Abstract:
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A group of students in the first class were found to utilize a type of classificatory recognition system that allowed them to correctly manipulate algebraic symbols through rote memorization or by referencing master examples. The students in the second class were more capable of producing mathematically correct work than the first group of students but were not able to develop an appropriate level of mathematical conceptualization for ideas associated with algebraic equations.

A model of classroom algebra learning was developed that postulates the following guidelines for effective classroom learning and instruction. It is necessary to (1) identify the levels of mathematical conceptualization of the intended subject matter, (2) provide an appropriate instructional match between the existing levels of the students' mathematical knowledge and the objects of study, and (3) use guided instructional activities that promote the active participation of the students in the learning environment. A model of participatory learning and a mathematical analysis of the learning structure of algebra were developed to address the implementation of the above guidelines.

CLASSROOM ALGEBRA INSTRUCTION AND THE LEARNING STRUCTURE OF ALGEBRA

by

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APPROVAL

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

This research consists of a qualitative study of the behavior and discourse of Native American students who participated in two consecutive, intensive, beginning level algebra classes at a western state university. The records of students' test responses and of their discourse related to the specific mathematical content of the course were analyzed in order to infer the types of mathematical meanings that the students were capable of assigning to the algebraic topics under study. Transcripts of classroom lectures were analyzed to determine the levels of mathematical conceptualization of the subject matter that was presented to the students. This information was used to assess the effectiveness of the particular classroom learning environment and to provide recommendations for the development of a model of effective classroom learning.

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INTRODUCTION

The purpose of this study was to identify the kinds of mathematical conceptualizations that were developed by a group of students during a first year course in algebra and to compare these conceptualizations with the types of information that were made available to the students within the classroom learning environment. Such an investigation provided a preliminary step towards the understanding and improvement of school mathematics learning.

The present study consisted of a qualitative investigation of the teaching-learning interaction that took place in a special algebra class for Native American students at a western state university. Observational records were collected during classes held in two consecutive summers. The report of this study presents descriptive accounts of the learning strategies employed by students in these classes and examines students' responses on quizzes to selected problems dealing with solutions to equations. This information has been interpreted in terms of a set of possible meanings that students could have assigned to the various collections of algebraic symbols with which they were working.

The report also examines transcriptions of certain class lectures in terms of the types of information that the instructor presented both implicitly and explicitly within the subject matter of each lecture. This information was related to the analysis that was performed on the students' quiz responses in order to identify the objects of study that students focused on and to understand the kinds of meanings that participants in the class assigned to the objects of study that were presented in these lectures.

The findings of the study are related to a postulated set of theories of mathematics concept learning and instruction. These theories include a detailed analysis of the subject of algebra and the description of a model of effective classroom instruction.

Organization of Report

Because of the length of this report, the descriptions of classroom observations and the presentation of the findings of the study have been organized into three parts. Part I contains general information relating to mathematics education research and a description of the first summer's research, Part II contains a description of the second summer's work, and Part III describes the theory developed from the findings of the two summers' observations.

Part I contains introductory material related to the focus of the research. A preliminary section presents the needs of current research in mathematics education within an historical perspective. A brief review of literature is then provided describing research studies in mathematics learning and mathematics education that focus on investigations of cognitive thinking and learning. The techniques of qualitative research are described and related to the special research requirements of the present study.

Following this introduction, information is presented concerning the first summer's work, including descriptions of the study site, the participants, and the methods of data collection and analysis that were used. Part I concludes with descriptions of the behavior of selected students in the first summer class, which serve as illustrations of a set of learning strategies that were employed by the students in their attempts to fulfill the requirements of the course.

Part II contains a detailed analysis of selected information collected during the second summer. This analysis consists of the careful examination of the students' written quiz responses to 19 different questions dealing with the solutions of various types of equations. These responses are compared to the types of mathematical information that were made available to the students in class in order to relate the students' levels of mathematical conceptualizations to the existing learning environment.

Part III describes the instructional and learning theories that were developed to explain the different types of student behavior that were observed in the two classes. Current theories related to mathematics concept learning are discussed, including those developed by Skemp and the van Hieles. A detailed analysis is presented of the subject of algebra in terms of its prerequisite components of manipulation skills and mathematical concepts.

A model of effective instruction is formulated to describe the implementation of a learning structure of algebra in actual instruction, and a model of participatory learning is developed to provide techniques for effective group learning of mathematical concepts. Part III concludes by presenting a summary of the findings of the study and a short discussion of the stages of analysis that were used to formulate the various theoretical models described.

PART I

INTRODUCTION AND FIRST YEAR'S REPORT

CHAPTER ONE

MATHEMATICS EDUCATION AND RESEARCH

A need exists for basic research investigations into fundamental questions of mathematics thinking and learning (Romberg and Carpenter, 1986). Two strands of research are currently involved in investigations into areas pertinent to mathematics education -- the work of cognitive scientists studying mathematical cognition, and the work of other researchers studying mathematical learning processes. Interest is focused on such work in terms of the ability of this research to provide answers for mathematics educators in their search for curricular and instructional reform (Romberg and Carpenter, 1986).

This chapter discusses current research trends in mathematics education. The subject is introduced by a brief historical review of the curriculum and instructional theories of school mathematics in the United States during the last 100 years. Recent research studies are then presented, and the goals of these investigations are compared and contrasted to the needs of contemporary mathematics education.

An Historical Perspective

Mathematics instruction in the United States has, throughout its history, reflected the current notions of the perceived usefulness of mathematics to the average citizen. This view, coupled with educational techniques reflecting prevailing learning theories has, throughout the years, influenced the content and methods of instruction in school mathematics (Bidwell and Clason; 1970, National Council of Teachers of Mathematics, 1970).

This section describes only a few instances of the long and varied history of mathematics education in the United States. The purpose is not to present a detailed account of curriculum changes and theories of learning, but to convey an impression of the range of educational viewpoints that have been espoused over the last 100 years, and to illustrate the changes that have occurred in the perception of what constituted an appropriate goal for students' school mathematics performance.

In the later half of the nineteenth century, the mind was viewed as a muscle, and learning consisted of those activities that were felt to "strengthen" this muscle. Mathematics was considered to be an ideal subject for inducing mental exercise. It was studied, not for its own sake, but for the mental strengthening it produced, with the expectation that this exercise would transfer mental power to other fields of study (Smith, 1923). Students during this period were deemed successful in mathematics if they could perform a series of lengthy mental calculations involving the operations of addition, subtraction, multiplication, and division.

Textbooks around the turn of the century contained problems such as the one below that were designed to test the mental powers of the student.

> If 5 horses eat as much as 6 cattle, and 8 horses and 12 cattle eat 12 tons of hay in 40 da., how much hay will be needed to keep 7 horses and 15 cattle 65 da.? (Smith, 1923, p. 6)

After the turn of the century, mathematics began to be viewed as a subject having social utility. It was perceived as important that students receive instruction in those operations of arithmetic that were necessary for the computations of the business world. Mathematics instruction was no longer regarded as a tedious activity to be performed for its own sake (Smith, 1923). Social utility became the concern of educators in the next several decades. Smith, writing in 1923, stated that

> The purpose of teaching arithmetic has come to be recognized as the acquisition of power to calculate within the needs of the average well-informed citizen. (p. 11)

In the early part of the period from 1920 to 1945, drill was emphasized in school mathematics (National Council of Teachers of Mathematics, 1970). The use of such a technique was based on the connectionist learning theories exemplified by Thorndike's theory of bonds. Thorndike (1922) postulated that learning was the result of connections or bonds in the brain that were formed under repeated exposure to external stimuli. He recommended that arithmetic instruction be presented by a series of sequential steps that, when memorized, would enable students to perform appropriate calculations. Thorndike stated

> We now understand that learning is essentially the formation of connections or bonds between situations and responses, that the satisfyingness of the result is the chief force that forms them, and that habit rules in the realm of thought as truly and as fully as in the realm of action. (p. v)

School mathematics continued to be viewed as having social utility during the period from 1920 through World War II (National Council of Teachers of Mathematics, 1970). During the 30s and 40s educators also began to be concerned with student readiness levels for the appropriate introduction of specific topics. The mathematics curriculum in the elementary grades was ordered to correspond to appropriate mental ages defined for each new concept (Bidwell and Clason, 1970).

The Second World War and the emergence of the United States into an era of rapidly changing technology produced a need for a large work force having a high degree of mathematical literacy. Mathematics became a subject to be studied as a necessary condition for future employment. Educators recognized the need to evaluate the existing mathematics curriculum in the light of a subject to be carefully, systematically, and purposefully taught (Bidwell and Clason, 1970).

During the 50s attention turned to reevaluating and upgrading the mathematics curriculum in the high schools. Concern was expressed in the early 50s over the degree of mathematical understanding that students were able to develop under the existing methods of school mathematics instruction.

> In the eyes of many thoughtful members of the mathematics community, the picture of mathematics education in American high schools in 1950 was not a pretty one. In particular, they were dissatisfied both with the content of course offerings and with the spirit in which the material was presented. They were convinced that the traditional subject matter was inappropriate to the times.... In their opinion there was undue emphasis being placed on skills, an unnecessary preoccupation with the immediate usefulness of what was taught, and an unfortunate distortion of the students' ideas as to the nature of mathematics. (Wooton, 1965, p. 5)

In response to criticisms such as those expressed by Wooton, several groups initiated curriculum and instructional changes in high school mathematics. Courses were redesigned to reflect an approach to mathematics based on an overview of the total abstract relatedness of the subject. New topics were introduced that were purported to be more in line with those topics currently studied at the forefront of mathematics (Kline, 1973). Emphasis on drill and efficient computation was replaced by instruction designed to allow students to discover for themselves the abstract generalizations of the number system. In the 60s and early 70s, success in school mathematics was regarded by many educators as the ability to emerge from the school system with an appreciation of the abstract structure of the field of real numbers.

An attempt was made to shift the goals of school mathematics learning from the mastery of facts to the development of the understanding of abstract mathematical concepts. Those educators advocating the "new math" approach to school learning recommended that the techniques of "discovering" fundamental concepts be utilized at all levels of school mathematics instruction.

> The mathematics program of the elementary school must help provide students with the knowledge, attitudes, and skills they will need to be mathematically literate. Through an inductive approach to learning, teachers can show students the excitement in discovering how to perform operations on numbers. Students can search for patterns among numbers and can develop an awareness of how patterns are useful in organizing and synthesizing ideas about numbers. Rather than the teacher or a textbook presenting generalizations of mathematics, students can be guided to formulate them in their own words. (Kennedy, 1970, p. 1)

The "new mathematics" promoted by mathematicians and educators in the 60s and 70s failed to meet the computational needs of the students. The emphasis on abstract generalizations provided learners with few computational skills and little ability to apply mathematics. The cry of "back to basics" heralded a renewed interest in the students' ability to use computational skills in order to perform everyday mathematics and to solve applied problems (National Council of Teachers of Mathematics, 1983). An expressed goal of education became the need for students to be able to make quantitative decisions as adults functioning successfully in a technical world.

Textbook writers readjusted their goals in the wake of this swing in instructional emphasis. Grossnickle et al (1983), in the introduction to an elementary level mathematics text, convey the dissatisfaction with past school mathematics learning that was felt by many educators at this time.

> Because the objectives of mathematical programs for the elementary school have changed drastically since the publication of the sixth edition in 1973 a seventh edition has become necessary.... Some of the topics dealing with the New Math that proved difficult for many pupils in the elementary school either have been deleted or abridged in this edition. (Grossnickle et al, 1973, p. vi)

The focus of mathematics education for the 80s shifted to an emphasis on problem solving. Students were expected to achieve more than simple competency in the basic skills. The hand-held calculator and the computer became recommended tools of instruction (National Council of Teachers of Mathematics, 1983).

Today, concern still exists over the level of competence achieved by American students in our school mathematics programs (Steen, 1988). This concern has prompted the statement that there exists a "current crisis in the teaching and learning of mathematics and [the need for a] resulting call for reform" (National Council of Teachers of Mathematics, 1987). Such a call appears to be echoing similar concerns that have been expressed periodically over the years by mathematics educators since the beginning of the century. The statement made in 1970 still holds true today, that "the continuing development of new mathematics, new uses of mathematics, new pedagogical devices, and changing goals for a changing society all demonstrate the need for continued change in mathematics education" (National Council of Teachers of Mathematics, 1970, p. 1).

Change, however, that will result in significant improvements in the teaching and learning of mathematics may be difficult to achieve. Despite the renewed interest in reform, House (1988) expresses concern over the ability of our contemporary school systems, based on past performances, to achieve significant changes in the existing state of our mathematics education programs.

> We have customarily looked to the records of the past for insight into making decisions for the future. Often the future we have projected has been largely a rearrangement of elements from the past. We have incorporated trigonometry into intermediate algebra, integrated solid with plane geometry, permuted the order of presentation and emphasis given to various topics, and introduced some new concepts with varying degrees of permanence. But for all that, the content of school mathematics in 1988 still bears a striking resemblance to the school mathematics of 1928, of 1948, or of 1968. (p. 2)

Romberg and Carpenter (1986) point out that the needs of educational reform designed to meet the accelerating technological changes of the approaching 21st century must be met with thoughtful policy grounded in knowledgeable considerations of the ways in which students learn mathematics. "Change is inevitable. If we can build upon a solid knowledge base derived from research on teaching and learning, the change will result in real progress in the teaching and learning of mathematics" (Romberg and Carpenter, 1986, p. 869)

The historical examples of mathematics instruction and learning theory that have been cited here serve to illustrate the varying ways in which educators have attempted to meet the instructional and curricular needs of classroom mathematics learning throughout the last century. These "records of the past" indicate the lack of a well developed and fully established body of related research and theory that can be used by present mathematics educators as the basis for making important decisions relating to today's classroom instruction. Research in the field of mathematics education, the study of how mathematics is taught and learned, is still attempting to investigate fundamental problems related to these areas of concern.

The Field of Mathematics Education

The field of mathematics education represents a fairly recent area of systematic study. Crosswhite (1987) states that "research in mathematics education may still be in its infancy" (p. 266). Freudenthal (1980) maintains that a large body of knowledge has yet to be developed from which theories concerning mathematics education can be derived.

Davis (1984) compares the study of mathematics education and mathematics thinking to the study of medicine. "Centuries ago there was very little sense of control, and more acceptance of fate, in both fields" (p. 3). He maintains that only recently has the field of mathematics education begun to make significant advances.

> We cannot overcome every educational problem related to the teaching and learning of mathematics, much less turn every student into a twentieth-century Archimedes, but we are beginning to get a more precise and detailed description of how human beings think about mathematical problems, and this can move us

> towards far more control, and far less need for fatalistic acceptance of everyday obstacles to learning. (Davis, 1984, p. 3-4)

There is much that remains to be studied in the field of mathematics education. Romberg and Carpenter (1986) cite the need for the establishment of new research studies designed to investigate the complex issues of mathematics learning and teaching. Only in this way will new and pertinent information become available to mathematics educators that can be used to develop significant improvements in classroom instruction. Without a systematic development of a relevant knowledge base and set of learning theories, the field of mathematics education will not be able to move forward beyond the pages of its past history.

The focus of research in the United States dealing with mathematics education has shifted since the 1970s from investigations of the external, or teaching and instructional components of education, to investigations aimed at developing models describing the internal, or cognitive activities of learners (Davis, 1983; Romberg and Carpenter, 1986). The type of data that mathematics education research typically utilizes has shifted from representing the measurement of instructional-related student output to representing descriptions of student behavior observed during the performance of a specific mathematics related activity. The goal of much of recent research utilizing these newer techniques is to develop models that postulate a certain set of cognitive processes that can be used to explain the mathematical thinking employed during the research task (Davis, 1983; Schoenfeld, 1986).

The following sections discuss two strands of research that are pertinent to the field of mathematics education. A general description is presented of the protocol research utilized by cognitive scientists interested in studying the area of mathematical thinking or cognition. This field of study is contrasted to the work being done by researchers investigating mathematics learning processes. These two fields of study are then compared and contrasted in terms of the applicability of their goals towards meeting the needs of contemporary mathematics education.

Research in Mathematical Cognition

Cognitive scientists and mathematics educators conducting research in mathematical cognition are concerned with investigating individuals' thought processes during the performance of some kind of mathematical task. The mental processes that individuals use during such thinking, however, cannot be observed directly (Ginsburg et al, 1983). The premise underlying current research is that such mental activities can be inferred by observations of the overt behavior of individuals by recording protocols of these subjects as they talk aloud through their solutions to certain mathematical problems.

Protocol Research

In research studies using protocol methods, subjects are individually recorded on audio or videotape as they talk aloud through their solutions to specific problems. Task-based interviews solicit additional information by asking subjects to describe the thought processes that they employed to arrive at solutions to particular problems. Such information is usually obtained in a laboratory-type environment where the subject is presented with the mathematics activity outside of any educational context. The actions of the subject are instigated by the researcher and cannot necessarily be assumed to be the same as those the subject would employ within a mathematics classroom environment.

Further data relating to an individual's performance on a mathematics based activity are collected through analyses of the particular mathematical tasks employed. The specific types of mathematical problems presented as the research activity are analyzed for problem difficulty and semantic structure. The format or method of representation of the mathematical activity may be varied. Specific subject responses may also be analyzed to establish particular error patterns. Further information on the subject's behavior is also collected by measuring the response times required by subjects as they solve different arithmetic problems, with the inference being that more complicated problems take longer to solve (Ginsburg, 1983).

Protocol methods are employed by cognitive scientists in their investigations of the general conditions of cognition and learning. Mathematics educators employ similar research techniques. However, in these investigations, the mathematical content of the activities becomes an important component of the study. The common thread linking these research studies is the use of a subject's discourse as the source of information from which mental thought processes are to be inferred.

Research in Cognitive Science

Research studies involving mathematical cognition vary in design according to the general purpose of the investigators. Cognitive scientists use the study of the performance of mathematical tasks as a vehicle for developing general theories of cognition and learning. The purpose of current research in this field is to develop models, based on the concepts of information processing, to explain the thinking that occurs when a subject solves a mathematics word problem (Davis, 1984; Greeno, 1980; Schoenfeld, 1987). Such models provide possible explanations of how each subject examines a problem, in terms of the information that is obtained from the problem; how this information is stored in memory and linked to previously stored information; what is retrieved from memory for use; and how solution strategies are arrived at. The emphasis of these models is on the analysis of the mechanics of information manipulation used by individuals.

These information processing models assume the existence of appropriately stored prior knowledge or concepts that individuals must have access to in order to attempt a solution to each experimental task. The study of the mental processes involved in the understanding of a problem and the processes of how such previously stored information is used to reach a solution to a new and unfamiliar problem is seen as a first step in the development of a theory of learning (Davis, 1984; Greeno, 1980).

Task Specific Research

Mathematics educators utilize protocol research in order to develop more content specific models of cognitive processes. In contrast to research in cognitive science, the particular choice of the mathematics task and the type of mathematics content become important aspects of the research situation. Investigators are concerned not only with subjects' thought processes during a task but also with the type of mathematical activity and the prevalence and types of errors exhibited by subjects (Romberg and Carpenter, 1986).

An example of such research, described by Carpenter and Moser (1983), consists of several projects that are concerned with developing descriptive lists of all the variations of mathematical tasks that can be generated from simple arithmetic word problems using addition and subtraction facts. These problems form the basis of protocol research in which subjects' responses are linked to the type of problem used.

Data for these studies are collected by recording the individual protocols of students describing their solution processes, by covert and overt observations of students, through probing interviews of subjects' choice of solutions, by error analysis of responses, and by the measurement of response times required for each solution. These data are then classified into types of solution strategies employed as a function of problem characteristics.

The goal of such research is to categorize student behavior as a function of problem characteristics. Descriptive stages have also been developed to describe the changes in behavior that occur by grade level as students proceed from counting strategies to the use of number facts. The categorization of the data is used to formulate models that describe possible knowledge structures that can be used to account for specific observed behaviors.

This type of research deals with very specific mathematics content and activity. Much of the research is of a static nature, describing student behavior related to a single task. The emphasis is on recording the subject's behaviors that lead to a specific output, and the data are analyzed toward the goal of describing mental structures that can account for the subject's immediate behavior. Research is geared to answering the question, "How does the subject do this task?" Thus the research is designed to develop models describing cognitive activities involving procedural knowledge or mathematical thinking rather than the acquisition of concepts and mathematical learning (Romberg and Carpenter, 1986).

Carpenter and Moser (1983) point out that this research on the acquisition of addition and subtraction facts is limited to a very narrow, specific domain. They cite specific criticisms that claim that larger scale models of cognition are needed to tie the performance of addition and subtraction into the larger context of the development of mathematical cognition. This research also needs to relate its findings on subject performance to the classroom and the problems of instructional improvement.

Other task specific research is concerned with conducting error analyses on the collected responses of subjects to more advanced arithmetic problems (Ashlock, 1986; Maurer, 1987; van Lehn, 1983). From these analyses it is possible to generate lists of mistakes that seem to indicate the presence of faulty or incompletely formed concepts that students develop while learning arithmetic. The identification of these errors, or "bugs," by teachers allows for careful remediation to correct students' mental misconceptions.

Much of the current research investigating mathematical cognition has focused on the construction of models describing the mental processes connected with specific content tasks. These processes describe mental procedures that are required to manipulate numbers and symbols according to a set of rules or concepts <u>previously</u> learned. Such research does not focus on the mental processes that are needed to initially learn these particular concepts.

<u>Mathematics learning Research</u>

A distinction needs to be made between the act of mathematical <u>thinking</u> engaged in during a specific task and the processes of mathematical <u>learning</u> that lead to the development of new concepts over time. The research reported by Greeno, Davis, Carpenter and Moser, Ashlock, van Lehn, and Schoenfeld is primarily concerned with describing the mental activities that occur during an isolated instance of time, and that are related to a single, isolated mathematics activity. Such research deals with subjects who, at the beginning of the task, have already acquired at least some basic procedural knowledge related to the specific task they are given.

Mathematics learning can be described as a process. It occurs over time and leads to the development of new concepts and related mental structures. Research into these learning processes must necessarily be conducted over a period of time studying subjects engaged in a series of related tasks. The behavior throughout the sequence of tasks is as important as the individual responses to each specific activity.

The following sections present descriptions of three different research studies that involve investigations into various aspects of the subject of mathematics learning. The first study is concerned with understanding a child's early development of the concept of number, and investigating the ways in which this concept develops over time. The second research project describes a clinical interview study of high school students' mastery of algebra topics over the course of a year. The third study reports on the use of classroom observations as a means of investigating the development of students' thought processes in response to a specific course in geometry.

Concept Development over Time

Resnick (1983) reports on research using protocol methods along with measurements of reaction time to study the acquisition and development of the concept of number in pre-school and early school age children. Studies of the protocols generated and analyses of interviews with subjects as they perform number related tasks in this research have led to the development of a theory describing the changes in number knowledge from a simple number line representation, to the more abstract concept of the base ten number system, and the use of algorithms of addition and subtraction.

Resnick uses an information processing model to describe the performance of procedural knowledge tasks in an effort to investigate the formation of concept knowledge. Her purpose is to integrate concept-formation theory with the study of procedural knowledge activities.

> It is important to note...that in stressing both procedural and schematic knowledge and their links, current theories of mathematical understanding offer promise of joining two hitherto separate and largely competing strands of research on mathematical development. These are (a) the behavioral, which has concentrated on number performance skills and has viewed growth in mathematical ability as the addition of successive performance skills; and (b) the cognitive-developmental, which has focused on changing concepts of number but has often paid little attention to the manifestation of these concepts in actual number performance. (Resnick, 1983, p. 110)

Resnick is careful to point out that her research proceeded from the "bottom-up." She did not begin with a hypothesized mental structure around which specific tasks were designed to demonstrate a particular theory. She instead developed her theory from a careful analysis of performance-based tasks. The theory was developed to infer a possible set of mental structures that would explain the observed behavior of the children in the study.

Resnick studied subjects over a wide range of ages and stages of development of their concept of number. Her work not only investigated the mental processes involved while performing specific tasks, but looked at how the concept of number must evolve as the complexity of the task changed. Thus her research was concerned with investigating learning processes rather than static thinking activities. This allowed her to develop a theory describing a particular concept development.

Clinical Interview Study

The research of Gerace and Mestre (1982) that is reported in this section is also concerned with the development of mathematical concepts over time. Unlike Resnick's work, however, which investigated different students at different levels of development, this study followed the development of individual students throughout the course of a year.

Gerace and Mestre utilized the clinical interview method of data collection to study the development of algebraic concepts and processes by a particular group of high school students. These investigators studied a group of 14 students consisting of nine Hispanics and five Anglos who were enrolled in ninth grade algebra. Students were presented with different sets of questions throughout the school year that were designed to elicit both written and verbal responses to mathematics problems related to topics covered in their algebra classes. The goal of the study was to investigate the cognitive processes and learning difficulties of this group of students and to investigate any possible differences that could be detected in the kinds of responses given by the Hispanic and Anglo students.

A team of four researchers was used to administer individual interviews on a monthly basis. In these interviews the students were presented with a set of questions that were designed to "monitor progress in mastering previously learned as well as newly learned concepts" (p. 38). These questions were developed by the research team in conjunction with the teacher of the algebra course and were based on the course syllabus and on the expressed goals of the study. "Probed were students' appreciation of algebra as an abstract logical system, command of the formal operations of algebra, ability to use algebra, and ability to solve problems" (p. i). The selected questions were hypothesized to measure these research goals.

The interviews were recorded on audio tape. The students were asked to write down their answers to each question as well as to explain the steps that they used. At times the interviewers would ask the students probing questions to elicit clarifications or help them when they appeared unable to proceed further with the given question. Immediately following each interview, the research investigator included on the student's written response paper a set of written comments concerning his impressions of the interview.

The interviewer analyzed each session by listening to the tape and reviewing the written responses made by the student. The complete set of individual interviews were then discussed by the team of researchers as a group in order to compare responses and develop hypotheses to explain the observed behavior.

Based on the responses to the eight sets of questions, Gerace and Mestre (1982) stated among their findings that

> students preferred not to use algebraic techniques in solving problems, were poor at verbalizing definitions and procedures and at translating problem statements into equations, did not use their textbooks very much except as a place to find assigned problems, and treated algebra as a rule-based discipline, not as a concept based one. (p. i.)

The nature of this study was such that it appeared to be measuring students' "mastery" of certain skills, procedures, and definitions of algebraic concepts. It is not clear from the information contained in the final report of this study whether it was possible to distinguish if the students' responses were based on material learned through rote-memory or from a conceptual understanding of the topic.

The interview questions were presented to the students in written form. In the case of the "problem solving" questions, it was possible that this approach was measuring students' abilities to translate from the representational system of "words" to that of algebraic symbols. Such questions do not necessarily measure a student's ability to model real-world situations. It was possible for students to have responded to the written word problems by a mechanical translation from words to symbols without ever understanding the mathematical relationships that were represented.

Although the study was designed to measure students' mastery of certain algebraic skills and concepts, little attempt was made to link this information to the kinds of information that was actually made available to the students in their day-to-day classes. Therefore, there is no way to determine if the students' failure to respond in an appropriate mathematical fashion to certain questions was due to a failure on their part to learn or a failure of instruction on the part of the teacher and the syllabus.

The level of questions presented to the students in the interviews were many times of a highly abstract nature. For example, the students were asked, "How would you define a fraction?" Following this question the students were then asked "Which of these are fractions? Why? 2, 1/2, 3/4, 5/3, 4/2, 9/3, 6/1, 10, 52/1." (p. I-2)

Gerace and Mestre do not make it clear whether they expected the students to respond appropriately to these questions as a result of their class exposure, or whether these questions were designed to test some arbitrary level of mastery. The investigators may have been expecting an unrealistic level of mathematical sophistication in the responses of these beginning algebra students.

The clinical interview technique used in this study enabled the researchers to determine the kinds of responses that these students tended to give to written questions concerning certain aspects of first year high school algebra. However, this technique does not provide a method of determining the extent that these responses were influenced by the learning environment presented in their algebra class. The researchers appear to have assumed that there was a correlation between the students' responses and the classroom instruction, but they obtained little direct information on the student-teacher-subject matter interaction.

Gerace and Mestre present a discussion of "pedagogy" at the end of their report. This section begins with the statement "we would like to ... briefly summarize some characteristics displayed by the students in this study which we believe are not conducive toward the learning of mathematics" (p. 83). No attempt was made, however, during the study to determine if these behaviors were the direct result of either implicit or explicit situations pertaining in the classroom environment.

Until the teaching-learning interaction is observed within its appropriate context, it is not possible to assign the responsibility for inappropriate mathematics learning to either the students or the instruction. It is first necessary to determine the actual cause of such behavior from an in-depth analysis of the total learning environment. The following section describes a study that includes the context of the actual classroom learning as part of the collected data.

Van Hiele Levels of Geometric Thinking

The research and theories of Pierre van Hiele and his wife Dina van Hiele-Geldof were reported in companion dissertations published in Dutch in 1957. Although this research was conducted over 30 years ago, it is extremely relevant to work currently being undertaken in the field of mathematics education. The work of the van Hieles utilized protocol techniques that not only investigated students' thought processes over time but that placed such information within the context of relevant classroom instruction.

Dina van Hiele-Geldof's dissertation presented a description of a year-long experimental class in geometry for 12 year-old students. The purpose of the class was to demonstrate the effectiveness of a particular form of instruction, developed by the van Hieles, to overcome students' difficulties with the traditional geometry instruction prevalent in Holland in the 1950s (Fuys et al, 1984).

A measure of the effectiveness of the new instruction was the demonstration of students' abilities to think within a specific geometric context. Van Hiele-Geldof maintained that the determination of the existence of these thought processes could only be obtained through a careful analysis of protocols of actual daily classroom activities. She claimed that in order to develop a psychology of mathematical thinking, it was necessary to observe and analyze the actual teaching processes during which a student's learning took place.

The van Hieles were concerned with the investigation of learning processes. According to the theory of geometry learning reported by Pierre van Hiele in his dissertation and later expanded on (van Hiele, 1986), learning is a process that involves the active participation of the learner. Part of this participation consists of two-way conversations between students or student and teacher. Learning is an inseparable part of a particular type of teaching and must be investigated within this teaching process. Thus, the actual protocols of specific classroom periods form the principal source of data for an investigation of the effectiveness of a learning situation.

This teaching-learning situation, or didactic experiment, differentiates the research of the van Hieles from the experimental settings of current research in cognitive science and mathematics education. Although such research utilizes mathematical tasks similar to those problems used in school instruction, subjects are observed outside of the classroom in a special setting. They are presented problems within an interview format and the performance of the problem is taken out of an instructional context. Because the didactic experiment of Dina van Hiele-Geldof consisted of actual classroom observations, her findings appear to be more available for classroom implementation. One of the criticisms of current protocol research utilizing mathematical activities is that it cannot be transferred directly into classroom use (Romberg and Carpenter, 1986).

The van Hieles' theory of a set of levels of geometric thought and instruction were developed following careful observations and analyses of actual classroom teaching experiences. Their arrangement of this level structure was in response to Dutch students' difficulties with formal geometric instruction (van Hiele, 1986). Learning difficulties provided information concerning the processes that were required for successful learning. (A more detailed description of this theory of geometric levels of thinking will be presented in Chapter Eighteen.)

Comparison of Research Goals

The preceding sections have presented examples of current research involved in investigations of mathematics thinking and mathematical learning processes. These two areas of research will be examined further in this section in terms of the ways in which such investigations have operationally defined the term "learning." The definitions developed in this section will be based upon an analysis of the ways in which various studies appear to be utilizing the word, as represented by their experimental goals and particular choice of research design. This section will also compare and contrast the ways in which these definitions of learning affect the applicability of such research to the needs of mathematics education.

Current research studies investigating cognitive processes and various aspects of mathematics education operate from the premise that learning consists of some kind of mental activity or set of mental processes that cannot be observed directly. Thus, any investigation of these postulated phenomena must necessarily rely on obtaining information from <u>inferences drawn from overt</u>

behavior. Such research also considers learning to be an activity that is carried out separately by each individual.

Any study of learning must, therefore, specify in what situations such learning is to be inferred to be taking place and what types of overt behavior must be monitored from which to draw these inferences. An examination of the kinds of activities that are used by various research studies reveals that investigators exhibit differences in the ways in which learning is operationally defined in their work. The following sections will examine the implications of these different research uses of the term "learning."

Cognitive Science and Learning

Weaver (1987) describes a set of research criteria that are common to most cognitive science studies on "mathematics learning." Such investigations study individuals, rather than groups, and pertinent information is obtained by observing these individuals as they perform some type of mathematical task. The overt behaviors that are focused on by the researchers consist of the individual's written responses to the task and the self-report of his or her thought processes during the performance of this activity. Such work is commonly referred to by authors in this field as research on mathematics learning.

As was described in a previous section, the data for this kind of research is based on written protocols obtained from observations of individuals as they work through some kind of mathematical task. The individuals are instructed to talk aloud as they proceed with their work, explaining the thoughts they use as they go along. The subject may proceed without investigator intervention, or the investigator may ask the individuals questions during the activity concerning their particular choice of procedures or present them with alternative choices to consider. Such tasks are performed in a "laboratory" setting, in that the individuals are presented with an activity that is not related to the context of the environment in which it is performed (Ginsburg et al, 1983).

Schoenfeld (1987) describes one of the goals of such research.

> A basic assumption underlying work in cognitive science is that mental structures and cognitive processes (loosely speaking, "the thinking that takes place in your head") are extremely rich and complex -- but that such structures can be understood, and understanding them will yield significant insights into the ways that thinking and learning take place. (p. 8)

(Notice the use of the word learning in the above statement.)

The protocol records of such research are analyzed in great detail in terms of developing descriptions of a sequence of steps or procedures that the individual utilizes to work through the given mathematical task. The thought processes that are postulated to account for these actions are described by information processing models. Davis (1983) describes such models in terms of mental structures and procedures which include "static representations of knowledge, the dynamic processing of information, collections or 'chunks' of information, storage and retrieval of entities, and 'real-time' construction of representational entities" (p. 257).

The behavior that researchers monitor in such investigations consists of the actions of individuals as they engage in the <u>doing</u> of mathematics. Such activities are indicative of an individual's current state of mathematical knowledge, and his or her ability to utilize this information to perform the task at hand. In this respect, such research appears to be measuring an individual's thought processes, rather than his or her learning processes.

The activities that the individuals engage in are initiated by the research setting. They do not occur within an instructional context. The focus of the data collection is on the particular mathematical problem under consideration and the individual's response that is given, rather than on eliciting information concerning the environment in which the individual first developed the knowledge that was used to perform the research activity.

Romberg and Carpenter (1986) point out that "most cognitive research provides a perspective on what children know and how they solve problems at isolated points in time" (p. 859). Furthermore, they add that "most current theories, however, have very little to say about transitions from one stage to another. In other words, they don't directly address issues of learning" (p. 859).

It appears that cognitive scientists use the word learning in a fairly loose sense connected to those mental activities that involve complex thought processes. For example, Robert B. Davis (1984) uses the title Learning Mathematics: the Cognitive Science Approach to Mathematics Education to refer to his book that describes in detail the use of an information processing model to describe the thought processes of individuals engaged in talk-aloud-type activities.

He describes the subject to be covered in his book as "an in-depth look at how people learn mathematics, and how people think about mathematics" (Davis, 1984, p. 8). However, the term learning is seldom mentioned. Instead, his work is concerned with a "discussion of human information processing as it relates to the solution of mathematical problems" (P. 28). Learning is never fully discussed.

Cognitive scientists appear to be approaching the term learning in an indirect manner. They postulate that once information is discovered about the way in which individuals think about mathematical problems, they can begin to develop models to address the development of mathematical understanding. Greeno (1980) states that "in order to understand the acquisition of such structures, we need to know the ways in which complex cognitive structures are modified and combined" (p. 726).

Davis (1983) places learning as secondary to the processes of thinking.

> Learning is regarded as definable in terms of transitions between one form of mathematical thinking and another and is in this sense not fundamental, but a kind of "derived" or "second-order" concept. The fundamental task is to describe the various forms of mathematical thought themselves, that form the "before" and the "after" of any change. (p. 254)

Davis' description of learning is consistent with the research objectives evidenced by cognitive scientists. Data are obtained by observing isolated instances of individuals engaged in activities that elicit complex mathematical thought. Such activities can detect the presence of learned knowledge but cannot investigate the situations that brought this about. Therefore, the primary source of information that can be obtained from such research concerns individuals' thought processes, rather than learning processes.

Learning can be described more easily than it can be examined. Weaver (1987) defines learning as a change in learned behavior, which is consistent with Davis' and Greeno's descriptions of learning as the transition from one state of knowledge to another. In the sense of this definition, learning can only be discussed or "measured" in terms of a behavioral output or product. It is possible to infer learning by comparing an individual's behavior at two different points in time. This definition of learning focuses the attention on the "before" and "after" mental states, not on the processes that may have occurred to bring about the change.

Such a definition of learning provides a criterion that enables learning to be observed through the talk-aloud protocols used in cognitive science research. Such investigations are designed to provide insights on the state of an individual's mental processes at any one point in time, and can determine levels of existing knowledge. This type of learning is therefore a second-order phenomenon to thinking.

Educational Research and Learning Processes

Cognitive scientists regard learning as a mental activity that can be associated with an individual's observed actions. This learning is related to other inferred mental activities that such researchers associate with the performance of a particular mathematical task. These tasks are presented to individuals within an unrelated context in which the purpose of the activity is to elicit mental actions that utilize pre-learned knowledge. There is no intent to teach new information through the individual's interactions with the research task.

Weaver (1987) is careful to make this distinction clear. He characterizes teaching as the behavior of the teacher, learning as a change in behavior of the learner, and instruction as the interaction between the pupil and the teacher. His definition of learning is consistent with those given in the previous section; that learning can be discussed without the necessity of including a parallel examination of a related teaching event. Such a view is also in line with the concept of learning as an observed <u>change</u> in knowledge state. Learning is the end product, and not the process.

Freudenthal (1980) presents another perspective on the term learning which is closer to Weaver's use of the word instruction.

> Learning processes distinguish themselves from what psychologists call learning by the fact that they are always also teaching processes. Learning processes do not go on spontaneously, they are influenced; and this influence should certainly not be eliminated in the experiments since it is an essential feature of learning processes as they occur in the real world. (p. 167)

Freudenthal also defines learning as a change in state. He refers to this differentiation between an individual's "knowing" and "not knowing" as a "discontinuity". However, Freudenthal (1980) stresses the importance of observing the learning processes, not just being aware of the end product or "learned" behavior.

> It does not suffice to state whether a learning process took place or not; one wants to know what prevented, impeded or facilitated it to occur, or whether one should pay attention to it and promote it consciously. (p. 180)

Thus, the teaching-learning interaction is the important area of focus for investigating the learning processes of individuals. Learned behavior is brought about through some kind of external influence, and the understanding of this behavior is not complete unless the nature of this influence is also taken into account. Such an approach is characteristic of an educational perspective on learning, and differs from that held by the cognitive scientists who are interested in the thought processes but feel no direct responsibility for the way these are formed.

This educational approach predicates the use of different research techniques from those employed in the talking-aloud studies. Since learning is a direct function of some kind of teaching influence, it must be studied within this context with both the teaching and the learning being observed. Neisser (1976) states the importance of paying "attention to the details of the real world ... and the fine structure of information which that world makes available" to the learner (p. 8).

A different set of research criteria from those described earlier by Weaver must be used when learning is considered to be a process, and to consist of a teaching-learning interaction. Since learning is influenced by some external source, the nature of this source becomes an important element. Thus, part of the research design is to specify the type of teaching-learning environment that will be investigated. Learning from this perspective becomes subject specific, with learned behavior redefined to represent specific types of knowledge systems.

If learning is regarded as a process, and the teaching-learning interaction as the vehicle through which such learning takes place, then observations of this phenomenon must be obtained over a period of time and from situations in which such interactions can be inferred to be taking place. These requirements preclude the use of "laboratory" investigations observing individuals performing static mathematics activities. Instead, research on the teaching-learning interaction must seek information within some kind of educational setting.

The end point of learning -- the change in state or the discontinuity -- is difficult to precisely define or measure. Such a condition may be detected after the fact in terms of some kind of learner outcome. The processes of learning, however, are postulated to occur whenever an individual is in contact with some kind of teaching influence. Thus, it is possible to investigate the conditions of the teaching-learning interaction.

Freudenthal (1980) maintains that information on the teaching-learning interaction must be obtained from observations within actual classrooms. Evidence of learning processes can be obtained from the systematic observations of the behavior of the teachers and students and from a careful analysis of the discourse that is carried out among the participants in this type of environment.

> A teacher who watches his pupils, has witnessed many learning processes and he knows that he has. Is there not a lot to be observed when people learn.... Only it is so difficult to organize the wealth of observations, to describe them: but not until we consciously set out to observe learning processes can we create the means to organize, describe and evaluate them. (Freudenthal, 1980, p. 164).

This educational approach to research on learning processes contains several similarities to that employed by cognitive scientists. Both types of investigations regard learning as an individual activity that occurs through mental actions and that can be inferred from observations of an individual's overt behavior. The cognitive scientists construct artificial situations that employ a talking-aloud technique to elicit certain behaviors. The educational researcher, however, must identify specific types of classroom behavior that can be used to draw similar inferences about an individual's mental activities.

Such an approach is different from that used by mathematics education research in the 60s and 70s (Davis, 1983). Although teaching and learning were both investigated, the focus was on measuring group behavior, usually in the form of some specified behavioral output, such as a set of test scores. Teaching and learning were regarded as separate entities. Many of the research studies attempted to assign statistical relationships between these "variables" on the basis of pre-and post-test scores comparing one teaching approach to another.

In order to focus on the teaching-learning process, today's educational research must identify appropriate behaviors within actual classroom settings that can serve as the source of pertinent data. These requirements place certain restrictions on the types of classrooms that can be used as rich sources of information.

The traditional classroom in which students passively listen to a teacher's lecture do not provide sufficient instances of student behavior from which mental processes can be inferred. It is important to select classrooms in which student-teacher or student-to-student discussions and questions are a regular part of the classroom activities.

According to van Hiele (1986), "the most important part of the teaching-learning process is discussion" (p.viii).

Such classroom situations provide many opportunities to collect observations pertaining to students' self-reports of their existing mental states and of the types of thought processes that they are engaged in using. Observations of the total learning environment provide the context in which such discourse is initiated and allow the researcher to investigate the influences that are brought to bear on each individual's thought processes. These situations provide the necessary information from which to infer the existence of the teaching-learning process. The didactic experiment described by Dina van Hiele-Geldof is an example of such research.

Relevance of Current Research to Mathematics Education

The focus of research used by cognitive scientists and that used by educational researchers are concerned with examining different aspects of a very complex field involving human cognition. The cognitive scientists focus on the mental processes involved, while the educational researchers focus on the factors at work influencing and forming these processes. These different approaches require different investigative objectives and research techniques. The two types of investigations also present results that have different degrees of applicability to the field of mathematics education.

Pollak (1987) raises a concern about the pertinence of research in information processing and artificial intelligence towards the needs of the mathematics teacher.

> All too often, there is a tendency to think of the research in artificial intelligence as a great advance by itself. It is, but educators and mathematicians also want to see it applied to education, and many researchers in AI stop short of that.... What does this piece of research, this alternative model, this new technology say about how to teach a specific mathematical topic? (p. 253)

As long as cognitive science research views learning and teaching as two separate disciplines, their findings will remain of less direct relevance to direct classroom implementation. Romberg and Carpenter (1986) provide evidence of the prevalence of this type of approach to current research with their article in the third annual Handbook of Research on Teaching that is titled, "Research on Teaching and Learning Mathematics: Two Disciplines of Scientific Inquiry." They state

> We need to rethink the content of the school mathematics program, but in doing so we need to take into account implications derived from two disciplines of scientific inquiry. The first comes from research on how students learn mathematics, and the second from research on teaching. (p. 850)

Crosswhite (1987), however, cautions against such a separation of learning and teaching in current research trends.

> Cognitive science may focus on a problem because it is both manageable enough to restrict the domain of their investigation and rich enough to supply the necessary examples. That focus is understandable. But they should not assume that the findings from this research will translate directly into practice. The problems that are of interest to cognitive scientists may not be the same as those that are of interest to classroom teachers. (p. 272)

Romberg and Carpenter point out the failure of cognitive science research to address issues of instruction. "Most of the implications are still in the potential stage, and much of the research directly addressing questions of instruction has remained untouched by the revolution in cognitive science" (p. 851).

The current trends in research exemplified by the work of the cognitive scientists have made significant contributions towards developing an understanding of the mental processes that individuals employ while doing mathematics. Their emphasis on the study of thought processes of individuals has provided educational research with an alternative research paradigm from that prevalent ten to 15 years ago (Romberg and Carpenter, 1986). Their use of talking-aloud protocols has provided a valuable source of data for the types of inferences that they wish to make.

However, research in mathematics education must take these research techniques further and apply them directly to the classroom teaching-learning interaction. Significant research that can be directly applied to classroom use can be designed. Pierre van Hiele and Dina van Hiele-Geldof demonstrated over 30 years ago that it was possible to develop significant educational theory from direct classroom observations. It remains now for current researchers to tie together research techniques from cognitive science and from the fields of sociology and anthropology in order to develop effective and systematic methodologies for the investigation of classroom learning. The following chapter presents a description of an appropriate methodology for such type of research.

CHAPTER TWO

QUALITATIVE RESEARCH

This chapter provides a brief overview of the techniques of qualitative research. This methodology is contrasted to that employed by quantitative, experimental research in the field of mathematics education. The specific techniques of the qualitative methodology are then examined in terms of their applicability to the requirements of current research needs in mathematics education. The chapter concludes with an overview of the research methodology employed by the present study.

Quantitative Research: Research as Testing

The techniques of quantitative research have been frequently employed in the field of mathematics education. Much of this research is characterized by carefully designed experiments that employ clearly defined and carefully controlled variables. The goal of such research is stated at the beginning of each investigation as a set of hypotheses that are tested by a statistical analysis of the collected data.

One important aspect of this type of research is that every experiment must be fully planned before data collection begins. During the planning phase, the sample population is carefully delineated and the variables to be investigated are identified. An experimental design is decided upon that includes a description of the type of data to be obtained and the method of collection to be used. This design must also include the specific steps that are necessary to control all of the variables under investigation. The anticipated results of the research are postulated before research begins and are stated in terms of null hypotheses to be tested.

Once research begins, data are collected only along the lines that were laid out during the planning stage of the experiment. This type of methodology is thus not designed to incorporate new or unexpected information that may be uncovered during data collection. Data are collected for the single purpose of accepting or rejecting the given hypotheses.

This type of experimental design is appropriate for investigations involving situations in which key variables and characteristic populations can be identified from prior work. Pre-existing theory can then be used to generate clearly stated research hypotheses. The purpose of such research is to verify theory that has been previously developed.

For such research to be valid, it is necessary that all the variables in the experimental and control groups be manipulated as theoretically stated. Furthermore, all assumptions must be met concerning the choice of a particular theoretical population distribution to be used. If these conditions cannot be met in actuality, no amount of statistical manipulation of the collected data will make the results meaningful.

The experimental techniques of quantitative research described above have, until recently, served as an important methodology for investigations in the field of mathematics education. These techniques were employed as a means of providing a "scientific" basis to studies in this field. However, current interest in research into human cognition has created a need for different investigative techniques (Davis, 1983).

> Considering the great complexity of educational phenomena and the substantial areas of real disagreement, it is not surprising that educators have sought a "scientific" basis for the study of teaching and learning. This search has led educators to look at the physical sciences--seeming models of scientific agreement, and objectivity -- and in some fashion to try to imitate them. The imitation has in many cases failed, and perhaps nowhere more than in the study of complex mathematical cognition. (p. 253)

Qualitative Research: Research as Exploration

The techniques of experimental, quantitative research present certain limitations when used to investigate activities involving human cognition. An extensive body of knowledge and theory does not at present exist that can account successfully for all the mental processes involved in mathematics thinking and learning (Ginsburg, 1981). Thus there are few existing theories that can be used to formulate experimental hypotheses.

It is difficult to design clean, quantitative experiments that can be used to study those aspects of human learning involved in mathematics education. Because of the existence of complex interactions of individuals with their environments and with each other, it is a difficult task to try to identify all of the variables that can account for a particular aspect of human behavior. Even among the variables that can be identified, many are not easy to isolate or control.

The techniques employed in qualitative research are designed to deal with the limitations that are inherent in any investigation involving human subjects. These techniques were developed by sociologists and anthropologists as a means of studying the characteristics or patterns of behavior of individuals or groups in an interactive setting. This methodology also allows the researcher to carry out investigations in areas in which a minimal amount of prior information currently exists (Bogdan and Biklen, 1982). Such techniques have made qualitative research a viable alternative for educational research.

The goal of qualitative research is discovery rather than verification. The techniques of data collection are designed to be flexible in order to allow the direction of research to change as new information is uncovered and to facilitate the emergence of unexpected results. Important variables and relationships emerge from the data as the research progresses, which are then used to generate theory that can account for the observed processes or patterns of behavior. "The researcher is interested primarily in describing, understanding, and explaining the activities of his hosts" (Schatzman and Strauss, 1973, p. ix).

Methodology of Qualitative Research

Qualitative data are collected from all aspects of the research situation. Much of this information is obtained by detailed observations of the subjects involved and the activities and conversations in which they engage. Other information is obtained from observations of the physical environment of the research setting, from personal interviews, personal documents, photographs, or other records (Bogdan and Biklen, 1982).

Various approaches are utilized to obtain the observations that constitute the research data or records. The researcher may obtain the desired information through formal or informal interviews, by observing the behavior of subjects within a particular setting in which he or she is not an active participant, or by participant observation within this setting.

Observations are recorded as field notes. These contain a detailed reconstruction of everything the researcher sees, hears and experiences and are written up after each period of observation. The researcher also includes observer comments in the field notes describing his or her reactions to particular situations, or comments on items of interest (Bogdan and Biklen, 1982). These reactions form an important aspect of the collected records, and provide a way for the researcher to become aware of and monitor his or her biases towards the research environment. Interviews or conversations within the research setting may be tape or video recorded and are transcribed and included in the field notes.

Unlike quantitative research, in which the analysis phase is performed after all data are collected, analysis of the information in the field notes is started soon after observations are begun. This is carried out initially by a preliminary classification of the subjects, environment, and activities into categories that are based on perceived patterns and regularities. Once the data collection begins, observation and analysis are carried out simultaneously. Under this continuing analysis, new information is included in the existing categories, and new classifications are added as patterns are identified.

The field notes are analyzed by continually comparing different aspects of the data against each other, and against patterns of interest that are identified through this analysis. Glaser and Strauss (1967) describe this as a constant comparative method. A list of patterns is identified by noting the presence of persistent similarities among the collected records or data. Once such patterns are developed, the researcher then searches for further situations that will yield similarities and differences on these patterns. This comparison of similarities in diverse situations, and differences in similar situations enables the researcher to develop a rich list of properties describing each pattern or category of interest (Glaser, 1978).

Analysis of the collected information is carried out in a sequence of several levels. Particular interactions and activities are placed into categories or described as patterns as the properties of these categories are identified. The analysis then, at a higher level of abstraction, focuses on the identification of relationships among the emerging categories. These relationships are then used to develop themes, which form the preliminary stages of grounded theory that can be used to account for the observed processes or behaviors.

Emerging theory is constantly tested by comparing it against new observations. To be valid, any theory must explain and fit the information collected. Theory is broadened or modified as new observations are found to be inconsistent or different from the previously formulated theory (Glaser and Strauss, 1967; and Glaser, 1978).

The direction of research becomes focused on a particular aspect of the setting as observations continue to be collected and analyzed. It is not necessary for preliminary data to be extremely detailed. Observation is funnelled from the very general, at the beginning of research, to more specific areas as analysis continues, and the researcher identifies categories of interest. There are many avenues of research possible, depending on which of the emerging categories the investigator chooses as a focus of his or her attention.

As observations and simultaneous analysis continue, a particular aspect of the group under investigation may become identified as a category around which certain activities and interactions appear to be related. Many such categories or themes may exist in a particular research situation. The researcher selects a particular theme as part of the analysis of the data and focuses further observations on those aspects of the research setting that have the potential for providing additional information relating to this chosen theme. Analysis of the collected observations then concentrates on developing the characteristics of this particular aspect of the group and the development of theory to account for the observed patterns (Glaser, 1978).

One of the differences in methodology between qualitative and quantitative research is this flexibility in the ability to alter the direction of research during the investigation. In this way, unanticipated, but important information can become a legitimate part of the research records. This flexibility also allows the researcher to incorporate other groups of people within the study as the focus dictates. This is in sharp contrast to the techniques of sampling used in quantitative research, where the study population is rigidly defined according to statistical principles before the research begins.

The researcher may begin observations in the field before investigating what has been published in the relevant literature (Glaser, 1978). A review of literature that is conducted after the research has begun is used to enhance the analysis of the data (Bogdan and Biklen, 1982). It is important that the researcher not be influenced by literature in the subject area to develop preconceived ideas of what to look for in the present study. The literature can provide guidelines and suggest avenues of investigation, but should not limit the ways in which the data is selected and analyzed (Bogdan and Biklen, 1982). After observations have begun and analysis has started, it is important for the researcher to conduct a literature search in order to determine how the present study fits in with other research published within the field.

If the particular study is designed to be exploratory, then it may not be possible to determine the exact direction of the study before observations begin. Until a central focus emerges from the analysis, it will not be possible to identify the specific nature of the research. In such a study a literature search is usually not feasible until after the investigation has been started. In contrast, the literature review forms part of the preliminary work that must be conducted before carrying out research designed to verify existing theory.

One criticism of qualitative research is its heavy reliance on the researcher as the instrument through which observations are made. All items recorded in the field notes are influenced by the subjectiveness of the observations. It is important, therefore, for the researcher to be aware of personal biases and limitations. These must be brought out during the course of observations as reflections on the researcher's own performance and reactions to situations. It is important to collect detailed records over an extended period of time in order to develop a dense set of data. The constant comparison and analysis of these data provide the researcher opportunities to confront his or her own opinions and prejudices and to continually assess them against the evidence of the collected records (Bogdan and Biklen, 1982). Agar (1980) advises the researcher to:

> Look for and be aware of contradictions. They can deepen insight and provide density to concepts and theory. People who disagree with you may have a different perspective. Use this to help you see your own biases and build a better insight. (p. 49)

Current Needs in Mathematics Education Research

The techniques of qualitative research described in the preceding sections provide a set of techniques that have useful applications to investigations in the field of mathematics education. This section will examine some of the aspects of classroom mathematics learning that lend themselves to this type of research methodology.

Educators concerned with contemporary mathematics instruction are engaged in searching for effective solutions to the problems of teaching mathematics (Romberg and Carpenter, 1986). Although many avenues of research are actively being pursued, there does not as yet exist a definitive body of knowledge about a science of mathematics education (Freudenthal, 1980).

> Everyone knows that mathematics teaching exists, but opinions are divided on the existence of mathematics education. By the latter we mean the disciplined study of the practical and the theoretical aspects of the teaching of mathematics, a study with its own intrinsic character. (Association of Teachers of Mathematics, 1977, p. xv)

Research in mathematics learning has, until recently, been concerned with investigating those aspects of the learning situation that lend themselves to carefully controlled experiments. Because mathematics thinking involves higher order, complex acts of cognition, it is difficult to investigate any but the simplest activities involving mathematics using experimental, quantitative techniques. Such research, because it simplifies the learning environment, yields results that are not always directly applicable to mathematics education.

Neisser (1976) provides a description of the level of complexity that is encountered as soon as one attempts to investigate situations involving actual learning experiences.

> First, cognitive psychologists must make a greater effort to understand cognition as it occurs in the ordinary environment and in the context of natural purposeful activity. This would not mean an end to laboratory experiments, but a commitment to the study of variables that are ecologically important rather than those that are easily manageable. Second, it will be necessary to pay more attention to the details of the real world in which perceivers and thinkers live, and the fine structure of information which that world makes available to them.... Third, psychology must somehow come to terms with the sophistication and complexity of the cognitive skills that people are really capable of acquiring. (pp.7-8)

Neisser's comments reflect the importance of designing investigations that are capable of studying the teaching-learning interactions that take place within actual classroom environments. Ginsburg (1981) describes such research as "naturalistic observation -- the careful examination of children's thoughts in various natural settings" (p.5).

Greeno (1980) recognizes the importance of the school environment as a source of data upon which to build learning theory. He states that:

> It seems quite certain, however, that instructional tasks constitute a domain of study and analysis that is potentially productive for psychological theory. Learning tasks in the school curriculum are complex enough to raise nontrivial theoretical questions.... A deep theoretical understanding of the psychological processes involved in school learning could become the keystone of a significant new psychological theory of learning. (p. 726)

Mathematics education is not yet at the point where there exists a significant body of theory concerned with the discipline (Freudenthal, 1980). Research in mathematics education must be careful about designing experiments testing theoretical hypotheses until adequate theory has been developed from which these hypotheses may be drawn. At the present time mathematics education is still in the early stages of trying to describe, categorize and relate those observed behaviors indicative of students engaged in mathematics learning processes. Once such careful descriptions have been developed and categorized, educators will have important information with which to begin to construct theories of classroom learning (Freudenthal, 1980).

Freudenthal (1980) and Ginsburg (1981) maintain that present investigations into mathematics learning and education should be exploratory in nature. The purpose of such research is not to test pre-stated hypotheses or seek to demonstrate the existence of variables that can be stated a priori. Investigations which state, before research begins, the types of results that will be found do not provide a vehicle which allows for the documentation of unanticipated results. By defining the scope of research in this way, such investigations "set artificial and unnecessary limits on one's research," (Ginsburg, 1981).

> In the area of mathematics education...at the beginning stages of research it makes no sense to define "mathematical thinking" in some arbitrary fashion: instead a process of discovery must be employed to determine the main developmental features of children's mathematical thought. As these are discovered -- and many findings will surprise us -- our conceptualization and definitions of mathematical thinking must necessarily evolve. (p. 5)

Summary

The recent shift in emphasis in mathematics education research to investigations of the cognitive behavior of learners has created a demand for different types of research techniques. These demands are reflected in the need to study complex, higher-order levels of thinking, that involve interactions between students and teachers utilizing real-life, school mathematics. The lack of a large body of theory related to the field places limitations on the extent to which hypothesis-testing research can be utilized. Investigations of an exploratory nature are needed that can provide the foundations for theories of mathematics education.

Qualitative research can be used to meet many of the above needs. This methodology provides a systematic way to study interactions within complex environments. The use of observational techniques allows the researcher to investigate situations within their natural settings, thus providing what Neisser (1976) calls "ecological validity" to the study.

Such research is also designed to be conducted effectively in areas where there is little existing theory that can be used to guide the direction of research. The exploratory nature of the qualitative methodologies allows the researcher to recognize and incorporate unexpected results into the study. The techniques of this research therefore provide a powerful tool for the effective advancement of the research aims of mathematics education.

Research Design of Present Study

This section presents an overview of the research design employed by the study described in this report. Included are descriptions of the focus of the study, the general research methodology employed, the criteria used to select the particular study site, and a brief summary of the methods of data collection and analysis that were used. Detailed information concerning each summer class, including examples of the collected records and the specific stages of analysis, will be presented in subsequent chapters.

Focus of Study

The general purpose of the present study was to conduct an exploratory, qualitative investigation of the teaching-learning interaction that takes place within an actual mathematics classroom. Once data collection began, the research was focused to concentrate on an investigation of the types of understanding that students were able to develop for the mathematical content contained in a specific course in algebra. The students' perceptions of the mathematical concepts presented in this course were investigated through an analysis of the classroom discourse and the students' written work that was generated for each topic of study. These perceptions were then compared to the material that was made available to the students within the learning environment.

Research Methodology

The techniques of qualitative research were selected as the most appropriate method of data collection based on the goals of the study. This methodology was chosen because of its ability to provide a systematic framework for the collection and analysis of information within the complex learning environment of the algebra classroom. The techniques of participant observation were employed to collect records of the interactions of the participants within the learning environment.

These qualitative techniques made it possible to identify patterns of interest related to the purposes of the study, thus enabling the research to be exploratory in nature. In this way, it was possible to discover and classify relevant themes that existed within the actual classroom setting.

Study Site

The criteria for the selection of an appropriate study site were determined by the general purposes of the study. In order to observe the teaching-learning interaction, it was necessary to select a mathematics classroom in which opportunities existed for information exchanges between the students and instructor. In order to observe students engaged in learning activities, it was necessary to use a classroom in which students had opportunities to ask questions, engage in discussions, and participate in actual problem solving activities. In order to obtain observational data from which to infer students' mental processes it was necessary to select an environment in which students were given the opportunity to talk aloud through their thought processes. More occasions for such observations were postulated to exist when students experienced difficulty understanding the subject matter, than when they found the material easy to master. Therefore, a further condition on the site selection was that it include such students.

The special university summer class in algebra for Native American students provided a rich environment which met the above criteria for site selection. This class consisted of an intense six-hour-a-day, four week exposure to the topics usually covered in beginning and intermediate algebra at the high school level. The instructional day was divided into periods of lecture, practice sessions in which students worked through algebra problems, testing periods, and laboratory activities. Students worked in groups of four and were encouraged to share ideas and help each other. An instructor and several assistants were available to provide one-on-one assistance throughout the day.

The students in the class had all exhibited math avoidance behaviors and experienced difficulty learning math in previous enrollments in remedial level mathematics courses at the university. The environment of the summer class was designed to provide a relaxed, non-threatening, atmosphere in which these students could ask questions, discuss learning problems, and seek peer support.

Although the course only lasted four weeks, the six hours a day of activities presented many varied examples of learning activities. Students were observed asking questions during formal lectures, working together in groups as they discussed solutions to algebra problems, and working individually during testing situations. Students were also observed during one-on-one sessions in which they asked for help with specific learning difficulties. Changes in student behaviors and responses were observed as students progressed through the course syllabus. The course was also offered again during the summer following the first year's observations. This second summer of observations provided an opportunity to compare and contrast the information obtained during the first year.

There was no difficulty in obtaining access to this class for research purposes. I participated in the planning and development of the special algebra class and served as an assistant instructor during two consecutive summer sessions of the class. This made it possible to collect records of the study site using the techniques of participant observation.

Data Collection and Analysis

As an assistant instructor, my duties were to provide half-hour lab activities once or twice a day and to be available to answer questions during group practice sessions. I also worked closely with the instructor coordinating the daily activities and reviewing the progress of the course. I was present in the classroom during the total six hour period each day.

I was involved in taking field notes of my observations of the class environment during the time I was not actually helping students in the class. Other information was gathered from informal interviews with the students during break sessions and before classes began, and from discussions with the instructor at the end of each day. Photocopies of homework assignments, quiz papers and final tests provided samples of students' written work. Each lecture session was also tape recorded during the second year's class. Records were collected for the first class during four weeks in July through August, 1986 and again during the second class that was presented over a five week period in June through July, 1987.

The field notes of classroom observations were typed up each evening and reviewed at the end of each week. The constant comparative method described by Glaser and Strauss (1967) was used to analyze these data. Patterns relating to the research purposes were identified during the weekly review of the field notes and then used to direct the focus of the observations to seek further instances where similar patterns might be found. The major analysis of the data was undertaken after the completion of each course.

The analysis of the first year's records was used to provide a further focus to the observations and data collection that occurred during the second summer class. In order to amplify the conclusions that had been made concerning the students' observed behavior during the first class, a tape recorder was used to collect information related to the instructional intent and the subject content of each lecture. Photocopies of the students' written quiz papers were also collected.

The focus of the second year's analysis was centered on the students' abilities to reproduce mathematically correct responses to selected problems during quizzes. These responses were also compared to the lecture transcripts in order to measure student behavior in terms of the information that was made available to them through the daily lectures.

Limitations of the Study

The unique nature of the summer algebra program did not provide a typical example of classroom algebra learning. Although the subject covered topics contained in high school algebra, the course did not follow the time schedule, or treat the subject in the way that is traditionally found in a high school mathematics course. The Native American students who participated in the course also represented a non-traditional student body.

The conditions of instruction of the course presented students with a special set of learning problems, not found in a typical algebra classroom. The students were expected to master six credits of material within only four weeks, which meant that they were exposed to new material at an accelerated rate that was not necessarily compatible with the learning rate of the students.

The research methodology employed to collect data provided a vehicle for focusing the analysis of the study on the behavior of a small group of individual students. This study does not, therefore, provide a statistical basis for the generalization of the findings to an overall population of high school algebra students. The ability to generalize the findings of the present research will be based instead on the specific nature of the information obtained and the types of conclusions drawn from the analysis of this information.

CHAPTER THREE

FIRST SUMMER'S RESEARCH

The purpose of this chapter is to present a detailed description of the first year's offering of the special algebra class for Native American students and describe the methods of data collection and specific steps in analysis that were used for this class. This chapter also includes a discussion of the personal biases that affected the manner in which the collected records were analyzed at the conclusion of the four-week course. The following chapter presents examples of the student interactions from this class that were used to develop the findings of the first year's study.

Special Summer Algebra Course

The research setting for the first year consisted of a class of twelve adult Native American Indian students participating in a special four-week, concentrated course in algebra presented during the 1986 summer session at a western state university. The class met six hours a day, five days a week and covered the material usually contained in three two-credit classes, Math 011 (linear equations), Math 012 (polynomials and fractional equations), and Math 140 (exponents and radical equations).

Need for Special Summer Course

At the beginning of the class all of the students exhibited varying degrees of anxiety towards the subject of mathematics and expressed doubts over their ability to perform well in the summer class. Those students who had previously attended the university had enrolled in other mathematics classes during past quarters. They had experienced difficulties in meeting the requirements of these classes, and had received failing grades or withdrawals on several occasions.

Math 011, 012, and 140 are taught during the year as tutorial-assisted courses without actual classroom sessions. Students proceed through the course syllabus at their own pace learning from a textbook. Exams are taken at prescribed intervals throughout the course, with repeat testing required for grades below a certain level. Tutors are available throughout the day to provide help in a specially designated help center.

Many of the Native Americans at the university who have enrolled in these tutorial-assisted courses have experienced difficulty in continuing through the course without the structure provided by a regular classroom approach. They have not been able to maintain the schedule necessary to complete the course in the required time.

Students have also found it difficult to ask for help from the tutors and have been reluctant to go to the help center.

Native American students at the university as a group have a lower success rate than other non-Indian students enrolled in these courses. In response to this failure to successfully complete their mathematics course requirements, the university offered a special, intensive four-week algebra course during the Summer Session of 1986 to Native American students.

Description of Summer Course

The purpose of the course was to provide intensive instruction within a structured environment, in an effort to enable students to successfully complete the requirements for six quarter credits of mathematics. The class was specifically designed to reduce levels of math anxiety and work on students' symptoms of math avoidance. Instruction was developed to fit the level of students who had in the past experienced difficulty learning arithmetic and algebra.

Because of the time limitations imposed, the course of instruction was extremely fast paced. It was necessary to introduce four new topics on a daily basis. Students were quizzed daily on material presented the day before. Homework assignments were made at the end of every class and collected the following day. (An outline of the course contents is presented in Appendix A.)

The class began each morning with a fifteen to twenty minute lecture by the principal instructor. Students sat in chairs facing a projection screen with their backs to a wall of windows. The instructor used an overhead projector and transparencies since there was no adequate blackboard space in the room.

The instructional emphasis in the course was on the mastery of procedural skills. The lectures were designed to illustrate manipulation techniques to be used to simplify or solve particular types of algebraic expressions or equations. The goal of this instruction was to provide students with the information needed for acquiring and mastering a wide variety of mathematical procedures, manipulation techniques, and algorithms related to the subject of algebra.

Following the lecture, students worked in groups of four at three long tables. The groupings were assigned on a weekly basis by drawing table assignments. After each lecture, students were assigned five to ten problems either from the textbook or from supplemental handout sheets. These problems were selected to illustrate the concepts discussed during the preceding lecture, and students were provided 30 minutes to work through these exercises. The textbook used in the class was Introductory Algebra by

Keedy and Bittinger, Addison Wesley publishers. The fourth edition was used during the first summer class, and the fifth edition used the following year.

The students were placed in the three small groups in order to provide a mutual support structure. Students were encouraged to ask each other for help and share solutions. If any student was observed to have finished the practice exercises early, he or she was asked to find someone else in the room whom they could help. During this time the principal instructor and two assistants were available for help where needed.

Following the practice period, the students participated in some form of math lab or math game. This 30 minute period was designed to provide a break from concentrated work and relieve tensions. It also gave the students a chance to participate in some form of mathematics activity that was perceived as fun rather than work. (A summary of the daily class schedule is presented in Appendix A.)

The room was a no-smoking area, and a short break from class was provided mid-morning. Those students who smoked usually went outside and sat on the lawn during this time. This break also provided students with the opportunity to get away from the math environment that they were in six hours a day. There was a full coffee pot and a daily supply of donuts and some form of fresh fruit available at all times. Students helped themselves throughout the day, as well as during the break.

After the break, the second lecture topic was introduced and students were provided another period of group practice on new problems. The morning ended with a half-hour quiz. For the first half of the course, the quiz was done as a group effort at each table. The students were encouraged to work the problems themselves and then ask for help from others in the group when they experienced difficulties. During the final two weeks of the course the quiz gradually became more individual. Students were provided a set of problems which were worked individually, and a second set of problems was then worked as a group. The student's quiz score was the sum of the grades from the individual and group problems. The quiz grades formed part of the total score on which the final class grade was assigned.

Following an hour and a half lunch break, the afternoon session was a repeat of the morning with two new topics introduced, followed by a practice period and a 30 minute group problem-solving activity. Instead of a quiz in the afternoon, the final half hour was used as a general working period. Students were assigned approximately 50 homework problems each day to be handed in at the beginning of the next class. During the final period of the day, they were allowed to begin work on these problems.

The homework assignment consisted of problems covering the four new topics introduced that day, plus problems reviewing those topics covered in previous days. By the end of the course, the homework would contain problems introduced at any time during the four weeks. Because the pace of the course was very rapid, it was important to provide students with daily practice on as much of the old material as possible. Each homework assignment also provided students with a journal question for them to respond to in writing. These questions asked students to reflect on their feelings and attitudes towards various aspects of the course.

The classroom was available for the students' use at night. A special tutor was available in the room for three hours each evening to help students with their homework. The students were encouraged to come back to the room and to work together, and with the tutor.

The homework was collected daily, graded, and handed back the following day. The quizzes were graded over the lunch break and handed back to the students after lunch. Grading on both the homework and quizzes was designed to encourage the students to continue working, rather than as a method of recording mistakes. Students were not penalized for handing in their homework late; however, they were encouraged to maintain the pace of the class and not fall behind.

The principal instructor for the course worked with Native American students at the university during the year through a special program designed for educationally disadvantaged students. She provided special instruction for students enrolled in the tutorial-assisted mathematics courses. She was previously acquainted with all but the new transfer students enrolled in the summer course. She designed the summer course syllabus, provided written hand out sheets supplementing the text book, assembled all the practice problem sheets, and selected the daily homework assignments.

Two assistant instructors and a tutor provided additional one-on-one help with students during the course. One assistant was a graduate student in mathematics at the university. She had previous experience teaching algebra at the high school level. I was the other assistant. My experience consisted of instruction in pre-algebra at the community college level, and, most recently, instruction in calculus at the university. The tutor had previous experience working one-on-one with students exhibiting high levels of anxiety towards the subject of mathematics.

Both the graduate student and I were available during the morning sessions to provide help during group work. I also attended the afternoon sessions. The tutor was available in the classroom for three hours each evening to provide assistance with homework problems. None of the personnel from the day were available during the evening hours.

The class met in the Native American Indian Club Room on campus. One half of the room was devoted to instruction. Twelve chairs were arranged around an overhead projector facing a projection screen, which was used during lecture sessions since there was no chalkboard in the room. Three long tables with twelve chairs were provided for group work activities. The remainder of the room was used for non-studying purposes and contained a sink area with coffee pot and microwave oven and an area containing a desk, typewriter, and couch.

Student Participants

The first summer class consisted of twelve Native American students -- nine women and three men. Four of these students were new students, either transferring from another college or entering as freshmen in the coming fall. The other eight students had all attended the university for several years. Four of the students were younger women who had graduated from high school within the past five years. The other eight students were all older than the traditional college age. Two students consisted of a husband and wife with a family of children living in town with them. Two older women had children living in town with them, and one man was married but living in town without his family. One woman brought her new-born infant to class with her each day.

The students were members of Indian tribes from South Dakota and Montana and maintained family ties to reservations. All students spoke fluent English. However, the Crow students tended to use Crow when talking among themselves. The students were all dependent on some form of financial aid in order to remain at the university taking classes during the summer.

The course was extremely intensive. With four new topics being introduced each day, students frequently expressed bewilderment and frustration at the speed with which they were required to master new material. The homework assignments were very long, and many students required more than three hours a night to finish the assignment. Both the students and the instructors became very tired as the course progressed.

The atmosphere in the classroom remained fairly relaxed. The politeness and good manners exhibited by the students made it possible for fifteen people to remain in close contact, working on a difficult and frustrating subject for six hours a day. The Indian students used humor and an ability to joke about things that bothered them as a way to reduce tensions and maintain a cooperative group feeling. At times the four younger women expressed more open feelings of resentment and frustration, but the older students helped them to maintain a level of acceptable group behavior.

The students made a tremendous commitment to complete the course. It was clearly a difficult situation for those students with families in town to be in class all day, and then spend many hours each night working on their homework either in the Club room or at home. Many of the students were able to hand in their homework assignments on a daily basis. Those students who fell behind were able to complete the majority of their work by the end of the four weeks. On weekends many of the students travelled back to their homes off campus. The pace of the course was very rapid, and the students were never given sufficient time to master any one topic. Yet, only one student dropped the course half way through the four weeks.

The level of instruction and the help provided were a major factor in the students remaining in the course. The lectures were designed to explain each topic clearly. The instructor was careful to write detailed problem solutions on the overhead. She proceeded slowly and carefully, stopping often to ask if there were points that weren't clear or needed to be repeated. The students were never led to think that their questions were "dumb" or that they were wasting class time by asking questions. Frequently the instructor simply stopped lecturing to allow the students time to get caught up in their note taking.

During group time with the practice exercises, the three instructors circulated from table to table on the lookout for students with problems. During the first week some students were reluctant to ask for help and the instructors would make a point of volunteering information. The instructors were willing to take the time to work carefully with any student who needed extra help. At all times the instructors approached the students as friends, expressing their confidence in the students' abilities to succeed.

The group work on the practice exercises and the group quizzes provided valuable peer support. As the class progressed, several students emerged as the "smart ones". They provided some of the tutoring initially given by the instructors. Students who had always thought of themselves as unable to do mathematics many times found themselves providing valuable contributions to group discussions. Anecdotal descriptions of student behavior will be provided in the following chapter and in subsequent chapters in Parts II and III to illustrate many of these situations.

Two groups of students were identified in the class. Five of the students, including the four younger women, appeared to have less difficulty with the subject matter. They did not ask as many questions, worked faster on assigned problems, and required less help during the practice sessions. The other students experienced more difficulty. They worked more slowly and usually asked for help as they worked through the practice problems on new material. They expressed more feelings of confusion and frustration with the material they worked with.

All the students, however, demonstrated their ability to perform the operations of addition, subtraction, multiplication, and division on whole numbers, decimal numbers, and integers. They relied heavily on calculators for these operations, but appeared able to quickly produce the required manipulations for any numerical problem.

