Sinc-Galerkin solution of second-order hyperbolic problems in multiple space dimensions

by Kelly Marie McArthur

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

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A fully Galerkin method in space and time is developed for the second-order hyperbolic problem in one, two, and three space dimensions. Using sinc basis functions and the sinc quadrature rule, the discrete system arising from the orthogonalization of the residual is easily assembled. (Two equivalent matrix formulations of the systems are given. One lends itself to scalar computation while the other is more natural in a vector computing environment. In fact, it is shown that passing from . one to the other is simply a notational change. In either setting the move from one to two or three space dimensions does not significantly affect the ease of implementation. Intermediate diagonalization of each matrix representing the discretization of a second partial leads to the diagonalization of the overall system.

The method was tested on an extensive class of problems. Numerical results indicate the method has an exponential convergence rate for analytic and singular problems. Moreover that is independent of the spatial dimension.

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ABSTRACT

A fully Galerkin method in space and time is developed for the second-order hyperbolic problem in one, two, and three space dimensions. Using sinc basis functions and the sinc quadrature rule, the discrete system arising from the orthogonalization of the residual is easily assembled. Two equivalent matrix formulations of the systems are given. One lends itself to scalar computation while the other is more natural in a vector computing environment. In fact, it is shown that passing from one to the other is simply a notational change. In either setting the move from one to two or three space dimensions does not significantly affect the ease of implementation. Intermediate diagonalization of each matrix representing the discretization of a second partial leads to the diagonalization of the overall system.

The method was tested on an extensive class of problems. Numerical results indicate the method has an exponential convergence rate for analytic and singular problems. Moreover that is independent of the spatial dimension.

CHAPTER 1

INTRODUCTION

The general second-order, linear partial differential equation

(1.1) $$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad ,$$

where A, B, C, D, E, F, and G are functions of x and y only, is classified at the point (x,y) via the discriminate $(B^2 - 4AC)(x,y)$. That is, when $(B^2 - 4AC)(x,y)$ is positive, zero, or negative the equation at (x,y) is hyperbolic, parabolic, or elliptic, respectively. The present chapter reviews classical results for a purely hyperbolic, constant coefficient problem corresponding to $A = -1$, $C = 1$, and $B = D = E = F = 0$. Including boundary and initial conditions, the specific partial differential equation examined is

(1.2) $$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= G(x,t) & (x,t) &\in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) &= 0 & t &\ge 0 \\ u(x,0) &= f(x) & 0 &\le x \le 1 \\ u_t(x,0) &= g(x) & 0 &\le x \le 1 \quad . \end{aligned}$$

This problem is referred to as a one-dimensional wave equation since it models the displacement of a vibrating string with initial displacement f, initial velocity g, and subject to an external force G.

The equation in (1.2) is representative of one of the two hyperbolic canonical forms due to the absence of the term u_{xt}. In contrast, the distinguishing feature of the alternative canonical form is that its only second-order term is the mixed partial u_{xt}. The change of variables $\varsigma = x + t$ and $\eta = x - t$ transforms the equation in (1.2) to

$$(1.3) \qquad -4u_{\varsigma\eta} = G((\varsigma + \eta)/2, (\varsigma - \eta)/2) \qquad (\varsigma,\eta) \in (0,\infty) \times (-\infty,1) \quad .$$

It is well-known that the correct change of variables is found by solving two ordinary differential equations which depend on the discriminate. For a complete discussion see Farlow [1]. By integrating (1.3), restoring the variables x and t, and applying the initial and boundary conditions, d'Alembert explicitly solved (1.2) [2]. His result of 1747 is

$$(1.4) \qquad u(x,t) = \frac{1}{2}\left\{ f(x + t) + f(x - t) + \int_{x-t}^{x+t} g(\alpha)d\alpha + \int_0^t \int_{x-(t-\beta)}^{x+(t-\beta)} G(\alpha,\beta)d\alpha d\beta \right\}$$

where f, g, and G are extended as odd periodic functions when necessary.

