



An engineering analysis of systems containing time-varying elements  
by Philip Alan Buckley

A thesis submitted to the graduate faculty in partial fulfillment of the requirements for the degree of  
DOCTOR OF PHILOSOPHY in Electrical Engineering  
Montana State University  
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**Abstract:**

In this thesis a general method of analysis is developed for single input, single output, linear, time-varying systems by use of familiar Laplace transform techniques. The method yields the pertinent information necessary for the engineering utilization of any system containing time-dependent parameters.

Contents of the thesis are as follows: First, electrical networks containing sinusoidally varying capacitors and excited by sinusoidal sources are investigated. By a unique block diagram representation these networks are visualized as feedback systems and from the diagrams a logical and natural series solution is derived in the complex frequency domain from a basic cause and effect relationship. Individual terms of the series contain the contributions of the various frequencies present in the solution. Second, two sufficient conditions for the absolute and uniform convergence of the series solutions are developed, one valid for the solution after steady-state conditions have been reached, the other valid for the total time solution including transient response. A different series for the steady-state solution is obtained by method of harmonic balance and the convergence criteria for this series is compared with the convergence criteria for the original series. Third, the method of analysis is applied to the general case of equations with time-varying coefficients and the method is extended to include the following cases: (1) nonsinusoidal yet periodic variation of system parameters, (2) more than one time-varying element in the system, (3) systems in which all the elements are time-varying, (4) arbitrary time variations of system parameters, and (5) time-varying systems excited by modulated signals.

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CONTAINING TIME-VARYING ELEMENTS

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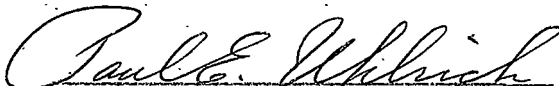
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## ABSTRACT

In this thesis a general method of analysis is developed for single input, single output, linear, time-varying systems by use of familiar Laplace transform techniques. The method yields the pertinent information necessary for the engineering utilization of any system containing time-dependent parameters.

Contents of the thesis are as follows: First, electrical networks containing sinusoidally varying capacitors and excited by sinusoidal sources are investigated. By a unique block diagram representation these networks are visualized as feedback systems and from the diagrams a logical and natural series solution is derived in the complex frequency domain from a basic cause and effect relationship. Individual terms of the series contain the contributions of the various frequencies present in the solution. Second, two sufficient conditions for the absolute and uniform convergence of the series solutions are developed, one valid for the solution after steady-state conditions have been reached, the other valid for the total time solution including transient response. A different series for the steady-state solution is obtained by method of harmonic balance and the convergence criteria for this series is compared with the convergence criteria for the original series. Third, the method of analysis is applied to the general case of equations with time-varying coefficients and the method is extended to include the following cases: (1) nonsinusoidal yet periodic variation of system parameters, (2) more than one time-varying element in the system, (3) systems in which all the elements are time-varying, (4) arbitrary time variations of system parameters, and (5) time-varying systems excited by modulated signals.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

## 1.1 INTRODUCTION

Most of the techniques that engineers apply to the solution of linear systems with constant parameters fail to be of much value to them in the solution to problems dealing with systems with varying parameters. If the parameters vary in such a way as to be functions of one or more of the system's dependent variables, then the system is immediately nonlinear and the method of solution is sensitive to the type and severity of the nonlinearity. If, on the other hand, the parameters vary in such a way as to be functions of the system's independent variable (most often time), then the system remains linear and is described by a linear differential equation with varying coefficients. In problems of the latter type, the engineer is equipped with several tools which he commonly applies to constant parameter systems which are also valid and powerful tools for the solution of systems with variable parameters. One of the most important of these is the principle of superposition.

It would be ideal if all systems with time-varying parameters could be characterized in complete mathematical detail by the appropriate differential equation and solved to yield all the necessary information. However, apart from simple first order differential equations with varying coefficients, little is known about such solutions. In an effort to obtain some sort of answer, the engineer usually attempts to simplify the system model or to solve the problem by numerical techniques, neither of which may yield the desired qualitative or quantitative result or

insight. It would be desirable if there were other techniques available which an engineer could apply to obtain useful information from time-varying parameter systems without having to resort to doubtful approximations or numerical techniques. In the following section, works from the technical literature directed toward this end are discussed.

