Indicability in products of groups

by Elizabeth Darragh Behrens

A thesis submitted in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

Abstract:
A group is indicable if and only if it has an infinite cyclic homomorphic image. We investigate the properties of indicability and local indicability in groups, and, in particular, in generalized free products of two groups and in standard wreath products of groups. G. Baumslag proved that any generalized free product of an indicable group with a finitely-generated torsion-free nilpotent group is indicable, and any generalized free product of two finitely-generated torsion-free nilpotent groups is locally indicable. We generalize his results to arbitrary finitely-generated nilpotent groups. We define a group G to be almost locally indicable if and only if every finitely-generated subgroup of G which is non-periodically-generated is indicable. We prove that any generalized free product of an indicable group with a finitely-generated nilpotent. group is indicable, and that the generalized free product of two locally nilpotent groups with a finitely-generated subgroup amalgamated is almost locally indicable.

We characterize indicability in the restricted wreath product as well as in unrestricted wreath products in which the "top" group is finite or non-periodic. We also obtain a complete characterization of local indicability in wreath products.

INDICABILITY IN PRODUCTS OF GROUPS

by

ELIZABETH DARRAGH BEHRENS

A thesis submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematics

Approved:

Chairman, Examining Committee

Head, Major Department

Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

August, 1974

ACKNOWLEDGEMENTS

The author wishes to express her appreciation to Professor Mitchell J. Billis for directing this thesis, and also to Professors Richard M. Gillette and Norman H. Eggert for their careful reading and helpful suggestions.

The thesis is dedicated to my husband, Dale, without whose cooperation and encouragement it could never have been completed.

TABLE OF CONTENTS

ABSTRACT

A group is <u>indicable</u> if and only if it has an infinite cyclic homomorphic image. We investigate the properties of indicability and local indicability in groups, and, in particular, in generalized free products of two groups and in standard wreath products of groups. G. Baumslag proved that any generalized free product of an indicable group with a finitely-generated torsion-free nilpotent group is indicable, and any generalized free product of two finitely-generated torsion-free nilpotent groups is locally indicable. We generalize his results to arbitrary finitely-generated nilpotent groups. We define a group G to be <u>almost locally indicable</u> if and only if every finitely-generated subgroup of G which is non-periodically-generated is indicable. We prove that any generalized free product of an indicable group with a finitely-generated nilpotent group is indicable, and that the generalized free product of two locally nilpotent groups with a finitely-generated subgroup amalgamated is almost locally indicable.

We characterize indicability in the restricted wreath product as well as in unrestricted wreath products in which the "top" group is finite or non-periodic. We also obtain a complete characterization of local indicability in wreath products.

CHAPTER 1
INTRODUCTION

The concept of indicability was introduced by G. Higman in his investigation of group rings [4]. He called a group _indicable_, or _indexed_, if and only if it has an infinite cyclic homomorphic image, and he called a group G _indicable throughout_ if and only if every subgroup of G is indicable. He proved that if G is a group which is indicable throughout and K is a ring with no zero-divisors, then the group ring, R(G,K), of G over K has no zero-divisors and only trivial units. G. Baumslag [1] noted that to obtain this result it is sufficient to require that every finitely-generated subgroup of G is indicable. Hence he directed attention to the class of locally indicable groups.

In this paper we investigate the properties of indicability and local indicability in groups, and principally in generalized products and wreath products of groups.

Gregorac [3] and Karrass and Solitar [5] obtained several results concerning indicability in generalized free products in which the amalgamated subgroup is cyclic. We construct examples to indicate limitations in generalizing their results to free products with non-cyclic amalgamations. Baumslag proved that any generalized free product of a non-trivial finitely-generated torsion-free nilpotent group with an indicable group is indicable. He then applied this result to demonstrate that any generalized free product of two finitely-generated torsion-free nilpotent groups is locally indicable. The main result of this paper is a generalization of Baumslag's theorems to arbitrary finitely-generated nilpotent groups.

With reference to standard wreath products, we obtain necessary and sufficient conditions that the restricted wreath product be respectively, indicable and locally indicable, necessary and sufficient conditions that the unrestricted wreath product be locally indicable, and, in the case that the "top" group is finite or non-periodic, necessary and sufficient conditions that the unrestricted wreath product be indicable.

CHAPTER 2

INDICABLLITY AND LOCAL INDICABILITY

2.1. <u>The Class of Indicable Groups</u>. A group G is defined to be <u>indicable</u> if there exists a non-trivial homomorphism of G into the infinite cyclic group of rational integers, which we denote by Z. Since every non-trivial subgroup of an infinite cyclic group is infinite cyclic, a group G is indicable if and only if there exists an epimorphism of G onto Z or, equivalently, if G has an infinite cyclic factor group. An indicable group, therefore, must be infinite and we note that such a group possesses at least one normal subgroup of index n for each positive integer n.

<u>Example 2.1</u>. Not every non-periodic group is indicable. Take, for example, the infinite dihedral group, which has presentation $D = gp\{a,b;a^2,a^{-1}bab\}$. If θ is a homomorphism of D into Z, then $a\theta = 0$ since $(a^2)\theta = 2(a\theta) = 0$, and $b\theta = 0$ since $(a^{-1}ba)\theta = b\theta = -(b\theta)$. Hence θ must be the trivial homomorphism. Since D may be mapped homomorphically onto the dihedral group of order 2s for each positive integer s, we note that D does possess normal subgroups of every possible even index. However D cannot be mapped non-trivially onto a group of prime order other than two.//

We proceed to establish several properties of indicable groups.

Lemma 2.1. If the group G is generated by subgroups $\{A_i\}_{i\in I}$ with each A_i not indicable, then G is not indicable.

Proof. If θ is a homomorphism of G into Z, then, since θ restricted to A_i is a homomorphism of A_i into Z, we must have this restriction equal to the zero map for each i in I. But then θ must be identically zero. Hence the only homomorphism of G into Z is the trivial homomorphism, so G is not indicable.//

Proposition 2.1. Every extension of a group by an indicable group is indicable.

Proof. If G/H is indicable, then there exists an epimorphism $\theta: G/H \longrightarrow Z$. Hence if ρ is the canonical projection of G onto G/H, the composition $\rho\theta$ is an epimorphism of G onto Z. Thus G is indicable.//

Proposition 2.2. The direct sum of a collection of groups is indicable if and only if at least one factor is indicable.

Proof. The direct sum of a family $\{A_i\}_{i\in I}$ of groups is generated by the A_i, so by Lemma 2.1 if the sum is indicable, then at least one factor must be indicable.

Conversely, by Proposition 2.1, if some factor is indicable, then the direct sum is indicable, since each factor is a homomorphic image of the direct sum.//

Higman [4] observed the following, which is a consequence of the universal mapping property of free groups.

Proposition 2.3. Free groups are indicable.
Since every group is isomorphic to a factor group of a free group, and non-indicable groups exist, the following corollary is evident.

