in the forward path of Figure 2.8 is equal to the impedance of the fixed part of the capacitor in Figure 2.7.

$$Z(s) = \frac{1}{sC_o} \tag{2.28}$$

The admittance in the feedback path of Figure 2.8 is defined by equation 2.29.

$$Y(s) = \tfrac{1}{2} sC_1 \tag{2.29}$$

For conciseness in the following equations, a transfer function G(s) is defined by equation 2.30.

$$G(s) = Z(s)Y(s) \tag{2.30}$$

Following the procedure set forth in Section 2.3, the individual terms of the series solution may be determined.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.31}$$

Suppose, in Figure 2.8 above, a current I(s) is applied to the system, and also suppose that the feedback path is for the moment open circuited. At the output will appear a voltage; call it $V_o(s)$.

$$V_o(s) = Z(s)I(s) \tag{2.32}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = Y(s)\left[\ V_o(s+jw_c) + V_o(s-jw_c)\ \right] \tag{2.33}$$

$$\begin{aligned} V_1(s) &= -Z(s)I_{co}(s) \\ &= -G(s)\left[\ V_o(s+jw_c) + V_o(s-jw_c)\ \right] \\ &= -G(s)\left[\ Z(s+jw_c)I(s+jw_c) + Z(s-jw_c)I(s-jw_c)\ \right] \end{aligned} \tag{2.34}$$

When the voltage $V_1(s)$ appears at the output it will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = Y(s)\left[\ V_1(s+jw_c) + V_1(s-jw_c)\ \right] \tag{2.35}$$

$$\begin{aligned} V_2(s) &= -Z(s)I_{c1}(s) \\ &= -G(s)\left[\ V_1(s+jw_c) + V_1(s-jw_c)\ \right] \\ &= G(s)\Big[\ G(s+jw_c)\left[\ Z(s+j2w_c)I(s+j2w_c) + Z(s)I(s)\ \right] \\ &\qquad + G(s-jw_c)\left[\ Z(s)I(s) + Z(s-j2w_c)I(s-j2w_c)\ \right]\Big] \end{aligned} \tag{2.36}$$

The above process should be repeated until the desired number of terms of the series solution is obtained. Of course, the more accuracy one desires, the greater is the number of terms of the series that must be calculated.

From the results obtained in this example thus far it is apparent that each successive term of the series introduces two additional frequency components into the output voltage. Table I illustrates that the output voltage V(s) will contain the frequency of the input current I(s) plus an infinite number of the sum and difference frequencies between the frequency of the input current and the frequency of the capacitor variation. Also apparent in this table is the fact that the odd numbered terms of the series contribute only odd sideband frequency components and the even terms contribute only even sideband frequency components.

Up to this point in the example, the solution for the output voltage v(t) has been in terms of a general input current i(t) to a network with unspecified circuit values. In order that a more complete picture of the performance of this time-varying system be obtained, v(t) was calculated for the particular i(t) and the particular set of circuit parameters shown below.

$$i(t) = \cos(w_o t) \text{ amps.}$$

$$w_o = 2\pi\ (10^7) \text{ rad./sec.}$$

$$w_c = 2\pi\ (10^5) \text{ rad./sec.}$$

$$C(t) = C_o + C_1 \cos(w_c t) \text{ farads}$$

$$C_o = 300 \text{ mmf.} \qquad C_1 = 75 \text{ mmf.}$$

TERMS IN SERIES	FREQUENCY COMPONENTS PRESENT
$V_o(s)$	s
$V_1(s)$	$s+jw_c$ $s-jw_c$
$V_2(s)$	$s+j2w_c$ s $s-j2w_c$
$V_3(s)$	$s+j3w_c$ $s+jw_c$ $s-jw_c$ $s-j3w_c$
$V_4(s)$	$s+j4w_c$ $s+j2w_c$ s $s-j2w_c$ $s-j4w_c$
⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮
$V_n(s)$	$s+jnw_c$ $s+j(n-2)w_c$ ... ... $s+jw_c$ $s-jw_c$ ... ... $s-j(n-2)w_c$ $s-jnw_c$
$V_{n+1}(s)$	$s+j(n+1)w_c$ $s+j(n-1)w_c$ .... $s+j2w_c$ s $s-j2w_c$ ... $s-j(n-1)w_c$ $s-j(n+1)w_c$

Table I Terms of the Series Solution V(s) and Their Respective Frequency Components.