The solution (1.4) is the sum of homogeneous and particular solutions, $u^H(x,t)$ and $u^P(x,t)$, respectively, where

$$(1.5) \qquad u^H(x,t) = \frac{1}{2}\left\{ f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\alpha)d\alpha \right\}$$

and

$$(1.6) \qquad u^P(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-\beta)}^{x+(t-\beta)} G(\alpha,\beta)d\alpha d\beta \quad .$$

From the form of the homogeneous solution, it is apparent that u^H at a fixed point (x_0,t_0) depends only on the initial conditions over the interval $[x_0 - t_0,\ x_0 + t_0]$ on the x-axis. As a result $[x_0 - t_0,\ x_0 + t_0]$ is called the interval of dependence. The particular solution may be used to find the domain of dependence shown by the shaded region in Figure 1 below. The curves representing the change of variables ς and η are called characteristics of (1.2). Notice that the characteristics cut the interval of dependence out of the x-axis and define the domain of dependence.

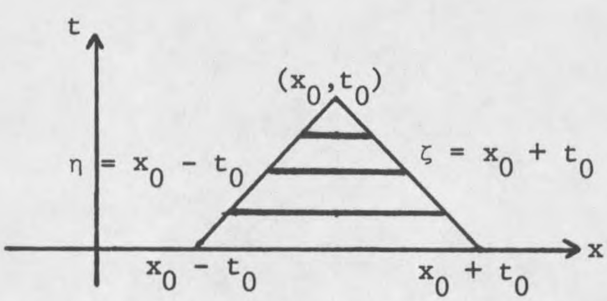

Figure 1. The Domain of Dependence.

The significance of the domain of dependence is the constraint it imposes on schemes used to numerically solve the one-dimensional wave equation (1.2). This constraint is called the Courant-Friedrichs-Lewy (or CFL) condition [3]. A scheme which provides a ready illustration of the condition is the centered finite difference method defined explicitly by the difference equation

$$(1.7) \qquad U_{i,j} = m^2(U_{i+1,j-1} + U_{i-1,j-1}) + 2(1 - m^2)U_{i,j-1} - U_{i,j-2} + (\Delta t)^2\, G(x_i, t_{j-1}) .$$

Here $U_{i,j}$ approximates u at $(x_i, t_j) = (i\Delta x, j\Delta t)$ and $m = \Delta t/\Delta x$ is the ratio of the stepsizes. A detailed derivation of the time-marching scheme (1.7) is included in Ames [4]. Stepping back in time determines the finite difference gridpoints which influence $U_{i,j}$ (see Figure 2 below). These gridpoints blanket the numerical domain of dependence outlined in Figure 2. The CFL criteria states that a necessary condition for the convergence of (1.7) (as Δt and $\Delta x \to 0$) is that the numerical domain of dependence must contain the analytic domain of dependence. This requires $m \le 1$. The computational impact of restricting $m \le 1$ is briefly discussed in Chapter 4.

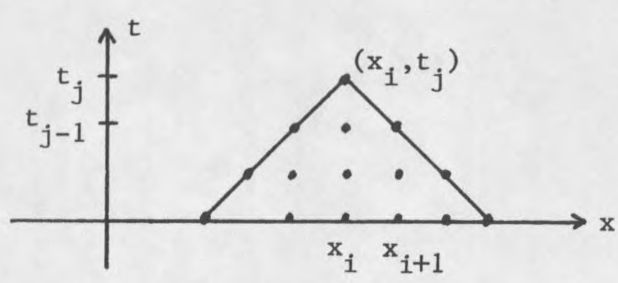

Figure 2. The Numerical Domain of Dependence for the Explicit, Centered Finite Difference Method.

The method of the present work satisfies the CFL condition trivially. This technique, called the Sinc-Galerkin method, builds an approximate solution for (1.2) valid on the entire domain. The approximate is a truncated generalized Fourier expansion of basis elements which are tensor products of sinc functions composed with suitable conformal maps. The support of each basis element is $(0,1) \times (0,\infty)$ hence, the method may be termed a spectral method and its numerical domain of dependence is identically the domain of the partial differential equation.

To ease the description of the Sinc-Galerkin method applied to (1.2), its counterpart for ordinary differential equations is derived in Chapter 2. Here the basis elements are single sinc functions composed with conformal maps. The pertinent sinc function properties needed to construct an approximate solution are reviewed. Further, it is shown that when $2N + 1$ basis functions are used to define this approximate, the optimal exponential order of convergence, $O(e^{-\kappa\sqrt{N}})$, $\kappa > 0$, occurs even in the presence of singularities. Stenger [5] discusses a more general setting than that considered in Chapter 2. The chapter closes with the formulation of the discrete linear system whose solution specifies the approximate. This system is symmetric, a result that Lund [6] has shown depends on the correct choice of weighted inner product.