Corollary. The class of indicable groups is not closed under factor groups.

If G is any group, we let G' denote the commutator subgroup of G. Hence $G' = gp\{[a,b] \mid a$ and b are elements of $G\}$ where $[a,b] = a^{-1}b^{-1}ab$ is the commutator of a with b. Then G/G' is the largest Abelian factor group of G. That is, if H is a normal subgroup of G, then G/H is Abelian if and only if G' is contained in H.

Proposition 2.4. A group G is indicable if and only if the Abelianized group G/G' is indicable.

Proof. If θ is an epimorphism of G onto Z, then $G/\mathrm{Ker}\theta$ is isomorphic to Z and is thus Abelian. Hence G' is contained in $\mathrm{Ker}\theta$, so $G/\mathrm{Ker}\theta$ is an infinite cyclic homomorphic image of G/G' and thus G/G' is indicable. Conversely, if G/G' is indicable, then, by Proposition 2.1, G is indicable.//

Our next result characterizes indicability in Abelian groups.

Proposition 2.5. An Abelian group G is indicable if and only if G is of the form $G = H + K$ where H is infinite cyclic.

Proof. If H is infinite cyclic then certainly H is indicable, so by Proposition 2.2, G is indicable. Conversely, if θ is a homomorphism of G onto Z then the exact sequence $E \longrightarrow K \longrightarrow G \xrightarrow{\theta} Z \longrightarrow E$ will be shown to split as follows. Since Z is free, there exists a homomorphism $\sigma : Z \longrightarrow G$ such that $\sigma\theta = id_Z$. Hence σ is injective, so $H = \mathrm{Im}\sigma$ is infinite cyclic. Now for each g in G, the element $(g - g\theta\sigma) = g'$ is in the kernel, K, so $g = g\theta\sigma + g'$ is an element of $gp\{H,K\}$. Hence G is generated by H and K. Further, if h is a non-zero element of H, then $h = z\sigma$ for some $z \neq 0$. So $h\theta = z\sigma\theta = z \neq 0$, and hence h is not in the kernel, K. Thus $H \cap K = E$. Consequently, since G is Abelian, we have $G = H + K$.//

Corollary. The class of indicable groups is not closed under subgroups.

Proof. For example, let K in Proposition 2.5 be any finite Abelian Group.//

The fundamental theorem of finitely-generated Abelian groups states that every finitely-generated Abelian group is equal to a direct sum of a finite number of cyclic groups of prime-power or infinite order. Hence by Proposition 2.5 every infinite finitely-generated Abelian group is indicable. We proceed to prove a generalization of this fact.

Proposition 2.6. A finitely-generated nilpotent group is indicable if and only if it is infinite.

Proof. Let G be a finitely-generated nilpotent group. It is well-known [7, p. 194] that G is infinite if and only if G/G' is infinite. Therefore, by Proposition 2.4 and the above remarks, we have the following chain of equivalences. G is indicable if and only if G/G' is indicable if and only if G/G' is infinite if and only if G is infinite.//

A complete group is a group in which extraction of roots is always possible. That is, G is complete if and only if, for each element g in G, and each positive integer n, there exists some element x in G such that $x^n = g$. A complete Abelian group is usually called <u>divisible</u>.

<u>Proposition 2.7</u>. No complete group is indicable.

<u>Proof</u>. Let G be a complete group. Suppose θ is an epimorphism of G onto Z. Then for some g in G, $g\theta = 1$. By the completeness of G there exists some h in G such that $h^2 = g$. Then $2(h\theta) = (h^2)\theta = g\theta = 1$, so $gp\{h\theta\} = gp\{1\} = Z$. Therefore $h\theta$ must be a generator and hence equal to +1 or -1. But we then have $2(h\theta) = 2(\pm 1) = \pm 2 = 1$. This is clearly false. Hence G is not indicable.//

This proposition provides a supply of infinite non-indicable groups which includes the torsion-free complete nilpotent groups, in which roots can always be extracted uniquely. Among these, for example, is the additive group of rational numbers.

2.2. <u>The Class of Locally Indicable Groups.</u> If P is a group property then a group G is called <u>locally P</u> if every non-trivial finitely-generated subgroup of G has property P. Hence a group G is <u>locally indicable</u> if and only if every non-trivial finitely-generated subgroup of G is indicable. Higman [4] called a group G <u>indicable throughout</u> if and only if every non-trivial subgroup of G is indicable.

Thus the class of locally indicable groups contains the class of groups which are indicable throughout. We proceed to establish several hereditary and closure properties of the class of locally indicable groups.

Proposition 2.8. If G contains a normal subgroup H such that H and G/H are locally indicable, then G is locally indicable.

Proof. (Higman demonstrated this property in similar fashion for the class of groups indicable throughout.) Let K be a non-trivial finitely-generated subgroup of G. If K is contained in H, then, by the local indicability of H, K is indicable. If K is not contained in H, then KH/H, which is isomorphic to $K/H \cap K$, is a non-trivial finitely-generated subgroup of G/H and is hence indicable. By Proposition 2.1, since $K/H \cap K$ is indicable, so is K.//

Proposition 2.9. Every subgroup of a locally indicable group is locally indicable.

Proof. If G is locally indicable and H is a subgroup of G, then any finitely-generated subgroup of H is a finitely-generated subgroup of G and is therefore indicable. Hence H is locally indicable.//

<u>Proposition 2.10</u>. The Cartesian product of a collection of locally indicable groups is locally indicable.

<u>Proof</u>. Let $\{A_s\}_{s\in S}$ be a collection of locally indicable groups. Let H be a non-trivial finitely-generated subgroup of $\prod_{s\in S} A_s$. If H is generated by the set $T = \{(a_s^1), \ldots (a_s^n)\}$ then, since H is non-trivial, there exists some σ in S for which the set $T_\sigma = \{a_\sigma^1, a_\sigma^2, \ldots, a_\sigma^n\}$ does not consist of the identity alone. Hence T_σ generates a non-trivial finitely-generated subgroup H_σ of A_σ which is thereby indicable. But H_σ is a homomorphic image of H under the restriction of the projection map of $\prod_{s\in S} A_s$ onto A_σ. Therefore by Proposition 2.1, H is indicable.//

<u>Corollary</u>. The direct sum of a collection of locally indicable groups is locally indicable.

<u>Proof</u>. The direct sum is a subgroup of the Cartesian product, so by Proposition 2.8 the direct sum is locally indicable.

Since no finite group is indicable, we note that locally indicable groups must be locally infinite and hence torsion-free. Higman noted that since every subgroup of a free group is free, free groups are indicable throughout. Thus we have the following proposition and corollary which

Proposition 2.11. Free groups are locally indicable.

Corollary. The class of locally indicable groups is not closed under factor groups.

Proposition 2.12. A locally nilpotent group G is locally indicable if and only if G is torsion-free.

