Inference procedures for fixed effects in multivariate mixed models

by Stephen Daniel Kachman

A thesis submitted in partial fulfillment of the requirements for the degree Doctor of Philosophy in Statistics
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Abstract:
Except in special cases, exact tests of fixed effects in multivariate mixed models are not available. Available approximate methods either apply techniques developed for balanced multivariate mixed models or treat unknown variance components as known. The objectives of this study were to develop a test for fixed effects in unbalanced multivariate mixed models and to examine the properties of the test.

The proposed test statistic is a generalization of the Lawley-Hotelling trace statistic. The sampling distribution of the test statistic was approximated by equating the first moment and a scalar function of the second moment of the two quadratic forms which make up the test statistic to the corresponding moments of two independent Wishart matrices. Four different scalar functions of the second moments were used to obtain four approximations to the sampling distribution. The approximations yield different critical values and, thus, give different tests. When the data are balanced, the test reduces to the usual Lawley-Hotelling test. Approximate simultaneous confidence intervals for linear functions of the fixed effects are given.

A Monte-Carlo study of the size and power of the proposed test was conducted. The model used in the simulation was based on a data set comparing different breeds of beef cattle. Three covariance structures for the random effects and seven effect sizes for the fixed effects were simulated. For each data set, all. four versions of the proposed test were conducted. In addition, an exact test, obtained by using a balanced subset of the data was made.

The Monte-Carlo based estimates of the size of the proposed test were within .003 of the nominal test size of .05. The estimated power curves of the proposed test were well above the estimated power curves of the exact test. The estimated power from the simulation agreed well with the computations based on the approximation to the non-null distribution. There appears to be no reason to prefer one method of approximating the distribution over another. Based on this simulation study, the proposed test is superior to an exact test which uses a subset of the data.

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ABSTRACT

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The proposed test statistic is a generalization of the Lawley-Hotelling trace statistic. The sampling distribution of the test statistic was approximated by equating the first moment and a scalar function of the second moment of the two quadratic forms which make up the test statistic to the corresponding moments of two independent Wishart matrices. Four different scalar functions of the second moments were used to obtain four approximations to the sampling distribution. The approximations yield different critical values and, thus, give different tests. When the data are balanced, the test reduces to the usual Lawley-Hotelling test. Approximate simultaneous confidence intervals for linear functions of the fixed effects are given.

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The Monte-Carlo based estimates of the size of the proposed test were within .003 of the nominal test size of .05. The estimated power curves of the proposed test were well above the estimated power curves of the exact test. The estimated power from the simulation agreed well with the computations based on the approximation to the non-null distribution. There appears to be no reason to prefer one method of approximating the distribution over another. Based on this simulation study, the proposed test is superior to an exact test which uses a subset of the data.

CHAPTER 1

INTRODUCTION

Multivariate mixed models arise in many animal science experiments. In these experiments, it is often of interest to test hypotheses involving the fixed effects. To illustrate some problems in the analysis of these experiments, a simple, restrictive, example will be used. Later, a less restrictive example will be used to illustrate the options that are currently available for testing hypotheses involving fixed effects.

Suppose that an experiment is to be conducted to determine the effect on their lambs' performance of different supplements given to ewes during gestation. Lamb performance will be measured by birth weight and by weaning weight. The model could include a fixed effect for the supplement fed to the ewe and random effects for the ewe nested within supplement and for the residual associated with the lamb.

Two questions which could be asked in this setting are 1) is there any effect of supplementation on lamb performance and, more particularly, 2) do the supplements have any effect on the birth weight of the lambs.

In the rest of the thesis, boldface capital letters will be used to denote matrices, boldface lower case letters will be used to denote vectors and normal lower case letters will be used to denote scalars.

Model

The model, in matrix notation, assuming p supplements, q ewes, and n lambs is given below:

$$\mathbf{Y} = \mathbf{XB} + \mathbf{Z}_1\mathbf{U}_1 + \mathbf{U}_2 \qquad (1.1)$$

where

Y is an n x 2 matrix where the ijkth row contains the birth weight and weaning weight of the kth lamb of the jth ewe that received supplement i,

X is an n x p matrix relating the effect of supplement on the birth and weaning weights of the lambs,

B is a p x 2 matrix where the ith row contains effect of the supplement i on birth weight and weaning weight,

$\mathbf{Z}_1$ is an n x q matrix relating the effects of the ewes on the birth and weaning weights of the lambs,

$\mathbf{U}_1$ is a q x 2 matrix where the ijth row contains the effect of the jth ewe receiving supplement i on the birth weight and weaning weight of her lambs,

and

$\mathbf{U}_2$ is a n x 2 matrix where the ijkth row contains the effect of the kth lamb of the jth ewe that received supplement i on its birth weight and weaning weight.

In addition, U_1 and U_2 are assumed to be distributed independently of each other. The rows of U_j, for j=1,2, are independently and identically distributed multivariate normal with mean vector 0 and covariance matrix V_j: 2x2. The number of ewes fed supplement i is denoted by m_i and the number of lambs produced by the j^{th} ewe is denoted by e_{ij}. It is assumed that m_i is greater than 0 for i equal to 1 through p and e_{ij} is greater than 0 for i equal to 1 through p and j equal to 1 through m_i. Therefore,

$$n = \sum_{i=1}^{p} \sum_{j=1}^{m_i} e_{ij} .$$

The two questions can now be written as hypotheses about the matrix of fixed effects, **B**. The first question, do the supplements fed to the ewes have any effect on lamb performance, can be written as follows:

$$H_o\colon \mathbf{CB} = \mathbf{0} \quad \text{versus} \quad H_a\colon \mathbf{CB} \neq \mathbf{0}, \qquad (1.2)$$

where

$\mathbf{C} = \mathbf{I}_p - (1/p)\mathbf{J}_p$,

$\mathbf{I}_p$ is a p x p identity matrix,

and

$\mathbf{J}_p$ is a p x p matrix of ones.

The second question, do the supplements have any effect on the birth weight of the lambs, can be written as,

$$H_o\colon \mathbf{CBa} = 0 \quad \text{versus} \quad H_a\colon \mathbf{CBa} \neq 0, \qquad (1.3)$$

where $\mathbf{a} = (\ 1\ 0\ )'$.

The question now, is how can the above hypotheses be tested. The options that are available depend on how much is known about the structure of the model.

Known Covariance Matrices

In the case where the covariance matrices are known, exact tests for both (1.2) and (1.3) are available. Rewriting (1.1) using the Vec operator yields

$$\mathrm{Vec}(\mathbf{Y}') = (\mathbf{X} \otimes \mathbf{I}_2)\mathrm{Vec}(\mathbf{B}') + (\mathbf{Z}_1 \otimes \mathbf{I}_2)\mathrm{Vec}(\mathbf{U}_1') + \mathrm{Vec}(\mathbf{U}_2'),$$

where Vec() is the usual Vec operator and $\otimes$ is the Kroneckor operator (Henderson and Searle 1979). The distribution of Vec($\mathbf{Y}'$) is given below

$$\mathrm{Vec}(\mathbf{Y}') \sim N_{2n}[\ (\mathbf{X} \otimes \mathbf{I}_2)\mathrm{Vec}(\mathbf{B}'),\ \mathbf{V}\ ],$$

where $\mathbf{V} = (\mathbf{Z}_1\mathbf{Z}_1' \otimes \mathbf{V}_1) + (\mathbf{I}_n \otimes \mathbf{V}_2)$. From the distribution it can be seen that Vec($\mathbf{Y}'$) can be described by a linear model with a known covariance matrix.

The best linear unbiased estimator (BLUE) of $\mathbf{CB}$ ($\widehat{\mathbf{CB}}$) is given by

$$\mathrm{Vec}(\hat{\mathbf{B}}'\mathbf{C}') = (\mathbf{C} \otimes \mathbf{I}_2)[(\mathbf{X} \otimes \mathbf{I}_2)'\mathbf{V}^{-1}(\mathbf{X} \otimes \mathbf{I}_2)]^{-}[\mathbf{X} \otimes \mathbf{I}_2]'\mathbf{V}^{-1}\mathrm{Vec}(\mathbf{Y}')'.$$

In model (1.1) the generalized inverse can be replaced by an inverse, because $\mathbf{X}$ is of full column rank. The likelihood ratio statistic for testing (1.2) is

$$\mathrm{Vec}((\widehat{\mathbf{CB}})')'\{(\mathbf{C} \otimes \mathbf{I}_2)[(\mathbf{X} \otimes \mathbf{I}_2)'(\mathbf{V}^{-1})(\mathbf{X} \otimes \mathbf{I}_2)]^{-}(\mathbf{C} \otimes \mathbf{I}_2)'\}^{-}\mathrm{Vec}((\widehat{\mathbf{CB}})'),$$

which has a noncentral chi-squared distribution with 2(p-1) degrees of freedom. The statistic for testing (1.3) is

$$\mathrm{Vec}(\widehat{\mathbf{CB}}\mathbf{a})'\{(\mathbf{C} \otimes \mathbf{a})[(\mathbf{X} \otimes \mathbf{I}_2)'\mathbf{V}^{-1}(\mathbf{X} \otimes \mathbf{I}_2)]^{-}(\mathbf{C} \otimes \mathbf{a})'\}^{-}\mathrm{Vec}(\widehat{\mathbf{CB}}\mathbf{a}),$$

which has a noncentral chi-squared distribution with p-1 degrees of freedom.

In general a chi-square statistic can be used to test any testable hypotheses about the fixed effect in a multivariate mixed model when the relevant covariance matrices are known. The problem is that it is unusual for the covariance matrices to be known.

Unknown Covariance Matrices

Univariate balanced data

In the univariate case with balanced data, exact tests can often be obtained when the covariance matrices are unknown by using the ANOVA approach (e.g. Searle 1971). However, there are hypotheses for which the ANOVA method will not yield an exact test. In those cases where the ANOVA method fails to yield an exact test there are two alternatives. One alternative is to develop an exact test using a subset of the available information (e.g. Khuri 1984, Seifert 1979). The other alternative is to construct an approximate F-test using either Satterthwaite's (1941) or Meyers and Howe's (1971) approximation.

In the case where the V_i's are unknown and the data are balanced, an exact test of (1.3) can still be obtained. Using that part of (1.1) related to the birth weights of the lambs yields the model

$$\mathbf{Ya} = \mathbf{XBa} + \mathbf{Z}_1\mathbf{U}_1\mathbf{a} + \mathbf{U}_2\mathbf{a}.$$

Letting $\mathbf{y} = \mathbf{Ya}$, $\mathbf{b} = \mathbf{Ba}$, $\mathbf{u}_1 = \mathbf{U}_1\mathbf{a}$ and $\mathbf{u}_2 = \mathbf{U}_2\mathbf{a}$ yields

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Z}_1\mathbf{u}_1 + \mathbf{u}_2, \tag{1.4}$$

which is a univariate mixed model. The hypothesis (1.3) becomes

$$H_o\text{: } \mathbf{Cb} = \mathbf{0} \text{ versus } H_a\text{: } \mathbf{Cb} \neq \mathbf{0}. \tag{1.5}$$

If the data are balanced, then $m_i=m$ for all i and $e_{ij}=e$ for all i and j. Therefore, n = pme. Reordering the data if necessary, the matrices X and Z_1 can be rewritten as follows,

$$\mathbf{X} = \mathbf{I}_p \otimes \mathbf{1}_m \otimes \mathbf{1}_e$$

and

$$\mathbf{Z}_1 = \mathbf{I}_p \otimes \mathbf{I}_m \otimes \mathbf{1}_e$$

where $\mathbf{1}_m$ is a m x 1 vector of ones.