Students shared aloud their feelings of dislike and anxiety for mathematics. At the beginning of the course most of the students' comments were negative. The instructors made a point of noticing and encouraging students when they made a positive comment. By the end of the course, all students had experienced success on some portion of mathematics that they had previously encountered with disastrous results. Students were encouraged to enroll in another math course during the following fall quarter.

Native American Background. The description provided above of the class participants is extremely brief. Very little information was obtained relating to the students' lives outside the classroom nor was much information obtained concerning the students' cultural backgrounds. The focus of this research was centered on the in-class behavior of each student in relation to the subject matter under study.

Although each student's past educational history, cultural background, and current life style all had important influence on his or her behavior, this study was not designed to collect information on these factors. It was not possible to observe a group of non-Indian students under a similar classroom situation to provide contrasting information. Therefore, it is not possible to address the degree to which the observed behavior of the Native American students was related to their various cultural backgrounds.

Data Collection

Data were collected mainly through observation and recorded as field notes. The main sources of data consisted of observations of the student-teacher exchanges that occurred during class lectures, observations of individual student's performances on specific mathematics problems during practice sessions, information relating to individual and group responses to specific quiz problems, and observed conversations that took place between students and between myself and other participants in the class.

I was provided access to students' work during practice sessions in my capacity as an assistant instructor. This enabled me to engage in one-on-one conversations relating to questions that individual students raised about their performance on specific mathematics problems. Following such interactions, I took notes on the substance of each conversation and wrote down the sequence of mathematical manipulations that the student had used during this time.

During lectures, I made notes of the questions that students asked on particular problems and noted the instructor's answers, writing down actual conversations where possible. I also wrote down conversations I observed among students and noted non-verbal behaviors of students as they participated in various class activities. Photocopies were made of representative student homework, daily journal entries that were handed in with homework assignments, selected quiz papers, and all final examination papers.

The rough field notes were transcribed into narratives each evening following the daily classes. (See Appendix A for a sample narrative.) The narratives contained quotations rendered as accurately as could be remembered of observed conversations. When the actual words were not recorded, the conversations were paraphrased to retain the flavor of the original interaction. The figures and calculations that were part of these interactions were written into the narrative. Part of the narrative consisted in recording the atmosphere of the classroom each day and noting the types of emotions expressed behind the words. My feelings and reactions to particular interactions were also recorded as observer comments.

Analysis of Data

The accumulated field notes were reviewed on the weekends while the course was in session. Particular conversations, comments, and non-verbal activities were indexed according to day and page of notes, and recorded on index cards. Each item was also underlined on the transcription. These examples of verbal and non-verbal interactions were initially classified under individual student names and under other relevant categories. Some of these preliminary categories of classification were: asking questions during lecture, asking for help at the tables, doing arithmetic mentally or with calculators, making comments about feelings on math, performance within groups, instances of mathematical insight or lack of understanding, past mathematical background, family or personal information, and instances of frustration or enjoyment. Different types of behavior by the same individual were cross referenced in the field notes, as were related interactions that occurred separated in time.

The final exam and quiz papers were examined by comparing and contrasting the various manipulation steps that students had written for each question. The photocopies of homework papers were examined in terms of the students' written responses to the assigned problems.

The focus of the analysis of the collected information became centered on the activities that the students engaged in as they were being exposed to new material during lectures and during the times that they were involved in working through assigned practice problems. Interactions were identified in which, by student actions or verbalizations, students could be considered as being engaged in some form of activity that was directed towards assigning meaning to algebraic symbols. This collection of examples was postulated to represent a set of mathematics learning strategies.

The observations taken during these interactions were used to investigate the various thinking strategies that students employed when they were engaged in relevant mathematics learning. The interactions involving student verbalizations with the instructors provided the most useful information, since it was found that it was not possible to identify the reasons that students had chosen particular approaches to specific problems unless verbalizations accompanied these activities.

The interactions that were selected for analysis consisted primarily of conversations that occurred during lecture sessions. Students asked questions or made comments about a particular algebraic expression, and the instructor gave a reply. This discourse was examined in terms of how each student was using the exchange to assign some kind of personal meaning to the algebraic expressions under discussion. The instructor's comments provided information pertaining to her perceptions of the same expressions.

The other main source of interactions used in this analysis consisted of my observations of students' activities during the practice sessions that directly followed each lecture. These interactions were analyzed in terms of how the student activities enabled each student to manipulate a given algebraic expression. These problems were also related back to the preceding lecture examples. My comments to students during these activities and my reactions to their choice of manipulations were analyzed in order to identify my own perceptions of the algebra involved in each problem.

All the examples collected for this analysis were recopied on sheets of paper and cut apart. Each item was then placed in a separate pile according to the type of activity displayed in the interaction. When the preliminary classifications were finished, the examples in each pile were contrasted among themselves to determine if they did indeed all exhibit that characteristic. Further reclassification and reassignment to categories were conducted until each collection of examples was determined to exemplify a particular type of interaction.

The characteristics of each type of interaction were generated from an analysis of its examples. These characteristics were then recorded on an index card and attached to each separate category. The organization of the final form of this analysis was facilitated by ordering these categories according to the notations on the index cards. This information, illustrated by examples of specific student interactions, is presented in the following chapter.

The examples that were used in this analysis were generated for the most part by a small group of students. These students appeared to have more difficulty mastering the subject matter than did the other students in the class. They were the ones that asked the most questions during lectures and required the most help during practice sessions. Since these students were more verbal and participated in more interactions with each other and myself, they provided the bulk of the recorded observations. The analysis of these records, therefore, does not reflect the behavior of all of the students in the class.

Personal Bias

The most important aspects of the analysis became possible only after I was able to separate my personal perceptions of the subject of algebra from those of the students. This allowed me to describe and categorize their actions in terms of their own perceptions of the subject matter. All words are charged with personal meaning and I found it necessary as I conducted the analysis of the classroom interactions to re-evaluate my vocabulary and become aware of the differences in connotations that the students and I assigned to the same words and phrases.

As I began the primary analysis of the field notes while the class was still in session, I had approached the task "thinking like a mathematician." This led me to measure all the actions of the students against my personal level of mathematical comprehension, which was based on a different and more extensive set of learning experiences than that of the students. Until I could separate my own set of definitions from the objects of study, I was not able to perceive that the student activities represented learning processes that were occurring within personal contexts at many different levels.

CHAPTER FOUR

EXAMPLES OF MATHEMATICS LEARNING STRATEGIES

The analysis that was conducted on the collected records was focused on identifying and describing classroom interactions in which students were engaged in assigning meaning to collections of algebraic symbols. The students' behavior and discourse that were observed to occur during periods in which the students were exposed to new and unfamiliar material was postulated to represent a set of learning strategies. These learning strategies were further classified as representing relevant mathematics learning because they focused only on those situations that involved the students' interactions with the algebraic subject matter.

The findings of the analysis of the first summer's research are reported in the following sections. These sections represent different types of strategies that were used by the students to meet specific learning needs. These strategies are classified as; searching for patterns, noting similarities and differences, identifying directions for the manipulation of symbols, and relying on the use of a familiar symbol format. Examples of classroom interactions are included in each section to illustrate the ways in which students used each type of strategy to interpret the actions of the instructor and to assign meaning to the collections of algebraic symbols under study. A section of examples is included that contrasts the different types of meanings that the students and the instructors assigned to the same objects of study.

In the examples that are reported, all the names of the participants have been changed. The interactions are not presented in the chronological order in which they occurred during the course, but are grouped under the particular learning strategy that they illustrate. The coded number before each example gives the specific week of the class, and the day of that week in which the interaction occurred. For example, the number 4-3 represents an interaction that took place during the third day of the last week of the course. (An index of the interactions described in this chapter is provided in Appendix A.)

Search for Patterns

During the class sessions, those students who were studied in this analysis were observed to orient themselves with unfamiliar algebraic expressions by identifying patterns of symbol placement. They organized these patterns spatially (i.e, "the expression on the right," or "the thing on the bottom"), and identified similarities or differences that existed with other symbol patterns. Symbol patterns were also linked to the particular manipulations that were used to simplify each expression. Students also linked unfamiliar symbols to previously learned material. It was observed that students were able to function more comfortably with new information when it was expressed in terms of familiar techniques and symbol patterns.

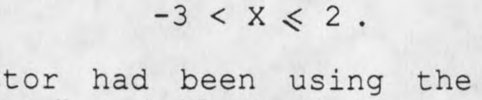
Students generated rules of manipulation for specific algebraic expressions in terms of symbol patterns they were able to identify, and expressed these rules in language reflecting the physical appearances of the symbols. They also compiled collections of master examples that served as references for the manipulations of similar expressions.

During lectures on new material students frequently asked questions of the instructor that indicated that they were trying to reformulate in their own minds the algebraic manipulations of the instructor. Their vocabulary dealt with the physical rather than the conceptual aspects of the expressions.

4-4 The lecture topic was graphing inequalities of the form:

$$-3 < X \leqslant 2 .$$

The instructor had been using the expressions "greater than," and "less than or equal to " as she graphed each example. The above was the third example she had illustrated. She graphed this on the number line as follows.



When she had finished with the graph, Shirley asked, "Couldn't you just look at the -3, draw an open dot, and at the 2 and draw a closed dot, and draw in between?"

Shirley was able to clearly express the actions of the instructor without using any mathematically technical terms. By removing these expressions and replacing them with the phrase "in between" she was able to make the connection between the algebraic expression and its graphical representation. When Shirley said, "Couldn't you just," she was taking the mystery out of the new information as she related it to actions that had meaning to her.

3-5a The lecture topic covered techniques for solving equations in which the variable appeared under a square root sign.

$$\sqrt{x + 1} = \sqrt{2x - 5}$$

$$x + 1 = 2x - 5 \quad \text{etc.}$$

When the instructor was finished with the problem, Valerie asked a question concerning the first step in which the instructor had squared both sides of the equation. "You can't do it to one. You got t^ do it to both of them?"

This student was expressing the actions of the instructor in terms of the changes in the physical appearance of the algebraic symbols. The mathematical manipulation of "squaring both sides of the equation" was referred to as "it."

Valerie was aware that the symbol pattern changed on both the right and left hand sides of the equal sign. This is a fundamental consequence of the concept of an equation, however, it is not explicitly stated. "You got to do it to both of them," is an important element in the sequence of manipulations needed to solve the equation for x. In the above exchange, the student was verifying that she had observed the important aspects of the manipulations used, through focusing on changes in symbol pattern. It appeared from her question that Valerie did not understand the concept of equality attributed to the equal sign. However, she had demonstrated her ability to reproduce the mathematically appropriate steps of symbol manipulations through an understanding of the physical and spatial change in symbol patterns.

2-4a The lecture topic was the division of fractional expressions. The instructor presented two examples similar to the following. The instructions included the reminder to factor each expression and try to cancel common factors.

1) $$\frac{x - 6}{x + 3} \div \frac{x + 5}{x^2-x-12} = \frac{(x - 6)(x - 4)(x + 3)}{(x + 3)(x + 5)}$$

2) $$\frac{x + y}{x - y} \div \frac{x + y}{x^2 - y^2} = \frac{(x + y)(x - y)(x + y)}{(x - y)\ (x + y)}$$

Bob asked the instructor if there could be problems in which the expression on the left of the division sign could be factored.

Bob was searching for meaningful patterns that would provide him with cues for the appropriate symbol manipulation. In the above example he demonstrated his observational skill by indicating that in each example, only the expressions on the right were factorable. He immediately tried to verify whether this rule would hold true in other examples.

4-3 The lecture topic covered a technique for sketching parabolas from equations written in the following form:

$$y = a(x-h)^2 + k.$$

Given: $y = -2(x-1)^2 + 3$, then $h = 1$ and $k = 3$.

The instructor was demonstrating how to determine the coordinates of the vertex of the parabola from the values of h and k represented in the above equation.

At this point Tom asked, "How come h is one?"

The instructor replied, "I was waiting for someone to ask that."

Tom: "Why isn't h minus one?"

Instructor: "h is the number that comes after the minus sign."

Donna: "What if there isn't a minus sign?"

Instructor: "We'll get to a problem like that later."

There continued to be some confusion on the sign of h, given the initial form of the equation. During work on practice problems following the lecture Tom soon devised his own rule for determining the sign of h. He explained to me that you just change the sign between x and h and you get the correct number.

In this exchange the instructor responded to Tom's question in terms of the physical arrangement of the symbols, "h is the number that comes after the minus sign."

However, this statement was too problem specific, as indicated by Donna's question. As the instructor found, it was difficult for an individual thinking in terms of the algebraic concepts attributed to the symbol pattern to formulate a rule based only on the physical arrangement of the algebraic symbols that was adequate for any given student. After working with examples of both symbol patterns, Tom was able to establish a rule with personal meaning that explained how to extract the appropriate information from the specific symbols present in the expression.

4-2b The lecture topic introduced two substitution techniques that could be used to change equations into the more familiar form of a quadratic equation.

First example: $x - 7\sqrt{x} - 8 = 0$

let $u = \sqrt{x}$ then $u^2 = x$

$$u^2 - 7u - 8 = 0$$

Tracy: "How do you decide on which is u and which is u^2?"

Instructor: "Usually the one in the middle is u."

The instructor finished the first example and answered another question.

Second example: $y^4 - 2y^2 - 15 = 0$

let $u = y^2$ then $u^2 = (y^2)^2 = y^4$

$$u^2 - 2u - 15 = 0$$

Shirley: "How come there is no radical sign?"

Instructor: "Because this is another example."

Shirley: "Heck, this is confusing."

Bob: "Every time you have a y^4 you substitute a u^2?"

The two examples were worked through, step-by-step on the overhead projector. At the same time, students were given hand-out sheets containing all the steps to these two examples to refer to while following the lecture. In spite of the care taken by the instructor to proceed slowly, allowing for questions, many students found the topic difficult.

If the concept representing quadratic equations is understood, then the specific substitutions used in the above examples can be seen to make the quadratic-ness of each of the equations explicit. However, if this concept cannot be attributed to the patterns of the algebraic symbols, it becomes more difficult to attribute meaning to the particular symbol manipulations used.

The students had difficulty in identifying meaningful patterns because the form of the algebraic symbols used in each expression were different. Because they were unable to attribute the general form of a quadratic expression to the specific symbol patterns, they were unable to attach any similarity to the two examples. A search for patterns using the physical appearance and placement of the algebraic symbols was hampered by the use of the different variables x and y, and by the different forms of each expression.

Further, the substitution of u in the first step in each example altered the original equation into yet another symbol pattern.

If the choice of substitution to use in each example cannot be linked to the quadratic-ness of the equations, then it must be based on some other kind of rule. The instructor was able to provide such a rule by relating the substitution to the physical rather than the conceptual attributes of the symbol patterns. She explained, "usually the one in the middle is u."

Bob indicated by his question that he was establishing a rule for substitution in the second example. "Every time you have a y^4 you substitute a u^2?" He was making a generalization by the use of the term "every time." However, his generalization was very problem specific, because he was not able to identify a pattern similarity between the two examples.

Shirley expressed the frustration of a student confronted with too many symbol changes. "Heck, this is confusing." She expected the pattern of the algebraic symbols to remain very similar from example to example. The differences in the symbols between the two examples were too great to attribute a relationship to them.

During practice work, Donna and Wanda did not match the symbol pattern of the practice problems to the lecture examples. Instead they assumed that, as had been the case in other practice sessions, the order of the practice problems would follow the order of examples in the lecture. They used the substitution $u = \sqrt{x}$ in the first problem, and the substitution $u = x^2$ in the second problem. Unfortunately, the order of the practice problems was reversed.

First problem: $x^4 - 7x^2 - 30 = 0$

Second problem: $x - 17\sqrt{x} + 18 = 0$

Tracy, Shirley and Bob completed the practice problems using the lecture examples and the accompanying notes as references. For each problem, they were able to locate a worked "master example" that contained a similar pattern of algebraic symbols.

The sequence of events described above can be used as an illustration of the students' abilities to produce mathematically appropriate sequences of algebraic manipulations by carefully matching forms of algebraic expressions to pre-worked examples. These students did not attribute the concept of á quadratic equation to the given problems. Their choice of substitutions to use for each problem was instead determined by the physical and spatial relationships of the pattern of symbols displayed.

3-4a The lecture topic was the simplification of expressions involving exponents and radicals.

First example: $\frac{\sqrt{9x^3y}}{\sqrt{3xy}} = \sqrt{\frac{9x^3y}{3xy}} = \sqrt{3x^2} = x\sqrt{3}$

Donna: "Could you go over that again?"

The instructor explained her steps.

Bob: "So you divide your numbers and you subtract your exponents?"

Second example: $\frac{\sqrt[4]{64xy^5}}{\sqrt[4]{2x^3y^{-3}}} = \sqrt[4]{\frac{64xy^5}{2x^3y^{-3}}}$

$$= \sqrt[4]{32y^{5-(-3)}x^{1-3}} = \sqrt[4]{32x^{-2}y^8}$$

$$= \sqrt[4]{2\cdot2\cdot2\cdot2\cdot2x^{-2}y^4y^4} = 2y^2\sqrt[4]{2x^{-2}}$$

After a pause, the instructor asked, "Is that OK? Is there a step you'd like me to go back and look at?"

There was another pause, and then Bob asked, "If there is a negative you leave the x, even though there is 2 of them?"

Instructor: "I need four copies."

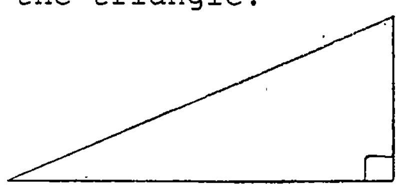
Bob: "OK, OK."

Bob's comment at the end of the first example indicated that he was able to extract a rule for the changing symbol patterns. The second example was much more complicated and involved a fourth root rather than a square root operation. The same rule of "dividing numbers and subtracting exponents" applied. However, in the last step of the problem, the pattern varied from the first example because of the presence of the fourth root symbol.

For the second example, Bob formulated a rule statement to explain the symbol changes. Since he did not focus on the fourth root symbol in the last step, he expected the expression x^{-2} to be altered in the same manner as x^2 in the first problem. The only difference that he attributed to the two patterns was the presence of the negative sign in front of the exponent. Therefore he attributed the different answers to the existence of this pattern. This led him to ask for a verification of the rule he had formulated. "If there is a negative, you leave the x, even though there is two of them?"

When the instructor said, "I need four copies," she was pointing out to him the reason why two factors of x could not be simplified in the second example.

3-4b The lecture topic was an introduction to the Pythagorean theorem. The instructor drew a right triangle, indicated by the small square in the lower right hand angle and discussed the hypotenuse of the triangle.



Shirley asked the instructor for another explanation of how to identify the hypotenuse. Then she said, "The pointy end points to the hypotenuse."

In this exchange, Shirley exhibited the creativity of her solution for locating the hypotenuse of a right triangle. The term "right angle" did not provide a reference for Shirley to use in locating the hypotenuse. Instead she focused on the physical aspects of the drawing.

She regarded the "pointy end" of the small square as a means of directing her attention to the hypotenuse, rather than as a means of designating a right angle. She was able to readjust the information presented to her into a format that carried personal meaning.

Summary

It was determined from the type of questions asked and the comments made that the students were not assigning a high level of mathematical conceptualization to the collections of algebraic symbols that were used as class examples. The students were relating manipulation procedures to the physical changes that they observed taking place in these examples, rather than to the mathematical relationships represented by the symbols. Students attempted to understand the actions of the instructor in terms of formulated rules that could be based on the physical attributes that they perceived in each separate problem. They were relying on the use of pre-worked master examples for guidance in manipulating similar problems.

Similarities and Differences

Students focused on the similarities or differences in symbol patterns they identified for algebraic expressions as a method of organizing new information. This strategy sometimes caused confusion when differences of pattern obscured the sameness of an underlying concept. In other situations, the similarities of two expressions caused students to apply mathematically inappropriate techniques of manipulation.

2-3a The lecture topic dealt with equations in two variables representing straight lines. After rearranging terms in an equation, the instructor ended up with two separate equations for the same straight line. They were written side-by-side on the overhead.

$$2.3x - y - 7.3 = 0, \quad -2.3x + y + 7.3 = 0$$

Tom: "It looks as if you have two different answers."

Instructor: "How can we make them look the same?"

Shirley: "Multiply through by minus one."

The instructor did this manipulation to one of the equations. Then she commented on how it was possible to have the same answer even though two expressions did not look alike. To illustrate this she wrote on the overhead:

$$1.5 = 3/2 = 1\ 1/2$$

Bob then wanted to know if what she had just written was part of the problem on straight lines.

Tom indicated by his question that he had examined the pattern of the two expressions very carefully. He was caught in a dilemma. When is the difference between expressions important, and when should it be ignored?

Bob's question indicated another strategy sometimes employed by the students. When students did not attribute mathematical concepts or relationships to particular symbol patterns, they relied on other types of clues to indicate the importance of various aspects of an algebraic expression. One such strategy for attributing meaning to symbols given in a problem is to require that all numbers and algebraic symbols be incorporated into the manipulations required to reach a solution. Bob was verifying whether this particular strategy applied in the lecture example. He had not connected the instructor's verbal explanations to the written symbols she had displayed on the screen.

A similar situation occurred during another lecture. After the instructor had arrived at an answer to a particular example, she proceeded to write down a similar expression to illustrate the significance of the last step in her manipulations. When she was finished, Wanda turned to me and asked me which expression was the correct answer to the original problem. She had focused on the written expressions on the screen and not the verbal explanations that had accompanied them. Both Bob and Wanda showed a dependence on the use of visual rather than verbal information in their ability to attribute meaning to algebraic expressions.

3-5b The lecture topic involved solving equations in which the variable appeared under a radical sign. The instructor worked through the following example.

$$\sqrt{5x + 4} + 3 = 12$$

$$\sqrt{5x + 4} = 12 - 3$$

$$\sqrt{5x + 4} = 9$$

$$5x + 4 = 81 \quad \text{etc.}$$

Following the lecture, students worked on practice problems of this type. I watched Donna as she began the first problem.

$$\sqrt{2x + 3} = x$$

$$\sqrt{2x} = x - 3$$

$$2x = (x-3)^2 \quad \text{etc.}$$

Donna was using her lecture notes as a reference. She focused on the similarity of the digit 3 in the lecture example with the practice problem. Since the first manipulation in the example was to move the 3 to the right side of the equation, this was the first step that Donna performed with the practice problem. Later in the day I described to the instructor what Donna had done and her immediate comment was, "So that's why they were all doing that!"

Donna and the other students were exhibiting behavior similar to that observed on 4-2 following the lecture on substitutions to form quadratic equations. In each case, students followed the sequence of algebraic manipulations displayed in similar examples when working through practice problems. Students selected specific manipulations on the basis of similarities they perceived in the symbol patterns of algebraic expressions with specific pre-worked problems and examples.

The most readily available set of cues for manipulating unfamiliar expressions are pre-worked examples similar to the problem at hand. Steve explained to me at the beginning of class one day that he was finding that the homework took a very long time to complete each night. He told me that it might take him as much as twenty minutes to search through his notes and the textbook in order to locate a worked example that was similar to the problem he was trying to solve. He would then use this "master example" as the model from which to generate the directions for completing the homework problem.

Such master examples were especially useful during the times when students were beginning practice with unfamiliar expressions. The practice problems that were assigned following each lecture usually contained expressions very similar to the lecture examples. During one class period I observed Donna as she worked through the first problem on a practice sheet. Her lecture notes were open beside her and her left index finger was following line-by-line through a worked example as she wrote down each step in the practice problem.

Bob expressed this dependence on access to master examples one day when he complained to the instructor about the textbook. "They never show you how to work one," was his comment. Master examples provided the necessary cues for manipulations when students were unable to otherwise identify directions within the symbol patterns they perceived.

Students who relied on master examples for providing directions for the manipulations of expressions were able to work through practice problems and assigned homework as long as they could locate appropriate examples to follow. However, test situations required such students to commit their collection of master examples to memory. Valerie said to me at one time, "I can do the homework, but when it comes to the quiz, I just can't remember." Wanda expressed the same sentiment when she remarked, "You think you know it until you take the quiz."

2-1 Bob had a question on the differences in notation between the instructor's lecture and an example in his book. He was explaining to her, "...in the book, only written backwards."

He showed the instructor the expression $(x + 2)^2$ written in the book. He perceived this as different from the expression $(x+2)(x+2)$ which the instructor had written.

The instructor told him they were the same.

Bob: "You said to be technical."

Instructor: "Either way is OK."

Bob: "Then either way is alright? The book writes them differently."

Bob perceived the two expressions $(x+2)(x+2)$ and $(x+2)^2$ as different, while the instructor perceived that they were the same. This interaction illustrates the different meanings that can be attributed to an algebraic expression by two different individuals.

When the instructor compared the two expressions, she focused on the multiplication operation expressed in two ways by the pair of parentheses and the presence of the exponent. She assigned the same meaning to two different symbol patterns. Bob, however, was correct from his learning system, that the two expressions were different collections of algebraic symbols. He expressed this as, "You said to be technical."

Part of his questions may have been the result of a concern for arriving at a unique "correct" answer. Such an answer is established by the specific format of an expression given in the answer section of a textbook, or by the form of an answer assigned to a multiple choice question on a test. Bob's criterion for establishing the similarity of the two expressions was based on the physical arrangement of the symbols, rather than on the mathematical relationship that could be attributed to this arrangement.

Summary

Students relied on the physical arrangement of symbols, and the presence of particular numbers or symbol patterns as a method of identifying specific types of expressions. These criteria were used to identify pre-worked master examples that could then be used to provide the correct sequence of manipulation steps for solving new problems. In this way, students could manipulate algebraic expressions in mathematically correct ways without having to understand the operations they were mimicking. This learning strategy relied on developing an awareness of the similarities and differences in the physical appearance of different collections of algebraic symbols.

Directions for Manipulations

1-2 The instructor performed the following operations.

$$\begin{array}{r} x - 6 = 14 \\ +\ 6 \quad +\ 6 \\ \hline x + 0 = 20 \\ x = 20 \end{array}$$

Instructor: "How do I know I did it right?"

Tom: "We took your word for it."

The instructor had performed the operation of adding six to each side of the equation because of the meaning she attached to the expression $x - 6 = 14$. She associated such a symbol pattern with an equation, and its solution with the rule expressed by the addition principle. She also attributed meaning to the answer of $x = 20$ in terms of its relationship to the original equation.

When she asked, "How do I know I did it right?" she was really testing to determine whether students attributed the same relationship as she did to the answer and the original equation. She anticipated that students would volunteer the information that the final step in the manipulations would be to substitute x = 20 into the original equation in order to check the answer.

Tom did not assign any of this meaning to the steps of the problem. He answered the instructor's question from within his own learning system. He perceived that the rules of manipulation and/or the final answer to a problem are established by authority, such as through the pronouncement of a teacher. His answer was a humorous way of stating that he had no other reason to assign to the manipulation that the instructor had performed.

1-5 The lecture topic covered the factoring of trinomial expressions. The instructor had worked through the following and wrote as her answer:

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

Valerie: "Can the final answer be written as (x + 3)(x+2)? Is there a right way?"

3-3 During the lecture, the instructor was demonstrating a technique for multiplying radical expressions. She set up a problem similar to:

given: $(\sqrt{5} + \sqrt{3})\ (\sqrt{5} - \sqrt{3})$

you set it up as:
$$\begin{array}{r} \sqrt{5} + \sqrt{3} \\ \underline{\sqrt{5} - \sqrt{3}} \end{array}$$

etc.

When the instructor was finished with the problem, Valerie asked: "Does it matter which binomial goes on the bottom?"

In both of the above examples Valerie was trying to establish a rule for the appropriate formatting of problems involving binomial multiplication. Once a rule was established, it could then be connected to a specific type of symbol pattern for future use. Her question "Is there a right way?" also reflected a concern held by some of the students for arriving at a "correct" answer.

Valerie had discussed the importance of a "correct" answer with me earlier in the course. She had previously taken a mathematics course at the university that used the tutorial-assisted approach. Testing in this course was done by computer scoring of multiple choice answers. On these tests, only a specific format was acceptable for each answer. Valerie needed to establish whether such restrictive conditions on answers would apply in the present course.

1-1 The topic was an introduction to scientific notation. During the practice session, students were asked to work through problems such as:

$78{,}000 = 7.8 \times 10^4$ and $0.0000057 = 5.7 \times 10^{-6}$.

Tracy was having trouble switching from one type of expression to the other. She was trying to formulate a rule that would give her the correct conversion format. She was muttering to herself, "right - negative, left - positive." I tried to show her how the sign in front of the exponent reflected whether the original positive number was greater or less than one. Tracy did not try to follow my reasoning, and when I left her she was at work generating a rule that held meaning for her.

The above example is a good illustration of the differences in type of thinking that can exist between an instructor and student. My explanation was a very workable one from my point of view. It allowed me to make sense out of two mathematical expressions of very similar format. My rule, however, was expressed in terms of the concepts I attributed to the exponential notation. Instead, Tracy was relating the right-and left-handedness of the movement of the decimal point to the resulting sign of the exponent. For each of us, our self-generated rule carried meaning, and allowed us to manipulate such expressions on future occasions.

Summary

The students demonstrated their reliance on the use of "rules" for determining the appropriate set of manipulations to be used in any given problem. Such rules were perceived to be established by authoritarian pronouncement, rather than as being the result of mathematical relationships and principles. These rules were related in many of the students' minds with the physical appearance of the collections of symbols, and the physical changes that occurred as these symbols were manipulated.

Familiar Format

One of the learning strategies employed by students was to search for a way to relate new material to information that they had used before. Information that could be linked to familiar patterns then became more accessible for use. Students also preferred to use manipulation techniques in the same format as they were when originally learned.

2-2 The lecture topic covered techniques for finding the slope of a line given two points on the line. Earlier in the day, students had been introduced to a method for graphing straight lines. They had spent some time finding solutions to equations for straight lines and plotting these values in order to draw the lines.

During the present lecture, the instructor had used the formula for finding the slope of a line for a particular example and had calculated a value of -8.

At this point Bob asked the instructor where this would be plotted on the graph. The instructor replied that they were just finding the answer to the problem at the moment.

Bob was trying to establish meaning for new information in terms of related, previously encountered material. Since the earlier exercises that day had involved plotting numbers on a graph, Bob tried to use this format for assigning meaning to the unfamiliar value of the slope. The instructor was not able to follow Bob, or determine his point of reference from the questions he asked. She was able to assign meaning to the number that she had calculated in terms of the symbols used in that calculation. Since Bob did not yet understand this meaning, he was attempting to relate her actions to the types of problems that he had manipulated earlier in the day.

2-3b The lecture topic dealt with equations for straight lines. The instructor performed the following manipulations in order to rewrite the given equation in another format.

$$y = -3/5x + 1$$

$$(5)y = 5(-3/5)x + 1(5)$$

$$5y = -3x + 5$$

$$5y + 3x - 5 = -3x + 3x + 5 - 5$$

$$5y + 3x - 5 = 0$$

Shirley: "Oh, so you just combine them so they add up to zero? Can you do another one?"

The instructor began to remove the transparency from the overhead, but several people said, "Wait, wait!"

Sue: "You should do it with the subtraction underneath."

Instructor: "Maybe we should redo this one in columns." She then proceeded to write the following.

$$y = -3/5x + 1$$

$$(5)y = 5(-3/5x) + (5)1$$

$$\begin{array}{rcl} 5y &=& -3x + 5 \\ \underline{+3x} && \underline{+3x\quad} \end{array}$$

$$\begin{array}{l} 3x + 5y = 0 + 5 \\ \underline{-5 \qquad\qquad\quad -5} \end{array}$$

$$-5 + 3x + 5y = 0$$

Instructor to Sue: "Is that easier when it is in columns?" Sue nodded her head.

Students had been exposed to a technique for manipulating the terms of linear equations of one variable the week before. During these lectures the addition and subtraction of equal values from each side of the equation had been demonstrated in a vertical format, and students had used this form in practice work. Once the instructor used this more familiar format, the students were able to follow the present lecture more easily.

2-4b The lecture began with the instructor writing the following:

$$\frac{5}{12} + \frac{7}{30} = \frac{5}{3\ 2\ 2} + \frac{7}{2\ 3\ 5}\ .$$

Tom: Why do you write them like that?"

Jane: "Because it's confusing."

The instructor wrote:

$$\begin{aligned} 12 &= 3\cdot 2\cdot 2 \\ 30 &= \underline{3\cdot 2\ \cdot\ 5} \\ &\ 3\cdot 2\cdot 2\cdot 5 = 60. \end{aligned}$$

Tom said something about using "trial-and-error" when finding the least common denominator.

Instructor: "The reason why I don't want to do this by trial-and-error is I am going to do this with polynomials and this gets too hard that way."

She then wrote:

$$\frac{5}{12}\cdot\frac{5}{5} + \frac{7}{30}\cdot\frac{2}{2} = \frac{25}{60} + \frac{14}{60} = \frac{39}{60}\ .$$

Bob: "You're losing me quickly!"

Instructor: "Where did I lose you?"

Bob: "Way, way back."

Shirley: "Why can't you just write 5/60 and 7/60?"

Instructor: "Is it OK, getting the LCM of 60?"

Bob: "Let 'er rip! OK so far."

Donna then asked the instructor if it wouldn't be easier to do the problem the way they were taught in grade school.

Jane: "Back in the buffalo days?" There was a big laugh.

Shirley: "12 goes into 60, 5 times. 5 times 5 is 25." Then she added loudly and jubilantly, "Aaaah! Sixth grade pays off!"

The instructor then set up the problem vertically.

$$\frac{5}{12},\frac{5}{5} = \frac{25}{60}$$

$$\frac{7}{30}.\frac{2}{2} = \frac{14}{60}$$

$$\frac{39}{60}$$

In the example given above, students felt much more comfortable with the instructor's symbol manipulations when the expressions were rewritten in a more familiar format. Some of the confusion during the lecture probably came from the students' general dislike of working with fractions and their difficulty with the techniques for finding the least common denominator. Shirley indicated by her question that she had not followed the step in which the instructor had converted 5/12 to 25/60.

It is interesting to note that as soon as Donna referred to the method of combining fractions used in grade school, Shirley was able to verbally describe a method for converting 5/12 to the appropriate number of sixtieths. She sounded amazed that her past mathematical schooling could have any use to her as an adult. Shirley was able to connect the appropriate mathematical techniques to the problem at hand, not through an understanding of the mathematical relationships involved, but through an association of ideas connected with her past school environment.

The students were more comfortable using the operations of addition and subtraction with fractions when the operations were presented in a vertical format. Once a familiar format was used, students were then able to follow the operations used by the instructor. The rules for operations on symbols seemed to be linked to a particular physical arrangement for each expression.

4-1 I was helping Shirley with a problem on her paper. Part of the solution required her to calculate the following:

$$-\frac{5}{2} + \frac{9}{4}.$$

She wrote: $-5 + 9 = 4$, then said, "Oh! Wrong way," and proceeded to write:

$$\begin{array}{l} -\frac{5}{2} \\ \\ +\frac{9}{4} \end{array} = \frac{4}{2}.$$

This incident occurred almost a week and a half after the class discussion using "sixth grade" mathematics described above. Shirley had remembered the importance of formatting the problem vertically, but had not been able to link this with the technique for finding the least common denominator. She had carried forward from the earlier lecture the physical placement of the symbols, but had not remembered the technique to be used when converting a number to an equivalent fraction that she had recited so glibly at that time.

3-1a The lecture topic was an introduction to a long division algorithm for dividing polynomial expressions by a binomial. The instructor worked through the following example.

$$\begin{array}{r@{\,}l} & \;y - 4 \\ y - 3 & \overline{)\, y^2 - 7y - 9} \\ & \underline{\;\; y^2 - 3y \phantom{{}-9}} \\ & \; - 4y - 9 \\ & \; \underline{- 4y + 12} \\ & - 21 \end{array}$$

The instructor worked carefully through the example, step-by-step. Then the students split up into groups to work through some practice problems. I worked carefully with Wanda, taking her through each step in the first problem. When we were done she turned to Steve and said, "This is fun!" She then proceeded to do the next problem completely on her own.

Wanda had found the class very difficult. She did not have as extensive a background in previous math classes as did some of the other students. She found new material confusing and usually reworked the lecture examples before she attempted the practice problems. Her comment, "this is fun!" was very unexpected.

I had anticipated that the students would have had difficulty proceeding through all the different steps using the division algorithm. Each successive step required a different operation of either division, multiplication or subtraction. However, once the sequence was explained, the students were able to proceed through the problems with little assistance. What I had not realized was that all the steps in the problems were identical to the division algorithm they had learned in grade school. Once the students got past the use of letters instead of numbers, they were on familiar ground.

Summary

Again, the students' ability to correctly manipulate algebraic symbols is observed to be related to the spatial, or physical arrangements of the symbols that are used in each expression. The examples cited in this section also illustrate the ways in which the students associate rules with the physical attributes of the symbol patterns, rather than with the mathematical concepts represented by the algebraic expressions. The students also demonstrated their need to link new information to knowledge systems that they had previously developed. The students relied on a familiar symbolic format as a way to assign meaning to new information.

Differences in Type of Algebraic Thinking

4-2a Tracy was working through the assigned practice problems during group work. She was complaining about how hard the work was and suggested that

> she could just make up a rule and use it. She said something like, "I can just make up a rule and call it algebra, right?"
>
> When I heard this, I was immediately struck by the image Tracy had created of a system of algebra that consisted of a collection of strange, arbitrary rules that have to be memorized and used in unrelated ways. My reaction to such a system followed quickly as I said to myself, "But that's the beauty of algebra; the meaning behind why we do things."

This interaction illustrates the different perceptions of the subject matter that were held by various participants in the class. Tracy and I were each being exposed to the same topics of study throughout the course. However, our approaches to algebra were not the same. We were each assigning different meanings and interpretations to the objects of study and the ways in which these objects were used in the learning environment. The following examples illustrate other situations in which the students and the instructor were observed to be assigning different meanings to the same subject matter.

2-4c The lecture topic covered the addition of fractional expressions having different denominators. The instructor had written the following:

$$\frac{x}{x^2-y^2} + \frac{y}{y^2-x^2}\cdot\frac{(-1)}{(-1)} = \frac{x}{x^2-y^2} + \frac{-y}{x^2-y^2}.$$

Shirley: "Why do you know how to multiply by minus one?""

Instructor: "Any time our denominators are additive inverses of each other."

> Shirley did not understand this. The instructor tried to explain, and finally said, "Today the denominators will either be equal or additive inverses of each other, and then you multiply by minus one."
>
> Shirley: "Tomorrow they'll be big numbers."

To the instructor, the expression "additive inverse" implied that one number was the negative of another. From her perspective it was an appropriate answer to Shirley's question. Given the different types of meaning attributed to the symbols by the instructor and the student, it was difficult for the instructor to formulate an explanation that would convey meaning to both herself and the student at the same time. Her final explanation that this was the method that would work for that day's assignment was her attempt to communicate "on the spot" without referring to any mathematical concepts or relationships.

One of the perceptions of algebra occasionally expressed by the students was that the learning experience consisted in an exposure to a large collection of disparate examples, each with its own separate set of rules of manipulation. Each new topic presented a new rule to learn. Tracy had expressed this perception when she commented, "I can just make up a rule and call it algebra."

Shirley may have been expressing a similar feeling when she stated, "Tomorrow they'll be big numbers." She was saying that today they would learn one thing, and tomorrow it would be something else.

2-4d The lecture topic covered the simplification of fractional expressions containing common factors. The instructor had written the following:

$$\frac{7x^2(6a+2)}{14(a+2)} = \frac{7\cdot x\cdot x\cdot 2(3a+1)}{7\cdot 2(a+2)}.$$

Shirley: "Can the a's cancel?"

Instructor: "Good question. I can cancel factors, but I can't cancel terms. Basically, when someone throws in a plus or a minus, I can't divide. Only if there is a times or a divide."

Shirley: "Oh, yes."

In this exchange the instructor also began her explanation in terms of mathematical concepts. To the instructor, the expressions "factors" and "terms" were abstractions which carried with them certain rules of manipulation. However, to Shirley, these words did not convey this meaning. The second explanation was more successful because the instructor was able to provide a rule in terms of the symbol pattern of the given expression. This presented the explanation within a context Shirley was familiar with.

2-3c The lecture topic covered techniques for solving two equations with two unknowns. After working through an example, the instructor demonstrated the importance of substituting the answers back into one of the original equations in order to check their correctness.

Shirley: "What if the check doesn't give you 5=5?"

Instructor: "Then you made a mistake."

Shirley: "Oh, they should always come out equal?"

This exchange illustrates the dilemma of trying to conduct a conversation between people who use two different types of algebraic thinking. Shirley asked a question that implied a "why?" to the instructor. Yet the instructor's answer, while conveying a great deal of information about the concepts she attributed to equations, did little to help Shirley. Her reply indicated that the two were still not communicating on common ground.

When Shirley asked, "What if the check doesn't give you 5 = 5?" she was probably expecting an answer in terms of, "Then you need to do thus-and-so." Instead, the instructor interpreted Shirley's question to refer to the mathematical concepts involved in the operation of checking the solution of an equation. Part of this concept expresses the relationship of an answer to the original form of the equation. When she said, "Then you made a mistake," she was implying this relationship.

Shirley's reply indicated that she did not attribute these kinds of meaning to the manipulations of checking the answer. She was surprised by the fact that the check must "come out equal."

Other Examples

The students were under heavy pressure throughout the four week course. They were exposed to four new topics each day, which gave them no time to become comfortable with any one technique. They were constantly having to confront new and strange symbol patterns and learn new rules of manipulation. Consequently there was very little opportunity to observe students working with material that was familiar and comfortable to them. However, on occasion, a student demonstrated his or her ability to function creatively, or was observed working within an area with which they were familiar.

2-3d The lecture topic covered techniques for deriving equations of straight lines when given coordinate points and values of slope. Following the lecture, the students had difficulty on the practice problems that required substitution into the general formula y = mx + b.

Steve devised a way to help him make sense out of all the symbols in the following problem.

Find the equation of the line with slope of -4 passing through the point (3,2).

On Steve's paper he had written:

$$m = -4$$

$$y = mx + b \qquad (3,2)$$

I was impressed with this technique for keeping all the substitutions straight and showed it to Sue who was having trouble with the same problem.

Sue turned to Steve and said, "Steve, you're a genius."

3-2 During group work, Shirley was working on a problem that asked her to find the average of the two numbers 0.678 and 0.6782. She punched the appropriate buttons on her calculator and got an answer of 0.6781. She pointed out to Tracy that this answer was true since 1 came between 0 and 2.

Shirley exhibited her ability to work confidently with numerical symbols. She was able to go beyond the symbol manipulation and explain her answer in terms of the abstract relatedness of the numbers in question. This ability to relate an answer back to the original problem and evaluate its validity was not demonstrated when students were manipulating algebraic rather than numerical symbols.

3-1b The students were working on an exercise designed as a problem-solving activity. The instructions were: to guess at the answer to a problem and use a calculator to evaluate the guess. From this trial, a second, better educated guess could then be made. The following was one of the problems.

I have a number in my head. First I add 2 to the number. Then I subtract 5. Then I multiply that answer by 10. My answer is 5. What is the number I am thinking of?

Donna proceeded to solve this problem by beginning with the answer. She worked backward, reversing all the operations and so arrived systematically at the correct number.

Donna found the material in the course very demanding and difficult. She required help getting started on practice problems following lectures, and was one of the slower workers. When confronted with numbers rather than algebraic symbols, however, she demonstrated her ability to handle a numerical problem with great ingenuity. Her method of solution demonstrated that she understood the behavior of the mathematical operations of addition, subtraction, multiplication and division when they were applied to numbers.

Students' Use of a Notational Focus

The interactions that were observed were taken from learning situations. On these occasions, students were being exposed to and working with new information that was unfamiliar to them. Unlike the protocol research of cognitive scientists, the students were not expected to be able to work through a given problem, talking through their solution as they worked. These students were first exposed to lecture examples illustrating the new techniques and then provided an opportunity to "practice" these manipulations on similar problems. Therefore, the discourse reported in the interactions cited in this chapter represent a different set of mental activities from those investigated by cognitive science research. It is postulated that these interactions represent parts of the learning process.

The examples of class interactions cited in this chapter were selected because they serve to highlight various learning strategies used by the students. The conversations between the students and the instructor were recorded along with many other interactions throughout the course. The significance of the examples selected as illustrations of student learning strategies was not apparent at the time notes were taken, but emerged through a careful analysis of all incidents in the field notes. Taken together, these interactions reveal many of the creative, purposeful learning strategies developed by the students to deal with overwhelming amounts of new and strange material.

The students used these learning strategies as a way to cope with the daily class requirements. In the interactions that were described in this chapter, the students were observed to be focusing on the notational aspects of each collection of symbols. They used this information as a way to assign meaning to the problem at hand and to manipulate the given algebraic expressions. It was apparent from the discourse reported, that the instructors were assigning different types of meaning from the students to these same collections of symbols.

The students and the instructors were each operating at a different level of conceptual understanding related to the mathematical information that was represented by the algebraic symbols under discussion. Since the students did not have appropriate prerequisite knowledge to follow the lectures at the same level of understanding with which the instructors were operating, they developed learning strategies based on the notational representations of each symbol collection in order to meet the manipulation tasks of the assignments.

As long as the students had access to appropriate master examples, they were able to produce mathematically correct homework assignments, and to correctly answer group quiz questions. If the only criterion for success in this class had been an analysis of the students' written work, then many of the students would have demonstrated the appearance of an appropriate mastery of many of the topics covered. It is apparent, however, that this mastery was made possible by the use of rote-memorization of notational changes in symbol patterns, and was not based on an understanding of the mathematical relationships and concepts that these symbols represented.

I was not aware at the time I observed many of the interactions reported on, that the students were assigning a notational type of meaning to the activities they engaged in. It was not until I assembled all the examples cited in this chapter, and began to examine and categorize the behavior of the students, that I became aware of the differences in meanings that the students and instructors were assigning to the same objects of study.

I had been making the erroneous assumption that the students would be able to develop the same sets of meanings for each example as that used by the instructor, simply by being exposed to information contained in lecture examples. The analysis of the student interactions shows, however, that the students did not have an adequate mathematical background to communicate with the instructors within the same conceptual level. They were only able to assign meanings to the lecture topics on the basis of their existing levels of knowledge, which did not extend beyond an awareness of the physical attributes of the symbol notation used.

The significance of the students' use of a notational focus during their work with new algebraic material will be discussed further after the results of the observations of the second summer class are reported in Part II. In Part III, the findings from the two classes will be compared and contrasted in terms of the levels of mathematical meanings that each group of students was able to assign to the subject material. These behaviors will be related to the mathematical nature of the subject of algebra, and to various theories describing mathematical concept learning. These analyses will be used to generate theory to explain the observed behavior of the students in both summer classes.

PART II

SECOND SUMMER'S REPORT

CHAPTER FIVE

DESCRIPTION OF SECOND SUMMER CLASS

Introduction

The special algebra class for Native American students was offered a second time during the summer of 1987. I participated in the class as an assistant instructor, and also collected observations during each class period, as I had the previous year. The purpose for continuing the study for a second year was to obtain additional information relating to the development of students' mathematical conceptualizations, to obtain information relating to the mathematical contexts in which this type of thinking took place, and to provide an opportunity to observe whether the conclusions drawn from the analysis of the first summer's records would remain valid when tested against further data from a similar situation.

The focus of the first summer's observations centered on interactions between students and the instructors in which some form of verbalization usually took place. Many of these instances occurred during lectures. However, since the actual discourse of these lectures was not recorded, it was not possible to place these question and answer exchanges within the mathematical contexts from which they arose. A primary reason for continuing observations for a second year was to obtain transcripts of each lecture session in order to provide more comprehensive data on the classroom learning environment. A focus of the analysis of the records from the second summer was to be centered on the comparison of the observed students' relevant mathematical behavior to the particular instructional contexts of each day's activities.

Description of Second Summer Class

The second year's algebra class consisted of eight students. Four students were single, or living in town without their families. The other students either had children, or children and spouses living with them. Five of the students were older than the traditional college age. There were four men and four women in the class. All the students except one were Native Americans with ties to tribes in Montana, South Dakota or Wyoming. One student was non-Indian, but was married to a Native American. Two of the students were transferring from other colleges, and one student was entering the university as a freshman the following fall. The other five students had attended the university prior to the summer class and had all enrolled in at least one previous math class on campus.

The principal instructor remained the same for the second year's class. I performed the same duties as an assistant instructor as I had the previous summer, providing one-on-one help to students during practice periods and conducting lab activities twice a day. Because the enrollment in the second class was smaller than the first year, a second assistant instructor was not employed. The evening tutor for the second summer was a Native American student from the university with a minor in mathematics who was married to one of the students in the class.

The class again met in the Native American Club Room on campus. The course syllabus remained essentially the same as that used the previous year. The textbook was the fifth edition of <u>Introductory Algebra</u> by Keedy and Bittinger. The instructor used the lecture examples, supplemental handout sheets, and homework assignments from the first class.

Both the instructor and I were now teaching the class for the second time. I had shared my observations and analysis of the data from the first summer with the instructor prior to beginning instruction the second year. We therefore made some alterations in the topics that were covered and the way in which some of the material was presented in class, based on the difficulties that I had observed that students had had with this material during the first year.

The length of the course was extended to five weeks, one week longer than the duration of the first course, in order to provide more class time to cover the material. Thus, it was possible to introduce three, instead of four new topics each day. The afternoon problem-solving labs that were used in the first year were eliminated. Instead, I generated a series of lab activities related to the daily topics, which were presented during two 30-minute lab periods each day. The arrangement of the daily class activities remained the same as in the previous summer. Students were again divided up into groups of four to provide support groups during the periods when they worked through practice exercises.

Differences Between Classes

The eight students in the second class exhibited a different set of behaviors from the students who formed the focus of observations for the first summer. The students in the second class did not rely on the use of master examples as they worked through practice problems. Neither did these students require as much help from the instructors, as had the students the previous summer, as they begin working through problems following each lecture.

The students in the second class appeared more confident in their ability to manipulate symbols than did the group of students from the first class whose learning strategies were described in Part I. In this respect, the eight students from the second summer resembled those students from the first summer that I did not observe very frequently. My attention the first year was focused primarily on those students who asked the most questions and required the most help during practice sessions. These were the students who provided the observations from which were drawn the conclusions that were reported in Part I.

During the first summer, the instructor and I had determined that, as a group, the students were not ready to respond to individual quiz questions until the end of the second week of classes. Even after the students were given questions to respond to individually, they were still permitted to use their notes and text while answering such questions. In contrast, the students from the second summer exhibited such confidence in their ability to respond to questions on an individual basis that they were given individual quiz questions on the fourth day of the first week of class. They were also prohibited from using any outside source of information except a calculator while they were taking these tests.

Data Collection

The differences in behavior between the first and second year's students affected the types of records that were collected during the second year of observations. During the first summer course, only those classroom interactions that were accompanied by student verbalizations were selected for analysis. Since at that time, students were observed to be using master examples and memorization techniques to perform algebraic manipulations, the presence of a mathematically correct solution to a problem was not interpreted, by itself, to be an indication of a student's ability to understand material at an appropriate mathematical level. Such work did not provide an adequate indication of this type of understanding unless the student provided some insight into his or her set of mental structures through some form of accompanying verbalizations.

I did not observe many instances during the second summer in which students asked questions during the lectures or engaged in verbalizations during practice sessions. It was necessary, therefore, to identify a different source of information that could be used to infer the students' mental processes and mathematical conceptualizations.

The students in the second year's class did not rely as heavily on direct copying techniques. Because these students did not avail themselves of master examples as they manipulated collections of symbols, they were observed to make more errors in their solutions to mathematical problems. It was postulated, therefore, that the students' written work that was produced under these circumstances, even without accompanying verbalizations, could be used as a source of information from which to infer the types of mathematical thinking that students employed during the course.

Records were collected of the students' written work that was produced in situations where students did not have access to such external resources as master examples or group produced work. The primary source of such information was obtained from photocopies of the students' individual quiz papers. Few examples were collected of the student's group quiz papers or daily homework assignments, since such work was produced in situations where students had access to many external resources and could avail themselves of pre-worked examples.

Each lecture was tape recorded in order to obtain a record of all the verbalizations that occurred during instructional periods. An exact reproduction of the information that the instructor wrote on each transparency was also included in the field notes. Those lectures that were later used for analysis were transcribed from the tapes, and the written material projected on the screen was placed within these transcripts to coincide with the instructor's narrative. These records were used to examine the mathematical learning environment that the students were exposed to in class.

Individual Quizzes

The copies of the individual quizzes provided the primary record of student behavior that was subsequently used to investigate relevant mathematics learning. These tests were administered on a daily basis starting with the fourth day of class. While taking the individual portion of the daily quizzes, students were not permitted to use any kind of outside resource except a calculator. The students were seated in chairs, facing the lecture screen, that were separated far enough apart that it was difficult for the students to see each other's papers. Sufficient time was provided during the half hour before lunch each day in which to answer all questions. The quizzes were usually graded during the afternoon session and handed back to the students before the end of the day.

In general, the questions on the individual portion of the quizzes covered material that had been presented in class at least two days prior to that test. Since students did not have access to their notes during individual quizzes, the instructor decided that it was inappropriate to test material that students had been exposed to only the day before.

A great deal of information was presented during the five week course. Each quiz, therefore, usually covered a different set of topics. This meant that each topic did not receive a great deal of exposure from re-testing on quizzes. Consequently, there were usually no more than three or four questions that were asked throughout the course on each topic. There were three "super quizzes" given on three separate Friday mornings. These were summative tests of twelve to fifteen questions that tested much of the material that had been covered during the preceding two weeks. These quizzes included both a group and an individual part and took the place of a single, comprehensive final exam.

Very little analysis was carried out on the data during the five week period that the class was in session. However, the students' written quiz responses were examined and compared to each other each day immediately after the quizzes were graded and before they were handed back to the students. This preliminary examination verified that the students usually produced a wide range of worked solutions to each problem and that these student responses would provide a rich source of information for investigating the types of mathematical thinking that students employed. Copies of the daily group quizzes were not made, since the responses to those questions did not reflect each individual student's own thinking. The preliminary review of the group quizzes indicated that the responses for all papers in each group were usually identical.

Not all of the quizzes were copied before they were handed back to the students during the afternoon sessions.

However, photocopies were made of thirteen different individual quizzes. This included a total of 110 different questions. A detailed analysis of these quizzes was not made until after the course was completed and all quizzes were assembled.

Analysis of Students' Quiz Responses

After the course was finished, the quizzes were initially analyzed by copying each separate problem onto an index card, and grouping these cards according to the mathematical content of each problem. This generated a list of problems dealing with equations, factors, fractional expressions, straight lines, simplification of fractions, powers, and radicals, inequalities, formulas, and parabolas.

Since the individual responses to each quiz question were to be examined as part of the analysis, it was necessary to narrow the field of investigation from the original 110 problems that were collected. A preliminary analysis of the types of questions in each category and the types of responses that were given to these questions was undertaken to select a single category to be used for further analysis. On the basis of this general survey, the category of equations was selected for an in-depth study.

An additional reason for selecting those problems that dealt with equations was based on my observations of the fact that the students in both classes had exhibited difficulties in mastering the appropriate types of checking procedures for each type of equation presented in the course. I was interested in using the collected examples of the students' written work to investigate the types of mathematical conceptualizations that the students were able to develop concerning each of these checking processes.

The focus of the analysis of the quiz responses was, therefore, centered on an examination of the students' use or non-use of checking. A total of 19 problems in the category of equations were selected from the collected pool of quiz responses for further study, using the criterion that each problem selected would contain some form of a checking response by at least one of the eight students in the class.

A preliminary examination was made of each student's checking response for each problem, and a classification was made as to whether the student did, or did not include a check, and whether, or not the check was mathematically correct. Table 11, Appendix B presents a listing of student checking responses using the above classifications. The 19 problems to be analyzed were then grouped according to the type of equation that each represented. This produced the following categories; linear, simultaneous, fractional, radical, and quadratic equations.

It was postulated that the conditions under which students provided answers to individual quiz questions were such that their written responses could be used to infer each student's set of existing mental structures. It was further postulated that the presence of mathematical errors in these responses would provide a rich source of data that could be used to reconstruct the particular mathematical conceptualizations that students applied to specific problems. These two assertions affected the direction of the analyses that were then performed to study the students' mathematical understandings of the checking process.

The analysis of the 19 quiz questions was carried out on the basis of an examination of the individual responses to selected problems. Broad characterizations of group behaviors were not investigated, as interest in this study was focused on developing an understanding of the ways in which individual students were thinking mathematically. Such a focus required that each quiz response be studied independently and regarded as a separate item of information.

Since one of the goals of the study was to investigate the teaching/learning interaction, it was important to present the analysis of each response within the instructional context relating to the topic being tested. Pertinent information was obtained from the comparisons of the ways in which each student's response varied from similar mathematical procedures that the instructor had presented as lecture examples. Another aspect of this analysis was concerned with noting changes in each student's behavior over time and in response to subsequent instruction. Such information was used to present conclusions regarding learning processes and the ways in which students developed concepts over time.

The above criteria for analysis, coupled with the qualitative nature of the data collected, precluded the use of summary descriptive statistics as a way of characterizing the findings of the analysis. Since the ways in which each individual student selected and manipulated various sets of mathematical procedures on each quiz response provided important information relative to the goals of the study, the report of the analyses performed on these data, and the report of the findings of these analyses are presented in detail in the following chapters.

Organization of Part II

The student responses that used some form of a checking procedure were examined in detail for each type of equation. Each response was compared to the standard mathematical procedure used for checking that particular kind of equation. Chapter Six presents an explanation of the different types of checking procedures that were introduced in class. For each student response, the differences that existed with the standard set of manipulations were carefully analyzed in order to determine the types of mathematical thinking that could be represented by that particular choice of manipulations.

Quiz responses were examined in this way in order to identify patterns of student behavior that existed as a function of each type of equation. The results of these analyses are presented in Chapters Seven through Eleven. All of the different responses written by each individual student were also analyzed as a single group in order to identify patterns of individual mathematical thinking. The results of this analysis are presented in Chapter Twelve.

Transcripts were made of those parts of each classroom lecture that involved narratives relating to the use of the checking procedures. These transcripts are presented at the beginning of each of the chapters that describe the analysis of the checking response for the different types of equations. Descriptions are also included in these chapters of all of the occasions throughout the course on which each type of checking procedure was discussed or practiced by the students. The lecture transcripts and the lists of activities are used to provide information concerning the mathematical contexts in which the students were operating. Table 12, Appendix B presents a chart describing the time frame of the various checking activities that occurred throughout the class.

The transcripts dealing with the descriptions of the various checking procedures were analyzed in order to identify the types of mathematical information that was made <u>explicit</u> in each lecture. Information that was present in <u>implicit</u> form was inferred from the types of examples and procedures that were used in each lecture. The analysis of these transcripts is presented in Chapter Thirteen.

The explicit and implicit forms of mathematical information formed the basis for the construction of two separate lists of characteristics that can be used to describe the concept of checking. Student quiz responses were then compared to these lists in order to match student behavior against both the explicit and implicit information that was made available to the students within the learning environment. This analysis is presented in Chapter Fourteen.

Chapters Six through Thirteen that present the different analyses of each type of equation, the individual students' performances, and the lecture transcripts involve a certain amount of overlap of information. These separate analyses represent the successive steps that I undertook in order to try to understand each student's behavior in relationship to their learning environment. It was necessary for me to analyze the quiz responses from these different perspectives in order to arrive at the specific conclusions that are reported in Chapter Fourteen. The chapters in Part II are presented in the order in which my analyses took place and provide a record of the evolution of my thinking processes.

CHAPTER SIX

CHECKING

The analyses performed on the 19 quiz problems indicate that the eight students in the second summer class were not able to understand the concept of checking at an appropriate mathematical level. Five of the students checked at least some of their work. However, they did not employ the checking processes in a consistent fashion nor did they always use the appropriate mathematical steps when they did check their answers. The three other students in the class did not use checking with any of the 19 problems examined.

It is apparent from the analysis of the lecture transcripts that the class instruction did not include sufficient material to enable students to develop an appropriate understanding of the concept of checking. The students in the class were unable to assimilate much of the information presented through the lecture examples because of the disparity that existed between the subject matter and their own levels of mathematical knowledge. Because of this, many students used the various steps of the different checking procedures in mathematically inappropriate ways on their individual quizzes.

The purpose of this chapter is to present descriptions of the various types of checking processes that were introduced to the students throughout the course. These descriptions form sets of standard mathematical manipulations against which the students' written quiz responses are compared. The levels of mathematical discussion included in these descriptions is, in most cases, above that which was presented to the students. However, this information is presented here in order to emphasize the complex nature of the subject of checking. The actual mathematical content that was presented to the students will be discussed in the analyses of the classroom lecture transcripts included in the following chapters.

Different Kinds of Checking

The checking process represents a method of establishing a mathematical relationship between a numerical solution set and the equation from which it is derived. This relationship represents an abstract concept that is a fundamental part of the mathematical ideas expressed in symbolic form by algebraic equations. In terms of manipulation procedures, this checking relationship is established by substituting a given number for the variable in a particular equation. If the resulting statement is a numerical identity, then the substituted value is said to satisfy the equation and is called a <u>root</u> or <u>solution</u> to the equation. If the resulting statement is not an identity, then certain decisions must be made concerning the status of the substituted numerical value.

The idea of "checking your solution" in order to establish the reasonableness of an answer or to reflect on the relationship that exists between this solution and the original problem represents another type of checking. The term "checking" is used in this chapter to refer to the processes of substituting the solution into the original form of the equation, and the evaluation of the numerical statement that this substitution produces. The following sections provide descriptions of this type of checking procedure, include discussions of the mathematical relationships that such checking can reveal, and compare the differences in checking for linear and quadratic equations, fractional equations, and radical equations.

Linear and Quadratic Equations

Linear equations always produce one or no solution, while quadratic equations produce one or two solutions that are either real or complex numbers. Checking is used to verify the accuracy of the algebraic and arithmetical manipulations that are used to find these solutions.

If the solution process for either of these two types of equations is mathematically correct, then substituting the answer that is found back into the original equation will always produce a statement of numerical equality as the final step in the check. If however, a mathematically incorrect solution is substituted, the final step in the check will produce a statement that is not a numerical identity.

For example, solve; $5x - 10 = 0$.

Student A	Student B
$5x = 10$	$5x = -10$
$x = 2$	$x = -2$
check:	check:
$5(2) - 10 = 0$	$5(-2) - 10 = 0$
$10 - 10 = 0$	$-10 - 10 = 0$
$0 = 0$	$-20 = 0$

In the above example, student A substituted the value $x = 2$ into the original equation and produced the mathematically true statement $0 = 0$. Student B, however, made an algebraic error during the solution process and arrived at an incorrect answer of $x = -2$. When this value was substituted back into the original equation, the check produced the non-true statement $-20 = 0$. Such a non-equality results because the value $x = -2$ is not a true solution of the original equation but was produced as the result of a manipulation error during the solution process.

The above example illustrates the way in which checking is used with linear or quadratic equations to evaluate the mathematical correctness of the set of manipulations that were used to solve the equation. The final step in the check will always produce a non-true statement when an incorrect solution is substituted for the variable in the original equation. If the solution to the equation, and the checking process itself, is performed without any mathematical errors, the check will always produce a true statement, or numerical identity.

Fractional Equations

The checking process serves a different function when used with fractional equations containing terms with a variable expression in the denominator. Such equations may not necessarily have solutions. The manipulations that are used to solve these equations may introduce solutions that are not roots to the original equation, and the checking process must be used to ascertain which of the solutions are true solutions of the original equation.

For example, solve: $\frac{1}{y - 1} = \frac{y^2}{y - 1}$

$$(y - 1)\frac{1}{y - 1} = \frac{y^2}{y - 1}(y - 1)$$

$$1 = y^2$$

$$y^2 - 1 = 0$$

$$y - 1 = 0 \quad \text{or} \quad y + 1 = 0$$

$$y = 1 \quad \text{or} \quad y = -1$$

check: $\frac{1}{1 - 1} = \frac{1^2}{1 - 1}$ $\qquad$ $\frac{1}{-1 -1} = \frac{(-1)^2}{-1 - 1}$

$\frac{1}{0} = \frac{1}{0}$ $\qquad$ $-\frac{1}{2} = -\frac{1}{2}$

y = 1 is <u>not</u> a true solution $\qquad$ y = -1 is mathematically valid solution

The manipulations used in the first step in the above example transform the original problem into a second equation which is not necessarily mathematically equivalent to the first equation. Such a situation occurs when both sides of the equation are multiplied by an expression that represents the value of zero or when both sides of the equation are divided by a variable expression. In such cases, the solution set for the original equation may or may not be a subset of the solution set for the transformed equation, and all solutions of the transformed equation are not necessarily solutions of the original problem. Therefore, when the answers derived from the transformed equation are substituted back into the original problem, some of them may produce numerical statements which contain fractional terms with zero in the denominator.

In the above example, it can be seen that the substitution of $y=1$ produces the mathematically undefined expression $1/0$, which indicates that this value of y is not a true solution of the original equation. The other answer of $y=-1$, however, yields the numerical identity $-1/2 = -1/2$ in the last line of the check, which indicates that this value is a solution of the original equation. The numerical identity also establishes that this solution represents the results of an error-free set of manipulations.

The checking process was used with the above fractional equation in order to identify the presence of an extraneous root that was created during the process of multiplying each term in the original equation by the least common denominator. This root is a solution of the transformed equation but not of the original problem.

The checking procedure is an essential part of the solution process for fractional equations. Unlike the solution processes for linear and quadratic equations, producing a set of "answers" is not considered to mean that the solution process is completed. It still remains to be established which members of this solution set are true solutions of the original equation. The final solution set for fractional equations is not valid until all answers have been checked in the above manner, and the extraneous roots have been identified and rejected.

It is also possible to determine the existence of extraneous roots for fractional equations by first establishing a system of equations to be solved. In the example presented in this section, it is possible to stipulate that the given equation is valid only if the denominators of each term do not take on the value of zero. This produces the following system of equations.

$$\frac{1}{y-1} = \frac{y^2}{y-1} \qquad \text{and} \qquad y - 1 \neq 0$$

The solutions to this system will automatically reject the one answer $y = 1$ that is generated from the transformed equation during the solution process, since this value is rejected by the second equation $y - 1 \neq 0$.

Radical Equations

Equations which contain the variable as part of a radicand may also produce extraneous roots that must be checked to determine if they are true solutions of the original equation.

For example, solve

$$3 = 2\sqrt{x} + x$$
$$3 - x = 2\sqrt{x}$$
$$(3 - x)^2 = (2\sqrt{x})^2$$
$$9 - 6x + x^2 = 4x$$
$$x^2 - 10x + 9 = 0$$
$$(x - 1)(x - 9) = 0$$
$$x = 1, \quad x = 9$$

Check:

$x = 1$	$x = 9$
$3 = 2\sqrt{1} + 1$	$3 = 2\sqrt{9} + 9$
$3 = 3$	$3 \neq 15$
$x = 1$ is mathematically valid solution	$x = 9$ is not mathematically valid solution

In the above example, the extraneous root $x = 9$ was identified through the checking process and rejected. The other solution of $x = 1$ was retained as a root of the original equation. Thus, even though the solution process generated two answers, only one of them was kept as a mathematically true solution to the original equation.

The technique of squaring both sides of the equation in the above example is used to produce a second equation that can be solved, in this case, by using the solution techniques for quadratic equations. This process of squaring, however, may produce a second equation that is not mathematically equivalent to the first. This is true because the statement $a^2 = b^2$, represented by the transformed equation, does not necessarily imply that $a = b$. The solutions that are derived for the transformed equation must therefore always be checked to determine whether or not they are true roots of the original problem. The solution set for the original problem will always be a subset of the solution set of the transformed equation. However, not all of the answers to the second equation will be true solutions of the first.

Once a set of answers has been obtained, a checking step must be performed by substituting these answers into the original equation, and examining the final step in each check for the presence of a numerical identity. If no manipulation or calculation errors have been performed, the presence of a non-equality in this step indicates that that particular solution is an extraneous root of the original problem. As with fractional equations, the checking procedure becomes an <u>essential part</u> of the solution process for radical equations. The problem is not finished until all extraneous roots have been identified.

Discussion of Checking Procedures

Similarities and Differences

The word check is used in the above sections to refer to two similar sets of manipulations which serve different mathematical purposes with different kinds of equations. The similarities between these checking procedures make it difficult to distinguish their differences, and to determine the appropriate situations in which each procedure is to be used.

Each checking process involves the initial action of substituting a numerical solution into the original form of an equation. If the result of this substitution is a numerical identity, then the checked answer is a mathematically true solution for any type of equation. The differences between the checking procedures are only apparent when the final step in the check does not produce a numerical identity. These differences are due to the mathematical purpose for which the check is performed.

The purpose of checking linear or quadratic equations is to establish the algebraic or arithmetic accuracy of the manipulations that are used in the solution process. Since the solution techniques for these types of equations always produce mathematically valid answers, checking is optional, and only used to determine if manipulation or calculation errors have been made during the solution process.[1]

The purpose of using checking with fractional or radical equations is to establish the mathematical validity of the answers that are produced by the solution process. The results of the checking are used to retain or reject these solutions. Thus, checking is an essential part of the solution process and must always be performed. The presence of calculation errors can also be determined by checking fractional or radical equations. However, care must be taken to interpret whether a non-identity in a check is due to the presence of extraneous roots, or to manipulation errors created in the solution process.

Problems in Implementing the Checking Procedure

The last line in the checking procedure will always produce a numerical identity if a true solution is substituted in the original equation, and no mathematical errors are made. However, it is possible to produce a solution through the correct set of manipulations, but arrive at a non-true statement in the last line of the check by making arithmetic errors during the checking process. It is also possible to check a solution that is

[1]The use of the word optional here does not imply that a more general form of checking should not always be employed at the conclusion of any problem in order to determine the reasonableness of the answer obtained.

mathematically incorrect, but produce a numerical identity in the last line of the check by reproducing the same errors in the checking process that were used to arrive at that particular solution. Therefore, <u>the checking process is only valid if it can be performed without mathematical errors</u>.

Both types of checking must begin with the substitution of the solution in question into the original form of the given equation. This is imperative when checking is used to establish the mathematical validity of solutions to fractional and radical equations. The purpose of this kind of checking is to identify the presence of extraneous roots that are solutions to the transformed equation but not to the original equation. If the substitution is performed into the transformed equation, instead of the original problem, the last line of the check will always produce a numerical identity if the process is error-free. In such a situation, checking becomes simply a method for detecting manipulation errors, rather than a process for identifying extraneous roots.

When checking is used to detect errors of manipulation with linear or quadratic equations, it is also necessary to substitute into the <u>original</u> form of the equation. Substitution of the answer into any other line in the solution process will not detect errors that are made prior to this step. This failure to substitute into the original problem was a common mistake made by many of the students on their quiz responses.

Checking Decisions

The checking procedure requires that a student make certain decisions in a particular sequence about his or her work. First, a student must examine the type of equation to be solved in order to determine if checking must be used to retain or reject solutions or can serve simply as a way to monitor computational errors. Secondly, a student must interpret the statement presented in the final step of the check. The presence of a mathematically false statement requires that the student either reject a solution or review his or her work for errors.

When a non-identity occurs with linear or quadratic equations, it indicates the presence of one or more algebraic or arithmetic errors in either the solution or the checking process. In such a situation, a student must review his or her work in order to locate the source of the errors. The non-identity in the check does not mean that the answer is to be rejected, since all answers, correctly arrived at, for linear and quadratic equations are automatically mathematically valid.

With fractional equations it is not necessary to perform all of the arithmetic calculations indicated by the check. Once the answer has been substituted in the original form of the equation, the checking process can be accomplished by simply inspecting for the presence of zeros in the denominators of each term. If any such zeros are found, then the answer used in the substitution must be rejected as a mathematically valid solution of the original equation.

If no zeros are present, the arithmetic calculations can be completed, and the check can be used to identify manipulation errors. If none were made in either the solution or the check, then the last line of the check will consist of a numerical identity. In this case, the checking process has demonstrated that the answer in question is both mathematically valid and "correct." If the last line in the check is not an identity, then both the solution and checking processes should be examined for manipulation errors.

The checking process for radical equations resembles that used for linear and quadratic equations. However, the presence of a non-equality in the last line of the check may indicate either the presence of computational errors or the existence of an extraneous root. If a student can ascertain that no manipulation errors were made in either the solution or the check, then the non-identity is used to indicate that the answer in question is not a mathematically valid root of the original equation. The presence of a numerical identity in the check indicates that the solution process is mathematically correct and that the solution is also a true root.

<u>Comments</u>

The concept of checking represents a complex set of mathematical relationships that exist between an equation and its solution set. The examples used in this chapter illustrate many aspects of these relationships and show the complicated nature of the information that checking reveals. These relationships need to be understood in order for students to be able to use checking appropriately with various types of equations.

The quiz responses of the students that are presented in the following chapters are used to investigate the degree to which the different mathematical uses of checking were understood by each student. Transcripts of the lectures explaining the various checking procedures are included in these chapters in order to relate each student's set of responses to the level of information that was made available to them in the classroom learning environment.

CHAPTER SEVEN

LINEAR EQUATIONS

This chapter presents a preliminary analysis of the ways in which students in the second summer class utilized the mathematical technique of checking by substitution when evaluating solutions to linear equations. Descriptions are provided of the occasions in which information on this procedure was made available to the students through some form of classroom instruction. The ways in which students utilized this information is then presented through an examination of the students' written responses to four different individual quiz questions that involved finding solutions to linear equations.

Classroom Instruction

On the first day of class students were introduced to the techniques for solving linear equations of one variable by using the addition and multiplication principles. During each lecture, the instructor presented numerous examples illustrating these techniques. However, she did not include any information concerning the checking procedure to be used with the solutions to these linear equations. Following each of the two morning lectures, the students were assigned problems from the textbook for practice work. The first set of problems in the text included a space labeled "check:" at the end of the space that was allotted on the page for the solution to each problem. The second set of assigned practice problems in the text did not have this feature. The directions to the problems, however, read "solve and check."

In mid-afternoon of the first day, the students were presented with a lab activity that illustrated the technique of checking solutions by substitution. I introduced this activity with a lecture in which I worked through the steps of a solution to a linear equation and then illustrated the substitution process that was used in order to check the mathematical accuracy of the answer. The students were then given a collection of pull-strip cards to work with. I did not make a tape recording of this lecture.

The pull-strip cards consisted of 3x5 index cards on which an equation was written. The variable in each equation appeared on a strip of paper that was visible through windows cut into the card. When the strip of paper was pulled, the variable was replaced by the numerical solution, and the equation appeared as it would look after the checking substitution was performed. (Table 13, Appendix B lists the equations that were used on each card.)

Students were seated in groups of four at two tables. Nine different cards were placed in the center of each table and students selected a card, evaluated it, and returned it to the central pile. The students were instructed to first solve the equation presented on the card. Then they were told to pull the strip and verify the answer that they had found. The students then performed the numerical calculations shown on the card in order to complete the checking procedure. If their calculations were performed correctly, they would arrive at a numerical identity in the last line of their check.