## 1.2 LITERATURE REVIEW

Recorded history of solutions to time-varying equations begins with the publication, in 1868, of the solutions to the Mathieu equation (50). In 1877, Hill (37) generalized the Mathieu equation and in 1883 Floquet (28) published a general treatment of linear differential equations with periodic coefficients. Mathieu and Hill equations are special cases in the Floquet theory (52). In 1887, by means of Hill's analysis, Lord Rayleigh studied the Melde experiment, which concerned the maintenance of vibrations in a body by forces of double frequency (47).

Variable parameter system analysis attracted scant attention among engineers and physicists prior to 1912. A possible explanation was the lack of physical application and the analytical difficulties. Whittaker (85) in 1912, and van der Pol (77) in 1928, studied the stability of solutions obtained by Mathieu, Hill, and Floquet. The phenomenon of parametric excitation was investigated in 1931 by Mandelstam and Papalexi (49) and later by Lazarev (44), Migulin (55), Barrow (8,9,10), and Gorelik (34,35).

In 1917 Nichols (60), and in 1921 Carson (18) investigated systems such as the variable capacitor microphone and the induction generator with variable mutual inductance. Carson showed that a system described by a differential equation with varying coefficients could also be written as an integral equation of the Volterra type, possessing an infinite series solution, which unfortunately and unexplainedly diverged in many cases of practical interest. Carson neglected to give any criteria relative to conditions under which his series solution would converge. Later in the same paper Carson suggested yet another series for steady-state solutions but did not develop the technique in detail. Bolle (15), in 1955, furthered this technique, commonly referred to as Harmonic Balance, by using phasor notation to obtain the first steady-state solution usable to an engineer. Later, Desoer (24) in 1959 and Sandberg (66) in 1964 contributed to Bolle's technique.

A parallel path of development has taken place since 1922, due to the increasing interest in frequency modulation. Analysis was done on constant parameter systems with variable frequency excitation and on the similarity between these and time-varying systems. Carson and Fry (19) in 1937, studied the conditions for a quasi-steady-state to apply, and their work was extended by several others including Armstrong (2), Stumpers (75), Weiner (80), and Baghdady (4,5). In 1964, Weiner and Leon (81,82) actually considered the quasi-steady-state plus distortion response of a simple resistor and time-varying capacitor network excited by an FM signal.

From 1950 to 1960, Zadeh (87,88,89,90,91,92,93,94,95,96) published many papers dealing with time-varying and stochastic parameter systems. In particular, he contributed a great deal to the theory of time-varying systems in the specific area of time-varying weighting functions (impulse responses) and their transforms. The papers of Zadeh contribute much to the theory of time-varying systems but the material contained therein is little comfort in that it does not provide a usable engineering analysis unless the system parameters vary slowly with time.

In 1960, Leon (45) reduced a second order differential equation describing a network containing a single sinusoidally varying element to a second order difference equation by transform techniques. In 1962, Schweizer (71,72) using phasor methods, solved essentially the same sort of problem. Confined to the steady-state response, both authors obtained an infinite series solution. The coefficients of the series were continued fractions. Although Leon developed convergence criteria for the continued fractions and estimated the truncation error, no physical significance is attributed to them.

Analytical techniques commonly used in the analysis of nonlinear systems have often also applied to linear time-varying systems with much success (97,98). Perturbation methods, for example, when suitably applied to the problems discussed within this thesis, yield results in the time-domain very similar to the frequency domain results obtained in this thesis.

### 1.3 PURPOSE OF THE THESIS

The purpose of the thesis is to contribute a sound engineering method of analysis applicable to linear systems containing elements which are functions of the system's independent variable. The types of systems considered are single input, single output, lumped linear systems containing parameters which are arbitrary deterministic functions of time.

A key term in the statement of the purpose of the thesis is the adjective 'engineering'. To be of any engineering value, the method of analysis must provide a solution in such a form that the quantities of engineering value are readily accessible and the numerical results require as little human effort as possible. The method should be mathematically sound and limitations, if any, should be clearly defined and related to the system's physical parameters whenever possible.



CHAPTER 2

A SERIES SOLUTION FOR LINEAR TIME-VARYING SYSTEMS

## 2.1 INTRODUCTION

Equations characterizing linear time-varying systems are similar to those characterizing linear time-invariant systems with the exception that in the time-varying case, one or more of the coefficients of the equation are functions of time. Figure 2.1 illustrates a time-varying linear system with a single input  $x(t)$  and a single output  $y(t)$ .