The covariance matrix of $\mathbf{y}$ ($\mathbf{V}^*$) is

$$\mathbf{V}^* = \mathbf{Z}_1\mathbf{Z}_1'\sigma_1^2 + \mathbf{I}_n\sigma_2^2$$
$$= (\mathbf{I}_p \otimes \mathbf{I}_m \otimes \mathbf{J}_e\sigma_1^2) + (\mathbf{I}_p \otimes \mathbf{I}_m \otimes \mathbf{I}_e\sigma_2^2)$$

where $\sigma_1^2 = a'V_1a$ and $\sigma_2^2 = a'V_2a$. From Searle (1971), an F test of (1.5) can be constructed by defining

$$ss_1=\mathbf{y}'\mathbf{Q}_1\mathbf{y}$$

and

$$ss_2=\mathbf{y}'\mathbf{Q}_2\mathbf{y}$$

where

$$\mathbf{Q}_1 = \{[1/(me)]\mathbf{I}_p \otimes \mathbf{J}_m \otimes \mathbf{J}_e\} - \{[1/(pme)]\mathbf{J}_p \otimes \mathbf{J}_m \otimes \mathbf{J}_e\}$$

and

$$\mathbf{Q}_2 = \{(1/e)\mathbf{I}_p \otimes \mathbf{I}_m \otimes \mathbf{J}_e\} - \{[1/(me)]\mathbf{I}_p \otimes \mathbf{J}_m \otimes \mathbf{J}_e\}.$$

The random variable

$$F = [ss_1/ss2][p(m-1)/(p-1)]$$

follows a noncentral F distribution with p-1 and p(m-1) degrees of freedom.

For balanced models, exact F tests for many of the hypothesis of interest can be constructed in a similar fashion. These ANOVA method tests can be obtained by taking the chi-squared statistic divided by its degrees of freedom that would be obtained if the covariance matrix for the data was known to be an identity matrix and dividing by a mean square from an ANOVA table whose expectation equals the expectation of the chi-square statistic divided by its degrees of freedom under the null hypothesis. When the data are balanced this mean square is the MINQUE0 estimator of the expectation of the chi-squared statistic under the null hypothesis (Searle 1971, 1979).

However, there are many cases where exact tests cannot be constructed using the ANOVA method (Seifert 1979). Exact F tests have been constructed for some of these situations at the cost of using a subset of the data (e.g. Khuri 1984; Seifert 1979). The subset methods involve taking a linear transformation of the data partitioned into two subsets of independent random variables. Sums of squares of the two subsets are taken to produce two independent chi-square random variables for use in a F test. The two chi-square random variables can be written as quadratic forms in the original data. Except in cases where the ANOVA method produces an exact F test there is more than one possible set of linear transformations that can be selected. Depending on which set of transformations are used, different values of the test statistic can be obtained. Where there is a choice as to which set of transformations to use, selection is made independently of the observed dependent variables. The subset methods lack appeal in that two different experimenters with the same data, model and hypotheses can arrive at different conclusions using the same test procedure.

In addition, approximate F tests can be constructed using either Satterthwaite's (1941) or Meyers and Howe's (1971) procedure. As with the ANOVA method, the approximate F tests are constructed using the mean squares from an ANOVA table. They differ from the ANOVA method in that the numerator or the denominator of the test statistic may contain a linear function of the mean squares. The test statistic is

$$F^* = \left(\sum_{i=1}^{k} c_i MS_i\right) / \left(\sum_{i=k+1}^{f} c_i MS_i\right) \qquad (1.6)$$

where

F^* is the test statistic,

k is the number of mean squares in the numerator,

f-k is the number of mean squares in the denominator,

c_i, i=1,..., f are known positive constants,

MS_i, i=1,...,f are mean squares from the ANOVA table and are distributed independently as γ_i/m_i times a chi-square random variable with degrees of freedom m_i,

γ_i, i=1,...,f are the expected values of MS_i,

and the expected value of the numerator is equal to the expected value of the denominator when the null hypothesis is true. As with the test statistic from the ANOVA method, the numerator and denominator are distributed independently of each other. In general the test statistic does not have an F distribution because the numerator and denominator are linear functions of chi-squared random variables and are not themselves chi-square random variables. The test statistic can be approximated by an F distribution.

Satterthwaite's (1941) approximation involves equating the first two moments of the numerator and denominator F^* in (1.6) with two independent chi-square random variables times a constant. The formulas for the approximate degrees of freedom depend on the unknown γ_i's. The degrees of freedom are then estimated using the MS_i's in place of the unknown γ_i's.

Meyers and Howe's (1971) approximation involves equating the approximate first moments of F^* in (1.6) and its inverse with the first moments of a F random variable and its inverse. The approximate first moments of F^* and $1/F^*$ are obtained by means of a Taylor series expansion. The first moment of a F random variable is a function of the denominator degrees of freedom and the first moment of the inverse of a F random variable is a function of the numerator degrees of freedom. The formulas for the degrees of freedom depend on the unknown first moments of the test statistic and its inverse. A function of the MS_i's in the denominator equal to the expectation of the test statistic and a function of the MS_i's in the numerator equal to the expectation of the inverse of the test statistic are found. The degrees of freedoms are then estimated using the two functions of the MS_i's in place of the two unknown first moments. Meyers and Howe's (1971) approximation involves more complex formulas than does Satterthwaite's (1941) approximation but, appears to yield superior results (Davenport 1975; Meyers and Howe 1971).

Multivariate balanced data

When the ANOVA method or the subset methods produce an exact test in the univariate case, then, as will be shown in chapter 2, exact tests in the equivalent multivariate case can be constructed by using the same matrices of the quadratic forms. Pavur (1987) gave conditions under which two multivariate quadratic forms will be distributed as independent central Wishart matrices. Under a true null hypothesis, Pavur's conditions are satisfied in the multivariate model whenever they are satisfied in the equivalent univariate mixed model. The quadratic forms can, therefore, be used in any of the usual test statistics [Wilk's Lambda, Pillai's trace, Roy's maximum root test, or Lawley-Hotellings trace statistic] (Seber 1984).

Generalizations of the approximate tests can be constructed by using the quadratic forms based on the same matrices as those used in the univariate approximate tests. In a manner similar to Satterthwaite (1941), approximations of the distribution of the test statistic can be found by equating the first moments of the quadratic forms to those of central Wishart matrices and equating a scalar function of the second moments of the quadratic forms to those of central Wishart matrices. Tan and Gupta (1983) approximated the distribution of a linear combination of independent central Wishart matrices with positive coefficients by a central Wishart matrix. Their approximation involved equating the first moment of the linear combination of Wishart matrices with the first moment of a Wishart matrix and equating the determinant of the variance of the Vech of the linear combination of Wishart matrices with the determinant of the variance of the Vech of a Wishart matrix. The Vech operator is described by Henderson and Searle (1979). They noted that the approximation could be used to produce approximate tests in balanced multivariate mixed models.

Unbalanced data

Except in special cases, exact tests are not available for unbalanced data. Several different approaches have been used to address the problem of testing fixed effects in the univariate setting. The approaches include both exact and approximate tests. The exact methods either involve treating all the random effects, except the residual as fixed effects (e.g. Thomsen 1971; Wald 1941) or using a subset approach (e.g. Khuri and Littell 1987). The approximate methods include treating the unknown variances as known (e.g. Henderson 1984; Hussein and Milliken 1978), performing the analysis as if the data was balanced (e.g. Tietjen 1974), and approximating the distribution using a Satterthwaite type approximation (e.g. Tan and Cheng 1984).

An exact test can sometimes be constructed by treating the random effects as fixed effects and using the test associated with the corresponding fixed effects model. In order for this approach to work the hypothesis must only involve estimable functions of the fixed effects in the corresponding fixed effects model. This approach was used by Wald (1941) and Thomsen (1975).

Khuri and Littell (1987) developed an exact test for testing main effects in an unbalanced two-way mixed model by using a subset approach similar to that used in balanced mixed models. The data was transformed into two independent sets of random variables. The first set consisted of the cell means for each combination of the main effects. The second set consisted of the residuals obtained by taking the difference between each of the observations and the associated cell mean. Orthogonal contrasts were then taken of each set. The orthogonal contrasts for the cell means were the usual ones associated with the main effects and interaction. A final transformation of the residuals was taken and the vector of transformed residuals was added to the vector of contrasts associated with the main effects and interaction to adjust for the unequal sample sizes. The vector of adjusted contrasts for main effects and interaction was then used to produce sums of squares associated with the main effects and interactions. The sums of squares produced are distributed as independent chi-square random variables and can be used to produce exact F tests for the main effects. As with the subset approaches used in balanced mixed models the resulting F test depends on the selection of transformations which are used. This procedure reduces to the usual ANOVA procedure if the data are balanced. The distribution of the test statistic under the null hypothesis does not depend on which set of transformations was chosen, assuming the selection is done independently of the data.

The resulting test statistics from both of the exact procedures are ratios of quadratic forms in the original data. Using the same matrices of quadratic forms in the equivalent multivariate mixed model exact multivariate tests can be obtained for testing main effects in an unbalanced multivariate mixed model.

Approximate F tests can be obtained by treating the unknown variance components as known. The test procedure involves first obtaining estimates of the unknown variance components and treating the estimated variance components as if they were the true variance components. The chi-square test statistic that would be used if the true variance components were equal to the estimated variance components is calculated. An approximate critical value for this test statistic is obtained by assuming that the true variance components were used. The procedure lacks appeal in that it does not reduce to an exact test if the data are balanced and it is reasonable only if the estimated variance components are close to the true variance components. This procedure was suggested by Hussein and Milliken (1978) and Henderson (1984).

Another approach is to conduct the test as if the data are balanced. An ANOVA table is constructed from the data. In general, with the data being unbalanced the sums of squares are no longer distributed as independent chi-square random variables. By using the corresponding sum of squares that would produce an exact F test if the data were balanced an approximate F test can be constructed. An approximate critical value is obtained by using the degrees of freedom from the ANOVA table. If the data are balanced this procedure produces an exact test. This procedure has the advantage that it is easier to implement than most of the other procedures and in certain cases appears to produce superior results (Tietjen 1974).