Two computer programs were written for the S.D.S. Sigma 7 computer which calculated the envelope of the response voltage for the input current i(t) and the circuit parameters shown above. The first of the two computer programs calculates the first four terms of the series for V(s) by the method described above and obtains v(t), the inverse transform of V(s) by evaluating the residues of V(s) using the Heaviside expansion method (7). In order to have a means for checking the accuracy of the series solution, the second computer program solved the time-varying differential equation 2.26 numerically for v(t) by the Runge-Kutta method (40). The results of the computer programs for this particular circuit are shown in Figure 2.9.

It should be emphasized at this point that in Figure 2.9, it is the envelope of the response voltage that is shown and not the total response voltage v(t). Notice that the series solution containing just three terms closely approximates the Runge-Kutta solution, and when the series contains four terms there is no visible difference between the series solution and the Runge-Kutta solution.

Figure 2.10 is a normalized illustration of the frequency content of the input current I(s) and the output voltage V(s) after the transients due to the input have subsided. Those frequencies contained within the dashed lines in Figure 2.10(b) are the frequency components present in the first four terms of the series solution shown in Figure 2.9. It is noteworthy that an illustration of the frequency characteristics such as this may only be obtained by the series method and is

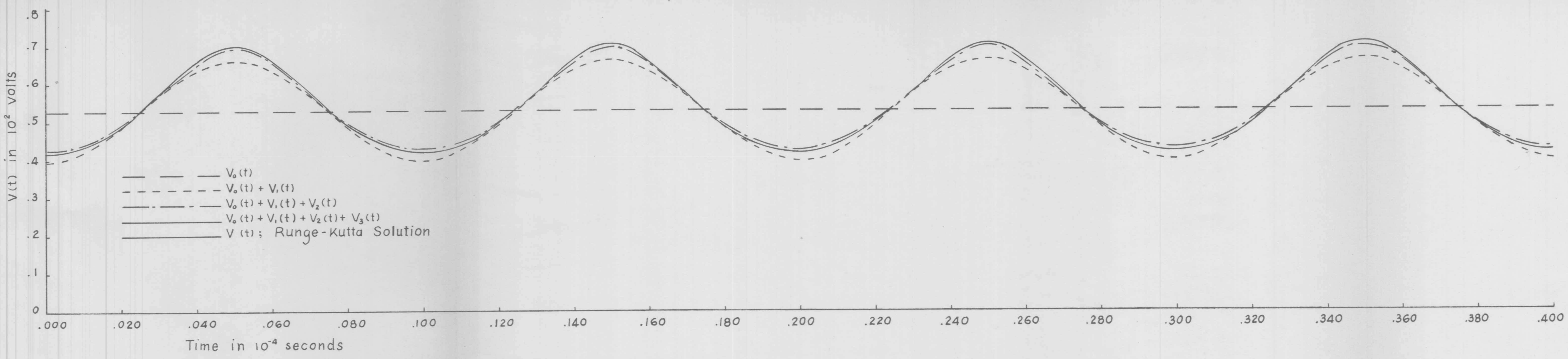

Figure 2.9 Plot of the Envelope of the Response Voltage of a Single Time-Varying Capacitor Excited by an Input Current $i(t) = \cos(\omega_0 t)$