The students in the class had only been introduced to the techniques for solving simple linear equations during that morning's lectures. They were still unsure of the appropriate techniques that they needed to use in order to solve the given equations. In several instances that I observed, students did not arrive at the correct mathematical solution that appeared on the card when the answer strip was pulled. When this occurred with a student that I was helping, I suggested that they ignore the solution they had arrived at, and perform instead the checking calculations shown on the card. In this way they could focus on the purpose of the lab activity, which was to illustrate that the substitution of the correct solution back into the original equation would result in a numerical identity.

Many of the students were unable to successfully solve the equations on the cards and produce the correct mathematical solutions. It was apparent from their work that many of the problems in the checking lab were too complex for the students at this point in the course. The solutions that appeared in fractional form (i.e., y = 5/9, x = 18/11) produced numerical calculations during the checking steps that were difficult for the students to perform. Under these conditions the emphasis during the lab shifted to the numerical and algebraic manipulations required to obtain a solution to each equation, away from a focus on the characteristics of the checking procedure.

One student asked me to explain the results of her checking statement to her. She had substituted the correct answer of z = 3 into her equation to yield a check of 14 = 14. She questioned how the answer to the equation of 3 could produce a different number of 14 in the check. Her attention was focused on the specific values of the numbers that she had generated during the solution and checking parts of the lab, rather than focused on the importance of producing a numerical equality in the final step of the check.

Aside from the lab activity, the mathematical procedure of checking by substitution was not emphasized during the first week of class. Other methods were used in the class to verify the accuracy of the students' work.

During the morning of the first day the instructor told the class that they could "check" their answers by referring to the answer section in the back of their textbook. Students monitored the accuracy of their work by frequently comparing answers with other people in their group during practice sessions. Students would also ask the instructor or me to check their work for mathematical errors. These methods provided quicker and more reliable ways to determine the mathematical accuracy of their work than if the students had relied on the results of the checking by substitution method.

The use of these alternative methods emphasized the way in which checking was viewed by the students and the instructors. At no time was class instruction focused on a discussion of the mathematical relationship that checking could reveal between a solution set and the equation from which it was derived. Checking was perceived as a way to verify that the solution found for any equation was identical to the correct answer for that problem.

The only other exposure that the students had during the first week of class to the mathematical procedures for checking linear equations was through the treatment of the subject in their textbook. In the sections dealing with linear equations, the textbook included a checking procedure for the answer to each equation that was presented as an example. At the end of the first day of class, the students were assigned homework from the text which contained the instructions "solve and check." However, they were not penalized if they did not check their answers either on their homework or on quizzes. The topic of linear equations was only discussed during the first day of class. Other topics that dealt with algebraic expressions rather than equations were covered on subsequent days.

Student Responses Using Checking

The preceding section described the instances in the classroom learning environment in which information was made available to the students concerning the checking procedure to be employed with solutions to linear equations. This procedure was introduced, but not emphasized as an important aspect of the solution process. In this section, students' written quiz responses will be examined to determine the ways in which the students utilized the information that was presented to them in class.

This section presents copies of the written work on individual quizzes in which students utilized some form of checking with their solutions to linear equations. Students were tested on this type of equation on 6-19, 6-23, and 6-26. A simple problem on quadratic equations that could be solved by two linear equations was included on the quiz on

6-26. The problems from these quizzes that are presented below are identified by the date on which the quiz was taken, and by the numbers they were assigned on each quiz. Only those student responses are shown that include some form of a checking procedure.

<u>6-19 #1</u> solve: $20 - 6(2x - 1) = 2$

Two of the eight students did not achieve any kind of solution to this problem. Therefore, they did not have an opportunity to use the checking process. (See Table 11, Appendix B for a complete description of which students checked each specific quiz problem.) Peter and Bill were the only students of the other six who solved this equation, who also included a check with their answer. Both students correctly solved the equation and produced a numerical identity in the final step of the check. Since Bill's and Peter's responses are similar, only Peter's work is shown below.

<u>Peter</u>
(6-19 #1)

$$20 - 6(2x - 1) = 2$$

$$20 - 12x + 6 = 2$$

$$\begin{array}{rcr} 26 - 12x &=& 2 \\ -26 \quad\quad && -26 \end{array}$$

$$\frac{-12x}{-12} = \frac{-24}{-12}$$

$$x = 2$$

$$\begin{aligned} 20 - 6(4 - 1) &= 2 \\ 20 - 24 + 6 &= 2 \\ 2 &= 2 \end{aligned}$$

6-23 #4 solve: $6 - 3(x+1) = 2(x + 3)$

All but one of the students in the class was able to achieve some kind of solution to this problem. However, Alice and Peter were the only students who also checked their solutions.

Alice
(6-23 #4)

$$6 - 3(x + 1) = 2(x + 3)$$

$$6 - 3(2x) = 2(4x) \qquad (1)$$

$$\begin{array}{r} 6 - 6x = 8x \\ -8x = -8x \\ \hline 6 - 2x = x \\ +2x = +2x \\ \hline \end{array} \qquad (2)$$

$$\frac{6}{2} = \frac{2x}{2}$$

$$3 = x$$

$$6 - 3(3 + 1) = 2(3 + 3)$$

$$6 + 12 = 2(6) \qquad (3)$$

$$18 = 18 \qquad (4)$$

Alice made an error on each side of the equation in line 1 when she incorrectly combined the variable x with a whole number. She also incorrectly combined 8x with -8x to get x in line 2. If the calculations in her checking procedure had been done correctly, she should have arrived at a non-true statement in the last line of the check, which would have indicated the presence of mathematical errors in her solution process. However, Alice made an error in line 3 when she lost the minus sign on the left side of the equation. She also wrote the product of six times two as 18 on the right side of the equation in line 4. This coincidentally gave her a numerical identity for her check.

It cannot be determined from the information that Alice wrote down, whether the right side of line 4 in the check represents an arithmetic error or whether Alice was intentionally changing the computation in order to produce the required identity in the final step. However, her work illustrates that the checking procedure fails to be an effective tool when used by students who make algebraic and arithmetic errors during the check.

<u>Peter</u>
(6-23 #4)

$$6 - 3(x + 1) = 2(x + 3)$$

$$3(x + 1) = 2x + 6 \qquad (1)$$

$$\begin{array}{rcl} 3x + 3 &=& 2x + 6 \\ -2x & & -2x \\ \hline x + 3 &=& 6 \\ -3 & & -3 \\ \hline x &=& 3 \end{array}$$

ck:

$$6 - 3(3 + 1) = 2(3 + 3)$$

$$3(4) = 2(6) \qquad (2)$$

$$12 = 12$$

In line 1, on the left side of the equation, Peter incorrectly subtracted the 3 from the 6, instead of first multiplying through the parentheses by the factor of 3. Because he made the <u>same</u> mathematical error in line 2 of his check, he arrived at a numerical identity in the final step. Peter's error illustrates the way in which the checking procedure fails as a means of locating computational errors when students repeat the same mistakes in checking as they used when solving the equation.

Elaine

Elaine correctly solved this problem. She then wrote down the following correct substitution as a check next to her solution. However, she did not proceed any further with the manipulations. It cannot be determined from her paper why she did not finish her check.

$$6 - 3(-3/5 + 1) = 2(-3/5 + 3)$$

6-26 #3 solve: $10 - 2(m - 1) = 5(m - 4)$

The quiz on 6-26 was the first of three "super quizzes" that were given throughout the course. This quiz was 14 problems long and was administered in the morning at the beginning of the class. Its purpose was to serve as a summative evaluation of the material covered during the first two weeks of the course.

All but one of the students in the class were able to achieve some kind of answer to 6-26 #3. However, Alice and Bill were the only students who included a check with their answers.

<u>Alice</u>
(6-26 #3)

$$10 - 2(m - 1) = 5(m - 4)$$

$$10 - 2m + 2 = 5m - 20$$

$$\begin{array}{llll}
 & 10 - 2m + 2 = 0 & 5m - 20 = 0 & (1) \\
 & 12 - 2m = 0 & 5m - 20 = 0 + 20 & \\
 & 12 - 2m + 12 = 0 + 12 & \frac{5m}{5} = \frac{20}{5} & \\
 & \frac{2m}{2} = \frac{12}{2} & m = 4 & \\
 & m = 6 & 5(4 - 4) & (2) \\
(4) & 10 - 2(6 - 1) & 20 - 20 = 0 & (3) \\
(5) & 10 - 12 + 2 = 4 & &
\end{array}$$

Three days before this quiz was given, the students were introduced to a technique for solving quadratic equations that involved the principle of zero products. This technique was used to rewrite an equation that was written in the form of the product of two binomial factors set equal to zero, as two separate equations, each set equal to zero. (See Chapter Eleven for more information on this technique.) It appears that Alice was attempting to use this technique in line 1 above when she set each side of the equation separately equal to zero. As a result of this incorrect use of the manipulation, she arrived at two separate solutions to the original linear equation.

Alice substituted the answer $m = 4$ into the expression on the right side of the original equation to form line 2, and the answer $m = 6$ into the expression on the left side of the equation to form line 4. It is not clear what her intentions were for these checking steps, since she did not include any equal signs to complete the two equations. She then made an arithmetic error in line 5. Lines 3 and 5 do not resemble the appropriate form of a numerical identity. However, Alice did not make any indication on her paper as to what conclusions she finally drew from the information she had generated by her checking substitutions.

The instructors had not included any examples in their lectures that presented students with a situation in which the last line in the checking procedure produced anything other than a numerical identity. The instructions in the checking lab had only stipulated that if the numerical identity was not present, a mistake had been made in the solution process. Given the level of classroom instruction and the lack of emphasis that was placed on the use of the checking procedure, it is not surprising that Alice was unable to finish her problem correctly or to reflect on the mathematical implications of the check she had written.

<u>Bill</u>
(6-26 #3)

$$10 - 2(m - 1) = 5(m - 4)$$

$$10 - 2m + 2 = 5m - 20$$

$$\begin{array}{rcl} 12 - 2m & = & 5m - 20 \\ -12 & & -12 \\ \hline -2m & = & 5m - 32 \\ -5m & & -5m \\ \hline \underline{-7}m & = & -\underline{32} \\ -7 & & -7 \\ m & = & 4.57 \end{array}$$

$$\text{CH:} \quad 10 - 2(4.57 - 1) = 5(4.57 - 4)$$
$$10 - 2(3.57) = 5(.57)$$
$$10 - 7.14 = 14.25 \qquad (1)$$
$$2.86 = 14.25$$

It is probable that Bill rewrote his solution of 32/7 as the decimal number 4.57 in order to use his calculator to perform the arithmetic calculations of the check. However, he made an error in line 1 of his check when he multiplied .57 by 25 instead of by 5. If his calculations had been correct he would have arrived at 2.86 = 2.85 in the last line of his check, which is a true statement within the limits of his rounding off procedures.

Bill did not include any information on his paper concerning the mathematical significance of the numerical non-equality in the last step of his check. According to the instructions that had been given during the checking lab, such a non-equality should have indicated to him that his solution to the equation was in error. It may be that, like Alice, Bill was only able at that time to reproduce a sequence of manipulations, but not able to interpret the results in an appropriate mathematical manner.

<u>6-26 #9</u> solve: $8a^2 - 2a - 15 = 0$

This problem illustrates a type of equation that can be solved by the process of factoring and using the technique involving the principle of zero products. Since these solution techniques produce two linear equations that can be solved easily, problem 6-26 #9 is included in this chapter rather than in the chapter describing quadratic equations.

Alice incorrectly applied part of this solution technique to the problem 6-26 #3, which was an earlier problem on the same test (see page 174). She was able to correctly use the technique with 6-26 #9. However, she did not perform a check on the solution that she found. All but one of the students in the class were able to solve 6-26 #9 correctly. However, only Bill checked his answer.

<u>Bill</u>
(6-26 #9)

$$8a^2 - 2a - 15 = 0$$

$$(2a - 3)(4a + 5) = 0$$

$$\begin{array}{rr} -12a & (1) \\ \underline{+10a} & (2) \\ -2a & (3) \end{array}$$

$$(8a^2 + 10a - 12a - 15) = 0 \qquad (4)$$

$$\begin{array}{lcl} 2a - 3 = 0 & \text{or} & 4a + 5 = 0 \\ \underline{\quad +3 \quad +3} & & \underline{\quad -5 \quad -5} \\ \dfrac{2a}{2} = \dfrac{3}{2} & & \dfrac{4a}{4} = \dfrac{-5}{4} \end{array} \qquad (5)$$

$$a = 3/2 = 1.5 \quad \text{or} \quad a = -5/4 = 1\ 1/4$$

$$\begin{array}{llll} \text{CH:} & 2(1.5) - 3 = 0 & \text{CH:} & 4(-1.25) + 5 = 0 \\ & 3 - 3 = 0 & & -5 + 5 = 0 \\ & 0 = 0 & & 0 = 0 \end{array}$$

Bill wrote down the product of the inner and outer terms of the two factors in lines 1 and 2, and found their sum in line 3, in order to verify that he had selected the correct binomial factors for the given problem. Then he doubly checked his factoring by multiplying them back together in line 4. Factoring trinomial expressions was still a relatively new technique for the students at the time this quiz was taken, and Bill therefore wrote out all the steps he used in the factoring process. Later on in the course, as the students gained proficiency in factoring trinomials, they began to perform the above steps mentally.

Bill checked his two answers by substituting his solutions into the two linear equations that were generated in line 5, rather than by substituting them back into the original equation. By converting each answer into a decimal number, he was then able to use his calculator to perform the arithmetic calculations in the checks. Bill arrived at the correct solution of -5/4 in line 6, and used the appropriate number of -1.25 in his subsequent check. He apparently neglected to carry along his negative sign when he converted his original answer to the mixed number 1 1/4.

It is informative to compare the way in which Bill used the checking substitution on problems #9 and #3 on the quiz of 6-26. Number 3 was a linear equation (see page 176), and in this problem, Bill substituted his decimal answer directly into the original equation. This produced a set of numerical calculations that could be solved in a series of simple operations on his calculator. However, he made an entry error which lead to a non-true statement in the last line of the check.

In problem #9, however, Bill elected to check his two solutions by substituting them into the two equations written in line 5. It may be that Bill expected the form of the checking substitution to resemble those used in past problems, which had all dealt with linear equations. It may also be that Bill elected to substitute his decimal answers into the simpler equations written in line 5 in order to avoid substitutions into the original quadratic equation, which would have produced a more complicated set of numerical calculations. If this were the case, his decision concerning which procedural steps were to be used was determined by the perceived complexity of the manipulations involved, rather than by the mathematical relationships involved in the checking procedure.

Discussion of Students' Work

The procedure for checking solutions to linear equations was introduced to the students but not emphasized during class work. The instructor did not include any checking procedures with the examples she provided during her lectures on finding solutions to linear equations. The checking procedure was presented to the students separately during a lab activity. However, following this lab, students were not encouraged to use this procedure with their work, nor were they penalized for failing to check their solutions to linear equations on homework problems and quiz questions.

Within this instructional context, it is not surprising that very few students included a check of their solutions on the quiz problems that dealt with linear equations. Of the eight students in the class, only Bill, Peter, and Alice used the procedure with their work. However, none of these three students checked more than two of the four problems they answered.

Of the eight checked responses that were examined, only the two written by Bill and Peter for problem 6-19 #1 were mathematically completely correct. On problem 6-23 #4, both Alice and Bill made a series of errors that produced a numerical identity in the final step of the check, which negated the usefulness of the checking process. Bill correctly performed all the manipulation steps in his solution and check for problem 6-26 #9. However, he did not substitute his solution into the original form of the equation when he began his check.

The examination of the ways in which Alice and Bill carried out their checking procedures indicates that they were able to reproduce a set of manipulations that closely resembled the appropriate steps of the checking process for the solutions to linear equations. However, because they failed to take any action following their non-equal checks for problem 6-26 #3, it is apparent that they did not understand the mathematical significance of these results. They were not able to take the appropriate mathematical action with these problems. The fact that Alice and Bill failed to substitute their solutions back into the original forms of their equations also indicates that they did not understand the mathematical significance of the operations they were performing.

The responses of Alice and Peter on problem 6-23 #4 indicate the difficulties that can occur when the checking procedure is used by students who are not yet proficient in the use of various manipulation techniques. The checking process is used with the solutions of linear equations for the purposes of detecting the presence of algebraic errors in the solution process. This purpose fails when students are unable to perform the steps in the checking process in an error-free manner.

It is a difficult instructional task to try to introduce the techniques of checking to students at a point in time when they are not yet able to manipulate algebraic symbols in an appropriate manner. This was illustrated by the problems that the students experienced when they tried to meet the objectives of the checking lab on the first day of class. The students' existing levels of numerical and manipulative knowledge were not such that they could be expected to understand the mathematical concepts represented by the checking process.

It is difficult to draw many generalized conclusions concerning the use of checking for the class performance as a whole, based on the limited examples that are presented in this chapter. However, the analysis of the individual student's performances indicates that there existed an instructional mis-match between the learning environment and the students' existing levels of mathematical competence. The level of mathematical response which the instructor and I had anticipated that the students could produce on their written quiz work was not commensurate with the actual learning conditions that the students were exposed to in the class.

CHAPTER EIGHT

FRACTIONAL EQUATIONS

The purpose of this chapter is to examine the ways in which students utilized some form of a checking procedure with solutions to fractional equations. The students' performances are examined within the context of the classroom learning environment. The first section of this chapter describes the topic of fractional equations in terms of its similarities and differences with other material involving fractional expressions. The next section presents descriptions of lecture transcripts and other class activities that involved the checking process for fractional equations. Following this section, examples are provided of the students' written work dealing with this topic.

Fractional Equations and Expressions

Students were introduced to fractional equations on Wednesday, July 1, of the third week of class. Previous lectures that week had dealt with the techniques of manipulating fractional expressions under the operations of multiplication, division, addition, and subtraction. The second lecture on Wednesday was the first time that students had been exposed to these fractional expressions within an equation. (Table 12, Appendix B presents the time schedule for all class activities dealing with the subject of checking.)

The techniques that are used for solving fractional equations are different from those used to simplify an algebraic expression containing similar fractions. These differences are illustrated by the following examples.

Equation

$$\frac{y^2 - y}{y^2 - 25} = \frac{2}{y + 5}$$

$$\cancel{(y+5)}\,\cancel{(y-5)}\,\frac{y^2 - y}{\cancel{y^2 - 25}} = \frac{2\,\cancel{(y+5)}\,(y-5)}{\cancel{y + 5}}$$

$$y^2 - y = 2y - 10$$

$$y^2 - y - 2y + 10 = 0$$

$$y^2 - 3y + 10 = 0$$

$$(y - 5)(y + 2) = 0$$

$$y - 5 = 0 \qquad\qquad y + 2 = 0$$

$$y = 5 \qquad\qquad y = -2$$

(By checking, the answer $y = 5$ is <u>not</u> a valid solution. The answer $y = -2$, is a valid solution.)

Expression

$$\frac{5}{y + 5} - \frac{y}{y^2 - 25}$$

$$\frac{(y-5)}{(y-5)} \cdot \frac{5}{(y+5)} - \frac{y}{y^2 - 25}$$

$$\frac{5y - 25}{(y-5)(y+5)} - \frac{y}{(y-5)(y+5)}$$

$$\frac{5y - 25 - y}{(y-5)(y+5)}$$

$$\frac{4y - 25}{(y-5)(y+5)}$$

It can be seen in the above example for the expression, that the fractions maintain their numerators and denominators throughout the problem, although they change form under the operation of multiplication. When similar fractions appear in the equation, however, they are subjected to a different set of manipulations. The first step in the solution process above multiplies each term in the equation by the least common denominator, to produce a second, transformed equation that no longer contains algebraic fractions.

The first lecture on Wednesday, July 1, covered techniques for adding or subtracting fractional expressions with different denominators. Students were instructed at this time to find the least common denominator and then to multiply the "top and bottom" of each fraction by the appropriate factors in order to produce equivalent fractions having the same denominator. These expressions were then combined to produce a simplified answer, as illustrated in the above example.

In the second lecture that day, similar fractional expressions were presented as examples in which the only symbolic difference from the previous work was the presence of an equal sign in the collection of symbols. The directions for manipulating these similar symbols was, however, different from those given during the first lecture. Students were now instructed to multiply only the "top" of each fraction by an appropriate expression. This step then produced a completely different set of symbols that no longer appeared as fractions, but resembled the equations that students had worked with during the first weeks of class.

Students were also instructed during this lecture to check all solutions to fractional equations. However, the form of this instruction was different from that which they had received concerning the checking procedure to be used with linear equations. The rationale for rejecting certain solutions of fractional equations was presented to the students as a rule of manipulation that needed to be applied as part of the solution process. (The mathematical justification behind this rule was presented in Chapter Six.)

The following section presents transcripts of the discourse of those parts of the class lectures that dealt with the subject of checking fractional equations. These records provide information concerning the learning context to which the students were exposed. These lectures took place in one corner of the classroom where the students' chairs were grouped in front of the projection screen. All the lecture examples were written on transparencies and projected on the screen since the room did not contain a blackboard. The style of the lectures was therefore somewhat dictated by the medium through which the instructor was required to present her material to the students.

A later section of this chapter examines the students' written responses to three different quiz problems that involve finding the solutions to fractional equations. In

14 of the 23 responses to these three questions, students were unable to reach any kind of solution to the given equation. Their difficulties in solving this kind of problem were related to their inabilities to understand the mathematical distinctions that exist between fractional expressions and fractional equations.

Classroom Instruction

Selected parts of the lecture transcripts that discuss checking procedures are presented in this section along with a discussion of some of the salient features of these lectures. The complete transcripts of these lectures are presented in Appendix B. The transcripts have been edited to produce a smoother flowing narrative than that usually found in extemporaneous speech. However, the mathematical content and intent of the instructor's discourse have been preserved.

Wednesday, July 1

The introductory lecture on fractional equations took place on Wednesday morning, July 1. The instructor provided two examples that were carefully written out, step-by-step, on a series of transparencies. The first problem was solved as follows, without a great deal of commentary by the instructor.

$$\frac{x}{x-1} = \frac{3x}{2} + \frac{x}{x-1} \qquad \text{LCD} = 2(x-1)$$

$$\frac{2\cancel{(x-1)}x}{\cancel{x-1}} = \frac{\cancel{2}(x-1)3x}{\cancel{2}} + \frac{2\cancel{(x-1)}x}{\cancel{x-1}}$$

$$2x = (x-1)3x + 2x$$

$$\begin{array}{rcl} 2x & = & 3x^2 - 3x + 2x \\ -\ 2x & & \qquad\ \ -\ 2x \\ \hline 0 & = & 3x^2 - 5x + 2x \end{array}$$

$$0 = 3x^2 - 3x$$

$$0 = 3x(x-1)$$

$$\frac{3x}{3} = \frac{0}{3} \quad \text{or} \quad \begin{array}{r} x - 1 = 0 \\ +1 \quad +1 \\ \hline \end{array}$$

$$x = 0 \quad \text{or} \quad x = 1$$

The necessity for checking fractional equations was introduced at the end of the first problem in the following transcript. The instructor is referred to as Ellen.

"Now it looks like I'm all done. But I should warn you that I am not." The instructor paused while she placed a clean transparency on the overhead projector.

"OK. If any answer that we get makes any denominator equal zero, then we have to throw it out. We <u>cannot</u> -- we cannot have fractions with zero in the basement. We're going to have to go all the way back to the beginning. We're going to look all the way back to our original equation, and we want to take a look at x = 0." (This refers to one of the two solutions to the equation.)

The instructor verbally substituted the value of $x = 0$ into the denominators of the original equation and arrived at values of -1 and 2. "I have a 2 there, and a -1 there. Zero's OK. I get to keep that answer."

"Let's check out $x = 1$. One minus one would be a zero. Uh oh! That's a no-no. You may not have a zero in the denominator. And so, we're going to have to throw this answer out. We're going to go back and we're just going to throw this answer out." She went back to the two solutions at the end of the problem and drew a large X through the line $x = 1$.

"What happened was, we solved the equation, and it looked like we had two solutions, but one of them didn't work. We throw it out."

The instructor then worked through a second example using similar procedures and terminology. The transcript for this portion of the lecture is presented in Appendix B. Following this example, Ellen instructed the students to go to their tables and work through six practice problems involving fractional equations.

After the students at one table had worked through several problems, I showed them a way to monitor the checking process using the following problem as an example.

$$\frac{y}{y+1} + \frac{3y+5}{y^2+4y+3} = \frac{2}{y+3}$$

Solving the above problem produces the following last steps.

$$(Y + 1)(y + 3) = 0 \qquad (1)$$

$$y + 1 = 0 \quad \text{or} \quad y + 3 = 0 \qquad (2)$$

$$y = -1 \quad \text{or} \quad y = -3 \qquad (3)$$

It can be seen that substituting either of these answers in the original equation produces a denominator of 0, and therefore, both solutions must be rejected. I pointed out to the students that they could use the results of step 2 above to predict whether or not the solutions obtained in the next step would be valid. Since in step 2, both equations represented the value of one of the denominators in the original equation under substitution, they could see that they would have both denominators equal to zero in this case.

The students appeared able to master the information I was giving them. Peter laughed and said, "Notice how Anne always shows us the trick <u>after</u> we have worked the problems?"

At the other table Sally wrote the following:

$$y^2 + 4y + 3 = 0$$

$$(y + 3)(y + 1)$$

Then she asked me, "Do I still need to keep it equal to zero?" I told her that she needed to keep the zero there. Sally's comment indicated that she did not assign any special significance to the symbol for the equal sign. She was attempting to work with the properties of equations without having previously developed the appropriate level of mathematical understanding. Her question indicated that she had probably not developed an adequate set of meanings for the concepts represented by the algebraic symbols.

The students were able to work through the assigned practice problems and arrive at solutions which they then checked. Both Ellen and I were available to help them master the proper manipulations required by this type of problem.

As part of their homework for the next day, the students were assigned eight problems from the text dealing with the addition and subtraction of fractional expressions and eight problems dealing with fractional equations.

Thursday, July 2

The next day the students were given the following as one of three problems on their group quiz.

$$\text{solve} \qquad \frac{m+1}{m-5} = \frac{20}{m^2-25} + \frac{2}{m+5}$$

The students were not able to solve the problem in their groups. Both Ellen and I tried to offer hints to the students without telling them outright how to solve the problem. Since this was a quiz, the students did not have access to their notes or the text. Without these resources, they were not able to remember from their pooled knowledge the correct sequence of steps that were to be used. Finally Ellen told the students to stop their work and come sit at the projection screen, where she carefully worked through the problem for them.

Instead of checking her solutions to the equation by substituting these values into the denominators of the original equation, she checked their validity by substituting them into the least common denominator used to clear the original equation of fractions. She rejected the solution that produced a zero value for one of the factors in the LCD.

Ellen emphasized the importance of rejecting any solution that produced a zero in the least common denominator repeatedly as she worked through the example and as she answered questions raised by one of the students. At the end of the discussion she summarized the rules for solving fractional equations.

"OK. Let's review the steps that we took. Step 1: we found the least common denominator. Now this is for equations. Keep in mind we're talking about equations with fractions. This is different from everything else we've done when we worked with expressions. We're now talking about equations."

"That's why we took this step 2. We multiply every term by the least common denominator, and we get rid of all denominators."

"Number 3. We have to solve the equation that we get and it usually is kind of long and full of parentheses. It certainly was today."

"OK. Step 4. Check. If any answer makes any denominator equal zero, you have to throw it out."

While Ellen was saying this review, she wrote the following on the transparency:

Equations

1. Find LCD
2. x every term by LCD and get rid of all denominators.
3. Solve equation.
4. Check, if answer makes any denominator = 0, throw it out.

The complete transcript of this lecture is presented in Appendix B.

<u>Further Instruction on Checking</u>

The following day, Friday, July 3 was a school holiday. Over the three day weekend the students were not assigned any homework problems dealing with fractional equations. On Monday morning the students worked through a lab activity using equation strips. Each student selected an envelope containing narrow strips of paper on which different steps of the solution of a fractional equation were written. The activity consisted in arranging the strips in a mathematically logical sequence preceding from a statement of the original problem through a check of the solutions. The final strip in the sequence was the checking step and was identifiable by a "$\surd$" placed on the left side of the strip. (Table 14, Appendix B presents a list of the equations that were used in this lab activity.)

An example of one of the checking strips is as follows:

$$\boxed{\surd \quad -\frac{10}{3} = -\frac{10}{3} \therefore\ x = 5 \qquad 2 - \frac{10}{0} = -\frac{10}{0} \therefore\ x \neq 2}$$

The students were given a review sheet on Wednesday morning, July 8, consisting of a side-by-side comparison of the manipulations required to simplify the subtraction of two fractional expressions and the steps necessary to solve a fractional equation containing similar fractional terms. (See the example on page 185 for these problems.) I explained the manipulations on this sheet using the overhead projector, and emphasized the differences that exist between the two types of problems. For the equation, the final step of checking was simply indicated by the phrases "not a solution by check" and "is solution" that were written next to the two answers.

The explanation of the review sheet was the last formal instruction on fractional equations. The topic had been introduced on Wednesday, 7-1. The equation strip lab was given on Monday, 7-6, and the review sheet was presented on Wednesday, 7-8. The majority of homework problems that were assigned on this topic were to be handed in on 7-3. Two additional problems were assigned on the homework for 7-7.

Discussion of Lectures

The checking process that was used with the solutions to fractional equations was stated during instruction as a rule. The instructor did not present any mathematical justification for this process, but simply presented the manipulation steps as a statement of fact: "If any answer that we get makes any denominator equal zero, then we have to throw it out."

The focus of the lectures on fractional equations was concerned with presenting a sequence of procedural steps, rather than with investigating the mathematical relationships expressed by the equations. If these relationships had been investigated, the objects of study would have been centered on the conditions of equality that differentiate the properties of fractional equations from those of fractional expressions. This aspect of the topic, however, was not emphasized during the class lectures. The focus of instruction was centered instead on providing descriptions of the manipulation procedures that were to be used to generate solutions to the equations.

No attempt was made to provide students with an appropriate level of mathematical understanding of the particular manipulation sequences that were employed in each lecture. Students reproduced these steps by memorizing each sequence in connection with an appropriate set of algebraic symbols.

The instructor used "authoritarian" rules, rather than rational explanations based on an understanding of underlying mathematical concepts, to present the checking information. This type of presentation can be observed by examining the instructor's choice of words that were used during the discussions of the checking procedure. She continually employed such phrases as, "We <u>cannot have</u> fractions with zeros in the denominator," "That's a <u>no-no</u>," "<u>You may not have</u>...," and "We <u>have to throw this answer out</u>."

This particular choice of words conveys the impression to the students that algebra is a mysterious subject that is made up of a collection of arbitrary rules. Peter expressed this attitude to me after I had demonstrated to his group a way to use the results of a solution in order to anticipate the presence of zeros during a check. He commented to the group, "Notice how Anne always shows us the <u>trick</u>...." His choice of words indicated that he had not understood the sequence of steps that I had demonstrated in terms of the mathematical relationships on which I had focused. Without this appropriate level of mathematical understanding, Peter was only able to interpret the information in terms of an algebraic "trick" to be mastered.

A more extensive analysis of the instructor's discourse on checking fractional equations will be presented in Chapter Thirteen. At that time the transcripts from Chapters Seven through Eleven will be re-examined in order to identify the different types of meanings that the instructor assigned to key words and phrases within each lecture. This information will be used to analyze the progression of information that was made available to the students within the learning environment.

Student Responses Using Checking

The students in the class were constantly being exposed to information at the rate of three new topics a day. Between the time that the topic of fractional equations was introduced and the time that the students were tested on this material on quizzes, they were also exposed to other concepts and techniques of manipulation. For example, on Thursday, July 9, the students were introduced to the topic of radical equations. (See Chapter Ten.) This lecture also presented a checking process for such equations that was different from the procedure that the students were already using with fractional equations. The information from this lecture may have influenced the type of responses that the students used when they were presented with a fractional equation on the following day's quiz.

It was difficult to assess the degree to which all of the students in the class were able to assimilate the procedure for checking fractional equations. Very few of the students were able to achieve any kind of solution to such quiz problems, and, therefore, only a few of them produced answers with which they could use the checking process.

Three problems dealing with fractional equations were included on various individual quizzes. This section presents copies of those responses in which the students employed some form of a checking procedure.

<u>7-6 #2</u> solve: $\frac{1}{y-1} = \frac{y^2}{y-1}$

<u>Alice</u> was the only student who was able to solve the above problem to arrive at an answer:

$$\frac{1}{y-1} = \frac{y^2}{y-1}$$

$$\frac{1}{\cancel{y-1}}(\cancel{y-1}) = \frac{y^2}{\cancel{y-1}}(\cancel{y-1})$$

$$\begin{array}{rcl} 1 & = & y^2 - 1 \\ -1 & & \\ \hline 0 & = & y^2 - 1 \end{array}$$

$$(y-1)(y+1) = 0$$

$$\begin{array}{rcl} \cancel{y - 1} & \cancel{=} & \cancel{0} \\ \cancel{+1} & & \cancel{+1} \\ \hline \cancel{y} & \cancel{=} & \cancel{1} \end{array} \quad \text{or} \quad \begin{array}{rcl} y + 1 & = & 0 \\ -1 & & +1 \\ \hline y & = & -1 \end{array}$$

Alice did not write out the substitution steps that she performed during her check. It is assumed that she performed these calculations mentally. She indicated that she rejected the solution of $y = 1$ by crossing out that answer.

<u>7-7 #2</u> solve: $\dfrac{x^2}{x+4} = \dfrac{4}{x+4}$

Peter and Bill were the only students who were able to reach a solution to this problem. Peter, however did not check his answer. Bill's solution and check are shown below.

<u>Bill</u>
(7-7 #2)

$$\frac{x^2}{x+4} = \frac{4}{x+4} \qquad (1)$$

$$\frac{x^2\,\cancel{(x+4)}}{\cancel{x+4}} = \frac{4\,\cancel{(x+4)}}{\cancel{x+4}} = x^2 = 4 \qquad (2)$$

$$\begin{array}{r} x^2 = 4 \\ -4 \quad -4 \\ \hline x^2 - 4 = 0 \end{array}$$

$$(x-2)(x+2) = 0$$

$$\begin{array}{rcr} x - 2 = 0 & \text{or} & x + 2 = 0 \\ +2 \quad +2 & & -2 \quad -2 \\ \hline x = 2 & \text{or} & x = -2 \end{array}$$

$$\begin{array}{rr} x^2 = 4 & (-2)^2 = 4 \\ 2^2 = 4 & (-2)(-2) = 4 \\ 4 = 4 & 4 = 4 \end{array}$$

They are all solutions!

Bill solved the problem correctly. However, when he began his check he did not substitute his answers into the <u>original</u> fractional equation in line 1. Instead he used the equation that he had generated at the end of line 2. His comment "They are all solutions!" appears to indicate that he was aware of the need to check his answers, and aware that the results of the check were to be used to reject or retain these solutions. However, he had not retained the information from the lectures concerning the need to examine the denominators of the fractional equation for zero values. His checking process resembles the steps that were used to check the solutions to linear equations, and the solutions to systems of equations. (See Chapter Nine.)

It is not clear what prompted Bill to write the phrase "They are all solutions!" after his two checks. During her illustrations of the checking procedure for fractional equations, the instructor had used verbal information rather than written words to signify those solutions that were found to be valid by the checking process. She had referred to these answers by such phrases as, "This is OK,' and "I get to keep this answer." In most of her discourse she used the word answer, rather than the word solution. Bill may have been influenced by the instructor's choice of words during her lecture on solutions to simultaneous equations that was given on June 25. At this time she used the term solution more frequently than answer. She wrote, "Yes it is a solution" next to one of the checks that she evaluated during that lecture.

7-10 #12 solve: $\frac{r}{r - 1} + \frac{4}{r^2 - 1} = \frac{-2}{r + 1}$

The individual quiz on 7-10 was twelve problems long and served as the second cumulative super quiz. Alice, Peter, Sue, Bill, and Elaine all achieved some sort of solution to problem 7-10 #12 on this quiz. All of them except Sue also performed a check on their solutions. Alice, Peter, and Bill solved the problem correctly but made errors in their checking procedures. Elaine did not solve the problem correctly, but she did check her answer.

Alice
(7-10 #12)

$$\frac{r}{r-1} + \frac{4}{r^2-1} = \frac{-2}{r+1}$$

$$\frac{r(r+1)}{r-1\ (r+1)} + \frac{4}{r^2-1} = \frac{-2(r-1)}{r+1\ (r-1)} \qquad (1)$$

$$\frac{r^2+r+4}{(r-1)(r+1)} = \frac{-2r+2}{(r-1)(r+1)}$$

$$\frac{r^2+r+4}{\cancel{(r-1)}\cancel{(r+1)}} \cdot \cancel{(r-1)}\cancel{(r+1)} = \frac{-2r+2}{\cancel{(r-1)}\cancel{(r+1)}}\ \cancel{(r-1)}\cancel{(r+1)}$$

$$r^2+r+4 = -2r+2$$

$$r^2+r+4+2r-2 = -2r+2+2r-2$$

$$r^2+3r+2 = 0$$

$$(r+2)(r+1) = 0$$

$$\begin{array}{ccc} r+2=0 & \text{or} & r+1=0 \\ -2\ \ -2 & & -1\ \ -1 \\ r=-2 & \text{or} & r=-1 \end{array}$$

$$\frac{-2(-2+1)}{(-2-1)(-2+1)} + \frac{4}{(-2-1)(-2+1)} = \frac{-2(-2-1)}{(-2+1)(-2-1)} \qquad (2)$$

$$\frac{4-2+4 = 4+1}{(-2-1)(-2+1)} \qquad (3)$$

$6 = 5$ does not check

$$\frac{-1(-1+1) + 4 = -2(-1-1)}{(-2-1)(-2+1)} \qquad (4)$$

$$\frac{1-1+4 = 2+1}{(-2-1)(-2+1)} \qquad (5)$$

$4 = 3$ does not check

When Alice performed her check, she did not substitute her answers into the original form of the equation. However, she did use line 1 for the substitution which still contained variable expressions in the denominator. She made a complete substitution of her answer in line 2, rather than simply examining the values of each denominator under substitution. She made an arithmetic error at the end of line 3 which produced the non-equal statement 6 = 5. Alice performed her second check in line 4. However, she neglected to change the value of the variable in the denominator. She repeated the same arithmetic error at the end of line 5 which produced a second non-equality.

A correct check shows that $r = -2$ does not produce any zeros in the denominators of the original equation, while the value of $r = -1$ produces a zero in the denominator of the term to the right of the equal sign. Therefore, $r = -1$ is not a valid solution to the original fractional equation. By neglecting to change the value of the variable in the denominator in line 4, Alice did not uncover the presence of a zero in the denominator.

It is not clear what conclusion Alice is drawing from her checking procedures by her comment "does not check," since she did not write any information next to her two solutions. At the beginning of the course, a non-equality was used to indicate the presence of calculation errors in the solution process. On 7-9, the day before this quiz was given, the topic of radical equations was introduced, and students were instructed to use the checking procedure to indicate the existence of extraneous roots. Such roots were to be rejected when the final step in the check produced a non-true statement. (See Chapter Ten.) Alice may have had this criterion in mind when she wrote down "does not check." However, she did not go back to her solutions and cross out the rejected values as she had done with one of her solutions to the fractional equation presented on the quiz of 7-6.

Peter
(7-10 #12)

$$\frac{r}{r-1} + \frac{4}{r^2-1} = \frac{-2}{r+1}$$

$$\cancel{(r-1)}(r+1)\frac{r}{\cancel{r-1}} + \frac{4\cancel{(r-1)}\cancel{(r+1)}}{\cancel{(r-1)}\cancel{(r+1)}} = \frac{-2(r-1)\cancel{(r+1)}}{\cancel{(r+1)}}$$

$$r^2 + r + 4 = -2r + 2 \qquad (1)$$

$$\begin{array}{r} +2r \qquad\quad +2r \\ \hline r^2 + 3r + 4 = 2 \end{array}$$

$$\vdots \qquad\qquad \vdots$$

$$r = -2 \quad \text{or} \quad r = -1$$

$$(-2)^2 + (-2) + 4 = -2(-2) + 2$$

$$4 - 2 + 4 = 4 + 2$$

$$2 + 4 = 6$$

$$6 = 6$$

$$(-1)^2 + (-1) + 4 = -2(-1) + 2$$

$$1 - 1 + 4 = 2 + 2$$

$$4 = 4$$

Peter solved the equation correctly, but he did not substitute his answers into the original equation when he checked them. By using line 1 instead for his substitution, he eliminated the crucial checking information that is contained in the denominators of the original equation. It is not clear whether he used the numerical identity in the last line of his check to draw any conclusions regarding the validity of his solutions, since he did not write any comments next to his answers.

During the previous week, on the day that the topic of fractional equations was introduced, Peter had made a comment on the "trick" that I had explained to his group concerning a way to check for zeros in the denominator. It appears from the way that Peter made his checking substitutions in the above problem that he had not retained the information from the lectures on fractional equations concerning the need to search for zeros in the denominators of the fractions during each check.

<u>Bill</u>
(7-10 #12)

$$\frac{r}{r-1} + \frac{4}{r^2-1} = \frac{-2}{r+1}$$

Bill proceeded to solve the equation in the same manner as Peter. He checked his answers using the following steps.

$$r^2 + 3r + 2 = 0$$

$$\vdots \qquad \vdots$$

$$r = -1 \quad \text{or} \quad r = -2$$

CH: $(-1)^2 + 3(-1) + 2 = 0$ $\qquad$ $(-2)^2 + 3(-2) + 2 = 0$

$1 - 3 + 2 = 0$ $\qquad$ $4 - 6 + 2 = 0$

$0 = 0$ $\qquad$ $0 = 0$

works! $\qquad$ They are a solution! $\qquad$ works!

Like Peter, Bill also failed to substitute his answers into the original equation. He was consistent in this regard with the way in which he used the checking procedure on the 7-7 quiz problem. He did indicate on the problem 7-10 #12 what conclusions he had drawn from the final step of his checking process. However, by failing to substitute his answers into the original form of the equation, he was not able to utilize the checking process correctly with the solutions to the fractional equation. The happy face that Bill drew on the last line of the problem is symbolic of emotions that students many times expressed when they were able to achieve a correct answer to a problem.

Elaine
(7-10 #12)

$$\frac{r}{r-1} + \frac{4}{r^2-1} = \frac{-2}{r+1}$$

$$\frac{r\cancel{(r+1)}\cancel{(r-1)}}{\cancel{r-1}\ \cancel{(r+1)}} + \frac{4\cancel{(r-1)}\cancel{(r+1)}}{\cancel{r^2-1}} = \frac{-2\cancel{(r-1)}\cancel{(r+1)}}{\cancel{r+1}\ \cancel{(r-1)}}$$

$$r + 4 = -2$$

$$r + 4 + 2 = 0$$

$$\begin{array}{rcl} r + 6 &=& 0 \\ -6 && -6 \\ \hline r &=& -6 \end{array}$$

Solution

CK

$$\begin{array}{rcl} r + 4 &=& -2 \\ -6 + 4 &=& -2 \\ -2 &=& -2 \end{array}$$

OK

Elaine did not use the correct procedure to clear her original equation of fractions. She did check her answer, but, like Peter and Bill, she failed to substitute this value into the original equation. She indicated by her comments that she used the numerical identity in the final step of the check to validate her answer. However, her checking procedure was unable to detect the errors she had made in the solution process because she failed to make the substitutions back into the original equation.

Bill and Elaine, as well as Alice, appear to have been influenced in their quiz responses to 7-10 #12 by part of the lecture on the previous day that dealt with the checking procedure for radical equations. At the conclusion of one example, the instructor wrote "OK" next to the last line of the check that produced a numerical identity, and "not a solution!" next to the check that did not produce an identity. Ellen did not use these kinds of written indications with the checking steps that she illustrated during her lectures on fractional equations.

Discussion of Students' Work

The most common mistake that was made by the students who checked their solutions to fractional equations was that they did not substitute their answers back into the original form of the equation. This meant that none of the answers that were checked in this way could ever be identified as a mathematically invalid solution to the fractional equation. By failing to substitute into the original equation, the students negated the mathematical purpose for which they checked their answers in the first place.

Alice was the only one of the four students who checked their answers, that did substitute into some form of the original equation. She appeared to have correctly performed this substitution mentally with problem 7-6 #2. (See page 198.) On 7-10 #12, she performed her substitution into an altered form of the equation which, however, still retained fractional expressions in the denominator. (See page 201.) Thus, on both occasions, Alice was able to utilize the checking process in the correct mathematical sense.

Peter, Bill, and Elaine failed to use the checking process in the mathematically correct manner. By substituting their answers into an altered form of the equation that had already been cleared of fractions, their checking procedures became simply a way to monitor the correctness of the manipulations that they had employed from that point on in their solution process.

The steps to be used in the checking procedure were presented very explicitly during the lectures on fractional equations. In the first example that Ellen gave on 7-1, she used the sentence "If there is a zero in a denominator, you throw out the answer" on four different occasions. In the second example she wrote out the rule "Check: If any answer makes any denominator = 0, just throw it out." On 7-2, Ellen used the phrases "zero in the denominator" and "throw it out" a total of eight times. Five of these occasions were in response to Sue's questions. She also wrote out her rule for checking.

Ellen did not emphasize that the checking process <u>must</u> be performed in the original equation in order to be valid. However, in each of the two examples that she worked through during the initial lecture on fractional equations, she made a point of mentioning that she needed to go back to the beginning of the problem to make her substitutions. (See page 188.) In each case she referred to the original equation as she performed a verbal check on her answers.

The fact that the original equation must be used is implicitly stated in the requirement that the checking process involves the search for <u>zeros in the denominator</u>. Ellen continually used this phrase when she referred to the checking process. Apparently, by the time the students answered the first quiz problem on fractional equations on 7-6, they had not retained the information from the lectures that connected the denominators with the checking process that they used.

It is not clear why Bill, Peter, and Elaine elected to check their answers into some form other than the original line of the equation. It may be that they were trying to avoid the complicated fractional calculations that would have resulted when their answers were substituted in the original fraction. They selected a different equation for checking that produced a set of numerical calculations that could be more easily evaluated.

The students may also have obscured the original form of the equation by the way in which they wrote their answers on the quiz papers. There was sufficient room on the quizzes for students to do their work right on the paper. They did not recopy the equation that was given at the beginning of each problem, but began their work by writing down the multiplication factors next to each term in the original problem. Thus the form of the original equation became obscured as students multiplied each term by various expressions and drew cancellation lines through appropriate factors. (Appendix B presents a facsimile of a student's written quiz response for this kind of problem.) It may be that this practice of writing over the original problem encouraged students to ignore this form of the equation. When they checked, students tended to select another equation farther along in the solution process that presented a less messy appearance.

It is apparent from the checking that was done that those students who used it did not understand the mathematical purpose for which such checking was performed. They were aware that checking should be used and that its outcome affected the final solutions of a problem. They were not, however, able to remember or apply the correct rules of checking to the solutions of fractional equations.

It is not possible to assess whether the procedure for checking fractional equations could have been utilized by more students than the four whose quiz responses were analyzed in this chapter. The techniques for solving fractional equations appeared difficult for students to master. On the first two times that fractional equations were tested (7-6 #2 and 7-7 #2), there were only three responses out of a possible 16 in which a student was able to work the problem through to an answer. Consequently, few students were in a position to utilize the checking procedure with these problems. The third problem on fractional equations was checked by the four students discussed earlier. Only one other student in the class found an answer to this problem.

From the analysis of the ways in which the students in the class utilized the checking process it is apparent that they did not understand the topic at the appropriate mathematical level. This is to be expected, since the techniques for solving and checking fractional equations were presented in class through a reduction in level of mathematical conceptualization as a collection of rules to be memorized.

No attempt was made to provide instruction dealing with the concepts inherent in the checking process. Checking was stated simply as one more rule of manipulation in the sequence of steps that were required to solve each problem. Those students who were able to successfully solve fractional equations demonstrated that they had mastered the rules for achieving a solution to this type of problem. However, they were not able to successfully master the final rule of checking that described the appropriate steps to be taken after solutions were found. Students did not understand the rationale for which checking was required. Those students who checked their solutions did not know why they were checking, but they knew they had to do it. Consequently they did not always use the appropriate mathematical procedure. The reasons why students were able to master the solution techniques, but not the checking procedures will be investigated further in Chapter Fourteen after the analysis has been completed on all the different quiz responses.

CHAPTER NINE

SYSTEMS OF EQUATIONS

This chapter examines the lecture content and quiz responses pertaining to the subject of systems of simultaneous equations. The lectures on this topic occurred during the week following the lectures on linear equations, but before the topic of fractional equations was introduced. (See Table 12, Appendix B.) The student responses that are examined in this chapter were written during the week following the lectures on fractional equations. It was decided, therefore, to place this topic within the chronological order of the written quiz responses, instead of within the order in which the topics were presented throughout the course. The student responses will be examined within the context of the lectures on systems of equations and also within the context of subsequent instructional material. The information in this chapter will be used to investigate the ways in which the students' concepts of the checking processes continued to develop as the course of instruction progressed.

Classroom Instruction

The portions of the lecture transcripts that are included in this section provide information on the types of words and phrases that the instructor employed during her discussions of the checking procedures. It is possible to trace the influence of these and other lectures on the students' work by comparing the words and phrases that they included on their written quiz responses to those used by Ellen in her lecture examples.

Thursday, June 25

The topic of systems of simultaneous equations was introduced during the morning lecture on Thursday, June 25. The instructor began her presentation by drawing a graph of two intersecting straight lines and describing their point of intersection as the solution of the system of equations that was represented by the two lines. She then discussed the different solutions that resulted if the system of equations represented two parallel lines or two equations representing the same line.

Then Ellen said, "I can also check if you tell me something is a solution. Let's suppose you tell me that (-1, 1) is a solution of this system."

$$x - y = 2$$

$$3x + 5y = 2$$

"Can you tell me that this is the solution of the system? I can check and find out if that is right. Is it really a system? And if it is a system, is it a solution of the system?"

"What I'll have to do, is that every time I come to an x, I'll put in a negative 1. Every time I that come to a y I'll put in 1. Do I get a true statement?"

ck;

$$x - y = -2$$
$$-1 - 1 = -2$$
$$-2 = -2 \qquad \text{true}$$

"That is true. Negative 2 equals negative 2. Now, if this is a system, there are two equations. So I'll have to make the same check in the other equation."

Ellen substituted her values of x = -1 and y = 1 into the second equation. When the check produced a numerical identity she said, "2 equals 2. That's true."

$$3(-1) + 5(1) = 2$$
$$-3 + 5 = 2$$
$$2 = 2 \qquad \text{true}$$

"If, when I put the number into the second equation, it comes out false, then it wouldn't be a solution. It has to work in both equations. It is supposed to be the point where the lines cross. Therefore, this point is supposed to be on both of those equations. We checked to find out if the statement was true. And the answer is yes, this is a solution. It is true." She then wrote, "yes, it is a solution," next to her last check.

"So if you asked me if it was a solution, I would say, yes. I checked it out, and they both came out true. If one of them came out false, or if both of them came out false, obviously, no, it is not a solution. I would say, hey, that's a false statement."

The students then worked through two practice problems that involved finding the solutions of systems of equations by graphing and two problems that asked the students to determine if a given pair of values represented a solution to a particular system of equations. I was absent from the class during the afternoon lecture when Ellen demonstrated techniques for solving systems of equations using a substitution method. I therefore have no record of the specific types of information that were presented to the students at that time.

<u>Comments</u>

The examples that Ellen presented during the morning lecture illustrate a technique that can be used to verify that a given pair of values is a solution to a particular system of equations. This information is based on the mathematical relationship that exists between an equation and its solution set. During her lecture, however, Ellen did not provide the students with any instruction concerning the relationships that were embodied in her examples. She justified her procedures to the students by stating them in terms of a given rule: a certain numerical value is regarded as a solution to an equation when it can be shown to produce a numerical identity as the result of substitution.

Ellen did not discuss in detail the mathematical relationships that existed between the graphical representation that she used in her lecture and the algebraic equations that represented the same system of equations. She simply stated that the coordinate points of the intersection of the two lines were used to provide the numerical values for the solutions of the two variables within the pair of equations.

Friday, June 26

Ellen continued instruction on the topic of systems of equations the following day, Friday, June 26. The first lecture illustrated the techniques to be used to solve simultaneous equations by addition and substitution methods. Ellen worked through the first example.

given: $x + y = 5$

$x - y = 1$

solution: $x = 3$, and $y = 2$, or $(3, 2)$

When she had arrived at her solution, she said, "We have a solution. I added the equations and we got an equation with a single letter. We solved for x. Now just take that value for x back in the equation and find the value for y. Let's check that and see if it works."

She substituted the values for x and y into the two equations and explained the checking steps as she wrote them on the transparency.

ck:	$x + y = 5$	$x - y = 1$
	$3 + 2 = 5$	$3 - 2 = 1$
	OK	OK

Ellen reviewed the steps she had used to solve the above problem. She then continued:

"We write down the point, and, if your arithmetic is like mine, you always check when you are done. Anytime you solve the system of equations you check."

Ellen then worked through a second example:

$$3x + 5y = 6$$
$$5x + 3y = 4$$

to arrive at the solution: $x = 1/8$ and $y = 9/8$.

After she had reviewed the steps that were used, she said, "We finally have a solution. Probably better check that and see if it works or not.... We want to put 1/8 in place of x and 9/8 in place of y to see if we get a true statement."

Ellen proceeded to write down the arithmetic calculations of the check. She was careful to explain all of the steps that involved manipulations of fractional terms. When she was done she commented on her work by saying, "So I really did OK."

$$3x + 5y = 6 \qquad\qquad 5x + 3y = 4$$

$$\frac{3}{1}\cdot\frac{1}{8} + \frac{5}{1}\cdot\frac{9}{8} = 6 \qquad\qquad \frac{5}{1}\cdot\frac{1}{8} + \frac{3}{1}\cdot\frac{9}{8} = 4$$

$$\frac{3}{8} + \frac{45}{8} = 6 \qquad\qquad \frac{5}{8} + \frac{27}{8} = 4$$

$$\frac{48}{8} = 6 \qquad\qquad \frac{32}{8} = 4$$

$$6 = 6 \quad \text{OK} \qquad\qquad 4 = 4 \quad \text{OK}$$

The complete transcript of this example is presented in Appendix B.

The students then worked through nine practice problems assigned from the textbook. In this book, the authors wrote out the steps for a check at the end of each example that was provided in the text. The instructions for working the problems in the exercise set that was assigned to the students, however, specified to solve but did not mention checking. The homework that was assigned at the end of that day included ten problems from the text that dealt with finding solutions to systems of equations by the methods illustrated in class. There were no other problems assigned as homework on this topic after the 6-26 assignment.

Comments

The checking procedure that Ellen used with the above examples served a different purpose from the way in which checking had been used during the morning lecture on the previous day. At that time, Ellen substituted specific numerical values into her equations in order to determine if they were true solutions to particular systems of equations. Checking was used to determine the mathematical validity of a proposed solution set. On the following day, as illustrated in the above examples, checking was used instead to verify the accuracy of the manipulations that were performed during the solution process. The values that were obtained for these equations were automatically true solutions if no manipulation errors had been performed. The checking steps were not required in order to determine the mathematical validity of these solutions.

Ellen did not make this distinction clear in her lecture on June 26. The fact that the checking steps were being used simply to verify the accuracy of the solution process was only implicitly stated when she said, "If your arithmetic is like mine, you always check when you are done." Ellen used a different form of comment following the last line of her check for these problems. On the day before, when she was determining if the given values were mathematically valid, she wrote "true" following the check. During the subsequent lecture, when she was checking the accuracy of her work, she wrote "OK" following each numerical identity.

The mathematical reasons were never explicitly stated as to why Ellen labeled checking statements "true" for the examples in one lecture and labeled the checking statements in the other lecture with the phrase "OK." Her choice of comments illustrates the implicit nature of much of the mathematical information that was presented to the students during the lectures. The mathematical information contained in each example was correct, but much of the implications of what was written on the transparencies was never made explicit to the students.

Student Responses Using Checking

On Tuesday, June 30, the students were given a problem involving simultaneous equations on both the group and individual parts of the quiz. No copies of the students' quiz papers were made that day, so there is no record of their responses to these problems.

The students were again tested on simultaneous equations on the super quiz given on Friday, July 10. The lectures on simultaneous equations had been given on June 25 and 26. By July 10, students had also been exposed to the unique checking requirements of fractional equations on July 1 and radical equations on July 9.

7-10 #5 solve this system $\begin{aligned} y &= 1 - x \\ 5x - y &= -13 \end{aligned}$

Only two students were not able to arrive at some kind of answer to this problem. Peter, Bill, and Sally solved the two equations correctly and also checked their solutions. Their checking steps are shown below.

Peter solution: (-2, 3)

ck: $\begin{aligned} 3 &= 1 - (-2) \\ 3 &= 1 + 2 \\ 3 &= 3 \quad \text{OK} \end{aligned}$ $\qquad \begin{aligned} 5(-2) - 3 &= -13 \\ -10 - 3 &= -13 \\ -13 &= -13 \quad \text{OK} \end{aligned}$

Bill $x = -2$, $y = 3$

CH: $\begin{aligned} 3 &= 1 - (-2) \\ 3 &= 3 \quad \text{works} \end{aligned}$ $\qquad \begin{aligned} 5(-2) - 3 &= -13 \\ -10 - 3 &= -13 \\ -13 &= -13 \quad \text{works!} \end{aligned}$

Sally $x = -2$, $y = 3$

$$\begin{aligned} 5(-2) - 3 &= -13 \\ -10 - 3 &= -13 \\ \hline -13 &= -13 \end{aligned}$$

Sally substituted her solutions only into the equation $5x - y = -13$. She did not perform a check on the other equation. She did not write any comment on her paper following the equality in the last line of her check.

7-10 # 7 solve this system: $\begin{aligned} 2x + 3y &= 1 \\ 5x - 2y &= -7 \end{aligned}$

Only three students in the class were not able to arrive at some kind of solution to this problem. Peter, Bill and Sally were again the only students to check their solutions.

<u>Peter</u>
(7-10 #7)

$$2x + 3y = 1$$
$$5x - 2y = -7$$

solution = (-1, 1)

$$5(-1) -2(1) = -7$$
$$-5 -2 = -7$$
$$-7 = -7 \quad \text{OK}$$

$$2(-1) + 3(1) = 1$$
$$-2 + 3 = 1$$
$$1 = 1 \quad \text{OK}$$

<u>Bill</u> Bill proceeded through the steps in the two checks in the same manner as Peter. He did not write any comment after the last line in each check. However, he wrote the following beside the solution to his problem, "They are a solution!"

Bill's comments on 7-10 #7 of "They are a solution!" and on 7-10 #5 of "works!" use the same phrases that Ellen used during her lecture examples on June 25 and on July 9 in connection with checking procedures that were used to determine the mathematical validity of her solutions. It is not clear, but it can be argued, that Bill did not regard his checks as simply a means of verifying the accuracy of his calculations. By using the same phrases that Ellen had used in a different mathematical context, his comments could be interpreted to mean that he did not consider the answers that he had derived to be legitimate solutions until the checking processes had revealed a set of numerical identities.

The instructor did not explain to the students during her lecture on June 26 that, given a system of simultaneous linear equations, if a solution exists, then it is automatically mathematically valid; i.e., such systems do not yield extraneous solutions. Therefore, it is not necessary to use the checking process to validate the solutions that are found. The purpose of checking this type of problem is to monitor the accuracy of the manipulations that are used during the solution process.

Students may have been mislead as to the purpose for which checking was used with simultaneous equations. The instructor introduced the topic on June 25 by including examples in which a set of solutions was presented along with a system of equations, and the question was posed as to whether they were solutions to the given equations. These values, however, were not derived by a direct solution of the system but were generated by some external means. If such values had been found by solution, then they would have automatically been considered valid.

By the time that the students responded to the quiz problems 7-10 #5, and 7-10 #7, they had also received instruction on fractional and radical equations and been introduced to the purposes that checking served in validating solutions to these types of problems. Bill's comments following the checks of his answers to the problems dealing with simultaneous equations can be interpreted in several ways. He may have been assigning the same meaning to the words "they are a solution," that Ellen used when she checked for mathematical validity. He may have been unaware, however, of the different mathematical purposes for which checking could be employed and thus, used the phrases unknowingly in an inappropriate context.

<u>Sally</u>
(7-10 #7)

$$2x + 3y = 1$$

$$\begin{array}{rcl} 5x - 2y & = & -7 \\ +2y & & +2y \\ \hline \frac{5x}{5} & = & \frac{-7}{5} + \frac{2y}{5} \end{array}$$

$$x = -1.4 + 0.4y \qquad (1)$$

$$2(-1.4 + 0.4y) + 3y = 1 \qquad (2)$$

$$2(-1.4) + 3.4y = 1 \qquad (3)$$

$$\begin{array}{rcl} -2.8 + 3.4y & = & 1 \\ +2.8 & & +2.8 \\ \hline \frac{3.4y}{3.4} & = & \frac{3.8}{3.4} \end{array}$$

$$y = 1.11$$

$$2(-1.4 + 0.4(1.11)) + 3(1.11) = 1$$

$$2(-1.4 + 0.444)$$

$$2(-0.956) +$$

$$-1.912 \quad + \quad 3.33$$

$$1.418 = 1$$

Sally elected to solve the system of equations by the substitution method. She solved the second equation for x to produce line 1, which she then substituted into the first equation. She made a multiplication error in line 3 which caused her solution for y to be mathematically incorrect. She did not substitute this value of y back into line 1 in order to solve for x, and thus, did not finish the problem.

It may be that she was attempting to find the solution for x when she substituted her value of y = 1.11 into line 2. This step, however, produced a checking statement rather than a numerical solution for x. She performed the arithmetic calculations of the subsequent steps correctly, but arrived at a false statement in the last line because of the error she had previously made in the solution for y. She did not include any written comment on her paper following the non-equality in the last line of her check. She did not attempt to change what she had written, nor did she go back and examine her solution process for errors. Sally's work seems to indicate that she was not clear on what sequence of steps she needed to use in order to arrive at a correct pair of solutions for the systems of equations. Her manipulations on this problem do not provide a good example to be used to assess her ability to apply the checking process to this type of problem.

Sue

Sue did not check her solutions. However, she made an interesting set of statements regarding her answers to the two problems involving simultaneous equations on the 7-10 quiz.

(7-10 # 5)

$$y = 1 - x$$

$$5x - y = -13$$
$$\underline{\quad +y \qquad +y}$$

$$\frac{5x}{5} = -\frac{13}{5} + y \qquad (1)$$

$$x = \frac{13}{5}$$

(2) $$y = \frac{5}{5}1 - x\left(\frac{13}{5}\right)$$

$$y = \frac{5}{5} - \frac{13}{5}$$

$$y = -\frac{8}{5}$$

Not A Solution

Sue failed to bring down the term involving y when she wrote line 1. In this line she also dropped the minus sign in front of the 13/5. Thus her solution for x = 13/5 is in error. Because she substituted this incorrect value of x into line 2, her answer of y = -8/5 is also in error. Her comment of "Not A Solution" will be discussed following the next example.

(7-10 #7)

$$2x + 3y = 1$$
$$5x - 2y = -7$$

$$2x + 3y = 1$$
$$\underline{\quad -3y \qquad -3y}$$

(1) $$\frac{2x}{2} = -\frac{2y}{2}$$

$$x = -1$$

$$5(-1) - 2y = -7$$

$$-5 - 2y = -7$$
$$\underline{+5 \qquad\qquad +5}$$

$$\frac{2y}{2} = -\frac{2}{2} \qquad (2)$$

$$y = -1$$

Is A Solution

Sue began her solution process by solving the first equation for x. However, in line 1 she incorrectly combined 1 - 3y to produce -2y. She then incorrectly simplified

-2y/2 to the value of -1. This error coincidentally produced the value of x = -1, which is the correct solution. She then dropped the minus sign in front of the term 2y when she wrote down line 2, which lead to an incorrect solution of y = -1.

On July 9, the day before this quiz, students had been given a lecture in which the instructor had introduced the rules that were to be used when checking solutions to radical equations. When the last line in such a check produced a non-equality, she wrote, "not a solution," following the checking steps. When the check produced a numerical identity, she wrote, "is a solution," on the transparency. (See Chapter Ten.)

It is possible that Sue had this information in mind as she solved the two problems dealing with simultaneous equations. For #5, in which her answers were y = -8/5 and x = 13/5 she wrote, "Not A Solution." For #7, in which her answers were x = -1 and y = -1 she wrote, "Is A Solution." Apparently, she regarded the two values of -1 as a numerical identity that was to be treated as if it represented the last line in the checking statement for a radical equation.

Such comments as "is a solution" should follow a checking process in which answers are substituted back into the original equation. Sue did not perform this kind of check with a single answer on any of her quiz papers. Yet on the quiz of July 10, she used some of the terminology associated with the checking process. Her response indicates that she was not able to attach an appropriate level of mathematical meaning to the phrase "is a solution," yet she was perfectly willing to use it in her written work.

Discussion of Students' Work

It is possible to draw only limited conclusions from the student responses analyzed in this chapter, since only two questions were examined. They were both present on the same quiz, which means that there was no information available that could be used to measure the students' responses over time. The students' responses would be expected to be consistent throughout a single test, so the analysis in this chapter is in effect over one set of problems.

An interesting aspect of the students' work is the way in which they utilized written comments based on the results of their checking procedures. It appeared that Bill and Sue were influenced in their responses by the information pertaining to radical equations that had been presented in a lecture the day before the quiz. The choice of comments that were made by these two students can be interpreted to indicate that neither student understood, at the appropriate level, the mathematical variations that the checking process represented with different types of equations.

It is not possible to determine from Peter's use of the words "OK" following each check, whether or not he was assigning the appropriate function to the checking steps he used. He could have been signifying that his check had demonstrated that his calculations were "OK," or that his answers were mathematically correct because his checks were "OK." It is always easier to attribute various types of thinking to students when they produce mathematically incorrect manipulations than when their work does not display any mistakes.

Of the eight students in the class, only Don and Elaine were unable to achieve some kind of solution to the two problems involving systems of equations. The majority of the students, however, did not employ any kind of checking procedure with these two problems. Only the three students Peter, Bill, and Sally went on to check their solutions. Alice did not use checking with her two problems, although she had been one of the few students in the past to check other types of problems. It is apparent from the students' written responses that, at the time they answered these questions, they had not developed an appropriate level of mathematical understanding for the use of the checking process.

CHAPTER TEN

RADICAL EQUATIONS

The students were provided additional exposure to the checking process during the lectures on radical equations that were presented on Thursday, July 9. The lecture material that day represented the students' first exposure to equations containing radical expressions. Throughout that week, the unitary operation of finding the square root of a number or expression had been treated in various lectures as students had been exposed to techniques for simplifying, adding, subtracting, multiplying, and dividing algebraic expressions containing square roots. Most of this information regarding roots appeared to be new to the students, and they required a great deal of practice with each type of manipulation that involved a radical expression.

The students did not appear to experience as much difficulty in mastering the techniques for solving radical equations as they had in mastering the appropriate techniques for solving equations containing fractional terms. (These techniques were described in Chapter Six.) The students were familiar from the lectures on fractional equations with the special use of the checking process as a means of determining whether or not solutions to certain equations should be retained or rejected.

Classroom Instruction

The following are excerpts from the transcripts of the introductory lecture on radical equations from Thursday, July 9. Ellen's comments at the beginning of this lecture are of interest because they present a mathematical justification for the procedures that she then demonstrates in the lecture.

Ellen introduced the topic of radical equations by saying, "What we're going to do is solve equations that have square roots or radicals in them, and we're going to do it by squaring both sides of the equation."

"Now, when we square things, some interesting things happen." She wrote on the transparency:

$$\text{if } a = b$$
$$\text{then } a^2 = b^2$$

"In fact that really is a law that is true and I think we should put it on a vocabulary card." There was a pause while one of the students wrote the law on a sticker for the vocabulary board.

"This law is interesting because it is true that if I have a quantity equal to another quantity and I square both sides, I'll still have a true statement. The reason this law is interesting is because it doesn't go the other way."

"That is, I start with the square of something. Maybe _a_ was negative 3 and _b_ was positive 3. The squares aren't equal to each other." She wrote on the transparency:

$$9 = 9$$
$$(-3)^2 = (3)^2$$
$$-3 \neq 3$$

"So it's fair for me to take an equation and square both sides. I'll get a true statement. However, when I have this true statement about squares, it does not guarantee that my original statement is true."

"In practical terms, what that means is that when I get done solving an equation, if I have squared both sides, I've got to check my answer and find out if it really is a solution. Maybe it doesn't work. Maybe I just have to throw it out."

"So we have the same sort of situation that we have to check that we had when we had the unknown in the denominator. We solved the equation. We went through the steps, but going through the steps didn't promise that the answer was going to work. Sometimes we looked at the answer and said, 'Uh oh, that makes _that_ denominator equal to zero,' and threw that answer out and didn't get to keep it."

"Well, the same thing can happen here. I can go through the steps and get an answer, but it doesn't necessarily work. Let's try some."

The first problem was solved as follows:

$$\begin{array}{r}
\sqrt{2x} - 4 = 7 \\
+4 \quad +4 \\
\hline
(\sqrt{2x}\,)^2 = (11)^2 \\
\dfrac{2x}{2} = \dfrac{121}{2} \\
x = \dfrac{121}{2}
\end{array}$$

When Ellen arrived at the solution to the equation, she said, "I am not done. I don't know that this answer is going to work. Let's check it." She then substituted her solution into the original equation and produced a numerical identity. She commented, "It is a solution. I get to keep that one. It does work." Her check is shown below.

$$\sqrt{2\left(\frac{121}{2}\right)} - 4 = 7$$

$$\sqrt{121} - 4 = 7$$

$$11 - 4 = 7$$

$$7 = 7$$

Ellen wrote, "is a solution," next to the answer x=121/2 on the transparency.

She then worked through the following example.

$$2\sqrt{y} = -2$$

$$\frac{2\sqrt{y}}{2} = \frac{-2}{2}$$

$$(\sqrt{y})^2 = (-1)^2$$

$$y = 1$$

When she had found her solution, she said, "OK. We'd better check that and see if it works." Her check, however produced the statement 2 = -2. When she arrived at this result, she said, "Uh oh! I'm in trouble. 2 doesn't equal negative 2!" Her check was written as follows.

$$2\sqrt{1} = -2$$

$$2 \cdot 1 = -2$$

$$2 = -2$$

Ellen then said, "This is <u>not</u> a solution. So I guess I'm just going to have to say there is no solution. It's true I got an answer, but my answer doesn't work." She wrote the phrase, "there is no solution," under the last step in the check and also wrote, "not a solution," next to the line y = 1.

Ellen presented a third example in which the solution produced a numerical identity in the check. The transcript for this work is given in Appendix B.

The students then split up into groups and worked through nine practice problems assigned from the textbook. The students had difficulties proceeding through all the steps that were required to reach a solution for the first problem. It was necessary for Ellen and me to circulate around the classroom and work with the students until they became more familiar with the manipulations that were performed with the radical expressions in the equation.

Following the practice period, Ellen presented a second lecture on radical equations that she read from a handout sheet that was also distributed to the students. It was written by Ellen and contained examples of more complex equations. She used the overhead projector as she read through each example in order to write out the intermediate steps of the problems. A description of this lecture is given in Appendix B.

After the lecture, the students worked through six fairly complicated practice problems that were similar to the two equations on the handout sheet. Each equation contained two separate radical terms and at least one other term that did not contain a variable expression. The students were required to square each equation twice in order to find a solution. An example of this type of problem is shown below.

$$\sqrt{p + 15} - \sqrt{2p + 7} = 1$$

It was the end of the afternoon and the students were tired. They spent a great deal of time working through lengthy algebraic manipulations in order to simplify each of these problems to the point where it was possible to obtain a solution. I observed that the students were reluctant to check those answers that appeared in fractional form because they lacked appropriate skills for adding and subtracting fractions.

Only one of the practice problems contained an answer that was an invalid solution. Consequently, there was little focus throughout this practice session on the checking process. The students' goal appeared to be the ability to successfully complete the lengthy series of multiplications of radical expressions in order to reach a solution to each problem. Under these circumstances, the checking process became a secondary part of the instruction.

At the end of the day, the students were assigned four homework problems from the text and six additional problems on a handout sheet. Only one other problem on radical equations was assigned as part of the homework that was due the following week.

Discussion of Lectures

An examination of the lecture transcripts indicates that Ellen presented the information concerning the special checking requirements for radical equations in terms of a sequence of manipulation procedures. The decisions to be made concerning the mathematical validity of any solution were stated in terms of rules relating to the physical appearance of the last line of the check.

At no time did Ellen focus her presentation on the mathematical relationships of equality expressed by the equations, or on the relationships between the solutions and the solution process that made checking necessary. The focus of her lectures was at a procedural level. The objects of study were the manipulations and procedures that were required to reach a solution to each problem.

Ellen presented the principle of squaring in terms of an abstract rule. She stated that this was a law that was true, but did not provide any material that could be used by the students to relate this information to their existing mental structures. Ellen acknowledged that the students probably could not understand this information at an appropriate mathematical level when she rephrased her introductory explanation to present checking simply as a procedural rule. She said, "...in practical terms, what that means is that when I get done solving an equation, if I have squared both sides, I've got to check my answer and find out if it really is a solution. Maybe it doesn't really work. Maybe I just have to throw it out."

The special checking requirements were basically presented to the students through this single statement. The students were never provided instruction, which they could relate to, that would have enabled them to understand the mathematical relationships that were being expressed between their answers and the true solutions to each radical equation. Students were instructed to "throw out" answers because the last line in the check was a non-equality, not because of the mathematical relationships that this represented. Since the students never understood the mathematical significance of their operations, it is not to be expected that they would be able to apply the checking information appropriately in all situations.

Student Responses Using Checking

The following interaction occurred during the practice session following the first lecture. It illustrates the way in which many of the students in the class were attempting to manipulate algebraic symbols without having first developed an appropriate mathematical understanding of the procedures they were using.

I was watching Sally as she checked her answer to one of the radical equations assigned as a practice problem. She substituted her solution into the original equation and then attempted to square both sides of the resulting numerical statement. However, she was unaware that such an operation would negate the purpose for which the checking procedure was used. It was apparent that she had not understood the mathematical implications of the instructor's introductory discussion concerning the equality of squared expressions.

Sally performed the following steps as she attempted to check her problem.

$$\text{Solve:} \quad \sqrt{x} = -5$$

$$\text{solution:} \quad x = 25$$

$$\text{Sally's check:} \quad \sqrt{x} = -5$$

$$\sqrt{25} = -5$$

$$25 = -5 \qquad (1)$$

In the line 1, Sally squared the left side of the equation to remove the square root operation, but she neglected to do the same to the number on the right side of the equation. Had she squared both sides, her check would have yielded an equality. The solution of $x = 25$, however, is not a valid solution, since the correct check produces $5 = -5$. Sally was apparently using the squaring technique following her substitution because that was the first step that was used in her solution process. Since she did not understand the mathematical implications involved in squaring her check, this technique appeared, from her point of view, as a perfectly legitimate step to take. She did not see that she could evaluate the numerical expression that was under the radical sign without having to square both sides first.

Sally's actions provide an illustration of the tendency of students in the class to utilize certain manipulation techniques in inappropriate ways through their lack of understanding of the mathematical principles behind such manipulations. Such situations can occur whenever instruction is presented to the students at a mathematically reduced level of conceptualization.