Another approach takes various linear functions of the sums of squares to construct the numerator and denominator sums of squares for an F test and then approximates the degrees of freedom using Satterthwaite's (1941) approximation (e.g. Tan and Cheng 1984). In multivariate mixed models Tan and Gupta's (1983) approximation of a linear function of central Wishart matrices can be used in place of Satterthwaite's (1941) approximation of a linear function of central chi-square random variables. A problem with using Satterthwaite's (1941) approximation or Tan and Gupta's (1983) approximation with unbalanced data is that the approximations assumes that the quadratic forms are independent and are distributed as either scalar multiples of chi-square random variables or Wishart random matrices. If the data are unbalanced, these assumptions are often violated.

Objectives

The objectives of this study are to develop a test statistic that can be used in unbalanced multivariate mixed models and to examine the properties of the test statistic. In particular, the following will be obtained: an approximation to the null distribution, a proof that the proposed test is exact in the balanced case, and an approximation to the power of the test under H_a. In addition, approximate simultaneous confidence intervals for the fixed effects will be obtained. An example will be used to illustrate the test procedure. A small Monte-Carlo study will be performed to determine the test size and to compare the power of the test procedure to an exact test using a balanced subset of the data.

CHAPTER 2

TEST PROCEDURE

Introduction

Unbalanced mixed models arise in many animal science experiments. In these experiments it is often of interest to test hypotheses about the fixed effects. Except in certain special cases, exact tests are not available in either univariate or multivariate unbalanced mixed models. Approximate tests procedures have been developed for use in unbalanced mixed models, but either they apply techniques developed for balanced mixed models or they treat unknown variance components as known.

In this chapter I will describe a multivariate mixed model and give a definition of a balanced multivariate mixed model. I will present a procedure for testing certain hypotheses about the fixed effects in balanced multivariate mixed models. The test procedure for balanced multivariate mixed models will be developed by showing that the multivariate statistics analogous to the univariate F ratios have the appropriate distributions. A generalization of the multivariate test for balanced data will be developed for testing hypotheses about the fixed effects in a general multivariate mixed model. For the generalized test, approximate null and non-null distributions for the test statistics will be found by extending Satterthwaite's (1941) results to multivariate quadratic forms.

Simultaneous confidence intervals for the fixed effects will also be developed.

Model

A multivariate mixed model can be written as follows:

$$Y = \sum_{i=1}^{r} X_i B_i + \sum_{j=1}^{s} Z_j U_j \qquad (2.1)$$

where

Y is an n by t matrix in which the k^{th} column contains the k^{th} dependent variable observed on the n individuals,

X_i is an n by p_i matrix relating the matrix of fixed effects B_i to the matrix of dependent variables Y,

B_i is a p_i by t matrix in which the k^{th} column contains the i^{th} fixed effects for dependent variable k,

Z_j is a n by q_j matrix relating the matrix of random effects U_j to the matrix of dependent variables Y,

and

U_j is a q_j by t matrix in which the k^{th} column contains the j^{th} random effects for dependent variable k.

The matrices of random effects are independently distributed as $Vec(U_j) \sim N(0, V_j \otimes I_j)$ where I_j is q_j by q_j identity matrix and V_j is a t by t covariance matrix. Thus, the rows of U_j are independently and identically distributed as $N(0, V_j)$. The covariance matrices V_j are positive semi-definite and V_s is positive definite. The matrix Z_s is an n by n identity matrix. The X_i and Z_j matrices can be less than full column rank.

The hypothesis to be tested is represented as follows

$$H_o: \mathbf{AXB} = \mathbf{0} \quad \text{versus} \quad H_a: \mathbf{AXB} \neq \mathbf{0}$$

where $\mathbf{A}$ is an h by n matrix, $\mathbf{X} = [\ \mathbf{X}_1 \mid \ldots \mid \mathbf{X}_r\ ]$, and $\mathbf{B}' = [\ \mathbf{B}_1' \mid \ldots \mid \mathbf{B}_r'\ ]$. This can be rewritten as

$$H_o: \mathbf{CB} = \mathbf{0} \quad \text{versus} \quad H_a: \mathbf{CB} \neq \mathbf{0} \qquad (2.2)$$

where $\mathbf{C} = \mathbf{AX}$ is an h by n matrix with rank $m \leq h$. To test this hypothesis a test statistic and critical value are needed. A definition of balanced data will be given next along with a class of hypotheses for which exact tests do exist.

Balanced Data

Following Seifert (1979), a balanced mixed model will be defined as a special case of model (2.1), where the $\mathbf{X}_i$'s and $\mathbf{Z}_j$'s have a Kronecker structure.

Definition 2.1: A set of matrices $\mathbf{X}_1, \ldots, \mathbf{X}_r, \mathbf{Z}_1, \ldots, \mathbf{Z}_s$ are said to have Kronecker structure if they can be written as

$$\mathbf{X}_i = \mathbf{G}_{i(1)} \otimes \ldots \otimes \mathbf{G}_{i(a)} \qquad \text{for } i = 1, \ldots, r$$

and

$$\mathbf{Z}_j = \mathbf{G}_{r+j(1)} \otimes \ldots \otimes \mathbf{G}_{r+j(a)} \text{ for } j = 1, \ldots, s$$

where $\mathbf{G}_{k(f)} = \mathbf{I}$ or $\mathbf{1}$ of order c_k, c_k is a known constant, $1 \leq f \leq a$, and a is known.

Univariate Tests

A univariate mixed model can be written as follows:

$$\mathbf{y} = \sum_{i=1}^{r} \mathbf{X}_i \mathbf{b}_i + \sum_{j=1}^{s} \mathbf{Z}_j \mathbf{u}_j \qquad (2.3)$$

where

y is an n by 1 vector which contains the dependent variable observed on the n individuals,

$\mathbf{X}_i$ is an n by p_i matrix relating the vector of fixed effects $\mathbf{b}_i$ to the vector of the dependent variable **y**,

$\mathbf{b}_i$ is a p_i by one vector containing the i^{th} fixed effects,

$\mathbf{Z}_j$ is an n by q_i matrix relating the vector of random effects $\mathbf{u}_j$ to the vector of the dependent variable **y**,

and

$\mathbf{u}_j$ is a q_j by one vector containing the j^{th} random effects.

The vectors of random effects are independently distributed as $\mathbf{u}_j \sim N(\mathbf{0}, \mathbf{I}_j\sigma_j^2)$ where $\mathbf{I}_j$ is a q_j by q_j identity matrix and σ_j^2 is the variance of the j^{th} random effects. The variances σ_j^2 are non-negative and σ_s^2 is positive.

Models (2.1) and (2.3) are said to be equivalent if the $\mathbf{X}_i$'s and $\mathbf{Z}_j$'s are the same in both models. The univariate hypothesis which is equivalent to the multivariate hypothesis (2.2) is

$$H_o\text{: } \mathbf{Cb} = \mathbf{0} \text{ versus } H_a\text{: } \mathbf{Cb} \neq \mathbf{0} \qquad (2.4)$$

where $\mathbf{b}' = [\ \mathbf{b}_1' \mid \ldots \mid \mathbf{b}_s'\ ]$.

When there is a single dependent variable, hypotheses can often be tested using the ANOVA method. The ANOVA method involves taking a ratio of two quadratic forms. The numerator, or hypothesis quadratic form, is obtained by taking the chi-squared random variable that would be used if the following was known to be true, $\sigma_s^2 = 1$ and the remaining σ_j^2's = 0. The denominator, or error, quadratic form is the MINQUE0 estimator of the expectation of the hypothesis quadratic form under the null hypothesis. Provided the hypothesis and error quadratic forms are distributed as independent chi-squared random variables the resulting test statistic has an F distribution. The test statistic is

$$F = y'Q_H y/y'Q_E y \qquad (2.5)$$

where Q_H and Q_E are known matrices.

Multivariate Tests

When hypothesis (2.4) can be tested using the test statistic (2.5) for the univariate model (2.3), then analogous tests can be constructed for testing hypothesis (2.2) in the equivalent multivariate model (2.1). The multivariate tests are found by establishing that the Q_H and Q_E that produced independent chi-square random variables in model (2.3) can also be used to produce independent Wishart random matrices in model (2.1).

Pavur (1987) showed that the conditions required for obtaining two independent central Wishart random matrices are similar to those required for obtaining two independent central chi-squared random variables. These results will be extended to the noncentral case, but first Theorem 2.1 will establish sufficient conditions for the independence of two multivariate quadratic forms. Theorem 2.1 will be useful for establishing that two Wishart matrices are independent and as a framework for finding a quadratic form which is independent of a given quadratic form.

Theorem 2.1: Given Y in model (2.1), symmetric n by n matrices Q_1 and Q_2 of rank r1 and r2 respectively, and $Q_1 Z_j Z_j' Q_2 = 0$ for $j=r+1,\ldots,s$, then $Y'Q_1Y$ and $Y'Q_2Y$ are independently distributed.

Proof:

It will be useful to rewrite the quadratic forms $Y'Q_iY$ for $i=1,2$. The matrices Q_1 and Q_2 can be written in diagonal form as follows

$$Q_i = P_i D_i P_i' \tag{2.6}$$

where $P_i'P_i = I_{ri}$ and D_i is a ri by ri nonsingular matrix. The quadratic forms can now be written as follows

$$Y'Q_iY = Y'P_iD_iP_i'Y.$$

The quadratic forms $Y'Q_1Y$ and $Y'Q_2Y$ will be independent if it can be established that $Y'P_1$ and $Y'P_2$ are independent. That is, if $Vec(P_1'Y) = (I_t \otimes P_1')Vec(Y)$ is independent of $Vec(P_2'Y) = (I_t \otimes P_2')Vec(Y)$, then $Y'Q_1Y$ is independent of $Y'Q_2Y$. The matrices $Y'P_1$ and $Y'P_2$ will be independent if the following holds

$$(I_t \otimes P_1')Var[Vec(Y)](I_t \otimes P_2) = 0. \tag{2.7}$$

Condition (2.7) will be satisfied if

$$\sum_{j=1}^{s} (I_t \otimes P_1')(V_j \otimes Z_jZ_j')(I_t \otimes P_2) = 0$$

$$\Leftarrow \quad \sum_{j=1}^{s} V_j \otimes P_1'Z_jZ_j'P_2 = 0$$

$$\Leftarrow \quad V_j \otimes P_1'Z_jZ_j'P_2 = 0 \text{ for all } j$$

$$\Leftarrow \quad P_1'Z_jZ_j'P_2 = 0 \text{ for all } j$$

$$\Leftarrow \quad D_1^{-1}P_1'P_1D_1P_1'Z_jZ_j'P_2D_2P_2'P_2D_2^{-1} = 0 \text{ for all } j$$

$$\Leftarrow \quad P_1D_1P_1'Z_jZ_j'P_2D_2P_2' = 0 \text{ for all } j$$

$\Leftarrow \qquad Q_1Z_jZ_j{}'Q_2 \qquad = 0$ for all j

QED.