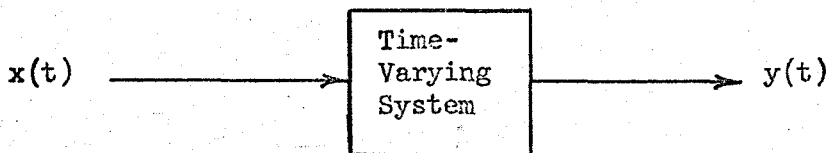


Figure 2.1 Single Input, Single Output Time-Varying System

The linear time-varying system shown in Figure 2.1 is assumed to be characterized by equations of the following form:

$$\begin{aligned} a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_0(t)y(t) \\ = b_m(t)x^{(m)}(t) + b_{m-1}(t)x^{(m-1)}(t) + \dots + b_0(t)x(t) \end{aligned} \quad (2.1)$$

Any set of  $n$  linearly independent functions  $y_1(t)$ ,  $y_2(t)$ , ...,  $y_n(t)$  satisfying the homogeneous, or unforced, part of equation 2.1, but not necessarily satisfying any particular boundary conditions, is

termed a fundamental set of solutions or a set of basis functions. Any solution satisfying the homogeneous or unforced part of equation 2.1 must be a linear combination of  $y_1(t)$ ,  $y_2(t)$ , ...,  $y_n(t)$ . Therefore, once any set of basis functions is known, the particular solution satisfying a given set of  $n$  boundary conditions can be found by algebraic manipulations of the known basis functions and their derivatives (1).

The classical approach to the solution of time-varying differential equations such as equation 2.1 hinges on the determination of a set of basis functions for the unforced portion of the solution, and the use of the superposition, or convolution, integral to determine the response of the system to particular inputs. The major problem herein is that the basis functions are generally difficult to determine and no general procedure exists for determining a set of basis functions for an arbitrary linear differential equation with time-varying coefficients. If a set of basis functions for a particular linear time-varying differential equation is not known (i.e., the differential equation doesn't belong to a class of equations for which the basis functions are known) or cannot be determined, then the solution to the equation cannot be obtained by the classical method and another technique must be found.

## 2.2 LAPLACE TRANSFORM METHOD

The use of Laplace transforms for analyzing linear time-invariant systems is attractive because a differential equation may be transformed into an algebraic equation and algebraic equations as a class tend to be

much easier to solve than do differential equations. On the other hand, the use of Laplace transforms in this usual way for analyzing linear time-varying systems is not so attractive. The reason lies in the fact that the transformed equation is rarely an algebraic equation as it is when analyzing time-invariant systems. Most often the resulting transformed equation is a differential equation in the complex variable  $s$ , or it may be a difference equation, or it may be a combination of both -- a differential-difference equation. In addition, even the resulting transformed equations generally contain varying coefficients and may or may not be easier to solve than the original equation. Fortunately, these difficulties in the use of the Laplace transform for analyzing time-varying systems may be circumvented by the method to be introduced in Section 2.3.

There may be, however, other difficulties associated with the use of Laplace transforms as an equation solving technique whether applied to linear time-varying or time-invariant systems. First of all, the Laplace transform must exist for each term  $a_i(t)x^{(i)}(t)$  of the differential equation, which means that the integral

$$\int_0^t a_i(t)x^{(i)}(t)e^{-st}dt \quad (2.2)$$

must converge for some positive, real value of  $s$  (46). It is quite possible that equations could be encountered for which the integral 2.2

would diverge for one or more terms of the equation, with the result that the Laplace transform method would fail to yield a solution.

Another difficulty in the use of the Laplace transform method is that after the differential equation is Laplace transformed and the resulting equation solved, the inverse transform of the result must be obtained. This inversion process may, depending upon the complexity of the transformed expression, involve lengthy calculations although the procedure in most cases is straightforward (7,46).

Despite the several difficulties mentioned above, there are still some advantages of the Laplace transform method over and above its sometime ability to yield a simpler-to-solve transformed equation. The first of these is that the transformed variables contain valuable information as to their frequency characteristics that is lost when the differential equations are solved by the classical method of Section 2.1. It is this information about frequency characteristics that is necessary so that the designer may determine such important information as the frequencies present in both the input and output of the system, the bandwidth of the system, the frequency dependent system gain, and the figure of merit  $Q$  of the system.

The second additional advantage is that when the inverse of the transformed solution is obtained, the solution is complete. Both the transient solution and the steady-state solution are obtained simultaneously. There is no need, as in the classical method of Section 2.1,



















































































































































































































































