Proof. If G is not torsion-free, then G contains a non-trivial finite subgroup which is thus finitely-generated but not indicable, so G is not locally indicable. Conversely, if G is torsion-free, then any non-trivial finitely-generated subgroup of G is an infinite finitely-generated nilpotent group which is indicable by Proposition 2.6.//

Every locally nilpotent torsion-free group G can be embedded in a complete locally nilpotent torsion-free group, m(G), called a Malcev completion of G. By Proposition 2.7, m(G) is not indicable, but by Proposition 2.12, m(G) is locally indicable. Thus Proposition 2.12 verifies the existence of groups which are not indicable, hence not indicable throughout, but in which every finitely-generated subgroup is indicable. So the groups which are indicable throughout form a proper sub-class of the class of locally indicable groups. (We will find further use for Malcev completions in Chapter 3, where they are helpful in our investigation of indicability in generalized free products of finitely-generated nilpotent groups.)

CHAPTER 3

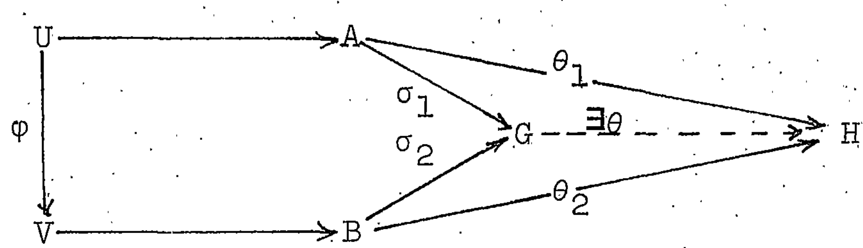
INDICABILITY IN GENERALIZED FREE PRODUCTS

3.1. Preliminaries. The free product of a family of groups with a single subgroup amalgamated was first introduced by Otto Schreier in 1927. The concept was generalized by Hanna Neumann in 1948, and in 1954 B. H. Neumann [9] collected their results, together with numerous applications and examples.

In this paper we are primarily interested in the generalized free products of two groups. We begin with the following definition. Let A and B be groups with presentations $A = gp\{a_\lambda; R_\alpha(a_\lambda)\}$, $B = gp\{b_\mu; S_\beta(b_\mu)\}$, where λ, μ, α, and β are elements of appropriate indexing sets. Let U be a subgroup of A, V a subgroup of B, and φ an isomorphism of U onto V. Define a group G with presentation equal to the union of the presentations of A and B together with an additional family of relations $\{T_u = (u\varphi)u^{-1} | u \in U\}$, called the amalgamating relations. Then G has presentation $G = gp\{a_\lambda, b_\mu; R_\alpha(a_\lambda), S_\beta(b_\mu), T_u\}$ and G is called the free product of A and B with the subgroups U and V amalgamated under φ. G will be denoted $(A*B;U,V,\varphi)$, or, if the isomorphism φ is understood, we will simply identify U and V as the same group and use the notation $(A*B;U)$.

It can be shown [8] that the subgroup A^* of G generated by the set $\{a_\lambda\}$ is isomorphic to A, the subgroup B^* generated by $\{b_\mu\}$ is isomorphic to B, and $A^* \cap B^* = U^*$ is isomorphic to U and V. Hence G is generated by isomorphic copies of A and B. If σ_1 and σ_2 are the embeddings of A and B into G, then we have, for each u in U, $u\sigma_1 = u\varphi\sigma_2$.

G is a free product in the sense that if H is a group and $\theta_1 : A \longrightarrow H$ and $\theta_2 : B \longrightarrow H$ are homomorphisms such that, for each u in U, $u\theta_1 = u\varphi\theta_2$, then there exists a unique homomorphism $\theta : G \longrightarrow H$ which extends θ_1 and θ_2 so that the following diagram commutes.



We note that this universal mapping property assures that the generalized free product of groups A and B is unique up to isomorphism, and is independent of the choice of presentations of A and B. We shall find the universal mapping property invaluable in our consideration of indicability in generalized free products.

Knowledge of the structure of subgroups, and especially that of finitely-generated subgroups, is essential to the study of local indicability. Karrass and Solitar [5] extended the results of Hanna Neumann to obtain a complete characterization of the subgroups of the generalized free product of two groups. In order to describe this subgroup structure we need first to define the notion of a tree product.

Let $\{A_i\}_{i \in I}$ be a collection of groups, and suppose that for certain pairs of groups A_i and A_j, $i \neq j$, there exist subgroups U_{ij} of A_i and U_{ji} of A_j and isomorphisms $\theta_{ij}: U_{ij} \rightarrow U_{ji}$ and $\theta_{ji}: U_{ji} \rightarrow U_{ij}$ such that $\theta_{ij} = \theta_{ji}^{-1}$. Let $J = \{(i,j) \in I \times I \mid \theta_{ij} \text{ exists}\}$. Then for each (i,j) in J, the amalgamated product $(A_i * A_j; U_{ij}, U_{ji}, \theta_{ij})$ exists. The <u>partial generalized free product of the A_i</u> is the group G with presentation being the union of the presentations of the A_i together with the presentations of the $(A_i * A_j; U_{ij}, U_{ji}, \theta_{ij})$ for each (i,j) in J, using a fixed presentation for each A_i throughout. It is not difficult to show that G is independent of the particular presentation used for each A_i.

Now with the partial generalized free product G we may associate a graph whose vertices are the groups A_i and whose edges join two vertices A_i and A_j if and only if (i,j)

and hence (j,i) is in J. If this graph is a tree--that is, if it is connected and contains no cycles--then G is called a <u>tree product of the factors A_i with the subgroups U_{ij} and U_{ji} amalgamated under θ_{ij}</u>.

Karrass and Solitar verified the following properties of the tree product G with corresponding graph Γ.

T1. G contains the vertices A_i as subgroups in the natural way--that is, G is generated by a family of subgroups which are isomorphic to the factors A_i.

T2. The vertices and edges of any subtree generate their tree product as a subgroup of G in the natural way.

T3. If the vertices of Γ are partitioned into subtrees Γ_α, then we may form a new graph Γ^* whose vertices are the Γ_α and whose edges are the edges of Γ which join vertices of Γ_α and Γ_β for $\alpha \neq \beta$. The graph Γ^* may be shown to be a tree. Let G_α be the tree product corresponding to Γ_α for each α. Then the tree product G* corresponding to Γ^*, whose vertices are the G_α, has the same presentation, and is hence isomorphic to G.

In characterizing the subgroups of the generalized free product of two groups, Karrass and Solitar showed that every subgroup H of (A*B;U) has the following properties.

S1. H is generated by two of its subgroups, F and S.

S2. The subgroup F is called the <u>free part</u> of H. F is a free group which has trivial intersection with S^H, the normal subgroup of H generated by S. Hence $H/S^H = FS^H/S^H \cong F/F \cap S^H = F$ and so F is a factor group of H.