In Theorem 2.2, it will be shown that if in model (2.3) $y'Q_1y$ and $y'Q_2y$ are independently distributed as noncentral chi-squared random variables, then in model (2.1) $Y'Q_1Y$ and $Y'Q_2Y$ are independently distributed as noncentral Wishart random matrices. Theorem 2.2 extends Pavur's (1987) results to handle noncentral Wishart matrices. Using Theorem 2.2, if an exact F test exists for H_o: $Cb = 0$ in model (2.3), then an exact test can be constructed for H_o: $CB = 0$ in model (2.1). Any of the usual test statistics [Wilk's Lambda, Pillai's trace, Roy's maximum root test or Lawley-Hotelling's trace statistic (Seber 1984)] can be used. A noncentral chi-squared distribution with degrees of freedom m and noncentrality parameter ϕ will be denoted by $\chi^2(m,\phi)$ (Muirhead 1982). A t dimensional noncentral Wishart distribution with degrees of freedom m, covariance matrix Ω, and noncentrality matrix $\Omega^{-1}\Phi$ will be denoted by $W_t(m,\Omega,\Omega^{-1}\Phi)$ (Muirhead 1982).

Theorem 2.2: If in model (2.3) $y'Q_1y$ and $y'Q_2y$ are independently distributed, for all σ_i^2, as

$$y'Q_jy \sim \omega_j\chi^2(m_j,\ (1/\omega_j)b'X'Q_jXb) \text{ for } j=1,2\ ,$$

where m_j is the rank of Q_j, Q_j is positive semidefinite,

$$\omega_j = \sum_{i=1}^{s} c_{ij}\sigma_i^2,$$

ω_j is positive, and c_{ij} is $tr(Q_jZ_iZ_i{}'Q_j)/tr(Q_j)$, then in model (2.1) $Y'Q_1Y$ and $Y'Q_2Y$ are independently distributed as

$$Y'Q_jY \sim W_t(m_j,\ \Omega_j,\ (\Omega_j^{-1})B'X'Q_jXB) \text{ for } j=1,2\ ,$$

where

$$\Omega_j = \sum_{i=1}^{s} c_{ij}V_i .$$

Proof:

From Theorem 2.1 $Y'Q_1Y$ and $Y'Q_2Y$ will be distributed independently, if $Q_1Z_iZ_i'Q_2 = 0$ for all i. Independence of $y'Q_1y$ and $y'Q_2y$ implies that $Q_1Var(y)Q_2 = 0$ for all σ_i^2. Expanding Var(y) yields

$$Q_1(\sum_{i=1}^{s} Z_iZ_i'\sigma_i^2)Q_2 = 0,$$

and

$$\sum_{i=1}^{s} Q_1Z_iZ_i'Q_2\sigma_i^2 = 0 \text{ for all } \sigma_i^2.$$

This implies that $Q_1Z_iZ_i'Q_2 = 0$ for all i. Therefore, $Y'Q_1Y$ and $Y'Q_2Y$ are independently distributed.

The matrix $Y'Q_jY$ will be distributed as a noncentral Wishart with m_j degrees of freedom, covariance matrix Ω_j, and noncentrality parameter $(\Omega_j^{-1})B'X'Q_jXB$, if $[Q_j \otimes \Omega^{-1}]Var[Vec(Y')]$ is idempotent (Boik in press). The matrix $Q_jVar(y)/\omega_j$ is idempotent because $y'Q_jy/\omega_j$ is a chi-squared random variable. This in turn implies that $Q_jVar(y)Q_j = Q_j\omega_j$. Expanding ω_j and Var(y) yields

$$\sum_{i=1}^{s} Q_jc_{ij}\sigma_i^2 = \sum_{i=1}^{s} Q_jZ_iZ_i'Q_j\sigma_i^2$$

for all σ_i^2. Therefore, Q_jc_{ij} and $Q_jZ_iZ_i'Q_j$ must be equal. Using this result, it will now be shown that $[Q_j \otimes \Omega_j^{-1}]Var[Vec(Y')]$ is idempotent, or equivalently that

$$[Q_j \otimes \Omega_j^{-1}] = [Q_j \otimes \Omega_j^{-1}]\mathrm{Var}[\mathrm{Vec}(Y')][Q_j \otimes \Omega_j^{-1}]. \quad (2.8)$$

Expanding Var[Vec(Y')] in the right hand side of (2.8) yields

$$[Q_j \otimes \Omega_j^{-1}]\mathrm{Var}[\mathrm{Vec}(Y')][Q_j \otimes \Omega_j^{-1}] = \sum_{i=1}^{s} [Q_j \otimes \Omega j^{-1}][Z_i Z_i' \otimes V_i][Q_j \otimes \Omega_j^{-1}]$$

$$= \sum_{i=1}^{s} [Q_j Z_i Z_i' Q_j] \otimes [\Omega_j^{-1} V_j \Omega_j^{-1}]$$

$$= \sum_{i=1}^{s} [Q_j c_{ij}] \otimes [\Omega_j^{-1} V_j \Omega_j^{-1}]$$

$$= \sum_{i=1}^{s} Q_j \otimes [\Omega_j^{-1} c_{ij} V_j \Omega_j^{-1}]$$

$$= Q_j \otimes [\Omega_j^{-1} (\sum_{i=1}^{s} c_{ij} V_j) \Omega_j^{-1}]$$

$$= Q_j \otimes \Omega_j^{-1}$$

QED.

Test Statistic

The Lawley-Hotelling test statistic used in the balanced case can be generalized for use in a general multivariate mixed model. The Lawley-Hotelling test statistic consists of the trace of the product of a hypothesis quadratic form and the inverse of an error quadratic form. Ignoring a constant multiplier these quadratic forms are distributed as independent central Wishart matrices with equal expectations when the null hypothesis is true.

The proposed test statistic consists of the trace of the product of a hypothesis quadratic form and the inverse of an error quadratic form. These quadratic forms will be distributed independently with equal expectations when the null hypothesis is true. The null distribution will be approximated by approximating the distribution of the quadratic forms by central Wishart distributions. The test statistic can be written as follows

$$T^2 = \mathrm{tr}(\mathbf{HE}^{-1}) \tag{2.9}$$

where **H** is a hypothesis quadratic form in Y and **E** is an error quadratic form in Y, both to be defined later.

The hypothesis quadratic form will be the usual quadratic form for testing the hypothesis **AXB** = **0** in a multivariate mixed model

$$\mathrm{Vec}(\mathrm{Y}) \sim \mathrm{N}_{nt}(\ \mathbf{XB},\ \mathrm{I}_t \otimes \mathrm{I}_n).$$

The matrix H is defined as follows

$$\mathbf{H} = \mathbf{Y'Q_HY}, \tag{2.10}$$

where

$\mathbf{Q_H} = \mathbf{M_XA'(AM_XA')^{-}AM_X}$,

$\mathbf{M_X} = \mathrm{X(X'X)^{-}X'}$ is the perpendicular projection operator onto the column space of X,

$\mathbf{AM_XY}$ is the ordinary least squares estimator for **AXB**,

and

$\mathrm{I}_t \otimes (\mathbf{AM_XA'})$ is the covariance matrix of $\mathbf{AM_XY}$ if $\mathbf{V}_j = \mathbf{0}$ for $j=r+1,\ldots,s-1$, and $\mathbf{V}_s = \mathrm{I}_t$.

The matrix H can also be written as

$$\mathrm{H} = \hat{\mathbf{B}}'\mathrm{C}[\mathrm{C(X'X)^{-}C'}]^{-}\mathrm{C}\hat{\mathbf{B}} \tag{2.11}$$

where

$\hat{\mathbf{B}} = (\mathbf{X'X})^{-}\mathbf{X'Y}$ is a least-squares estimator of **B**.

The expected value of H, under general $\mathbf{V}_j$, is

$$E(\mathbf{H}) = \mathbf{B'X'A'AXB} + \sum_{j=1}^{s} \operatorname{tr}(\mathbf{Z}_j\mathbf{Z}_j'\mathbf{Q}_H)\mathbf{V}_j,$$

and when H_o is true this reduces to

$$E(\mathbf{H}|H_o \text{ is true}) = \sum_{j=1}^{s} \operatorname{tr}(\mathbf{Z}_j\mathbf{Z}_j'\mathbf{Q}_H)\mathbf{V}_j.$$

The error quadratic form will be selected so that it is distributed independently of the hypothesis quadratic form and under the null hypothesis will have the same expected value as the hypothesis quadratic form. Theorem 2.1 provides the necessary framework for guaranteeing that **E** is independent of **H**.

The matrix **E** is a multivariate MINQUE0 *type* estimator of $E(\mathbf{H}|H_o$ is true) and is defined as follows

$$\mathbf{E} = \mathbf{Y'Q}_E\mathbf{Y}, \tag{2.12}$$

where $\mathbf{Q}_E$ is a symmetric matrix which satisfies the following conditions:

(1) $\mathbf{Q}_E\mathbf{Z}_j\mathbf{Z}_j'\mathbf{M}_X\mathbf{A}' = \mathbf{0}$ for $j=1,\ldots,s$,

(2) $\mathbf{Q}_E\mathbf{X} = \mathbf{0}$,

(3) $E(\mathbf{E}) = E(\mathbf{H}|H_o$ is true),

and

(4) $\operatorname{tr}(\mathbf{Q}_E^2)$ is the minimum of $\operatorname{tr}(\mathbf{Q}^2)$ for all **Q** that satisfy the above conditions.

Condition 1) guarantees the independence of **H** and **E** by Theorem 2.1. Conditions 2), 3), and 4) are the usual conditions for obtaining the MINQUE0 estimator of $E(\mathbf{H}|H_o$ is true). The **E** matrix is the MINQUE0 estimator of $E(\mathbf{H}|H_o$ is true) in the augmented model

$$Y = WB^{*} + \sum_{j=r+1}^{s} Z_j U_j \qquad (2.13)$$

where

$W = [\ X \mid Z_1 Z_1' M_X A' \mid \ldots \mid Z_s Z_s' M_X A'\ ]$ is the augmented X matrix

and

$B^{*\prime} = [\ B' \mid B_{r+1}' \mid \ldots \mid B_{r+s}'\ ]$ is the augmented B matrix.

Conditions 1) and 2) can be written as $Q_E W = 0$. Following Searle (1979) and using model (2.13) Q_E can be written as

$$Q_E = \sum_{j=1}^{s} \tau_j (I_n - M_W) Z_j Z_j' (I_n - M_W) \qquad (2.14)$$

where

$\tau = [\ \tau_1, \ldots, \tau_s\ ]' = \Gamma\delta,$

$\delta = \{tr(Z_j Z_j' Q_H)\},$

$\Gamma = \{tr(Z_i Z_i' (I_n - M_W) Z_j Z_j' (I_n - M_W))\}^{-1},$

and

$M_W = W(W'W)^{-}W'$ is the perpendicular projection operator onto the column space of W.

MINQUE0 type estimators of V_j, $\hat{V}_j$, can be obtained by using e_j in place of δ, where e_j is the j^{th} column of an n by n identity matrix.

Distribution Under the Null Hypothesis

To use the test statistic (2.9) to conduct a test, its distribution under the null hypothesis is needed. In general its distribution will depend on the unknown covariance matrices. An approximate null distribution will be obtained using estimates of the covariance matrices. The distributions of **H** and **E** under the null hypothesis will be approximated by distributions of independent central Wishart matrices, W_H and W_E having distributions, $W_H \sim W_t(m_H, (1/m_H)\Omega)$ and $W_E \sim W_t(m_E, (1/m_E)\Omega)$.