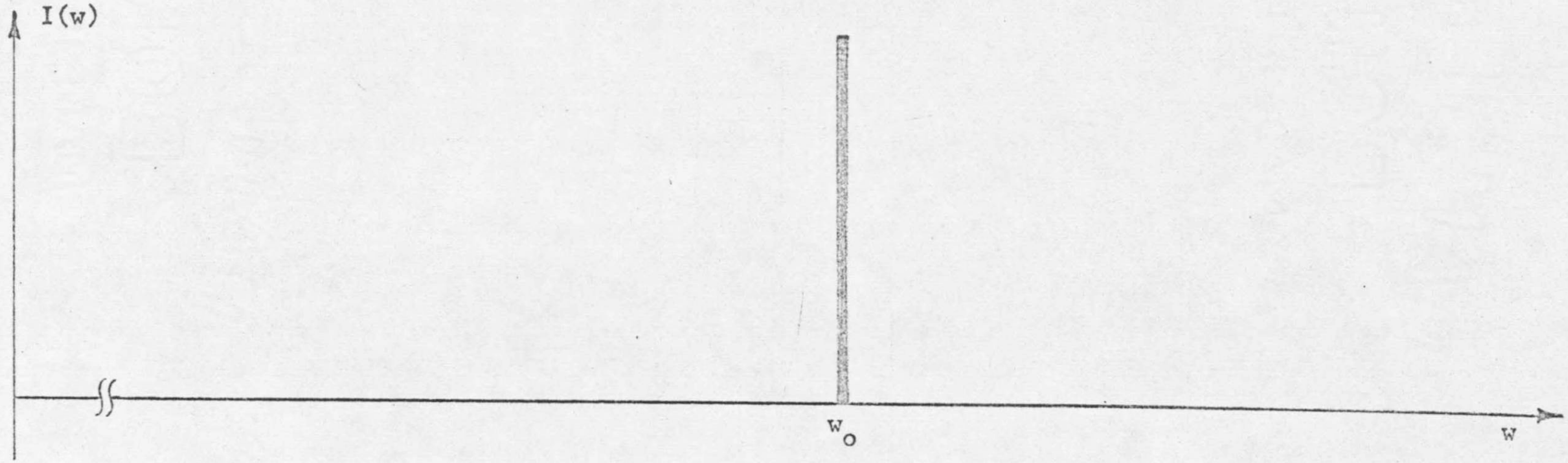

Figure 2.10(a) Normalized Plot of the Frequency Content of i(t).

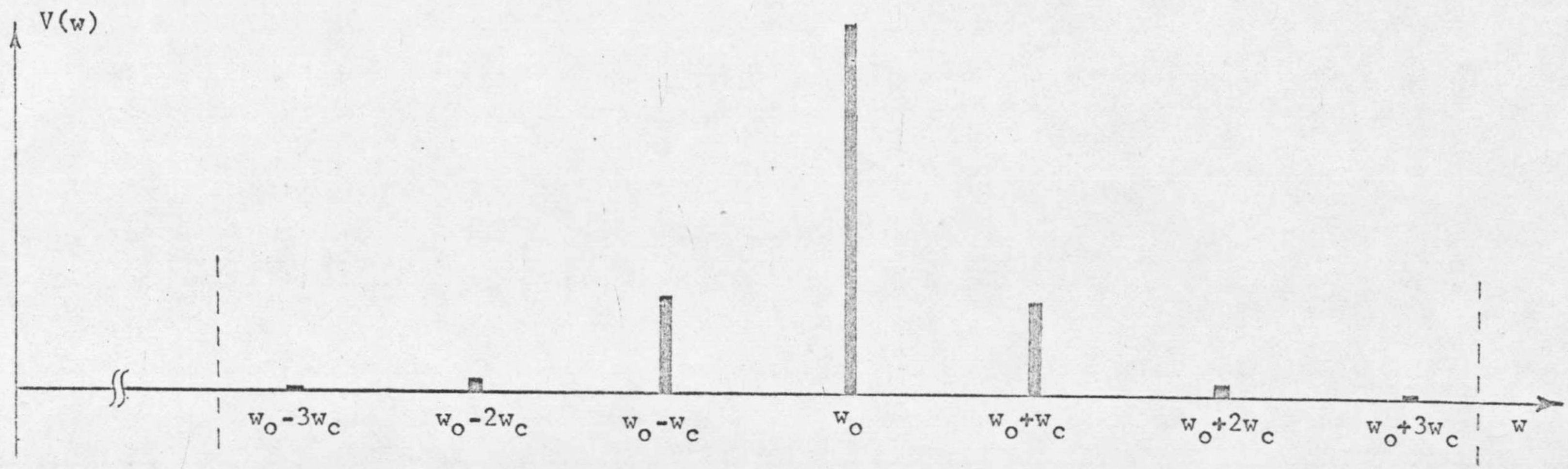

Figure 2.10(b) Normalized Plot of the Frequency Content of v(t).

not at all possible by using the Runge-Kutta method.

As a second example, consider the network shown in Figure 2.11. This network is similar to the one in the first example except that there is now a fixed resistor R in parallel with the time-varying capacitor C(t). The problem remains the same: Determine the voltage response v(t) to a driving current i(t).