Sally was working on a practice problem that was similar to the second lecture example provided by Ellen. These two problems were included because they presented simple radical equations whose solutions were not mathematically valid. However, a careful inspection of the original problem indicates that this type of equation does not represent a true statement to begin with.

$$\sqrt{x} = -2$$

By definition, the symbol $\sqrt{}$ represents the action of taking the principal square root, which is always represented by a positive number. Therefore, the statement $\sqrt{x} = -2$ cannot be true.

The above definition of the principal root was included in the introductory lecture on square roots that was presented to the students on Thursday, July 2. However, this information was ignored during the lecture on July 9 when the instructor presented the problem $2\sqrt{y} = -2$ as an illustration of the topic under study.

In this regard the instructor appeared to be following the textbook. The authors included a definition of the principal square root in their introductory section on square roots. Then in the section on radical equations they included the problem $\sqrt{x} = -5$ in the exercise set. This lack of instructional continuity between classroom topics helped to foster the students' perceptions of algebra as a collection of discrete and unrelated topics.

Student Quiz Responses

The two lectures on July 9 represented the only formal class time that was spent on radical equations. Between that day and the final day of class, 15 additional topics were taught. Therefore, by the time the students were tested on radical equations on Monday, July 13, and on Friday, July, 17, they were also responsible for the mastery of many other types of manipulation techniques. This intensive exposure to different topics may have affected the students' abilities to remember and correctly apply the solution procedures that they were introduced to during the two lectures on radical equations. The following written quiz responses indicate the various ways in which students utilized the instructional material that was presented throughout the course.

The quiz questions 7-13 #6 and 7-13 #7 that covered radical equations were not very difficult for the students to solve. Sally was the only student in the class who was not able to find an answer to either question. However, Elaine was the only student who checked the solutions to both problems. Peter and Bill checked their answers only to 7-13 #7.

<u>7-13 #6</u> solve: $\sqrt{x} = -2$

<u>Elaine</u>
(7-13 #6)

$$\sqrt{x} = -2$$
$$(\sqrt{x})^2 = (-2)^2$$
$$x = (-2)(-2)$$
$$x = 4 \qquad (1)$$
$$x = \sqrt{4} \qquad \sqrt{x} = -2$$
$$x = 2 \qquad \sqrt{2} \neq -2$$

no solution

Elaine solved the problem correctly to produce x = 4 in line 1. However, she then took the square root of this number, and reported her answer as x = 2. She checked her solution using the incorrect value of x = 2 and used the resulting non-equality to reject her solution.

A check performed with the correct answer of x = 4 yields a statement of 2 = -2, which indicates that the equation $\sqrt{x}$ = -2 does not have a mathematically valid solution. Elaine's final comment of "no solution" is the appropriate answer for the original problem. However, she arrived at this checking conclusion because of an error in her solution process.

The situation that Elaine produced by her manipulation error following line 1 illustrates a difficulty that exists in the implementation of the checking procedure with radical equations. The presence of a non-equality in the final step of the check is not an automatic guarantee that a mathematically valid solution for the original equation does not exist. Such a non-equality may also result when manipulation errors occur in the solution process. If checking is to be used appropriately with radical equations, it is important that students be aware of the two functions that this process performs.

The responses written by the students to problem 7-13 #6 indicate that, as a group, they did not regard checking as an essential part of the solution process for radical equations. Five of the eight students answered this problem correctly, but none of them checked their solution to determine its mathematical validity. Of the remaining three students, only Sally was not able to achieve some kind of answer to the problem. Yet, of these eight students, only Elaine included a check with her solution. None of the students in the class was able to recognize that, by the definition of the principal square root, the original equation in 7-13 #6 represented a mathematically false statement which, therefore, should not have a mathematically valid solution.

Problem 7-13 #6 was solved by a very straightforward squaring manipulation. It may have been that this solution was so easy to obtain that the students did not feel required to check their answers. The next problem on the quiz, 7-13 #7, presented a more complicated radical equation, and it is interesting to note that two of the students who did not check #6, did check #7.

<u>7-13 #7</u> solve: $\sqrt{3m - 4} = \sqrt{m + 2}$

Six students achieved some kind of an answer to this problem with three of these solutions being correct. Peter, Bill, and Elaine were the only ones who included a check with their answers.

<u>Peter</u>

$$\sqrt{3m - 4} = \sqrt{m + 2}$$

$$(\sqrt{3m - 4})^2 = (\sqrt{m + 2})^2$$

$$\begin{array}{rcl} 3m - 4 &=& m + 2 \\ -m & & -m \end{array}$$

$$\begin{array}{rcl} 2m - 4 &=& 2 \\ -2 & & -2 \\ \hline 2m - 6 &=& 0 \end{array}$$

$$2(m - 3) = 0 \qquad\qquad \sqrt{3(3) - 4} = \sqrt{3 + 2}$$

$$2 \neq 0 \text{ so } \begin{array}{rcl} m - 3 &=& 0 \\ +3 & & +3 \\ \hline m &=& 3 \end{array} \qquad\qquad \begin{array}{rcl} \sqrt{9 - 4} &=& \sqrt{3 + 2} \\ \sqrt{5} &=& \sqrt{5} \end{array}$$

OK OK

<u>Elaine</u> solved the problem in a similar fashion to Peter and also made a direct substitution into the original

equation to obtain the same check. She wrote "OK" after the last line in her check.

Bill (7-13 #7)

$$\sqrt{3m - 4} = \sqrt{m + 2}$$

$$(\sqrt{3m - 4})^2 = (\sqrt{m + 2})^2$$

$$(3m-4)(3m-4) = m + 2 \qquad (1)$$

$$\begin{array}{r} 9m^2 - 24m + 16 = m + 2 \\ -m \quad -2 \quad -m \quad -2 \\ \hline 9m^2 - 25m + 14 = 0 \end{array}$$

$$(9m - 7)(m - 2) = 0 \qquad (2)$$

$$\begin{array}{ll} 9m - 7 = 0 & m - 2 = 0 \\ \quad : \quad\; : & \quad +2 \quad +2 \\ \quad m = 7/9 & \quad m = 2 \end{array}$$

CH: $\sqrt{3(2) - 4} = \sqrt{2 + 2}$ $\qquad$ $\cancel{\sqrt{3(7/9) - 4} = \sqrt{(7/9) + 2}}$

$\sqrt{6 - 4} = \sqrt{4}$ $\qquad$ $\cancel{\sqrt{\frac{21}{9} - \frac{9}{9}} = \sqrt{2\ 7/9}}$

$\sqrt{2} = 2$ $\qquad$ $\cancel{\sqrt{\frac{12}{9}} = \sqrt{2\ 7/9}}$

Bill introduced an error into the solution in line 1 when he squared $\sqrt{3m-4}$ to produce (3m-4)(3m-4) instead of the expression 3m-4. Bill was fortunate that this error produced a trinomial expression in line 2 that could be factored, which allowed him to find a solution to the problem.

On his paper Bill had crossed out the check for the answer x = 7/9 and had made no indication of the non-equality of $\sqrt{2} = 2$. According to the procedures that the instructor had presented in class, the existence of these non-equalities in the last line of the checks were used to indicate that there were no solutions to the original equation. However, Bill did not include any written comments as to what conclusions were to be drawn from his checking statements regarding the validity of the two answers that he had found.

7-17 # 6 solve: $\sqrt{m - 3} = 3 - \sqrt{m}$

The individual quiz on Friday, July 17 consisted of thirteen problems, and was used as a summative evaluation of the work that had been done during the final two weeks of the course. This quiz was given on the last day of class and was the only activity scheduled for that day. Students were finished with the course after they had completed both the individual and group quizzes.

Problem #6 on this test was a more complicated problem than the other two radical equations that the students had been tested on previously. Five of the eight students in the class were able to achieve some kind of answer to 7-17 #6, none of which, however, was correct. Alice was the only student who included a check with her answer.

Alice
(7-17 #6)

$$\sqrt{m - 3} = 3 - \sqrt{m}$$

$$(\sqrt{m - 3})^2 = (3 - \sqrt{m})^2$$

$$m - 3 = (3 - \sqrt{m})(3 - \sqrt{m})$$

$$m - 3 = 9 - 6m + m^2 \qquad (1)$$

$$\vdots \qquad \vdots$$

$$0 = 12 - 7m + m^2$$

$$\vdots \qquad \vdots$$

$$(m - 4)(m - 3)$$

$$\begin{array}{rcl} m - 4 = 0 & \text{or} & m - 3 = 0 \\ +4 \quad +4 & & +3 \quad +3 \\ m = 4 & & m = 3 \\ \text{keep} & & \text{throw out} \end{array}$$

Alice made an error at the end of line 1 when she multiplied $\sqrt{m}$ by $\sqrt{m}$ to obtain m^2. However, this mistake produced an equation that she was still able to solve. Alice apparently performed her checking substitutions mentally. The form of the original equation was simple enough that this could be done in a straightforward manner.

If the manipulations are correctly performed on the above problem, the equation produces a single solution of $m = 4$. This answer yields a numerical identity when checked, which presumably Alice verified mentally. The value of $m = 3$ is the result of Alice's manipulation error in line 1 and does not represent an extraneous root. However, it is not possible to discover this distinction from the non-equality produced by the check. Alice's comment "throw out" probably indicates that she was not aware of the reason for her checking non-equality. Alice's comments following her two answers, however, do indicate that she was aware that the checking process was to be used to evaluate the status of the two answers that she found.

Comments

The fact that the answers to radical equations are to be retained or rejected on the basis of a checking procedure is similar to that used with fractional equations. Thus, by the final week of class, students had received two separate exposures to this particular aspect of the checking process. However, the number of students who used checking with radical equations did not change appreciably from the number who had checked fractional equations earlier in the course.

For the fractional equation in problem #12 on the July 10 quiz, four of the five students who achieved some sort of an answer checked their results. Yet, for the radical equation in problem #7 on the July 13 quiz, only three of the six students who found an answer also included a check of their solution. It appears, therefore, that the procedure of checking equations for mathematically invalid solutions did not become more widely adopted by the students as they received additional instruction and further practice.

Those students who checked their solutions to radical equations substituted their answers back into the original form of the equation each time. This was not a practice that was used as consistently when the students checked fractional equations. One reason why such substitutions were correctly performed with radical equations may have been that the resulting numerical statement was easier to evaluate than that produced by substitution of an answer into a fractional equation.

It is not clear that any deeper understanding of the checking process had been achieved by the students at the end of the course than that which they were able to demonstrate at the beginning. Throughout the course, students appeared able to correctly reproduce the required sequence of manipulations in order to solve particular types of problems. However, their inconsistent use of the checking processes indicated that they had not developed an appropriate level of mathematical understanding of the procedures involved. Such behavior is commensurate with the level of mathematical instruction that was presented to them throughout the course.

CHAPTER ELEVEN

QUADRATIC EQUATIONS

The lectures on solving different types of quadratic equations represent the final exposure that students were given regarding the concepts relating to equations, their solution processes, and the checking procedure. The student responses to quiz questions following these lectures are of interest because they reflect the sum of all their learning experiences throughout the course regarding the use of checking. This work can be used to investigate the types of concepts concerning the checking process that had been developed by the students by the end of the five week course.

Checking does not form an essential part of the solution process for quadratic equations. Unlike the solutions to fractional and radical equations, the answers that are found to quadratic equations are always mathematically valid. Checking is used instead with these values as a way to determine the accuracy of the manipulations that are performed during the solution process. If no calculation errors are made, these solutions will always produce a numerical identity when substituted back into the original quadratic equation. (See Chapter Six for a discussion of the checking requirements for quadratic equations.)

Lectures were presented on quadratic equations at two different times during the course. During the second week of class, on Tuesday, June 23, the students were introduced to techniques for solving quadratic equations by factoring. The topic was not dealt with again until Friday, July 10. On that day, students were introduced to the general form of a quadratic equation ($ax^2 + bx + c = 0$), and provided techniques for solving equations of the form $ax^2 = c$. The lectures on Monday and Tuesday, July 13 and 14, provided techniques for solving other types of quadratic equations by factoring, completing the square, or by using the quadratic formula.

At the time that the first lecture on quadratic equations was presented on June 23, the students had been introduced to only one kind of checking process. Up until the first lecture on fractional equations on July 1, checking was regarded by the students solely as a technique for verifying the accuracy of their work. The only formal lecture that had been given to them concerning this process was the pull strip lab on the first day of class. (See Chapter Seven.)

By the time that the students had received further lectures on quadratic equations on July 10, 13, and 14, they had also been exposed to a second kind of checking procedure. This type of checking was an essential part of the solution process for fractional and radical equations and was used to determine the mathematical validity of all solutions. The lecture on quadratic equations on July 10, occurred one day after the students had been exposed to information concerning solution techniques for radical equations. (See Table 12, Appendix B for a schedule of the class checking activities.)

Classroom Instruction

Tuesday, June 23

The first lecture on June 23 introduced the students to a specific technique for solving certain quadratic equations. This involved factoring the given equation and applying the principle of zero products in order to generate two simple linear equations. Many of the examples used in the lecture were already in factored form, thus obscuring the fact that the students were dealing with second degree or quadratic equations. The focus of the lecture was on manipulating each equation into the form $a \cdot b = 0$, so that the next step could be written as $a = 0$ or $b = 0$.

The instructor worked through the first two examples in this lecture without mentioning the checking process. The following transcript begins at the solution to the third example.

$$m^2 - 8m = 0$$

$$m(m - 8) = 0$$

$$\begin{array}{llcr} \text{either} & m = 0 & \text{or} & m - 8 = 0 \\ & & & \underline{+8 \quad +8} \\ & m = 0 & \text{or} & m = 8 \end{array}$$

"Can you remember back on the very first day? The little equations are the same. But the original equation is different, because we end up with two answers: $m = 0$ or $m = 8$."

"We should see if they both work, shouldn't we? We'd better check that out. I've just been assuming they work, and you've been assuming that because I did it, they work. But what if I made a mistake? Let's check this and see."

Ellen substituted the values for her two solutions into the original equation and produced two numerical identities in the last line of each check. She then said, "Oh, good, it really does check. We can solve the problem and we get two answers. We can check both answers and they usually do work." Her check was written on the transparency as shown below.

$$\text{CK:} \quad 0^2 - 8 \cdot 0 = 0 \qquad \begin{array}{r} 8^2 - 8 \cdot 8 \stackrel{?}{=} 0 \\ 64 - 64 = 0 \end{array}$$

The instructor worked another example through to a pair of solutions. Then she said, "This is the solution. You can put them back in and it turns out they check. We're running a little late, so rather than write the check down, I'm going to ask you to go to section 5.7 and do the margin exercises."

The next lecture was taken directly from the textbook. The topic dealt with the process of translating word problems into equations similar to those discussed in the preceding lecture. At the beginning of this lesson in the text, the authors presented a list of "problem solving tips" which included, "3. <u>Solve</u> the equation, 4. <u>Check</u> the answer in the original problem."

Ellen worked through three examples from the text, checking her solutions to only the first problem. The transcript of this part of the lecture is presented in Appendix B.

<u>Comments</u>

These two lectures were presented to the students prior to the lectures dealing with the specialized checking requirements of fractional and radical equations, and also prior to the lectures on systems of linear equations Ellen's use of checking on June 23, therefore, represents only the second time that checking was formally included in a classroom lecture; the first instruction consisted of the checking lab on the first day of class.

In the lectures described above, Ellen implied that her checking was used to verify the accuracy of her solution process when she said, "...what if I made a mistake? Let's check this." However, checking was presented to the students as an optional activity during this lecture. Ellen did not check the example that she presented following her checking illustration. She justified this decision by commenting that she was running out of lecture time, and thus would not include a check with her last problem. Instead, she told the students that if the checking manipulations were performed, the answer would "check."

Ellen's attitude towards checking was consistent with the goals of the course. Her purpose throughout the five weeks was to present examples of a sufficient number of procedures to enable students to manipulate a wide range of different types of algebraic expressions and equations. The emphasis was placed on achieving mastery of these techniques, not on developing an understanding of underlying mathematical relationships and concepts. The type of checking process that was used with linear and quadratic equations was, therefore, within this perspective considered to be only a minor, simple technique, compared to the large amount of more complicated manipulation procedures that needed to be covered in the syllabus. Under this condition, the straightforward checking techniques of linear equations was allotted minimum class time.

Monday, July 13

During the lecture on quadratic equations presented on July 10, the instructor did not check the answers to the problems that she used as examples. However, similar examples given in the textbook for equations of the form $ax^2 = c$ did contain a check for each answer.

Ellen did not include a check of her solutions for the first example she presented during the lecture on July 13. The following equation was the second example that she explained to the students.

$$4x(x - 2) - 5x(x - 1) = 2$$
$$4x^2 - 8x - 5x^2 + 5x = 2$$
$$\vdots \qquad \vdots$$
$$(x + 2)(x + 1) = 0$$
$$\vdots \qquad \vdots$$
$$x = -2 \quad \text{or} \quad x = -1$$

Ellen worked this problem through to a solution but did not discuss checking her answer. At this point Bill asked her a question.

Bill: "Are you supposed to check all those?"

Ellen: "We should go back and put them in and check them. That's right. To be sure they work."

Peter: "Do you check that against the original?"

Ellen: "You always check against your original equation as it first shows up. That's right. We would go all the way back to 4x times (x - 2) minus 5x times (x - 1) equals 2 and start by putting in negative 2 in all these places.... Rather than spend a lot of time checking the problem, we are going to look at a word problem that involves an equation that I am going to have to factor."

Ellen then presented a lengthy word problem whose solution was found by solving a quadratic equation. This example is of interest because Ellen evaluated the reasonableness of each answer by comparing the values she obtained against the conditions of the original word problem. This kind of verification represented a different type of checking than the substitution procedures that were used with other classroom examples. The transcript of this portion of the lecture is given in Appendix B.

At the conclusion of this lecture, the students were assigned six problems in the text and four additional word problems on a handout sheet to practice on. All the problems resulted in quadratic equations that could be factored into the product of two binomials. After Paul had solved his first practice problem, he asked me if he had to check the answers. I told him that these problems were not like the fractional and radical equations we had done before. If the work had been done correctly, both solutions to a quadratic equation were automatically valid answers. Checking was only used to catch mistakes.

Then Paul looked at me and asked hopefully, "I don't have to check?" Paul's comment is interesting because he did not once use checking in any of his quiz responses. Yet apparently, he was aware that checking was part of the techniques that they had been taught.

My comments to Paul were directed at the algebraic problems that he was working on at the time. I did not discuss with him the second use of "checking" that he would encounter when he began the word problems at the end of the assignment. For these problems, according to the lecture example, students had been instructed to check their solutions to see if they presented a reasonable answer to the original word problem. At no time in class did Ellen or I discuss with the students the various types of checking procedures that they had been exposed to or help them to distinguish the differences in their application.

The subsequent two lectures that day introduced techniques for completing the square of an algebraic expression and solving equations by this method. These techniques are used to solve quadratic equations that cannot be factored conveniently and produce answers in the form of $b \pm \sqrt{c}$. Such answers produce complicated numerical calculations when they are checked by substitution into the original equation. Neither the text nor Ellen mentioned the use of checking with these answers. The final lecture on quadratic equations was on Tuesday morning, July 14, and covered the use of the quadratic formula. Again, checking was not mentioned.

<u>Discussion of Lectures</u>

The checking process did not form a focus of instruction during the lectures on quadratic equations on

July 10, 13, and 14. It is not clear whether Ellen would have even mentioned checking on July 13 if Bill and Peter had not asked their questions during the lecture that day. Peter's question, "Do you check that against the original?" may have been prompted by the comments that Ellen wrote on his quiz response on Friday, July 10. In problem #12 Peter checked his two solutions, but failed to substitute these values in the original equation. Ellen wrote "Ck. in original equation!" on his paper following this work.

It was never made explicit to the students that checking was not a mandatory part of the solution process for quadratic equations. Bill's question, "Are you supposed to check all those?" was probably motivated by the emphasis that had been placed on checking during the lectures on radical equations the previous week, on July 9. This concern with including checking as part of the required manipulation steps was probably also behind Paul's question during the practice session later that day.

Bill and Paul asked their questions in order to verify that they were using an acceptable set of manipulation techniques to solve their problems. They were not aware of the mathematical reasons behind the answers they were given, nor did they ask for such an explanation. What was important for them to ascertain was whether or not the checking process formed part of the required set of manipulations for such problems.

Ellen was correct from her point of view, in not placing an emphasis on checking during her lectures on quadratic equations. She was aware that the solutions to such equations were mathematically valid, and since she knew that she had performed the correct manipulations to solve each problem, she did not need the checking step to monitor her work. Ellen was always under constant pressure to present a great deal of information within the limited amount of time that was allotted for each topic throughout the course. For these reasons, she did not include checking as part of her lecture presentations on quadratic equations.

The students, however, did not have access to the same concept of checking that had motivated Ellen's choice of lecture material. Their mental constructs relating to the checking process were based on the information and experiences that were made available to them in class. Ellen did not provide sufficient information during her lectures on quadratic equations to make explicit to the students her mathematical reasons for not employing checking.

Ellen and I were not aware at the time these lectures were given that students in the class were not being provided with adequate sets of experiences from which to develop appropriate mathematical concepts of the checking process. The focus of each lecture was centered on enumerating the various procedures that formed the solution techniques for different kinds of quadratic equations. The primary goal was to provide students with sufficient examples from which they could develop skill in solving such problems. It was not until I began analyzing the students' quiz responses that I realized that the students had not been provided with sufficient instruction in class to develop an appropriate set of concepts regarding the checking process.

Student Responses Using Checking

The students were given a quadratic equation to solve on the quiz of June 26. The responses to this problem are presented in Chapter Ten on linear equations. (See page 177.) Three other quadratic equations were presented as quiz problems on July 15, 16, and 17. All the students but Don were able to achieve some form of solution for each of these problems. However, few of the students also included a check with their answers.

7-15 # 6 solve: $2m^2 - 14m = 0$

Only Don was not able to achieve some kind of answer to this problem. Four students, Peter, Bill, Sally, and Elaine, found the correct solution. However, only Bill and Elaine checked their answers.

Bill (7-15 #6)

$$2m^2 - 14m = 0$$

$$2m(m - 7) = 0$$

$$\begin{array}{lcl} 2m = 0 & \text{or} & m - 7 = 0 \\ & & \underline{+7 \quad +7} \\ m = 0 & \text{or} & m = 7 \end{array} \qquad \text{ck: } \begin{array}{r} 2(7)^2 - 14(7) = 0 \\ 2(49) - 98 = 0 \\ 98 - 98 = 0 \\ 0 = 0 \end{array}$$

Elaine Elaine solved the problem in a similar manner. However, unlike Bill, she included a check for both solutions on her paper. Bill may have checked his answer of m = 0 mentally. Neither student wrote any comments on their papers following their checking steps.

7-16 #5 solve: $(x - 2)^2 = -5$

All the students in the class achieved some kind of solution to this problem. Paul, Alice, Peter, and Bill solved the problem correctly to produce the answer of $x = 2 \pm i\sqrt{5}$. The form of this answer is such that it is not easy to check by substitution, and none of the students had ever performed such a check before. Therefore, it is not surprising that none of the students who found this answer attempted to check their work by substitution.

Elaine made an error in her solution which produced answers that were whole numbers. She then checked these values.

Elaine (7-16 #5)

$$(x - 2)^2 = -5 \qquad (1)$$

$$(x - 2)^2 = (-5)^2 \qquad (2)$$

$$(x - 2)(x - 2) = 25 \qquad (3)$$

$$\begin{array}{r}
x^2 - 4x + 4 = 25 \\
-25 \quad -25 \\
\hline
x^2 - 4x - 21 = 0
\end{array}$$

$$(x + 3)(x - 7) = 0$$

$$\begin{array}{ccccc}
\begin{array}{r} x + 3 = 0 \\ -3 \quad -3 \\ \hline x = -3 \end{array} & \begin{array}{c} \text{or} \\ \\ \text{or} \end{array} & \begin{array}{r} x - 7 = 0 \\ +7 \quad +7 \\ \hline x = 7 \end{array} & \begin{array}{r} 7 - 2 = -5 \\ 5 \neq -5 \end{array} & (4)
\end{array}$$

No Solution

Elaine introduced a squaring operation on the right side of the equation in line 2 without performing the same operation on the other side of the equal sign. Because of this error, her answer of x = 7 is mathematically incorrect. It appears from her comment of "no solution" that she was rejecting her answer using the criterion from the checking process for radical equations, rather than regarding her non-equality as an indication of the presence of mathematical errors in her solution process.

Elaine substituted her value of x = 7 incorrectly into line 1 when she performed her check. She dropped the squaring operation from the left side of the equal sign when she wrote down line 4. It is not clear whether she checked her other answer of x = -3 mentally, or simply failed to check it. If Elaine made the same incorrect substitution as she had used in line 4 with a mental check of x = -3, she would have arrived at an equality in the check. It can be argued that she was retaining her answer of x = -3 and rejecting the answer of x = 7 from the comments she had written following her two solutions. The value of -3, however, does not produce a true statement in a correct check because of Elaine's error in line 2.

<u>7-17 #7</u> solve: $a^2 = 15a - 36$

This problem was included on the final super quiz that was given on the last day of class. Seven students were able to obtain some kind of solution to this problem. Alice, Peter, Bill, and Sally all arrived at the correct solution, but only Sally included a check with her answer.

<u>Sally</u>
(7-17 #7)

$$a^2 = 15a - 36$$
$$a^2 - 15a + 36 = 0$$
$$(a - 12)(a - 3) = 0$$

$a - 12 = 0$	$a - 3 = 0$
$a = 12$	$a = 3$
doesn't work	yes
$(12)^2 = 15(12) - 36$	$(3)^2 = 15(3) - 36$
$144 = 2700 - 36$	$9 = 45 - 36$
$144 = 2664$	$9 = 9$

Sally solved the problem correctly and substituted her answers into the original equation. However, she multiplied 15 incorrectly by 12 to produce the number 2700 in the second line of her check of x = 12. She apparently then used the criterion from the checking procedure for radical equations and rejected this answer because of the non-equality that was generated in the last line of her check.

Sally's written work on this problem illustrates the kind of situation that can occur when a student has been exposed to a variety of different mathematical situations that are linked by some common element. By the time that she responded to 7-17 #7, Sally had received instruction on various sets of techniques for checking solutions to five different types of equations. The common elements of these lectures were the word "check" and the technique of substituting each solution into the original equation. However, the purpose for which the checking process was used differed for different types of equations. Sally's responses indicate that she had mastered the technique of checking by substitution, but that she had not developed an appropriate level of mathematical understanding that enabled her to apply this technique in mathematically correct ways.

Other Use of Checking

The term checking covers a wide range of techniques that can be used to establish the acceptability of a solution to a given equation. On the first day of class, Ellen reminded the students that they could verify the correctness of the answers to the textbook problems by comparing them to the values given in the answer section in the back of the book. Checking was then presented to the students throughout the course primarily as a substitution technique. The lecture that Ellen gave on July 13 that dealt with the problem of constructing a box from a sheet of paper represented one of the few times that she discussed another aspect of checking. In that example, Ellen presented the criterion of the reasonableness of an answer as a way to validate a solution to a word problem.

In the examples described below, Peter illustrates another method that can be used to establish the correctness of a calculated solution. He uses the technique of comparing solutions that are derived in two different ways as a criterion for establishing the accuracy of his work. This particular method of checking was not discussed by Ellen during her lectures or used by other students in the class.

Peter checked his solution to problem #6 on the quiz of July 16 by reworking the equation using a different set of techniques. The directions for the problem specified that the given quadratic equation was to be solved using the quadratic formula. Once Peter had completed this work, he then resolved the problem using the technique of completing the square. This second set of solution steps was labeled "CK:".

Peter checked the accuracy of his answers to two equations on the final test in a similar fashion. In problem 7-17 #8 the students were specifically asked to find the solution to a quadratic equation by using the quadratic formula. Peter solved the problem correctly, and then, under the label "ck:", he solved the problem again using the technique of completing the square. Peter reversed this procedure for problem 7-17 #12. The directions specified that the problem was to be solved by the technique of completing the square. Following this solution, Peter then used the quadratic formula to solve the problem a second time. For some reason, this check was then erased on his paper.

7-17 #10

Find an equation whose solutions are -1/2 and 3/5.

Peter solved this problem correctly. He then wrote down the word "CK" and proceeded to work the problem backward. He started with the equation he had just derived, factored it, and solved the resulting equations to arrive at solutions of $x = 3/5$ and $x = -1/2$. He wrote "OK." following each of these values.

Comments

The procedure for checking answers to quadratic equations was not emphasized during the lectures and subsequent practice work on this type of equation. Most of the answers to the problems that were solved by the technique of completing the square or by the quadratic formula produced solutions of the form $a \pm \sqrt{b}$ that were not easy to check by substitution. It is not surprising, therefore, that few students attempted to check their answers to the quiz problems that dealt with quadratic equations.

Elaine's checking on 7-16 #5 and Sally's checking on 7-17 #7 are of interest because of the way in which they used the results of the last line in their checks. In both cases, these students used the statement of non-equality to reject their answers. Elaine indicated this decision by writing "no solution" on her paper, while Sally wrote "doesn't work" next to her answer.

At no time during the lectures on quadratic equations did Ellen ever present an example in which the solution to an equation was rejected as a result of the checking process. This, however, was not a sufficiently explicit statement of the fact that the purpose for checking quadratic equations differed from that for fractional and radical equations. It appeared that Elaine and Sally generalized the rules of checking from these two types of equations to apply to quadratic equations as well.

If Elaine and Sally had performed their calculations correctly, they would not have generated statements of non-equality in their checks. Their errors provided me with an insight into the types of concepts that they had developed for the checking process. If their solutions had been mathematically correct, I would not have been aware that they held mathematical misconceptions. For this reason, it is always difficult to accurately assess a student's set of mental structures from a correctly worked problem.

Ellen and I were not aware that some students were transferring information from the lectures on fractional and radical equations at the time that the lectures on quadratic equations were given. An analysis of the lecture transcripts and students responses described in this chapter suggests that additional instruction needs to be provided at this point in the course in order to enable students to understand that the checking requirements for quadratic equations are different from those of fractional and radical equations.

Peter's alternative method of checking is of interest because it demonstrates that he understood the basic concept of checking as a method of verifying the mathematical accuracy of his work. A criterion that can be used to demonstrate understanding is to observe an individual applying the concepts in question in a new and unfamiliar situation, or in a different fashion (van Hiele, 1986). Peter fulfilled this condition when he verified the accuracy of his solutions by resolving each problem using a different type of method.

It is not clear from the checking responses that were written by other students in the class whether they had developed comparable levels of understanding for the checking processes that they used on their quizzes. It is possible that they were simply reproducing the manipulation steps that they had observed the instructor using with similar equations. For such a situation, it is possible for students to produce mathematically correct responses by learning the proper sequence of manipulation steps and patterns of symbol placement without ever having to develop a parallel level of conceptual understanding. Using paper and pencil tests, it is only possible to assess the students' levels of mathematical understanding from analyses of incorrect responses or from their performance on applied problems.

This chapter concludes the analysis of the students' written quiz responses in terms of the ways in which checking was utilized with the different types of equations that were presented in the course. This analysis also investigated the ways in which these responses were affected by the chronological presentation of the topics and by the instructional content of each lecture.

The following three chapters in Part II will examine the overall checking behavior of each student individually and investigate the nature of the classroom instruction in more detail. These analyses will then be used to postulate a specific set of checking concepts that form a possible explanation of the students' behavior on their written quiz responses.

CHAPTER TWELVE

INDIVIDUAL PERFORMANCE

The purpose of this chapter is to present an analysis of the students' behaviors concerning their use or non-use of the various checking procedures that were presented throughout the course. The frequency with which checking was employed and the specific nature of the quiz responses are used to draw inferences concerning the types of meanings that students likely assigned to the checking procedures that they used. This analysis will first be presented in terms of statements about the class as a whole, and then, later, in terms of each student's individual performance. Subsequent chapters will place these student behaviors within the context of the actual classroom learning environment.

Group Behavior

There was a wide variation among the students in the frequency with which each individual employed checking with their solutions to the 19 quiz problems. Some of the students in the class used this process on a regular basis, while other students checked rarely, or not at all. A direct relationship was found to exist between the frequency with which students employed checking and their final course grades (Table 1).

Table 1 below compares each student's checking performance with their composite grade. This grade is the average of the three numerical grades for each of the three two-credit classes that were covered during the course. These two-credit grades were calculated on the basis of weighted averages of each student's individual quiz scores, group quiz scores, homework grades, and journal responses. These grades are based on many other sources of evaluation besides the 19 quiz questions that are examined in this study. Thus it is possible for students to do poorly on these specific problems and still do well enough on other material to achieve a passing grade in the course.

Table 1. Student Checking vs. Class Grade

Name	# Correct[a]	# Checked[b]	Grade
Don	0	0	63.3
Paul	4	0	72.4
Sue	4	0	72.9
Sally	7	3	76.4
Alice	10	5	78.4
Elaine	5	6	79.1
Bill	13	9	89.7
Peter	15	10	91.2

[a]Number of mathematically correct quiz responses

[b]Number of responses including some kind of check

There is a correlation of .91 (r = .954) between the students' grades shown in Table 1 and the number of times that students arrived at mathematically correct solutions to the 19 quiz problems. There is also a correlation of .88 (r = .937) between these grades and the number of times that students employed checking with these problems. These figures indicate that those students who tended to do better in class, as measured by pencil and paper responses, were more apt to employ checking on a regular basis.

A comparison in Table 1 of the number of correct responses to the number of checking responses indicates that the students in the class did not employ checking with every problem in which they were able to achieve a solution. The checking process was discussed during class lectures but not emphasized throughout the course. The frequency with which students employed checking on the quiz responses reflects this class approach to the subject.

Table 2 below lists the number of students per question that utilized some sort of checking process on each of the 19 quiz questions. The low frequency with which some of the questions were checked is an indication of the degree of emphasis that was placed on the checking process throughout the class. As can be seen in Table 2, there were 11 different occasions in which only one student out of a possible eight checked the answer to a specific quiz problem. The infrequent use of checking by all the students in the class also reflects the difficulty that certain students experienced in producing solutions to fractional equations, which limited the number of times that they were able to employ checking on these particular quiz questions.

Table 2. Frequency of Checking Response

# of Students checking/problem	# of problems
1	11
2	3
3	4
4	1

Individual Responses

It is difficult to make many generalizations concerning the performance of the class as a whole regarding the use or non-use of the various checking procedures because of the wide range of responses that were produced on the individual quizzes. For this reason, the analysis in this section centers, instead, on the behavior of each individual student, through an examination of their written responses to each of the 19 problems involving checking. The students' abilities to apply the various checking procedures in an appropriate manner to the equations presented on the individual quizzes will be assessed in terms of the frequency with which they used checking, and in terms of the specific responses they employed. This behavior, since it occurred over a period of time and in response to classroom instruction, will be used to infer the students' development of a concept of checking.

Checking Index

Specific instruction was provided to the class on July 1 and July 9 on the special checking requirements pertaining to the solutions of fractional and radical equations. It is therefore anticipated that the students will employ checking more frequently on quiz problems following these lectures. For this reason, a checking index was devised to provide a measure of each student's use or non-use of the checking process on those quiz problems that were given to the students following the lecture on checking on July 1. This index covers 10 problems examined from 7-6 #2 through 7-17 #7, with the exclusion of the two equations 7-16 #5 and 7-16 #6, which could not be easily checked by substitution. The quiz problems that were administered before July 1 were not included in the index because they occurred at a time when checking was not emphasized in class.

The checking index is calculated for each student by the following formula:

$$\text{checking index} = C/S$$

S = # of problems containing some kind of solution
C = # of problems which include a check

The checking index is designed to represent the ratio of the number of times that a student arrived at a solution to an equation and included a check with this work to the total number of solutions achieved. The numbers in the index do not include those problems for which a student was not able to generate some kind of solution. For example, a student who produced solutions (not necessarily correct ones) to seven of the 10 problems in the index, and included a check of some kind to three of these, would be assigned an index of 3/7. This number indicates that there were three problems out of a total of seven solutions for which this particular student did include a check with his or her work.

The relative size of the numerator and denominator of the index provide a comparison of the frequency with which students employed checking on the quizzes with equations that they were able to solve. The checking index for each student is included in Table 11, Appendix B, which presents a compilation of each student's checking responses.

Special Checking Requirements

Checking was used for two different purposes with the various types of equations that were presented during the class lectures. Checking was presented by the instructor as an essential part of the solution process for fractional and radical equations and was presented during lectures as a method for determining the mathematical validity of these solutions. On the other hand, checking was considered an optional activity with the solutions to linear, simultaneous, and quadratic equations, and was used by the instructor to monitor the accuracy of her solution processes.

One of the criteria that will be used to assess each student's checking performance in this chapter is the ability to apply these two different kinds of checking with the appropriate types of equations. If, for example, it is found that a student does not include checks with the solutions to fractional or radical equations, it will be inferred that he or she is not aware of the special checking requirements pertaining to these types of problems.

Information will also be obtained regarding the level of a student's knowledge of the various checking requirements by examining their comments written on the quiz papers following the final step in each check. Phrases such as "works," "keep," or "is not a solution" will be used to infer the type of purpose that each student was assigning to the checking procedure for a given problem. The use of these statements will not, however, be considered as a measure of the students' understanding of the underlying mathematical concepts. Such statements as "keep" or "throw out" can only be used as an indication that a student was relating his or her work to specific lectures in which the instructor employed those same phrases.

The following sections consist of analyses of the collected responses of each individual student in the class. These responses will be examined in terms of the nature of specific problems and also in terms of changes in the student's behavior over the entire course. A compilation of each student's behavior on the 19 quiz problems including a listing of correct solutions, use of checking, and the checking index is presented in Table 11, Appendix B.

Peter

Peter was the most successful student in the class on the basis of his written responses to the 19 quiz problems. He produced mathematically correct solutions to 15 of these problems and was unable to find an answer to only one equation. He checked six of his solutions by substitution and checked the answers to four additional problems by reworking each equation through an alternative method.

Nine of Peter's checks were performed on problems that had been solved correctly. All but two of these checks were error-free. On 6-23 #4, Peter made the same manipulation error in his checking process that he had used when solving the original linear equation, and in the check on 7-10 #12 he failed to make his substitution back into the original fractional equation. (See page 172 and page 203.)

Despite the fact that Peter responded correctly to 79% of the 19 quiz questions and checked over 50% of his answers, he only checked two of the six problems on the quizzes that dealt with fractions and radicals, equations for which checking was a mandatory part of the solution process. His checking index of 4/9 reflects this lack of a widespread use of checking. Although Peter appeared to have mastered the various solution techniques required for each type of equation, he did not show evidence in his quiz responses that he had understood the various mathematical reasons for which checking was used.

There does not appear to be enough evidence from Peter's written quiz responses to determine whether or not he had mastered the distinctions that existed among the different functions for which checking was used. All the checks that he performed produced numerical identities. Thus, Peter was never provided an opportunity to indicate what kind of conclusions he would have made had his checking steps produced a non-equality.

Peter did not apply the checking process correctly when he checked his solutions to the fractional equation in problem 7-10 #12. He incorrectly substituted his answers into a line of the problem other than the original equation. On his quiz, he lost one point out of five for this step, and the instructor wrote on his paper, "Ck. in original equation! $r = -2$ is OK. $r = -1$ makes a denom. = 0, so not a solution." These comments may have prompted Peter to ask Ellen in class the following Monday, "Do you check that against the original?" (See page 256).

Since problem 7-10 #12 was the last question in which fractional equations were tested, there was no other occasion on which to evaluate whether Peter's question and the instructor's subsequent comments would have had any effect on the way that Peter used checking with such equations. Peter checked only one of the three radical equations on later quizzes. In this problem his substitution was correctly made in the original equation.

Peter did demonstrate, however, that he was aware that the accuracy of his solutions could be verified by some sort of checking procedure. On four separate occasions on 7-16 #6, 7-17 #8, 7-17 #10, and 7-17 #12, he reworked a quiz problem by an alternative method to confirm his original answer. He was the only student in the class to use this technique of checking (See page 266).

Three of these checks occurred on the quiz given on the last day of class. This was a long quiz and was regarded in the light of a final exam by the students, since it was the last activity that they performed in the class. It could be argued that Peter had sufficient time during that quiz, and felt that the grade on his paper was important enough, so that he took extra care over the accuracy of his work and was more conscientious in using checking.

There is no evidence of erasures or changes in any of these solutions on his quiz papers. Since his original solutions were in each case correct, the checking procedures served as a means of reaffirming his work. It would have been interesting to have observed what steps he would have taken if his checking procedure had yielded a different answer.

Peter's checking responses do not reveal more than the fact that he used the procedures frequently. The absence of errors indicates that he had mastered the required manipulations for those problems he had checked but gives no indication of the level of mathematical comprehension on which he was operating.

Overall, Peter checked more answers than any other student. However he received a checking index of 4/9, which means that there were five opportunities in which he could have checked his answers and did not. He was the only student in the class who took the time to check four problems by alternative methods. However, he did not also make it a practice to check every answer for which he had generated a solution.

Bill

Bill correctly answered 13 of the 19 problems and found some kind of solution to all but two of the rest of the equations. He used the checking by substitution method more frequently than any other student, checking nine of his solutions. His checking index was 6/8, which indicated that he failed to include a check on only two occasions after he had found an answer.

There were three instances of checking in which Bill did not substitute his answers back into the original equation but into another, more simplified line of the problem. In two of the three fractional problems on 7-7 #2 and 7-10 #12 (he was unable to solve the third), Bill made his substitution into the simplified quadratic equation that was produced when the original equation was cleared of fractions. (See pages 199 and 204.) According to the checking requirements of fractional equations, such a substitution fails to determine the presence of invalid solutions. Bill's improper checking substitutions indicate that he did not understand the mathematical purpose for which checking was used with solutions to these fractional equations.

In both problems 7-7 #2 and 7-10 #12, Bill wrote the statement "They are a solution" following his checking procedures. He also included the word "works!" following the numerical identity in the last line of each check for 7-10 #12. He used this same terminology on the two quiz problems that dealt with simultaneous equations. On problem 7-10 #5 he wrote "works!" following the last line in each check, and on 7-10 #7 he wrote "They are a solution!" beside his answer to this problem. (See page 221.)

The phrase "they are a solution" was used by the instructor during her first lecture on systems of equations on June 26, and again during her lectures on radical equations on July 9 when she was checking problems in which there was a question about the validity of the solution. It is not clear from the comments on Bill's checks of fractional and simultaneous equations whether he was simply reproducing what he had observed the instructor to use, or whether he understood the fact that solutions to some equations are not valid for specific reasons.

During the introductory lecture on simultaneous equations, the instructor had illustrated a problem in which a solution was presented and the question was to determine by substitution whether it was valid for a given system of equations. Under such circumstances, the presence of an equality in the check verifies the solution, and the phrase "works" implies an acceptance of this solution. However, If a system of equations is given and the directions are to find its solution, a check of the answers is used instead to verify the accuracy of the calculations. If a solution can be found, then it is automatically a mathematically valid solution.

The fact that Bill used the phrases "works" and "they are a solution" following the checks of his answers to the two problems on simultaneous equations implies that he was not aware of these mathematical distinctions. His use of checking indicated that he was aware that such procedures should be performed, and that a statement of equality was the desired result. However, it was not clear to him that different mathematical reasons existed for checking solutions to different types of equations.

There were two occasions on which Bill did not arrive at an equality in the last line of his check. These situations are used to provide further insights into the kinds of meanings that Bill had developed regarding the checking process. On the first occasion, in problem 6-26 #3, the non-equality was produced by a calculation error in the checking procedure, rather than from an error in the solution process. (See page 175.) On this problem Bill did not write any comments on his quiz paper following the final step in the check. It is therefore difficult to determine what meaning Bill was attributing to the non-equality that he had produced. However, it could be inferred from the lack of written comments that Bill did not understand the mathematical implications of the non-equality.

At the time that this quiz was given on June 26, the students had not yet been introduced to the specialized requirements of checking that were used with fractional and radical equations. The instruction on checking up to this point consisted of the checking lab on the first day of class, instruction on quadratic equations on June 23, and the first lectures on systems of equations on June 25 and 26.

The second occasion on which Bill encountered a non-equality in his check occurred with the radical equation in the problem 7-13 #7. Bill obtained two incorrect answers due to a manipulation error in his solution process. (See page 245.) Because of this error, both his checks produced statements of non-equality. On his paper, Bill crossed out the checking steps used with his fractional answer, but did not include any written comments as to what conclusions were to be drawn from his checks.

It is interesting to note that the week before, on problems 7-7 #2 and 7-10 #12 dealing with fractions, and on problems 7-10 #5 and #7 dealing with simultaneous equations, Bill had used the phrases "works" and "they are a solution" as commentary following checking statements of equality. However, the following week on 7-13 #7, he did not use the phrase "they are not solutions" when he encountered a non-equality in his check.

Later in that same week, on the quadratic equation in problem 7-15 #6, Bill checked one of his two solutions to produce a numerical identity in the last line of the check. However, he did not write any comment on his paper following this work. (See page 262.) His use of statements regarding the status of each solution were limited to the responses produced during only one week of class from July 6 through July 10. During this week, fractional equations had been re-emphasized with a lab activity and a review sheet. It may be that Bill associated the option of accepting or rejecting solutions only with work done at that time. He did not carry this behavior forward to be used with solutions of equations during the following week.

Bill used the procedure of checking by substitution more often than any other student in the class. His checking index of 6/8 indicates that he used checking on a fairly regular basis. In all but two problems that he solved successfully, his solutions and checking calculations were error-free. He appeared to be aware that checking was part of the solution process for equations, but from his comments following the checking procedures, it is apparent that he did not understand the various mathematical purposes for which checking was to be used.

<u>Alice</u>

Of the 19 quiz problems, Alice found some kind of solution to all but one, and answered 10 of these problems correctly. She included a check on five of the problems, three of which were on incorrect solutions. Alice had a checking index of 3/10 which contained one of the lowest numerators of any student who checked. Although she tended to use the checking procedures correctly, there were seven instances in which she could have checked her solutions and did not.

Alice checked two linear equations at the end of the second week of the course. She used an incorrect sequence of manipulations to solve problem 6-23 #4, but then made two calculation errors in her check which coincidentally produced a numerical identity in the final step. (See page 171.) For the problem 6-26 #3, she used an incorrect technique to solve the equation but used this same technique to check her answer, which negated the purpose of the checking process. (See page 174.)

Alice's use of checking with these two problems illustrates the situation in which a student's ability to use the checking procedure is frustrated by his or her lack of appropriate manipulative skill. The checking process cannot be used to monitor the accuracy of a student's set of calculations until the student has gained a basic level of competence in using various solution techniques. Alice's work on problem 6-26 #3 demonstrated that she did not have a sufficient understanding of the mathematical manipulations at that point in time to be able to interpret the information presented by her check at an appropriate mathematical level.

The three other checks that Alice performed can be used to investigate the types of meanings that she was able to develop for the checking process. In each of these situations Alice generated a non-equality in the final step of a check and included a written comment on her paper based on this result. These checks were performed on two fractional equations and one radical equation that were presented on quizzes throughout the last two weeks of the course.

Alice was the only student in the class who was able to achieve any kind of solution to the fractional equation in problem 7-6 #2. (See page 198.) She solved this problem correctly and performed a mental check to correctly assess the mathematical validity of each of her two solutions. She also checked the fractional equation in problem 7-10 #12, which was the only problem in the course that was checked by four different students. (See page 201.) On this problem, Alice was the only student in the class who performed her checking substitutions into a line of the equation that still contained fractional expressions. She made arithmetic and substitution errors in both her checks, which resulted in two statements of non-equality. Following each check, Alice wrote "does not check". She did not make any other notations on her paper, however, to indicate whether her statements of non-equality were to be used to reject her answers to the problem.

The final problem that Alice checked was the radical equation 7-17 #6 on the individual quiz presented on the last day of class. (See page 246.) She was the only student who included a check with this problem. During her solution process, Alice made an error that produced one incorrect answer and a second answer that was coincidentally the correct solution to the original problem. She performed the checking procedures mentally, and on her paper wrote "keep" below the correct answer and "throw out" below the incorrect answer.

Because of her solution errors, Alice was provided opportunities on the quizzes in which she was able to display her knowledge of the appropriate action that was to be taken when a non-true statement was generated in the final step of the check. In two of the three situations in which this occurred she indicated correctly that the appropriate answer was to be rejected as a solution. It is not clear why Alice failed to comment on her answers for the problem 7-10 #12. On both problems 7-6 #2 and 7-17 #6, she used the presence of a non-equality to reject her solutions. Her behavior on these problems indicates that she was aware that certain decisions regarding the status of solutions to equations were to be made on the basis of the checking process.

Alice was the only student in the class who correctly substituted her answers into the original fractional equation when performing a check. When Peter, Bill, and Elaine checked this kind of problem, they made their substitutions into equations that had been generated after the original equation had been cleared of fractions. This incorrect application of the checking process indicates that these three students were not aware of the mathematical necessity for performing all checking substitutions into the original form of a fractional equation. Alice's correct use of the checking procedure, however, cannot be interpreted farther than to observe that she had correctly copied the manipulations used by the instructor for this type of equation. There is not enough other evidence in her quiz response to indicate whether she was also aware that this was a mathematically essential part of the checking procedure. (The following chapter will examine the instructional contexts within which the mathematical properties of each type of checking were presented to the students in order to assess the degree to which this information was made available for their use.)

Alice checked three of the six quiz problems on fractional and radical equations which required checking as part of the solution process. She was the only student who achieved a solution of some kind to all six problems. However, because she did not check all six sets of solutions, it can be inferred that she had not understood that checking was a mandatory part of the sequence of manipulations required to complete the solutions to these kinds of problems. Alice's responses on her quiz problems indicates that she had assimilated the appropriate sequences of manipulations required for solutions to fractional and radical equations, but her lack of consistency in checking her answers indicates that she did not understand the mathematical reasons underlying the use of these manipulations.

Elaine

Elaine was only able to answer five of the 19 problems correctly. There were four other problems for which she was not able to achieve any kind of solution. She checked a total of six problems, three of which were on correct solutions. Elaine had a checking index of 4/6 which indicates the frequency with which she checked her problems. There were only two occasions on which she was able to obtain solutions to equations and did not also include a check with her work.

Elaine began a check for the linear equation in problem 6-23 #4. She solved the problem correctly and correctly substituted the fractional answer x = -3/5 into the original equation. (See page 173.) However, she did not complete any further calculations. It may have been that the subsequent arithmetic calculations involving fractional numbers discouraged her from completing the check.

Elaine was not able to find solutions to the next four quiz problems involving equations. Problem 7-10 #12 was the next question for which she obtained an answer. She included a check with this problem and also checked the next four problems that dealt with equations on subsequent quizzes. Following this spate of checking, she did not check her answers to any other equations. All but one instance of Elaine's checking took place at the end of the course, with most of her responses occurring from July 10 to July 16.

Elaine was able to achieve a solution to only one of the three problems dealing with fractional equations. She used an incorrect set of manipulations on problem 7-10 #12 in order to clear the original equation of fractions, which produced an incorrect but simple linear equation. (See page 205.) For her check, she substituted her solution into this linear equation instead of into the original form of the fractional equation. Consequently, her check produced a numerical identity. Elaine wrote "OK" below her check and "Solution" below her answer. These comments indicate that she was aware that the results of her checking procedure on this problem were to be used to pass some kind of judgement on her solution.

Elaine checked two of the three radical equations that were given on the individual quizzes. She was the only student in the class to include a check with the very simple radical equation in problem 7-13 #6. (See page 242.) She made an error in her solution process which automatically produced a non-equality in her check. She wrote "No Solution" on her paper below her answer for this problem.

The next problem on the same quiz, 7-13 #7 also involved a radical equation. Elaine solved and checked this problem correctly. (See page 244.) She wrote "OK" below the numerical identity in the last line of her check, but did not write any other comments on her paper. It can be inferred that the absence of any comments relating to the status of her answer implies that she accepted her answer as a valid solution to the equation.

Elaine was the only student who checked both of the problems dealing with radical equations on the quiz of July 13. These problems were written consecutively on the same page of the quiz. The first equation was the easier problem to solve and was the one that other students failed to check. If the students had been aware that checking was required of all solutions to radical equations, it could be argued that they would have checked both of these problems on the quiz. It could also be argued that if the students had been careless in not checking the first problem, the action of checking the second equation might have reminded them to return and add a check to their work on the first problem. The fact that Elaine was the only student in the class who included a check with both these problems indicates that the other students did not regard the checking process as a very necessary part of their work.

However, the fact that Elaine checked both of these problems cannot be used as evidence that she was aware that checking was an essential part of the solution process to radical equations. Her behavior can be interpreted to indicate that she was probably either being careful to check all problems dealing with equations, or that she recognized the similar features between the two problems and so checked both of them.

Further evidence that Elaine was probably not aware of the mathematical necessity for checking all solutions to radical equations can be found in the fact that she did not check her answer to this type of equation in problem 7-17 #6. Although she generated a solution that would have produced a straightforward set of checking calculations, she did not include a check with her work on this problem.

Elaine included a comment with her check for one other problem. She incorrectly solved the second degree equation in problem 7-16 #5 to produce two simple solutions. (See page 262.) She wrote out the check for only one of these answers, but failed to make her substitution into the correct form of the original equation. Because her check produced a statement of non-equality, she drew a slanted line through the equal sign in the last line of the check to indicate this fact, and wrote "No Solution" below her corresponding answer.

There was no other written work on her paper, but she may have mentally checked the second answer to her problem. If she had used the same incorrect substitution as she had for her other solution, she would have arrived at a numerical identity for her check. It appears from her written work that she was rejecting only the one solution for which she had written down a check.

When she wrote "No Solution" following one of her answers to this problem, Elaine was incorrectly applying the criterion of a non-equal check for solutions of radical equations to the two answers that she had generated to the quadratic equation in problem 7-16 #5. Elaine indicated by her comments that she was not aware that she could not apply the checking conditions from radical equations to the solutions of quadratic equations. The presence of a non-equality in the last line of her check should have been interpreted as an indication of the presence of computational errors rather than used as a criterion for rejecting her solution.

Elaine's response to problem 7-16 #5 indicates that she was not aware of the distinctions that existed among the various checking procedures. She had carried forward the instructions from radical equations into an area in which they no longer applied. Even though Elaine had not observed the instructor using checking when she had demonstrated problems similar to 7-16 #5 during her lectures on quadratic equations earlier in the week, Elaine continued to apply those techniques from previous lectures on her later quiz responses.

There is little evidence from other students' quiz responses that could be used to determine whether many other students were similarly confused about which checking criteria were to be employed with particular kinds of equations. Elaine and Sally were the only students in the class who performed checks on equations other than a radical or fractional equation in which the result was a statement of non-equality. No other students were put in a similar situation in which such information could have been tested.

Elaine's checking index of 4/6 is evidence that she employed the checking process more frequently with problems for which she generated solutions than did the more "successful" students Alice and Peter. She appeared to have mastered the appropriate manipulations for checking certain kinds of equations and applied them consistently throughout the period from July 10 through July 16. She used the correct checking criteria to accept or reject appropriate solutions for fractional and radical equations. However, since she subsequently applied these criteria to a quadratic equation, it is apparent that she was not aware of the mathematical reasons behind her manipulations.

Sally

Sally checked only three of the quiz problems dealing with equations. She generated a total of seven correct answers, two of which she then checked. Sally did not include a check with any of the four problems on linear and quadratic equations that were given on quizzes during the first two weeks of the course. She was not able to find an answer to any of the three problems dealing with fractional equations, and was only able to solve one of the three questions on radical equations. Thus, there were few opportunities among her quiz responses that could be used to observe her applications of the various checking procedures. She received a checking index of 3/5.

Two of the three problems that Sally checked dealt with simultaneous equations. She correctly solved the two equations in problem 7-10 #5 but checked the answers she found for x and y in only one of the two given solutions. She did not write any comments on her paper following her check. (See page 221.) In her solution of the other two simultaneous equations in problem 7-10 #7, she introduced an arithmetic error during her solution for one of the two variables and then failed to solve for the other variable. (See page 224.) It is not clear whether her subsequent substitution of y was intended to serve as a check, or whether she was attempting to solve for her second variable. Because of her original computational error, this substitution produced a non-equality in the final step of her manipulations. She did not include any written comments on her paper regarding this outcome.

Sally used checking only once more on the final day of class. She was the only student in the class who checked the quadratic equation in problem 7-17 #7. (See page 264.) She solved this problem correctly and substituted both solutions into the original equation. However, she made a computation error in one of her two checks, which resulted in one true and one non-true checking statement. She wrote "doesn't work" under the answer whose check produced the non-equality, and "yes" under the answer that generated the numerical identity.

Sally did not provide many quiz responses that could be used to infer her depth of knowledge regarding the various checking requirements that were introduced throughout the course. She was able to solve only one of the six problems that dealt with fractional and radical equations, and she did not include a check with this solution. However, her incorrect application of the checking process to the quadratic equation in problem 7-17 #7 indicates that she had assimilated some of the checking information that was presented in class. Her comments on this problem show that she was aware that the results of the check were to be used to affect the final status of the solutions of an equation. She was not able, however, to apply this knowledge to the appropriate types of equations. Sally received a checking index of 3/5, which is somewhat misleading. Two of the three problems that she checked were simultaneous equations on the same quiz. She checked only one other problem throughout the course. It is not possible to determine whether or not Sally would have used the checking procedures more frequently if she had been able to solve more of the fractional and radical equations presented on the quizzes. There is not enough evidence to be able to accurately assess the extent of Sally's knowledge about checking, given her responses to the 19 problems examined.

Sue, Paul, and Don

Sue did not include a check with any of the 19 quiz problems. She did, however, employ some of the terminology associated with the checking process when she found her solutions to the two problems on simultaneous equations that were present on the quiz of July 10. She wrote "not a solution" on her paper following her answers of $x = 13/5$ and $y = -8/5$ to the problem 7-10 #5, and "Is A Solution" following her answers of $x = -1$ and $y = -1$ to the problem 7-10 #7. (See page 225.) It appears that she was associating the presence of equal numerical values with the term "is a solution."

Paul also did not check any of the 19 quiz problems. However, he was aware of the existence of the checking process. On July 13 during the practice work with quadratic equations, he asked me "I don't have to check?" (See page 257.) His use of the word "have" indicates that he was aware that the checking processes existed and were associated with the solutions to equations. He was not aware, however, of the mathematical reasons behind these processes that made checking mandatory or optional.

Don was the other student in the class who did not use checking. He experienced a great deal of difficulty in solving the equations on the individual quizzes. There were only four instances in all of the 19 problems in which he was able to achieve some kind of answer to the given equation, none of which, however, was mathematically correct.

These three students were exposed to the same lectures, group practice work, and lab activities as the rest of the class. The fact that they did not use checking on the solutions to equations does not indicate that they were unaware of the checking processes, or unable to perform any of its manipulations. The reasons for this lack of use, however, cannot be determined from examinations of their quiz responses.

Summary

The use of checking was found to be correlated to the students' abilities to perform well in class on pencil and paper tests. The students with the higher grade averages tended to check their answers more frequently on the individual quizzes. None of the students, however, used checking consistently with each of the 19 problems examined. The class as a whole did not include checking on a regular basis with the answers generated from the fractional and radical equations in which checking was considered to be a mandatory part of the solution process.

The analysis of the students' written quiz responses that were reported in the preceding chapters emphasizes the inadequacy of paper and pencil tests as a means of evaluating the development of underlying mathematical concepts. The conclusions that were drawn in this chapter rest on the analysis of students' incorrect responses. The presence of mathematically correct work cannot necessarily be interpreted as an indication of an appropriate level of mathematical understanding.

Sue's use of the terms "solution" and "no solution" is a reminder that it is possible for students to memorize sequences of manipulations, patterns of symbol placement, and key phrases as a way to correctly reproduce mathematical procedures on paper and pencil tests. The problems that were used on the quizzes given throughout the course were very similar in pattern to those used in lecture examples and during practice sessions. Thus, they do not provide an effective method for evaluating a student's true level of mathematically understanding. The conclusions of this chapter are therefore based, for the most part, on an analysis of the students' errors, not on their mathematically correct work.

It is apparent from the analysis of the students' errors that they performed the required checking steps without understanding the mathematical reasons behind such manipulations. This lack of an appropriate level of understanding prevented the students from being able to apply the various checking procedures appropriately with the different types of equations that they encountered.

This chapter documents that the students were unable to reproduce an appropriate mathematical level of response on the written work that was designed to examine their mastery of various checking procedures. The students' performance, however, must be placed in context within the existing learning environment in order to investigate the conditions that brought about this level of behavior. The following chapter examines the classroom instruction that served as the primary source of the students' information concerning checking.

CHAPTER THIRTEEN

SUMMARY OF CLASSROOM LEARNING AND INSTRUCTION

Introduction

The information presented in this chapter provides a summation and integration of the classroom records and student responses that were studied in detail in Chapters Seven through Eleven. The purpose of this review is to relate the students' quiz responses to the content of specific classroom lectures and to the chronological sequencing of lectures and testing periods. Such information is used to assess the effectiveness of the instruction in promoting an appropriate level of mathematical concept development for this subject.

The effect that specific lectures had on student responses is measured by comparing the choice of words that are written by the students at the end of their checking processes to the particular words and phrases that the instructor presented to the class during her lectures on checking. The time frame in which student responses were written following specific lectures is examined in order to trace the influence of successive classroom instruction on the students' uses of specific types of checking procedures.

Part of the analysis of the students' written responses that is presented in this chapter will assess the degree to which students altered their perceptions of checking as the instructional emphasis on the subject changed. Before July 1, checking was used by the students and instructors solely for the purpose of verifying the accuracy of the manipulations that were used in the solution process. During this part of the course, checking was presented by the instructor as an optional activity and was not emphasized during her lectures on linear and quadratic equations. However, beginning with the lecture on July 1, checking was presented by the instructor as a mandatory part of the solution process that was to be used with fractional and radical equations. The checking procedure became a much more integral part of the classroom instruction at this time. Subsequent sections of this chapter will examine the relationship between this shift in instructional emphasis and the way in which students employed checking procedures on their quiz responses throughout the course.

The exact nature of the information that the students were exposed to in class is obtained by examining selected portions of the transcripts of the classroom lectures that dealt with the checking processes. Attention is paid to certain key words and phrases that the instructor used repeatedly in each lecture to indicate manipulative actions or mathematical relationships. The meanings that were assigned to these words are determined from the context in which they were used within each transcript. Many of the key words and phrases were used by the instructor in several different lectures. The students in the class not only assigned meanings to these words within the context of each lecture but developed broader meanings for these phrases as they were used throughout the course.

In the sections that follow, individual student responses are compared chronologically to pertinent classroom lectures. The identification of specific key words within each lecture is used to measure the influence of this material on the types of student responses that were written. These key words are underlined in the those portions of the transcripts that are presented below to emphasize their presence and to indicate the frequency with which they were employed throughout each lecture.

Early Instruction on Checking

The lab activity presented on the first day of the course represented the only formal instruction that the students received on checking during the first week of class. Since the actual discourse of the lecture introducing this lab was not recorded, a further analysis of this class session is not presented here. A description of this instruction and the accompanying lab activity are given in Chapter Seven.

June 23

The instructor presented her first lecture examples that employed the checking process on Tuesday, June 23. At that time she provided examples of quadratic equations that could be solved by factoring techniques. (See pages 252 through 254.) A portion of the transcripts of this lecture is presented below. The key words check and work that represent various aspects of the checking procedure are underlined in these transcripts.

Checking was not mentioned until the third example was presented in this lecture. As the instructor indicated the two solutions that she had found for this problem, she said, "We should see if they both work, shouldn't we? We'd better check that out. I've just been assuming they work, and you've been assuming because I did it, that they work. But what if I made a mistake? Let's check this."

She checked the two solutions by substituting their numerical values in the original equation, commenting at the end of the first check, "That one works," and at the end of the second check, "Good, it really does check." She then added, "We can solve the problem and get two answers. We can check both answers and they usually do work."

The instructor worked a fourth example through to a pair of solutions and said, "This is the solution. You can put them back in, and it turns out they check. We're running a little late, so rather than write the check down, I'm going to ask you to go to section 5.7 and do the margin exercises."

The discourse presented above provides an indication of the emphasis that the checking process received during the June 23 lecture. During that time, the instructor checked the solutions to only one of her four examples. Further, her comments following the last problem gave the students the impression that she considered checking to be an optional activity and one that was not an important part of the solution process.

It can also be seen in the above discourse that the instructor at no time explicitly stated that she was using the checking by substitution method to verify the accuracy of her manipulation procedures. This information, however, was implied when she stated, "I've just been assuming they work .... But what if I made a mistake? Let's check this." The students were also able to use the context of the lecture to associate the word "checking" with the word "works" and to infer this purpose for the checking manipulations that the instructor had used.

Simultaneous Equations

The two lectures dealing with solutions to simultaneous equations were presented on June 25 and 26.

The complete transcripts of these lectures are presented in Chapter Nine and in Appendix B.

June 25

During the morning lecture on June 25, the instructor presented an example in which a system of equations was given, and the question was to determine if a particular set of values represented its solution. The instructor used the following key words during this lecture: check, solution, true, false.

The instructor introduced her lecture example by saying, "Can you tell me that this is the solution of the system? I can check and find out if that is right." She then proceeded to substitute the given numerical values into one of the two equations and arrived at a numerical identity.

Ellen then said, "That is true. Negative two equals negative two. Now if this is a system, there are two equations. So I'll have to make the same check in the other equation ... Does that come out equal to two? ... Minus three plus five equals two and two equals two. That's true. If, when I put the number into the second equation, it comes out false, then it wouldn't be a solution ... If you asked me if it was a solution, I would say, yes, I checked it out, and they both came out true."

The context of the discourse provided the students in the class with information from which they could assign a specific meaning to the word true. In the lecture example that the instructor displayed on the overhead projector, both checking substitutions produced statements of equality, and Ellen wrote, "true," next to these lines on her work. The meaning of false could only be inferred in this case, since the example that was used did not include a non-equal check.

Ellen used the word check to introduce the processes in which she substituted the numerical value of an answer into one of the original equations. The word solution was used in the same sentence with either check or with the words true and false. These contexts imply that the solution is dependent on the outcome of the check. "True" means there is a solution, and "false" means there is no solution. The instructor conveyed this information by writing the word "true" next to each check and then verbally linking this result with the word solution.

<u>June 26</u>

A second lecture was presented during the afternoon of June 25 that dealt with a technique for solving very simple sets of simultaneous equations. I was not present during this lecture and have no records of the discourse. The following day, June 26, the instructor discussed the procedures for solving systems of equations using addition and substitution techniques. During this lecture, she used the following key words: <u>check</u>, <u>solution</u>, <u>works</u>, <u>OK</u>.

The instructor proceeded through the appropriate manipulation steps in her first example to produce a solution to the given system of equations. Then she said, "Let's <u>check</u> that and see if it <u>works</u>." She proceeded through the arithmetic of the two checks, writing down the steps as she went to arrive at two numerical identities. She wrote, "<u>OK</u>," next to these calculations on her transparency.

She used the following commentary as she worked through one of the checks. "Does three minus two equal one? Yes, that's <u>OK</u>. So, it does <u>work</u>. ... If your arithmetic is like mine, you always <u>check</u> when you are done. Anytime you solve the system of equations you <u>check</u>..."

The instructor then proceeded through the steps of a second example. When she had reached a solution she said, "We finally have a <u>solution</u>. Probably better <u>check</u> that and see if it <u>works</u> or not." She wrote out the steps of the checking process and wrote, "<u>OK</u>," next to the statements of equality produced in the last line of each check.

As in the lecture the previous day, Ellen used the word check to introduce the process whereby she substituted the numerical values of her solutions for the two variables displayed in the given equations. The word works was used in connection with "check" and "OK," and was associated with the presence of a statement of equality in the final step of the check.

Comments

The purpose for which checking was used during the lecture on June 26 was different from that used during the lecture on June 25. Checking was employed on the first day in order to verify that a given pair of values that were not obtained by a direct solution process were mathematically valid solutions to the system of equations. On the following day, the pair of values that formed an answer were obtained by appropriate mathematical manipulations of the two equations and were therefore automatically solutions to the given system of equations. The checking process in this case was used to verify the accuracy of the manipulations employed in the solution process.

The checking process was a necessary part of the manipulations that were required to obtain an answer to the example presented on June 25. Until the checking substitution had been performed, it was impossible to state whether the given values were indeed true solutions to the system of equations. On June 26, however, checking was considered by the instructor to be an optional activity that could be performed at the discretion of the person solving the problem in question. The mathematical validity of the solution did not depend on the outcome of the check.

The above distinctions that existed in the different mathematical purposes for which checking was used were not explicitly stated in either lecture. Information concerning these purposes was implied by the instructor's actions and her choice of commentary but never deliberately stated. In the lecture on June 26, the instructor implied that checking was to be used with the problem in question as a means of verifying her calculations by the statements she made before she began her check: "If your arithmetic is like mine, you always check when you are done," and "Probably better check that and see if it works or not." Ellen did not make it clear, however, that this type of checking was an optional activity. She first said, "...you always check," but then later rephrased this as, "...probably better check."

An examination of the discourse that occurred during the checking procedures on June 25 reveals that Ellen did not provide the students with any information relating to the specific mathematical purpose for which checking was used during that lecture. The fact that checking was required to establish the mathematical validity of each solution could only be implied from her statement: "So if you asked me, if it was a solution, I would say, yes, I checked it out, and they both came out true."

The students were dependent on the form in which pertinent information was made available to them in class. During the period that they were developing their concepts on checking, this information needed to be explicitly presented at an appropriate mathematical level in order for it to be accessible for inclusion in the students' existing mental structures. It is apparent from the preceding analysis that much of the information concerning the mathematical purposes for which checking was used was not in a form that could be clearly understood by the students.

From the examples provided in each lecture, the students were aware that the checking process was used to pass some kind of judgement on the solutions of a given system of equations. On June 25, the instructor wrote, "true," following the final step of the check. On June 26, she wrote, "OK," following each check. However, since on both occasions each checking procedure produced a true statement, the students were only exposed to examples of the appropriate behavior to be used when the checking statement "worked." They were not provided with any examples of the actions that were required when a non-equality was produced in the last line of the check.

One other important aspect of the checking process was present implicitly in each lecture but never explicitly stated. This was the requirement that all checking substitutions must be performed into the original form of the given equations. The instructor performed this substitution correctly in each example that she presented to the students but never drew their attention specifically to the importance of this action.

Student Response

I did not monitor student performance on problems dealing with solutions to simultaneous equations until the quiz on July 10. By that time, other instruction dealing with different aspects of the checking process had also been given. There is, therefore, no way to determine what information specifically contained in the two lectures on simultaneous equations was utilized by the students when solving such problems. Their responses to questions of this type written on the later quizzes were influenced by information from subsequent lectures and will therefore be analyzed within the context of the instructional topics that were presented later in the course.

Table 3 below presents a summary of the frequency and nature of the students' checking responses to the four individual quiz problems that were given to the students prior to and following the lectures on June 23, 25, and 26. This information is an excerpt from Table 11, Appendix B, which displays the students' checking responses to all 19 quiz problems. The first three problems listed in the table below represent linear equations and the last problem represents a quadratic equation.

It can be seen in this table that only three of the eight students in the class employed checking on a fairly regular basis. Alice and Peter checked two equations, and Bill included a check with three of the four problems listed. None of the students in the class included written comments on their papers following their work. (See pages 170 through 178.)

Table 3. Student Checking for Linear, Quadratic Equations

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
6-19 #1	E-N	Na	C-C	E-N	Na	C-C	C-N	C-N
6-23 #4	E-N	E-C	E-C	E-N	Na	E-N	C-N	C-Ce
6-26 #3	E-N	E-C	C-N	C-N	Na	C-Ce	E-N	E-N
6-26 #9	C-N	C-N	C-N	C-N	Na	C-Ce	C-N	C-N

<u>Key</u>

C-C = Correct solution - Check
C-Ce = Correct solution - Error in Check
C-N = Correct solution - No Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved

Problem 6-19 #1 was presented on a quiz at the end of the first week of class, five days after the checking lab. Problem 6-23 #4 was given on the quiz directly following the morning lectures on quadratic equations in which the instructor had included a check with two of her lecture examples. There is not a sufficiently large variation in the student responses for these two quiz problems to be able to determine if the lecture on the morning of June 23 had any effect on the students' inclusion of a check with their solutions to problem 6-23 #4. As can be seen in Table 3, two students checked problem 6-19 #1, while three students included a check with problem 6-23 #4.

The two equations 6-26 #3 and 6-26 #9 were included on the super quiz that was given to the students at the beginning of class on June 26. The responses to these questions had the potential of being influenced by the lectures on simultaneous equations that were presented the day before. However, since only two students included a check with the first problem and only one student checked the second, there is limited information available to assess such influence.

Problem 6-26 #3 on the super quiz presented a linear equation which required a certain amount of manipulation in order to reach a solution. Alice selected an inappropriate method of solution, and then incorporated this technique in her check as well. She thus arrived at statements of equality in her checks. She did not write any comments on her paper following her work. (See page 174.)

Bill solved this equation correctly to produce a fractional answer. He then expressed this value as a decimal number in order to perform his checking substitutions. He made a computational error in his check, which resulted in a non-true statement in the final step. Bill did not include any written statement on his paper commenting on this result. (See page 175.)

Bill was the only student who checked the quadratic equation in problem 6-26 #9 on the super quiz. He solved this problem correctly. However, he did not make his checking substitution into the original form of the equation. He did not include any comments on his paper concerning the numerical identities produced by his checks. (See page 177.)

The fact that neither Bill nor Alice wrote any comments next to their checks seems to indicate that they were not applying any of the different checking mannerisms from the lecture of June 25 to their responses on the quiz of June 26. If such had been the case, one could have assumed that these two students would have written down the words "true" or "false" following the numerical identities or non-true statements that they produced in the last line of their checking procedures.

Fractional Equations

The techniques for solving fractional equations and their specialized checking requirements were presented in class on July 1 and 2. The complete transcripts of these lectures are presented in Chapter Eight and Appendix B. During her lectures on the subject, the instructor used the following key words when referring to the checking process: check, denominator, zero, keep, throw out, work, OK.

July 1

The instructor introduced information concerning the special nature of the checking process during her first lecture on fractional equations. After she had found the solution values of zero and one to the first lecture example, she commented, "Now it looks as if I am all done, but I should warn you that I am not. If any answer that we get makes any <u>denominator</u> equal <u>zero</u>, then we have to <u>throw it out</u>."

The instructor then verbally substituted the answer of zero into the denominators of the original form of the given equation to produce values in each denominator of two and minus one. She commented, "Zero's <u>OK</u>. I get to <u>keep</u> that answer." She then verbally substituted the second answer of one into the denominators of the original equation, saying, "Let's <u>check</u> out x equals one. One minus one would be a <u>zero</u>. ... You may not have a <u>zero</u> in the <u>denominator</u>. And so we're going to have to <u>throw</u> this answer <u>out</u>."

She accompanied these statements by drawing an X on the transparency through her solution of x = 1. Then she summed up her checking by saying, "...we solved the equation, and it looked like we had two solutions, but one of them didn't <u>work</u>. We had to <u>throw it out</u>."