The matrices **H** and **E** must be positive definite or positive semidefinite with probability close to one in order for a Wishart distribution to be a reasonable approximation for the their distribution. Satterthwaite (1941) in developing his approximation for the distribution of univariate quadratic forms also cautioned against its use in cases where the quadratic form may be negative. The **H** matrix will always be a positive semidefinite matrix because Q_H is a positive semidefinite matrix. In general there is no guarantee that **E** will be a positive semidefinite matrix. One approach to dealing with cases where **E** is not positive semidefinite is to remove enough traits from the analysis until the new **E** matrix is a positive semidefinite matrix. This approach will be taken here.

The distribution for a general quadratic form in **Y** , **Y'QY**, can be approximated by a central Wishart distribution in several ways, four of which will be examined here. The i^{th} method involves equating the first moment of **Y'QY** to the first moment of a central Wishart, W_i, having distribution

$$W_i \sim W_t(m_i, (1/m_i)\Omega).$$

The methods differ in which function of the second moments is used.

<u>First Moment</u>. Under the null hypothesis the expected value of **QY** is zero. The expected value of **Y'QY** is

$$E(\mathbf{Y}'Q\mathbf{Y}|H_o \text{ is true}) = \sum_{j=1}^{s} tr(\mathbf{Z}_j\mathbf{Z}_j'Q)\mathbf{V}_j.$$

The expected value of $\mathbf{W}_i$ is $\boldsymbol{\Omega}$. Equating $E(\mathbf{W}_i)$ to $E(\mathbf{Y}'Q\mathbf{Y})$ yields

$$\boldsymbol{\Omega} = \sum_{j=1}^{s} tr(\mathbf{Z}_j\mathbf{Z}_j'Q)\mathbf{V}_j. \qquad (2.15)$$

The **E** matrix was selected in such a way that its expected value was equal to the expected value of **H** under the null hypothesis. For both **E** and **H** the covariance matrix, $\boldsymbol{\Omega}$, is equal to $E(\mathbf{H}|H_o$ is true). The MINQUE0 type estimator of $\boldsymbol{\Omega}$ is **E**.

<u>Second Moment</u>. The variance of $Vec(\mathbf{Y}'Q\mathbf{Y})$ can be used to obtain the degrees of freedom to use in the approximation. The resulting matrix is a t^2 by t^2 matrix, but only a scalar parameter is needed. Therefore, a single valued function will be used to obtain an approximate degrees of freedom. There are many possible functions that could be used. Four different functions will be examined. They are

$$f_1\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\} = Var(tr(\mathbf{Y}'Q\mathbf{Y})),$$

$$f_2\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\} = Var(tr(\mathbf{Y}'Q\mathbf{Y}\boldsymbol{\Omega}^{-1})),$$

$$f_3\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\} = tr\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\},$$

and

$$f_4\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\} = tr\{Var[Vec(\mathbf{Y}'Q\mathbf{Y}\boldsymbol{\Omega}^{-1})]\}.$$

The first function can be rewritten as follows

$$f_1\{Var[Vec(\mathbf{Y}'Q\mathbf{Y})]\} = Var[tr(\mathbf{Y}'Q\mathbf{Y})]$$

$$= Var[\sum_{i=1}^{t} (\mathbf{y}_i'Q\mathbf{y}_i)]$$

$$= \sum_{i=1}^{t} \sum_{j=1}^{t} Cov(y_i'Qy_i, y_j'Qy_j),$$

where y_i is the i^{th} column of Y. From Searle (1971) $Cov(y_i'Qy_i, y_j'Qy_j)$ can be rewritten as

$$Cov(y_i'Qy_i, y_j'Qy_j) = tr[QCov(y_i, y_j')QCov(y_j, y_i') + QCov(y_i, y_j')QCov(y_j, y_i')]$$

$$= 2tr[Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q(\sum_{f=1}^{s} Z_fZ_f'\sigma^2_{f(i,j)})]$$

where $\sigma^2_{k(i,j)}$ is the ij^{th} element of V_k. Summing over i and j and bringing the summations over k and f to the outside yields

$$f_1\{Var[Vec(Y'QY)]\} = \sum_{k=1}^{s} \sum_{f=1}^{s} 2tr(QZ_kZ_k'QZ_fZ_f')tr(V_kV_f).$$

Applying the same function to W_1 yields

$$f_1\{Var[Vec(W_1)]\} = 2tr(\Omega^2)/m_1.$$

Equating $f_1\{Var[Vec(W_1)]\}$ to $f_1\{Var[Vec(Y'QY)]\}$ and solving for m_1 yields

$$m_1 = \frac{tr(\Omega^2)}{\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f')tr(V_kV_f)}. \quad (2.16)$$

In a similar fashion an estimate for the degrees of freedom can be obtained from the second function by noting that the second function involves a linear transformation of the t dependent variables. Applying (2.16) to the transformed dependent variables yields

$$m_2 = \frac{t}{\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f')tr(V_k\Omega^{-1}V_f\Omega^{-1})}. \quad (2.17)$$

The third function can be rewritten as follows

$$f_3\{\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}'\mathbf{QY})]\} = \mathrm{tr}\{\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}'\mathbf{QY})]\}$$

$$= \sum_{i=1}^{t} \sum_{j=1}^{t} \mathrm{Var}(\mathbf{y}_i'\mathbf{Q}\mathbf{y}_j).$$

From Searle (1971), $\mathrm{Var}(\mathbf{y}_i'\mathbf{Q}\mathbf{y}_j)$ can be rewritten as

$$\mathrm{Var}(\mathbf{y}_i'\mathbf{Q}\mathbf{y}_j) = \mathrm{tr}[\mathrm{QCov}(\mathbf{y}_i,\mathbf{y}_j')\mathrm{QCov}(\mathbf{y}_i,\mathbf{y}_j')] +$$

$$\mathrm{tr}[\mathrm{QCov}(\mathbf{y}_i,\mathbf{y}_i')\mathrm{QCov}(\mathbf{y}_j,\mathbf{y}_j')]$$

$$= \mathrm{tr}[\mathrm{Q}(\sum_{k=1}^{s} \mathrm{Z}_k\mathrm{Z}_k'\sigma^2_{k(i,j)})\mathrm{Q}(\sum_{f=1}^{s} \mathrm{Z}_f\mathrm{Z}_f'\sigma^2_{f(i,j)})] +$$

$$\mathrm{tr}[\mathrm{Q}(\sum_{k=1}^{s} \mathrm{Z}_k\mathrm{Z}_k'\sigma^2_{k(i,i)})\mathrm{Q}(\sum_{f=1}^{s} \mathrm{Z}_f\mathrm{Z}_f'\sigma^2_{f(j,j)})]$$

Summing over i and j and bringing the summations over k and f to the outside yields

$$f_3\{\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}'\mathbf{QY})]\} = \sum_{k=1}^{s} \sum_{f=1}^{s} \mathrm{tr}(\mathrm{QZ}_k\mathrm{Z}_k'\mathrm{QZ}_f\mathrm{Z}_f')\cdot$$

$$[\mathrm{tr}(\mathbf{V}_k\mathbf{V}_f) + \mathrm{tr}(\mathbf{V}_k)\mathrm{tr}(\mathbf{V}_f)].$$

Applying the same function to W_3 yields

$$f_3\{\mathrm{Var}[\mathrm{Vec}(W_3)]\} = [\mathrm{tr}(\boldsymbol{\Omega}^2)+\mathrm{tr}(\boldsymbol{\Omega})^2]/m_3.$$

Equating $f_3\{\mathrm{Var}[\mathrm{Vec}(W_3)]\}$ to $f_3\{\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}'\mathbf{QY})]\}$ and solving for m_3 yields

$$m_3 = [\mathrm{tr}(\boldsymbol{\Omega}^2)+\mathrm{tr}(\boldsymbol{\Omega})^2]/\{\sum_{k=1}^{s} \sum_{f=1}^{s} \mathrm{tr}(\mathrm{QZ}_k\mathrm{Z}_k'\mathrm{QZ}_f\mathrm{Z}_f')\cdot$$

$$[\mathrm{tr}(\mathbf{V}_k\mathbf{V}_f) + \mathrm{tr}(\mathbf{V}_k)\mathrm{tr}(\mathbf{V}_f)]\}. \tag{2.18}$$

In a similar fashion an estimate for the degrees of freedom can be obtained from the fourth function by noting that the fourth function involves a linear transformation of the t dependent variables. Applying (2.18) to the transformed dependent variables yields

$$m_4 = (t + t^2)/\{ \sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f') \cdot$$

$$[tr(V_k\Omega V_f\Omega) + tr(V_k\Omega)tr(V_f\Omega)]\}. \qquad (2.19)$$

As was noted earlier the true values of Ω and the V_k's are unknown and must be estimated to obtain critical values. To obtain a critical value, the MINQUE0 type estimates of the covariance matrices will be used. That is, **E** will be substituted for Ω and $\hat{V}_k$ for V_k. The test statistic will have an approximate Lawley-Hotelling generalized T distribution with degrees of freedom obtained by substituting Q_H in for Q for the numerator degrees of freedom and substituting Q_E in for Q for the denominator degrees of freedom and multiplying the test statistic by the numerator degrees of freedom. Clearly, if the test statistic times the numerator degrees of freedom does have a Lawley-Hotelling generalized T distribution, then the four methods will all yield the same degrees of freedom and their values will not depend on what positive definite matrices are selected for Ω and V_k, assuming the value of Ω is consistent with the selection of the V_k's.

Distribution under the Alternative Hypothesis

Under the alternative hypothesis, the assumption that $Q_HXB = 0$ can no longer be made. The distribution of **H** will be approximated by a noncentral Wishart matrix, W_H, having distribution

$$W_H \sim W_t(m_H, (1/m_H)\Omega, m_H\Omega^{-1}\Phi).$$

The distribution for a general quadratic form in $\mathbf{Y}$, $\mathbf{Y'QY}$, can be approximated by a noncentral Wishart distribution in several ways, four of which will be examined here. As in the central case, each method involves equating the first moment of $\mathbf{Y'QY}$ to the first moment of a noncentral Wishart matrix, $\mathbf{W}_i$, having distribution

$$\mathbf{W}_i \sim W_t(m_i,\ (1/m_i)\mathbf{\Omega},\ m_i\mathbf{\Omega}^{-1}\mathbf{\Phi})$$

with the additional assumption that the same $\mathbf{\Omega}$ is used for both the hypothesis and error quadratic forms. As in the central case, the methods differ in which function of the second moments is used.