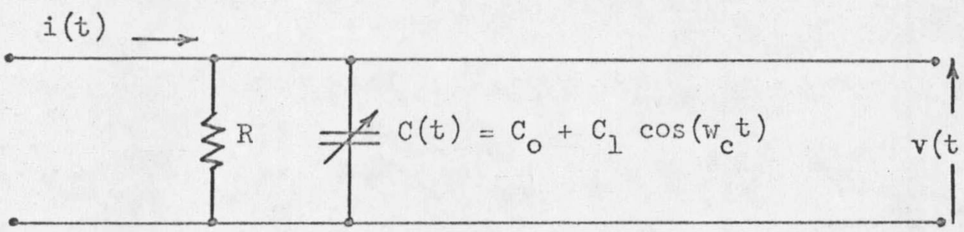

Figure 2.11 Network containing Fixed Resistor and Time-Varying Capacitor

This network is described by the time-varying differential equation 2.36.

$$i(t) = \frac{1}{R} v(t) + \frac{d}{dt}\left[C(t)v(t) \right] \tag{2.36}$$

$$= \frac{1}{R} v(t) + C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\left[\cos(w_c t) \cdot v(t) \right]$$

$$= \frac{1}{R} v(t) + C_o \frac{dv(t)}{dt} + \tfrac{1}{2} C_1 \frac{d}{dt}\left[v(t)e^{jw_c t} + v(t)e^{-jw_c t} \right]$$

The procedure to be followed to obtain a solution v(t) for the above differential equation is exactly the same procedure explained in

Section 2.3 and used to obtain a solution in the previous example. The first step is to Laplace transform equation 2.36 to obtain equation 2.37.

$$I(s) = \left[\frac{1}{R} + sC_o\right] V(s) + \tfrac{1}{2}sC_1\left[V(s+jw_c) + V(s-jw_c)\right] \qquad (2.37)$$

The above transformed equation suggests the use of the following complete system block diagram.

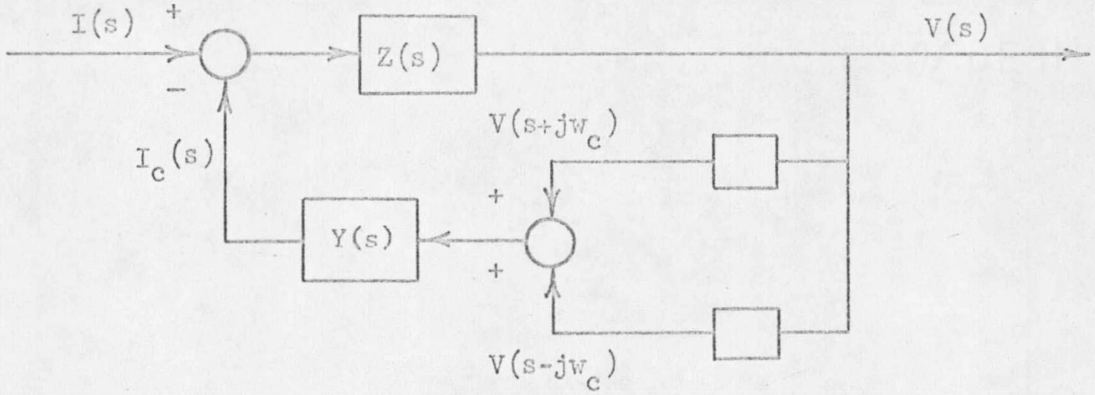

Figure 2.12 Complete Block Diagram of the System Described by Equation 2.37

The block diagram of Figure 2.12 is identical to the block diagram for the single capacitor shown in Figure 2.8 except in the above diagram the impedance of the fixed part of the network Z(s) has changed due to the introduction of the fixed resistance. All the remaining parameters are the same as in Figure 2.8.

$$Z(s) = \frac{1}{C_o} \cdot \frac{1}{s + \frac{1}{RC_o}} \tag{2.38}$$

As before, for conciseness, let G(s) be defined by equation 2.39.

$$G(s) = Y(s)Z(s) \tag{2.39}$$

Following the same procedure as before, it is possible to obtain the individual terms of the series solution.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.40}$$

Suppose, in Figure 2.12 above, a current I(s) is applied to the system, and also suppose that the feedback path is, for the moment, open circuited. At the output will appear a voltage; call it $V_o(s)$.

$$V_o(s) = Z(s)I(s) \tag{2.41}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = Y(s)\left[V_o(s+jw_c) + V_o(s-jw_c) \right] \tag{2.42}$$

$$\begin{aligned} V_1(s) &= -Z(s)I_{co}(s) \\ &= -G(s)\left[V_o(s+jw_c) + V_o(s-jw_c) \right] \\ &= -G(s)\left[Z(s+jw_c)I(s+jw_c) + Z(s-jw_c)I(s-jw_c) \right] \end{aligned} \tag{2.43}$$