S3. The subgroup S, called the <u>base group</u> of H, is a tree product whose vertices are conjugates in (A*B;U) of A or B intersected with H and whose amalgamated subgroups are conjugates of U intersected with H.

S4. If H is finitely-generated, then F is finitely-generated and S is the tree product of finitely many of its vertices.

S5. If S is finitely-generated and each amalgamated subgroup is finitely-generated, then each vertex is finitely-generated.

3.2. <u>Indicability in the Generalized Free Products of Two Groups</u>. We begin with two useful lemmas.

<u>Lemma 3.1</u>. If neither A nor B is indicable, then (A*B;U) is not indicable.

<u>Proof</u>. (A*B;U) is generated by isomorphic copies of A and B, so by Proposition 2.1, the product is not indicable.

<u>Lemma 3.2</u>. If A is indicable and U is not indicable then (A*B;U) is indicable.

Proof. Since A is indicable, there exists an epimorphism θ_1 of A onto Z, and since U is not indicable, θ_1 restricted to U is the zero map. Therefore, if $\theta_2:B \longrightarrow Z$ is the zero map, then θ_2 agrees with θ_1 on U, and hence there exists an epimorphism $\theta:G \longrightarrow Z$ which extends the two maps.//

Gregorac obtained the following results in the case that the amalgamated subgroup is cyclic.

Theorem 3.1. [3] If A and B are finitely-generated non-trivial torsion-free nilpotent groups and U is cyclic, then (A*B;U) is indicable.

Corollary. If A and B are groups with normal subgroups M and N, respectively, such that A/M and B/N are non-trivial finitely-generated torsion-free nilpotent groups and U is cyclic, then (A*B;U) is indicable.

He observed that the generalized free product of two free groups with a cyclic subgroup amalgamated is therefore indicable. More generally, a direct result of his corollary is the following proposition.

Proposition 3.1. If A and B are indicable and U is cyclic, then (A*B;U) is indicable.

Thus in the case of a cyclic amalgamation indicability of both factors is sufficient to assure indicability of the generalized free product. We note that it is not, in general, sufficient that just one factor be indicable, as the following example shows.

Example 3.1. Let $A = gp\{a\}$, an infinite cyclic group. Let $B = gp\{x,y;x^2,y^2\}$. Since B is periodically-generated B is not indicable. Let $U = gp\{a^2\}$, and $V = gp\{xy\}$. Since both U and V are infinite cyclic we may define an isomorphism $\varphi: U \longrightarrow V$ by $a^2\varphi = xy$. Let $G = (A*B;U,V,\varphi)$. Now if θ is a homomorphism of G into Z, then, since x and y have finite order, $x\theta = 0$ and $y\theta = 0$. Hence $(xy)\theta = a^2\theta = 2(a\theta) = 0$, and thus $a\theta = 0$. Therefore the only homomorphism of G into Z is the trivial homomorphism, so G is not indicable, although the factor A is indicable.//

We consider next whether the condition in Proposition 3.1 that the amalgamation be cyclic can be weakened. In general, the generalized free product of two indicable groups is not indicable. In Example 3.2 we construct a generalized free product of two free groups which is not indicable. Hence it is clear that the condition on U may be relaxed only when conditions stronger than indicability are placed on the factors.

Example 3.2. Let $F_1 = gp\{a,b\}$ and let H_1 be the subgroup of F_1 generated by the set $S = \{a^3,b^3,[a,b],[a^2,b^2]\}$. F_1 is a free group on two generators and we will show below that the set S freely generates H_1. Similarly, let $F_2 = gp\{x,y\}$, and $H_2 = gp\{x^3,y^3,[x,y],[x^2,y^2]\}$. Define an isomorphism $\varphi: H_1 \longrightarrow H_2$ by $a^3\varphi = [x,y]$, $b^3\varphi = [x^2,y^2]$, $[a,b]\varphi = x^3$, and $[a^2,b^2]\varphi = y^3$. Then the generalized free product $G = (F_1 * F_2; H_1, H_2, \varphi)$ has the presentation $G = gp\{a,b,x,y;\ a^{-3}[x,y], b^{-3}[x^2,y^2], [a,b]x^{-3}, [a^2,b^2]y^{-3}\}$. One can readily observe that the commutator subgroup of G is of finite index in G, and hence G is not indicable.

To show that the subgroups H_1 and H_2 are freely generated by the given generators, we must show that no non-empty word in the four generators is trivial. The procedure is well-known and tedious. For this reason we present here only an outline of the proof.

1. $S_1 = \{[a,b],[a^2,b^2]\}$ is free in F_1. (One shows by induction that if W is a non-empty, freely-reduced word in $[a,b]$ and $[a^2,b^2]$, then W is non-trivial and, further, W begins in one of $a^{-1}b^{-1}$, $b^{-1}a^{-1}$, $a^{-2}b^{-2}$, $b^{-2}a^{-2}$, and ends in either ab, ba, a^2b^2, or b^2a^2.)

2. $S_2 = \{a^3,b^3\}$ is free in F_1. (If W is a freely-

reduced word in a^3 and b^3, then W is also freely-reduced with respect to a and b. Hence if W is non-empty, then W is non-trivial.)

3. Any word of the form $X_1X_2\ldots X_{2n}$ is non-trivial if n is greater than zero, where X_{2i} is a freely-reduced word in a^3 and b^3, and X_{2i-1} is a freely-reduced word in $[a,b]$ and $[a^2,b^2]$, and each X_j, except possibly X_1, is non-empty. (One shows by induction that W is non-trivial, and, when freely-reduced with respect to a and b, ends in one of a^3, a^{-3}, b^3, b^{-3}, a^{-1}, b^{-1}, a^{-2}, b^{-2}.)

4. Any non-empty word in the generators $S = S_1 \cup S_2$ is either freely-equal (with respect to S) to a word of the type described in 1 or 2, or is freely-equal to a conjugate of a word of the type described in 3. Hence no non-empty word in these generators is trivial, and consequently the subgroup H_1 which they generate is free of rank 4.//

Baumslag considered the case in which the factors are finitely-generated torsion-free nilpotent groups and obtained the following theorem.

<u>Theorem 3.2</u>. [1] If A is an indicable group and B is a finitely-generated torsion-free nilpotent group, then (A*B;U) is indicable.

In proving this theorem, Baumslag used several properties of the Malcev completion of a finitely-generated torsion-free nilpotent group. It may be shown that every locally nilpotent torsion-free group G can be embedded in a complete locally nilpotent group [6, p.256]. A Malcev completion of G is defined to be a minimal element in the collection of all complete locally nilpotent groups which contain G. We include the following properties of completions for reference. Proofs of the first three may be found in the sources cited.

Lemma 3.3. [6, p. 256] If M and M^* are two Malcev completions of the locally nilpotent group G, then there exists an isomorphism between them which extends the identity automorphism of G.