In the discourse accompanying her work, the instructor used the word check to refer to the operation of substituting the numerical value of an answer into the denominators of the original equation. The meaning of check was extended from its use in previous lectures to include a restricted form of substitution, where the presence of nonzeros rather than a statement of equality was the goal of the process.

Ellen used the phrase "throw it out" to indicate her action of crossing out the particular solution that had generated a value of zero in one of the denominators of the original equation. The word work was linked to an acceptable solution through the negative association of "does not work" and "throw it out."

During the lectures on simultaneous equations, Ellen had used the word work to designate solutions that had generated statements of equality in the final checking step. In the present lecture, the meaning of this word was extended to refer also to solutions that were acceptable through a new definition of the checking process.

The idea that an answer might not work even after it had been produced by a set of correct mathematical manipulations was unfamiliar to the students. One of the assumptions that was held by the students at that time was the assurance that, if a problem was solved by the correct techniques, then the answer would always be correct. The presence of an incorrect answer was due to mistakes made by the student.

In her lecture on July 1, the instructor did not emphasize this change in mathematical perspective regarding the solution of an equation. A great deal of mathematical information was therefore implied in her statement, "Now it looks like I'm all done, but I should warn you that I am not."

Ellen did not provide any mathematical reason to explain why an answer was to be thrown out if it did not work. The steps in the checking process were presented to the students in the form of a given rule, illustrated by examples. Again, a great deal of mathematical information was implied by her statement, "If any answer that we get makes any denominator equal zero, then we have to throw it out."

The instructor continued her explanations of the checking process using a second example. She again checked her solutions by examining for zeros in the denominators of the original equation. Ellen performed these steps verbally, saying, "That would make ... a six. That's an <u>OK</u> thing.... Minus one plus one makes <u>zero</u>. Uh oh! That's not <u>OK</u>."

Here, Ellen was associating the phrase "OK" with those answers that met the checking requirements for accepting a solution. "OK" appeared to be used in a similar fashion to the word "works."

In both of her lecture examples, Ellen performed the checking substitutions and subsequent evaluations mentally. She provided verbal descriptions of these processes but did not write down any of the steps she used. For the first example, Ellen crossed out the one answer that produced a zero in a denominator during the check. However, she did not do this with the one rejected answer in her second example.

The instructor did not at any time explicitly state that the checking substitutions must always be made into the original form of the equation. This information could be inferred, however, from the pattern of manipulations that were used in each example. The use of the original equation was also implied by Ellen's checking statements concerning the presence of zeros in the denominator. Once manipulations were performed on the original fractional equation, subsequent steps in the solution process presented equations that no longer contained variable expressions in fractional form. In order to perform the checking substitutions into denominators, it was always necessary to use the original equation.

Another action that was implied but not explicitly stated was the kind of label that was to be attached to a particular answer as the result of the checking process. If an answer did not produce a zero in the check, the instructor used such phrases as "keep" and "OK." In her first example, she indicated her actions regarding the answer that produced a zero by crossing out this value on the transparency. For the second example she used the phrases "throw it out" and "not OK," when referring to such an answer. She did not, however, make any notation on her written work with this problem.

The instructor used the word "solution" only once during the lecture when she said, "Sometimes I have to <u>throw</u> both my answers <u>out</u>, and that always hurts after all that hard work to end up having to say there is no <u>solution</u>. I have to <u>throw</u> them both <u>out</u>."

The instructor provided very explicit instructions concerning the nature of the checking process that was to be used with fractional equations. These were included in a list of four general statements that described the manipulations that were required to find the solutions to fractional equations. The fourth statement in this list was written as follows:

4. <u>Check</u>: If any answer makes any <u>denominator</u> equal <u>zero</u>, just <u>throw it out</u>.

This was the most explicit statement that Ellen made concerning the special checking requirements for this type of equation. The fact that this information was written on a transparency made it very accessible to the students. They were able to copy this statement verbatim into their notes for future reference. The instructor wrote down the above rule as she reviewed the steps that she had used to solve the first example and then again as she worked through the second example.

July 2

Further instruction was provided on checking during a follow-up lecture on fractional equations presented the next afternoon on July 2. During this lecture, the instructor employed a different method of checking. Rather than substituting her answers into the denominators of her original equation, she substituted these values into an expression that represented the least common denominator of the original fractions.

Ellen checked the first solution by saying, "Remember, we agreed we always had to check and see if we get to keep the answers or if we have to throw them out the window.... Backing up ... we'll take a look at our least common denominator. OK If m is one, this factor is one minus five, which is negative four. That's OK."

Ellen checked the second answer by saying, "... but over here, I get minus five plus five. That makes a zero, and that's a no-no! .... I can't have a zero, so I throw that one out."

These two checks were performed mentally. The instructor did not write down any of her calculations, but she did draw lines through the answer on the transparency that she "threw out." She did not write down any comment on the transparency next to the answer that she had described as "OK."

One of the students had questions concerning the steps that Ellen had used during the checking process. The instructor replied by saying, "Any time it puts a <u>zero</u> in the least common denominator, that means that one of those factors someplace in that <u>denominator</u> is going to be <u>zero</u>, and you can just chuck it out. Or if you want, you can go back to the original problem and just look at the <u>denominators</u>."

Ellen did not provide any additional comments as to why the checking substitution could be made into the least common denominator of the fractions instead of into the denominators of the original equation. Her reference to "the original problem" in the above discourse was the first time she had explicitly stated this important aspect of checking.

After she had finished responding to students' questions, the instructor then reviewed all of the steps that she had used to achieve her solution to the lecture example. She again wrote down step #4 on a transparency: 4. <u>Check</u>, if any answer makes any <u>denominator</u> equal to <u>zero</u>, <u>throw it out</u>.

In the two lectures on July 1 and July 2, the checking process for fractional equations as expressed in step #4 was explicitly stated many times and written several times on various transparencies. The words "throw it out" were accompanied in two of the three examples in these lectures by the physical action of crossing out one of the answers to the problem. The actual checking processes were always done verbally by referring to the problem displayed on the screen but none of the calculations were written down.

These lectures on fractional equations represent the first time that the students had been exposed to equations whose answers were not automatically valid solutions. This fact was not explicitly stated at any time. However, equations presented during the two lectures did contain examples of solutions which did not "work" under checking. The fact that answers could be rejected or retained as solutions to a particular equation was implied by the instructor's use of the phrases "OK," "keep," and "throw it out." At no time, however, did the instructor explicitly state that the checking procedures were an essential part of the solution process for these equations.

Student Responses

Table 4 below presents a summary of the frequency and nature of the students' checking responses to the three individual quiz problems that dealt with fractional equations. All three of these equations were on quizzes presented during the week that followed the initial lecture on July 1. During this week, further class work was done on this type of equation through a lab activity on July 6, and a review sheet on July 8. The day before problem 7-10 #12 was given, the instructor introduced the students to the special checking requirements of radical equations.

Table 4. Student Checking with Fractional Equations

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
7-6 #2	Na	C-C*	Na	Na	B	Na	Na	Na
7-7 #2	Na	E-N	E-N	Na	--	C-Ce*	Na	Na
7-10 #12	Na	C-Ce*	C-Ce	E-N	Na	C-Ce*	Na	E-Ce*

<u>Key</u>

C-C = Correct solution - Check
C-Ce = Correct solution - Error in Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved
B = Problem Left Blank
* = Student Comment Follows Check

It can be seen in Table 4 that most of the students in the class experienced difficulty in achieving any kind of solution to the fractional equations on the daily quizzes that were presented immediately following the initial lecture on the subject. As the students received additional instruction and practice on this type of equation, more students were able to achieve solutions to the quiz problems involving this topic. As can be seen above, most of the students who were able to find solutions also checked their problems. In every instance but one, the students also wrote some kind of comment with this check.

One important aspect of the students' responses to the three questions on fractional equations was the manner in which the checking substitutions were performed. Of the six checks that were made, only the two written by Alice included a substitution into a form of the equation that contained variable expressions in the denominators of fractions. Peter, Bill, and Elaine all began checking by substituting their answers into a form of the equation that no longer contained fractional denominators.

The performance of the checking substitution into a step in the problem other than the original equation is counter to the information provided by Ellen during her lectures on fractional equations. She continually used the phrase "zero in the denominator," when referring to her checking process. She also provided the students with a written rule specifically emphasizing this point. Since the manipulations used to solve this type of problem produced equivalent equations that did not contain terms with algebraic denominators, Ellen's statement "zero in the denominator" could only refer to the original form of the equation. Her procedure for checking, therefore, implied that the substitution must be made at this point.

The fact that Peter, Bill, and Elaine did not follow these instructions can be taken as an indication that they were unable to understand the significance of this procedure. They were unable to enlarge their mental structures relating to the checking process to include a different type of procedure. When confronted with the requirement of checking fractional equations, they continued to apply the techniques of checking that had been previously used, which consisted of the evaluation of the numerical statement produced by the substitution of their answers into some form of the equation.

It can be seen from an examination of the written comments following their checking procedures, that these three students had been able to expand their concepts of checking to some degree. They were aware that the results of the checking process could now be used to retain or reject a solution. However, they continued to use the final step in the check for this criterion. They were not able to incorporate a search for zeros into this mental construct.

It is postulated that the students were unable to utilize the checking information contained in the lectures on fractional equations because they lacked the appropriate level of mathematical understanding. The instructor acknowledged this possible lack of appropriate background when she presented the checking criterion to the students as a rule. They were given no information concerning the mathematical justifications behind the statement. Therefore, they had no way of determining the importance of this information, or way to place the rule appropriately within their existing mental structures.

It is difficult to determine from Alice's two quiz responses whether she was more successful than the other three students in her ability to appropriately utilize the checking processes for fractional equations. For problem 7-6 #2, she used mental substitution to check her answers and drew lines through the answer that produced a zero in the denominator under an appropriate check. (See page 198.) However, with problem 7-10 #12, even though she substituted her answers into a form of the equation that contained denominators, she carried her calculations through to completion and did not use the criterion of zeros in the denominator to evaluate her answers. (See page 201.)

Alice was able to closely imitate the checking procedures and notation that Ellen had used in her lectures on fractional equations when she solved the equation on July 6. However, by July 10, Alice's checking response more closely resembled that used with radical equations than fractional equations. Apparently no student in the class was able to retain the special checking criterion for fractional equations through succeeding lectures that covered other, more familiar types of checking procedures.

Bill was the only student in the class to check his solutions to the fractional equation in problem 7-7 #2. He wrote out his checking substitutions but incorrectly used an equation other than the original one for his checks. Both checks produced statements of equality and Bill wrote "they are all solutions" next to his problem. (See page 199.) It is apparent that Bill had not retained the special checking criterion for fractional equations. He used the form of the last line in his check to establish the validity of his solution. It is true that a numerical identity in the check does mathematically validate solutions to fractional equations, but this information was not included in the checking introduction for this type of equation.

Bill's comment of "they are all solutions," is of interest because this choice of words was not used during the lectures on fractional equations. Ellen only used the word "solution" once during her lectures on July 1 and July 2. She referred to answers that were solutions by saying they "worked," or they were "OK." The phrase "they are all solutions" can be traced back to the instructor's discourse during her lectures on simultaneous equations presented two weeks earlier.

Ellen provided high visibility for the manipulation procedure used for checking fractional equations throughout her lectures on this topic. ("If any answer makes any denominator equal to zero, throw it out.") She wrote down the rule describing this manipulation step several times and stated the rule verbally many times as well. The fact that no student used this specific manipulation later in the course illustrates the minimal impact that Ellen's instruction had on the students' subsequent behavior. Simply telling the students what to do did not induce a change in student performance. It is postulated that without an accompanying level of understanding of the mathematical reasons prompting the use of this rule, the students were unable to assess its importance and so failed to utilize it.

The students' responses to the fractional equation in problem 7-10 #12 will be examined at the end of the following section on radical equations. The nature of the responses to this question seems to indicate that they were influenced by the instructional context and discourse of the lecture that was presented on July 9, the day before this particular quiz question was administered.

Radical Equations

The techniques for solving radical equations and the special checking requirements of their solutions were presented during lectures on July 9. The complete transcripts of these lectures are presented in Chapter Ten and Appendix B. During these lectures, the instructor used the following key words: <u>check</u>, <u>solution</u>, <u>keep</u>, <u>throw out</u>, <u>work</u>, <u>OK</u>.

The instructor began her lecture on this topic by presenting a mathematical justification for the importance of checking solutions to radical equations. She ended this introduction by saying, "... In practical terms, what that means is that when I get done solving an equation, if I have squared both sides, I've got to <u>check</u> my answer and find out if it really is a solution. Maybe it doesn't work. Maybe I just have to throw it out."

"So we have the same sort of situation that we have to check that we had when we had the unknown in the denominator. We solved the equation. We went through the steps, but going through the steps didn't promise that the answer was going to work. Sometimes we looked at the answer and said, uh oh, that makes that denominator equal to zero, and threw that answer out, and didn't get to keep it."

The instructor began working through an example. When she arrived at a solution to this equation, she said, "I don't know that this answer is going to work. Let's check it." She proceeded to write down the substitution steps in the check and arrived at a numerical identity at the end of her calculations. She then said, "Eleven minus four is seven. OK. So it is a solution. I get to keep that one. It does work." Ellen wrote, "is a solution," next to the answer to the original equation that was displayed on the transparency.

The instructor then proceeded through the manipulation steps to arrive at a solution to a second example. When this value was substituted into the original equation, it produced a non-true statement in the last line of the check. At this point, Ellen said, "Uh oh! I'm in trouble. Two doesn't equal negative two. This is not a solution. So I guess I'm just going to have to say, 'there is no solution.' It's true that I got an answer, but my answer doesn't work." She wrote, "there is no solution," under the last step in her check on the transparency and the phrase, "not a solution," next to her answer to the problem.

Ellen continued her lecture from a hand-out sheet which contained two additional examples of radical equations. For the first example, the phrase "x = 4 is a solution" was written on the sheet following the checking steps. For the second example, which contained two solutions, the phrase "OK" was written following the numerical identity produced in the check for the first solution, and "not a solution!" was written following the non-true statement produced in the final step of the check for the second answer.

During her lectures on July 9, Ellen used the word check to refer to the process of substituting the answers to each problem into the original form of the equation. The checking steps were completed when either a numerical identity or a non-true statement was produced. This process differed from the steps that Ellen had used to check the solutions to fractional equations but was similar to the substitution techniques that were used when checking the answers to simultaneous equations.

Even though the actual checking manipulations were different for solutions to fractional and radical equations, the checking steps were used in each case to determine the mathematical validity of all solutions. The instructor pointed out this similarity in purpose at the beginning of her lecture on radical equations. However, Ellen had used a different set of phrases to refer to this process when she lectured on fractional equations.

When a zero was identified in the denominator during the checking process with a fractional equation, the instructor used the phrase "throw it out," and crossed out the corresponding answer on the transparency. When the check of the answer to a radical equation produced a non-equality in the final step, the instructor wrote down the phrase "not a solution" next to the appropriate answer. Ellen used the phrase "OK" when referring to valid solutions to fractional equations, and used the phrases "works" and "is a solution" with such solutions to radical equations.

There is not sufficient information available from the quiz responses and the analyzed discourse to be able to determine to what degree students were able to make appropriate mental comparisons among the various terms that were used in the lectures on fractional and radical equations in order to abstract the similarities that existed in the two checking processes. The phrases "true," "OK," "keep," "works," and "is a solution" were all used by the instructor to refer to the acceptance of an answer as a valid mathematical solution to an equation. The phrases "false," "doesn't work," "throw it out," and "not a solution" were used to refer to the process of rejecting an answer as a valid mathematical solution. The actual manipulation steps that Ellen used in each checking process also varied with the type of equation, but the concept of rejecting or retaining a solution on the basis of some checking criterion remained the same.

Student Response

There were three problems that dealt with equations on the quiz that was given on July 10. Although none of the questions covered radical equations, the students' responses to these three problems are examined within the context of the information on radical equations that was presented to them in class on the previous day. Table 5 below lists the student responses for these three equations. Problems 7-10 #5 and 7-10 #7 were sets of simultaneous equations and problem 7-10 #12 was a fractional equation.

The types of written comments that the students included with their answers for problems 7-10 #5 and 7-10 #7 are used to assess the degree to which the material presented in previous lectures influenced these students' quiz responses. The specific form of the comments that were written down by Peter, Bill and Sue at the end of each of these problems can be traced to information used by the instructor during earlier lectures.

Table 5. Student Checking on July 10

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
7-10 #5	E-N	E-N	C-C*	E-N*	B	C-C*	C-C	Na
7-10 #7	Na	C-N	C-C*	C-N*	B	C-C*	E-C	Na
7-10 #12	Na	C-Ce*	C-Ce	E-N	Na	C-Ce*	Na	E-Ce*

<u>Key</u>
C-C = Correct solution - Check
C-Ce = Correct solution - Error in Check
C-N = Correct solution - No Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved
B = Problem Left Blank
* = Student Comment Follows Check

Peter correctly solved each system of equations, substituted these answers into the original equations, and wrote, "OK," on his paper at the end of each check. (See page 221.) The use of this phrase can be traced back to several lectures throughout the course. The instructor had written, "OK," following her checks of lectures examples of systems of equations on June 26, had used it verbally to refer to valid solutions to fractional equations on July 1 and 2, and had written it next to valid solutions for lecture examples dealing with radical equations on July 9.

Bill checked his answers to problem 7-10 #5 by appropriate substitutions, wrote, "works!" and drew a happy face next to each check. In 7-10 #7, he correctly checked his answers and wrote "they are a solution!" on his paper beside his work. (See pages 221 and 222.) Bill's use of the phrases "works" and "they are a solution" can be traced to terminology employed by the instructor in the previous day's lecture. Ellen had also used the word "works" when checking simultaneous equations.

Sue did not check either problem 7-10 #5 or 7-10 #7, but she used the checking comments that Ellen had employed in the previous day's lecture to label her solutions to these two questions. In 7-10 #5, she arrived at the answers of x = 13/5 and y = -8/5. Following her work, she wrote, "not a solution," on her paper. For problem 7-10 #7 she produced answers of x = -1 and y = -1. Following this, she wrote on her paper, "is a solution."

The use of these two phrases can be traced back to the instructor's comments during the previous day's lecture. At that time, Ellen used such statements in connection with the presence of an equality or non-equality in the last line of the check of the solution to a radical equation when she commented on the mathematical validity of this solution. Sue misapplied these phrases on her quiz of July 10 to comment on the fact that she had produced solutions of systems of equations which were either equal or non-equal in value. (See page 226.)

Sally did not include any written comments with her work for problems 7-10 #5 and 7-10 #7 even though she produced a non-true statement in the check of the second problem. In this respect, she did not appear to be influenced by the instructor's lecture the previous day in which Ellen identified solutions that produce non-true statements in the check by writing, "there is no solution," following her work.

The third problem displayed in Table 5, 7-10 #12, represents a fractional equation. Five of the eight students in the class were able to achieve some kind of solution to this question. This problem was also checked by more students than any other of the 19 problems examined. Four students included a check with their work. However, none of them was able to perform this check without some kind of mathematical error.

The techniques for checking fractional equations like problem 7-10 #12 were described during the lectures on July 1 and July 2. However, the four students listed in Table 5 who included a check with 7-10 #12 used some of the terminology and checking procedures for radical equations that had been presented in the previous day's lecture. These students employed the techniques illustrated during the lecture on fractional equations in order to solve the problem, but then used the checking techniques presented in the lecture on radical equations to check their work.

Alice was the only student who checked her answers to this problem by substituting them into a form of the equation that still contained variable expressions in the denominators. Her solution process was correct. However, she made arithmetic errors in her two checks which resulted in non-equal statements in the last line of each check. She wrote "does not check" next to each of these statements, but did not write any comments on her paper next to her answers. (See page 201.)

Peter solved problem 7-10 #12 correctly but substituted his answers into the equation that was produced after he had cleared his original problem of fractions. These checks produced numerical identities. Peter did not include any written comments on his paper following these checks. (See page 203.)

Bill also failed to substitute his answers to this problem into the original equation. Since his solutions were correct, he arrived at two statements of equality in the last line of his checks. He wrote, "works," on his paper next to each check, and, "they are a solution!" at the end of his problem. He also drew a small happy face following this comment. (See page 204.)

Elaine failed to solve this problem correctly. However, because she made her checking substitution into the line of the problem that followed her mistake, she arrived at a numerical identity in the last line of her check. She wrote, "OK," next to this statement on her paper and, "solution," under her answer. (See page 205.)

None of the four students described above employed the specific checking techniques that the instructor had used with her lecture examples on fractional equations presented on July 1 and July 2. Ellen did not at any time during this instruction completely evaluate the expressions that were produced when she substituted her answers into the original equation. She simply examined this substitution step for the presence of zeros in each denominator, and the calculations were never carried beyond this point. However, when she checked her solutions to radical equations during the lecture of July 9, she carried through all the checking manipulations to form either a numerical identity or a non-true statement in the final line of the check.

The four students who checked problem 7-10 #12 all used this second method of checking when they evaluated their answers to the fractional equation. Alice was the only student who made her substitution into a form of the equation that still contained fractions, but she did not then evaluate this step for zeros in the denominator. It is not clear whether the four students, Alice, Bill, Peter, and Elaine had failed to assimilate the specialized checking procedure for fractional equations, or whether, the later instruction on radical equations superseded this information in the minds of the students.

The influence of the previous day's lecture over the earlier lecture on fractional equations can also be seen in the types of comments that the students used to evaluate their checks. During the lectures on fractional equations, the instructor did not write any comments next to the acceptable solutions but simply referred to the results of the checking by saying, "OK." However, during the lecture on radical equations, she wrote, "solution," next to such answers. Bill and Elaine used these "solution" phrases when they evaluated the results of their checking processes for problem 7-10 #12.

Table 6 below presents the students' performances on those quiz problems that dealt with radical equations. All three of these questions were presented on quizzes that were given during the last week of the course, and that followed the lecture on radical equations on July 9.

Table 6. Student Checking with Radical Equations

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
7-13 #6	C-N	C-N	C-N	C-N	E-N	C-N	Na	E-C*
7-13 #7	E-N	C-N	C-C*	E-N	B	E-C	Na	C-C*
7-17 #6	Na	E-C*	E-N	Na	E-N	Na	E-N	E-N

<u>Key</u>

C-C = Correct solution - Check
C-N = Correct solution - No Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved
B = Problem Left Blank
* = Student Comment Follows Check

It can be seen in the above table that all of the students in the class except Sally and Don were able to achieve some kind of solution to problems 7-13 #6 and 7-13 #7. However, very few of these students also included a check with their work. Only Elaine checked both problems, while Peter and Bill included a check with the second problem.

Fewer students were able to solve the third problem, 7-17 #6. However, of the five students who achieved some kind of solution to this question, only Alice included a check with her work. The overall lack of checking for the three problems involving radical equations indicates that, for the most part, the students in the class had not retained the information that was presented in the lecture on July 9 pertaining to the necessity of including a check with the solutions to all radical equations.

Elaine incorrectly solved problem 7-13 #6 and thus arrived at a statement of non-equality in the last line of her check. She wrote "no solution" on her paper following her problem. She correctly solved the following problem, 7-13 #7, and produced a numerical identity in her check. She wrote "OK" next to the last line of this check. (See pages 242 and 244.)

Peter correctly solved 7-13 #7 and wrote "OK" following the last line in the check of each of his two solutions. (See page 244.) Bill made an error in his solution to this problem which produced two non-true statements in the final step of his checks. He did not write any comments on his paper following his work, although he crossed out the checking steps for one of his two solutions. (See page 245.)

Alice made an error in her solution to problem 7-17 #6 which resulted in two solutions, one of which was the correct answer. She did not write out any checking steps but apparently performed them mentally because she wrote "keep" below the correct solution, and "throw out" below the incorrect answer. (See page 246.)

The written responses that were used by Elaine, Peter, and Alice consisted of phrases that the instructor had used throughout the course during her lectures on simultaneous, fractional, and radical equations. Even though these students were able to reproduce the correct responses from their exposure to the instructor's classroom examples, they were probably supplying answers to the quiz questions through rote-memorization rather than through understanding. The low frequency with which the students in the class employed checking with these problems indicated that, as a group, they did not understand this checking process at an appropriate mathematical level.

Quadratic Equations

Solutions to quadratic equations were discussed during lectures on July 10, 13 and 14. The instructor did not check any of the examples that she used during her lectures on July 10. On July 13, however, she employed checking during the lecture that introduced techniques for solving certain equations that could be factored into the product of two binomials. The complete transcripts of this lecture are presented in Chapter Eleven and Appendix B. The following key words were used during this lecture: <u>check</u>, <u>work</u>, <u>original equation</u>.

After the instructor had finished working through the second example of this lecture, Bill asked, "Are you supposed to <u>check</u> all those?" The instructor replied, "We should go back and put them in and <u>check</u> them. That's right. To be sure they <u>work</u>."

Peter then asked, "Do you <u>check</u> that against the <u>original</u>?" Ellen replied, "You always <u>check</u> that against your <u>original equation</u> as it first shows up."

The above conversation represents only the second time in class that Ellen had made an explicit statement during a lecture concerning the necessity of performing the checking substitutions in the original equation of a problem. Peter's question during the July 13 lecture may have been prompted by the fact that on the previous class day Ellen had written the phrase "check in the original equation" on both Peter's and Bill's quiz papers under their work for 7-10 #12. In this problem, both students had failed to substitute their solutions to the fractional equation into the original form of the problem.

It is not clear from the lecture discourse on July 13 whether the instructor would have mentioned checking with her work with quadratic equations if Bill had not asked his question. Ellen considered checking an optional activity with this type of equation. However, she did not make a point of explicitly stating this fact in her lecture. This information was only implied by the fact that the instructor did not perform any checks on the solutions she derived from her lecture examples.

The answers to quadratic equations, if found correctly, are automatically valid solutions. The checking process in these situations is used to determine the correctness of the manipulations that are used to find an answer rather than to determine which solutions are mathematically valid. These equations, thus, have different checking requirements from those of the fractional and radical equations that were introduced to the students in previous lectures. However, this point was never explicitly stated to the students during Ellen's lectures on quadratic equations.

Ellen only made an oblique reference to the purpose for which checking was used with quadratic equations as she completed her explanation for the third lecture example. "If we take these two numbers and put them back in our <u>original equation</u>, they really will <u>check</u>. They'll both <u>work</u>...I assure you these two numbers will <u>work</u> if you put them back in the <u>original equation</u>."

The above statement did not make it clear to the students why the instructor could offer such assurance that the solutions would "work." Based on the previous lectures on fractional and radical equations, students could only assume that the instructor had performed the check on her two answers at some previous time, and these values had been acceptable by some criterion of the checking process.

Students were not provided sufficient information at this time to be aware that the purpose for checking quadratic equations was different from that of fractional and radical equations. During the work on practice problems following the lecture on July 13, Paul asked me, "I don't have to check?" In this question, he probably was attaching a meaning to the word check that was derived from his exposure to earlier lectures on fractional and radical equations. I told Paul and other students during this work period that the solutions to quadratic equations were automatically valid, and that checking was used to verify the accuracy of their work. However, my comments were in the nature of a statement of fact. I did not amplify this information with any explanation of the mathematical reasons why this was so.

Subsequent lectures on July 13 and 14 dealt with techniques for solving quadratic equations by completing the square or by using the quadratic formula. Solutions for the lecture examples were of a complicated form and did not lend themselves to straightforward checking by substitution. The instructor did not mention checking during these lectures, and students did not ask any questions concerning the use of checking at that time.

Student Response

Table 7 below presents the performance of the eight students on the three quiz problems dealing with quadratic equations that were given following the lectures on July 10, 13, and 14.

Table 7. Student Checking with Quadratic Equations

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
7-15 #6	E-N	E-N	C-N	E-N	Na	C-C	C-N	C-C
7-16 #5	C-N	C-N	C-N	E-N	E-N	C-N	E-N	E-C*
7-17 #7	E-N	C-N	C-N	E-N	Na	C-N	C-Ce*	E-N

Key
C-C = Correct solution - Check
C-N = Correct solution - No Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved
* = Student Comment Follows Check

Elaine checked both of her answers to problem 7-15 #6 by substitution but did not write any comments on her paper following her checks. Bill checked only one of his two solutions to this problem, and like Elaine, did not write any comments on his paper. (See page 262.)

The answer to 7-16 #5 was a complex number, $2 \pm i\sqrt{5}$, which did not lend itself to being checked by the substitution method. It is therefore not surprising that the instructor had not checked any of her lecture examples that contained this type of answer and that none of the four students who solved this problem correctly on the quiz included a check with their work.

Elaine checked her solution to 7-16 #5 only because she made an error which produced an equation similar to that used in the simpler problem 7-15 #6. Elaine wrote down the check to only one of her two answers. Because of her errors in the solution process, she arrived at a statement of non-equality in the last line of the check. She wrote "no solution" next to this answer. She may have checked the second answer mentally. If this had been done by the same method that she used for her first check, it would have produced a statement of equality. Since she had no comment written by the second answer, it is difficult to determine if this was indeed the case. (See page 262.)

Elaine's comment of "no solution" indicates that she was using the criterion of checking that had been demonstrated with radical equations during the lecture on July 9. Elaine had not received sufficiently explicit instruction during the later lectures on quadratic equations to understand that the presence of a non-equality in the check of a quadratic equation did not mean that the solution in question was to be rejected.

The phrase "no solution" when used with answers to radical equations implies that, mathematically, a solution does not exist. However, the "no solution" conclusion drawn from a non-equality to a check of a quadratic equation should be taken to mean that a solution exists, but its mathematically correct value has not yet been determined. It can be assumed from the type of instruction given during the class lectures, that Elaine was not aware of this distinction when she wrote "no solution" for her answer to 7-16 #5.

The final question on quadratic equations that could be easily checked by substitution was problem 7-17 #7. Of the seven students who achieved some kind of solution to this problem, only Sally included a check with her answers. Sally correctly solved the problem but made an arithmetical error in one of her two checks. This produced a numerical identity and one non-true statement. She wrote, "yes," and, "doesn't work," next to the appropriate solutions. (See page 264.)

If Sally was using the phrase "doesn't work" in the same way that the instructor had applied these words to non-valid solutions of radical equations, then Sally's response of "doesn't work" is similar to that given by Elaine for problem 7-16 #5. Since neither student had been given sufficient instruction to understand the different purposes for which checking was used with different types of equations, it is not surprising that they were not able to interpret their checking results at an appropriate mathematical level.

Summary

The analyses of the lecture discourse and the students' quiz responses that are presented in this chapter provide an overall description of the teaching-learning interactions that were centered around the subject of checking. By placing the students' responses within the chronological context of each specific lecture, it is possible to assess the degree of influence that these learning environments had on subsequent student behavior. The specific form of the information that was contained in each lecture can also be compared to the specific form of each student's later quiz response.

An analysis of each lecture transcript reveals that the instructor presented very little explicit information to the students concerning the checking process. Checking was presented during lectures at a procedural level through the use of example problems. Information on the subject consisted of lists of rules describing sequences of manipulations and the use of certain key words. At no time did Ellen present checking explicitly in terms of its mathematical relations with the solution set and the original equation. Students, therefore, received a very low conceptual level of instruction concerning the mathematical ideas inherent in the checking procedures.

Ellen's use of a procedural approach to instruction was consistent with the tremendous time pressure that she was under to present a great deal of information within the five week period. The lecture method, providing mostly procedural information, is an efficient way of "covering" material describing algebraic manipulations. Unfortunately, such an instructional approach cannot adequately present information on the underlying mathematical concepts that justify the types of procedures used.

Ellen did not provide much information during lectures that compared and contrasted the different checking processes that she used. Checking was presented as a separate entity within each lecture that was related only to the particular type of equation under discussion. However, these lectures were all linked together by the use of the common term "checking."

It might have been easier for the students to perceive the mathematical differences in the various checking procedures if these could have been individually characterized by different words than the common term "check." During the five week course, this word was used to signify the different but similar procedures of ascertaining the correctness of the solution process (June 23), identifying the correctness of given solutions to systems of equations (June 25), and establishing the mathematical validity of derived solutions (July 1 and July 9). It can be postulated from the analyses of the students' written responses that they were not aware that these different meanings were being assigned to this single word as the instructor presented her different lectures throughout the course.

The written quiz responses provide evidence that the students were not developing an appropriate level of mathematical understanding to be able to apply the various checking procedures appropriately with each type of equation. It can be seen from the lecture transcripts that this type of information was never made explicit to the students. The students' misapplication of checking was demonstrated in the way that all of the students, except Alice, employed the checking techniques that Ellen had demonstrated with radical equations with their checks of fractional and quadratic equations.

On their quizzes, the students also tended to employ the checking procedures that had been presented in the most recent lecture prior to that particular quiz. They used the checking method described in this lecture regardless of whether or not the equations on the quiz were of the same type as those illustrated in the preceding lecture.

It did not appear that the various types of instruction on checking that were given throughout the course appreciably increased the students' overall use of this procedure. Those students who employed checking before an instructional emphasis was placed on the subject continued to use the process throughout the course. Those students who did not check early quiz problems failed to employ checking even after they had received direct instruction on specific procedures.

The specific types of conceptual structures concerning checking that students could have developed from the classroom environment will be discussed in more detail in the following chapter. These structures will be contrasted to the degree of mathematical information that was presented both explicitly and implicitly within each lecture as a way to measure the effectiveness of the classroom instruction.

CHAPTER FOURTEEN

DISCUSSION OF FINDINGS

General Comments

The primary focus of the classroom instruction on checking was centered around the introduction of the special, mandatory checking requirements of fractional and radical equations. Although checking was never heavily emphasized during the course, it received more direct instruction during the lectures on these two types of equations. However, it did not appear from the students' quiz responses that this instruction appreciably changed the overall checking behavior of the students in the class. Those students who checked problems before such instruction was given continued to do so throughout the course. Those students who did not check problems at the beginning of the course failed to include checking with their problems after specific instruction on the subject was presented.

The influence that the classroom checking instruction did exert on those students who included checks with their work on the 19 quiz problems can be seen from the analysis of the different ways in which they employed written comments with their work. Before instruction on checking was included in the classroom lectures, these students did not provide any written comments on their quiz papers. Following such instruction, those students who checked their work employed the same kinds of phrases that the instructor had used in her lectures relating to the checking process. However, the students tended to employ these phrases in an inconsistent fashion and frequently applied them in mathematically incorrect ways. These checking responses can be interpreted to indicate that the students who used checking did not understand, at an appropriate mathematical level, the lecture information pertaining to the different checking processes. The types of meanings that the students likely assigned to the checking process will be discussed further in a later section of this chapter.

<u>Lecture Discourse</u>

It is apparent from the analysis of the instructor's discourse that she did not provide sufficient explicit information during her lectures to enable the students in the class to utilize the various checking procedures appropriately with the different types of equations that were presented as quiz questions. A detailed analysis of the level of mathematical information that was made accessible to the students within these lectures will be presented later in this chapter in a comparison with the students' perceptions of checking.

It is important to place the instructor's lecture discourse in perspective within the conditions that existed for the implementation of the summer course. Ellen was expected to cover the same amount of material within a five week period that was usually spread out over three academic quarters during the school year. The extremely rapid pace of the course did not allow the instructor time to develop topics thoroughly. Each topic was limited to a single lecture of thirty minutes or less of instruction. There was little time available to redo or review material that appeared difficult for students to learn within one lesson. Within this time constraint, instruction was necessarily limited, and usually consisted of a description of a set of rules of manipulations and several worked examples that were used to illustrate these techniques.

The only requirement that was placed on the students during each day was that they were expected to be able to successfully reproduce the demonstrated techniques during the period immediately following each lecture. The conditions placed upon successful completion of the course required only that the students be able to answer pencil and paper tests of the material that was covered in class.

The fact that the analysis of the students' uses of checking has uncovered certain problems in the lecture presentations should not be taken as an indictment of the instructor or of the students' abilities to learn mathematics. One of the pitfalls of teaching is the inability to know what is effective instruction until after such teaching has taken place. Most instructors do not have the resources or the leisure to perform the minute kind of analysis that was done in this study. Yet, unless such an analysis is performed, it is difficult to understand exactly what information the students are able to utilize from the materials that are presented in the classroom lectures.

Level of Instruction

The nature of the students' quiz responses provides information on the students' relative abilities to both solve and check equations. Many of the students in the class were able to employ the correct sequence of manipulations in order to solve the different types of equations presented as quiz problems. However, these same students demonstrated that they were not able to apply the various checking procedures correctly, once they had achieved a solution to the problem.

The examples that the instructor presented in each lecture on fractional and radical equations apparently provided sufficient information to enable the students to correctly solve many of these equations. However, the classroom explanations and examples that illustrated the different checking processes that were to be used in each case did not provide the same degree of useful information.

As can be seen from the analysis of the quiz responses, the students were not always able to retain this checking information in an appropriate form for future application.

This variation in level of response between the solution and the checking process indicates that there was a difference in the "learnability" of various sections of each lecture. Students could master and reproduce the appropriate solution techniques, but did not appear to be able to master the accompanying checking steps that were included at the end of each example.

The classroom instruction apparently provided the students with sufficient information and practice on procedural skills that they were able to associate particular solution techniques with specific collections of symbols. The manipulations that were used to solve each equation were a direct function of the operations represented by the symbols in each equation. For example, the process of clearing each equation of fractions could be associated with the presence of fractional expressions, and the process of squaring both sides of the equation could be associated with the presence of the square root operator.

The appropriate cues for the use of each checking process were not as visible within the form of each equation. Throughout the course, the students apparently were representing the checking process by a substitution procedure which utilized a formula of responses to evaluate the form that appeared in the final step of the check. The students were not able to link this evaluation formula in their minds to the form of the original equation but only to the final checking step, which appears similar for _all types_ of equations.

It was apparent from the ways in which the students in the summer class utilized the various checking procedures, that they did not have the necessary level of mathematical understanding to employ checking properly. The process of checking solutions to various types of equations consists of more than a simple set of procedural rules. These steps are based on mathematical relationships that exist between the solution set and the given equation. The mathematical status of the solution set is also a function of the type of manipulations that are used to generate these values. _These relationships must be dealt with at an appropriate level of mathematical conceptualization in order for students to be able to apply the different processes correctly with each type of equation_.

The instructor recognized that the students in the class did not have an appropriate knowledge base to be able to understand information at this level. Checking was presented during the classroom lectures at the procedural level as a set of manipulations that were to be performed at the conclusion of each problem. The mathematical relationships that defined the different purposes for which checking was employed were reduced to a set of statements presented to the class in the form of rules to be memorized.

It can be seen from the ways in which the students applied the various checking procedures with the solutions to the quiz questions, that such instruction did not provide them with sufficient information to be able to utilize the different types of checking appropriately. Even when the process was explicitly stated as a clearly defined rule, as in the case with fractional equations, the students did not have an appropriate knowledge base to be able to retain the rule for future use.

Formation of Concepts

Analyses of the lecture examples and discourse used by the instructor and the students' responses to the 19 quiz questions provide descriptions of a specific learning environment and a set of student behaviors that occurred within this environment. This information can be used to identify those aspects of the learning situation that students were able to utilize in their development of concepts relating to the subject of checking.

According to Skemp's (1987) model of concept formation, students develop their concepts of the checking process by abstracting certain generalities from the learning environment that they perceive as characterizing the similarities of the many different examples of checking encountered throughout the course. These generalities are based on the types of symbol patterns with which each example of checking is used and based on the results that are obtained from each set of checking manipulations. The specific nature of this set of attributes or generalities is a function of the type of information that the students are able to focus on within the learning environment. The analysis of the discourse of the classroom lectures indicates that certain information concerning the checking procedures was presented to the students through explicit statements, while other information was present only implicitly through an analysis of the instructor's actions and/or discourse. Thus, there were several layers of instruction within each lecture. The specific attributes that characterized any one student's concept of checking were, therefore, based on each student's ability to extract meaning from either the explicit or implicit layer of each lecture.

An example of this duality of instruction can be found in the instructor's lecture on fractional equations that was presented on July 1. After she had found a solution to the equation in her first example, Ellen said, "Now it looks like I'm all done. But I should warn you that I am not." These comments imply but do not explicitly state that the solution process for fractional equations is not finished until each answer to the problem has been checked. At no time during this lecture, however, did Ellen explicitly state this fact.

The analysis of the students' use of inappropriate and/or inconsistent checking on quiz problems indicates that students did not develop a very mathematically detailed concept of the checking process. The more detailed information on checking that was presented at the implicit level of instruction did not appear to have been utilized by the students. Their quiz responses were consistent with the simplified nature of the information that was found in the explicit layer of classroom instruction.

Table 8 below presents a list of attributes that are postulated to describe a concept of checking that matches the students' behavior on their quiz responses and is consistent with the level of information that was present explicitly in the classroom instruction. Such a concept represents a model of the type of learning that students appeared to be capable of accomplishing under the existing conditions of the course. The choice of vocabulary describing the attributes in this list is a reflection of the general level of knowledge that the students had developed relating to their use of the checking process.

This list represents one of a possible set of meanings and does not exhaust other meanings related to checking that students could have been developing from the learning environment. Individual student interviews might have provided other and different insights into the student's thought processes. The attributes described in Table 8 are based on a structural analysis of the collected data. This list represents the least minimal set of conditions shared by the students in the class. Certain students, on different quiz responses, demonstrated more mathematically appropriate knowledge of specific checking processes.

Table 8. Students' Apparent Perceptions of Checking

1. The checking procedure is used only with solutions to equations.

2. A check involves the substitution of the numerical value of the solution into some form of the equation.

3. Checking is used for the same purpose with all types of equations. This purpose is to show that the solution "works."

4. There is a "correctness" or feeling of satisfaction connected with arriving at an equality in the last line of the check.

5. If the last line of the check is not an equality, then that answer should not be used. This is a rule that must be memorized.

6. Checking is not a mandatory part of the manipulations that must be used to get the "right" answer to a problem.

7. There is no penalty assigned to not using the checking procedure on quiz problems.

Items 3, 4, and 5 in the above table represent an extension to the concept of checking that was derived from class lectures dealing with fractional and radical equations. The attributes listed in these three items are stated in non-specific terms since the students' written responses on quiz problems occurring after instruction on these topics indicated that, although students used and made comments on their checking procedures, they were not able to consistently apply the processes correctly. Items 6 and 7 are included in the above list because of the inconsistent use of checking that was displayed by most of the students. The idea that checking was an optional process appeared to be substantiated also by the behavior of Paul and Sue who did not use checking at all on their quiz problems.

Table 9 below presents a second set of attributes for the checking process. This list was generated by examining those aspects of checking that were present as <u>implicit</u> information within the classroom examples, and from an analysis of the information that was implied but not explicitly stated during the instructor's discourse. The conceptual level of each statement in Table 9 reflects the level of presentation that was used by the instructor during her classroom lectures. The attributes in this list represent one possible set of meanings and was derived from the above mentioned analysis of the data. This set of meanings does not, however, necessarily represent the types of attributes that the instructor was assigning to the subject of checking.

Table 9. Implicitly Stated Attributes

1. Checking is performed by substitution into the original equation.

2. Not all solutions to certain equations automatically "work." These need to be checked.

3. Checking is mandatory for some kinds of equations. There are different requirements for rejecting a solution for different types of equations. A mathematical reason exists for doing this with radical equations.

4. The solutions to quadratic and linear equations do not use the same kind of checking as radical equations.

5. It is necessary to make some kind of judgment statement regarding the status of a solution based on the results of the checking process.

6. Checking is sometimes used to catch mistakes in the solution process.

Summary

It can be seen from a comparison of the two separate lists that a knowledge of the attributes of the second list is necessary in order to correctly apply the checking processes to the different types of equations that were included as quiz problems. The students' knowledge, however, appeared to be based on a concept of checking that can be characterized by the attributes presented in the first list. Using this type of a concept, it is apparent that the students in the class would not be expected to be able to utilize checking appropriately at all times.

Because the students were unfamiliar with the material that was being presented to them in the class, they focused their attention primarily on the explicit aspects of information contained in each lecture. For the most part, the students concentrated on observing the particular techniques that were demonstrated with specific kinds of mathematical examples and on noting the physical aspects of each symbol arrangement. The objects of study of each lecture were perceived by the students at a procedural level of understanding.

The two instructors in the class, however, perceived each lecture from a different mental framework with a different knowledge base. Because they were already familiar with the subject, they were aware of procedures, rules, and relationships that linked various lectures together. This information was, however, only implied within the context of the lectures. It was present within each lecture, and affected the particular techniques required for each example, but never explicitly stated.

The comparison of the two checking lists indicates that most of the students in the class were utilizing only that information in the learning environment that was available within their existing knowledge bases. The fact that their checking responses on the quizzes did not meet certain criteria for mathematical correctness was due to the nature of the instruction to which they were exposed. It would be necessary to provide more appropriate instruction in order to expect the students to develop a more mathematically detailed concept for the checking process.

One of the problems that impeded the presentation of effective mathematics instruction was the instructors' high level of conceptual understanding of the subject matter. Ellen and I tended to assign meanings to the lecture discourse that were not always apparent to the students in the class, who were operating from a different knowledge base. Unfortunately, we were not aware at that time that the students were unable to understand the lecture material from our perspective. We did not have access to the kind of detailed analysis that is presented in this study to enable us during the course to better understand the existing teaching-learning environment.

It was found from the analysis of the students' quiz responses that they did not have the appropriate level of mathematical understanding to be able to effectively use the limited amount of checking information that was made available to them in class. The different checking processes should not have been included in the course unless sufficient time and appropriate learning activities could have been assigned to the subject to enable the students to develop an appropriate set of concepts. The subject of checking is sufficiently complex that it cannot be presented successfully by a few examples within each lecture. This topic should be dealt with at a higher degree of mathematical conceptualization than the level of instruction that was employed during the summer course.

The lectures on checking represented a situation in which the topic under study was being presented to the students outside of its appropriate level of mathematical conceptualization. Because this material was <u>displaced</u> from its appropriate learning context, it could not be learned with understanding. The level of instruction at which the subject was presented to the students during the summer class could not address the mathematical framework on which the checking procedures are founded.

The following chapters in this study (Part III) discuss a learning model that addresses the problems of instructional mismatch and displacement that were observed to be operating during classroom lectures in the summer course. This model analyzes the subject of algebra in terms of the types of mathematical conceptualizations that are required to develop appropriate levels of understanding of the topics involved. A specific theory of instruction is also presented that provides guidelines for the implementation of effective classroom mathematical concept learning.

PART III

THEORETICAL DISCUSSION

CHAPTER FIFTEEN

LEARNING AND INSTRUCTION

Part III presents descriptions of various theoretical models of learning and instruction that are developed to discuss and integrate the findings from the two summers' research. Chapter Fifteen discusses individual models of mathematical concept learning and effective instruction. This information is used to develop a prescriptive set of instructional activities that are outlined in Chapter Sixteen. Chapter Seventeen discusses the relevance of the teaching/learning models to current trends in mathematics education research. Further information relating to the findings of the present study are presented in Chapter Eighteen which describes a mathematical analysis of the learning structure for the subject of algebra. Chapter Nineteen presents a summary of the findings of the study, summarizes and integrates the various theories that are developed in Part III, discusses the significance of the present study, and presents recommendations for future research.

Mathematics Learning

The purpose of this chapter is to discuss a set of theoretical models describing mathematics learning and instruction. In the first part of the chapter, a model of learning is presented through a set of definitions of such terms as "concept learning," "understanding," and "meaning." These definitions are based on learning theories developed by Skemp and other authors in the fields of mathematics education and cognitive science. This model is used to explain certain aspects of the behavior of the students in the two summer classes. In the second part of the chapter, a model of effective instruction is presented that discusses the implementation of the learning theories described in the previous section.

The model of mathematics learning that is presented here is concerned with describing postulated conditions under which an individual is capable of developing mathematical concepts through the construction of certain types of mental structures. This model is centered on describing the overt behavior of individuals and those conditions in the learning environment that are postulated to represent situations in which mathematics learning processes are taking place. Unlike the focus of much of cognitive science research, the emphasis of this model is not on the mental processes that account for mathematical thinking but on the conditions that pertain to learning during the period that individuals are engaged in the initial stages of developing new mathematical ideas and concepts. Selected interactions and patterns of student behavior are provided that illustrate the applicability of this model of mathematics learning to the findings that were reported in Parts I and II of this report.

Concept Learning

The term mathematics learning will be used in this chapter to refer to the complex cognitive behavior that is associated with the development of mathematical concepts. Such concepts are formed as individuals abstract common characteristics that they perceive to be shared by a collection of examples. Skemp (1979) defines this abstraction as "the process by which certain qualities of actual objects and events are internalized as concepts while other qualities are ignored" (p.24). Once concepts are formed, an individual is able to distinguish between examples and non-examples based on the abstracted characteristics of these concepts.

According to Skemp (1987), a person is able to classify new experiences on the basis of an internalized concept long before he or she is able to successfully formulate a definition of this concept. An individual cannot develop a concept from its definition but must build the mental construct by experiencing its elements. Higher order concepts may be built out of lower order or primary concepts by abstracting specific characteristics of these original concepts. The necessary lower order concepts must be developed first, however, before an attempt is made to introduce higher order abstractions.

Mathematical concepts may be based on concrete examples drawn from physical reality, or they may represent highly abstract ideas that cannot be illustrated by physical examples. Mathematical ideas may represent second order concepts that are formed from generalizations based on the abstractions of certain properties of lower order concepts.

> Not all mathematical concepts are at the same level of abstraction. Some concepts such as number and length appear to arise out of the child's first-hand experience of reality. These are mainly the concepts of arithmetic. Later, concepts develop through reflection on these first-order concepts and an entirely new concept such as proportionality or functionality becomes available to the pupil. Moreover, mathematical concepts, like other concepts, do not usually develop suddenly into their final form. They widen and deepen with experience and are often available in specific situations at first. The process of concept formation thus differs from the learning of facts and isolated details. (Lovell, 1966, p. 210)

The term concept learning can be applied to the process of mastering algebraic skills and procedures as well as to the development of abstract ideas. Procedures must assume a set of properties in the minds of the learners if they are to be placed within an appropriate mathematical network of relations that permit such procedures to be applied in unfamiliar situations. Thus, each set of manipulations must be accompanied by a set of abstract characteristics that describe the contexts within which such procedures are used, the patterns of symbol manipulations that are employed, and the algebraic changes that such manipulations produce within each context. These properties must also be placed within an appropriate network of relations with other procedures and algebraic concepts.

Narrow and Meaningful Learning

Much of the learning that occurred during both summer classes consisted of narrow learning that was not based on an understanding of underlying concepts. Such learning occurs when individuals are not able to relate new material to their existing mental structures or to expand or restructure existing knowledge systems to accommodate this information (Skemp, 1987). Material learned in this way can only be retained by being stored as collections of unrelated items in memory. Rote learning that is based on memorization without underlying comprehension is one form of narrow learning (Meyer, 1977).

In contrast, meaningful learning takes place as students are able to form relationships between new material and previously acquired knowledge in ways that allow the new information to become incorporated into or expand upon existing mental structures. Studies have found that material learned through comprehension is retained more effectively than rote learned material, and can be applied more successfully (Meyer, 1977; Skemp, 1987).

If the outcomes of a learning experience are measured in terms of such behavioral objectives as the responses on paper and pencil tests, it may be difficult to distinguish whether narrow or meaningful learning has occurred (Meyer, 1977) since it is possible for such responses to be produced through memorization as well as through learning that is based on underlying comprehension. Certain behavioral objectives may be achieved more quickly through rote learning than through understanding. However, such behavior has serious consequences for students' abilities to develop meaningful learning in the future.

> If a task is considered in isolation, schematic learning may take longer. For example, rules for solving a simple equation can be memorized in much less time than it takes to achieve understanding. So if all one wants to learn is how to do a particular job, memorizing a set of rules may be the quickest way. If, however, one wishes to progress, then the number of rules to be learnt becomes steadily more burdensome until eventually the task becomes excessive.... Time spent in acquiring [the main contributory ideas] is...of psychological value (meaning that present and future learning is easier and more lasting). (Skemp, 1987, p. 27)

The problems inherent in learning without comprehension that Skemp describes can be seen in the patterns of the student quiz responses from the second summer class. The analysis of the ways in which the students used the various checking procedures provides evidence that these responses were the result of narrow learning.

During this class, the students were able to master the procedure of substituting the numerical value of a solution into some form of the given equation and to memorize an accompanying collection of phrases that were to be applied to the results of this substitution. This allowed the students to be able to respond adequately to quiz questions given within days of their exposure to the various checking procedures. However, such learning was not based on an underlying level of comprehension that allowed the students to retain specific information concerning the specialized checking requirements for fractional and radical equations in a form that could be applied appropriately to the quiz questions given at the end of the course.

The students did not have an appropriate set of mental structures to which the mathematical rationales for the new types of checking could be related. Their existing mental structures concerning the basic substitution process of checking provided a set of concepts that they were unable to restructure in a mathematically appropriate way to incorporate the different checking techniques. Consequently, the unfamiliar information related to checking fractional and radical equations was either not retained or used in mathematically inappropriate ways.

The learning strategies that were employed by the students in the first summer course provide another example of a form of narrow learning. These students used master examples as a coping mechanism that relied on a classificatory recognition strategy rather than on the development of understanding as a way to meet the course objectives. Such learning was employed by many of the students as they attempted to master manipulation techniques for particular algebraic expressions and equations. Since these students did not have an appropriate set of mental structures that would allow them to relate these techniques to relevant concepts, each algebraic problem was perceived as an unrelated item that could only be linked to an appropriate course of action through an accompanying master example.

The students relied on the use of master examples, rather than on developing an understanding of the mathematical relationships expressed by the algebraic symbols, as a way to successfully perform manipulations on given problems. Such practice enabled the students to meet the requirements of the course as long as they were allowed access to sources of master examples. On individual quizzes and tests, however, they were required to rely on the rote memorization of specific techniques and procedures, which placed a burden on the amount of information that they could effectively retain. Such narrow learning did not enable the students to incorporate the information contained in the summer course into their existing mental structures or to expand and develop mathematically adequate concepts for the topics of study.

If students cannot relate new material to some item of previously learned information, new topics can only be retained through memorization or through the reliance on the copying of pre-worked master examples. Such narrow learning is very short term, however, and does not provide a foundation on which to base future learning. Much of the learning that is needed to progress successfully through an algebra course requires the development of a succession of concepts upon which further learning is based. The use of such narrow learning as memorization and classificatory recognition strategies do not provide individuals with adequate concept development for the successful understanding of this subject.

Mastery

The term mastery is concerned with measurements of student achievement or output. Such output is usually measured by the performance of students on paper and pencil tests of their ability to meet certain behavioral objectives; i.e., students will demonstrate the ability to solve linear equations of one variable. Such a criterion represents the most predominant method currently employed to measure achievement in high school algebra courses.

Bloom (1984) uses the term "mastery learning" to refer to a type of instruction in which progress from one topic to the next is predicated on the successful execution of a series of achievement tests. Students who fail to respond to certain questions correctly are provided remediation on those items that they missed. Such remediation continues until correct test responses are produced.

Such a reliance on the results of paper and pencil tests to determine the criterion of success typifies the meaning that educators associate with the term mastery. This was the criterion that was used to determine whether or not the students received a passing grade in each of the two algebra classes reported on in this study. However, as the analyses of the collected records from the two summers indicates, students were able to reproduce correct mathematical responses without necessarily developing a set of underlying concepts that represented the manipulations they performed.

Meaningful learning can be said to have taken place when an individual is able to demonstrate mastery of a particular subject in such a way that it is apparent that he or she has also developed an underlying understanding of the concepts involved. It is also possible for individuals who have achieved meaningful learning to produce responses to specific tasks in which no conscious thought has been made concerning the ideas or principles on which the task is based. Skemp (1987) describes such behavior in terms of the "automatic performance of routine tasks."

> Once we have understood a mathematical process, it is a great advantage if we can run through it on subsequent occasions without having to repeat every time...the conceptual activities involved. If we are to make progress in mathematics it is, indeed, essential that the elementary processes become automatic, thus freeing our attention to concentrate on the new ideas which are being learnt -- which in their turn, must also become automatic.... This automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols, which is not mathematics. (pp. 62, 63.)

Skemp's emphasis on the importance of developing the ability to automatically perform routine tasks highlights some of the instructional difficulties that existed during both summer classes. Students in the two classes were expected to be able to reproduce many mathematical procedures in a routine fashion in order to manipulate and investigate other information presented in subsequent lectures. The rapid pace at which the courses were conducted, however, did not provide the students time in which to master each set of manipulations with understanding. Consequently, the students were not able to free their attention to grapple with new ideas as the focus of instruction changed from day to day.

Such a situation occurred during the introductory lectures presenting techniques for solving fractional equations. The focus of this instruction was centered on the conditions of equality expressed by each equation and the types of procedures that needed to be used to deal with this equality. Unfortunately, at this time, the students had not yet developed underlying concepts for the idea of a fractional expression and could only manipulate such symbols in a mechanical fashion. They were therefore not able to understand the differences between the new information pertaining only to fractional equations and previously encountered material related to fractional expressions. Without the ability to manipulate fractions automatically, but with understanding, the students could only proceed with the manipulations of the new material relating to equations in a mechanical and meaningless way.

Much of the subject matter involving manipulation procedures and mathematical concepts that is covered in an introductory course in algebra is of a sequential nature. Many of the algebraic procedures that are used consist of combinations of individual mathematical operations. These procedures, in turn, are used to manipulate algebraic expressions and relationships in order to find solutions to particular equations. It is imperative, therefore, that each topic be mastered with understanding in order for subsequent instruction to be effective. Students must be able to develop the ability to perform many algebraic manipulations in a routine manner as these topics become prerequisites for the development of other algebraic procedures and concepts.

The danger of measuring the mastery of a subject in terms of behavioral objectives is that such a criterion may not be able to distinguish whether students are performing these task through rote memory or through understanding. Students may be misled into thinking that the successful completion of paper and pencil tests is an indication that learning has taken place because of the undue emphasis that typically is placed on the results of such testing. Until mastery with understanding is measured and emphasized, students will continue to experience difficulties in learning subjects such as algebra in which progression through the course depends on the ability to achieve understanding rather than on the ability to produce mechanical manipulations of each topic of study.

<u>Meaning</u>

Skemp (1987) defines meaning as the concept or idea that is attached to any symbol. "A symbol is a sound, or something visible, mentally connected to an idea. This idea is the <u>meaning</u> of the symbol" (p. 47). If this meaning represents a concept, then it is distinguished by a set of characteristics or attributes that represent a particular set of generalizations about the symbol.

In the traditional, lecture-format, learning environment, meaning is often assigned to symbols through definition. Algebra texts often begin each section by presenting a definition of the topic under study, which is then illustrated by a set of examples. It is assumed that the students in the class will then be able to assign this given meaning to all future representations of the objects of study. However, if care is not taken to provide learning situations that match the focus of instruction with the students' existing levels of mathematical development, students may develop alternative sets of meanings to those assigned by the instructor to the presentation of the subject matter.

Students in any learning environment may develop meanings from "hidden" information that is available to the students from their perspective, but that is not apparent from the perspective of the instructor who brings a different knowledge base to the situation. Such a situation is illustrated by Tracy's frustrated comment, "I can just make up a rule and call it algebra!"

My immediate reaction when I heard Tracy's comment was to picture an image of a system of algebra that consisted of a collection of strange, arbitrary rules that had to be memorized and used in unrelated ways. I contrasted this to my own image of algebra in which the "rules" that Tracy objected to were perceived as the result of basic mathematical principles that provided a satisfying connectiveness to the subject.

The rule-based image of algebra that was conjured up by this interaction represents a possible set of meanings that Tracy could have been assigning to this subject from her collected experiences within various learning environments. Such meaning was not purposefully taught as part of the classroom instruction but could nevertheless have been perceived by Tracy from a hidden curriculum. Tracy's comment provided me with an insight into how vastly different the types of meanings must have been that she and I applied to the same objects of study within the classroom. Given the existence of such hidden curricula, it is important to provide students with learning experiences that will enable them to develop more mathematically appropriate sets of meanings for their classroom environments.

Learning, Instructional Contexts

The meaning actually assigned to any object of study is a function of both the instructional context and the learning context of each individual student. The term learning context is used to refer to the existing knowledge base that any learner brings to a situation. This knowledge base consists of his or her experiential background, functional vocabulary related to the objects of study, and the set of concepts and relationships that have been previously developed.

The learning context affects the ways in which each individual is able to utilize the information that is made available in the instructional context. This instructional context refers to the types of meanings that are both explicit and implied within the learning environment. These meanings are a function of the focus of instruction and the types of examples that are employed with each object of study.

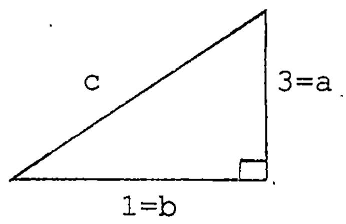
If an individual's learning context matches the prerequisite requirements of the instructional context, then the learner will be able to utilize the information in the learning environment within the intended focus of instruction. However, if the information in the learning environment is based on concepts and skills which the learner has not yet developed, he or she will not be able to construct meanings that parallel this focus of instruction. In such a situation the learner will either resort to rote memorization or develop an alternative set of meanings that are based on his or her existing learning context. In order to understand the objects of study in such a situation, the learner will attempt to identify aspects of the learning environment that he or she can relate in a meaningful way to his or her present set of mental structures.

The discussion of the findings (Chapter Fourteen) that were based on the analyses of the students' quiz responses present many examples of classroom situations in which the students' learning contexts did not match the instructional context. These differences can be seen by comparing the list of the students' perceptions of checking presented in Table 8, page 363 to the list of implicitly stated attributes described in Table 9, page 365.

The different knowledge bases possessed by the students and the instructors produced situations in which two different sets of meanings were being assigned to the same set of objects of study. The result of this disparity was that much of the intended information in the instructional context was not available in a form that could be learned by the students.

The learning strategies that were described in Chapter Four include examples of students attempting to identify familiar material within an instructional context that does not match their existing learning contexts. The following is an example of such behavior and illustrates the way that two of the students in the first class were attempting to relate unfamiliar information to similar objects of study within their past experiences.

> The lecture topic covered the Pythagorean formula and a formula for finding the distance between two given points. To illustrate the Pythagorean formula, the instructor gave the following as one of her examples.

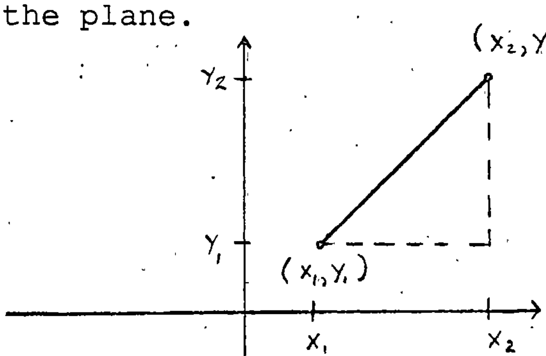


$$a^2 + b^2 = c^2$$
$$9 + 1 = c^2$$
$$10 = c^2$$
$$\sqrt{10} = c$$

Following this example, the instructor introduced the formula for the distance between two points (x_1, y_1) and (x_2, y_2). This information was contained on a handout sheet (shown below) which she read through with the students.

DISTANCE FORMULA

Let's once again look at (x_1, y_1) and (x_2, y_2) in the plane.



The rise $= y_2 - y_1$ is the length of side A of the triangle.

The run $= x_2 - x_1$ is the length of side B of the triangle.

How long is side C?

According to the Pythagorean Theorem:

$A^2 + B^2 = C^2$ but $A = y_2 - y_1$ and $B = x_2 - x_1$

so $(y_2 - y_1)^2 + (x_2 - x_1)^2 = C^2$

so $C = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Now we not only know the length of side C but we also know the distance between the points $(x_1 y_1)$ and $(x_2 y_2)$. Using d (for distance) in place of C we get the distance formula

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

After the instructor had discussed the above derivation of the distance formula, she proceeded to work through the following example.

$P_1 = (2, -3)$
$\quad\quad x_1 \quad y_1$

$P_2 = (-2, 1)$
$\quad\quad x_2 \quad y_2$

$$d = \sqrt{(1 - (-3))^2 + (-2 - 2)^2}$$

The instructor performed the required calculations and arrived at

$$d = 4\sqrt{2}$$

Bob: "So where does that go? Is that your rise over run?"

The instructor drew a visual aid.

Bob: "Where do you put the $4\sqrt{2}$?"

Instructor: "This is the distance between the points."

Bob: "Is that your c?"

Tom: "Do they usually draw the right triangle?"

Instructor: "When finding the distance between two points they usually don't give you anything except the two points."

The above example was presented in a lecture that occurred during the period in which various topics concerning the use of the square root operator were discussed. The intended focus of instruction for the two lecture examples was to provide students with illustrations of applied situations in which calculations involving roots of numbers were used. However, the type of application used in the lecture represented a higher level of concept development for the objects of study than that which had been utilized during the introductory lectures describing manipulation techniques for radical expressions. In order for the students to have appropriately understood the focus of instruction of the present lecture, they would have had to have previously developed facility in using the skills that were being applied. This knowledge base would have allowed them to turn their attention to the study of the applications of these skills.

The particular applications that were used in each example in the interaction described above are based on certain sets of prerequisite knowledge. In order for students to understand the use of the Pythagorean formula, it is first necessary for them to possess knowledge about the properties of a right triangle, including the ability to identify its hypotenuse. The derivation of the distance formula, which provides an application of the use of the square root operator, involves techniques used in analytic geometry. In order for the students to have been able to assign significance to this derivation, it was necessary for them to have previously developed an understanding of the ways in which algebraic symbols could be manipulated to determine geometric information. The manipulation of the generalized symbols (x_1,y_1) and (x_2,y_2) also represented an abstraction which students could only have followed after they had developed sufficient properties for specific examples of such a technique.

It is apparent from Bob's questions that he was not able to immediately assign meanings to the symbols that were analogous to those that the instructor had intended. His existing set of mental structures did not include appropriate prerequisites to enable him to follow the instructor's lecture at an appropriate level of understanding. He used his questions in an attempt to relate the new material to those objects he perceived within the lecture for which he was able to assign some kind of significance. He was trying to assign meaning based on past experiences; meaning, however, that did not coincide with the meanings that the instructor was assigning to the same objects of study.

The terms "rise" and "run" that appeared in the handout sheet represented objects which Bob had previously encountered and for which he had developed a set of meanings. His question "Is that your rise over run?" was an attempt to identify a conceptual link between the new material and his existing learning context.

When the instructor answered this question by drawing two points connected by a straight line, she was abstracting what appeared to her to be the salient feature of the object of instruction, i.e., that the shortest distance between any two points is measured along a straight line path. Unfortunately, such an abstraction removed the information in the lecture even farther from Bob's level of understanding.

Bob's question "Is that your c?" and Tom's question "Do they usually draw the right triangle?" indicated that they were able to use the straight line that the instructor had drawn to focus their attention on the graphical representation illustrated at the beginning of the distance formula derivation on the handout sheet.

Unfortunately, neither the instructor nor myself were able, at that time, to perceive from the questions asked that Bob was not able to identify meaningful material within the lecture. Both Ellen and I were assigning significances to the various aspects of the distance formula derivation which enabled us to assign meaning to the value of $4\sqrt{2}$ within the context of the intended focus of instruction.

After Tom asked, "Do they usually draw the right triangle?", the instructor replied, "When finding the distance between two points, they usually don't give you anything except the two points."

My attention was focused on the straight line that the instructor had drawn. Instead of using this to reference the original graph, I, in a similar fashion to the instructor, abstracted the property of "the straight line representing the shortest path that can be drawn between two points." Using this meaning I commented to the students, "That's how the crow flies."

Unfortunately, this comment prevented us from following up on Bob's confusion during the lecture. I had used a common analogy that was unfamiliar to Indian students, especially to the Crow students in the class. The instructor immediately realized that my comment was culturally out of context and explained the allusion I had meant by my comment. She then asked the students if they had any equivalent sayings in their various cultures.

One of the Crow students replied, "We don't have any straight lines; everyone is crooked," which produced a big laugh among the students. After this exchange the students were asked to begin work on a set of practice problems involving fractional exponents.

This lengthy interaction can be used to illustrate the ways in which different meanings are assigned to the same collection of objects by different participants in the learning environment. These meanings are based on each individual's existing set of mental structures and the particular perspective that such knowledge brings to the information in the environment. As can be seen in the discourse that was reported, it is difficult to communicate when various individuals are assigning different meanings to a shared experience. The differences that existed between the students' learning contexts and the intended instructional context also made it difficult for effective learning to take place.

Understanding

Skemp (1987) defines understanding as the assimilation of new information into an existing schema. Students can be said to be operating with understanding when they are able to relate information in an appropriate way to their existing mental structures. The following examples provide illustrations of such a definition of understanding.

During the first summer class Shirley demonstrated her understanding of the operation of taking the average value of two numbers. (See page 126.) She had just finished determining the average of the two values 0.678 and 0.6782 by entering the appropriate sequence of numbers and operations on her calculator. She explained to Tracy that the resulting value of 0.6781 was true, since 1 came between 0 and 2. Shirley was relating the results of the specific calculation to her existing mental concepts representing the relative ordering of numbers. The value that she obtained from her calculation was consistent with that predicted by her conceptualization.

Donna exhibited her ability to operate from understanding when she developed a solution strategy for an unfamiliar problem situation described by the following word problem. (See page 127.)

> I have a number in my head. First I add 2 to the number. Then I subtract 5. Then I multiply that answer by 10. My answer is 5. What is the number I am thinking of?

Donna solved this problem by beginning with the answer and working backwards, reversing each operation as she combined the given numbers. This procedure demonstrated that Donna had developed a mathematically appropriate set of concepts representing the general properties of the ways that the operations of addition, subtraction, multiplication, and division behaved with real numbers. She was able to understand the mathematical relationships that were expressed in the original problem based on her existing set of mental structures.

Skemp makes a distinction between understanding and accommodation or restructuring in which existing schemas are altered to allow for the inclusion of new information. Such a situation produces a similar feeling of understanding in the mind of the learner.

In the following interaction, Tom exhibits behavior that can be postulated as representing a restructuring process. Unlike Donna and Shirley who demonstrated understanding in the interactions previously described, Tom could not immediately relate the information presented during a certain lecture to his existing knowledge base. However, he was able to restructure his present set of mental structures to include new information relating to the changes in symbol patterns that he perceived in the practice examples. (See page 97.)

4-3 The lecture topic covered a technique for sketching parabolas from equations written in the following form

$$y = a(x-h)^2 + k$$

given: $y = -2(x-1)^2 + 3$, then $h = 1$, $k = 3$

The instructor was demonstrating how to determine the coordinates of the vertex of the parabola from the values of h and k represented in the above equation.

At this point Tom asked, "How come h is one?"

The instructor replied, "I was waiting for someone to ask that."

Tom: "Why isn't h minus one?"

Instructor: "h is the number that comes after the minus sign."

Donna: "What if there isn't a minus sign?"

Instructor: "We'll get to a problem like that later."

There continued to be some confusion on the sign of h, given the initial form of the equation. During work on practice problems following the lecture Tom soon devised his own rule for determining the sign of h. He explained to me that you just change the sign between x and h and you get the correct number.

Tom did not have an adequate learning context to understand the instructor's explanations based on the one example that she demonstrated. However, he was able to generate an alternative rule that had meaning for him after he had experienced several examples. From observing the pattern of symbol changes in several different examples, Tom was able to generate a generalized statement of the rule that provided correct results in each situation. At this point Tom probably would have claimed that he "understood" how to determine the value of h.

Tom's understanding was based on a mental structure that he was able to generate from within his existing learning context. This understanding was not based on a knowledge of the correspondence between the algebraic and graphical representations of the mathematical relationship described by the parabola. A mental structure based on this information would provide a schema that would allow for more effective growth and mathematical development than that on which Tom based his "rule." However, Tom did not have the necessary knowledge base to extract such information from the learning environment. Such information was also not made available to the students during the class.

Tom's rule was adequate for the limited application of the problems on hand. The characteristics of this rule were a function of the present level of Tom's learning context. At this point, his concepts relating to algebra were largely based on observations of the physical changes that occurred in the symbol patterns of each problem under different types of manipulations.

<u>Student Learning</u>

Tom's behavior demonstrates a type of experiential learning. Tom could not understand the reasons for assigning a certain value to h simply because he was provided with a definition of a workable rule. It was necessary for Tom to focus his attention on the values of h from several different examples before he could reproduce the appropriate behavior with understanding. This behavior was then based on his personal experiences. It is interesting to note that when he did achieve understanding, it was based on an alternative rule from that provided by the instructor.

This interaction illustrates a fairly typical situation in an algebra class. Topics are introduced to the students by providing them with a definition or description of a set of procedures. These are then illustrated by an example. After such instruction students are provided with problems, similar to the lecture example, with which they practice using the rules or definitions given at the beginning of the lesson. The assumptions behind such a presentation is that the initial rule statement and its subsequent illustration provide the learning experience. The problems that students work on are for "practice" or reinforcement.

In Tom's interaction, the actual learning took place during the time following the lecture when the students were working through the practice problems. What was learned, in Tom's case, turned out to be different from the specific information that was presented in the lecture. In such a situation the instructor does not control the actual learning that takes place.

Tom's "learned" rule was appropriate to the specific situation, and formed a viable alternative to the rule presented by the instructor. However, it is also possible for students in such a situation to extract inappropriate information from the learning environment and develop mental structures that are not mathematically correct. Mauer (1987) recognizes that such situations can very easily occur in learning environments in which little control is placed over the actual "learning" situation.

> What students get out of a lesson may be quite different from what the teacher intends or even what the teacher actually presents. The regularities the students pick up may be quite incidental to the intended lesson, e.g., apparent surface structure rather than meaning.... Consequently, if students get problems wrong, it may not be because they "haven't understood" some part of the procedure, but rather because they have already invented an alternative to that part. (p. 171)

Tom's learning represents an example of concept learning that occurred as he perceived certain common properties in the collection of practice problems. In Tom's case, this type of learning was not actively promoted by specially guided classroom activities, but occurred as a spontaneous event prompted by Tom's need to discover meaning in the given learning activity.

It makes sense to promote such learning and capitalize on students' abilities to perceive and organize patterns and similarities in their work. Such activities should be the focus of instruction rather than an unrecognized by-product. Wenger (1987) has proposed a set of assumptions based on his experiences with work with high school and college students enrolled in algebra courses that addresses this issue of by-product learning.

> Assumption 1: Students learn mathematics primarily from examples and practice tasks--typically from textbooks. That is, they do not usually learn by understanding the explanations of procedures and using those explanations. Rather, they figure out what the procedures are about by working through them.
>
> Assumption 2: An important force driving student behavior is the need to make sense of the things by creating simple, straightforward procedures that work. As students work problems, they invent rules for the procedures that seem to fit the expected answers. Often those rules are as simple as possible, and often they are incorrect. (p. 221)

Constructivism

Wenger's assumptions and Tom's rule making can be explained by a constructivist theory of learning. Much of current research in mathematics learning is based on this type of model. Although the following statement is related to the learning behavior of children, the description of the constructivist learning model can be applied to adult students as well.