When the voltage $V_1(s)$ appears at the output it will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = Y(s) \left[V_1(s+jw_c) + V_1(s-jw_c) \right] \tag{2.44}$$

$$\begin{aligned} V_2(s) &= -Z(s)I_{c1}(s) \\ &= -G(s) \left[V_1(s+jw_c) + V_1(s-jw_c) \right] \\ &= G(s) \Big[G(s+jw_c) \left[Z(s+j2w_c)I(s+j2w_c) + Z(s)I(s) \right] \\ &\quad + G(s-jw_c) \left[Z(s)I(s) + Z(s-j2w_c)I(s-j2w_c) \right] \Big] \end{aligned} \tag{2.45}$$

The above process should be repeated until the desired number of terms of the series solution is obtained. As in the previous example, each successive term of the series introduces two additional frequency components in the output voltage and Table I is again appropriate as it illustrates the terms of the series solution and their respective frequency components.

As a numerical example of this sort of time-varying network, v(t) was calculated for the particular i(t) and the particular set of circuit parameters shown below.

$$i(t) = \cos(w_o t) \text{ amps.}$$

$$w_o = 2\pi\ (10^7) \text{ rad./sec.}$$

$$w_c = 2\pi\ (10^5) \text{ rad./sec.}$$

$$C(t) = C_o + C_1 \cos(w_c t) \text{ fds.}$$

$$C_o = 300 \text{ mmf.} \qquad C_1 = 75 \text{ mmf.}$$

$$R = 265 \text{ ohms.}$$

Again, as in the first example, the first four terms of the series for the response voltage V(s) were evaluated by digital computer and this solution was compared to the computer solution of the differential equation 2.36 by the Runge-Kutta method. The plots of the envelope of the response voltage v(t) are shown in Figure 2.13. Notice the series solution containing just three terms closely approximates the Runge-Kutta solution, and when the series contains four terms there is no visible difference between the series solution and the Runge-Kutta solution.

Figure 2.14 is a normalized illustration of the frequency content of the input current I(s) and the output voltage V(s) after the transients due to the input have died away. Those frequencies contained within the dashed lines in Figure 2.14(b) are the frequency components present in the first four terms of the series solution shown in Figure 2.13.

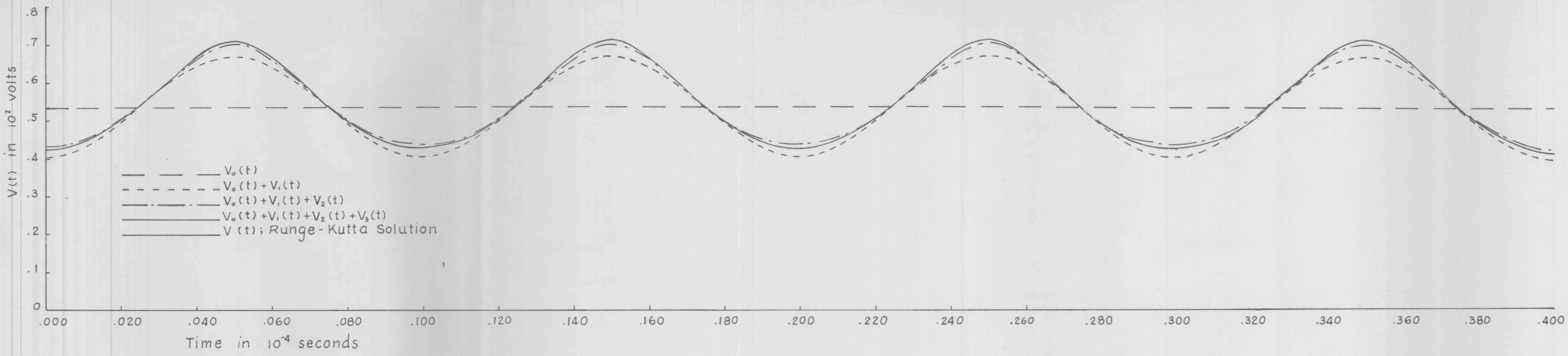

Figure 2.13 Plot of the Envelope of the Response Voltage of a Network Containing a Fixed Resistor and a Time-Varying Capacitor to an Input Current $i(t) = \cos(\omega_o t)$