First Moment. Under the alternative hypothesis the expected value of $\mathbf{QY}$ is $\mathbf{QXB}$. The expected value of $\mathbf{Y'QY}$ is

$$E(\mathbf{Y'QY}) = \sum_{j=1}^{s} \mathrm{tr}(\mathbf{Z}_j\mathbf{Z}_j'\mathbf{Q})\mathbf{V}_j + \mathbf{B'X'QXB}.$$

The expected value $\mathbf{W}_i$ is $\mathbf{\Omega} + \mathbf{\Phi}$. Equating $E(\mathbf{W}_i)$ to $E(\mathbf{Y'QY})$ and using (2.15) yields

$$\mathbf{\Omega} = \sum_{j=1}^{s} \mathrm{tr}(\mathbf{Z}_j\mathbf{Z}_j'\mathbf{Q})\mathbf{V}_j$$

and

$$\mathbf{\Phi} = \mathbf{B'X'QXB}.$$

Second Moment. The same four functions will be examined to obtain an approximate degrees of freedom for the hypothesis quadratic form. The first function can be rewritten as follows

$$f_1\{\mathrm{Var}[\mathrm{Vec}(\mathbf{Y'QY})]\} = \mathrm{Var}[\mathrm{tr}(\mathbf{Y'QY})]$$

$$= \mathrm{Var}[\sum_{i=1}^{s} (\mathbf{y}_i'\mathbf{Q}\mathbf{y}_i)]$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{s} Cov(y_i'Qy_i, y_j'Qy_j).$$

From Searle (1971), $Cov(y_i'Qy_i, y_j'Qy_j)$ can be rewritten as

$$\begin{aligned} Cov(y_i'Qy_i, y_j'Qy_j) = {} & 2tr[QCov(y_i, y_j')QCov(y_j, y_i')] + \\ & \mu_i'QCov(y_j, y_i')Q\mu_j + \\ & \mu_i'QCov(y_j, y_j')Q\mu_i + \\ & \mu_j'QCov(y_i, y_i')Q\mu_j + \\ & \mu_j'QCov(y_i, y_j')Q\mu_i \end{aligned}$$

$$\begin{aligned} = {} & 2tr[Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q(\sum_{f=1}^{s} Z_fZ_f'\sigma^2_{f(i,j)})] + \\ & \mu_i'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q\mu_j + \\ & \mu_i'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(j,j)})Q\mu_i + \\ & \mu_j'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,i)})Q\mu_j + \\ & \mu_j'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q\mu_i. \end{aligned}$$

Where μ_i is the i^{th} column of the expected value of Y. Summing over i and j and bringing the sums over k and f to the outside yields

$$\begin{aligned} f_1\{Var[Vec(Y'QY])\} = {} & \sum_{k=1}^{s} \sum_{f=1}^{s} 2tr(QZ_kZ_k'QZ_fZ_f')tr(V_kV_f) + \\ & \sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXB)tr(V_k) + \\ & \sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXBV_k). \end{aligned}$$

Applying the same function to W_1 yields

$$f_1\{Var[Vec(W_1)]\} = [2tr(\Omega^2) + 2tr(\Phi)tr(\Omega) + 2tr(\Omega\Phi)]/m_1.$$

Equating $f_1\{Var[Vec(W_1)]\}$ to $f_1\{Var[Vec(Y'QY)]\}$ and solving for m_1 yields

$$m_1 = [tr(\Omega^2) + tr(\Phi)tr(\Omega) + tr(\Omega\Phi)]/$$

$$[\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f')tr(V_kV_f) +$$

$$\sum_{k=1}^{s} tr(B'X'QZ_kZ_k'QXB)tr(V_k) +$$

$$\sum_{k=1}^{s} tr(B'X'QZ_kZ_k'QXBV_k)]. \qquad (2.20)$$

In a similar fashion an estimate for the degrees of freedom can be obtained from the second function by noting the second function involves a linear transformation of the t dependent variables. Applying (2.20) to the transformed dependent variables yields

$$m_2 = [\ t + t\cdot tr(\Phi\Omega^{-1}) + tr(\Phi\Omega^{-1})]/$$

$$[\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f')tr(V_k\Omega^{-1}V_f\Omega^{-1}) +$$

$$\sum_{k=1}^{s} tr(B'X'QZ_kZ_k'QXB\Omega^{-1})tr(V_k\Omega^{-1}) +$$

$$\sum_{k=1}^{s} tr(B'X'QZ_kZ_k'QXB\Omega^{-1}V_k\Omega^{-1})]. \qquad (2.21)$$

The third function can be rewritten as follows

$$f_3\{Var[Vec(Y'QY)]\} = tr\{Var[Vec(Y'QY)]\}$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{s} Var(y_i'Qy_j).$$

From Searle (1971), $Var(\mathbf{y}_i'Q\mathbf{y}_j)$ can be rewritten as

$$
\begin{aligned}
Var(\mathbf{y}_i'Q\mathbf{y}_j) = {} & tr[QCov(\mathbf{y}_i,\mathbf{y}_j')QCov(\mathbf{y}_i,\mathbf{y}_j')] + \\
& tr[QCov(\mathbf{y}_i,\mathbf{y}_i')QCov(\mathbf{y}_j,\mathbf{y}_j')] + \\
& \boldsymbol{\mu}_i'QCov(\mathbf{y}_j,\mathbf{y}_j')Q\boldsymbol{\mu}_i + \\
& \boldsymbol{\mu}_j'QCov(\mathbf{y}_i,\mathbf{y}_i')Q\boldsymbol{\mu}_j + \\
& 2\boldsymbol{\mu}_i'QCov(\mathbf{y}_j,\mathbf{y}_i')Q\boldsymbol{\mu}_j \\
= {} & tr[Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q(\sum_{f=1}^{s} Z_fZ_f'\sigma^2_{f(i,j)})] + \\
& tr[Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,i)})Q(\sum_{f=1}^{s} Z_fZ_f'\sigma^2_{f(j,j)})] + \\
& \boldsymbol{\mu}_i'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(j,j)})Q\boldsymbol{\mu}_i + \\
& \boldsymbol{\mu}_j'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,i)})Q\boldsymbol{\mu}_j + \\
& 2\boldsymbol{\mu}_i'Q(\sum_{k=1}^{s} Z_kZ_k'\sigma^2_{k(i,j)})Q\boldsymbol{\mu}_j .
\end{aligned}
$$

Summing over i and j and bringing the summations for k and f to the outside yields

$$
\begin{aligned}
f_3\{Var[Vec(Y'QY)]\} = {} & \sum_{k=1}^{s}\sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f')\cdot \\
& [tr(V_kV_f) + tr(V_k)tr(V_f)] + \\
& \sum_{k=1}^{s} 2tr(\mathbf{B}'\mathbf{X}'QZ_kZ_k'Q\mathbf{X}\mathbf{B})tr(V_k) + \\
& \sum_{k=1}^{s} 2tr(\mathbf{B}'\mathbf{X}'QZ_kZ_k'Q\mathbf{X}\mathbf{B}V_k) .
\end{aligned}
$$

Applying the same function to W_3 yields

$$f_3\{Var[Vec(W_3)]\} = [tr(\Omega^2) + tr(\Omega)^2 + 2tr(\Phi)tr(\Omega) + 2tr(\Phi\Omega)]/m_3$$

Equating $f_3\{Var[Vec(W_3)]\}$ to $f_3\{Var[Vec(Y'QY)]\}$ and solving for m_3 yields

$$m3 = [tr(\Omega^2) + tr(\Omega)^2 + 2tr(\Phi)tr(\Omega) + 2tr(\Phi\Omega)]/$$

$$[\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f') \cdot$$

$$[tr(V_kV_f) + tr(V_k)tr(V_f)] +$$

$$\sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXB)tr(V_k) +$$

$$\sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXBV_k)]. \qquad (2.22)$$

In a similar fashion, an estimate of the degrees of freedom can be obtained from the fourth function by noting that the fourth function involves a linear transformation of the t dependent variables. Applying (2.22) to the transformed dependent variables yields

$$m_4 = [t + t^2 + 2t \cdot tr(\Phi\Omega^{-1}) + 2tr(\Phi\Omega^{-1})]/$$

$$\{\sum_{k=1}^{s} \sum_{f=1}^{s} tr(QZ_kZ_k'QZ_fZ_f') \cdot$$

$$[tr(V_k\Omega^{-1}V_f\Omega^{-1}) + tr(V_k\Omega^{-1})tr(V_f\Omega^{-1})] +$$

$$\sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXB\Omega^{-1})tr(V_k\Omega^{-1}) +$$

$$\sum_{k=1}^{s} 2tr(B'X'QZ_kZ_k'QXB\Omega^{-1}V_k\Omega^{-1})\}. \qquad (2.23)$$

Distribution when Q has Rank 1

When testing a hypothesis where **C** has rank one, the resulting Q_H will also have rank one. Theorem 2.3 will establish that the matrix **Y'QY** will have an exact Wishart distribution, with a single degree of freedom, whenever the rank of Q is equal to one. From Theorem 2.3 the matrix **H** will have an exact Wishart distribution with a single degree of freedom. Furthermore, the degrees of freedom for the numerator of the test statistic, resulting from the approximations, will also be one.

Theorem 2.3: If a positive semidefinite matrix Q has rank one, then in model (2.1) **Y'QY** is distributed as

$$\mathbf{Y'QY} \sim W_t(1,\ \boldsymbol{\Omega}^{-1},\ (\boldsymbol{\Omega}^{-1})\mathbf{B'X'QXB})$$

where

$$\boldsymbol{\Omega} = \sum_{i=1}^{s} c_i \mathbf{V}_i$$

and $c_i = \mathrm{tr}(Q\mathbf{Z}_i\mathbf{Z}_i')$.

Proof:

If $[Q \otimes \boldsymbol{\Omega}^{-1}]\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}')]$ is idempotent then **Y'QY** will be distributed as $W_t(1,\ \boldsymbol{\Omega}^{-1},\ (\boldsymbol{\Omega}^{-1})\mathbf{B'X'QXB})$ (Boik in press). The idempotency condition is equivalent to $[Q \otimes \boldsymbol{\Omega}^{-1}] = [Q \otimes \boldsymbol{\Omega}^{-1}]\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}')][Q \otimes \boldsymbol{\Omega}^{-1}]$, because $\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}')]$ is positive definite. Expanding $\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}')]$ yields

$$[Q \otimes \boldsymbol{\Omega}^{-1}]\mathrm{Var}[\mathrm{Vec}(\mathbf{Y}')][Q \otimes \boldsymbol{\Omega}^{-1}] = \sum_{i=1}^{s} [Q \otimes \boldsymbol{\Omega}^{-1}][\mathbf{Z}_i\mathbf{Z}_i' \otimes \mathbf{V}_i][Q \otimes \boldsymbol{\Omega}^{-1}]$$

$$= \sum_{i=1}^{s} [Q\mathbf{Z}_i\mathbf{Z}_i'Q] \otimes [\boldsymbol{\Omega}^{-1}\mathbf{V}_i\boldsymbol{\Omega}^{-1}]. \quad (2.24)$$

The matrix Q can be written as $\mathbf{qq}'$ where $\mathbf{q}$ is n by one matrix, because Q has rank one. The matrix $QZ_iZ_i'Q$ can be written as

$$\begin{aligned} QZ_iZ_i'Q &= \mathbf{qq}'Z_iZ_i'\mathbf{qq}' \\ &= \mathbf{q}\mathrm{tr}(\mathbf{q}'Z_iZ_i'\mathbf{q})\mathbf{q}' \\ &= \mathbf{qq}'\mathrm{tr}(\mathbf{qq}'Z_iZ_i') \\ &= Q\mathrm{tr}(QZ_iZ_i') \\ &= Qc_i. \end{aligned} \tag{2.25}$$

Substituting (2.25) into (2.24) yields

$$\begin{aligned} [Q \otimes \Omega^{-1}]\mathrm{Var}[\mathrm{Vec}(Y')][Q \otimes \Omega^{-1}] &= \sum_{i=1}^{s} Q \otimes [\Omega^{-1}c_iV_i\Omega^{-1}] \\ &= Q \otimes [\Omega^{-1}(\sum_{i=1}^{s} c_iV_i)\Omega^{-1}] \\ &= Q \otimes \Omega^{-1}. \end{aligned}$$

QED.