> One of the basic assumptions underlying much current research is that children actively construct knowledge for themselves through interaction with the environment and reorganization of their own mental constructs. Although instruction clearly affects what children learn, it does not determine it. Children are not passive recipients of knowledge; they interpret it, put structure into it, and assimilate it in light of their own mental framework. There is a growing body of research that suggests that children invent a great deal of their own mathematics and that they come to school with well developed informal systems of mathematics. (Romberg and Carpenter, 1986, p. 853)

This theory places the responsibility for the learning act on the student as opposed to the teacher. It is the student's actions, based on his or her existing set of mental structures, in reaction to a learning environment that produce learning. The teacher acts as a guide, or resource person, but not as the vehicle through which knowledge is transferred.

Summary

The model of mathematics learning presented in this section is concerned with understanding conditions that promote the formation of mental constructs representing mathematical concepts. It is postulated that these mental structures are formed as individuals abstract certain characteristics from collections of examples of the objects of study. This model is also based on the constructivist theory that states that learning takes place as individuals construct concepts and related mental structures through an active interaction with their learning environment. Because the types of concepts that are developed in any environment are a function of the level of conceptualization of each individual's existing knowledge base, they may not necessarily coincide with the level of conceptualization intended by the formal instruction present in the environment. Information presented through instruction that cannot be related in any way to an individual's present knowledge base is not utilized or else retained through some form of narrow learning.

A significant aspect of the model of mathematics learning that is presented in this chapter is the importance of providing an appropriate match between students' present levels of knowledge and the level of mathematical conceptualization intended in the learning environment. According to this model, the ideal mathematics classroom learning situation occurs when students are provided with activities in which the instructional context is designed to match the level of each students' learning context. In such a situation, students are able to focus their attention on the intended objects of study and can engage in meaningful learning at that level of conceptualization. Manipulation skills and algebraic ideas are then developed within a conceptual framework that allows the students to assimilate new information in the learning environment into their existing mental structures.

Such a learning environment is carefully organized to provide meaningful experiences that actively and effectively engage each student in appropriate learning experiences. The resultant learning is based on conceptual understanding rather than rote memorization and provides a rich mental network of related concepts. Such learning enables students to successfully progress to more advanced levels of mathematical study and enables them to apply their acquired knowledge in diverse and unfamiliar situations.

Effective Instruction

The purpose of this section is to develop a model of effective instruction that utilizes the theories of mathematics learning that were presented in the previous section. Examples of student interactions from the two summer classes will be used to illustrate various aspects of this model.

Romberg and Carpenter (1986) describe the limitations of mathematics instruction as it appears in a traditional mathematics classroom. Information is primarily delivered in lecture format as the instructor presents the subject matter through verbal explanations illustrated with blackboard examples. At the conclusion of each lecture, students are assigned a collection of problems that are designed to provide them with practice on the manipulations that were illustrated during each lecture. "Traditional instruction is based on a metaphor of production in which students are seen as 'raw material' to be transformed by 'skilled technicians'" (Romberg and Carpenter, 1986, p. 850).

This type of classroom environment is the most efficient in terms of its ability to present a maximum amount of information in a minimum amount of time. However, such instruction is not very effective in terms of providing an environment that is conducive to the development of complex, higher order concepts. As Romberg and Carpenter (1987) observe:

> The daily lessons of the traditional classroom are obviously geared to absorption and not inquiry. Yet, current research indicates that acquired knowledge is not simply a collection of concepts and procedural skills filed in long-term memory. Rather the knowledge is structured by individuals in meaningful ways, which grow and change over time. (1986, p. 851)

A different approach to instruction is called for to replace the traditional lecture format, teacher-centered method of information delivery. A three-component model of effective instruction is described below that is postulated to promote the development of mathematical concept learning.

The three components of this model consist of (1) identifying the intended objects of study within the subject material, and the levels of mathematical meaning that are to be assigned to them; (2) matching an appropriate set of objects of instruction to the students' existing mental structures; and (3) providing instruction through guided activities that focus the students' attention on the desired objects of study, enabling the students to develop appropriate mathematical concepts.

First Component

The subject of algebra at the level of instruction encountered in a first year high school course consists of the study of manipulation procedures and related algebraic concepts. The topics of study are of a sequential nature, and successful comprehension of each topic depends upon an individual's ability to master with understanding the prerequisite information for each area of study. It is important, therefore, to be able to identify the particular manipulation techniques and mathematical concepts that form the prerequisite knowledge for each topic and to be able to order the instruction of these topics in an appropriate sequential fashion.

A learning structure of algebra will be presented in Chapter Eighteen that describes a specific ordering of algebraic topics. The organization of this structure is based on the analysis of the types of information, mathematical manipulations, and algebraic concepts that are considered to be prerequisite knowledge for the development of appropriate mathematical concepts for each selected area of study. The level of mathematical conceptualization of each object of study is identified in order to place the subject matter in an appropriate order from lower order to higher order concepts.

This learning structure presents an organization of the subject of algebra which can be used to address the first component of effective instruction. Identifying the location of a specific topic within the structure enables an instructor to determine the types of mathematical skills and concepts that characterize the particular topic. The learning structure also provides a set of guidelines for determining the necessary prerequisite material that must be learned before students are capable of developing appropriate mathematical meanings for each new topic of study.

Second Component

The second component is very often ignored in actual instruction because of the difficulty that exists in ascertaining accurate information concerning the level of mathematical conceptualization of each student's set of existing mental structures in relation to the requirements of the present learning environment. Paper and pencil tests are sometimes employed as pre-tests to provide a measure of a student's mastery of prerequisite material. However, such tests may be measuring a student's ability to memorize correct sequences of manipulations rather than his or her development of prerequisite mathematical concepts.

It is usually assumed that because a student is in a particular class, he or she must automatically have developed the correct set of prerequisite knowledge. However, students who are able to reproduce required manipulations may not have the necessary levels of understanding that accompany such behavior and that form the prerequisites for further mathematics learning. If the intended objects of instruction are not appropriately aligned to the students' existing mental structures, it may not be possible for the students to make effective use of the learning environment.

The following examples illustrate the importance of providing an appropriate instructional match between the student and the intended learning environment. The first example represents a <u>mismatch</u> between the student's learning context and the instructional context of the lesson. The second example represents a <u>displacement</u> of the objects of instruction from an appropriate conceptual context.

During the checking lab that was presented on the first day of the second summer class, a difference was found to exist between the intended instructional focus and the objects of study that were perceived by the students. (See page 165.) I had designed this activity to draw the students' attention to certain attributes of the checking process that were used with linear equations. The objects of study of this activity, from my perspective, consisted of the numerical statement that was displayed on each card when the answer strip was pulled and the numerical identity that was produced at the end of each checking calculation. The meaning that was to be assigned to this focus of instruction consisted of the relationship of the presence of a numerical identity in the final step of the check to the existence of a mathematically correct solution to the equation.

When I assigned this activity, I was not aware that the students in the class did not have an appropriate level of mathematical knowledge to be able to focus their attention on this aspect of the lab. Because the students had not yet developed sufficient skill in manipulating linear equations and were still weak in numerical computational skills, they were unable to "see" the intended properties of checking as they worked through the activity. Instead, the students' perceived objects of study consisted of the actual manipulations that were used to solve each equation and the numerical calculations that were performed after the answer strip was pulled. The students' attention was focused on the procedures that they employed during the activity, instead of on the form that the symbols acquired at the beginning and end of each set of checking manipulations, and on the relationships that existed between these aspects of the checking process.

The checking lab should have been presented to the students later on in the course, after they had gained proficiency in solving linear equations. At such a time, students would have been able to focus their attention on other aspects of the activity than that of obtaining an "answer." Students would not be able to abstract the properties of the checking process from the lab activity as long as they were still developing the prerequisite concepts that involved the procedural manipulations of each equation.

The checking lab illustrates the way in which various participants in the learning environment can identify different objects of study within the same activity, based on their present level of mathematical understanding. Although the symbols and manipulation steps encountered by the students and the instructor were identical, each was perceiving a different focus to their actions and assigning a different set of meanings and significance to the symbol patterns that they perceived. In such a situation it was not possible for learning to occur at the intended level of instruction. This lab activity represented an example of instructional <u>mismatch</u>, in which an appropriate match did not occur between the intended object of instruction and the students' existing mathematical abilities and levels of conceptualization.

Another example of a situation involving the second component of effective instruction is illustrated by the lectures on fractional equations that were presented during the second summer class. In these situations the inappropriate alignment between the students' existing mental structures and the level of instruction was due to displaced instruction.

During the lectures on this topic, the instructor presented the checking process as a manipulation rule that described the form of substitution to be used on each fractional equation; "If any answer makes any denominator equal to zero, we have to throw it out." (See page 188-193.) This rule was written on the overhead transparencies several times, and demonstrated by different examples. It represents a straightforward set of manipulations in which the answers to each equation are substituted in the denominators of each expression of the original equation. There are no difficult algebraic or numerical manipulations involved in this substitution. However, only one student out of the eight in the class appeared to use this method on quiz problems involving fractional equations that were presented throughout the rest of the course.

The lectures involving checking with fractional equations provides an example of <u>displaced instruction</u>. The focus of instruction in these lectures was centered on mastering the <u>manipulations</u> of solving the equations and not on examining those aspects of the symbols that were related to the concepts of equality, equivalent equations, and solution sets. In other words, the lecture material treated the objects of study as if they were within the focus of instruction of a procedural treatment of the manipulations of algebraic expressions.

The concepts involved in the checking process, however, should be developed within a focus of instruction centered on the concept of the equivalence relationship expressed by algebraic equations. This topic of study should not be presented until students have developed prerequisite skills and concepts related to the basic uses of algebraic notation and its applications to complex algebraic expressions. By placing checking into such a learning environment, <u>it was displaced from the set of concepts that provided the mathematical justifications for its uses</u>. (See Chapter Seventeen and Appendix C for a more detailed description of the placement of the subject of checking within the learning structure of algebra.)

It was apparent from the students' subsequent behavior that they were not able to utilize the very specific manipulation instructions regarding checking, because they were not provided with an opportunity to develop the necessary parallel concepts related to the topic. By taking checking out of its learning context within the conceptualization framework of algebraic equations and presenting it as a series of manipulation steps within the framework of algebraic procedures, the topic became reduced to a set of procedural steps that could not be related to appropriate material within the students' existing sets of mental structures.

A general rule of thumb to follow, is that if it is found that any item of instruction is presented as a rule without justification to the students, it is probably being used out of its appropriate learning context. As was demonstrated by the checking lectures, simply <u>telling</u> the students what to do, however explicitly it was done, did not induce an appropriate level of mathematical learning. Such topics should be delayed until they can be presented within their appropriate focus of instruction.

The two examples presented above that represent mismatched and displaced instruction serve to illustrate the importance of aligning the level of the students' existing knowledge base to the appropriate objects of instruction. As the examples illustrate, it is easier to determine <u>after the fact</u> that ineffective instruction has taken place, than to pre-test to determine the students' existing set of mental structures. The present study has provided numerous examples in which it is possible to determine from the students' responses and behavior that they did not have the appropriate mental tools for the learning task in hand. However, this study has not addressed techniques that can be used before instruction takes place to determine the appropriate placement of students within such instruction.

Such testing involves the examination for the presence of those concepts and skills that are considered to be part of the prerequisite knowledge base that is required to successfully understand the present level of instruction. Paper and pencil tests may not be sufficient to measure such mental development, since written problems can sometimes be answered correctly through rote-memorization of a set of manipulation steps. A more appropriate test would involve a student interview in which the student was presented with a set of objects of study representing prerequisite knowledge and asked to comment on them, describing any properties or relationships that they perceived in the given collection. The particular form of the response would provide information regarding the types of concepts and relationships that the student had already developed for these objects of study and provide the instructor with information as to the appropriate placement of the student within the sequence of instruction.

Third Component

The third component of effective instruction concerns the type of learning environment that is most conducive to concept learning. Such an environment utilizes student activities as the primary vehicle of instruction. Through active participation in these activities, students develop manipulation skills and formulate concepts as they familiarize themselves with collections of examples and non-examples of the objects of instruction. These activities will be examined in more detail in the following chapter.

Summary

The behavior of the students in the two summer courses in response to specific types of instruction have been interpreted in terms of the models of mathematics concept learning and effective instruction that were developed in this chapter. This student behavior was postulated to be the result of situations in which individuals were being presented with instructional materials representing mathematical conceptualizations that were beyond the students' present levels of mathematical knowledge. Students were found to construct alternative sets of meanings for the objects of study and to utilize memorization and other forms of narrow learning in response to these particular learning situations.

A model of effective instruction was proposed to address the areas of instructional mismatch and displacement that were identified in the learning environments of the two summer courses. Such a model is concerned with providing a learning environment commensurate with the constructivist theory of learning that was described in this chapter. This model is also concerned with developing a learning structure of algebra to identify prerequisites sets of knowledge and levels of concept development within the topics covered in a course of high school algebra.

The following chapters in this report provide a more detailed examination of the instructional implications expressed by the first and third components of the model of effective instruction. Chapter Sixteen presents a description of a participatory learning approach to instruction that utilizes active student participation in guided activities as a method to promote an appropriate level of mathematical concept learning. Chapter Eighteen presents an analysis of the subject of algebra in terms of descriptions of levels of prerequisite knowledge and concept development that are postulated to be required for the formation of appropriate levels of mathematical conceptualization for the topics of study.

CHAPTER SIXTEEN

PARTICIPATORY LEARNING

This chapter examines the third component of effective instruction that involves the use of guided student activities. A model of mathematical concept learning is postulated to place these activities within an appropriate learning context. A set of instructional strategies described as participatory learning is then developed to meet the needs of this learning model.

Model of Mathematical Concept Learning

In the preceding chapter, the classroom interactions were described from an instructional perspective based on a three-component model of effective instruction. In order to examine the third component consisting of student activities, it will now be necessary to reverse this perspective and examine the same classroom environment from a learner's point of view through a model of mathematical concept learning. The similarity between these two models reflects the interdependence of the teaching/learning interaction.

The postulated model of mathematical concept learning consists of three components; (1) a description of the type of material that is to be learned, (2) a model that describes how the learning of this material takes place, and (3) a description of those types of activities that promote such learning. Components 1 and 2 are based on the assumption that the appropriate type of learning for any subject is a function of the kinds of material to be learned; that the learning of facts, a series of algorithms, or a set of abstract concepts all require different kinds of learning techniques (Lovell, 1966). Therefore, the first two components of the model must be clearly identified before an appropriate set of learning activities can be described in the third component.

1) The material that is to be learned, in this case, consists of algebraic concepts, procedures that manipulate symbols according to these concepts, and the skills and algorithms that represent these procedures. Lovell (1966) defines such mathematical concepts as mental abstractions or "terms that exist in thought indicating generalizations about systematic patterns of relations" (p. 210). The specific concepts, procedures and skills that make up the subject matter to be learned will be discussed in detail in Chapter Eighteen.

2) A constructivist model, applied to Skemp's model of concept learning is used to describe the way in which this subject matter is learned. According to this model, individuals develop mental concepts through the processes

of identifying patterns and noting similarities and differences among a collection of examples and non-examples. The abstract components of these concepts are constructed as students form generalizations about the perceived patterns. Students are able to assign meanings to these abstractions on the basis of their existing mental structures. These structures are then expanded and altered as new information is assimilated into existing schemas. Such learning is dependent on an individual's ability to perceive important information in the learning environment and to relate it to his or her existing knowledge base.

3) According to the constructivist model, learning is carried out by each individual as he or she interprets information in the learning environment. Learning is not externally imposed, as through a lecture presentation, but comes about through the interactions of each individual with the subject matter. The student occupies the central role in such a process. A detailed description of an appropriate learning environment and set of student activities will be presented later in this chapter.

The importance of such active student participation in the learning process is stressed in the recommendations recently formulated by the <u>Curriculum and Evaluation Standards for School Mathematics</u> (1989).

> Active student participation in learning through individual and small-group explorations provides multiple opportunities for discussion,

> questioning, listening, and summarizing. Using such techniques, teachers can direct instruction away from a focus on recall of terminology and routine manipulation of symbols and procedures toward a more in-depth conceptual understanding of mathematics. (p. 140)

This active, participatory approach to learning mathematics provides students opportunities to develop and utilize skills that are an important part of the <u>doing</u> of mathematics. In the following definition, Davis's (1984) use of action verbs emphasizes the non-passive nature of mathematical thinking.

> Mathematics is ... [not] a matter of learning dead 'facts' and 'techniques,' ... its true nature...involves processes that demand thought and creativity; confronting vague situations and refining them to a sharper conceptualization; building complex knowledge representation structures in your own mind; criticizing these structures, revising them and extending them; analyzing problems, employing heuristics, setting sub-goals and conducting searches in unlikely (but shrewdly-chosen) corners of your memory. (p. 347)

<u>Role of Teacher</u>

The third component of the model of concept learning stresses the importance of active student participation in the learning process. This student-centered focus places the teacher in a different instructional role from that used in the traditional, lecture-format style of teaching. In the constructivist model of learning, the teacher does not represent the primary vehicle through which knowledge is "transferred" to the student. "Teachers should view their role as guiding and helping students to develop their mathematical knowledge and power" (National Council of Teachers of Mathematics, 1987a, p. 1). Freudenthal (1980) describes teaching as "the intentional promotion of the learning process" (p. 198). Skemp (1987) eloquently describes, however, the awesome responsibility that ultimately rests on the shoulders of every teacher.

> [The responsibility of a teacher is] to find for oneself, and help one's pupils to find, the basic patterns; secondly, to teach them always to be looking for these for themselves; and thirdly, to teach them always to be ready to reconstruct their schemas -- to appreciate the value of these as working tools, but always to be willing to replace them by better ones. The first of these is teaching mathematics; the second and third are teaching children to learn mathematics. Only these last two prepare children for an unknown future. (p. 34)

Also, according to Skemp, the teacher must perform a careful analysis of the subject matter to be learned, in order to identify the order in which material must be presented to the students for the optimal development of appropriate mathematical concepts. "Then when in direct contact with the students, the teacher is responsible for general direction or guidance of the work, for explanation, and for correction of errors" (Skemp, 1987, p.83). It is not enough to simply present students with a set of activities from which all necessary information can be "discovered" by appropriate actions; these activities must be carefully guided by the teacher.

It is important for the students' attention to be directed towards the appropriate focus of instruction within any learning situation. It may be necessary to employ strategic questions that help students to identify the types of attributes that are to be abstracted from each activity. Many times, discussions between students and/or the instructor provide students with insights into the relationships under study. The teacher can provide important guidance in helping the student to consolidate new information and integrate this knowledge into a larger frame of reference.

> It is here that the teacher, with his broader knowledge and richer store of related references, can help the student refine the significance of what he has apprehended on his own and help him construct an orderly scheme of meaning in which new knowledge can take its proper place. Without these refining and conserving operations that can stem from the teacher's guidance the discovery itself is likely to be of limited value. Promoting the concept of discovery alone as a method of instruction is likely to be a deceptive and a vain pursuit because it is incomplete. We should pay as much attention to the question of consolidating the student's new insights as to the methods for eliciting these insights. (Friedlander, 1965, p. 30)

The teacher's role of a guide and facilitator can only be effective if the teacher also acts as a diagnostician. The teacher must constantly monitor the interactions within the learning environment to ascertain that an appropriate match exists between the level of mathematical conceptualization of the objects of study and the students' present set of mental structures. It is important to provide opportunities in which students are encouraged to discuss their ideas, ask questions, and explain their thinking processes in their own words. Besides providing effective learning environments, such situations can be used by the teacher as a method of gaining insights into the students' existing mental constructs. Such information can then be used to move the students from their present knowledge bases towards the level of mathematical conceptualization appropriate for the objects of study.

Traditional Classroom Mathematics Learning

The importance of developing a model of mathematics learning that involves active student participation can be emphasized by examining some of the conditions that prevail under traditional mathematics instruction. In the typical, traditional classroom, a lecture-type presentation is followed by a period in which students work through "representative" problems. In these classes, it is usually possible for students to fulfill the class requirements by relying on rote memorization that involves a minimum amount of concept development.

> To most people, 'mathematics' means the arithmetic they studied in school, plus a little high school algebra, geometry or trigonometry. In

> the past, these subjects have commonly been taught by rote. The result is that people hold the erroneous view that mathematics is mainly a matter of learning a few rules and then following them precisely. This is hardly a correct view of what mathematics really is, or of how it is learned, or of how it is used later in life. It leads to quite a wrong notion of what it is that our students really need to learn. (Davis, 1984, p. 8)

Traditional textbooks base much of their development of topics on the "absorption" model of learning that is based on the assumption that students can learn mathematics by being "told" what to do and what to think. Such a model relies heavily on the use of definitions to convey information and knowledge. Mathematical procedures, algorithms, and concepts are usually introduced in each section of the text by a definition, followed by several examples. According to Skemp's (1987) model of schematic learning, this is in the reverse order from that which is necessary for the formation of mathematical concepts.

> [This principle] is broken by the vast majority of textbooks, past and present. Nearly everywhere we see new topics introduced not by examples but by definitions, definitions of the most admirable brevity and exactitude for the teacher (who already has the concepts on which they refer) but unintelligible to the student. (p. 19)

The use of passive learning situations that do not promote active student involvement in the learning process and the inadequate development of conceptual understanding do not provide students with sufficient skills or understanding to apply the mathematical knowledge that they do learn. Without a firm basis of underlying concepts and principles, students are only capable of performing mathematical computations and procedures that are closely related to the narrow range of examples that they were exposed to in class.

Problem solving skills are not developed "when students do a page of computations, when they 'follow the example at the top of the page,' or when all the word problems practice the algorithm presented on the preceding pages" (National Council of Teachers of Mathematics, 1987a, p. 54). Such "mathematics is done _to_ students, not _by_ them" (Research Advisory Committee, 1987, p. 341).

Summary

The model of mathematical concept learning described in this chapter utilizes the constructivist view that students build their own sets of mental structures by assimilating, from the learning environment, certain information that can be related to their existing levels of knowledge. Such assimilation takes place as students actively participate in this environment. Mathematical concepts are developed as students identify and abstract common properties from collections of examples and non-examples of the objects of study.

The role of the teacher is that of a guide, facilitator, and diagnostician. The primary vehicle of instruction is the guided activity that is designed to provide a specific focus of instruction from which students can abstract mathematically appropriate properties. The objects of study that are placed in the learning environment and the focus of instruction that each activity presents are developed according to a careful analysis of the subject matter, such as that established by the learning structure of algebra. (See Chapter Eighteen.)

The National Council of Teachers of Mathematics Commission on Standards for School Mathematics summarizes this approach to instruction in the working draft of their Standards report (1987a).

> This constructive, as opposed to passive view of the learning process must be reflected in the way mathematics is taught. Instruction based on this conception of learning is different from that in the typical classroom where the teacher is transmitting lessons through exposition to a captive audience. Instruction from this perspective should include:
>
> - project work rather than exercises;
> - group assignments as well as independent work;
> - discussions about the origin of and relationships among concepts, rather than independence of concepts; and
> - intrinsic motivation through curiosity rather than extrinsic reward. (p. 8)

Participatory Learning

This section postulates a type of teacher/student/ subject matter interaction called participatory learning that is presented as a theoretical model of instruction for the implementation of the third component of the model of mathematical concept learning that is concerned with the development of effective classroom instruction. The three way interactions utilized by participatory learning involve the active participation of students with the objects of study and with the instructor and other students through the use of carefully guided small group activities.

This model is derived from the learning and instructional models developed in the preceding chapter and represent a set of theoretical postulations. Examples taken from the collected records of the two summer courses that describe students engaged in activities characteristic of participatory learning will be presented in later sections to illustrate the ways in which this model can be used. However, many of the components of the model have not yet been tested in an actual classroom environment.

Participatory learning utilizes many of the classroom management techniques that are associated with the cooperative learning model of instruction. Students are assigned to work together in groups of four in order to foster a sense of cooperative rather than competitive learning, to provide peer tutoring, and to provide within-group monitoring and evaluation of individual student's work. The use of small groups encourages discussion and provides individual students with more opportunities to contribute on a regular basis to classroom activities. The use of such groups has been shown to increase student achievement levels and improve students' attitudes (Sharan, 1980).

Participatory learning employs many of the aspects of the cooperative learning group investigation model. This model involves "classroom learning through cooperative group inquiry and/or discussion [that] emphasis data gathering by pupils, interpretation of information through group discussion, and synthesis of individual contributions into a group product," (Sharan, 1980, p. 250). Students using this type of group activity assume the responsibility for the direction of their work, make decisions regarding the types of activities and procedures that they will employ, and engage in higher-order thinking skills as they organize, interpret and synthesize data.

The use of these groups provides a major change in the roles played by the teacher and the students in a traditional classroom. Johnson (1984) describes a set of three principles that are employed to foster group interdependence.

> (1) students are responsible for their own work and behavior; (2) students must be willing to help any group member who asks, and (3) students may only ask for the teacher's help when all four in the group have the same question. (Johnson, 1984, p. 7)

These three principles encourage students to be accountable for their own work, to provide mutual assistance and tutoring within the group, and to seek validation of their work independently of the teacher. If the students understand and abide by these principles, it is possible to make effective use of this type of cooperative group learning within a large class of students (Johnson, 1984).

Although participatory learning utilizes many of the organizational techniques of cooperative learning including small group discussions, peer-tutoring, and the group validation of student work, an important difference exists between the two instructional approaches. Many cooperative learning groups begin each topic of study with a teacher-presented explanation or with the reading of a prepared work sheet (Sharan, 1980). Such an activity implies that knowledge can be acquired by transfer from teacher to student through the passive mechanism of the lecture. In contrast, participatory learning is based on a constructivist model and regards the <u>guided student activity</u> as the primary vehicle through which learning takes place.

The participatory learning model postulates that learning occurs as students interact with sets of representative examples of the objects of study (Skemp, 1987), and that a principal method of learning is the student/teacher or student-to-student discussion (van Hiele, 1986). The use of small group, cooperative learning strategies are postulated to provide an effective method of facilitating this approach to learning by providing an environment conducive to group discussion and a free exploration of ideas.

Role of Students

Participatory learning places the main responsibility for the learning act with the students. Groups of three or four students are used to provide the learning environment within which the guided activities take place. Students are encouraged to reflect on their actions through the use of group discussions that share and compare each individual student's work.

The students usually begin each activity by working individually with sets of manipulatives or interactive materials. Once the students have achieved preliminary results, they are encouraged to share their findings with other members of their group. If different answers are achieved by the students, they are then asked to compare and contrast these solutions.

During this group phase of the learning process, the students are asked to establish to their own, and to the group's satisfaction, which sets of results are to be accepted, and on what basis these decisions are to be made. This type of internal, group monitoring encourages the students to develop confidence in their own abilities to detect and correct errors, and to determine the mathematical appropriateness of their own work. "There must be a shift of authority for the validation of conjectures, ideas, and solutions from the teacher or the answer key to the students' own mathematical reasoning" (National Council of Teachers of Mathematics, 1987a, p. 14).

The sharing and discussion of student work within each group provides an opportunity for peer tutoring to take place. Those students who have developed a greater understanding of the objects of study are encouraged to explain their thought processes to the students who appear to be experiencing difficulty with the activity. This type of group interaction serves to reinforce the concept development of the more able students at the same time that it provides the slower learners with valuable assistance.

The use of group sharing enables the students to become their own teachers. At times, students may discover the need for more information in order to complete the required activity. Before the teacher is called upon to supply this, the students are encouraged to solicit the information from other members in the class. The "smarter" students can many times supply the needed information. However, as with the teacher's pronouncements, the other students are encouraged to question the validity of the supplied information, and to accept nothing unless it can be understood.

The student directed, group activity provides an effective environment in which students are able to assimilate new information and develop appropriate concepts. The use of this constructivist approach to learning allows the students to become the primary vehicle for instruction. Active participation in the learning activities enables each individual to establish a personally understood basis for newly acquired information. This permits the students to "have the opportunity to explore the properties of empirical mathematics and to see for themselves the relationships, rules of transformations, extensions, and structures derived from these investigations" (National Council of Teachers of Mathematics, 1987a, p. 6).

Role of Teacher

The group monitoring of the results of each activity must be carefully fostered and developed within the groups by the actions of the teacher. This is accomplished by asking strategic questions and providing suggestions that promote a spirit of uncritical sharing. The teacher's role is to facilitate ways in which the students can effectively discuss their findings within each group.

The results of each group's findings are also shared with the class as a whole. In this way, the learning experiences become enriched as the students are exposed to different ways of thinking about the same activity.

Students are encouraged to express their ideas in their own words. During such class discussions, the teacher can also introduce the need for the use of a common vocabulary and encourage the students to work together to develop a uniform description of the ideas involved. Any definitions that are provided are produced through group discussions and are not externally imposed by the teacher.

Throughout each activity and the discussions that accompany it, students are required to provide explanations, to describe what they did, and why their choice of procedures worked. In this way, the students are encouraged to develop deeper understandings of each activity and to develop the ability to explicitly state the underlying principles as they are identified. The teacher can also use the student generated explanations as a way to monitor the progress of each student's understanding as the activity progresses.

The teacher plays a coordinating and facilitating role in the group dynamics. He or she must be alert to identify and utilize unexpected results that may occur as students proceed through each activity. Students may uncover additional applications of the activity or different ways in which the activity can be used to develop the concepts under study. These discoveries are shared with the class as a whole to provide a richer experience. Such contributions, because they arise out of the students' interactions with the subject matter, are usually easily grasped by other students who are operating within a similar knowledge base. Many times, students who do not consider themselves "smart" may find that they are able to contribute in this way.

The teacher must also be alert to indications that students are having difficulties grasping certain intended concepts. This may indicate that the activity is inappropriate for the students at that particular stage in their learning development or may indicate that the activity does not produce the intended learning experience. It may be possible to make such an activity more effective by supplying additional information to the students or having them manipulate the materials in a different manner.

The teacher monitors each activity by observing the degree of ease with which the students follow directions and develop appropriate sets of properties. The depth of understanding of each activity can be determined by listening to the types of comments that students use during their discussions. The ways in which the students use new vocabulary can also give an indication of the types of understanding that is being developed. In many cases, the understanding of the students can also be tested by providing them with activities that develop the same concept through a different type of representation.

The information that the teacher gathers from the monitoring of the students' performance is used to coordinate the sequencing of each activity. The teacher uses his or her judgement of when to provide a reinforcing activity and when to advance to related topics. The teacher also provides the integration that helps students to relate information learned through different activities. The teacher can also utilize appropriate questions and discussions as a means of developing the students' ability to reflect on their learning experiences. These opportunities allow the students to exercise metacognitive skills and develop the facility to become more effective learners.

Affective Aspects

Participatory learning provides many opportunities to develop positive, affective responses to mathematics. It is important that the group experiences be used to relieve feelings of anxiety rather than foster them. Sharing should become a non-threatening experience rather than a potential source of embarrassment or ridicule. The total contributions of the group enable the less able students to participate in richer mathematical experiences than would be possible on an individual basis.

If the group dynamics are carefully developed, the "smart" and the "dumb" students are able to constructively interact with each other. In many cases, these labels quickly disappear as each member of the group learns and contributes equally. Students are able to develop the ability to work cooperatively as teams in problem solving situations. This group work also provides students with opportunities to develop sensitivities to the needs and feelings of each other.

The use of group and class discussions, and the insistence that each student be able to explain an activity in his or her own words, helps students to develop confidence in their abilities to communicate mathematically. Since these contributions are based on each individual student's own experiences, they are perceived as evidence of the student's intellectual worth. "Continually encouraging students to clarify, paraphrase, or elaborate is one means by which teachers can acknowledge the merit of students' ideas and the importance of their own language in explaining their thinking. (National Council of Teachers of Mathematics, 1989, p. 140).

Summary

Participatory learning is facilitated by the use of small group activities, and the exchange of ideas within these groups and with the class as a whole. Activities that involve manipulatives and that allow the students to investigate different ways of looking at the objects of study facilitate learning. Learning is expanded and enriched by the sharing of ideas and the use of verbal discussions. Students are encouraged to explain their actions and to tell why they drew specific conclusions.

Skillful questioning can be used to help students develop critical thinking skills, to critically assess sources of knowledge, and to reflect on their thinking.

Such activities foster the development of problem solving skills. Some activities can be provided with a minimum of structure that allow students to approach the topic from a wide variety of ways. By sharing these approaches and analyzing their actions, students become familiar with a variety of problem solving skills. These acquire meaning to the students as they see the ways in which different strategies are used to elicit specific types of information or order within each activity.

The teacher plays a crucial role in the direction and guidance of the group dynamics and in the progression of each activity. He or she must also constantly monitor the interactions in the learning environment to assess the degree to which the activities are able to provide an appropriate match between the intended objects of instruction and the students' present levels of mathematical knowledge. Such monitoring allows the teacher to adapt the classroom activities to the existing mental constructs of the students.

Students develop a wide range of thinking skills from participatory learning. At no time do they sit passively while information is dispensed through lectures. They are constantly required to interact with the subject matter. It is necessary for them to learn to classify, identify patterns, look for similarities and differences in comparisons, and to continually verbalize their actions and thoughts. They are encouraged to continually assess information against their own experiences, and to form judgments about the appropriateness of any "externally delivered" knowledge.

Students are encouraged to ask, "Why did I get a different result?" rather than to assume that if the answer differs from another student's that their own answer is "wrong." In this way, mistakes, misunderstandings, and instances of non-comprehension become learning experiences. Students are able to detect which parts of their actions are incorrect, and to build on the pieces of appropriate information they have assembled. A "wrong " answer becomes only a step in the whole process of reaching a solution, not the signal to stop the learning process.

The overall goal of participatory learning is to provide an appropriate environment in which learning with understanding can take place. By providing the students with opportunities to actively interact with appropriate mathematical materials, it is possible to take advantage of each individual's drive to establish order and pattern out of his or her environment. The resultant learning is rich, deep, and meaningful, and is based on appropriate levels of mathematical understanding.

The model of participatory learning can be used to facilitate the instructional approach for grades 9-12 that is described by the National Council of Teachers of Mathematics (1987a) in their working draft of the Standards.

> In order for students to internalize the view of mathematics as a process, a body of knowledge, and a human creation, they need many opportunities to experiment with ideas, develop strategies, formulate and communicate conclusions, apply fundamental skills, and interact in groups. Productive student interaction includes group projects and problem-solving investigations, peer instruction, and total class discussions.... The importance of informal experiences for developing primitive conceptual understanding, a prerequisite for all students to their formal study and abstraction of mathematical ideas, should be recognized. (p. 90)

Examples of Participatory Learning

The organization of subject matter and methods of instruction used in the two summer classes were based on the traditional classroom format. New information was presented to the students through lectures that usually began with a definition and included illustrative examples. Students were then assigned sets of practice problems that were very similar in nature to the lecture examples.

A major difference, however, from this traditional instructional format, was the use of group learning. The students in the class were divided into groups of four that worked together at separate tables. Students were encouraged to share ideas and solutions as they proceeded through each set of practice problems. At the beginning of each course, a majority of the work on each quiz was also performed collectively within the groups. Students were also encouraged to work together outside of class when doing their homework assignments.

The students utilized these resources of group thinking in a constructive manner, especially while taking group quizzes. On the quizzes, the students were instructed to work through the problems individually, and compare notes only when they were stumped, or had finished the problems. The following interactions that were observed during the first summer's class provide an illustration of how participatory learning can occur under such conditions. These examples are presented as evidence of the efficacy of those techniques of participatory learning that were postulated in the preceding sections.

Group Quiz

On Monday, at the beginning of the second week of classes, the group quiz was given as the last activity of the afternoon. Tom, Bob, Donna, and Tracy were sitting together at the same table. For some time there was no conversation as everyone worked quietly by themselves. I noticed that Bob appeared stumped. He was just sitting there not writing, and occasionally looking over at Donna's paper. After a while he asked Tracy a question, and she got up from her seat and went around the table to help him. I noticed that Tom, who worked by himself the whole time, appeared to be listening to what Tracy had to say.

A little later, Tracy got together with Donna so that they could compare each other's work. I watched them as they worked through one of the problems and found each other's mistakes. At one point, Tracy came up to me to ask a question about one of the problems, and I tried to help her by giving hints rather than direct answers.

At another table, Wanda turned to Shirley and asked her a question. Shirley looked up at the instructor and asked if she could go to another table for help. Ellen told her to first check with the other people at her table, so Shirley turned to Sue and asked a question. As Sue provided an explanation, Wanda watched and listened.

Towards the end of the quiz, Shirley asked Ellen, "Can we go to another table and ask a smart one?"

Ellen's reply was, "Are you stuck?" In response, Shirley grinned and pointedly walked over to Lori sitting at the third table. Tom and Bob watched her cross to the other table, then looked at each other and laughed.

At this point, Tracy went over to Sue at her table and began discussing a problem. When Tracy returned to her own table, I saw Sue's face light up with wonder and joy. Tracy sat down and turned to Tom and Bob, saying, "OK you guys. Don't listen to me." Apparently Sue had the correct answer.

Comments

This series of interactions illustrates the way in which a group of students can utilize their own resources to promote understanding. All of the students in the class exhibited a willingness to try to answer the quiz questions on their own before seeking help from others. When students decided that they could not generate the correct answer on their own, they asked other students for help. These students were always willing to provide them with assistance.

Tracy, who at the beginning of the class, considered herself a "dummy" was often called upon for assistance as the course progressed. On the first day of class, after she worked through a problem to a correct answer, she commented, "I got it right! That's pretty rare." However, as the class progressed, she discovered that she was often right, and was also able to help others when they had difficulties. The day before the mentioned quiz, Lori had told Tracy, "You should be a teacher."

Tracy's help was always provided in a friendly, positive way. Even though she was recognized by the other students as one of the smarter ones, she always provided assistance in a spirit of sharing, and was willing to be found wrong. On two occasions during the afternoon quiz, Tracy was able to uncover errors in her own thinking as she worked first with Donna, and then with Sue.

The interaction with Sue presents a good example of the way in which group sharing of each other's work provides opportunities for all members of the group to be contributors. Sue, even more so than Tracy, thought of herself as not math able. It was therefore a very image-boosting occasion when she found herself providing the "right" answer to Tracy.

The instructors very carefully fostered a cooperative atmosphere in the classroom. Ellen ensured that all students were provided opportunities to be teachers as well as students by her proviso that students must ask for help at their own tables first. In this way, the students were prevented from always going to Lori or Beth, the two best prepared students in the class. Even though the other students might not have had ready answers to each other's problems, they were able to understand their difficulties in most cases by discussing these problems within their own group.

During the quizzes, the processes of asking questions, checking each other's work, and explaining their work to others all provided positive learning opportunities for the students. Throughout the course most of the students were able at one time or another to provide assistance to someone else, and be in Sue's position of being "right." These interactions enabled the students to engage in participatory learning. During quiz times they were never told the information that they needed by the instructors, but were provided opportunities to develop the necessary understandings by discussions among their peers.

Learning by Doing

The students in both summer classes exhibited a desire to do their own learning. They were well acquainted with the traditional lecture format, and expected this to be the primary method of acquiring new information. However, when they were engaged in working through the sets of practice problems following each lecture, they were less willing to sit passively and let one of the instructors write out a problem for them. They insisted in working through the steps themselves.

Wanda found the class very difficult and usually required a lot of help getting started on these practice problems. On many occasions, one of the instructors would have to write out a complete problem for her as it was explained step by step. I noticed that when we were done with our explanations, she would usually take out a fresh sheet of paper and begin to work this problem over again from the beginning. The process of actually doing the problem by herself was where the significant learning took place, not when she watched one of the instructors telling her what to do.

The actual doing of problems many times provided students with more feelings of understanding than they developed by listening to a lecture and asking questions. As in Tom's situation described in the preceding chapter, much of the students' learning took place as they worked with the sets of practice problems. This is illustrated in the following interaction, where Wanda is able to understand the material only as she works through it.

3-1a The lecture topic was an introduction to a long division algorithm for dividing polynomial expressions by a binomial. The instructor worked through the following example.

$$
\begin{array}{r@{\,}l}
 & \underline{\;y \;-\; 4\;} \\
y - 3 & \big)\, y^2 - 7y - 9 \\
 & \underline{\,y^2 - 3y\,} \\
 & \quad\; - 4y - 9 \\
 & \quad\; \underline{- 4y + 12} \\
 & \qquad\quad - 21
\end{array}
$$

The instructor worked carefully through the example, step-by-step. Then the students split up into groups to work through some practice problems. I worked carefully with Wanda, taking her through each step in the first problem. When we were done she turned to Steve and said, "This is fun!" She then proceeded to do the next problem completely on her own.

Wanda's comment illustrates the importance of utilizing participatory learning. Watching and listening do not provide the learning that can come from an active interaction with the subject matter.

Validation of Solutions

The students in the class provided many examples of situations in which they indicated by their verbalizations that they were not able to internally validate mathematical information. Such validation was sought from answers printed in the text, or from pronouncements made by the instructors in the class.

During one interaction, I was helping Shirley with a problem. She solicited my help by asking, "Did I do this right?" I responded by saying, "I don't know, what did you do?"

Shirley was unable to respond to my promptings. She was seeking an external, authoritative way to check her problem, and did not know how to initiate an investigation of her work by herself. Many times throughout the course, Shirley would ask for a check on her work, even when an examination of her problem revealed that the solution was correct. She was not able to develop confidence in her own abilities to reflect on her work and to make judgments about the appropriateness of her actions.

This unquestioning reliance on an "expert's" statement of fact was exemplified by an interaction that took place the second day of the first summer class. The instructor had just finished solving a simple linear equation. As she wrote down the solution she asked the class, "How do I know I did it right." Tom jokingly replied, "We took your word for it."

The ground rules laid down during the group quiz periods provided a way for the instructors to help the students develop confidence in their own abilities to determine mathematical validity. By working among themselves, asking each other questions, and comparing results, they were able to answer to their own satisfaction the question, "How do I know if I did it right?" A much deeper understanding comes with such internal validation than can be achieved by simply checking an answer with those printed in the answer key in the back of a textbook.

The following example illustrates the power of participatory learning. Throughout the interaction, the students engaged in verbalizations, cooperative learning, affective support, and achieved internal validation of the mathematically correct solution. This interaction took place on Thursday of the first week of classes. Again, the need to correctly answer the group quiz questions initiated the students' discourse.

Shirley had finished her quiz before the other students and wanted to hand in her paper. In order to foster more interaction with the group, the instructor asked her, "Do your answers agree with everyone?" Ellen had noticed that Shirley's answer to one problem was correct, but that the other students at her table all had a different answer to this problem.

Shirley took her paper over to Valerie, noticed that their two answers were different, and began to rework the problem on a separate sheet of paper. When she was finished Shirley exclaimed, "You know what? She's right!"

Valerie's reaction was similar to that used by Sue on another occasion. She clapped her hands and smiled. At this Shirley couldn't resist teasing her a little and said, "For some crazy reason, she's right." Then she softened this statement by adding, "Just kidding. I had to say that."

Tracy then leaned over to see how Valerie had arrived at her solution. As Valerie explained her work, Tracy discovered a mistake in Valerie's work. By this time, Shirley had also reworked the problem and independently discovered the mistake. She now had arrived back at her original (and correct) solution. Tracy went through the problem with Valerie, and when she was done she asked her, "Does that make sense?"

The collective minds of the three students were sufficient to finally validate the solution to the problem. As individuals, none of them initially had the mathematical expertise to recognize the correct approach to the problem. However, by comparing answers, and by attempting to explain or justify their actions to each other, they were forced to critically assess their work. At the end of the exchange, they could probably have stated to the instructor with some assurance that they had worked the problem correctly.

Student Contributions

The use of participatory learning within a group format enables students to share a diverse set of methods and techniques for working through each problem. Many times students are able to come up with different techniques that meet the students' needs better than those suggested by the teacher. One of the responsibilities of the teacher within a participatory learning environment is to be alert to recognize such student contributions and make sure that they are shared with the rest of the class. The following interaction illustrates such a situation.

2-3d The lecture topic covered techniques for deriving equations of straight lines when given coordinate points and values of slope. Following the lecture, the students had difficulty on the practice problems that required substitution into the general formula y = mx + b.

Steve devised a way to help him make sense out of all the symbols in the following problem.

Find the equation of the line with slope of -4 passing through the point (3,2).

On Steve's paper he had written:

$$m = -4$$

$$y = mx + b \qquad (3,2)$$

I was impressed with this technique for keeping all the substitutions straight and showed it to Sue who was having trouble with the same problem.

Sue turned to Steve and said, "Steve, you're a genius."

Group sharing of different students' approaches to problems provides a much richer learning environment than that which the students would experience working only by themselves. When students listen to other students' explanations of their work, they are liable to encounter ideas and techniques that they have not seen before. Many times they are exposed to important information of which they should have been aware, but might have missed while working by themselves.

By explaining solutions to each other and asking questions of the group, students are provided opportunities to reflect on their own work in comparison to what others have done. Such opportunities for reflection help students to develop important metacognitive skills.

It is important for the teacher to be aware that there are alternative methods of approaching the solution to many types of problems. Students may discover different ways of working through practice problems than those illustrated by the teacher. Because these methods are generated by the students themselves, they probably are more meaningful to them than the externally imposed techniques that were presented during the lectures.

On several occasions I found that individual students were not following the "prescribed" method for solving certain problems. My initial reaction was to interrupt their work and explain the "right" method of solution, since I was not able on a cursory examination of their papers to follow what they were doing. Such an attitude on the teacher's part is contrary to the precepts of participatory learning.

An illustration of such a situation occurred during the practice session devoted to work with the formula for factoring algebraic expressions representing the differences of perfect cubes. The students were provided the following formula and expected to use its format in order to factor given expressions.

$$(AX)^3 - (BY)^3 = (AX - BY)(A^2X^2 + ABXY + B^2Y^2)$$

I was helping Alice with a problem, and at first glance, did not think that she was doing the problem correctly, since I never saw her refer to the given formula. Fortunately, before I tried to explain what I thought she should be doing, I asked her to tell me what it was that she had done. Alice explained that she had discovered a pattern in the way the numbers appeared in the solutions to earlier problems, and was using this pattern to factor the other problems in the assignment.

She had observed that the second factor in the answer could be generated simply by manipulating the two terms in the first factor, given by AX and BY. First, square the first term $(AX)^2$, then add the product of these two terms $(AX)(BY)$, and then add on the square of the second term $(BY)^2$.

I was impressed with the nice, economical rule that Alice had developed. I asked her if Ellen or the evening tutor had showed her this way of remembering the complicated formula. She replied that she had just found it for herself and sounded surprised that I thought it was something special. She did not realize that the other students were laboriously referring to the formula at each step and had not realized that there was a simple pattern that could be followed.

Constructive Use of Mistakes

It is important to make the students' mistakes part of the learning process. In the non-threatening atmosphere of the group quiz work, Shirley, Valerie, and Tracy found that there was no stigma attached to finding the "wrong" answer. When students discover that making mistakes is a natural part of the learning process, they are more willing to confront their errors, and examine them critically to understand at what point their choice of solution process went awry.

At the beginning of the course, Wanda was very apprehensive about her work, and very conscious of not being able to do the problems correctly. I noticed that when I helped her with a practice problem, she would immediately erase all of her work when I pointed out an incorrect step. She didn't examine her work first, but simply erased the complete problem, not just the step containing the error.

By the middle of the third week of class, her attitude toward her work had changed. During one period of work on practice problems she asked for my help. She showed me her work and said, "Look, it gives me a simple answer. Did I do something wrong?"

She had written: $\sqrt[3]{\frac{a}{b}} = \sqrt[3]{\frac{a}{b}\cdot\frac{a\cdot a}{b\cdot b}} = \frac{a}{b}$

My immediate reaction to the problem was the same as hers; how could she have gotten such a "friendly" answer? When I realized what she had done, I replied, "Yes, you cheated." Then Wanda also understood her error (she had not multiplied the top and bottom of the original fraction by the same number) and she thought this was so funny, in light of my comment, that she burst out laughing.

Wanda exhibited an approach to her work that is an important part of participatory learning. Unless students can develop the ability to critically assess their work and to regard mistakes as non-threatening, they cannot utilize this aspect of the doing of mathematics as a learning experience.

Wanda indicated that she was applying some kind of critical assessment to her solution when she stated that it gave her a simple answer. Even though it is not a mathematically sound criterion, Wanda had recognized by that point in the class that most of the problems did not yield solutions of a very simple form. She was able not only to work the problem through to a solution, but then to examine the suitability of this solution.

When she decided that there were grounds for questioning her solution, her question, "Did I do something wrong?" indicated that she was curious as to why her answer took on the form that it did. She did not feel threatened by the possibility that the answer might turn out to be mathematically incorrect. Thus, the interaction became a positive learning experience. She was able even to see the humor in the way in which her particular mistake had lent itself to the formation of a simple solution.

It is important for students to learn that making mistakes is part of the learning process. Many times a careful analysis of errors will uncover mathematically inappropriate concepts or incorrect perceptions of mathematical processes. It is important for the teacher to allow students to make mistakes, and then to help them to trace the reasonings that lead them there.

Comments

As was mentioned earlier, both summer classes relied heavily on many of the instructional techniques that are to be found in a traditional mathematics classroom. This was particularly true in the arrangement of the subject matter and the method of its presentation. Information was delivered through lectures, usually via definitions and a selection of two or three very similar example problems. The students were then provided with additional problems to solve that were very similar in format to those presented in the lecture. The solutions to these questions could be found by closely mimicking the actions the instructor had used in the preceding lecture.

A very effective group learning environment was created for the periods in which students worked on their solutions to the group quiz questions. Unfortunately such times were regarded by the instructors as periods of assessment, rather than of learning. The learning portions of the class were considered to be the time during which students sat in front of the projection screen, listening to each lecture, and the period following this when they worked as groups on limited collections of examples.

The instructor organized the use of group work during practice sessions and group quizzes as a method of providing a non-threatening, peer supportive atmosphere for the highly math anxious students. The analysis of the group interactions described in this section indicates that this system provided a much richer learning environment than we had intended. Because we were not aware at the time we conducted the class of all the interactive possibilities that group work offered, and because of the time constraints placed on the organization of the course, we did not utilize this learning environment to its potential.

Much of the success of the group interactions is due to the fact that the students in each class were made up of highly motivated adult learners. The students also exhibited great patience, courtesy, and friendliness to each other and to the instructors, which made the positive group dynamics possible. Ellen was able to further influence the group interactions by her insistence that work be shared and compared by tables.

Ellen made a point of rotating the composition of students within each group so that all the students were given a chance to interact with each other. This prevented some students from relying too heavily on the assistance of one smart student at their table. By sitting with different students, each individual found themselves in both teaching and learning situations.

An attempt was made during the second summer to provide some guided student activities during the two half-hour lab periods each day. The equation pull strip lab that was described in Chapter Seven was one such activity. Unfortunately these labs did not always closely meet the needs of the students. Most of these activities required a prerequisite facility in interpreting and manipulating algebraic notation. Because this necessary level of mathematical understanding was lacking, in many cases, the students found that the directions for the labs were difficult to follow, and that the activities did not clarify their thinking or help them develop appropriate levels of understanding for the particular objects of study. Under such conditions, it was not possible to develop the kinds of constructive participatory learning in these labs that the students were able to exhibit at other times in class. The group quizzes and practice sessions provided more opportunities for the students to actively interact with the learning environment and to assign their own sets of meanings to the objects of study.

Guided Student Activities

The purpose of this section is to present guidelines for the construction of effective student activities. These guidelines are based on the model of participatory learning that was previously described, and on observations of the ways in which the students in the two classes interacted with the subject material throughout each course. Until the activities proposed in this section can actually be tested within a learning environment, these guidelines must remain theoretical postulations.

Examples and Non-Examples

Material for the proposed student activities should consist of appropriate collections of objects of study that represent both examples and non-examples of the focus of instruction. One effective way to present such collections is to employ a sorting activity that requires students to contrast similar, but different sets of objects.

For example, students can be presented with slips of paper containing either algebraic expressions or equations of one variable and asked to sort these slips into two piles according to some common property that the students are able to perceive. The comparisons of the non-examples (expressions) to the examples (equations) help to accentuate the common property that equations share and that algebraic expressions lack; namely the presence of an equal sign.

Verbalizations

After the students have performed this type of sort, they are asked to explain the criterion that they used. This verbalization provides a way for the students to organize their thinking and further reflect on the activity. It also provides the teacher with a method of monitoring the student's understanding of the activity.

The ability of students to identify meaningful patterns within the focus of instruction and to express these verbally was demonstrated during the first summer class by Tom and Alice. Tom was able to formulate a rule to identify the numerical value of the x-coordinate of the vertex of a parabola, given the quadratic equation of the curve. Alice was able to describe a sequence of steps that she had devised to write out the appropriate factors for the sums and differences of cubes. In both cases, these verbalizations provided important insights into the types of meanings that the two students were able to assign to the objects of instruction.

<u>Different Representational Systems</u>

It is important that students are provided examples that clearly present the desired properties of the objects of study. If the examples and non-examples are not varied enough or do not present sufficient special cases, it is possible that students may form their abstractions from the observations of properties that are as much a function of the representational system, as the object of study. Whenever possible, it is important to present students with activities that utilize the same concept across different representational systems.

Usiskin (1988) cites such a situation in his discussion of the concept of a variable.

> [Students] tend to believe that a variable is always a letter. This view is supported by many educators, for
>
> $$3 + x = 7 \quad \text{and} \quad 3 + \Delta = 7$$
>
> are usually considered algebra, whereas
>
> $$3 + \underline{\quad} = 7 \quad \text{and} \quad 3 + ? = 7$$
>
> are not. (p. 10)

By changing the representational system from that of a letter in the alphabet to the use of blank spaces and other symbols, students are able to abstract the concept of a variable as a symbolic representative of an unknown quantity, rather than as a letter to be manipulated by certain rules in any collection of symbols.

Lesh, Landau, and Hamilton (1983) discuss the importance of varying the representational system in the development of mathematical concepts. In their conceptual model of mathematical learning, understanding can be measured by the student's ability to recognize and utilize objects expressed in different representational frameworks. Such frameworks consists of written symbols, pictures, concrete materials, and symbolic representations.

It is also important in activities that use symbolic algebraic notation to present students with examples that present variations in the symbol format. This assures that students focus on the underlying properties of the objects of study, and not simply on the specific format used by each example.

During the first summer class, it was apparent that students were basing their concepts of the procedures required to manipulate equations on specific symbol patterns rather than on underlying mathematical principles. On one occasion, students requested the instructor to rewrite her manipulations "with the subtraction underneath" in order for them to understand the lecture. (See page 116.)

Ellen's lecture:

$$5y = -3x + 5$$

$$5y + 3x - 5 = -3x + 3x + 5 - 5$$

yields: $$5y + 3x - 5 = 0$$

Students' preferred format:

$$\begin{array}{rcl} 5y & = & -3x + 5 \\ +3x & & +3x \\ \hline 3x + 5y & = & 5 \\ -5 & & -5 \\ \hline \end{array}$$

yields: $3x + 5y - 5 = 0$

The following is presented as an example of an activity that can be used to counteract such learning. Groups of students are presented examples of the solutions to linear equations that are written in the two different formats that were shown above. Students within each group are requested to discuss the similarities and differences that they perceive in each set of manipulations. The students are then asked to use this analysis to develop a statement describing the underlying mathematical operations that they perceive to be the same in each example. Such an activity can be used to discuss some of the mathematical concepts inherent in the equivalence relationship represented by the sets of equivalent equations generated by each solution process.

Structured and Open-Ended Activities

It is important to provide both structured and open-ended activities for the students. Structured activities can be used to draw the students' attention to specific properties of the objects of study. These serve as appropriate introductory activities at the beginning of an area of study, and provide students with access to new information, vocabulary, and descriptions of mathematical procedures.

An example of a structured activity consists of the following study of the relationships that exist between the equation $y = mx + b$ and its graphical representation. During this activity, students work in pairs. The first student provides the second student with two numbers to be substituted for the constants m and b in the above equation. The second student then uses these values to sketch the graph by locating the y-intercept (b), and by determining another point from the given slope (m). The students then reverse roles. After many graphs have been generated in this manner, the students are asked to discuss the effect that m and b have on the form of each graph.

In contrast to the very narrow focus of structured activities, open-ended activities allow students to explore relationships, to tie together information and concepts, and to apply knowledge. When these activities are shared by a group of students they provide opportunities to develop rich, in-depth conceptualizations and understandings. As students verbalize, ask questions, and explain their work, they further integrate their thoughts and actions.

An open-ended activity consists of the following: A group of students are given a set of index cards that each contain a graph depicting a linear equation, with the algebraic representation of the equation written below. The students are asked to sort these cards into piles based on some criterion for differentiating among them.

The development of this criterion is based on the type of analysis that the group performs on the entire collection of cards. One possible sort can be based on the identification of different types of slopes, i.e., positive, negative, vertical or horizontal. Another kind of sort may be based on whether the lines intersect the y or x-axis. By leaving the choice of this criterion up to the students, they become placed in a position where each individual's actions must be based on reflection and the ability to communicate one's ideas. As the students work to a group consensus, they find it necessary to reflect on many of the properties of linear functions and their graphical and algebraic representations, thus integrating previously learned material.

Homework

Assigned homework problems provide students with opportunities to practice manipulative skills and to apply concepts and principles investigated in class. By working through collections of related problems, students are exposed to additional examples of particular objects of study. The processes of examining each problem, selecting an appropriate solution strategy, and manipulating symbols according to these precepts provide students with valuable learning experiences that help them to further their concept development.

Homework problems can be used to extend the use of group learning strategies beyond the regular class seat-time. Students should be encouraged to work together on these assignments using the same rules of group dynamics that they employed during class. In this way, they can enrich the learning time they spend on out-of-class assignments and provide each other with valuable peer tutoring. Such practices, which occur in the absence of the teacher, also serve to reinforce the students' confidence in their abilities to think mathematically on their own.

Summary

The components of participatory learning described in this chapter provide a potentially powerful set of instructional techniques for the development of mathematical concept learning. Some of these components were observed in operation in an actual classroom environment and indicate that this type of learning can be applied in other similar situations.

The model of participatory learning utilizes the instructional strategies of guided student activities as well as small group and whole class discussions to promote active student participation in the learning environment. The teacher serves as a guide and facilitator to help the students effectively participate in the classroom activities. Learning is postulated to take place as the students construct meanings from their perceptions of the objects of study and relate them to their present levels of mathematical conceptualizations. This learning is extended and enriched by group sharing, peer tutoring, and classroom discussions. Participatory learning presents an effective way to meet the instructional challenges outlined in the working draft of the National Council of Teachers of Mathematics (1897a) <u>Standards for School Mathematics</u>.

> The study of mathematics should stimulate and increase our CURIOSITY so that we FORMULATE and SOLVE problems that expand our COMPREHENSION and APPRECIATION of the underlying structures of the universe. In the process, we experience the ENJOYMENT of a challenge, the EXCITEMENT of success, and the DEVELOPMENT of a good self-image. (p. iii)

CHAPTER SEVENTEEN

RELEVANCE OF TEACHING/LEARNING MODELS

The purpose of this chapter is to examine the relevance of the theories of learning and instruction, developed in the preceding chapters, to current trends in mathematics education research. The need for the development of a systematic learning structure of algebra is postulated and related to current ideas about mathematics instruction.

The Need for Adequate Theories of Mathematics Education

Chapter Fifteen presented a three-component model of effective instruction while Chapter Sixteen described a comparable model related to mathematical concept learning. The similarities between these two models is related to the fact that they each present a description of the same teaching/learning interaction, but discuss this event from different perspectives. Each model consists of three components that (1) identify the levels of mathematical conceptualization within the objects of study, (2) stress the importance of an appropriate match between these levels and the types of learning/instruction that take place, and

(3) provide for the implementation of an appropriate environment to promote mathematical concept learning.

The learning theory postulated by these models was developed to explain the observed classroom behavior of the students in the two summer classes and is based on a constructivist model and Skemp's theories of mathematical concept learning. Examples of specific classroom interactions were provided in the two preceding chapters to demonstrate the applicability of these theories to explain students' mathematics learning.

Such theory is extremely relevant to the contemporary needs of mathematics education. Concern exists over the lack of adequate theory that is available to today's classroom mathematics teacher (Romberg and Carpenter, 1987). "In this traditional classroom, the teacher's job is related neither to a conception of mathematical knowledge to be transmitted nor to an understanding of how learning occurs" (p. 851).

At present, most classroom teachers do not have resources available that can provide them with teaching methodologies related to current research in the areas of mathematics thinking and learning (Crosswhite, 1987). Schoenfeld (1987) provides an eloquent picture of the plight of many mathematics teachers within the traditional educational setting.

> A large part of our teaching practice is based on incorrect assumptions, in particular the assumptions that characterize the absorption model of instruction: "The mathematics [we teach] is assumed to be a fixed body of knowledge, and it is taught under the assumption that learners absorb what has been covered" (Romberg and Carpenter, 1986, p. 26). In other words, most teaching assumes that the student is a blank slate, a *tabula rasa*. We show the students a procedure (how to subtract....) and hope the message gets across. Since the message doesn't usually take on the first try, we show them again.... By the time we're done, we may have shown them the same thing five or six different ways -- all in the hope that the message will eventually sink in. (p. 25)

This description of classroom mathematics instruction presents a picture of a situation in which the learning processes are left to chance. Information is presented to the student "in the hope that the message will sink in." There are no underlying theoretical constructs directing the teaching/learning processes. Mathematics educators recognize the need to develop theories of mathematics learning based on research that can provide a foundation for the development of more effective instructional practices (Davis, 1984; Romberg and Carpenter, 1987).

A necessary component of research in mathematics learning must be to directly address the issues and concerns of actual classroom learning. "Research is needed that blends the strengths of current cognitive science research with a concern for the realities of the classroom and focuses on students' learning from instruction over extended periods of time" (Romberg and Carpenter, 1986,

p. 268). Booth (1988) points out the importance of analyzing actual classroom mathematics as part of this research.

> Some seemingly simple ideas are not always as simple for students as they may seem to adults. A continuing assessment of exactly what is involved in the learning of new mathematical topics, assisted by an analysis of the errors that students make and the reasons for them may provide us with extremely useful tools for deciding on ways to help children improve their understanding in mathematics. (Booth, 1988, p. 31-32)

Crosswhite calls for research that can be applied more directly to the day-to-day mathematics instruction in the classroom. He maintains that much valuable information concerning student learning can be developed from "insights born of experience" (p. 267)

> [Such research] is based on careful observation of what is happening in classrooms, in negotiations between teachers and students.... Capturing [the intuitions of researchers] in a format teachers can understand may be as important to mathematics education as scientifically tying them down in research terms. (p.273)

The models of effective instruction and mathematical concept learning that were described in the two preceding chapters provide important theoretical guidelines that can be used to address the above concerns. Because the research for the present study was based on observations of actual classroom behavior collected over an extended period of time, the theoretical models developed to explain the study's findings represent an important contribution to the development of a set of theories related to mathematics learning and instruction. The classroom observations that were collected in this study were used for the development of instructional theories that are directly useful to the "practitioner of mathematics education -- the teacher" (Crosswhite, 1987, p. 267).

Importance of Subject Matter Analysis

Romberg and Carpenter (1986) maintain that mathematical content should be included as an important component in the developing body of learning theory in mathematics education.

> Dynamic models are needed that capture the way meaning is constructed in classroom settings on specific mathematical tasks.... A theory of learning must include a central role for what is to be learned. In neither research on learning nor research on teaching has there been an adequate consideration of mathematical content." (p. 868)

The first and second components of the instructional and learning models developed in this study address the crucial role that specific subject matter plays in any classroom learning situation. It is recognized that the specific nature of the mathematical topics studied and the sequences in which they are encountered during instruction are important variables in the total learning environment.

Evidence was obtained from the analyses of the records of the two summer classes to substantiate the importance of postulating the first two components of these models. The learning strategies that were discussed in Chapter Four provide evidence that a mismatch existed between the intended level of mathematical conceptualization of the instruction and that which was developed by many of the students during the first summer class. The analyses that were performed on the collected quiz responses of the students in the second class indicate that many of the students were unable to develop an appropriate level of mathematical conceptualization for the subject of checking, in part because this topic was presented through instruction that was displaced from its appropriate conceptual context. These conclusions indicate that attention needs to be paid to the types of mathematical conceptualizations inherent in each topic of algebra in order to determine an appropriate instructional sequencing of the subject matter.

Any subject matter analysis that is to be conducted must take into consideration the types of knowledge and skills that are to be learned. House (1988) describes the study of high school level algebra as being made up of both procedural knowledge and mathematical concepts.

> Basic algebra skill must be conceived as encompassing more than symbol manipulation. Of fundamental significance are an understanding of

> concepts such as variables and functions, the representation of phenomena in algebraic and graphical form; and facility in the presentation and interpretation of data, estimation and approximation, prediction, and problem formulation as well as problem solving. (p. 4)

House also recognizes the importance of concept development in any algebra learning.

> Although appropriate levels of factual knowledge and skill are important outcomes of the algebra program, what students need even more is a sound understanding of algebraic concepts and the ability to use knowledge in new and often unexpected ways. (p.2)

The analysis of the subject of algebra that is presented in the following chapter investigates the range of algebraic material discussed by House and addresses the research need for specific subject matter analysis. This analysis is concerned with identifying the levels of mathematical conceptualization and types of prerequisite skills that are required for an appropriate level of understanding of each topic of study. It is postulated that a learning structure of algebra exists that is characterized by a sequencing of each topic according to its required prerequisite skills and concepts. Such a structure provides students with appropriate material from which abstract concepts can be constructed and also provides the necessary levels of skill development that allow students to express and manipulate these concepts in problem solving situations.

Existence of Subject Matter Structure

The learning structure of algebra that is presented in Chapter Eighteen divides the subject matter into elements of instruction that are characterized by different conceptual foci, i.e., algebraic notation, manipulations of algebraic expressions, statements of equality, algebraic functions, etc. These divisions are determined from an analysis of the different types of mathematical meanings that are associated with the objects of study in each element.

It is possible to develop various analytic divisions for the topics of algebra depending on the criteria used in the analysis. Usiskin (1988) presents an example of a structure of algebra that is based on an examination of the different uses for the concept of a variable. His divisions include four "conceptions" of the purpose for which algebra is used.

Conception 1 describes "algebra as generalized arithmetic" (p. 11), in which the variables are used as pattern generalizers. Conception 2 describes "algebra as a study of procedures for solving certain kinds of problems" (p.12), in which variables appear as constants and unknowns. Conception 3 investigates "algebra as the study of relationships among quantities" (p. 13), in which variables appear as arguments or parameters that express relationships. Conception 4 examines "algebra as the study of structures" (p. 15), in which variables may be regarded as representing arbitrary marks on paper that may be manipulated according to particular rules and properties.

In both Usiskin's set of algebraic conceptions and the learning structure to be described in Chapter Eighteen, the division of the subject matter is based on an analysis of the types of meaning that are to be assigned to the algebraic symbols under study. The learning structure takes such an analysis one step further, however, and arranges the identified elements of instruction on the basis of their required levels of prerequisite knowledge and concept formation.

The organization of the algebraic topics within the learning structure provides for a systematic development of a set of mathematical procedures and concepts. It is recognized that learning must proceed in a sequential fashion, allowing for careful development of ideas and skills within each instructional element. Of particular importance is the necessity of providing a firm foundation for the notational system of algebraic symbols that form the medium of communication for much of the subject matter. Freudenthal (1973) recognizes the significance of providing such instruction.

> All language instruction requires a vast and well-directed reading drill. Algebraic expressions are a linguistic matter with a peculiar structure, which can be much more complex than ordinary linguistic matter. Reading

> algebraic texts should not be left to haphazard learning. It should be guided by a well-designed plan which proceeds according to scales of increasing difficulty. Yet material which has been designed in this way is rarely found. (p. 309)

Higher Order Thinking Skills

In the analysis of any learning structure, it is important to include an examination of the types of thinking skills that are required to perform each learning activity. The acquisition of facts, the mastery of manipulation procedures, and the application of mathematical principles and concepts require different types of mental activities and cognitive skills. Not only is it necessary to appropriately sequence the topics of study according to prerequisite skills and concepts, but also to order the activities related to these topics in terms of the levels of thinking skills that they utilize.

It is important to provide instruction in algebra that is designed to develop higher order thinking skills. The typical high school algebra course concentrates on very problem specific learning, in which each different solution or manipulation technique is compartmentalized into its own separate unit of instruction (Davis, 1984). Romberg and Carpenter (1986) note that "this fragmentation of mathematics has divorced the subject from reality and from inquiry. Such essential characteristics of mathematics as abstracting, inventing, proving and applying are often lost" (P. 851).

The processes of abstracting, generalizing, and integrating, however, must be developed by each student in a systematic fashion. Students can only learn problem solving skills meaningfully if they are incorporated within a learning structure in terms of a set of prerequisite skills and concepts. The necessity of developing a sequential set of instruction in problem solving skills can be illustrated by an analysis of the ways in which such techniques are currently regarded by mathematics educators.

In an editorial commentary, Steen (1988) expresses the concern of American mathematics educators over recent evidence of the lag of mathematics achievement in American schools behind those measured in other industrialized countries. Such concern has led to an increased emphasis on providing school programs that foster the students' abilities to <u>think</u> mathematically and to engage in creative problem solving.

> Classes in which students are told how to solve a quadratic equation and then assigned a dozen homework problems to practice the approved method will rarely stimulate as much lasting knowledge as ones in which students encounter such equations in a natural context; explore approaches to solutions including estimation, graphing, computers and algebra; then compare various approaches and argue about their merits. (Steen, 1988, p. 21)

In the preceding paragraph, Steen is stressing the educational importance of providing students with opportunities to utilize "the processes of creating and discovering mathematics" (p. 21). The activities that he describes, however, represent a high level of abstraction. In order for students to discover generalizations about problem solving situations, they must first experience collections of specific examples from which such generalizations can be abstracted. In order for students to be able to focus their attention on these abstract properties, they must already have developed the ability to work easily with each specific type of problem in a given collection of examples.

Steen (1988) recognizes that students learn by <u>doing</u>, not by listening to teachers <u>talking</u>. He describes mathematics learning "as a process of constructing and interpreting patterns, of discovering strategies for solving problems, and of exploring beauty and applications of the discipline." (p. 21) However, it appears from his comment on the study of quadratic equations that he equates such "discovery" learning only with very high level abstract activities.

The organization of a learning structure of algebra is designed on the premise that students engage in processes of discovery and exploration within <u>all levels</u> of abstraction of the subject matter. It is not necessary to provide students with general problem solving situations in order for them to experience these learning techniques. The activities included in the initial instruction on each topic should consist of situations in which the student initiates the learning process by observing patterns and forming rules and abstractions. Such activities promote the development of higher order thinking skills while providing students with instructional materials appropriate to their developing levels of mathematical conceptualization. Under such activities, students can indeed find the solutions to quadratic equations exciting and meaningful.

It is important to stress that simply providing students with opportunities to learn by discovery does not necessarily provide effective instruction. It is imperative that each activity be placed within a proper instructional sequence. Students must be provided with appropriate prerequisite skills and lower order concepts before they can utilize problem solving strategies with meaning. The development of students' appreciation for Steen's "beauty and application of the discipline" is part of the learning of algebra. It is developed gradually through successive elements of instruction in a systematic fashion, but only as appropriate prerequisite concepts and thinking skills are developed.

An important aspect of higher order mathematical thinking is the ability to apply concepts and principles with new and unfamiliar types of problems. Much interest has currently been focused on the development of such problem solving skills. Schoenfeld (1987b) describes recent research in cognitive science that is involved in investigations of the ways in which expert and novice problem solvers attempt to solve algebraic problems.

> The strategies used by the mathematician cut across problem types: 'getting rid of nasty functions' applies to equations with radicals, with trigonometric functions, with logarithms, and so on. This type of strategy is not domain specific, as are the techniques for solving problems in each of those areas. In a serious sense, such strategies transcend the domain-specific techniques. With them one can solve problems that have not been encountered before and for which one does not have prepackaged solutions. Moreover, many of these strategies may not have been explicitly taught. Thus they represent an integration and restructuring of equation-solving knowledge: what the mathematician knows about solving equations is not only more than, but different from, a collection of the techniques mastered in individual domains. (p. 16)

In contrast to experts, novice problem solvers, usually students who have had a traditional series of high school or college level mathematics courses, are not able to bring the same level of knowledge of integrated problem solving strategies to bear on similar problems (Schoenfeld, 1987c). Schoenfeld advocates the use of special courses designed to teach students these higher level strategies as a way of developing their problem solving abilities.

It is important to provide instruction within beginning level mathematics classes to develop the types of higher order thinking skills that facilitate problem solving. Students need to be trained in methods of inquiry and the abilities to discover and organize patterns throughout each instructional element within an appropriate learning structure. Such techniques are directed to the development of abstractions that transcend specific objects of instruction and go beyond domain-specific references to enable students to develop appropriate thinking skills for general problem solving analysis.

Summary

The three-component models of effective instruction and mathematical concept learning provide important contributions to the area of developing theories of mathematics education. Because these theories are based on observations of actual classroom behavior, they can provide mathematics teachers with directly applicable classroom material and instructional techniques. The inclusion in these models of a learning structure of algebra based on an analysis of the subject matter represents an important contribution to mathematics education research.

Such a subject specific analysis makes it possible to provide effective instruction in an area where much of the learning is of a sequential nature. This learning structure also provides an appropriate instructional guide for the systematic development of mathematical concept learning and important problem solving skills. Through appropriate instruction, students are able to develop higher order thinking skills and are provided with opportunities to gradually develop and expand their abilities to think mathematically and to uncover and appreciate the beauty of mathematics. A detailed description of such a learning structure for the subject of algebra is presented in the following chapter.

CHAPTER EIGHTEEN

LEARNING STRUCTURE OF ALGEBRA

The purpose of this chapter is to present an organizational model called the learning structure of algebra that is designed to provide appropriate instructional sequencing for the subject matter. The general outline of this model is described, including the specific types of concept learning that are utilized within the framework of the structure. The information contained in the learning structure is then related to the findings of the present study in order to discuss the types of instructional mismatches that were observed in the learning environment. The specific properties of the learning structure of algebra are also compared to the van Hiele levels of geometric thinking which represent a different type of mathematical learning structure.

The learning structure of algebra was postulated to exist on the basis of the instances of instructional mismatch and displacement that were observed to be occurring in the two summer courses. The determination of the specific arrangement of topics in this structure is based on an analysis of the mathematical nature of each object of study and on the identification of the prerequisite concepts and manipulative skills that are required to understand each topic at an appropriate level of mathematical conceptualization. It is postulated that this structuring of the subject matter can be used to provide effective instruction that will enable students to successfully develop concepts from other, prerequisite concepts and to develop and expand manipulation skills within a framework of relevant mental structures.

The results of the mathematical analysis was used to divided the topics of study into five <u>elements</u> of instruction that are identified by their predominate objects of study as (1) algebraic notation, (A-1), (2) algebraic expressions, (A-2), (3) statements of relationship (equations and inequalities), (A-3), (4) representations of relations, (F-1), and (5) properties of relations, (F-2). The objects of study within each element represent the focus of each learning activity and are identified by specific content material and by the <u>levels of mathematical meaning that are assigned to them</u>. Elements A-1, A-2, and A-3 in this learning structure comprise the study of the algebra of numbers, while elements F-1 and F-2 present introductory topics in the study of the algebra of functions.

The purpose of developing a learning structure is to provide a sequence of instruction that allows students to focus their attention on each object of study within an appropriate mathematical context. Even though each instructional element utilizes the same set of algebraic symbols, the focus of attention and the types of meanings that are assigned within this focus to the objects of study are different.