Hence we may denote by m(G) the Malcev completion of G, since it is unique up to isomorphism.

Lemma 3.4. [6, p. 256] A complete locally nilpotent group H is the Malcev completion of the locally nilpotent group G if and only if, for each x in H, there exists a positive integer n such that x^n is an element of G.

Lemma 3.5. [2, p. 18] If φ is a homomorphism of a finitely-generated torsion-free nilpotent group G into a complete group H, then there exists a homomorphism Φ mapping m(G) into H which extends φ.

Lemma 3.6. If G is a finitely-generated torsion-free nilpotent group and U is a subgroup of finite index in G, then m(G) = m(U).

Proof. Let x be an element of m(G). Then by Lemma 3.4 there exists an integer n, greater than zero, such that x^n is an element of G. Since U is of finite index in G, there exists a normal subgroup K of G which is of finite index in G and is contained in U [12, p. 53]. Thus x^nK has finite order in the finite group G/K, so there exists an integer m, greater than zero, such that $(x^n)^m$ is in K and hence in U. Therefore m(G) is a completion of U. Hence by Lemma 3.3 we have m(G) = m(U).//

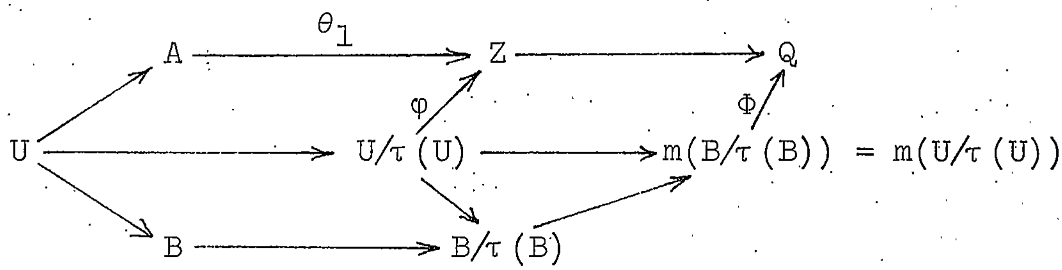
Using an approach similar to Baumslag's, we prove that the requirement in Theorem 3.2 that B be torsion-free may be omitted. We use the fact that in a locally nilpotent group g, the elements of finite order form a normal subgroup $\tau(G)$ such that $G/\tau(G)$ is torsion-free [6, p.229]. We obtain the following theorem.

Theorem 3.3. If A is an indicable group and B is a finitely-generated nilpotent group, then $(A*B;U)$ is indicable.

The proof of Theorem 3.3 is divided into two cases and we present each in the form of a lemma.

Lemma 3.7. If A is an indicable group, B a finitely-generated nilpotent group, and U a subgroup of finite index in B, then $(A*B;U)$ is indicable.

Proof. Consider the following diagram, where the unlabeled arrows represent the natural injections and canonical epimorphisms.



Since A is indicable, there exists an epimorphism θ_1 of A onto Z. Note that if U is contained in the kernel of θ_1 then the zero map of B into Z agrees with θ_1 on U. Hence by the universal mapping property there exists an epimorphism of $(A*B;U)$ onto Z and we are done. In any case, we do have the torsion subgroup of U, $\tau(U)$, contained in the kernel of θ_1. Hence there exists an induced homomorphism φ mapping $U/\tau(U)$ into Z. By a fundamental isomorphism theorem, we have $U/\tau(U) = U/U\cap\tau(B) \cong U\tau(B)/\tau(B)$, and hence, since [B:U] is finite, $U/\tau(U)$ is isomorphic to a subgroup of finite index in $B/\tau(B)$. Now $U/\tau(U)$ and $B/\tau(B)$ are finitely-generated torsion-free nilpotent groups, so their Malcev completions, $m(U/\tau(U))$ and $m(B/\tau(B))$, exist, and they are isomorphic by Lemma 3.6. Consider Z as a subgroup of Q, the additive group of rational numbers. Then φ is a homomorphism of $U/\tau(U)$ into the complete group Q. Thus, by Lemma 3.5, φ has an extension Φ mapping $m(U/\tau(U))$ into Q which makes the diagram commute. Now let θ_2 be the composition $B \to B/\tau(B) \to m(B/\tau(B)) \xrightarrow{\Phi} Q$. By the commutativity of the diagram we see that θ_2 is a homomorphism which agrees with θ_1 on U. Hence there exists a homomorphism, θ, mapping (A*B;U) into Q which extends θ_1 and θ_2. The image of θ, being generated by $A\theta_1 = Z$ and $B\theta_2$, is a finitely-generated non-trivial subgroup of Q and hence is isomorphic to Z. Therefore the generalized free product is indicable.//

<u>Lemma 3.8</u>. If A is any group, B a finitely-generated nilpotent group, and U a subgroup of infinite index in B, then (A*B;U) is indicable.

Proof. Baumslag [1] has shown that if B is finitely-generated and nilpotent then a subgroup U is of infinite index in B if and only if UB' is of infinite index in B. Thus B/UB' is an infinite finitely-generated Abelian group which is therefore indicable. Hence there exists an epimorphism θ_2 of B onto Z with U contained in the kernel. Letting $\theta_1: A \twoheadrightarrow Z$ be the zero map, we have a pair of homomorphisms which agree on U and hence there exists a homomorphism $\theta: (A*B;U) \twoheadrightarrow Z$ which extends the two. Moreover, θ is an epimorphism since its restriction to B is an epimorphism.//

The proof of Theorem 3.3 now follows immediately from Lemmas 3.7 and 3.8. In addition we mention the following corollary which is an obvious consequence of Theorem 3.3 and Lemma 3.1.

Corollary. If A and B are finitely-generated nilpotent groups, then (A*B;U) is indicable if and only if A and B are not both finite.

3.3. Local Indicability in the Generalized Free Product of Two Groups. Since subgroups of locally indicable groups are locally indicable, and the groups A and B are identified with subgroups of (A*B;U), it is clear that local indicability of the factors is a necessary condition for the generalized free product to be locally indicable. However this condition is not, in general, sufficient. In Example 3.2 we constructed a generalized free product of two groups which is finitely-generated but not indicable, and hence it is not locally indicable. However the factors which are free, are locally indicable by Proposition 2.11.

Baumslag used the structure theorems of Karrass and Solitar to prove the following theorem.

Theorem 3.4. [1] If A and B are finitely-generated torsion-free nilpotent groups, then $(A*B;U)$ is locally indicable.

This theorem is clearly false for generalized free products of finitely-generated nilpotent groups which contain non-trivial torsion elements, because such groups are not locally indicable. To accommodate this situation, we make the following definition. Let P be a group property. A group G is called almost locally P if and only if, for each non-trivial finitely-generated subgroup H of G, H has property P if and only if H has no system of generators which are all periodic.