Simultaneous Confidence Intervals

Building confidence intervals for linear functions of the fixed effects is closely tied to testing hypotheses about linear functions of the fixed effects. Associated with test statistic (2.9) is a set of simultaneous confidence intervals for linear functions of the form $\boldsymbol{\lambda}'\mathrm{Vec}(\mathbf{CB})$. The confidence intervals will be found by maximizing a scaled function of $\boldsymbol{\lambda}'\mathrm{Vec}(\mathbf{CB})$ over all $\boldsymbol{\lambda}$. In constructing the confidence intervals it will be assumed that the approximations for the null distribution of the test statistic are reasonable and that E is positive definite.

The scaled function of $\boldsymbol{\lambda}'\mathrm{Vec}(\mathbf{CB})$ to be maximized is

$$\theta(\mathbf{B}) = [\boldsymbol{\lambda}'\mathrm{Vec}(\mathbf{CB})]^2/\{\boldsymbol{\lambda}'[\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']\boldsymbol{\lambda}\}. \tag{2.26}$$

If $\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}$ was positive definite it would be a simple matter to maximize (2.26). However, the rank of $\mathbf{C}$ (i.e., m) may be less than the number of rows in $\mathbf{C}$ (i.e., h). It will be convenient to decompose $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{PR} \qquad (2.27)$$

where

$\mathbf{P}$ is an h by m matrix such that $\mathbf{P}'\mathbf{P} = \mathbf{I}$

and

$\mathbf{R}$ is an m by p matrix of rank m.

Substituting (2.27) into $\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}'$ yields $\mathbf{E} \otimes \mathbf{PR}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'\mathbf{P}'$. Expanding $\mathbf{E} \otimes \mathbf{PR}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'\mathbf{P}'$ yields

$$\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}' = (\mathbf{I} \otimes \mathbf{P})[\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'](\mathbf{I} \otimes \mathbf{P}'). \qquad (2.28)$$

Because $\mathbf{C} = \mathbf{AX}$, $\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}'$ has the same rank as $\mathbf{C}$ and therefore $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'$ is a positive definite matrix. Substituting (2.27) into $\mathrm{Vec}(\mathbf{CB})$ yields $\mathrm{Vec}(\mathbf{PRB})$. Expanding $\mathrm{Vec}(\mathbf{PRB})$ yields

$$\mathrm{Vec}(\mathbf{CB}) = (\mathbf{I} \otimes \mathbf{P})\mathrm{Vec}(\mathbf{RB}). \qquad (2.29)$$

Substituting (2.27) and (2.28) into (2.26) yields

$$\theta(\mathbf{B}) = [\boldsymbol{\lambda}^{*\prime}\mathrm{Vec}(\mathbf{RB})]^2/\{\boldsymbol{\lambda}^{*\prime}[\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}']\boldsymbol{\lambda}^*\} \qquad (2.30)$$

where $\boldsymbol{\lambda}^* = (\mathbf{I} \otimes \mathbf{P}')\boldsymbol{\lambda}$. The matrix $\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'$ is positive definite, because $\mathbf{E}$ and $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}'$ are positive definite. From the Cauchy-Schwarz inequality, the function $\theta(\mathbf{B})$ will be maximized over all $\boldsymbol{\lambda}^*$ at $\boldsymbol{\lambda}^* = [\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}']^{-1}\mathrm{Vec}(\mathbf{RB})$ with a maximum value of

$$\max[\theta(\mathbf{B})] = \mathrm{Vec}(\mathbf{RB})'[\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}']^{-1}\mathrm{Vec}(\mathbf{RB}) \qquad (2.31)$$

(Seber 1984). This will be the maximum over all $\boldsymbol{\lambda}$, if a $\boldsymbol{\lambda}$ can be found such that $(\mathbf{I} \otimes \mathbf{P}')\boldsymbol{\lambda} = [\mathbf{E} \otimes \mathbf{R}(\mathbf{X}'\mathbf{X})^{-}\mathbf{R}']^{-1}\mathrm{Vec}(\mathbf{RB})$. Because $\mathbf{P}$ was selected such that $\mathbf{P}'\mathbf{P} = \mathbf{I}$,

$$\boldsymbol{\lambda} = (I \otimes P)[E \otimes R(X'X)^{-}R']^{-1}\mathrm{Vec}(RB)$$

will maximize $\theta(B)$ over all $\boldsymbol{\lambda}$. The maximum value of $\theta(B)$ can be written as

$$\max[\theta(B)] = \mathrm{Vec}(RB)'[E^{-1} \otimes [R(X'X)^{-}R']^{-1}\}\mathrm{Vec}(RB).$$

Simplifying this yields

$$\max[\theta(B)] = \mathrm{Vec}(RB)'\mathrm{Vec}\{[R(X'X)^{-}R']^{-1}RB(E^{-1})\}$$

Since $\mathrm{Vec}(A)'\mathrm{Vec}(B) = \mathrm{tr}(A'B)$ the maximum value of $\theta(B)$ can be written as

$$\max[\theta(B)] = \mathrm{tr}\{B'R'[R(X'X)^{-}R']^{-1}RB(E^{-1})\}.$$

Because $P[R(X'X)^{-}R']^{-1}P'$ is a generalized inverse for $C(X'X)^{-}C'$, the maximum value of $\theta(B)$ becomes

$$\max[\theta(B)] = \mathrm{tr}\{B'C'[C(X'X)^{-}C']^{-}CBE^{-1}\} \qquad (2.32)$$

which is the proposed test statistic for testing H_o: $CB = 0$, if $\hat{B}$ was substituted in place of B.

Simultaneous confidence intervals for $\boldsymbol{\lambda}'B$ can be constructed by substituting $(\hat{B} - B)$ for B in (2.26). Making this substitution yields

$$\theta(\hat{B}-B) = \{\boldsymbol{\lambda}'\mathrm{Vec}[C(\hat{B}-B)]\}^2/\{\boldsymbol{\lambda}'[E \otimes C(X'X)^{-}C']\boldsymbol{\lambda}\}. \qquad (2.33)$$

Because (2.32) is the maximum value of $\theta(B)$ this implies that

$$\theta(\hat{B}-B) \leq \mathrm{tr}\{(\hat{B}-B)'C'[C(X'X)^{-}C']^{-}C(\hat{B}-B)\}. \qquad (2.34)$$

The right hand side of (2.34) will be distributed the same as the null distribution of T^2. Let T^2_α be the 100(1-α) percentile of the null distribution of T^2. From (2.34) and the distribution of the right hand side of (2.34) the following result is obtained

$$\mathrm{Prob}[\theta(\hat{B}-B) \leq T^2_\alpha : \forall\ \boldsymbol{\lambda}] = 1-\alpha. \qquad (2.35)$$

Assuming **E** is positive definite and bringing the denominator of $\theta(\hat{\mathbf{B}}\text{-}\mathbf{B})$ to the right hand side of the inequality yields

$$\text{Prob}\{\boldsymbol{\lambda}'\text{Vec}[\mathbf{C}(\hat{\mathbf{B}}-\mathbf{B})]^2 \leq T^2_{\alpha}\boldsymbol{\lambda}'[\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']\boldsymbol{\lambda}: \forall\ \boldsymbol{\lambda}\} = 1-\alpha.$$

Bringing Vec(**CB**) to the right hand side yields the following 100(1-α) simultaneous confidence interval for $\boldsymbol{\lambda}'$Vec(**CB**)

$$LB \leq \boldsymbol{\lambda}'\text{Vec}(\mathbf{CB}) \leq UB$$

where LB is the lower bound of the confidence interval and is equal to

$$LB = \boldsymbol{\lambda}'\text{Vec}(\hat{\mathbf{CB}}) - \{T^2_{\alpha}\boldsymbol{\lambda}'[\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']\boldsymbol{\lambda}\}^{\frac{1}{2}}$$

and UB is the upper bound of the confidence interval and is equal to

$$UB = \boldsymbol{\lambda}'\text{Vec}(\hat{\mathbf{CB}}) + \{T^2_{\alpha}\boldsymbol{\lambda}'[\mathbf{E} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']\boldsymbol{\lambda}\}^{\frac{1}{2}}.$$

As with the proposed test statistic the reasonableness of the simultaneous confidence intervals depend on how closely the null distributions of **H** and **E** follow a central Wishart distribution and on **E** being positive definite.

<u>Example</u>

Data collected on 37 calves born to mature Hereford cows in 1977 will be used to illustrate an hypothesis test of fixed effects in an unbalanced multivariate mixed model. The data are presented in Table 1. In this experiment Angus (A) and Simmental (S) bulls were mated to mature Hereford cows. Weaning weight and yearling weight were recorded as measures of the calves' performance along with the calves' birth date, sire and sex. A complete description of the experiment, from which this subset of data was taken, can be found in Lawlor et al. (1984). Twenty one of the calves were sired by 9 Angus bulls and the remaining 16 of the calves were sired by 10 Simmental bulls. The model for the dependent variables, weaning weight and yearling weight is

$$Y = \sum_{i=1}^{3} X_i B_i + \sum_{j=1}^{2} Z_j U_j, \qquad (2.36)$$

where

Y is a 37 by 2 matrix where the first column consists of the 37 weaning weights in pounds and the second column consists of the 37 yearling weights in pounds,

X_1 is a 37 by 2 matrix consisting of ones and zeros with a one in the first column if calf was sired by an Angus bull and a one in the second column if the calf was sired by a Simmental bull,

B_1 is a 2 by 2 matrix of breed effects,

X_2 is a 37 by 2 matrix consisting of ones and zeros with a one in the first column if the calf is a female (F) and a one in the second column if the calf is a male (M),

B_2 is a 2 by 2 matrix of sex effects,

X_3 is a 37 by 1 matrix containing the day of the year that the calf was born,

B_3 is a 1 by 2 matrix containing the regression coefficients for birth date,

Z_1 is a 37 by 19 matrix of ones and zeros in which the i^{th} column contains a one if the calf was sired by the i^{th} sire,

U_1 is a 19 by 2 matrix containing the sire effects,

Z_2 is a 37 by 37 identity matrix,

Table 1. Weaning Weight and Yearling Weight Data for Angus and Simmental Sired Calves.