In order for students to concentrate their learning activities on the correct set of objects of study, it is important for these activities to be directed by an appropriate focus of instruction. Under such instruction, specific properties of the field of study become the primary focus of attention. Learning activities are provided that emphasize particular attributes of the objects of study to the exclusion of others in order to present these topics within an appropriate mathematical context.

The learning structure of algebra uses the above criteria to arrange the topics usually covered in beginning and intermediate high school algebra into a logical sequence in which the level of mathematical meaning, objects of study, and focus of instruction in each element are designed to build on the information developed in the preceding element. The following section presents a brief description of this structure in terms of the different objects of study and focus of instruction that characterize each instructional element. Appendix C presents a more detailed description of the material, including learning activities that can be used to emphasize each focus of instruction. Table 15 in Appendix C presents an organizational outline of the various components of the learning structure.

Elements of Instruction

Algebraic Notation (A-1)

According to the learning structure, students begin their study of algebra within a notational focus of instruction. They are introduced to the set of symbols that is employed in algebraic notation, investigate the types of symbolic representations that are utilized by different sequences of mathematical operations involving variables, and develop notational conventions to deal with complex combinations of operation and quantity. Within this focus, students develop a preliminary concept for the idea of a variable as a symbolic representation of an unknown or variable quantity. Students also develop a preliminary concept for the utility of algebra as a notational system that can be used to represent relationships between quantities and mathematical operations.

The objects of study for this element consist of the individual symbols that make up algebraic notation, with an emphasis on those symbols that represent literal variables. The focus of instruction enables students to investigate the ways these symbols are combined and to develop skills in interpreting and manipulating representations of mathematical operations on variables. The knowledge developed in this element provides students with the rudiments of a necessary prerequisite algebraic language that is required for all further study of the subject.

Algebraic Expressions (A-2)

This element begins as a natural extension of the investigations of the preceding element. Students direct their attention to investigating the behavior and properties of specific collections of algebraic symbols. Concepts are developed from these properties to represent such symbol collections as factors, terms, polynomials, and fractional expressions.

Students develop skill in manipulating and simplifying algebraic expressions and develop algorithmic procedures for such operations as factoring and the multiplication of binomial factors. Concepts are developed to represent these procedures in terms of the types of expressions with which they are used and the symbolic form of the results they produce. Students also expand their concept of the utility of algebra as they begin to develop algebraic expressions that represent more complex relationships between quantity and operation.

The focus of instruction has shifted from the examination of individual symbols in the preceding element A-1 to the study of specific collections of symbols that are each examined as a unit having a set of distinguishing properties. The development of these properties enables students to recognize specific types of expressions in unfamiliar situations and to apply appropriate series of manipulations to them. The skills developed in this element allow students to free their attention from a preoccupation with symbol manipulations in order to investigate other objects of study in subsequent elements.

Statements of Relationship (A-3)

In this element students investigate statements of relationship such as algebraic equations and inequalities. Properties of these objects of study are used to develop concepts for such entities as equations and solution sets. Students acquire skills in solving various types of statements and expand their concept of the utility of algebra to include its ability to use these statements to model and manipulate physical situations.

Students expand their concepts for literal variables and unknowns to include the behavior of such variables when they are placed in equations. They develop concepts for solution sets and checking procedures in relationship to their associated equations. Students develop skills in solving different types of equations based on their knowledge of procedural manipulations and the conditions of equality that are imposed by such algebraic statements.

Although the symbols and procedures utilized are similar to those used in the two preceding elements, the focus of instruction is unique to this element and is centered on investigating the relationships expressed by equations and inequalities. Rather than emphasizing the manipulations that are used to solve various types of equations, instruction is directed at the examination of the relationships themselves and their resulting solutions. The manipulation skills that were developed in the preceding element enable students to turn their attention to this different focus of instruction and set of objects of study.

Representations of Relations (F-1)

In this element students begin their study of the algebra of functions by investigating the concept of a mathematical relation or mapping. Various representational systems are studied in terms of the ways in which they are able to display important information pertaining to mathematical relations. Students are introduced to the Cartesian coordinate system and develop skill in plotting ordered pairs representing given relations. Students also develop skill in expressing relations algebraically by equations of two variables. Students expand their concept of the utility of algebra to include the use of these representational systems as devices for displaying mathematical relations.

The objects of study for this element are various representational systems such as the Cartesian coordinate system and the use of algebraic equations. The focus of instruction draws students' attention to the ways in which these systems can display various items of information concerning relations; i.e., the rule of correspondence, the domain and range of the variables, and the identification of the independent and dependent variables. This element serves to introduce students to necessary prerequisite notational skills that will be used in succeeding elements that investigate the algebra of functions.

Properties of Relations (F-2)

This element begins as an extension of the preceding element F-1. The focus of instruction, however, shifts from the examination of the various representational systems to the investigation of specific types of relations and the ways in which their properties can be displayed in different representational systems. The concept of a function is introduced as one of the properties that can be used to distinguish different types of relations. The actual study of the general properties of functions, however, forms the focus of instruction of its own separate instructional element.

Students develop skill in graphing techniques and investigate the graphical characteristics of linear and quadratic functions, certain conic sections, continuous and non-continuous functions, and other unique types of relations. Students develop properties to describe the general appearance of the graphs of such relations and use this information to sketch generalized graphs when presented with specific algebraic representations of various types of relations.

Students expand their concept of the utility of algebra to include its ability to represent complex functional relationships both algebraically and graphically. Real world situations are modeled in graphical and algebraic form through word problems and the use of tables of data.

Other Elements

The five instructional elements described above represent the majority of the topics that are covered in beginning and intermediate high school algebra. The organizational criteria that were used to formulate the learning structure of algebra can be applied to other topics in order to extend learning activities beyond these five elements to include the study of more advanced areas of mathematics that deal with trigonometric functions, the manipulations of functions, and the topics of calculus.

Concept Development

Concepts are developed in two different directions within the learning structure of algebra. Concepts relating to such entities as variables and the utility of algebra are developed and extended horizontally across the elements as students proceed in their study through the learning structure. Higher order concepts are built out of lower order concepts through a vertical level structure within each element.

Concept learning within each element takes place as students assemble information, discover properties of the objects under study, and formulate a set of abstractions to represent these objects. This type of learning and the way in which it is employed within each element will be discussed in more detail in a later section of this chapter.

Strands

Although the objects of study and focus of instruction are different for each element within the learning structure, there are certain strands of subject matter that carry through from one instructional element to another. The meanings that are assigned to the objects of study in these strands are a function of the focus of instruction within each element and are altered and expanded by the students as they progress through the learning structure.

It is important for instruction to address any strand concept that reappears throughout different elements at the appropriate level of development in order for students to be able to relate this information to their existing set of mental structures. In this way, students are able to develop highly complex concepts for the objects of study in a strand as they investigate their properties under the changing focus of each successive element of instruction.

The concept of a variable represents one such strand. Initially, it is perceived by the students as a "place holder" representing some unknown quantity. Students investigate the properties and behavior of this unknown as it is represented by various literal symbols and combined and manipulated by other algebraic symbols. The concept of a variable takes on new meanings when it becomes associated with other symbols in algebraic expressions. When collections of algebraic symbols are used to represent any real number, students learn to investigate the range of values that variables in these expressions can assume when they are present in radical expressions or as an expression in the denominator of a fraction.

The concept of a variable is extended further when variables are present within an equation. Students learn to associate those values with the variable that make the given equation a true statement. This use of a variable contrasts to the meanings that were assigned to it within the first two elements of the learning structure. An additional extension to the concept of literal variables occurs as students begin to examine the dependent and independent relationships expressed by functions of two or more variables.

Overlap

The instructional elements are not necessarily discrete units that must be taught separately, each one being completed before the next is begun. There is a certain amount of overlap of subject matter and focus from one adjacent element to the next. The initial stage or entry level of any element can be developed as the students progress through the final, application stage of the preceding element. Students may continue to perfect skills and investigate more and more complex relationships within one instructional element at the same time that they begin their investigations of certain aspects of the next element.

This overlap occurs, for example, during the sequencing of instruction dealing with the subject of equations. Students may begin the investigation of the relationship of equality expressed by simple linear equations at the same time that they are investigating such collections of symbols as fractional and radical expressions. However, the focus of instruction for each activity must reflect the appropriate object of study within the learning structure. It is inappropriate to present the solution techniques for various equations as a vehicle for practicing manipulations on collections of symbols. These solution techniques should be addressed only after students have begun to assign certain properties to equations within element A-3 in the learning structure.

Implications for Instruction

Learning must proceed in a sequential fashion through the five instructional elements. Students cannot focus their attention on the investigations of the properties of algebraic expressions until after they have developed a familarity with the form and properties of each type of individual symbol used in algebraic notation. Similarly, they will not be able to focus their attention on investigations of the special properties of equations while they are still developing their abilities to read and manipulate algebraic expressions.

If the objects of study from one of the later elements are presented to students while they are still developing the skills and concepts of a preceding element, these students will be unable to focus their attention outside of the level of their present set of mental structures. In such a situation there is an instructional mismatch between the level of understanding required to learn the objects of study and the present level of knowledge of the students. These students will not be able to understand such instruction within its intended focus and will only be able to mimic the actions of the instructor through some form of narrow learning.

Inappropriate instruction may also occur when a particular subject from a later element is presented within the focus of instruction of a preceding element in the learning structure. In such displaced instruction, the properties and behavior of this subject are no longer based on a set of relevant concepts, since these will not be developed until the next instructional element is reached. Students will not be able to investigate these properties with understanding at this point in the instruction and will only be able to reproduce the actions of the instructor through some kind of rote-learning.

The focus of instruction of each element and its related level of mathematical conceptualization form an essential part of the learning structure. It is not enough to simply order the subject matter of algebra in terms of a set of increasingly complex concepts. Unless the students' learning is directed towards the specific focus that characterizes each element, the students may not be able to identify and assemble appropriate properties to describe the subject matter under study. Effective instruction occurs when the focus of each element is identified within the subject matter and appropriate activities are presented to direct the students' learning to this information.

Levels of Concept Development

The learning structure of algebra is organized into two different areas of learning. The five instructional elements divide the subject matter into separate areas of study on the basis of the objects study and different focus of instruction in each element. Instruction is further organized into three levels of concept development within each element in order to promote the formation of a succession of concepts and abstractions representing the objects of study. These levels of concept development provide instruction that allows students to develop an appropriate sequencing of higher order concepts from lower order ideas within each focus of instruction.

This section examines a type of concept learning that presents a rationale for the existence of these levels. Details of this level structure is then presented which include examples of the types of concepts that are developed as students move from investigations of lower order to higher order abstractions of the subject matter.

The inclusion of levels of concept learning within each element is based in part on Skemp's (1987) model of the learning of mathematical concepts. This model describes the accumulation of layers of abstract knowledge through the development of concepts and related mental structures called schemas. Learning consists of the construction of concepts and the gradual expansion and alteration of schemas through systematic exposure to increasing levels of abstraction. Skemp (1987) states the following principles related to such learning.

> 1) Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.
>
> 2) Since in mathematics these examples are almost invariably other concepts, it must first be insured that these are already formed in the mind of the learner. (p. 18)
>
> Before we try to communicate a new concept, we have to find out what are its contributory concepts; and for these, we have to find out <u>its</u> contributory concepts; and so on, until we reach either primary concepts, or experiences which we may assume. When this has been done, a suitable plan can then be made which will present to the learner a possible, and not an impossible task. (p. 19, 20)

Skemp's model of schematic learning and concept formation requires a specific approach to the teaching of mathematics. Instruction must begin at the lowest level, developing primary concepts from appropriate concrete examples based on the student's physical reality. Higher order concepts cannot be introduced until lower order concepts are in place. Students must be provided the opportunities and experiences that will allow them to recognize and develop appropriate relations among concepts and develop appropriate schemas.

Level Structure

Skemp's concept learning model is used to separate instruction within each element of the learning structure into a hierarchy of three levels in which information studied at any level is based on abstractions of concepts developed in the preceding level. In level I students are introduced to the objects of study of each particular element. In level II students begin to investigate the properties of these objects within an appropriate focus of instruction. In level III students investigate the uses and applications of the objects of study and integrate what they have learned.

Study begins within any instructional element at level I. At this entry level, the students familiarize themselves with the symbolic forms that represent the objects of study. They learn to distinguish objects on the basis of the differences that are present in the type and arrangement of specific symbols within each representation. Students are also presented with the specialized vocabulary representing these objects of study. Instruction at this level consists primarily of information that is presented by the instructor to the students in order to identify the objects that will be studied within each element. The following example, taken from level I of element A-2 covering the study of algebraic expressions, illustrates the types of concepts that are developed for the objects of study within this entry level.

Students are introduced to collections of algebraic symbols that represent, for example, a specific fractional expression, such as $(3x^2 - 5)/(7 - x)$, or a specific binomial, given by $190 - 54y^2$. These objects are initially categorized by noting the differences that exist in their respective symbol patterns. Students assign names to these patterns in terms of specific examples rather than on the basis of generalizations about the symbolic patterns that such algebraic expressions possess.

The second level of concept development, level II, is concerned with assigning each object of study, identified by a name in level I, with a set of properties that describe the behavior and general appearance of any one of many specific representations bearing this label. The label then becomes a name for a broader conceptual representation of the object of study than that initially developed in level I. This type of concept development is illustrated by a description of specific activities related to the study of algebraic expressions.

Within level II of this element, students work with the objects of study in order to become familiar with the algebraic behavior of various expressions as they are manipulated through mathematical operations or operate on other objects of study. Students characterize this behavior by noting the changes that occur in the symbolic patterns of the original expressions, by observing the contexts in which each expression appears, and by noting the effect that this type of expression has on other objects of study within the element.

The binomial expression $190 - 54y^2$ that was mentioned earlier can be used to illustrate the type of properties that are investigated at this level of concept development. Students no longer need to refer to this specific example in order to discuss the concept of a binomial. By comparing and contrasting many examples of binomial expressions, students are able to develop properties that define the general symbolic form of such an object of study; namely, that it is made up of two unlike terms, separated by a plus or a minus sign. It is important in this type of concept development that students are exposed to many standard and non-standard examples as well as non-examples of binomial expressions in order to abstract the most general set of properties to define this concept.

Further properties for this object of study are developed as students recognize that binomials can appear as factors, may be added, subtracted, or multiplied together, make up factors of trinomial expressions, can themselves sometimes be factored, and can be cancelled from fractional expressions. These properties are developed as students manipulate various collections of symbols with the purpose of focusing their attention on the behavior of the objects of study within each particular mathematical context. Again, many different examples and non-examples must be investigated in order for the students to develop the most general set of properties possible that represent their developing concept of binomial expressions.

By assigning the collection of general properties to any specific example of a particular object of study, students are able to manipulate each example in mathematically appropriate ways when it is present in any given mathematical context. When students have reached this level of concept development, they are ready to examine the entire field of objects of study within the instructional element in order to apply and integrate their acquired knowledge.

Level III is concerned with the integration of the entire field of study within each instructional element. At this level of concept development, students investigate the uses and applications of the objects of study. They expand their concept of the utility of algebra as they explore problem solving situations and word problems that require them to use the manipulation skills and concepts that they have developed for all of the objects of study within the element. At this level, students also abstract common properties that they perceive within the entire set of concepts that they have developed for the focus of instruction. Such abstractions can consist of general manipulation and solution strategies that transcend specific types of procedures or collections of symbols and that allow students to apply the information from each instructional element within unfamiliar and non-standard situations.

Implications for Instruction

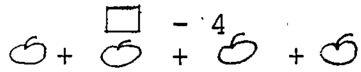
The type of concept learning that is described in this section occurs as students form abstractions concerning the objects of study that relate to the particular focus of instruction of each element. According to the model of mathematical concept learning, this process takes place as individuals abstract common properties that they perceive to be shared by collections of examples. It is important, therefore, that instruction be provided that allows this type of learning to take place.

Students must be provided the opportunities to investigate, compare, and contrast examples of the objects under study. These examples must be in such a form that the students' attention is drawn to those attributes of the subject matter that are representative of the types of concepts to be developed. These attributes are a function of the level of concept development required and of the specific focus of instruction of each element. If care is not taken to present an appropriate collection of instructional examples, students may develop alternative sets of abstractions that are based on different perceptions of the objects of study.

It is important to provide students with non-standard examples of the objects of study that utilize different representational forms. Non-examples also provide an effective way to direct the students' attention to the intended focus of instruction. An example of this type of instruction is presented in Table 10 which displays a collection of symbolic expressions representing examples, non-standard examples, and non-examples that can be given to students who are attempting to develop a concept for the idea of a variable within the instructional element A-1.

The collections of expressions given below are shuffled together and presented to groups of students with the directions that the examples be sorted into different piles on the basis of some distinguishing characteristic. The students are instructed to discuss among themselves the type of sorting criterion that they will use and be ready to explain their decision at the end of the activity.

Table 10. Learning Activity for the Concept of a Variable

Examples	Non-Standard Examples	Non-Examples
$3x$	$\square - 4$	$15 - 9$
$2y - 1$		Bobby
A	"boy"	$\sqrt[4]{(10056 + 4)^3}$
$5/z^2$	"dog"	
$p^3 - 7q$	3? + ___	

The instructional techniques of participatory learning that were described in Chapter Sixteen provide an appropriate way to implement the type of activity displayed in Table 10. Students actively participate in the investigations of such collections of examples as they compare and contrast the symbolic form of each example and make decisions about the properties that are to be used to categorize and sort the activity. The teacher plays an important role in helping the students to focus on an appropriate set of properties by asking strategic questions and encouraging the students to discuss their choice of sorting criterion. Other examples of this type of concept development activity are outlined in the detailed description of the learning structure that is presented in Appendix C.

Relationship of Learning Structure to Present Study

The information contained in the learning structure adds an important dimension to the findings of this study by providing a mathematical content analysis that can be used to identify instances of instructional mismatch and displacement. The organization of topics within this structure is based on an analysis of a logical sequencing of the subject of algebra in terms of the identification of prerequisite topics and the presence of lower and higher order concepts within this material.

The subject matter analysis was prompted by the observations of the present study that found that certain students experienced difficulties in developing appropriate levels of mathematical understanding for the topics of study. It is postulated that these learning difficulties were in part due to inappropriate matches between the levels of mathematical conceptualization of the intended instruction and the levels of the students' existing mental structures. The purpose of this section is to examine the ways that the material in the learning structure can be used to explain the observed behavior of the students in the two summer classes.

The behavior of many of the students in the first summer class indicated that they had not developed the appropriate sets of prerequisite skills and knowledge to be able to manipulate algebraic expressions with understanding. They used master examples to provide them with directions for performing required algebraic manipulations. This type of behavior places these students at a mathematical knowledge level commensurate with level I or level II of the instructional element A-1.

The students' discourse provided evidence that they had not as yet developed an appropriate set of meanings for the algebraic notation that was used in each problem. Bob's comment to the instructor that the two expressions $(x + 2)^2$ and $(x + 2)(x + 2)$ were different indicates that he was not able to abstract the property of multiplication from the two symbol arrangements. His focus of meaning was on the physical differences in the patterns and not on the mathematical properties that these symbols represented. He was not able to develop a level II concept for the symbols that could assign properties to the objects of study on the basis of the types of mathematical operations that they represented. The instructor, however, assigned such a mathematical meaning to the two sets of symbols when she told Bob that either way of writing the algebraic expression was "OK."

Another example of a mismatch between the students' perceptions of the objects of study and that assigned by the instructor to the same objects was observed during the lecture on substitutions to form standard quadratic equations. (See page 98.) The instructor was able to abstract the property of a quadratic equation from different collections of algebraic symbols on the basis of the relationship that she assigned to the two variable terms in each of the two examples,

$$x - 7\sqrt{x} - 8 = 0 \quad \text{and} \quad y^4 - 2y^2 - 15 = 0.$$

This property was an abstraction of the generalized pattern of any quadratic equation, represented as $ax^2 + bx + c = 0$.

The students, however, were not able to regard the two examples from this focus of instruction. They had not yet developed the prerequisite mathematical conceptualizations from the instructional element A-1 that would have allowed them to free their attention from a focus on the appearance of individual symbols to be able to form abstractions about collections of symbols and mathematical operations within the focus of instruction of element A-2. Shirley expressed the sentiment of many of the students when faced with this instructional mismatch when she said plaintively, "Heck, this is confusing."

The learning structure of algebra provides a framework that can be used to diagnose students' existing levels of mathematical conceptualizations. The responses of the students in the two examples cited above provide evidence that they lacked the conceptual framework necessary to develop an appropriate level of understanding for most of the topics of algebra that were covered in the course. They had not as yet been able to develop the types of concepts that would allow them to assign an appropriate level of mathematical meaning to the basic ideas of algebraic notation used throughout the course. Since this knowledge is investigated in the first instructional element and forms the basic prerequisite to all further learning, according to the learning structure, it is postulated that these students lacked the required foundation to achieve an appropriate level of mathematical understanding of the topics covered in the course.

It was observed that these students developed learning strategies based on their ability to recognize specific symbol patterns as a way to cope with the requirements of the course. Even though they were not able to develop the appropriate levels of conceptual development to understand the topics under study, they could still reproduce the actions of the teacher through rote learning and the use of master examples.

The level of mathematical conceptualization that was developed by the students during the second summer course can also be discussed within the framework of the learning structure of algebra. These students were able to manipulate algebraic expressions without the aid of master examples. They were able to correctly reproduce required mathematical operations on an individual basis during testing periods in which they were not allowed to use such external references as their textbooks or notes. It is postulated that this difference in behavior from that of many of the students in the first course is due to the fact that the second summer's students had developed an appropriate level of mathematical understanding of many of the properties and procedures that are investigated in the instructional element A-2 that deals with algebraic expressions.

The analysis of the quiz responses indicated that the students were in many cases able to produce correct solutions to given equations but were not able to utilize appropriate checking procedures with these answers. The placement of the subject of checking within the learning structure provides an explanation of this observed behavior.

The checking process of substituting values of a solution set into an equation is related to the mathematical concepts investigated within the instructional element A-3 that deals with equations and inequalities. The focus of instruction is centered on examining the conditions of equality (or inequality) expressed symbolically by algebraic equations of one variable. Within this focus, the checking procedure represents a way of demonstrating the mathematical relationship that exists between the numerical values of a solution set and the original equation from which it was derived.

The lecture discourse that was examined in this study indicates that the classroom instruction related to checking did not use such a focus of instruction. This subject was presented instead from a procedural perspective in which various checking processes were represented as series of manipulation steps that were to be applied with specific types of equations. The instructional focus was on manipulating symbols rather than on investigating mathematical relationships. The subject matter was displaced from its appropriate mathematical context within element A-3 and presented instead within the instructional focus of element A-2 that dealt with algebraic expressions and procedures.

It is postulated that the findings of the second summer are a consequence of this displaced instruction. The students demonstrated their ability to appropriately manipulate algebraic expressions when they correctly solved various types of equations on their quizzes. However, their inability to correctly apply the checking procedures is evidence that they were not able to develop an appropriate level of mathematical conceptualization for the process. By teaching the subject of checking outside of its conceptual context, the instructor did not provide the students with a sufficient understanding of this object of study so that they could utilize its various properties in a consistent and mathematically correct fashion.

The disparity that was observed to exist between the students' existing levels of mathematical knowledge and the levels of mathematical conceptualization of the intended instruction in each summer class provided information that was used to identify the different instructional elements of the learning structure. Although the final analysis of the subject of algebra was conducted from a logical examination of the mathematical nature of each topic of study, the existence of such a learning structure was postulated on the basis of the findings developed from each summer course. The learning structure of algebra, although primarily developed from a theoretical analysis, is based, in part, on empirical evidence obtained from the present study.

The behavior of the students and the instructors in the first class indicated that different types of mathematical conceptualizations were required to develop appropriate concepts for the ideas of algebraic notation and the objects of study represented by algebraic expressions and procedures. The analysis of the quiz responses from the second summer indicated that a different form of mathematical conceptualization was also required to develop an appropriate level of understanding for the subject of equations. The fact that students were not able to assign appropriate mathematical meanings to the objects of study was used as evidence of the sequential nature of the levels of concept development.

It was postulated that a logical sequencing of skills and concepts existed in the subject of algebra that prevented students from developing appropriate levels of mathematical understanding if certain topics were not encountered in a specific order and within a certain focus of instruction. This postulation led to the establishment of the following generalizations about the subject matter. (1) Certain topics require the development of prerequisite skills in order for students to perform the required manipulations. (2) Certain mathematical concepts are based on a preceding level of concept and skill development. (3) Students are unable to focus their learning attention on topics for which they have not yet developed the necessary prerequisite learning.

These generalizations were used to initiate an in-depth analysis of the subject of high school level algebra. Such information was then related to the theoretical constructs of the model of mathematical concept learning to produce the detailed description of instructional elements and levels of concept learning that represent the learning structure of algebra.

Implications for Instruction

The learning structure of algebra provides an analytical framework that can be used to order the topics of algebra into an appropriate learning sequence. By placing each topic within the appropriate instructional element, it is possible to provide students with instruction that proceeds in a sequential fashion, developing all necessary prerequisite skills and concepts for each successive topic of study. The learning structure also provides information relating to the level of mathematical conceptualization that should be developed for each object of study. The location of topics within this structure can be used to identify the particular focus of instruction that should be utilized for the development of appropriate mathematical concept learning.

The learning structure can also serve as a diagnostic tool. Once a student's existing level of mathematical knowledge is determined, it is possible to place this information within the conceptual framework of one of the instructional elements. This location can then be used to determine whether or not that student possesses the appropriate level of knowledge and skills to begin learning the topics of study in any other instructional element in the learning structure.

The learning structure was developed, in part, to fulfill the recommendations made by the models of effective instruction and mathematical concept learning. The first and second components of each of these models stipulate that effective learning cannot take place unless there is an appropriate match between the level of mathematical conceptualization of the objects of study and the existing knowledge base of the learner. The learning structure provides a set of theoretical instructional guidelines that can be used to place the student within an appropriate instructional context and to provide a sequencing of topics and focus of instruction that meet the requirements of the two teaching/learning models.

Van Hiele Levels of Geometric Thinking

The learning structure of algebra is a very subject specific theory of learning and instruction. It was developed as an analytic tool to be used not only to explain the existence of students' particular mathematical learning difficulties but also to provide a framework on which to design more effective algebra instruction. In this respect, the learning structure is very similar to the van Hiele theory of levels of geometric thinking which consists of a subject-specific analysis of a hierarchy of learning levels relating to the study of geometry. The purpose of this section is to present a brief outline of the van Hiele theory and to compare and contrast its instructional ideas with those representing the learning structure of algebra.

The Dutch educators Pierre van Hiele and his late wife Dina van Hiele-Geldof developed this theory of levels of learning based on their experiences teaching geometry to 12 and 13 year-old students (Fuys et. al., 1984). This theory was presented in companion dissertations that were written in Dutch in 1957.

According to this theory, geometry learning, and hence instruction, proceeds from a basic level of subject matter through four additional levels that are concerned with developing increasingly abstract concepts related to this material (which the van Hieles originally identified as levels 0 through 4) (Fuys et al, 1984; van Hiele, 1986). At the basic level a student recognizes geometric figures only globally as shapes. At the first level a student goes beyond the shape and is able to identify specific properties of each figure, such as the fact that a square is made up of equal sides. At the second level a student begins to study these properties of the figures and investigate their relationships. At the third level, these relationships become the object of study and the student is ready to investigate the system of proofs that make up much of the formal study of geometry. The final or fourth level involves the study of abstract, axiomatic systems.[1]

It is necessary for students to pass through these levels in order. What is implicit in a lower level becomes an object for study at the next higher level. The first level is based on student experiences with actual shapes, while each succeeding level becomes more abstract. Each level develops its own set of symbols and language. Thus students at a lower level are not able to communicate or understand someone at a higher level.

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[1]Pierre van Hiele has recently revised the way in which the level structure is organized. His original "basic level" is currently referred to as the visual or physical level. The next two levels (1 and 2 in the original numbering system) are grouped into a single level called the descriptive level. The next level which deals with formal geometric proofs is referred to as the theoretical level. Van Hiele places less stress on the highest level in his original structure that investigates axiomatic systems. He is more concerned with the implementation of the visual, descriptive and theoretical levels which cover the material typically studied through high school geometry (Geddes, 1987).

There are two major aspects of the van Hiele theory of levels. First, the van Hieles recognized that geometry learning must proceed from the concrete to the abstract, and must proceed in an orderly and sequential fashion. Their analysis of the entire field of geometry led them to develop the original sequence of five levels.

The second important aspect of learning contained in their theory is the assumption that learning must be a process within each level. New abstractions, language and a network of relations among concepts must be formed at each level by each student, before he or she can progress from one level of abstraction to another.

Within each level a student must pass through five phases of learning. These phases are specifically designed to enable the student to attain the appropriate mental structures necessary to pass to the next level of thinking. These phases are guided by the teacher, and the student must pass through them in the following order.

The initial phase is one of information in which the teacher presents the objects of study in an appropriate context. Specific activities are then provided to allow the students to begin careful, guided work during the second phase of bound orientation or exploration. The third phase consists of a period of explicitation developed through class discussions in which students begin to express perceived relationships and utilize the language of the level appropriately. At this point the student then becomes ready to explore the level on his own, and engages in free orientation activities. Networks are formed among the relations studied and students perceive many ways to examine the objects of study. The final phase consists of a period of integration in which the student develops an overview of the subject. The student is then ready to begin studying the subject at the next level of abstraction.

As with Skemp, van Hiele maintains that learning cannot be imposed externally by providing students with definitions. Students must pass through the learning process and proceed through the levels in order. Van Hiele postulates that the passage through the series of five sequential levels will allow students to develop the ability to think mathematically in a geometrical context.

Implications for Instruction

Even though students at a lower level cannot fully understand explanations presented to them at the next higher level, it is possible for them to appear to work with material at a level above themselves. This is referred to as a reduction in level. The students however, must rely on rote-memorization in order to perform the appropriate actions, since they do not have the appropriate mental structures necessary for understanding. According to van Hiele (1986):

> It is very usual, though always condemnable, to speak to pupils about concepts belonging to a level that they have not at all attained. This is the most important cause of bad results in the education of mathematics. The result of such instruction is that the pupils are obliged to imitate the action structure of the teacher. By doing so they usually succeed in mastering operations belonging to the level. But because the action structure does not result from a real understanding (i.e., not by analysis of lower structures), it must result from a global structure of acting. (p. 66)

Since students may appear to be acting at a higher level by recourse to rote memorization, teachers may be misled into thinking that the students are actually at that level in understanding. If the testing of their knowledge consists of questions for which memorization is adequate, the fact that a reduction of level has taken place will not be uncovered. Testing for comprehension and understanding requires careful planning (van Hiele, 1986).

Skemp (1987) defines understanding as the ability to assimilate new information into an existing schema and as the ability to accommodate, or change schemas to incorporate new information. Van Hiele (1986) utilizes the term <u>insight</u> as a measure of understanding. "Insight exists when a person acts in a new situation adequately and with intention." (p. 24)

Thus, a student can be considered to have grasped the concepts and network of relations at a particular level when he or she can apply the information successfully to a new situation. "With intention" means that the student did not simply have a lucky guess, but used the appropriate tools and thinking strategies of that level in order to solve the problem.

The van Hieles, like Skemp, stress in their model of mathematical learning that learning must proceed from the concrete to the abstract. Learning must be based on what is experiential to a student at that particular point in his or her mental process and must be presented in a format that is easily understandable at that level. Understanding cannot proceed from definitions or from information presented at a level above the student's present state. If this occurs, the student will operate within a level reduction and produce apparently correct manipulations through rote memorization.

Pierre van Hiele and his wife first presented their theories of geometric thinking in 1957. Pierre van Hiele also presented a paper describing this theory in France that same year, which was subsequently published in French in 1959. The Soviets became interested in his work and organized research in the early 1960s to study and validate the theory of levels, the result of which led to an extensive reorganization of the Soviet school geometry curriculum (Wirszup, 1976).

It was not until 1974, however, that mathematics educators in the United States became aware of the work of the van Hieles when Izak Wirszup presented a paper on their theory at the Annual NCTM Meeting in Atlantic City (Hoffer, 1983). Since the beginning of the 1980s American interest in the van Hiele levels of geometric thinking has increased and much research has recently been conducted investigating and validating the theory (Crowley, 1987).

Although research reports and recent journal articles have made information concerning the van Hiele levels accessible in the United States, much work remains to be done in order to implement these ideas in the classroom. However, this model of geometric thinking is now recognized as an important theory of learning and instruction that presents a potentially significant contribution to the development of effective school instruction. (Crowley, 1987)

Comparison with the Learning Structure of Algebra

The van Hiele levels and the learning structure of algebra each present a model of learning and instruction that is based on an analysis of a specific area of mathematics. Both theories describe a sequencing of instructional segments that are characterized by different objects of study that build on concepts and knowledge developed in previous segments. Both theories describe a type of concept learning that requires active student participation in the learning process.

It was not the intent during the analytic development of the learning structure of algebra to produce a theory of "van Hiele levels" for algebra. The van Hiele levels provided a model that demonstrated that it was possible to base a theory of instruction around a learning-type structural analysis of a specific area of mathematics. The actual organization of the subject of algebra into elements containing levels of concept development occurred because of the nature of the specific topics that were analyzed. The differences in the two learning structures are the result of the differences in the subject matter and the goals of instruction in each discipline.

The goal of instruction using van Hiele's levels of geometry is <u>to develop students' abilities to think geometrically</u>. Such thinking should reach the level of geometric proof by the time students have completed a high school course in geometry and be concerned with an understanding of the thinking required to elaborate formal deductive geometric proofs. Each level in this theory is based on the development of higher order concepts that are formulated by abstracting properties developed at the preceding level. Thus, the subject of geometry is organized into levels of increasing abstraction.

The goal of the learning structure of algebra is not to develop students' abilities to utilize a deductive system of proofs for the subject of algebra, but <u>to develop</u>

a notational system and set of mathematical procedures that can be used to model and investigate complex physical relationships. The structuring of the subject matter into elements is therefore based on the development of a series of concepts and manipulation skills that have this goal in mind.

The sequencing of instructional elements in the learning structure is not based primarily on the development of higher order concepts as in the van Hiele model, but on a logical ordering of prerequisite skills and concepts. The material that is developed in each element represents parallel learning rather than vertical learning that is based on abstractions of existing knowledge. Such vertical learning takes place within each element rather than horizontally from one element to the next.

This contrast in the structuring of the learning is reflected in the vocabulary used to identify various aspects of each theory. Van Hiele describes each segment as a "level," denoting a move from lower order to higher order concepts. In the learning structure, each segment is designated as an "element" to distinguish the development of parallel rather than hierarchal concepts. The term "level" is reserved for the learning processes that occur within each element.

The passage through the three levels within each element of the learning structure contains many of the kinds of learning activities that van Hiele describes in his phases of instruction, since both theories are presenting a model of instruction based on a similar model of concept learning. In each instance, such instruction places students in activities that proceed from basic information, through the development of a set of abstractions regarding the objects of study, to an overview and generalization about the entire field of study. The level structure in the learning structure is presented in greater detail than van Hiele uses to describe his phases of learning, with each set of three levels being specifically oriented to the special focus of instruction within that element.

Both the learning structure of algebra and the van Hiele levels of geometric thinking present models of learning and instruction that have direct application to the school curriculum. These theories were developed in response to observations of students' difficulties with existing school mathematics curricula. Because the theories are based on observations of actual student learning and involve the analysis of those subjects that make up the school curriculum, each theory provides a set of learning activities that is capable of promoting significant mathematical learning within the school environment.

Summary

The learning structure of algebra presents an organizational framework that divides the topics usually covered in beginning algebra into a set of five instructional elements. Topics are assigned to each element on the basis of the types of mathematical conceptualizations that they represent and on the basis of the types of prerequisite skills and concepts that are considered to be necessary for the development of an appropriate level of mathematical understanding for each succeeding element. The five instructional elements are characterized by the objects of study they contain and by the instructional emphasis placed on these objects as (1) algebraic notation, (2) algebraic expressions, (3) equations and inequalities, (4) representations of relations, and (5) properties of relations.

Within each instructional element, learning proceeds through three levels of concept development that provide students with opportunities to investigate the mathematical properties of the objects of study and to abstract generalized concepts form these properties that can be used to apply the knowledge and skills that are learned. Learning proceeds from lower order to higher order concepts within each element and through parallel concept and skill development from one element to the next.

Learning must proceed in a sequential fashion through the elements in the structure since the mathematical concepts and algebraic procedures that are developed in one element form important prerequisites for the successful mastery of the objects of study in the following element. Students who have not developed these appropriate prerequisites will not have the necessary mental constructs to be able to focus on the objects of study of any element within the intended instructional emphasis. Under such an instructional mismatch, students will be unable to develop an appropriate level of mathematical conceptualization for the objects of study. Instruction also fails to be successful when the objects of study of any element are presented outside of their appropriate mathematical context through displaced instruction.

The learning structure of algebra provides a potentially powerful tool that can be used to promote effective instruction. The mathematical analysis of the subject matter provides a framework that can be used to order each topic of algebra in terms of its mathematical conceptualizations and sets of prerequisite concepts and skills. This framework can also be used to place students at an appropriate instructional level that matches their identified mental constructs to the intended level of conceptualization of the subject matter. Utilized in this way, the learning structure has the potential for providing an important contribution to the improvement of school mathematics instruction.

CHAPTER NINETEEN

SUMMARY AND CONCLUDING COMMENTS

This chapter presents a review of the findings of the study described in Parts I and II and a brief overview of the various theories that were developed in Part III of this report. The steps in analysis that were used to develop this theory are presented, along with a discussion of the ways in which this analysis evolved from an initial classification of the classroom records to the final theoretical statements. Comments are presented on the general significance of the study and directions for future investigations are suggested.

Summary of Findings

This study represents a qualitative investigation of the three-way teacher/student/subject matter interactions that took place within a special, intensive summer algebra class for Native American students at a western state university. The records of these interactions were collected over two consecutive summers through the techniques of participant observation. Additional records were obtained from tape recordings of classroom lectures and from photocopies of students' work.

The focus of the data collection was centered on observing the behavior of the classroom participants that occurred during lectures, during periods that the students worked on practice problems, and during testing periods in order to investigate the ways that the instructors and the students interacted with the subject material. The discourse and written work that were recorded and collected were used to infer sets of possible meanings that the participants in the two classes could have been assigning to the mathematical topics under study.

The study of the records from the first summer centered primarily on an analysis of those situations in which students provided some kind of discourse related to the subject matter, either in the form of questions to the instructor during lecture periods or in the form of questions and comments about their work on practice problems. This analysis centered on a group of eight students and led to the following conclusions.

(1) These students employed a type of classificatory recognition system that enabled them to reproduce correct mathematical responses on their written work without having to develop an appropriate level of mathematical understanding for the concepts and procedures involved. (2) This system, described as a set of learning strategies, involved the students' use of master examples and a reliance on familiar procedural format and specific symbol

pattern as a way to identify and utilize specific algebraic procedures with particular types of mathematical problems. The students were assigning meanings to these collections of algebraic symbols within a notational level of conceptualization. (3) It was postulated that the students developed these learning strategies because of the existence of a mismatch between the intended level of mathematical conceptualization of the instruction and the students existing mental structures.

The students that participated in the second summer course exhibited different types of behavior from that reported on in the findings of the first summer. These students did not utilize master examples and were capable of writing mathematically correct work without the aid of textbooks and notes. It was postulated that the students' written quiz responses that were produced under such conditions could be used to infer a possible set of meanings that the students were developing for the topics covered by the test questions.

The students' written responses to 19 quiz questions dealing with equations were analyzed to determine the ways in which the students employed the various checking procedures with their solutions. These responses were compared to the level of mathematical conceptualization that was made available to the students through the class lectures. These analyses led to the following conclusions.

(1) The students were more capable of producing mathematically correct solutions to various types of equations than they were in using the specific checking procedures in mathematically appropriate ways. (2) Students were inconsistent in their use of the special checking procedures for fractional and radical equations and applied these techniques and related terminology to their checks of other types of equations. (3) The analysis of the types of mathematical information that was made available to the students through the classroom lectures revealed that the topic was being presented at a procedural level that did not contain much material related to the underlying mathematical concepts. (4) It was postulated that the students' inappropriate use of checking indicated that they had not developed an appropriate level of mathematical conceptualization for this subject. This type of learning was postulated to be the result of instructional displacement in which the subject of checking was presented outside of its appropriate mathematical context.

Summary of Theory

The models of mathematical concept learning, effective instruction, participatory learning, and the learning structure of algebra were developed to address the findings of this study. The subject matter analysis provided a mathematical framework that was used to compare and contrast the types of mathematical conceptualizations that were postulated to be representing the mental constructs of the participants in the two learning environments. These comparisons were used to identify the instances of instructional mismatch and displacement that occurred.

The model of mathematical concept learning was developed to describe the types of mental activities that are required to appropriately learn the different objects of study that were identified by the learning structure of algebra. The model of effective instruction was developed to address the instructional problems that were observed within the two summer classes. The techniques of participatory learning were devised as a way to implement these models within the classroom.

The theories developed in Part III of this report can be summarized as a three-component model of <u>classroom algebra learning</u> that integrates learning theory, instructional recommendations, and the mathematical analysis of the subject of algebra. The three components of this model consist of (1) the identification of the objects of study, focus of instruction, and level of mathematical conceptualization of each topic contained in the subject material to be studied, (2) the existence of an appropriate match between the student's present knowledge base and the level of mathematical conceptualization of the objects of study, and (3) the implementation of appropriate, guided student activities that promote the active participation of the students in the learning environment.

First Component

The learning structure of algebra described in Chapter Eighteen provides information needed to meet the requirements of the first component, which recognizes that mathematics learning is subject-specific. An analysis of the topics of algebra that are usually covered at the beginning and intermediate high school level was used to categorize the subject matter on the basis of prerequisite levels of knowledge and skills and on the basis of the types of mathematical conceptualizations that the various objects of study represent. An organizational structure was developed from this analysis to divide the subject matter into five elements of instruction that are characterized by the objects of study included and the types of mathematical conceptualizations that form each focus of instruction.

According to the model of classroom algebra learning, the learning structure is used to arrange instructional topics in a specific sequence that allows students to systematically develop skills and concepts within appropriate mathematical contexts. Students are placed within this sequencing so that their present levels of mathematical knowledge match the prerequisite conditions of any instructional element. Such placement meets the conditions stipulated by the second component of the model which postulates that this type of match must be present in order for students to develop appropriate levels of mathematical understanding.

Second Component

One facet of the second component of the model of classroom algebra learning is that effective learning not only requires a match between the student's existing knowledge base and the level of knowledge represented by the subject matter, but also requires that a match exist between the subject matter and the type of learning necessary to understand its objects of study. It is important, therefore, to be able to identify and describe the specific types of learning that are needed to understand those topics of algebra that are included in the learning structure.

The learning model described in Chapters Fifteen and Sixteen, which uses Skemp's model of schematic learning within a constructivist framework, postulates that mathematical concepts are developed as students form abstractions representing sets of properties that they perceive to be shared in common by a collection of examples. Higher order concepts are developed from lower order concepts by this same process of abstraction. Through the use of appropriate activities, students are guided by the teacher to develop higher order thinking skills to compare, contrast, abstract, generalize and integrate their experiences in order to develop appropriate mathematical concepts and networks of related information.

Third Component

The third component of the model of classroom algebra learning addresses the implementation of the ideas postulated by the learning structure and the models of effective instruction and mathematical concept learning. It is postulated that effective instruction can be implemented through the use of the participatory learning strategies that were described in Chapter Sixteen. These techniques provide a constructivist form of concept learning that utilize the sequencing of subject matter outlined in the learning structure of algebra. Participatory learning employs cooperative, small group learning strategies that enable students to interact actively with activities designed to focus their attention on specific attributes of collections of examples of the objects of study. The teacher plays an active role in the learning process by encouraging group discussions and individual student participation and by carefully monitoring these activities to identify the types of meanings that students develop from the learning environment.

It is postulated that effective learning takes place only if all the components of the model of classroom algebra learning are utilized. This involves the inclusion of a detailed description of the learning structure of the subject matter to be studied, a teaching/learning method that promotes active student participation in the learning environment, and active coordination and direction of the learning environment by the teacher in order to provide appropriate instructional matches between the students and the subject matter. This model not only provides a theoretical framework on which to discuss the behavior of the participants in the two summer classes but provides a prescriptive set of theories that can be used to develop and direct other learning situations.

Development of Theory

The various components of the theory of classroom algebra learning were developed in response to the instructional issues that were raised by the analyses of the collected records from the two summer classes. This section traces the evolution of these theories as they were developed during the different stages of those analyses.

The processes that were involved in the completion of this study occurred during three different phases. The first phase consisted of the observational periods during the two consecutive summers that involved the collection of the records of the classroom behavior, discourse and students' written work. The second phase consisted of the examination, classification, and analysis of these records. This phase began shortly after the beginning of the observational period and continued after all records were obtained. The third and final phase consisted of the development of a set of theories that could be used to explain the analyzed classroom behavior.

Analyses of Collected Records

The initial focus of the study began to emerge half-way through the first summer's course as the records were reviewed and initial classifications of interactions were made. At that time, the decision was made to concentrate observations on those aspects of classroom behavior in which students were engaged in some kind of interaction with the subject matter.

The records that were selected from the first summer for an in-depth analysis based on this focus consisted of those situations in which students were observed to be asking questions concerning the lecture topics, were engaged in working through the practice problems, and/or were engaged in some kind of discourse involving algebra. These selection criteria were based on the decision to investigate the kinds of mental structures that the students could possibly be developing from their interactions with the classroom learning environment. It was postulated that it would be possible to infer these mental constructs from an analysis of the students' discourse during those times that they were engaged in investigating algebraic topics. During the second summer, copies of the students' written quiz responses were collected in order to gather information from which to infer the students' thought processes related to the objects of study.

The material from the first summer was analyzed in terms of the ways in which certain students utilized various learning strategies in order to cope with new material. The students' interactions with the subject of algebra was further narrowed during the analysis of the second summer's records to the examination of the ways in which students responded to specific questions involving the solution of algebraic equations.

A second level of analysis was undertaken to compare and contrast the results of the two summers' analyses. The description of the set of learning strategies that had been applied to the students' behavior from the first summer was found to be an inappropriate categorization with which to discuss the types of quiz responses that were examined from the second summer. The differences in types of responses from the two summers prompted the need for a broader description of the students' thinking than that produced by each individual summer's analysis. This led to the development of the theory described by the learning structure of algebra.

Learning Structure of Algebra

The initial analyses that were carried out on the two summers' records involved various aspects of the students' interactions with the subject matter. In the first summer, many of the students were observed to be having difficulty developing an appropriate level of meaning for the notational symbols used in the course. The second summer's analysis of the quiz responses indicated that the students were not able to develop an appropriate level of mathematical understanding of the uses of checking.

These two areas of non-understanding provided an initial differentiation of the subject of algebra into three separate elements. As a preliminary stage in the development of a theory to explain the students' interactions with the subject matter, the topics under study were identified as belonging to either a "notational level," a "procedural level," or a "relational level." It was postulated that these levels represented a sequential set of prerequisite skills and concept learning.

It was then postulated that those students in the first class who were observed to be having difficulties manipulating algebraic expressions did so because of a lack of an appropriate level of understanding of the symbolic language implied by the algebraic notation. It was also postulated that those students in the second class who exhibited difficulty responding in a mathematically correct way to quiz questions dealing with equations had not developed an understanding of important prerequisite concepts that dealt with algebraic expressions.

These learning difficulties were regarded as being the result of the inherent sequential nature of the subject of algebra rather than the result of deficiencies in the capacity of the students to learn. It was postulated that students could not understand specific information until they had developed certain prerequisite sets of concepts and skills that would free their attention to focus on other types of mathematical conceptualizations. These conclusions motivated the development of a learning structure of algebra that could be used to provide an analytical as well as empirical framework within which to discuss the mathematical conceptualizations of the various participants in the two summer classes.

Effective Instruction and Mathematics Concept Learning

The learning structure of algebra was developed in response to the types of observations that became the focus of the study and provided a framework around which the perceived learning difficulties of the students could be discussed. These learning difficulties were explained in terms of instructional mismatches that were postulated to exist between the students' present levels of mathematical understanding and the level of mathematical conceptualization of the intended instruction. The model of effective instruction was developed as a prescriptive set of guidelines that could be used to remediate such instructional difficulties.

This set of guidelines was developed as a model of effective "instruction" rather than of "learning" because of the perspective that was used in the analysis of the collected records. Although I was in the class as an observer, I also served in the capacity of assistant instructor, and my initial response was to analyze the students' learning difficulties from an instructional point of view. I was concerned with addressing the question of what needed to be done to "teach" these subjects more effectively.

Once the model of effective instruction was postulated, it was apparent that the developing theory also needed to address the learning processes that were involved in the classroom interactions. A learning model based on current theories related to mathematics concept learning was assembled to meet this need. The particular selection of learning theories that was incorporated into this model was based in part on the types of student behavior that were observed in the two summer classes. In many cases, it was possible to use specific classroom interactions to illustrate significant aspects of the model of mathematical concept learning. Although such theory was not grounded in the records, its applicability to the present study is demonstrated by testing it against observed student behavior.

Participatory Learning

The techniques of participatory learning were developed from criteria contained in the model of mathematical concept learning. This model postulates the importance of promoting active student involvement with the learning process and advocates the use of activities involving collections of examples and non-examples as an effective method of developing appropriate mathematical concept learning.

Once these criteria were incorporated into a set of instructional/learning techniques, it was possible to test the appropriateness of the ideas of participatory learning by re-examining the collected records. This testing forms an important part of qualitative research, since the validity of developing theory is established by examining the ways in which it can be successfully used to explain observed behavior. The classroom examples that were cited in Chapter Sixteen indicate that the theory behind participatory learning is consistent with what was actually observed to be happening in the classroom.

Appropriateness of Theory

The original purpose of this study was to conduct an exploratory investigation of the types of teaching/learning interactions that are found in a regular mathematics classroom. Qualitative techniques were selected to provide an appropriate methodology for such discovery-type research. The findings of the study were used to develop a set of theories to explain the behavior of the participants in the learning environment rather than to test hypotheses based on existing theories. The relevance of the developed theories was established by their ability to explain the information contained in the analyzed records.

The theories that were described in this study were based on both empirical data and existing theory and were developed to explain the observed classroom behavior. These theories "work," as stated by Glaser (1978); "by work, we mean that a theory should be able to explain what happened, predict what will happen and interpret what is happening" (p. 4). These theories also provide a way to integrate the findings of the present study with other current theories of mathematics education. Glaser (1978) describes the ways in which such theory should be related to current thinking in the field of study.

> The linkage, at minimum, can place the generated theory within a body of existing theories. More often, as we have said, it transcends part of it while integrating several extant theories. It may shed new perspectives and understandings on other

> theories and highlight their powers. Other theories are neither proved or disproved, they are placed, extended and broadened. (Glaser, 1978, p. 38)

The function of a theory as described by Skemp (1979) is to "understand, predict and sometimes to control events" (p. 217). Glaser (1978) states that "the credibility of the theory should be upon its integration, relevance and workability" (p. 134). The theories comprising the model of classroom algebra learning that are developed and reported in this study fit these criteria of good theory.

Levels of Thinking

The ways in which the different aspects of this theory were developed reflect Skemp's descriptions of the thought processes that occur as higher order concepts are formed from lower order concepts. The analysis of the collected records began with an examination of specific interactions. These were classified and contrasted to identify collections of examples and non-examples that represented emerging categories of interest. These categories then became the abstract representations of those sets of properties that had been identified from the individual classroom interactions.

Once the initial classification of the records was completed, a higher level of analysis was begun in which the properties that were assigned to the categories representing learning strategies and quiz responses became the objects of a second analysis. Such processes are representative of Skemp's description of the formation of higher order concepts from lower order concepts. Van Hiele (1986) describes such processes in terms of successive levels of abstraction, in which what is only implicitly perceived at one level becomes explicit and the object of study at the next level of thought.

An Awareness of Mathematics Education

Van Hiele's concept of a hierarchy of levels of thinking can be used to perform an analysis of the mathematics teaching/learning environment. Van Dormolen (1975) reports on the use of this type of thinking as part of a course for pre-service mathematics teachers that focuses on improving the abilities of these students to reflect on and improve their future classroom performances. Van Dormolen's sequence of levels of thinking closely parallels the sequence of thought processes that were employed in the present study to develop the theory of classroom algebra learning.

Van Dormolen states that teachers pass through three levels of thinking regarding their awareness of the teaching/learning interaction. At the ground level of thought, teachers provide their students with classroom activities that appear to be effective, but these teachers do not understand why these activities work. I found that I was operating at this level of thinking as I began to categorize the classroom records and develop a preliminary focus to the study. I was able to identify an area of interest for further study but was not able at that time to state why such a selection was important.

At the next, or "first" level, of van Dormolen's sequence, teachers develop the ability to begin to talk objectively about their classroom interactions. They are able to observe which student activities are more effective than others and to identify reasons why this is so. These teachers begin to relate their observations to various sets of principles that are able to explain the behavior they observe in the classroom. This situation represented my level of thinking as I completed the preliminary analyses of the records of each summer class. I had developed a set of properties for the objects of the first analysis and was beginning to formulate a set of theories related to these properties.

The final, or "second" level, is reached as teachers turn their thinking to the examination of the relationships that they became aware of at the preceding level of thinking. I found that I was functioning at this level as I began to develop my overall theories of classroom algebra learning. These were formulated to examine and explain a set of three-way relationships that existed in the mathematics classroom among the learner, the instructional mechanisms, and the subject matter.

Higher order concepts and networks of relationships between these concepts are developed as individuals become aware of their own thinking and reflect on these thoughts (Skemp, 1987). However, the explicitation of these newly formed concepts is a difficult task. According to Skemp (1987), an individual understands a concept at the intuitive level long before he or she can state its properties in a formal way.

> The process of becoming aware of one's concepts for the first time seems to be quite a difficult one.... But even in persons with a highly developed reflective ability, it is still a struggle to make newly formed, or forming, ideas conscious. (p.57)

Skemp's comment sums up the difficulties that I encountered as I attempted to present my analyses within the written format of this study. Not only was it necessary to progress through the successive stages of concept development described by van Hiele and van Dormolen, it was also necessary to struggle with the formidable task of making a set of very complex and interrelated theories explicit. The formal descriptions of the theories presented in this study represent the end product of much thought and a great deal of writing and re-writing.

Significance of the Study

The methodology of qualitative research made it possible for this study to develop an extensive body of theory that has direct applicability to actual classroom learning. The techniques employed provided an effective way to systematically investigate a very complex, real-life learning environment. The findings of this study demonstrate the power and versatility of the observational techniques and methods of analysis that were used.

These research techniques made it possible to collect and analyze information pertaining to three important aspects of the classroom environment -- the students, the instruction, and the subject matter. As the chapters describing the theory indicate, the analyses of these factors identified an interrelated set of interactions, which needed to be examined in relationship to each other in order to begin to understand the complex dynamics of the students' observed behaviors. Such a complex set of variables could not have been studied as effectively or as extensively without the use of the open-ended, discovery-oriented techniques of qualitative research. The results of these analyses led to the postulation of instructional models and learning theories related to the development of effective classroom algebra learning.

A need exists in the secondary schools for appropriate models of effective algebra instruction. In 1978, Sells

(1978) drew attention to the critical importance that students' failures in high school mathematics courses exerted on their subsequent choices of college majors and future careers. She described such high school mathematics courses as a "critical filter." Concern exists today over the low achievement scores measured for American high school students on the Second International Assessment of Mathematics Education (Steen, 1988).

Further concerns are expressed over the ability of today's students to cope with the rapid technical changes that will confront them as adults. "No one doubts the importance of mathematics. Of concern, however, is whether students who are currently being taught mathematics in school will have adequate preparation for the scientific world of the twenty-first century" (Romberg and Carpenter, 1986, p. 850). Algebra plays a critical role in a student's ability to study mathematics. Unless students are able to develop an appropriate set of algebraic concepts, further mathematics learning may be in jeopardy. "School algebra is the area in which symbolic conventions needed for more advanced mathematics are first developed" (House, 1988, p. vii).

The present study provides an in-depth analysis of one particular instance of classroom algebra learning. The significance of the findings of the study and the models of instruction and learning theory that were developed lie in their ability to address some of the instructional concerns expressed by today's mathematics educators. In particular, this study was directed to examining the reasons why students fail to develop appropriate levels of mathematical conceptualization for their course of study. The model of classroom algebra learning was postulated as a prescriptive set of teaching/learning principles related to this area of concern.

The recommendations of the model of effective instruction can be used to investigate the reasons why otherwise capable students fail to learn high school mathematics with understanding. In many cases, as was observed with the students in the two summer classes, these individuals encounter topics before they have developed appropriate prerequisite knowledge. It is postulated that such situations can be corrected by ordering the course topics according to their respective learning structures, matching instruction to the students' levels of understanding, and providing appropriate, guided student activities within each learning environment.

The learning structure of algebra provides a detailed sequencing of many of the topics that are covered in first and second year high school algebra. It can therefore serve as an important outline for the organization of these courses. The learning structure can also be used as a diagnostic tool. Once a student's present level of mathematical knowledge is ascertained, he or she can be placed within an appropriate sequence of materials based on prerequisite learning, as described by the learning structure.

The model of classroom algebra learning developed in this report provides a potentially important set of guidelines that can be used to enable students to develop a firm basis of mathematical understanding for the important subject of algebra. This teaching/learning model represents an initial exploration into the field of classroom mathematics education. Many more such investigations are necessary at this point in order to gather a large body of information on which future theories of mathematics education can be built.

Directions for Further Study

The present study was based on a small and very unique population of Native American students who participated in a non-standard algebra class. Important information could be obtained by conducting a similar qualitative study within a more representative high school algebra classroom in order to verify the extent to which the theories and findings of the present study can be generalized to a wider population. The records from such a study would also provide a comparative framework that could be used to address the degree to which the behavior of the students in the present study was a function of their Native American backgrounds. No attempt was made to investigate these influences in the present study.

The original field notes that were recorded from the two summer classes contained much information that was not analyzed because it lay outside the selected focus of the study. Such information, however, could provide the foundations for other, different types of research investigations. The interactions that occur between the participants in any mathematics classroom and between the participants and the subject matter provide a diverse and complex field of information. The techniques of qualitative observation can be used to generate a multitude of important studies relating to mathematics education and classroom learning. Three examples of potential areas of investigtion are given below.

(1) The lecture transcripts from a mathematics class can be analyzed from a linguistic perspective in order to investigate the frequency with which the instructor utilizes directive and imperative statements in order to examine the effect that such a style of instruction has on students' perceptions of the study of mathematics. (2) It was observed that the students in the two summer courses made many statements regarding their self-concept or related to their math abilities. An important area of research would be to investigate the ways in which students alter these perceptions as they experience instances of

success with mathematics. (3) In the present study, the use of cooperative learning groups during practice and testing periods provided many examples of effective group dynamics. An investigation into this area of learning could provide important insights into the ways that such learning can be employed to promote effective instruction.

The theories developed in this study include the prescriptive organization of the learning structure of algebra and the instructional techniques of participatory learning. This model of classroom algebra learning does not represent an end product but rather a beginning. The next logical step to be taken is to implement the findings and recommendations of the present study. Actual classroom activities need to be developed that can put the generated theory into practice.

An important area for further work is the development of a set of guided student activities that can be used to provide appropriate kinds of learning situations. These must be developed within the framework of the learning structure and be designed to present each topic of study within the appropriate focus of instruction. Once such activities are developed, it will be necessary to field test them to determine how well they can be utilized in the format of participatory learning. Such field testing can be evaluated very effectively using techniques of qualitative observation.

The ultimate goal of future work based on this study is to develop a complete set of classroom materials that can be implemented directly into the high school algebra classroom. The results reported in this study can be used to provide the initial steps in such a development and can point the way towards further research. The findings and theories that are generated by such work have the potential of providing important contributions to the developing fields of mathematics education and student learning.

BIBLIOGRAPHY

BIBLIOGRAPHY

Agar, Michael H. The Professional Stranger: An Informal Introduction to Ethnography. New York: Academic Press, 1980.

Ashlock, Robert B. Error Patterns in Computation: a Semi-Programmed Approach. Columbus: Charles E. Merrill Publishing Company, 1986.

Association of Teachers of Mathematics. Notes on Mathematics for Children. Cambridge: Cambridge University Press, 1977.

Ausubel, David P. "In Defense of Advance Organizers: A Reply to the Critics." Review of Educational Research 48, no. 2 (Spring 1978): 251-257.

Bidwell, James K., and Clason, Robert G. Readings in the History of Mathematics Education. Washington, D. C.: National Council of Teachers of Mathematics, 1970.

Bloom, Benjamin S. "The Search for Methods of Group Instruction as Effective as One-To-One Tutoring." Educational Leadership May 1984: 4-17.

Bogdan, Robert C., and Biklen, Sari Knapp. Qualitative Research for Education: An Introduction to Theory and Methods. Boston: Allyn and Bacon, Inc., 1982.

Bogdan, Robert, and Taylor, Steven J. Introduction to Qualitative Research Methods: A Phenomenological Approach to the Social Sciences. New York: John Wiley and Sons, 1975.

Booth, Lesley R. "Children's Difficulties in Beginning Algebra." In The Ideas of Algebra, K-12: 1988 Yearbook. Reston, Virginia: The National Council of Teachers of Mathematics, 1988.

Bruner, Jerome S. The Process of Education. Cambridge: Harvard University Press, 1961.

Bruner, Jerome S. Toward a Theory of Education. Cambridge: Harvard University Press, 1967.

Carpenter, Thomas P., and Moser, James M. "The Acquisition of Addition and Subtraction Concepts." In Acquisition of Mathematical Concepts and Processes. Eds. Richard Lesh and Landau, Marsha. New York: Academic Press, 1983.

Crosswhite, Joe F. "Cognitive Science and Mathematics Education: A Mathematics Educator's Perspective." In Cognitive Science and Mathematics Education. Ed. Alan H. Schoenfeld. Hillsdale, New Jersey: Lawrence Erlbaum Associates, 1987.

Crowley, Mary L. "The van Hiele Model of the Development of Geometric Thought." In Learning and Teaching Geometry, K-12. Reston, Virginia: The National Council of Teachers of Mathematics, 1987.

Davis, Robert B. "Complex Mathematical Cognition." In The Development of Mathematical Thinking. Ed. Herbert P. Ginsburg. New York: Academic Press, 1983.

Davis, Robert B. Learning Mathematics: The Cognitive Science Approach to Mathematics Education. Norwood, New Jersey: Ablex Publishing Co., 1984.

Freudenthal, Hans. Mathematics as an Educational Task. Dortrect, Holland: D. Reidel Publishing Co., 1973.

Freudenthal, Hans. Weeding and Sowing: Preface to a Science of Mathematical Education. Dortrect, Holland: D. Reidel Publishing Co., 1980.

Friedlander, Bernard Z. "A Psychologist's Second Thoughts on Concepts, Curiosity, and Discovery in Teaching and Learning." Harvard Educational Review 35, no. 2 (Winter 1965): pp. 18-38.

Fuys, David; Geddes, Dorothy; and Tischler, Rosamond, Eds. English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele. Brooklyn, New York: Brooklyn College, City University of New York, 1984.

Gagne, Robert M. The Conditions of Learning. New York: Holt, Rinehart and Winston, Inc., 1965.

Gagne, Robert M. The Conditions of Learning and Theory of Instruction. New York: Holt, Rinehart and Winston, Inc., 1985.

Geddes, Dorothy. Private Conversation. 12 June, 1987.

Gerace, William J. and Mestre, Jose P. "A Study of the Cognitive Development of Hispanic Adolescents Learning Algebra Using Clinical Interview Techniques." Columbus, Ohio: ERIC Document Reproduction Service, ED 231 613.

Ginsburg, Herbert. "The Clinical Interview in Psychological Research on Mathematical Thinking: Aims, Rationales, Techniques." For the Learning of Mathematics 1, no. 3 (March 1981): 4-11.

Ginsburg, Herbert P., Ed. The Development of Mathematical Thinking. New York: Academic Press, 1983.

Ginsburg, Herbert P., et al. "Protocol Methods in Research on Mathematical Thinking." In The Development of Mathematical Thinking. Ed. Herbert P. Ginsburg. New York: Academic Press, 1983.

Glaser, Barney. Theoretical Sensitivity: Advances in Methodology of Grounded Theory. Mill Valley, Calif.: The Sociology Press, 1978.

Glaser, Barney, and Strauss, Anselm L. The Discovery of Grounded Theory: Strategies for Qualitative Research. Chicago: Aldine, 1967.

Greeno, James G. "Psychology of Learning, 1960-1980: One Participant's Observations." American Psychologist 35, no. 8 (August, 1980): 713-728.

Grossnickle, Foster F., et al. Discovering Meanings in Elementary School Mathematics, Seventh Edition. New York: Holt, Reinhart and Winston, 1983.

Hoffer, Alan. "Van Hiele-Based Research." In Acquisition of Mathematical Concepts and Processes. Eds. Richard Lesh, and Landau, Marsha. New York: Academic Press, 1983.

House, Peggy A. "Reshaping School Algebra: Why and How?" In The Ideas of Algebra, K-12. Reston, Virginia: The National Council of Teachers of Mathematics, 1988.

Johnson, Lizbeth. "The Effects of the Groups of Four Program on Student Achievement." Columbus, Ohio: ERIC Document Reproduction Service, ED 254 399, 1984.

Kennedy, Leonard M. Guiding Children to Mathematical Discovery. Belmont, Ca.: Wadsworth Publishing Co., 1970.

Klausmeier, Herbert J. and Harris, Chester W. Eds. Analyses of Concept Learning. New York: Academic Press, 1966.

Kline, Morris. Why Johnny Can't Add: The Failure of the New Math. New York: St. Martin's Press, 1973.

Lesh, Richard, and Landau, Marsha, Eds. Acquisition of Mathematical Concepts and Processes. New York: Academic Press, 1983.

Lesh, Richard; Landau, Marsha; and Hamilton, Eric. "Conceptual Models and Applied Mathematical Problem-Solving Research." In Acquisition of Mathematics Concepts and Processes. Eds. Richard Lesh, and Landau, Marsha. New York: Academic Press, 1983.

Lovell, Kenneth. "Concepts in Mathematics." In Analysis of Concept Learning. Eds. Herbert J. Klausmeier, and Harris, Chester W. New York: Academic Press, 1966.

Mayer, Richard E. "Different Rule Systems for Counting Behavior Acquired in Meaningful and Rote Contexts of Learning." Journal of Educational Psychology 69, no. 5 (1977): 537-546.

Maurer, Stephen B. "New Knowledge about Errors and New Views about Learners." In Cognitive Science and Mathematics Education. Ed. Alan H. Schoenfeld. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987c.

National Council of Teachers of Mathematics. The Agenda for Action. Reston, Virginia: National Council of Teachers of Mathematics, 1983.

National Council of Teachers of Mathematics. Curriculum and Evaluation Standards for School Mathematics. Reston, Virginia: National Council of Teachers of Mathematics, 1989.

National Council of Teachers of Mathematics. Curriculum and Evaluation Standards for School Mathematics: Working Draft. Reston, Virginia: National Council of Teachers of Mathematics, 1987a.

National Council of Teachersof Mathematics. A History of Mathematics Education in the United States and Canada. Reston, Virginia: National Council of Teachers of Mathematics, 1970.

National Council of Teachers of Mathematics. The Ideas of Algebra, K-12. Reston, Virginia: National Council of Teachers of Mathematics, 1988.

National Council of Teachers of Mathematics. Learning and Teaching Geometry, K-12. Reston, Virginia: National Council of Teachers of Mathematics, 1987b.

Neisser, Ulric. Cognition and Reality. San Francisco: W. H. Freeman, 1976.

Parker, Ruth. "What does Research Say about Cooperative Learning?" Columbus, Ohio: ERIC Document Reproduction Service, ED 242 065, 1984.

Pollak, H. O. "Cognitive Science and Mathematics Education: A Mathematician's Perspective." In Cognitive Science and Mathematics Education. Ed. Alan H. Schoenfeld. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987a.

Research Advisory Committee. "NCTM Curriculum and Evaluation Standards for School Mathematics: Responses from the Research Community." Journal for Research in Mathematics Education 19, no. 4 (July 1988): 338-344.

Resnick, Lauren B. "A Developmental Theory of Number Understanding." In The Development of Mathematical Thinking. Ed. Herbert P. Ginsburg. New York: Academic Press, 1983.

Romberg, Thomas A., and Carpenter, Thomas P. "Research on Teaching and Learning Mathematics: Two Disciplines of Scientific Inquiry." In Handbook of Research on Teaching. Ed. Merlin C. Wittrock. New York: MacMillan Publishing Co., 1986.

Schatzman, Leonard, and Strauss, Anselm. Field Research: Strategies for a Natural Sociology. Englewood Cliffs, N. J.: Prentice Hall, Inc., 1973.

Schoenfeld, Alan H., ed. Cognitive Science and Mathematics Education. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987a.

Schoenfeld, Alan H. "Cognitive Science and Mathematics Education: an Overview." In his Cognitive Science and Mathematics Education. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987b.

Schoenfeld, Alan H. "What's all the Fuss about Metacognition?" In his Cognitive Science and Mathematics Education. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987c.

Sells, Lucy. "Mathematics - a Critical Filter." The Science Teacher 45, no. 2 (February, 1978): 28-29.

Sharan, Shlomo. "Cooperative Learning in Small Groups: Recent Methods and Effects on Achievement, Attitudes, and Ethnic Relations." Review of Educational Research 50, no. 2 (Summer 1980): 241-271.

Shulman, Lee S. "Psychological Controversies in the Teaching of Science and Mathematics." The Science Teacher September 1968: 34-38, 89-90.

Silver, Edward A. "Foundations of Cognitive Theory and Research for Mathematics Problem-Solving Instruction." In Cognitive Science and Mathematics Education. Ed. Alan H. Schoenfeld. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987.

Skemp, Richard P. The Psychology of Learning Mathematics. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987.

Skemp, Richard P. Intelligence, Learning and Action. Chichester: John Wiley and Sons, 1979.

Slavin, Robert E. "Cooperative Learning and Individualized Instruction." Arithmetic Teacher April 1985: 14-16.

Smith, Donald E. The Progress of Arithmetic in the Last Quarter of a Century. Boston: Ginn and Company, 1923.

Steen, Lynn Arthur. "A 'New Agenda' for Mathematics Education." Education Week May 11, 1988, 28.

Thorndike, Edward. The Psychology of Arithmetic. New York: The MacMillan Company, 1922.

Tobias, Shelia. Overcoming Math Anxiety. Boston: Houghton Mifflin Company, 1978.

Usiskin, Zalman. "Conceptions of School Algebra and Uses of Variables." in The Ideas of Algebra, K-12. Reston, Virginia: The National Council of Teachers of Mathematics, 1988.

Van Dormolen, Joop. "Mathematics Learning and Learning Mathematics." Columbus, Ohio: ERIC Document Reproduction Service ED 133 184, 1975.

Van Hiele, Pierre M. Structure and Insight: A Theory of Mathematics Education. Orlando: Academic Press, 1986.

Van Lehn, Kurt. "On the Representation of Procedures in Repair Theory." In The Development of Mathematical Thinking. Ed. Herbert P. Ginsburg. New York: Academic Press, 1983.

Wax, Rosalie H. Doing Fieldwork: Warnings and Advice. Chicago: The Univ. of Chicago Press, 1971.

Weaver, J. F. "What Research Says: The Learning of Mathematics." School Science and Mathematics 87, no. 1 (January 1987): 66-70.

Wenger, Ronald H. "Cognitive Science and Algebra Learning." In Cognitive Science and Mathematics Education. Ed. Alan H. Schoenfeld. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1987.

Wirzup, Izaak. "Breakthroughs in the Psychology of Learning and Teaching Geometry." In Space and Geometry: Papers from a Research Workshop. Ed. J. L. Martin. Columbus, Ohio: ERIC Document Reproduction Service, ED 132 033, 1976.

Wooten, William. SMSG: The Making of a Curriculum. New Haven: Yale University Press, 1965.

APPENDICES

APPENDIX A

INFORMATION ON FIRST YEAR'S COURSE

COURSE CONTENT

1. Solving Linear Equations of One Variable

   Topics covered: addition principle, multiplication principle, equations containing parentheses, word problems, simple formulas.

2. Polynomial Expressions

   Topics covered: identification of polynomials, addition, subtraction, and multiplication of polynomials.

3. Factoring Polynomial Expressions

   Topics covered: finding common factor, factoring differences of squares, cubes, factoring trinomial expressions, general strategies of factoring, solving equations by factoring.

4. Graphs of Straight Lines

   Topics covered: Cartesian coordinate system, plotting points, graphing equations of straight lines, x- and y-intercepts, slope of line, slope-intercept form of equation, finding equation from slope and point.

5. Systems of Linear Equations

   Topics covered: solving by substitution, solving by addition method, word problems.

6. Statements of Inequality

   Topics covered: solving inequalities of one variable, graphing inequalities of one and two variables.

7. Fractional Expressions and Equations

   Topics covered: multiplying and dividing fractional expressions, finding LCM, adding and subtracting fractional expressions, solving fractional equations, word problems, proportions, formulas involving fractional expressions, division of polynomial expressions.

8. Radical Expressions and Equations

   Topics covered: square roots and radical expressions, multiplying, factoring and simplifying radical expressions, division, rationalizing denominator, addition and subtraction of radical expressions, Pythagorean formula, distance formula, solving equations with radical expressions.

9. Quadratic Equations

   Topics covered: solving by factoring and completing the square, quadratic formula, word problems, graphing quadratic equations.

10. Miscellaneous Topics

    Topics covered: scientific notation, set notation, functional notation.

DAILY CLASS SCHEDULE

(times approximate)

9:00 - 9:20	Lecture
9:20 - 9:50	Practice problems
9:50 - 10:20	Lab or games
10:20 - 10:35	Break
10:35 - 11 00	Lecture
11:00 - 11:30	Practice problems
11:30 - 12:00	Group and individual quizzes
LUNCH	
1:30 - 1:50	Lecture
1:50 - 2:20	Practice problems
2:20 - 2:40	Problem solving lab
2:40 - 2:55	Break
2:55 - 3:20	Lecture
3:20 - 3:50	Practice problems
3:50 - 4:30	Individual help, start on homework

The above schedule served as a guideline. On certain days, the lectures required more than the allotted 20 minutes to complete and it was necessary to delete one of the lab periods in order to fit in the necessary practice periods.

INDEX OF LEARNING STRATEGIES

SAMPLE FIELD NOTES

The following is a transcription of a section of field notes from the second day of the second week of the class. The names of the students and the instructor have been changed. The instructor is Ellen, and the researcher is Anne. Observer comments are designated by "OC."

The instructor is just finishing a lecture, following which, the students work together in groups of four at separate tables.

Ellen has been lecturing on a section. Then she says, "Let's start number 4. Oops! I'm ready to start number 4, but that doesn't mean that everybody else is ready. Are there any questions?" Silence. Ellen goes over to look Jane in the face. "Is that a grin? Jane isn't sure she likes this yet." Jane giggles.

Lori is sitting with her chair back from the table with her arms folded.

Beth leans over and points something out to Sue. Wanda is leaning back and sitting on her hands.