We turn our attention to the class $\mathfrak{L}$ of almost locally indicable groups. We note first that an almost locally indicable group is locally indicable if and only if it is torsion-free. We mention two hereditary properties of the class $\mathcal{L}$ which parallel Propositions 2.8 and 2.9.

Proposition 3.2. The class $\mathcal{L}$ is closed under subgroups.

Proposition 3.3. If G has a normal subgroup H such that H is almost locally indicable and G/H is locally indicable, then G is almost locally indicable.

Proof. Let K be a finitely-generated subgroup of G. If K has some system of generators which are all periodic, then clearly K is not indicable. If K is not generated by a set of periodic elements, then either K is a subgroup of H and is thereby indicable, or K has a non-trivial homomorphic image in the locally indicable group G/H. Since K is finitely-generated, this image must be indicable. Hence K is indicable.//

It is not sufficient to require in Proposition 3.3 that G/H is almost locally indicable as the following example illustrates.

Example 3.3. Let $G = gp\{a,b;a^2,a^{-1}bab\}$. Then G is the infinite dihedral group which is the semidirect product of Z by Z_2. If $H = gp\{b\}$ then H is a normal subgroup of G which is locally indicable, and hence almost locally indicable. Also $G/H \cong gp\{a\} = Z_2$, so the factor group is almost locally indicable. We make the following observations.

1. G is not indicable, as shown in Example 2.1.

2. Since a is the only non-trivial element of finite order in G, G has no system of generators which are all periodic. To prove this assertion, we note that as a consequence of the relation $a^{-1}bab = 1$, we have $a^{-n}b^m a^n = b^{\pm m}$ for all integers n and m. Consequently, every word in G is equal to a word of the form $a^r b^s$. Thus if any word is of finite order we obtain a relation of the form $a^r b^s = 1$. We show that s must be equal to zero as follows. Let D_s denote the dihedral group of order 2s. Then D_s has presentation $D_s = gp\{x,y;x^2,y^s,x^{-1}yxy\}$, and there exists a homomorphism φ of G into D_s such that $b\varphi = y$ and $a\varphi = x$. Then $1 = (a^r b^s)\varphi = (a\varphi)^r = x^r$, so r must be even and hence $a^r = 1$. Thus the relation is of the form $b^s = 1$. But b is of infinite order in G, so s must be equal to zero. Hence the only words of finite order in G are 1 and a.

By these remarks and the fact that G is finitely-generated, it follows that G is not almost locally indicable.

Karrass and Solitar proved the following theorem using their characterization of the subgroup structure of a generalized free product of two groups.

Theorem 3.5. [5] If A and B are locally indicable groups and U is cyclic, then (A*B;U) is locally indicable.

The following example shows that the condition on the factors may not be weakened to almost local indicability.

Example 3.4. If G is the non-indicable group constructed in Example 3.1, then G is the generalized free product of the groups A = gp{a}, and B = gp{x,y;x^2,y^2} with cyclic subgroups U = gp{a^2} and V = gp{xy} amalgamated. We make the following observations.

1. A is infinite cyclic, so clearly A is almost locally indicable.

2. B is almost locally indicable. We observe that the elements of finite order in B are exactly those of the form $(xy)^m x$ or $(yx)^m y$, while the elements of infinite order are exactly those of the form $(xy)^n$. Let H be a finitely-generated subgroup of B. It is not difficult to show that if H contains at least one element of finite order then H has a system of generators which are all periodic. However,

if H is torsion-free, then H is infinite cyclic and therefore indicable. Hence B is almost locally indicable.

3. G is not almost locally indicable. G is not indicable, as we showed in Example 3.1. However G is finitely-generated and has no system of generators which are all periodic. This follows from the fact that periodic elements of the generalized free product of two groups are conjugates of periodic elements in the factors [8, p.208]. If a is a product of periodic elements, then, since A is torsion-free a must be in the normal subgroup, B^G, of G generated by B. But if φ is the homomorphism of G onto Z_2 defined by $x\varphi = y\varphi = 0$ and $a\varphi = 1$, then B is contained in the kernel of φ and thus B^G is contained in $\mathrm{Ker}\varphi$. But a is not in the kernel and hence is not in B^G. Therefore G is not periodically generated, and, since G is not indicable, G is not almost locally indicable.//

We return to consideration of Baumslag's Theorem 3.4, which deals with the generalized free products of finitely-generated torsion-free nilpotent groups. We first state the following result.

<u>Proposition 3.4</u>. If G is locally nilpotent, then G is almost locally indicable.

Proof. If $\tau(G)$ is the torsion subgroup of G, then $\tau(G)$ is a locally nilpotent group all of whose elements are periodic. Hence every subgroup of $\tau(G)$ is periodically-generated and non-indicable. Thus $\tau(G)$ is almost locally indicable. Further, $G/\tau(G)$ is a locally nilpotent torsion-free group which is locally indicable by Proposition 2.12. Thus by Proposition 3.3, G is almost locally indicable.//

We prove the following generalization of Theorem 3.4.

Theorem 3.6. If A and B are locally nilpotent groups and U is finitely-generated, then (A*B;U) is almost locally indicable.

Proof. Let H be a finitely-generated subgroup of (A*B;U) and suppose H is not periodically-generated. If the free part of H (S2, p. 17) is non-trivial, then H has a free factor group and is therefore indicable by Propositions 2.3 and 2.1.

If the free part of H is trivial, then H is equal to a tree product. Now U is finitely-generated and nilpotent, so every subgroup of U is finitely-generated. Hence the amalgamated subgroups in the tree product H are finitely-generated. Thus by Properties S4 and S5 of tree products, H is a tree product of finitely many finitely-generated groups. Further, these groups are isomorphic to subgroups of the locally nilpotent groups A and B and are thereby nilpotent. Therefore, by Proposition 2.6, the vertices of H are indicable if and only if they are infinite. Since H is not periodically-generated, at least one of the vertices of H must be infinite. We prove that H is indicable by induction on the number, n, of vertices in H.

If $n = 1$, then H is isomorphic to a subgroup of A or B so H is an infinite finitely-generated nilpotent group and is hence indicable.

Suppose such a group is indicable if its number of vertices is less than n.

Now suppose H has n vertices, $A_1, A_2, \ldots, A_n$. Choose an extremal vertex, say A_n, which is joined by an edge to exactly one other vertex. Then the vertices $A_1, \ldots, A_{n-1}$ form a subtree and hence generate their tree product H_{n-1}. If $A_1, \ldots, A_{n-1}$ are all finite, then, by Lemma 2.1, H_{n-1} is not indicable. However the amalgamated subgroup U_n corresponding to the edge which connects A_n to the subtree H_{n-1} must also be finite since it is a subgroup of one of the vertices of H_{n-1}. Therefore U_n is not indicable. Further, since not all the vertices of H are finite, A_n must be infinite and thus indicable. Now by property T3 of tree products, H is equal to the free product of A_n and H_{n-1} with U_n amalgamated. Hence by Lemma 3.2, H is indicable. If some vertex in H_{n-1} is infinite, then, by the induction hypothesis, H_{n-1} is indicable, so by Theorem 3.3 H is indicable.