Weaning Weight	Yearling Weight	Sex of Calf	Sire Breed	Sire ID	Birth Date
540	1056	M	A	6216	081
430	0728	F	A	6216	110
375	0614	F	A	6217	096
550	1062	M	A	6217	072
475	0954	M	A	6219	074
485	0758	F	A	6219	075
540	1034	M	A	6220	081
465	0766	F	A	6220	102
485	1022	M	A	6221	082
510	0751	F	A	6222	068
475	0687	F	A	6222	092
475	1102	M	A	6223	072
470	1044	M	A	6223	082
465	0706	F	A	6223	082
540	1026	M	A	6223	070
470	1046	M	A	6223	094
410	0660	F	A	6224	071
475	1028	M	A	6224	093
465	0696	F	A	6224	077
530	0815	F	A	6229	075
445	0665	F	A	6229	100
480	0766	F	S	0003	079
475	0752	F	S	0006	090
540	0772	F	S	0006	093
435	0725	F	S	0008	091
540	1160	M	S	0008	089
415	1095	M	S	14	076
490	1058	M	S	14	077
505	0709	F	S	15	110
495	1046	M	S	SM39	095
470	0776	F	S	0048	110
430	0930	M	S	0048	096
475	1030	M	S	0048	091
470	1020	M	S	1004	096
490	0990	M	S	1004	116
360	0569	F	S	8603	117
440	0670	F	S	8607	091

and

U_2 is a 37 by 2 matrix of residuals.

The null hypothesis to be tested is that the breed of the sire has no effect on either the weaning weight or the yearling weight of calves. The hypothesis quadratic form was constructed using equation (2.11) as follows

$$\hat{\mathbf{B}} = \begin{bmatrix} 282.95 & 503.66 \\ 283.92 & 527.78 \\ 271.74 & 358.84 \\ 295.14 & 672.61 \\ -1.03 & -1.73 \end{bmatrix}$$

$$\mathbf{C} = (\ 1\ -1\ 0\ 0\ 0\ )$$

$$\mathbf{C}\hat{\mathbf{B}} = (\ -.97\ -24.12\ )$$

$$\mathbf{H} = \begin{bmatrix} 6.8 & 170.3 \\ 170.3 & 4236.5 \end{bmatrix}$$

It should be noted that $\hat{\mathbf{B}}$ is not unique, because **B** is not estimable. However, $\mathbf{C}\hat{\mathbf{B}}$ is unique, because **CB** is estimable. The error quadratic form was constructed from equation (2.12) and yielded

$$\mathbf{E} = \begin{bmatrix} 1805.2 & 1430.0 \\ 1430.0 & 2760.0 \end{bmatrix}$$

Substituting **H** and **E** into the test statistic (2.9) yielded

$$T^2 = 2.4441.$$

To determine if these differences are statistically significant approximate p-values are needed. These p-values along with the approximate degrees of freedom are presented in Table 2. Based on this data set it cannot be concluded that the breed of the sire has an effect on either the weaning weight or yearling weight of their calves.

Table 2. Approximate Degrees of Freedom and P-values.

	Degrees of Freedom		
Formula	Hypothesis	Error	p-value
2.16	1	14.558	.3485
2.17	1	14.802	.3476
2.18	1	14.615	.3483
2.19	1	14.838	.3474

Balanced Data

When the data are balanced it can be shown that the proposed test is equivalent to the Lawley-Hotelling test statistic for multivariate balanced data. The only difference between the proposed test and the test for balanced data is that the proposed test uses a MINQUE0 *type* estimator for Ω while the balanced data test uses the ordinary MINQUE0 estimator for Ω. Therefore, the tests will be the same if it can be shown that the MINQUE0 *type* estimator for Ω is the ordinary MINQUE0 estimator for Ω when the data are balanced.

The difference between the MINQUE0 *type* estimator and the ordinary MINQUE0 estimator is that the MINQUE0 *type* estimator has the condition that $\mathbf{QW} = 0$ while the ordinary MINQUE0 estimator has the condition that $\mathbf{QX} = 0$. If it can be shown that the column space of $\mathbf{W}$ is the same as the column space of $\mathbf{X}$, then the estimators and, therefore, the tests are equivalent. Theorem 2.4 will show that the column space of W and the column space of X are the same when the data are balanced and therefore provide the necessary framework for showing that for balanced data the MINQUE0 *type* estimator and the ordinary MINQUE0 estimator are equivalent.

Theorem 2.4: If the matrices $X_1, \ldots, X_r, Z_1, \ldots, Z_s$ have Kronecker structure (Definition 2.1), then the column space of W and the column space of X are the same.

Proof:

The matrix W can be partitioned as $[W_0 \mid W_1 \mid \ldots \mid W_s]$ where $W_0 = X$ and $W_j = Z_jZ_j'M_XA$. The column space of X is contained in the column space of W because $W_0 = X$. The column space of W will be contained in the column space of X if the column space of W_j, for $j=1,\ldots,s$ is contained in the column space of X. This will be true if $Z_jZ_j'X$ is contained in the column space of X. This will be true if $Z_jZ_j'X_i$, for $i=1,\ldots,r$, is contained in the column space of X_i. The matrix $Z_jZ_j'X_i$ can be written as

$$\begin{aligned} Z_jZ_j'X_i &= [G_{r+j(1)} \otimes \cdots \otimes G_{r+j(a)}]\cdot \\ &\quad [G'_{r+j(1)} \otimes \cdots \otimes G'_{r+j(a)}]\cdot \\ &\quad [G_{i(1)} \otimes \cdots \otimes G_{i(a)}] \\ &= \bigotimes_{k=1}^{a} [G_{r+j(k)}G'_{r+j(k)}G_{i(k)}]. \end{aligned}$$

The matrix $Z_jZ_j'X_i$ will be contained in the column space of X_i if $G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ is contained in the column space of $G_{i(k)}$, for $k=1,\ldots,a$. There are two possible values for $G_{i(k)}$, I and 1.

If $G_{i(k)}$ is equal to I, then $G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ is equal to

$G_{i(k)}G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ and is therefore, in the column space of $G_{i(k)}$.

If $G_{i(k)}$ is equal to 1, then the value of $G_{r+j(k)}$ must be considered, which also must be equal to I or 1. If $G_{r+j(k)}$ is equal to I, then $G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ is equal to $G_{i(k)}$ and is therefore, in the column space of $G_{i(k)}$. If both $G_{r+j(k)}$ and $G_{i(k)}$ are equal to 1, then $G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ is equal to $G_{i(k)}G'_{r+j(k)}G_{i(k)}$ and is therefore, in the column space of $G_{i(k)}$. Therefore, $G_{r+j(k)}G'_{r+j(k)}G_{i(k)}$ is in the column space of $G_{i(k)}$. Therefore, **W** is in the column space of **X**.

QED.

Conclusion

The test developed in this chapter provides an approximate procedure for testing fixed effects in an unbalanced multivariate mixed model. When the data are balanced the proposed test is exact. Several different approximations to both the null and non-null distributions of the test statistic were developed. In addition, an example provided an illustration of how the test is implemented. Simultaneous confidence intervals based on the test statistic were also constructed. In the next chapter, a Monte-Carlo study is conducted in order to examine the accuracy of the approximation.

CHAPTER 3

MONTE-CARLO STUDY

In the last chapter, an approximate test was developed for testing hypotheses about fixed effects in a multivariate mixed model for unbalanced data. Four approximations to the null distribution of the test statistics were used to obtain critical values. An exact test can also be constructed for testing the same hypotheses. The exact test is constructed by selecting a balanced subset of the full data set. Two questions that arise are what are the true sizes of the approximate tests and how does the power of the approximate test compare to an exact test. A Monte-Carlo study was conducted to answer these questions.

Data Set

Model

To select a reasonable model for use in the simulation, data from research comparing different breeds of cattle were used. A complete description of this experiment can be found in Lawlor et al. (1984). From this data set a subset of the data was selected. This subset contained the 76 calves born to mature Hereford cows in 1977. As in the example from the last chapter, weaning weight and yearling weight were recorded as measures of the calf's performance along with the calf's birth date, sire and sex. The data on the 37 calves sired by Angus and Simmental bulls are presented in Table 1. The data on the remaining 39 calves sired by Polled Hereford (PH), Horned Hereford (HH), and 1/2 Simmental-1/2 Hereford (SH) bulls are presented in Table 3. The number of sires in each of the sire breeds ranged from a minimum of four for the Horned Hereford to a maximum of 10 for the Angus. The number of calves produced by a particular sire ranged from a minimum of one to a maximum of five with an average of just over two calves per sire.

The model for the dependent variables, weaning weight and yearling weight, is that of (2.36) with the addition of effects for the three additional sire breeds and is presented below

$$Y = \sum_{i=1}^{3} X_i B_i + \sum_{j=1}^{2} Z_j U_j \qquad (3.1)$$

where

Y is a 76 by 2 matrix where the first column consists of the 76 weaning weights in pounds and the second column consists of the 76 yearling weights in pounds,

X_1 is a 76 by 5 matrix consisting of ones and zeros with a one in the first, second, third, fourth or fifth column if calf was sired by an Angus, Simmental, Polled Hereford, Horned Hereford, or 1/2 Simmental-1/2 Hereford bull respectively,

B_1 is a 5 by 2 matrix of breed effects,

X_2 is a 76 by 2 matrix consisting of ones and zeros with a one in the first column if the calf is a female (F) and a one in the second column if the calf is a male (M),

B_2 is a 2 by 2 matrix of sex effects,

Table 3. Weaning Weight and Yearling Weight Data for Polled Hereford, Horned Hereford, and 1/2 Simmental-1/2 Hereford Sired Calves.

Weaning Weight	Yearling Weight	Sex of Calf	Sire Breed	Sire ID	Birth Date
440	0940	M	PH	9501	083
570	1134	M	PH	9501	084
450	0965	M	PH	9501	078
480	1012	M	PH	9503	092
470	0710	F	PH	9503	107
470	0747	F	PH	9503	073
505	1014	M	PH	9504	082
450	0920	M	PH	9504	109
415	0645	F	PH	9505	088
425	0990	M	PH	9505	105
415	0982	M	PH	9505	106
380	0586	F	PH	9506	104
510	0750	F	PH	9507	072
375	0621	F	PH	9507	082
495	0968	M	HH	954	097
340	0948	M	HH	954	093
405	0629	F	HH	963	108
355	0909	M	HH	963	097
425	0595	F	HH	9001	090
445	1030	M	HH	9002	076
580	1146	M	HH	9002	087
545	0978	M	SH	840	081
540	1066	M	SH	840	090
475	1050	M	SH	842	078
430	0715	F	SH	842	083
370	0605	F	SH	842	113
415	0629	F	SH	843	114
425	0660	F	SH	843	104
415	0650	F	SH	843	096
400	0607	F	SH	844	086
490	0738	F	SH	846	098
505	1024	M	SH	846	098
405	0684	F	SH	847	095
460	0701	F	SH	847	102
390	0698	F	SH	848	076
390	0568	F	SH	848	084
490	0710	F	SH	925	082
405	0651	F	SH	925	103
425	0992	M	SH	925	117