Ellen is defining the term constant. "The length of this math class is a constant 6 hours." Shirley: "Oh, gosh, I know!" Tracy: "Or longer."

Bob to Tracy: "Could you help me?" He seems willing to take help from her. He does not like to get help from me. Tracy looks at his work and questions what he is doing. "Why do you want to do this?"

Bob: "It's my paper." He says it with a "smarty-pants" kind of voice.

Anne: "Are you going to let him get away with murder?"

Tracy: "Cause his biceps grew more than mine." (This is in reference to a math lab they did recently.) She can joke and banter with Bob successfully. He will not allow me to joke in this way with him.

Ellen to Shirley: "Are you done with the pink sheet already?"

Shirley: "It's easy!"

Wanda: "I hate it when people say it is easy."

Ellen to Wanda: "I have a question for you. Is it hard, or are you just slow?"

Wanda: "I'm just slow."

Ellen: "You have to remember, you haven't done this before. It should take time."

Sue: "I'm slow."

OC: I wonder if, besides giving Wanda some encouragement, Ellen was hoping that Shirley was listening and would learn not to say such things.

Sue is asking Beth questions about calculations and Beth is always ready with the answer.

OC: I wonder if Sue is using Beth the way she used me in the scientific notation dominoes to get a ready answer without having to go through the final struggle on her part.

Wanda drew a fancy coordinate system on her paper, drawing in all the grid lines the way they are shown in the book. By late afternoon she was just drawing in the axes and marking the scale on each.

QUIZ

Today the front side of the quiz was individual and the back was a group effort. Both sides had to be done without notes, but you could ask someone about the problems on the back. After about 5 minutes everyone was working on the back.

Shirley knows the rules about what to do if you finish early (don't hand it in, find someone to help). But she holds up her quiz as if to hand it to Ellen. Ellen was not looking, so she put it down in the center of her table.

Sue is helping Wanda. Wanda is erasing her whole problem. Sue: "No, that's right." Wanda looks up and laughs.

OC: I found this interesting, because this probably happened just like that when I was helping Sue many times. The first week I noticed that when students didn't get the right answer, they would erase all the work they had done. It seems to me that this isn't happening as much any more, but I need to look consciously for this to see if it is true.

Shirley is telling Wanda how to complete the square on a problem from the group part of the quiz. She works through the whole problem and wants to get Wanda to put the answer down in the space provided. Wanda, however, goes back and reworks the problem herself. Sue gets up and stands over Wanda's back and helps.

At 11:50 I notice that everyone is done with the quiz and working on other things except Valerie and Wanda. Valerie and Wanda are back on the first page finishing it up. Beth has put her stuff away. She tells Sue, "I don't want to start on my homework." A few minutes later, however, I notice she has taken her book out of her backpack and is working in it.

I was helping Steve with the factoring problem. $125x^6 - 9z^3$ I was trying to get Steve to identify A and B and plug them in the formula $A^2 - B^2$. He just couldn't make the connection and I was not giving him the answer he wanted. Ellen had just told everyone to pack up and go for lunch. He finally just put the problem away in frustration and said he'd come back to it later.

OC - I should have sensed that we were getting nowhere and either told him how to do it or let him put it aside. I cannot use the same teaching techniques on these students as I can on my calculus students. If I make leading suggestions to these students, chances are they won't pick up on them. I wonder if my teaching technique is based more on assuming comprehension on the part of the student. If they operate only from a vague pattern recognition, too much noise may make it difficult for them to understand.

APPENDIX B

INFORMATION ON SECOND YEAR'S COURSE

Table 11. Student Checking Response on Quiz Problems

Problem	Paul	Alice	Peter	Sue	Don	Bill	Sally	Elaine
6-19 #1	E-N	Na	C-C	E-N	Na	C-C	C-N	C-N
6-23 #4	E-N	E-C	E-C	E-N	Na	E-N	C-N	C-Ce
6-26 #3	E-N	E-C	C-N	C-N	Na	C-Ce	E-N	E-N
6-26 #9	C-N	C-N	C-N	C-N	Na	C-Ce	C-N	C-N
7-6 #2	Na	C-C*	Na	Na	B	Na	Na	Na
7-7 #2	Na	E-N	E-N	Na	--	C-Ce*	Na	Na
7-10 #5	E-N	E-N	C-C*	E-N*	B	C-C*	C-C	Na
7-10 #7	Na	C-N	C-C*	C-N*	B	C-C*	E-C	Na
7-10 #12	Na	C-Ce*	C-Ce	E-N	Na	C-Ce*	Na	E-Ce*
7-13 #6	C-N	C-N	C-N	C-N	E-N	C-N	Na	E-C*
7-13 #7	E-N	C-N	C-C*	E-N	B	E-C	Na	C-C*
7-15 #6	E-N	E-N	C-N	E-N	Na	C-C	C-N	C-C
7-16 #5	C-N	C-N	C-N	E-N	E-N	C-N	E-N	E-C*
7-16 #6	E-N	C-N	C-C	E-N	E-N	E-N	E-N	E-N
7-17 #6	Na	E-C*	E-N	Na	E-N	Na	E-N	E-N
7-17 #7	E-N	C-N	C-N	E-N	Na	C-N	C-Ce*	E-N
7-17 #8	C-N	C-N	C-C	E-N	Na	E-N	C-N	E-N
7-17 #10	Na	E-N	C-C*	Na	Na	C-N	E-N	E-N
7-17 #12	E-N	E-N	C-C	E-N	E-N	C-N	E-N	E-N
Ck Index	0/5	3/10	4/9	0/7	0/2	6/8	3/5	4/6

Key

C-C = Correct solution - Check
C-Ce = Correct solution - Error in Check
C-N = Correct solution - No Check
E-C = Error in solution - Check
E-N = Error in Solution - No Check
Na = Problem started, No Answer Achieved
B = Problem Left Blank
* = Student Comment Follows Check
--- = Instruction on Checking Has Taken Place

6-19 #1 - Linear Eq.
6-23 #4 - Linear Eq.
6-26 #3 - Linear Eq.
6-26 #9 - Quadratic Eq.

7-6 #2 - Fractional Eq.
7-7 #2 - Fractional Eq.
7-10 #12 - Fractional Eq.

7-10 #5 - Simultaneous Eq.
7-10 #7 - Simultaneous Eq.

7-13 #6 - Radical Eq.
7-13 #7 - Radical Eq.
7-17 #7 - Radical Eq.

7-15 #6 - Quadratic Eq.
7-16 #5 - Quadratic Eq.
7-17 #7 - Quadratic Eq.
7-16 #6 - Quadratic Form.
7-17 #8 - Quadratic Form.
7-17 #10 - Find Equation
7-17 #12 - Complete Sq.

Table 12. Time Schedule for Classroom Checking Activities

Mon	Tues	Wed	Thurs	Fri
(June) 15 Checking Lab	16	17	18	19 6-19 #1
22	23 6-23 #4 Quadratic Eq.	24	25 Simult. Eq.	26 6-26 #3 6-26 #9
29	30	(July) 1 Intro. to Fract. Eq.	2 Intro. to Roots Group Quiz #2	3 No Class
6 Equation Strips 7-6 #2	7 7-7 #12	8 Review Sheet on Fract.	9 Intro. to Radical Eq.	10 Intro. to Quad. Eq. 7-10 #5 7-10 #7 7-10 #12
13 More Quad. Eq. 7-13 #6 7-13 #7	14	15 7-15 #6	16 7-16 #5 7-16 #6	17 7-17 #6 7-17 #7 7-17 #10 7-17 #12

Table 13. Equations Used in Pull Strip Lab Activity

1. $3x + 5x = 34 - 10$
2. $57 = 8y + 25$
3. $0 = 19n - 57$
4. $5b = 28 - 2b$
5. $12 - 3r = 3r$
6. $7y - 9 = 2y$
7. $b + 28 = 6b$
8. $4z + 2 = 2z + 8$
9. $12a - 3 = 4 - 2a$
10. $4r + 2 = 2r + 2$
11. $24x - 24 = 12 + 2x$
12. $12n + 8 = 18 - 6n$
13. $5(a - 3) = 3 - 2(a + 2)$
14. $3(1 - z) + 4(z + 2) = 0$
15. $2t = 5(t - 3)$
16. $5b + 4(b + 1) = 10 - 2(b + 3)$
17. $-3(k + 2) = 5(k + 2)$
18. $p + 2 = 9p + 18$
19. $0 = 4(2 - x) + 3(x - 1)$
20. $\frac{2y}{3} - 3 = \frac{1y}{3}$
21. $8 - \frac{5a}{7} = \frac{2a}{7} + 5$

Table 14. Equations Used in Equation Strips Lab Activity

1. $\dfrac{x + 3}{x + 5} + \dfrac{2}{x - 9} = \dfrac{-20}{x^2 - 4x - 45}$

2. $\dfrac{2x + 3}{x^2 - 5x + 6} = \dfrac{2}{x - 2} - \dfrac{5}{x - 5}$

3. $\dfrac{4}{y^2 - 8y + 12} = \dfrac{y}{y - 2} + \dfrac{1}{y - 6}$

4. $\dfrac{2}{a + 2} - \dfrac{a}{a - 2} = \dfrac{-13}{a^2 - 4}$

5. $\dfrac{x^2 + 7x}{x - 2} = 4 + \dfrac{36}{2x - 4}$

6. $x - \dfrac{5x}{x - 2} = \dfrac{-10}{x - 2}$

7. $1 + \dfrac{30}{x^2 - 9} = \dfrac{5}{x - 3}$

8. $\dfrac{1}{k^2 - k} = \dfrac{3}{k} - 1$

9. $\dfrac{r + 1}{r - 1} = \dfrac{6}{r^2 - r}$

10. $x + \dfrac{x + 3}{x - 9} = \dfrac{12}{x - 9} - 4$

11. $\dfrac{6}{a + 2} + \dfrac{3}{a^2 - 4} = \dfrac{2a - 7}{a - 2}$

12. $1 + \dfrac{3}{y + 2} = \dfrac{y + 4}{y + 2} - y$

13. $\dfrac{4x - 1}{x + 3} = \dfrac{2}{x + 3} + \dfrac{x^2 + 11}{3x + 9}$

14. $\dfrac{2x + 1}{x - 1} - \dfrac{3x}{x + 2} = \dfrac{5x - 2}{x^2 + x - 2}$

STUDENT WRITTEN RESPONSE

Shown below is a facsimile of the written quiz response of a student to problem 7-10 #12. The instructor's grading comments are included, and consist of the check marks written at the end of every line, the lines drawn through the student's check, and the written comments at the bottom of the page: "check in original equation, r = -1 makes a denominator = 0 so it is not a solution."

This example illustrates the way in which the student's work obscures the original form of the equation. The problem was originally written on the quiz paper as follows:

$$\frac{r}{(r-1)} + \frac{4}{r^2-1} = \frac{-2}{(r+1)}$$

12 Solve: $\frac{r(r+1)}{\cancel{(r-1)}} + \frac{4}{\cancel{r^2-1}} = \frac{-2(r-1)}{\cancel{(r+1)}}$ ✓

LCM: $(r-1)$

$(r-1)(r+1)$

$(r+1)$

$(r-1)(r+1)$

$r(r+1) + 4 = -2(r-1)$ ✓

$r^2 + r + 4 = -2r + 2$ ✓

$-2 \qquad -2$

$r^2 + r + 2 = -2r$

$+2r \qquad +2r$

$r^2 + 3r + 2 = 0$ ✓

$(r+1)(r+2) = 0$

$r + 1 = 0$ or $r + 2 = 0$

$-1 \quad -1 \qquad -2 \quad -2$

$r = -1$ ✓ $\qquad r = -2$ ✓

Ck: ~~$(-1)^2 + 3(-1) + 2 = 0$~~ $\qquad$ ~~$(-2)^2 + 3(-2) + 2 = 0$~~

~~$1 - 3 + 2 = 0$~~ $\qquad$ ~~$4 - 6 + 2 = 0$~~

~~$0 = 0$~~ $\qquad$ ~~$0 = 0$~~

~~works!~~ $\qquad$ ~~works!~~

check in original equation!

r = -1 makes a denom = 0 so is not a solution

only one solution! ☺

LECTURE TRANSCRIPTS DEALING WITH CHECKING PROCEDURES

This appendix presents transcripts from the classroom lectures that deal with the subject of checking. The entire discourse of those portions of the lecture sessions that were partially presented in Part II are included in the following sections, however the dialogue has been edited to present a smoother flowing narrative. The instructor is referred to as Ellen. The transcripts are arranged by type of equation and presented below in chronological order.

Early lecture on Quadratic Equations, June 23

The following transcript describes the second lecture on June 23 dealing with solution techniques to quadratic equations. The lecture material was taken directly from the textbook. The topic dealt with the process of translating word problems into equations similar to those discussed in the first lecture. At the beginning of this lesson in the text, the authors presented a list of "problem solving tips" which included, "3. Solve the equation, 4. Check the answer in the original problem."

Ellen worked through the second example in this lecture, filling in steps as she went.

Example 2 Solve this problem.

The square of a number minus twice the number is 48.

$$x^2 - 2x = 48$$

The text reported the solutions to this problem as follows:

"There are two such numbers, 8 and -6. They both check."

Ellen then proceeded to write out the checking steps. "They said 'they both check.' They didn't do it, so they left it up to us. We have x squared minus 2x = 48."

"OK. Let's check the 8 first of all. 8 squared minus 2 times 8 is supposed to come out equal to 48. The 8 squared is 64. 2 times 8 is 16. Does it work? Good. It comes out to be 48." Her check was written on the transparency as follows.

$$8^2 - 2\cdot 8 = 48$$
$$64 - 16 = 48$$
$$48 = 48$$

"OK, let's check the other one. We've got x squared minus 2x is supposed to come out to 48. Now instead of x we're going to use negative 6 squared minus 2 times negative 6 is supposed to come out to 48. Minus 6 times minus 6 is 36. Minus 2 times minus 6 is plus 12. Sure enough, it comes out to 48."

$$x^2 - 2x = 48$$
$$(-6)^2 - 2(-6) = 48$$
$$36 + 12 = 48$$
$$48 = 48$$

Ellen worked through two more examples, but did not check her answers to either of these problems. The text simply mentioned at the end of each problem that the answers checked.

Systems of Equations, June 26

The following transcript presents the second example from the lecture. Ellen worked through the problem:

$$3x + 5y = 6$$
$$5x + 3y = 4$$

to arrive at the solution: $x = 1/8$ and $y = 9/8$.

She reviewed the steps she had used. Then she continued, "We finally have a solution. Probably better check that and see if it works or not. We've got 3x plus y equals 6. And we've got 5x plus 3y equals 4. So we want to put 1/8 in place of x and -9/8 in place of y to see if we get a true statement."

"This is the same thing as 3 over 1, so I'm going to multiply numerators and get 3, and multiply denominators and get 8. This is the same thing as 5 over one. I'm going to multiply numerators and get 45 and multiply denominators and get 8. I can add those now. We're going to get 48 over 8, and that does equal 6. So we're OK here. Whew! All right. OK. Let's try over here. Maybe it won't work here. 5 times 1/8 plus 3 times 9/8 is supposed to come out to 4. Multiplying again, I'm going to get 5 over 8. Multiplying numerators that'll be 27. Denominators will give me 8. I add those up and get 12 (carry the one) I get 32 over 8. Oh, that does equal 4. OK. So I really did OK."

$$3x + 5y = 6 \qquad\qquad 5x + 3y = 4$$

$$\frac{3}{1}\cdot\frac{1}{8} + \frac{5}{1}\cdot\frac{9}{8} = 6 \qquad\qquad \frac{5}{1}\cdot\frac{1}{8} + \frac{3}{1}\cdot\frac{9}{8} = 4$$

$$\frac{3}{8} + \frac{45}{8} = 6 \qquad\qquad \frac{5}{8} + \frac{27}{8} = 4$$

$$\frac{48}{8} = 6 \qquad\qquad \frac{32}{8} = 4$$

$$6 = 6 \quad \text{OK} \qquad\qquad 4 = 4 \quad \text{OK}$$

Fractional Equations

Wednesday, July 1

The following transcription presents the second example that the instructor presented during the lecture. The solution was written on a series of transparencies as follows.

$$\frac{2x}{x^2+3x+2} - \frac{x}{x^2+x-2} = \frac{4}{x^2-1}$$

1. Find LCD

$$\begin{array}{lcl} x^2 + 3x + 2 & = & (x+2)(x+1) \\ x^2 + x - 2 & = & (x+2)(x-1) \\ x^2 - 1 & = & (x+1)(x-1) \\ \hline \text{LCD} & = & (X+2)(x+1)(x-1) \end{array}$$

2. Multiply the LCD x every term and get rid of all denominators

$$\frac{2x\cancel{(x+2)}\cancel{(x+1)}(x-1)}{\cancel{(x+2)}\cancel{(x+1)}} - \frac{x\cancel{(x+2)}(x+1)\cancel{(x-1)}}{\cancel{(x+2)}\cancel{(x-1)}} = \frac{4(x+2)\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x-1)}\cancel{(x+1)}}$$

$$2x(x-1) - x(x+1) = 4(x+2)$$

3. Solve the (usually messy) equation that we get

$$2x^2 - 2x - x^2 - x = 4x + 8$$
$$\vdots \qquad \vdots$$
$$x^2 - 7x - 8 = 0$$
$$\vdots \qquad \vdots$$
$$(x-8)(x+1) = 0$$

"I've got 2 little equations I need to solve. X minus 8 equals zero, or x plus one equals zero." Ellen then solved each equation separately.

$$x = 0 \quad \text{or} \quad x = -1$$

"I'm not done now. After all the hard work, and I'm still not done! I have step number 4 to go."

She wrote the following on the transparency below her two solutions.

4. Check: If any answer makes any denominator = 0, just throw it out.

"I can't have a zero in the denominator. "We'll have to go back here one panel, 2 panels, 3 panels back and find out what our equation looked like".

She looked through her written transparencies until she located the first line of the problem. She talked as she rewrote the original problem on another transparency. She miswrote the signs in the first denominator and did not catch this until she worked through the check.

$$\frac{2x}{(x-2)(x-1)} - \frac{x}{(x+2)(x-1)} = \frac{4}{(x-1)(x+1)}$$

"This is the equation we started with. Now I have to find our answer again." (laugh) "That's the trouble with moveable blackboards. OK. Let's try x equals 8." She mentally substituted this value in each denominator of the equation. "That would make 8 minus 2 equals 6. That's an OK thing. 8 minus one would make 7. That's OK."

At this point students pointed out that she needed a plus sign in her denominator. She rewrote:

$$\frac{2x}{(x+2)(x+1)} - \frac{x}{(x+2)(x-1)} = \frac{4}{(x-1)(x+1)} .$$

"Did I lose a sign? It's supposed to be plus? OK- like that, there. Thank you."

Ellen rechecked the value x = 8 in each denominator. "OK, let's try 8 here. 8 plus 2 is 10. That's an OK number. 8 plus one is 7. That's an OK number. Let's see, we've already checked that. We haven't checked 8 minus one, that's 7 and 8 plus one is 9."

They're all going to be OK, aren't they? They're all going to be fine. Let's check minus 1. Minus 1 plus 2 is 1. That's OK. Minus 1 plus 1 makes zero. Uh oh! That is <u>not</u> OK."

Bill had a question on where Ellen was in her checking process. She explained which expression she was checking.

"I'm checking the negative one now. The 8 is OK. I'm looking at negative one now." She mentally substituted this value into the first denominator on the left side of the equation. "That would make this into a zero. It might make it into a zero somewhere else, but all that matters is that it makes one of these denominators equal to zero. So we have to go back and we have to throw this answer out."

"Sometimes I have to throw both my answers out. That always hurts after all that hard work -- to end up having to say there is no solution. I have to throw them both out. Sometimes I get to keep them both, and sometimes I get to keep one and have to throw the other one out."

During the checking process, Ellen talked through the steps she was using, referring to the written equation, but she did not write out any of the substitutions as she discussed them. She did not go back to her written solutions on the transparency and physically cross out the one answer that was rejected.

<u>Thursday, July 2</u>

The next day the students were given the following as one of three problems on their group quiz.

solve $$\frac{m+1}{m-5} = \frac{20}{m^2-25} + \frac{2}{m+5}$$

The students were not able to solve the problem in their groups. Both Ellen and I tried to offer hints to the students without telling them outright how to solve the problem. Since this was a quiz, the students did not have access to their notes or the text. Without these resources, they were not able to remember from their pooled knowledge the correct sequence of steps that were to be used. Finally Ellen told the students to stop their work and come sit at the projection screen, where she carefully worked through the problem for them.

$$\frac{m+1}{m-5} = \frac{20}{m^2-25} + \frac{2}{m+5}$$

$$\vdots \qquad \vdots$$

$$m^2 + 4m - 5 = 0$$

$$(m+5)(m-1) = 0$$

$$\vdots \qquad \vdots$$

$$m = -5 \quad \text{or} \quad m = 1$$

After she had finished the problem, Ellen asked the students, "Am I all done?"

"Remember, we agreed we always had to check and see, if we get to keep the answers or if we have to throw them out the window. Let's look and see."

"Backing up, all the way up, we'll take a look at our least common denominator." She substituted the value m = 1 into each of the factors in the least common denominator. "OK. If m is 1, this factor is 1 minus 5 which is negative 4. That's OK. This would be 1 plus 5 which is 6. That's OK."

"But if m is minus 5, I get here minus 5 minus 5 is minus 10. That's OK. But over here, I get minus 5 plus 5. That makes a zero, and that's a NO-NO! Absolutely not. I can't have a zero, so I have to throw that one out."

She crossed off the statement m = -5 on the transparency, and said, "No zeros permitted."

"I have 2 answers, but when I go back and look at them, I don't get to keep one of them. One of them has to go out."

Sue: "Can you go back to that other sheet? I got lost."

Ellen: "OK. This one, or this one, or the one before?"

Sue: "Yeah, that one."

Ellen: "All right."

Sue: "OK. Where you check it, if any place has a zero, it means that the zero..."

Ellen: "Any time it puts a zero in that least common denominator, that means that one of those factors someplace in that denominator is going to be zero and you can just chuck it out. Or, if you want, you can go back to the original problem and just look at the denominators. Of course as soon as I put a negative 5 in this denominator, I have a zero here." She made a chucking sound. "Throw it out. I can't do that."

Sue: "So if both work, is that what you said...?"

Ellen: "No. I'm saying the negative 5 doesn't work... Ellen and Sue are talking together at this point.

Sue: "Oh, we stuck it into the, um, where we ..."

Ellen: "You can go all the way back to the first denominators and look right there, or you can look at your least common denominator. If any one of those factors turns out to be zero, you chuck your answer."

Sue: "Oh, good. If the 5 -- the minus 5 equals.."

Ellen: "The minus would make that zero, wouldn't it? And I have to throw it out. I don't get to keep it."

"OK. Let's review the steps that we took. Step 1: We found the least common denominator. Now this is for equations. Keep in mind we're talking about equations with fractions. This is different from everything else we've done when we worked with expressions. We're now talking about equations."

"That's why we took this step 2: We multiply every term by the least common denominator, and we get rid of all denominators."

"Number 3. We have to solve the equation that we get and it usually is kind of long and full of parentheses. It certainly was today."

"OK. Step 4. Check. If answer makes any denominator equal zero, you have to throw it out."

Bill had a question on Ellen's writing. She clarified what was written on the screen.

Ellen: "We have to check. If the answer makes any denominator zero, we've got to throw it out."

"I think we've been doing so many expressions that it's hard to remember about equations."

While Ellen was saying this review, she wrote the following on the transparency:

Equations

1. Find LCD.
2. x every term by LCD and get rid of all denominators.
3. Solve equation.
4. Check, if answer makes any denominator = 0, throw it out.

Radical Equations, July 9

The following transcript presents the third example given during the lecture on radical equations.

$$\sqrt{x+1} = \sqrt{2x-5}$$
$$(\sqrt{x+1})^2 = (\sqrt{2x-5})^2$$
$$\begin{array}{rcl} x + 1 &=& 2x - 5 \\ -x & & -x \\ \hline 1 &=& x - 5 \\ +5 & & +5 \\ \hline 6 &=& x \end{array}$$

When Ellen had reached her solution, she said, "OK. I am not done. I haven't checked it. Let's go back and check this and see if it works."

"The square root of 6 plus 1 is the square root of 7. Does that equal the square root of 2 times 6 minus 5? Now let's see. 2 times 6 is 12 minus 5."

$$\sqrt{6 + 1} = \sqrt{2 \cdot 6 - 5}$$
$$\sqrt{7} = \sqrt{12 - 5}$$
$$\sqrt{7} = \sqrt{7}$$

"Oh, it does work! The square root of 7 is the square root of 7. So it is a solution. We get to keep that one."

Second Lecture - Handout Sheet

The following transcript refers to the steps that Ellen used as she checked the solution to the first problem written on the handout sheet.

given: $\sqrt{x - 3} + \sqrt{x + 5} = 4$

solution: $4 = x$

After she had solved the problem, Ellen said, "4 equals to x. Now we can't assume that it is going to work, so we have to check and see. We go back and put 4 in place of x."

"The square root of 4 minus 3 plus the square root of 4 plus 5. Does that come out equal to 4? That's the square root of 1 plus the square root of 9. That's the same as 1 plus 3. That does equal 4. So it is a solution."

The check appeared on the handout sheet as follows:

Check: $\sqrt{4 - 3} + \sqrt{4 + 5} \stackrel{?}{=} 4$

$$\sqrt{1} + \sqrt{9} = 4$$
$$1 + 3 = 4$$

so $x = 4$ is a solution.

Ellen then began lecturing on the second problem on the sheet. "In the next example, we'll end up with a problem where there will be 2 answers. One is a solution and one isn't."

I was not able to record her discussion of the checking procedure on this problem since the tape recorder came to the end of a cassette just as she reached the solution. The checking procedure was written on the handout sheet as follows.

$$\text{solve} \quad \sqrt{x+1} + \sqrt{2x} = 1$$

$$\text{solution:} \quad x = 0 \quad \text{or} \quad x = 8$$

$$\text{(ck)} \quad \sqrt{0+1} + \sqrt{2(0)} = 1 \qquad \sqrt{8+1} + \sqrt{2(8)} = 1$$

$$\sqrt{1} + 0 = 1 \qquad \sqrt{9} + \sqrt{16} = 1$$

$$1 + 0 = 1 \qquad 3 + 4 \neq 1$$

$$\text{OK} \qquad \text{not a solution!}$$

Later Lecture on Quadratic Equations, July 13

The following transcript represents the second half of the lecture in which Ellen illustrated how to solve a quadratic equation derived from a story problem. This example is of interest because it represented a different type of checking that involved examining each solution to determine if it represented a reasonable answer to the given word problem.

Ellen set up and worked through an example involving a rectangular piece of paper whose four corners were cut out to allow the paper to be folded into a box. The problem was to determine the size of the square cut on each corner of a 20 in. by 10 in. rectangle that would produce a box with a bottom face of 96 sq. in. The equation for this problem was quadratic and produced two possible solutions.

$$96 = (20 - 2x)(10 - 2x)$$

solution: $$4(x - 2)(x - 13) = 0$$

$$x = 2 \quad \text{or} \quad x = 13$$

Ellen explained that the greater solution could be ignored since it was meaningless for the problem in question. This concept of rejecting an answer, not on the basis of a check, but because it did not make physical sense was yet another criterion for rejecting solutions to equations.

Ellen explained this in the following transcription.

"This equation says to me that either 4 is equal to 0, which I know never happens, or x - 2 equals 0. Well that's OK, or x -13 equals 0."

$$x = 2 \quad \text{or} \quad x = 13$$

"If we take these two numbers and put them back in our original equation, they really will check. They'll both work. However, if we go back and look at our box, they don't both make sense."

"Let's go back and look at our box again. I assure you these two numbers will work if you put them back in the original equation. However, while I can cut corners that are 2 inches by 2 inches out of here, if I cut a corner that's 13 inches out of here, when this is only 10 to start with, I think I just performed a miracle. It's not going to work."

"The point is that I cannot cut out a corner that is 13 by 13 when my whole box is only 10 by 20. So this is a logical, good answer for the equation, but it is a mighty poor answer for the story problem. Whereas the x equals 2 is both a good answer for the equation and a good answer for this kind of problem."

"Sometimes when we solve word problems, we solve the equation and we get two answers that are perfectly good answers, except that in terms of the story problem they don't make sense. If I end up with positive 2 people got on the bus or positive 3 and one-half people got on the bus, that's not too logical. Half people just don't walk on the bus. If I say I traveled negative 60 miles, we're going to pick the answer that makes sense in terms of the distance. The same thing here. These both solve the equation, but they're not both answers to the problem that we started looking at."

APPENDIX C

LEARNING STRUCTURE OF ALGEBRA

Table 15. Learning Structure of Algebra

ELEMENTS

ALGEBRA OF NUMBERS

A-1. Algebraic Notation

A-2. Algebraic Expressions

A-3. Statements of Relationship
(equations and inequalities)

ALGEBRA OF FUNCTIONS

F-1. Representations of Mathematical Relations
(graphs, equations of two variables, ordered pairs)

F-2. Properties of Relations
(functions, characteristics of graphs of functions)

HORIZONTAL, VERTICAL DIRECTIONS OF LEARNING WITHIN L.S.A.

Concept Development ↑

Level III - Integration, Application

Level II - Properties of Objects of Study

Level I - Information, Introduction of Objects of Study

Elements: (A-1) → (A-2) → (A-3) → (F-1) → (F-2)

Development of Prerequisite Skills and Concepts →

ELEMENTS OF THE LEARNING STRUCTURE

The information contained in this appendix presents a more detailed description of the learning structure of algebra than that which was provided in Chapter Eighteen. The sequencing of topics and focus of instruction are organized according to the types of prerequisite skills and levels of mathematical knowledge that are postulated to be necessary conditions for the development of appropriate levels of understanding of each sequential element in the learning structure.

The five instructional elements are examined in detail in the following sections. Descriptions are provided of the different types of objects of study and levels of mathematical conceptualization that characterize the focus of instruction within each element. Various activities are also outlined that can be used to promote appropriate learning within each instructional set of three levels of concept development. The activities and types of student behavior that are discussed in this appendix represent a <u>prescriptive</u> set of teaching/learning guidelines that can be used in the implementation of the model of classroom algebra learning that was presented in this report.

Algebraic Notation (A-1)

Overview of Element

The object of study for this element is the set of symbols used in algebraic notation. The focus of instruction is on developing the students' abilities to interpret and use this notation to represent mathematical relationships of operation and quantity. The mastering of these skills by the students, is accompanied by the parallel development of the concept for the idea of a variable as the representation of an unknown number. This element begins the student's study of the algebra of numbers.

Within this element, students investigate the symbolic representations of different sequences of mathematical operations on variables. Using these experiences, students develop notational conventions and rules of order of operations that can be used to interpret symbolic representations of complex sequences of mathematical operations.

The learning activities in this element are carried out within a hierarchy of three levels of concept development. At the beginning, or entry level, students investigate the ways that letters, or literal variables, can be used to represent unknown quantities. Students are introduced to the symbol notation that is used when discussing such objects.

At the second level, students investigate the symbolic notation that is used to represent mathematical operations with unknowns. Students gain skill in simplifying sequences of operations by such techniques as combining like terms under addition or subtraction. Students learn to use grouping symbols such as pairs of parentheses in order to indicate a specific order of operations. Students also develop skill in applying the operation of taking the root of a variable expression, and using exponential notation with variables.

At the third level of concept development, students identify a generalized set of statements concerning the various types of mathematical operations and notational conventions that they have worked with throughout the element. These properties form a set of manipulation strategies that can be applied to many different situations involving algebraic notation.

The following list presents key words representing some of the objects of study for this element, identified either as "objects" or as mathematical "operations." The two groups of words are independent listings that do not necessarily represent related objects of study.

Object		Operation	
variable	index	add, subtract, etc.	
unknown	exponent	take root of	substitute
constant	radical	evaluate	factor
expression	term	simplify	
equation	factor	combine like terms	

It is assumed that the students who begin their study of this element have developed an appropriate set of skills and concepts relating to the study of numbers. That is, they know how to apply the operations of addition, subtraction, multiplication, and division to whole numbers, integers, fractions and decimal numbers. This knowledge base is considered a prerequisite entry level to begin study in the first element of the learning structure.

Students expand and alter their present level of mathematical knowledge through their participation in activities within this first element. New information relating to the concept of an unknown or variable quantity is applied to their existing set of concepts relating to the behavior of numbers under mathematical operations, to include mathematical operations on undetermined quantities represented in general terms by letters.

The following sections describe each of the three levels of concept development in detail. Examples of student activities are included to illustrate the specific focus of instruction that characterizes this element and to illustrate the stages of concept development that are expected to occur.

<u>Level I</u>

At this entry level, students are presented with information introducing them to the concept of a variable as the representation of an unknown numerical quantity.

Specific examples are used to illustrate the ways in which different types of unknowns can be represented using algebraic notation. Such notation includes the use of letters, usually at the end of the alphabet, such as "x" to represent variables, letters such as "r" (for radius) to represent unknown measures of specific entities, letters such as "c" to represent unspecified, but constant values, and such symbols as " " to represent unique numbers.

Activities are provided in which students gain experience in modeling physical situations using these symbolic representations. For example, students may be given a list of phrases such as the following, asked to represent each quantity by a letter, and to explain their choice of notation.

A bag of marbles.
The length of a piece of string.
The length of a 12 inch ruler.
The number of trees in a forest.
The average temperature at the North Pole.
The part of a cake that is left.
The amount of money in the cash register.

The particular set of situations that are described above were selected in order to familiarize students with the idea that an unknown or variable may represent decimal numbers, fractions and negative numbers, as well as whole numbers. By involving the students in a group discussion in which they are asked to explain their choices of literal symbol for each problem, students also begin to recognize the need for a set of standard notational conventions in order to communicate effectively with each other.

Students are also introduced to the other types of symbols that are used along with literal variables in algebraic notation. Such symbols can be classified into four categories as 1) numerical quantities and variables, 2) mathematical operations on numbers, 3) relational symbols, and 4) grouping symbols. The following list presents a representative set of the types of symbols that are included in each of these categories.

Quantity	Operational	Relational	Grouping
3	$+$	$=$	( )
x	$-$	$\neq$	[ ]
a	$\div$	$<$	$\overline{\quad}$
π		$>$	

Level II

In level II, students investigate the ways in which algebraic notation is used to indicate mathematical operations on unknown quantities represented by literal variables. In such expressions many of the mathematical operations cannot actually be performed because the exact values of the numbers are unknown. In students' previous experiences with operations on numbers, all such problems could always be replaced by a single number after the appropriate calculations were performed. Closure was achieved by combining the given numbers according to the stated mathematical operations, so that, for example, the problem 3 + 5 - 2 was always replaced by its answer of 6.

The presence of literal variables in combination with mathematical operations produces statements, however, that cannot be replaced by calculated numbers. Students must learn to regard such algebraic expressions as "3 + x" as the representation of a number, rather than as a problem in addition. The fact that closure on algebraic expressions cannot be reached is an idea that differs from the students' previous experiences with operations on number.

Complex sequences of mathematical operations on literal variables produce expressions that utilize a specialized set of algebraic notation. Students develop general rules for this notation from their experiences with specific examples of various combinations of operations. These rules are then used to interpret and manipulate other examples of algebraic expressions.

Operations on Variables. One of the groups of notational conventions that students investigate concerns the various ways to symbolically represent multiplication involving variables. In many cases, this operation may be interpreted as the result of repeated addition. In such a situation, the expression x + x + x can be represented symbolically as 3x. In this example, the 3 is customarily written to the left of the variable, and the multiplication operation is represented by placing the two factors next to each other. Multiplication can alternatively be represented by placing sets of parentheses around each factor, as (3)(x), or by a raised dot placed between the factors, as $3 \cdot x$. When repeated factors are multiplied together, this operation can be expressed using exponential notation. For example, the expression $x \cdot x \cdot x \cdot x$ can be represented as x^4. It is important for students to be provided with many examples of the different types of notation used to represent multiplication, and given the opportunities to discuss the similarities and differences of this notation with their previously developed concepts of the multiplication operation.

Students are also provided activities at this level in order to develop the properties of the exponential and radical operators when applied to variables. These properties enable students to develop the skills to simplify such expressions as x^4/x^{-5}, $(y^{-2})(y^3)$, and $\sqrt[3]{y^6}$.

Complex Expressions. It is important for students to spend time in activities exploring algebraic notation in order for them to develop an understanding of both the mathematical operations involved, and the ability to read each expression fluently. Once students have become familiar with the basic uses of the symbols, they can then begin to investigate more complex algebraic relationships that require the use of grouping symbols.

For example, students can use a modeling activity as an introductory way to explore more complex symbol collections. In such an activity, the students are asked to represent written descriptions of combinations of operations using whatever symbol notation they think is appropriate. Students share their results, and through a class discussion, develop a uniform set of notational conventions. The following represents a typical sentence for this type of activity:

> Five is subtracted from some mystery number and the result is multiplied by 3. The sum of 9 and 5 times the mystery number is then subtracted from this.
>
> Using conventional algebraic notation, this can be written as $3(x - 5) - (5x + 9)$.

Such an activity should also involve the use of spoken algebra. Students must not only develop the ability to interpret in written form the appropriate sequencing of manipulations on any collection of variables but must also be able to communicate this information verbally. One activity that can be used for this necessary verbal practice is for pairs of students to take turns dictating algebraic expressions to each other. In such a situation, students also develop the ability to receive information aurally as well as visually.

Besides developing the abilities to read, write, speak and hear complex relationships of operation and quantity, students must also develop the skills required to simplify certain basic combinations of variables. These include the ability to combine like factors by adding or subtracting their numerical coefficients, the ability to apply the distributive rule to simplify such expressions as $7(9 + 3x)$, and the ability to identify common factors in such expressions as $25-10x$ and to rewrite this expression in the form of $5(5 - 2x)$.

Substitution. The concept of a variable as an unknown quantity is explored further as students investigate the process of evaluating algebraic expressions by substituting a specific numerical value for the variable. This activity can also be used to investigate the necessity for specifying the domain of the given variable in any expression, which presents to the students a slightly different aspect of their developing concept of the idea of a variable.

For example, the students may be asked to evaluate the following expressions by assigning the value of one to the variable in each case.

1) $x - 9$ 2) $\frac{2}{1 - x}$ 3) $\sqrt{3x - 10}$ 4) $x^5 + x^2$

It can be seen that this substitution produces the expression $2/0$ in problem 2 and the expression $\sqrt{-7}$ in problem 3. Since these values are not in the real number system, such an activity provides an opportunity to discuss the importance of specifying the domain of any variable that is present in an algebraic expression.

Students are also provided with activities that acquaint them with the techniques of substituting algebraic expressions for any given variable, as illustrated by the following example.

Given the expression: $Ax^2 + Bx + C$

Evaluate when x is replaced by 3x+2

Result: $A(3x + 2)^2 + B(3x + 2) + C$

This activity provides students with the opportunity to explore the depth of information that is conveyed by the original collection of symbols. The substitution in this problem utilizes the students' concept of a variable, the idea that the expression 3x+2 represents a number, and the students knowledge of the notational conventions represented by the multiplication of unknown quantities such as "A" and "x." The use of these skills and knowledge need to be brought to the students' attention through appropriate group discussions following such activities.

In summary, within level II, students expand their concept of a variable to include the notion of a domain and a set of properties describing the behavior of variables when operated on by a sequence of mathematical operations. The students learn to think of algebraic expressions as representations of numerical quantities. Students also expand their knowledge about the use and notational conventions of algebraic symbols and develop certain skills to manipulate, simplify, and evaluate certain basic types of algebraic expressions. Students also gain fluency in their abilities to communicate in symbolic notation through visual and aural means. Throughout the activities at this level, examples are provided of the ways in which algebraic notation can be used to model physical relationships between quantity and mathematical operations.

Level III

Through a set of appropriate activities, students abstract a general set of properties to describe the various types of notational conventions that they worked with in the preceding level. These properties form a set of notation and manipulation strategies that are employed whenever students are required to manipulate or simplify an unfamiliar algebraic expression.

A general rule can be established for reading the order in which manipulations in a complex expression are to be performed. This order includes the evaluation of operations inside grouping symbols, such as parentheses and radicals, before other operations are performed. Without the presence of such grouping symbols, multiplication and division are performed before addition and subtraction, and addition and subtraction are performed from left to right.

Another set of generalizations concerns the strategies to be employed when students are required to simplify algebraic expressions. Students must identify all like terms when simplifying addition and subtraction operations. Similarly, common factors must be identified before appropriate division can be performed or such factors can be factored from collections of terms. Like factors may be grouped using exponential notation or simplified in the presence of radical operators.

Since variables represent numbers of unknown value, students at this level can formally apply the properties of the real number system to situations in which the quantities to be operated on are represented by variables. Thus, the associative, commutative, and distributive properties must automatically apply in all such situations where the variable's domain is the set of real numbers.

The above generalizations are developed by the students as they participate in activities interpreting, manipulating, and simplifying complicated and unfamiliar collections of algebraic symbols. As they perform these activities, students are asked to reflect on their actions in order to develop a descriptive list of strategies. The list of these strategies includes those generalizations that were described above.

Students are also asked to reflect on the ways that algebraic notation can be used to represent relationships between quantity and mathematical operations and on the ways that the generalized manipulation strategies can be employed with unfamiliar and non-standard algebraic expressions. This type of activity enables students to begin the development of a concept of the utility of algebra as a modeling tool.

Level III provides students with the opportunity to integrate the objects of study and the focus of instruction for the element. At this level the students are required to form abstractions and higher order concepts that describe the common properties of the concepts that were developed in Level II. As in the two preceding levels, this concept development takes place as students participate in activities that are designed to highlight the properties under study.

Once students have developed the set of generalized notation and manipulation strategies, they can use such information in applied problem solving situations. This set forms part of the general set of problem solving strategies that will be developed throughout their study of algebra. By providing the students with activities from which they themselves develop the components of these strategies, students develop the necessary skills to become effective problem solvers.

Comments

It is imperative that sufficient instructional time be provided for a thorough study of the first element in the learning structure. Students must be provided with specific activities within an appropriate focus of instruction in order to develop the concepts and skills that make up this important area of instruction.

The objects of study investigated in element A-1 form a set of essential, prerequisite knowledge that is required for further study of the algebra of numbers. Too often, instructors simply assume that students possesses such skills because they have been exposed to a few introductory lectures on variables and algebraic notation. Students are expected to begin solving equations and simplifying complex algebraic expressions before they have fully developed the necessary skills and concepts to understand the very language that they are asked to use. If adequate time is spent within the first element, students will be well prepared to move with confidence through the increasingly complex topics developed in succeeding instructional elements.

Algebraic Expressions (A-2)

Overview of Element

The objects of study for this element are specific types of algebraic expressions and specific combinations of mathematical operations and procedures. The focus of instruction is on developing the students' abilities to manipulate these expressions and use the mathematical procedures in appropriate situations. The mastering of these skills by the students is accompanied by the parallel development of an understanding of the mathematical properties that characterize the objects of study. The concepts and skills of this element are investigated through student activities within three levels of concept development.

The entry level for this element consists of certain objects of study that were placed into instruction within level II of the element A-1. Within level I of the present element, these objects are presented as collections of algebraic symbols that are identified by specific symbol configurations and an accompanying name or label.

In level II, students investigate the properties of the different types of expressions in terms of their behaviors under manipulation and the contexts in which they appear. Students also investigate the ways in which specific mathematical procedures operate on algebraic expressions and the contexts in which they are used.

In level III, students apply their skills from level II to investigate unfamiliar symbol collections in applications and modeling situations. Students develop a set of manipulation strategies for general use, and further investigate the utility of algebra to represent complex relationships of operation and quantity.

The following list presents some of the key words that describe objects and operations within this element.

Object	Operation
fractional expression	factor
radical expression	simplify
polynomial	multiply
binomial	divide
trinomial	cancel common factors
factor	FOIL
least common multiple	factor trinomials

Level I

At this level, specific collections of symbols are placed into instruction. The focus of instruction is shifted from that used in element A-1. Instead of reading these symbol patterns to determine the relationships of operation and quantity expressed by each symbol, each collection is examined as a unit or entity in itself.

Students are presented with specific examples of each object of instruction and provided with a label or name to be used to identify the different types of symbol collections. The properties that students associate with each label consist of generalizations on the nature of the symbol patterns that are abstracted from collections of examples and non-examples of each type of algebraic expression.

For example, the following collections of symbols all represent fractional expressions.

$$\frac{5 - x}{7x + 4} \qquad \frac{x}{x^2 - 2x} \qquad \frac{3x + x^3}{x}$$

$$\frac{5}{x} \qquad \frac{1}{x^3 - 2x^2 + x} \qquad \frac{(8x - 9)^5}{(7 - 3x)^2}$$

By examining such a collection of examples, students are expected to be able to determine that a symbolic property of fractional expressions is that they contain algebraic symbols above or below, or on both sides of a dividing bar. Exposure to the following non-examples could lead students to develop a more specific description of those expressions that are classified as algebraic fractions.

$$\frac{40x - 16}{11} \qquad \frac{4 + 9}{20 - 6} \qquad \frac{549071}{27105} \qquad \frac{36x}{5}$$

<u>Level II</u>

The purpose of study at this level is to develop a set of properties for each type of algebraic expression that is based on generalizations of their behavior under different types of mathematical manipulations. The properties that were developed in level I are used to identify the objects of study so that they can be studied further as they are placed in specific contexts in conjunction with other algebraic symbols and mathematical operations.

Specific collections of algebraic symbols such as fractional expressions are studied as they are combined under the operations of addition, subtraction, multiplication and division. As students observe the behavior of these expressions under various operations, they extend their previous concepts concerning the behavior of numerical fractions under mathematical operations. Appropriate activities are designed to draw students attentions to the similarities and differences that exist between numerical and algebraic fractions. From their experiences with these objects of study at this level, students continue to develop the set of properties that they began to formulate in level I to describe the concept of an algebraic fraction.

Other collections of symbols such as binomials and factors are examined. These objects of study are identified because of their reoccurring use as objects of manipulation in certain mathematical procedures. The properties of these objects are developed as students observe the different types of symbol contexts in which they appear, and relate their presence to specific types of mathematical operations.

In level II, students are also introduced to collections of manipulations, procedures, and algorithms that are used in the manipulation of specific symbol patterns. Through appropriate activities, students develop properties for these objects of study in terms of the types of expressions with which they are used and the types of changes that occur in the expressions that are being operated on.

For example, the division operation takes on several different forms when used with various types of expressions. In the following example, the division of a trinomial by a binomial expression is performed using an algorithm similar to the algorithm used with the long division of numbers.

$$
\begin{array}{r|l}
 & \underline{3x \quad + 8 \qquad\qquad} \ \text{R } 7x \\
x^2 - 2x & 3x^3 + 2x^2 - 9x \\
 & \underline{3x^3 - 6x^2} \\
 & \qquad\quad 8x^2 - 9x \\
 & \qquad\quad \underline{8x^2 - 16x} \\
 & \qquad\qquad\qquad 7x
\end{array}
$$

A different procedure is required to perform division with fractional expressions as illustrated by the following problem.

$$\frac{x^2 + 4x + 4}{x - 3} \div \frac{x + 2}{x^2 - 9}$$

This example employs the division process that is used with fractional numbers. The fractional expression to the right of the division sign is inverted and the division operation is changed to multiplication. This operation is then performed by multiplying together the algebraic expressions in the numerators and denominators. In this particular example, this is accomplished by factoring each trinomial expression and then cancelling the common binomial factors that occur in the numerator and denominator.

$$\frac{x^2 + 4x + 4}{x - 3} \div \frac{x + 2}{x^2 - 9}$$

$$\frac{x^2 + 4x + 4}{x - 3} \cdot \frac{x^2 - 9}{x + 2}$$

$$\frac{\cancel{(x + 2)}\ (x + 2)\ \cancel{(x - 3)}\ (x + 3)}{\cancel{(x - 3)}\ \cancel{(x + 2)}}$$

$$(x + 2)\ (x + 3)$$

$$x^2 + 5x + 6$$

The above example illustrates how complex the division operation can become when used with certain fractional expressions. By analyzing their work, students can see that such manipulations involve a knowledge of the basic division process with fractional numbers, a knowledge of factoring techniques for trinomial expressions, and a knowledge of the multiplication operation on binomial expressions.

This example also illustrates the amount of sequential knowledge that must be developed within level II. Prerequisite instruction must be presented on the use of mathematical operations with polynomial expressions before multiplication and division of fractional expressions can be introduced. Students should be provided opportunities to reflect on their experiences as they progress through such a learning sequence. The work with polynomial expressions gains a new set of behavioral properties when it is incorporated into the multiplication and division procedures for fractional expressions.

It is important that instruction be presented in such a way that students are able to realize the interconnectedness of the information and skills that they are developing within this level. Although much information must be presented in a sequential order, each topic is carried forward when its skills and knowledge are incorporated into a later topic. Students must be provided examples that enable them to continually enlarge and develop the behavioral properties for the expressions and procedures under study.

Besides developing properties for specific algebraic expressions and procedures, instruction in level II is also designed to provide students with appropriate manipulation skills. Not only should students be presented with experiences from which they are able to develop sets of properties for the objects of study, but they should also be provided with opportunities to practice specific types of manipulations in order to master these skills.

Students are able to apply their mastery of manipulation skills in unfamiliar situations through their developed knowledge of the properties of expressions and mathematical procedures. Algebraic manipulations are thus utilized with understanding rather than executed through the use of a collection of memorized sequences of symbol changes.

Level III

Once students have developed appropriate properties for the algebraic expressions and procedures within this element and have developed skill in manipulating these objects, they are at an appropriate level of knowledge to use this information with unfamiliar types of examples. Level III exposes the students to situations in which they are required to apply their knowledge to unfamiliar expressions that are presented out of the learning context in which they were previously encountered. Activities are presented in which students model physical situations by manipulating symbolic representations using the skills and knowledge developed in level II.

These experiences provide students with examples from which they can extract general statements regarding manipulation strategies to be used with any type of algebraic expression. For example, the word "simplify" is extended from its use in level III of the first element to cover such procedures as combining fractional expressions, identifying common factors by grouping terms, and factoring or multiplying together polynomial expressions.

The general properties of the operations of addition, subtraction, multiplication and division are extended to cover the ways in which these operations are used with various types of algebraic expressions such as fractions and polynomials. The overall goal of the work within this element is perceived as being directed to increasing the students' abilities to manipulate very complex collections of algebraic symbols in order to express the result of such manipulations in its simplest form. Students extend their concept of the utility of algebra to include the new power of manipulation that is developed in this element.

Appropriate activities are presented in level III that require students to integrate the field of study within the whole element. At this level of concept development, students reflect on the types of activities, skills, and knowledge that they have been employing as they progressed from one level of development to the next. Such activities enable students not only to relate their learning experiences but also to develop critical higher order thinking skills.

Statements of Relationship (A-3)

Overview of Element

The objects of study for this element consist of equations, and inequalities of one variable that are identified as algebraic sentences that contain one of the relational symbols. The focus of instruction is on developing the students' ability to interpret and solve equations and inequalities. In order to develop this skill, students must develop concepts for equations and solution sets and develop an understanding of the ways in which the properties of the relational statements affect the solution processes.

The entry level for this element consists of certain objects of study that were placed into instruction within level II of the notational element A-2. Further investigation of these specific types of symbol collections was postponed until they could be studied within their own focus of instruction. In level I of the present element, students begin a further study of the collections of algebraic symbols containing some kind of relational symbol by investigating the specific symbol patterns that make up these objects of study.

In level II, students investigate the properties of the objects of study in order to develop concepts for the ideas of equality, and inequality, and solution set. These properties are used to determine the kinds of mathematical manipulations that may be performed on algebraic statements of relationship. At this level, students also develop specific skills for solving different kinds of equations, such as linear, quadratic, fractional, and radical equations.

In level III, students apply the knowledge developed in level II in order to solve complex equations and inequalities. Students model physical situations by constructing various algebraic statements which are then manipulated in order to achieve a solution to the original problem. These applied problems provide students with experiences from which they are able develop a generalized set of solution strategies. Such activities also enable students to expand their concept of the utility of algebra to apply to the manipulation and solution of a wide range of problems.

The following list presents some of the key words that describe objects and operations within this element.

Object	Operation
equality	solve
equation	substitute
inequality	check
equivalent equation	
solution set	
variable	
true statement	
numerical identity	
addition property of equality	

Level I

This level is used to place the objects of study into instruction. Students learn to identify specific collections of algebraic symbols, called algebraic statements of relationship, by the type of relational symbol that is present in each statement. Students develop a set of properties to describe the specific symbol configurations that make up such statements; i.e., there must be at least one quantity symbol to both the left and the right of a relator symbol, such statements must include at least one variable term, etc.

Level I of this element is characterized by a shift in focus of instruction from the entry levels of the two preceding elements. Students no longer focus their study on individual symbols or on specific algebraic expressions. Instead, their attention is directed to the identification of the presence of a relator symbol within a collection of algebraic symbols. The following examples are illustrations of the objects of study that are addressed in this element.

$$x - 3 = 4, \quad 2 - x + 2x = 9x + 3, \quad 3x + 5 > x + 7,$$
$$2x^2 - 4x + 3 = 0, \quad \frac{5 - x^3}{2 - x} = 3 + \frac{5}{2 + x}$$

Students investigate the above examples as well as the non-examples listed below in order to develop level I concepts for the symbolic representations of algebraic statements of relationship.

$$\sqrt{9} = \sqrt{3 \cdot 3} = 3, \quad 8x - 4x^2 = \;, \quad 0 \neq 25, \quad 5 = 5, \quad 3 < 7,$$
$$3x - 9 + 2x + 7 = 3x + 2x - 9 + 7 = 5x - 2$$

Level II

Concept of Equality. At this level students investigate the concept of equality (or inequality) that is implied by the collections of symbols identified in the previous level. (The discussions that follow will focus only on statements of equality for brevity of presentation.) The following activity provides an example of instruction at this level.

Students are presented with a group of equations and told to perform the following activities with each of them.

a) $3x - 9 = 5$ b) $x + 2 = 0$ c) $4 = \frac{1}{x - 1}$

d) $x^2 = 3$ e) $-9 = x^2$

1. What do you produce if you assign x the value of -1?
2. Can you guess what value of x makes each statement true?
   Is there more than one value for each equation?
   Are there some equations that cannot be made true statements?

The above activity provides experiences from which students can begin to assemble a set of properties for the concepts related to algebraic equations. Class discussion is used to investigate the idea that the equal sign in each statement represents a symbolic way of indicating that the expressions on each side of this sign are taken to represent the same quantity. This representation of equality becomes apparent as the students replace the variable in each equation by certain numbers.

Students use these observations to begin the formulation of a definition for the solution set of an equation. The presence of a true statement under the substitution of certain values into the original equation leads students to adjust their previously developed properties for the concept of a variable to include the special case when it is present in an equation.

The above example is an illustration of the type of activities that are appropriate for beginning instruction in level II. Care should be taken to present, along with standard examples, other examples which are non-typical, as well as non-examples. It is important that students experience equations whose solutions are fractions, decimal numbers and negative numbers, as well as whole numbers.

Solving Equations. The above example provides an introduction to investigations into the solution process. Students can see that guessing the solution set to any equation may not be easy to accomplish if the equation contains complex algebraic expressions. Students can use the same set of equations as described above in order to begin this type of investigation, as illustrated by the following activity.

1. What happens if you add (or subtract) the same number from each side of the equation?
   What can you say about the new equation you produce?
   What do you produce if you add a number to only one side of the equation?

2. What happens if you multiply (or divide) each side of the equation by the same number?
   What happens if you multiply just one term on each side of the equation by the same number?
   What can you say about the new equations that you produce?

3. Are there any other operations that can be used with these equations that will produce a new equation that represents the same relationship?

4. Can you generate some rules to describe what works and what doesn't work?

This activity introduces students to the concept of equivalent equations. Once students have grasped the basic rules for manipulating equations, they are ready to begin investigating solution techniques for specific types of equations. The goal of solving an equation, that is, "getting x all by itself on one side of the equation," can be introduced by an activity that allows students to solve simple linear equations by applying the rules of manipulation that they previously generated. Such an activity provides the experiences from which a more formal investigation into the solution process can be conducted.

As was found in level II of the element A-2, much of the knowledge and skills developed within level II investigating equations is of a sequential nature. Once the solution techniques for linear equations have been developed from the information previously assembled at this level, students are ready to investigate solution techniques for other types of equations containing fractional or radical expressions.

Up through the time that solutions to linear equations are investigated, all the information that has been developed within the first two levels of this element can be based solely on knowledge that was developed within the element A-1. However, once solution techniques for fractional and radical equations are investigated, students will be required to use prerequisite knowledge that was presented within level II of the element A-2.

Because of this staggering of prerequisite knowledge, it is possible to begin instruction in element A-3 before all instruction is completed in the second element. In such a situation, instruction would be proceeding concurrently within two elements. However, such instruction should always be distinguished by the different focus placed on each set of objects of study. The learning structure can be implemented by many different overlapping of elements as long as the prerequisite structures are preserved, and the appropriate focus of instruction is employed.

Besides the development of concepts for equations, solution sets, and the solving process, students spend time within level II developing skill in utilizing the solution techniques for different types of equations. These techniques provide students with additional examples of contexts in which particular algebraic expressions and procedures are utilized. The process of solving different equations provides students with examples of the ways in which the knowledge developed within element A-2 can be applied.

It is important to provide students with opportunities to reflect on the similarities and differences that exist between the manipulations that are performed on complex equations and the manipulations that were used on algebraic expressions of a similar nature in the preceding element. The conditions of equality imposed by the presence of the equal sign places restrictions on the types of manipulations that can be used to simplify and solve equations. Activities like the following can be used to compare the types of manipulations required for similar equations and algebraic expressions.

a) Simplify:

$$\frac{x^2}{x+4} - \frac{4}{x+4}$$

b) Solve:

$$\frac{x^2}{x+4} = \frac{4}{x+4}$$

1. What similarities do you see between a) and b), what differences are there?

2. Discuss the reasons why different sequences of manipulations are used for each problem.

<u>Checking</u>. The checking process is introduced to students after they have become familiar with the solution techniques required for linear equations. Appropriate activities involving checking enable the students to continue to enlarge their sets of attributes representing the concepts of equation, solution set, and variable, since the checking process provides a symbolic link among these ideas.

Students should be provided with activities involving situations in which the checking substitutions produce non-true statements as well as numerical identities. Such experiences illustrate the way in which the checking process can be employed as a tool for monitoring the accuracy of manipulations used in the solution processes.

The checking process takes on additional meaning with fractional and radical equations. Students should be provided with activities involving these types of equations in which specific checking examples are employed for the different purposes of detecting calculation errors, identifying mathematically non-valid solutions, and verifying the validity of other solutions. It is important that students be provided opportunities through class discussions and appropriate activities to reflect on the different purposes for which checking is used and the mathematical significance of each type of process. The use of checking to identify the presence of extraneous roots in the solutions of certain radical equations can also provide students with the opportunity to investigate the mathematical reasons for this particular occurrence.

Modeling activities should be presented with each type of equation that the students study within level II. Such experiences provide them with practice in translating such situations into algebraic symbols and also provide specific examples of practical applications of the topics under investigation.

Level III

Students begin their study at this level by exposure to unfamiliar types of problems that are taken out of the context in which specific solution techniques were investigated. These problems may include purely abstract equations, as well as problem solving situations that involve modeling.

These activities provide the students with experiences from which they are asked to identify and abstract a set of general solution strategies. This set of generalized strategies includes such items as identifying the type of equation, using substitution to produce a standard type of equation, reducing the noise in the equation by simplification of its expressions, combining like terms, factoring out the variable, placing all terms containing the variable on one side, and manipulating the equation until it is in the form of a linear equation.

At this level, students also develop generalizations concerning the behavior of different types of equations. For example, the number of values in a solution set is related to the highest power of the variable in the original equation. Solutions to linear and quadratic equations are always mathematically valid. The checking process has two uses with different types of equations; to determine the accuracy of the solution process and to determine the mathematical validity of all solutions.

Students engage in activities that enable them to form an overview of the objects of study in this element and integrate the field of study, relating new information to the mental structures developed in preceding elements.

Students also develop a set of generalizations on the modeling processes that they have been using throughout the first three elements of the learning structure. Modeling is perceived as a process that abstracts certain aspects of quantity and relationship from a physical situation and represents these by collections of algebraic symbols. This symbolic representation is then manipulated through algebraic techniques to identify a specific relationship. The algebraic solution is finally related back to the original physical situation in order to provide an answer to the problem.

Students use this analysis of the modeling process in order to develop a set of general modeling strategies. Throughout these activities students are continually called upon to utilize higher order thinking skills such as analyzing, finding patterns, generalizing, and making conjectures.

Representations of Relations (F-1)

Overview of Element

The objects of study for this element are various systems of representation that can be used to display relations or mappings between two variables. The focus of instruction is on developing the students' ability to represent specific relations using algebraic symbols, graphing techniques, and other systems of notation. The mastering of these skills is accompanied by the parallel development of the idea of a mathematical relation as the mapping of values from one set to another by some form of rule.

At the entry level for this element, students are presented with specific examples of ordered pairs of numbers in order to explore the concept of a relation or mapping that transforms one set of values into another. Students investigate the domain and range of these mappings and investigate the ideas of dependent and independent variable.

In level II, students explore several different representational systems that are used to indicate the nature of the relation. Students are introduced to the Cartesian coordinate system, and the techniques of plotting ordered pairs of numbers in this system. Students investigate ways to express relations represented by tables of values using algebraic notation. Students also explore ways to express relations that are given in one representational system in the notational format of another system.

In level III, students examine the different representational systems of algebraic notation, coordinate graphing, and the listing of ordered pairs in terms of the types of information about the relation that each system can display effectively. Students investigate which types of relations lend themselves best to particular representational systems and investigate the advantages of using each system for the representation of different mathematical relations.

The following list presents some of the key words that describe objects and operations within this element.

Objects	Operations
Ordered pair	map
relation	list
dependent variable	plot
independent variable	graph
coordinates	evaluate
rule correspondence	
image set	
graph	
solution set	
formula	

Level I

Students begin their introduction to the study of mathematical relations by examining specific relations that are presented in table form. Students are asked to identify the relation from the patterns that they perceive in each table and express this relation verbally. The following provides an example of such an activity.

Given the following sets of pairs of numbers, determine a relation that transforms, or maps, the value on the left to the value on the right.

A		B		C	
20	21	-7	49	-7.8	15.6
22	23	1	1	-3	6
24	25	5	25	4.09	-8.18
26	27	10	100	12	-24

1. Is there enough information present in each example to determine a pattern?

2. If the listing of four values does not represent the entire set of values, can you decide from the numbers given what is the complete set of values that is used in each map?

3. Does the mapping produce the same range in values for the numbers on the right as those represented by the numbers on the left?

4. Select at least one other value that can be placed in the column on the left in each example and find the value that it maps into.

5. Give this mapped value to a friend and have him or her determine the value you used to produce this number. Which is easier to do - find the value on the right or on the left? Can you arbitrarily select one of the two values in each pair? Do the restrictions on the allowed values on the left affect your ability to arbitrarily pick any number for this column?

The above exercise serves to introduce students to the concept of a relation or mapping. The questions can be used to introduce the concepts of domain and range. The limited number of values present in each example serve as a motivation for determining a more precise method of representation for all the components of a mathematical relation.

Students are also presented with examples of relations that model physical situations as in the following problem.

> At the beginning of a trip, a car's odometer read 7590 miles. After the first hour of travel, the reading was 7650. Three hours after that, the reading was 7830. At the end of the trip, which lasted a total of five hours, the odometer read 7890.
>
> How far did the car travel? Was there a relationship between the length of time driving, and the distance traveled? Represent this by a table of values, and describe the relationship.

This example can also be used to discuss the ideas of dependent and independent variables. Students are asked to indicate which sets of numbers represent each type of variable and to explain their answer. They are also asked to determine the domain and range of these variables. Given the nature of the problem, they could be asked to explain if mapping into the set of negative numbers would make sense. (Can the car go in the opposite direction?) Students can also be asked to formulate an algebraic expression for the mapping. Given this relation out of the context of the problem, would its domain and range have to remain the same as for the given problem?

This and similar examples can be used to illustrate the ways in which mathematical relations are used to model physical situations. Students investigate the use of formulas in this context. The use of such models provides students with practice in perceiving patterns and relations.

Level II

In level II, students examine the rules expressing mathematical relations in more detail. Activities are provided that allow students to explore the ways that these relations can be displayed through different representational systems.

Students are presented with tables of values and listings of ordered pairs and asked to represent each relation using algebraic symbols. Such an exercise provides the motivation for the introduction of algebraic equations in two variables. For example, the listing {20, 21}, {22, 23}, {26, 27} produces a situation in which the second paired number is always one more than the first. This relation can be represented algebraically by the equation $y = x + 1$.

In the discussion that accompanies such an activity, students associate the x with the first number in each pair of values, and the y with the second number. These x and y values are also used to illustrate the concept of independent and dependent variables. Students are asked to specify a domain and range for the relation as it is represented by the original set of ordered pairs. They are then asked to comment on whether the algebraic equation conveys the same kind of information pertaining to the original domain and range. Students are also asked to compare the equation y = x + 1 that they generated for this activity with equations such as 5 = x + 1 and explain the difference between the two equations.

This exercise presents an introduction to the algebraic representational system for displaying mathematical relations. Students are provided with other activities in which equations in two variables are used to generate tables of values for these same relations. If the domain and range are not specified with the original equation, students are asked to determine from the specific algebraic format if any restrictions exist on the actual values of the domain and range.

Such activities provide students with opportunities to explore the ways in which the notation and concepts of the preceding element A-3 are utilized to express information within the new context of a mathematical relation. The concept of a solution set is expanded in such activities to include the representations of ordered pairs.

Students are introduced to the Cartesian coordinate system and the process of representing ordered pairs as points on the coordinate axes. Students use this plotting technique to represent the pairs of values that they generate from story problems. They also generate tables of values from equations in two variables, which are then plotted on the coordinate system. Through a series of activities, students are provided experiences in using the different representational systems as a means of displaying mathematical relations.

Students are presented activities in which they use tables of values to form graphs and graphical representations of relations from which they generate tables of values and/or equations. These activities familiarize students with the process of moving from one representational system to another.

Level III

At this level, students examine the advantages and disadvantages that each representational system presents for displaying information concerning mathematical relations. Students address the following questions by investigating different examples.

1. How easy is it to display the domain and range efficiently in each system?
2. Is the mathematical relation easier to see in some systems than in others?
3. How easy is it to predict other values in the relation in each system?
4. Are there some situations in which one representational system is more useful than others? (Is a table of values or an equation easier for finding sales tax at the sales counter?)

Students apply their knowledge of representational systems through applied problems in which they are asked to represent relations that are expressed by unfamiliar and non-standard examples. They are asked to identify the relation in each situation and select an appropriate representational system with which to display the important information. Such activities are used to develop generalizations concerning the processes required to use each system.

Students expand their concept of the utility of algebra to include its ability to represent mathematical relations between two variables. The various representational systems provide ways to use this type of information to model physical situations in order to identify and study such relations. Students also engage in activities that enable them to integrate the entire field of study and place it in a relational context to the preceding elements.

Properties of Relations (F-2)

Overview of Element

The objects of study for this element are specific types of relations. The focus of instruction is on developing the students' abilities to recognize each type of relation in either graphical or algebraic form, and to be able to sketch graphs of the relations from a knowledge of their properties. In order to develop these skills students must develop an understanding of the mathematical properties that characterize the objects of study. The information contained in this and the preceding element begin the student's study of the algebra of functions.

The entry level for this element utilizes the representations that were developed in level II of the preceding element. In level I of the present element, the algebraic and graphical representations of relations are assigned names based on their specific algebraic symbol patterns, or on the particular form of their graphical representations.

In level II, students investigate the mathematical properties that distinguish each type of relation. Many of these properties can be characterized by their particular physical appearance in the graphical representation. Functions are identified as a special class of relations in this element, but the investigation of their properties is placed in a separate element where this general category becomes the object of study.

Manipulations on specific functions and relations are also studied at this level. Techniques are introduced for solving systems of linear equations. Algebraic methods are employed to locate the places where linear and quadratic functions intersect coordinate axes when graphically represented. These manipulations are also used to investigate properties of the objects of study.

In level III, students investigate the general strategies that are employed to identify specific relations and functions in either algebraic or graphical form. Students examine what types of information can be used from a specific algebraic representation in order to quickly sketch a representative graph of this relation or function.

The following list presents some of the key words that describe objects and operations within this element.

Objects	Operations
line	sketch graph of
parabola	find intercepts
circle	solve system
function	locate vertex
continuous function	write in standard form
slope	
intercepts	
conic sections	
systems of equations	
inverse relation	

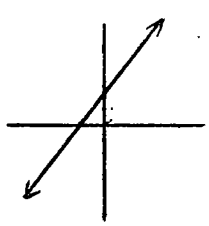
Level I

At this level, specific relations are placed into instruction. They are assigned a name and identified by a specific collection of algebraic symbols and by an accompanying graphical representation.

In the following example, specific algebraic equations and graphs introduce three different types of relations.

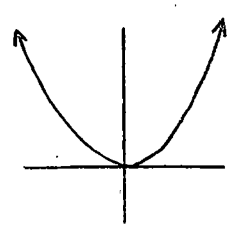
linear

$y = x + 1$



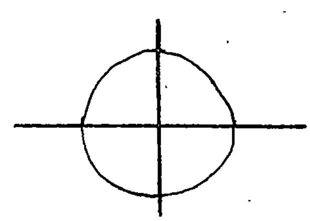
quadratic

$y = x^2$



circle

$x^2 + y^2 = 9$



Other relations that are examined may include the constant function, continuous and non-continuous functions, inverse variations, the absolute value function, the greatest integer function, conic sections, the logarithmic and exponential functions, and periodic functions. Within Level I, these objects are presented to the students as items of information. However, it may be appropriate from an instructional point of view to begin investigations of the properties of each function as it is introduced, rather than presenting an overwhelming list of all the objects of study at once.

Level II

One of the properties of relations that will be investigated at this level is that which distinguishes a function; i.e., that every value in the domain maps onto a unique value in the range. Once this property is identified as belonging to a particular relation, it will be used as part of the label identifying that relation. The focus of instruction within this element is on the investigations of properties of specific functions and relations. Although specific functions will be identified as such, the properties of functions in general will not be studied until a following element.

Instruction commences at level II as the students begin to investigate those properties of each relation that distinguish it from other types of relations. Such an investigation can begin by visual comparisons of the symbolic and graphical representations of collections of different relations.

For example, in one such activity, students are presented sets of cards that contain both the graphical and algebraic representations of various functions and relations. They are asked to sort them into piles on the basis of whatever classification scheme they feel is appropriate. Once the sort is accomplished, students are asked to generate a list of properties that describes each separate collection of cards.

The collection of cards should contain four or five examples of each category being compared. One possible grouping is to compare continuous, non-continuous, and constant functions and relations that are represented by examples containing curves as well as straight lines.

Students are also presented activities that enable them to systematically investigate specific types of relations in greater detail. Within a typical beginning or intermediate level algebra class, the primary emphasis within this element would be centered around investigations of linear and quadratic functions and relations.

The properties of linear functions and relations, or straight line graphs, are developed by investigating the types of information that are required to fully describe the location and orientation of any straight line on the

Cartesian plane. Such properties include the slope and y-intercept of any line.

Students use the properties of straight line graphs to develop the ability to quickly sketch graphical representations of such relations expressed in algebraic form. Students investigate ways to identify the graphical properties of straight lines through specific algebraic representations such as the abstract point-slope form, $y = mx + b$.

Systems of linear equations are investigated to determine if they share a common ordered pair. Students are introduced to the algebraic and graphical techniques that are used to identify possible solutions to such simultaneous equations.

Similar activities are presented to the students in order for them to investigate the properties of the quadratic relations and functions that represent parabolas. Students develop the ability to graph such relations from a knowledge of these properties. Students investigate ways to manipulate algebraic representations into a standard form that displays easily identified information concerning the graphical properties of the relation.

Students also investigate the ways in which specific relations change their graphical appearance as various parameters of the algebraic representations are altered. Such alterations also involve shifting the lines of symmetry of the curve away from the coordinate axes. Such information can be displayed in general form in such representations as $y = (x - h)^2 + k$.

The properties that are generated at this level that relate algebraic notation to particular orientations and shapes in the Cartesian plane provide information and skills that can be further investigated within the elements that focus on analytic geometry. This link to another area of mathematical study can be used to provide modeling examples that motivate further study in other areas.

The properties of linear and quadratic functions and relations are illustrated by examples that require students to model physical situations. Once students are able to identify the type of mathematical relation that characterizes each given situation, they are able to utilize their knowledge of its algebraic and graphical properties in order to make statements about the behavior of the variables in the original physical situation.

Other relations and functions are studied within this focus of instruction and level of concept development, but within other mathematics classes. Such objects are the periodic or trigonometric functions and the logarithmic and exponential functions.

Level III

At this level, students develop a set of generalizations for the types of activities that they have been engaged in throughout the two preceding levels. One of the skills that they developed was the ability to sketch characteristic graphs from specific algebraic representations. This skill is examined in terms of overall strategies that relate to the sketching process for any type of curve. Students examine these strategies through activities involving unfamiliar examples containing many different types of relations and functions.

Students gain an appreciation of the usefulness of developing a set of generic graphical representations that allow them to represent basic properties without having to specify any particular relation in terms of a specific collection of algebraic symbols. This knowledge can be used to predict the behavior of any pair of related variables once the type of relation is known.

For example, students may be presented with a physical situation in which an inverse relationship exists between the cost per item and the number of items manufactured. Students are asked to state whether the cost per item increases or decreases as the number of items manufactured increases. Students are also asked to represent this by some form of graphical representation in which the units of each variable are identified. Similar activities can be used in which a general graph is presented and students are asked to state the type of relationship using the units of measure displayed on the graph. Such activities enable students to extend their concept of the usefulness of algebra to model, manipulate, and analyze physical reality.

Structure of the Learning Structure

The logical organization of the subject of algebra that has been described by the learning structure produces a repetitive pattern in the composition of the elements. Elements F-1 and F-2 form a related structure similar to that exhibited by elements A-1 and A-2. In each case, the pair of elements provide an introduction to the notation and uses of different mathematical systems of algebra.

Element A-1 serves as an introduction to the algebra of numbers. The purpose of instruction within this element is to acquaint students with a set of notational symbols, and their uses and conventions, centered around the idea of a variable as an unknown quantity. In a similar fashion, element F-2, dealing with representational systems, serves as an introduction to the study of the algebra of functions. This element acquaints students with the notational and representational conventions related to the idea of two variables connected by a mathematical relation.

Elements A-2 and F-2 extend the study begun in the element immediately preceding each of these. In element A-2, algebraic expressions, which were present in element A-1, now become the specific focus of instruction. Students engage in activities designed to investigate properties of specific types of expressions, utilizing notation developed in the preceding element. In element F-2, students turn their attention away from the examination of the various forms of notation employed to the investigations of the properties of specific relations.

This structural comparison among elements can be extended further if additional elements relating to functions are included in the learning structure. An analysis of this type of sequencing can also provide a theoretical guideline to allow for the extension of the learning structure to include both the study of the arithmetic of numbers, and the study of calculus topics.

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