Thus if H is a finitely-generated subgroup of G which is not periodically-generated, then H is indicable. Hence G is almost locally indicable.//

CHAPTER 4

INDICABILITY IN WREATH PRODUCTS

4.1. Preliminaries. In defining the wreath product of a group A by a group B, we employ the construction of a semidirect product. For the reader's convenience we include some remarks on this construction. For further details, see Scott [12].

Let G and H be groups, and θ a homomorphism of H into the group of automorphisms of G. The semidirect product P of G by H which realizes θ is defined to be the set of all ordered pairs (g,h) with g in G, h in H. P is a group under the following operation: $(g_1,h_1)(g_2,h_2) = (g_1[(g_2)(h_1\theta)],h_1h_2)$. It is not difficult to show that P contains subgroups G^* and H^*, isomorphic to G and H, respectively, which have the following properties: G^* is normal in P, P is generated by G^* and H^*, and $G^*\cap H^* = E$. If we identify G and H with these subgroups G^* and H^*, we may drop the ordered pair notation. Then each element of P has a unique representation of the form gh with g in G, h in H, and the element $(g)(h\theta)$ of G is naturally identified with the element $g^{h^{-1}} = hgh^{-1}$ of G^*. Hence the operation in P is defined by $(g_1h_1)(g_2h_2) = g_1h_1g_2h_1^{-1}h_1h_2 = g_1g_2^{h_1^{-1}}h_1h_2$.

We will adhere largely to the notation of Peter Neumann [11] in our discussion of standard wreath products of groups. Let A and B be groups. Then one may view the Cartesian power of A, $A^B = \prod_{b\in B} A_b$, where $A_b = A$ for each b in B, as the set of all mappings of B into A. To simplify notation, let $\bar{K} = A^B$. Multiplication in $\bar{K}$ is defined coordinate-wise--that is, if f and g are elements of A^B, then $(fg)(b) = f(b)g(b)$. Now B may be identified with a group of automorphism of $\bar{K}$ as follows. For b in B, f in $\bar{K}$, let f^b denote the image of f under the automorphism associated with b, where $f^b(\beta) = f(\beta b^{-1})$. It is easily verified that the mapping that takes f to f^b is an automorphism of $\bar{K}$. Hence we may form the semidirect product of $\bar{K}$ by B. This group, denoted A Wr B = $\bar{W}$, is called the <u>unrestricted (standard) wreath product of A by B</u>.

To construct the restricted wreath product, let $K = A^{(B)}$, the direct power of A. That is, if we define, for f in $\bar{K}$, the set $\sigma(f) = \{\beta\in B \mid f(\beta) \neq 1\}$ to be the <u>support</u> of f, then K consists of those f in $\bar{K}$ for which $\sigma(f)$ is a finite set. Now since K is a characteristic subgroup of $\bar{K}$, each element of B induces an automorphism of K and we may form the semidirect product of K by B. The group obtained is denoted by W = A wr B and is called the <u>restricted (standard) wreath product of A by B</u>.

In the case of the restricted wreath product, A wr B, we find the following map useful. Choose a linear ordering on (the set) B. (This is possible by the axiom of choice.) We may define a mapping $\pi: K \longrightarrow A$ by $(f)\pi = \prod_{b \in \sigma(f)} f(b)$, where the product is ordered by the ordering on B. Since $\sigma(f)$ is finite, this product is allowable and hence π is a well-defined mapping of K onto A. The following properties of the mapping π are easily verified.

<u>Lemma 4.1</u>. (i) $(fg)\pi = (f)\pi(g)\pi \pmod{A'}$.

(ii) $(f^{-1})\pi = ((f)\pi)^{-1} \pmod{A'}$.

(iii) $(f^{b})\pi = (f)\pi \pmod{A'}$.

By Lemma 4.1, then, if A is Abelian, π is a well-defined epimorphism of K onto A.

We also have need of the following lemma.

Lemma 4.2. [10] If φ is an epimorphism of A onto a group A^* then there exists an epimorphism Φ of A Wr B onto A^* Wr B with the following properties.

(i) Φ restricted to W = A wr B is an epimorphism onto A^* wr B.

(ii) Φ restricted to A_b is equal to φ for each b in B.

(iii) Φ restricted to B is the identity mapping.

Proof. Define $(fb)\Phi = f'b'$ where $f'(\beta) = (f(\beta))\varphi$, and $b' = b$. It is easy to verify that Φ defines a homomorphism with the desired properties.//

4.2. Indicability in Wreath Products. Both $\tilde{W}$ and W are extensions of a group by the group B. Hence by Proposition 2.1, if B is indicable, then both $\tilde{W}$ and W are indicable. In the case of the restricted wreath product, we prove the following theorem.

Theorem 4.1. The restricted wreath product of A by B is indicable if and only if either A or B is indicable.

Proof. If B is indicable, then by the remarks above W is indicable. If A is indicable, then there exists an epimorphism φ of A onto Z. Hence by Lemma 4.1 there exists an epimorphism Φ of A wr B onto Z wr B. Now the mapping $\pi : Z^{(B)} \to Z$ is a well-defined homomorphism since Z is Abelian. We extend π to a mapping $\bar{\pi} : Z \text{ wr } B \to Z$ by defining $(fb)\bar{\pi} = (f)\pi$. $\bar{\pi}$ is well-defined by the uniqueness of representation of elements of W, and the following simple computation verifies that $\bar{\pi}$ is, in fact, a homomorphism.

$$(fb)(gc)\bar{\pi} = (fg^{b^{-1}}bc)\bar{\pi} = (fg^{b^{-1}})\pi = (f)\pi(g^{b^{-1}})\pi =$$
$$= (f)\pi(g)\pi = (fb)\bar{\pi}(gc)\bar{\pi}.$$

Further, the mapping $\bar{\pi}$ is an epimorphism because it extends the epimorphism π. Hence the composition $\Phi\bar{\pi}$ is an epimorphism of W onto Z, so W is indicable.

Conversely, suppose W is indicable. Since W is generated by K and B, by Lemma 2.1, either K or B is indicable. By Proposition 2.2, K is indicable if and only if A is indicable. Hence if B is not indicable then A must be indicable.//

Our results concerning indicability in the unrestricted wreath product are somewhat less comprehensive due to the intractability of the Cartesian power. However we are able to characterize indicability when B is finite or not periodic.

Theorem 4.2. If B is not periodic, then the unrestricted wreath product of A by B is indicable if and only if B is indicable.

Proof. Peter Neumann [11] showed that if B contains at least one element of infinite order then every element of $\bar{K}$ is a commutator. Thus $\bar{K}$ is contained in $\bar{W}'$ and hence $\bar{K}B'$ is contained in $\bar{W}'$. But by fundamental isomorphism theorems, $\bar{W}/\bar{K}B' = \bar{K}B/\bar{K}B' = (\bar{K}B')B/\bar{K}B' \cong B/\bar{K}B' \cap B = B/B'$. So $\bar{W}/\bar{K}B'$ is Abelian and therefore we must have $\bar{W}'$ contained in $\bar{K}B'$. Hence $\bar{K}B'$ is equal to $\bar{W}'$, and we have $\bar{W}/\bar{W}'$ isomorphic to B/B'. By Proposition 2.4, therefore, $\bar{W}$ is indicable if and only if B is indicable.//

<u>Theorem 4.3</u>. If B is finite, then the unrestricted wreath product of A by B is indicable if and only if A is indicable.

<u>Proof</u>. If B is finite, then $A^B = A^{(B)}$, so the unrestricted wreath product, A Wr B, is equal to the restricted wreath product, A wr B, and the result follows by Theorem 4.1.//

If B is infinite and periodic and A^B is not indicable, then clearly A Wr B cannot be indicable. Hence, for example, if A is of bounded exponent or divisible, $\bar{W}$ is not indicable. It appears to be non-trivial to find an example of an indicable wreath product where neither A nor B is indicable.

4.3. <u>Local Indicability in Wreath Products</u>. We now turn our attention to local indicability in wreath products. Using the stronger inheritance properties of local indicability, we prove the following comprehensive theorem and corollary.

<u>Theorem 4.4</u>. The unrestricted wreath product of A by

B is locally indicable if and only if A and B are both locally indicable.

Proof. A and B are identified with subgroups of $\bar{W}$. Hence, by Proposition 2.9, if $\bar{W}$ is locally indicable, then both A and B are locally indicable. Conversely, suppose A and B are locally indicable. By Proposition 2.10, $\bar{K}$, a Cartesian power of A, is locally indicable. Thus the local indicability of $\bar{W}/\bar{K} = B$ implies that of $\bar{W}$ by Proposition 2.8.//

Corollary. The restricted wreath product of A by B is locally indicable if and only if A and B are both locally indicable.

Proof. If W is indicable, then A and B inherit the property of local indicability as subgroups of W. Conversely, if A and B are locally indicable, then $\bar{W}$ is indicable by Theorem 4.4, and hence the subgroup W is locally indicable.//

CHAPTER 5

TOPICS FOR FUTURE CONSIDERATION

5.1. The Class of Groups Which Are Indicable Throughout. Higman observed that, since free groups are indicable and since every subgroup of a free group is free, free groups are indicable throughout. We note that finitely-generated torsion-free nilpotent groups are also indicable throughout since every non-trivial subgroup of such a group is again a finitely-generated torsion-free nilpotent group and hence is indicable. What sort of characterization may be obtained for groups which are indicable throughout? Is it true that an Abelian group is indicable throughout if and only if it is free Abelian? (For example it is known that an infinite Cartesian product of infinite cyclic groups is not free. Is such a group indicable throughout?)

5.2. Generalized Regular Products of Two Groups. Let $G = (A*B;U)$ be the generalized free product of A and B with the subgroup U amalgamated. Then $[A,B]^G$ denotes the normal closure in G of the subgroup generated by all the commutators of the form $[a,b]$ where a is an element of A and b is an element of B. If φ is a homomorphism of G onto a group P such that the kernel of φ is contained in $[A,B]^G$, then P is called a <u>generalized regular product of A and B with U amalgamated</u>. Generalized regular products were introduced by Wiegold [13] and various aspects of their structure and properties have been considered by such group theorists as R. B. J. T. Allenby, R. J. Gregorac, and B. H. Neumann.

We note that if P is the image of $(A*B;U)$ under the homomorphism φ, then we have $\text{Ker}\varphi \subset [A,B]^G \subset G'$. Hence there exists an epimorphism of $G/\text{Ker}\varphi = P$ onto G/G'. Thus P is indicable if and only if G is indicable.

Regarding local indicability, we obtain the following rather specialized result which relates local indicability in a generalized regular product to local indicability in the generalized free product.

<u>Theorem 5.1</u>. If A and B are groups, U is central in A and B, and if some generalized regular product of A and B with U amalgamated is locally indicable, then the generalized free product, $G = (A*B;U)$ is locally indicable.

<u>Proof</u>. Let P be a locally indicable generalized regular product, and let $\varphi:(A*B;U)\longrightarrow P$ be the canonical map. Then $\text{Ker}\varphi \subset [A,B]^G$. Wiegold [13] has shown that $[A,B]^G \cap A = E = [A,B]^G \cap B$. Hence $[A,B]^G$ is a free group [5, p. 243], so $\text{Ker}\varphi$ is free and therefore locally indi-

cable. Thus, since P = (A*B;U)/Kerφ is locally indicable, we have (A*B;U) locally indicable by Proposition 2.8.//

Can this theorem be generalized to an arbitrary amalgamation? What other relationships between local indicability in generalized regular products and local indicability in generalized free products might be found? In light of the results in Chapter 3, it would be natural to begin by considering generalized regular products in which the factors are finitely-generated nilpotent groups.

BIBLIOGRAPHY

1. Baumslag, G. "On Generalized Free Products of Torsion-Free Nilpotent Groups I," Illinois J. Math., 16 (1972), 526-528.
2. ________. Lecture Notes on Nilpotent Groups, (Amer. Math. Soc., 1971).
3. Gregorac, R. "A Note on Certain Generalized Free Products," Proc. Amer. Math. Soc., 18 (1967), 754-755.
4. Higman, G. "The Units of Group Rings," Proc. London Math. Soc. (2), 46 (1940), 231-248.
5. Karrass, A. and Solitar, D. "The Subgroups of a Free Product of Two Groups with an Amalgamated Subgroup," Trans. Amer. Math. Soc., 150 (1970), 227-255.
6. Kurosh, A. The Theory of Groups, Vol. II, (New York: Chelsea Publishing Company, 1960).
7. MacDonald, I. The Theory of Groups, (Oxford: Clarendon Press, 1968).
8. Magnus, W., Karrass, A., and Solitar, D. Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, (New York: Interscience, 1966).
9. Neumann, B. H. "An Essay on Free Products of Groups with Amalgamations," Philos. Trans. Roy. Soc. London Ser. A, 246 (1954), 503-554.
10. Neumann, H. Varieties of Groups, (New York: Springer-Verlag, 1967).
11. Neumann, P. "On the Structure of Standard Wreath Products of Groups," Math. Z., 84 (1964), 343-373.
12. Scott, W. Group Theory, (Englewood Cliffs: Prentice-Hall, 1964).
13. Wiegold, J. "Some Remarks on Generalized Products of Groups with Amalgamations," Math. Z., 75 (1961), 57-78.

DATE	ISSUED TO