Chapter 3 extends the Sinc-Galerkin method to (1.2) (with f = g = 0) and then further to the analogous wave equations in two and three space dimensions. At this writing, the most common procedure for solving partial differential equations with a semi-infinite time interval is a Galerkin discretization of the spatial domain with time dependent coefficients. The result is a system of ordinary differential equations usually solved via difference techniques. Botha and Pinder [3] develop Galerkin discretizations of space using finite element basis functions. In contrast, Gottlieb and Orszag [7] use globally defined spatial basis elements. They show that in space many of these spectral methods exhibit an exponential convergence rate. However, because Gottlieb and Orszag consider only finite difference techniques for the temporal domain, the time solution has finite-order accuracy. They acknowledge the incompatibility of the error statement in time versus space with the following remark:

> No efficient, infinite-order accurate time-differencing methods for variable coefficient problems are yet known. The current state-of-the-art of time-integration techniques for spectral methods is far from satisfactory on both theoretical and practical grounds...[7].

The point of view taken here differs from the two sources just cited by carrying the Galerkin discretization into the time domain. Chapter 5 reports numerical results which attest to the success of this notion.

Besides developing the Sinc-Galerkin method, Chapter 3 introduces notation to facilitate the description of the resulting discrete systems. These systems are posed in two algebraically equivalent matrix forms. The choice of form to use depends somewhat on available computing facilities. Chapter 4 discusses this topic along with algorithms for the solution of the linear systems in either form.

The nine examples included in Chapter 5 are broken into groups of three. Each group highlights a feature of the Sinc-Galerkin method for problems in one, two, and three space dimensions. For instance, the first three examples have analytic solutions while the second three have combinations of both algebraic and logarithmic singularities. The numerical results show that the rate of convergence is not affected by this singular behavior. The last three examples show the dramatic reduction in the size of the discrete system solved when care is exercised in parameter selections. Finally, each group indicates that the asymptotic error $O(e^{-K\sqrt{N}})$, $K > 0$, is attained independent of the dimension of the wave equation.

CHAPTER 2

THE SINC-GALERKIN SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The goal of this chapter is to derive the discrete Sinc-Galerkin system necessary to build an approximate to the solution of

$$\text{(2.1)} \qquad \begin{aligned} &Lf(x) \equiv f''(x) + \nu(x)f(x) = \sigma(x) \qquad a < x < b \\ &f(a) = f(b) = 0 \end{aligned}$$

valid on the interval (a,b). A symmetric matrix formulation for the system can be posed and is, in fact, easy to solve numerically. The resulting approximate solution converges to the true solution on (a,b) at the rate $O(e^{-K\sqrt{N}})$ where K is a positive constant and $2N + 1$ basis functions are used to build the approximate. Further, the convergence rate is maintained in the presence of singularities (the solution has an unbounded derivative) on the boundary. To prove these statements a background in general sinc function theory is necessary. In particular, the foundation for the error analysis of the Sinc-Galerkin method is the error associated with the truncated sinc quadrature rule.

Interpolation on $(-\infty,\infty)$

Numerical sinc function methods are rooted in E.T. Whittaker's [8] work concerning interpolation of a function at the integers. Rather than using well-known expansions based on polynomials, Whittaker sought an expansion whose properties are far more distinguished when applied in the proper setting. The foundation of the series is the sinc function

$$\text{(2.2)} \qquad \operatorname{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbf{R}$$

shown in Figure 3 below. The resulting formal cardinal series for a function f is

$$\text{(2.3)} \qquad \sum_{k=-\infty}^{\infty} f(k)\operatorname{sinc}(x-k) \quad .$$

To generalize (2.2) and (2.3) to handle interpolation on any evenly spaced grid define for $h > 0$

$$\text{(2.4)} \qquad S(k,h)(x) = \operatorname{sinc}\left(\frac{x-kh}{h}\right)$$

and denote the Whittaker cardinal function by

$$\text{(2.5)} \qquad C(f,h)(x) \equiv \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(x)$$

whenever this series converges. In engineering literature (2.5) is often called a band-limited series.

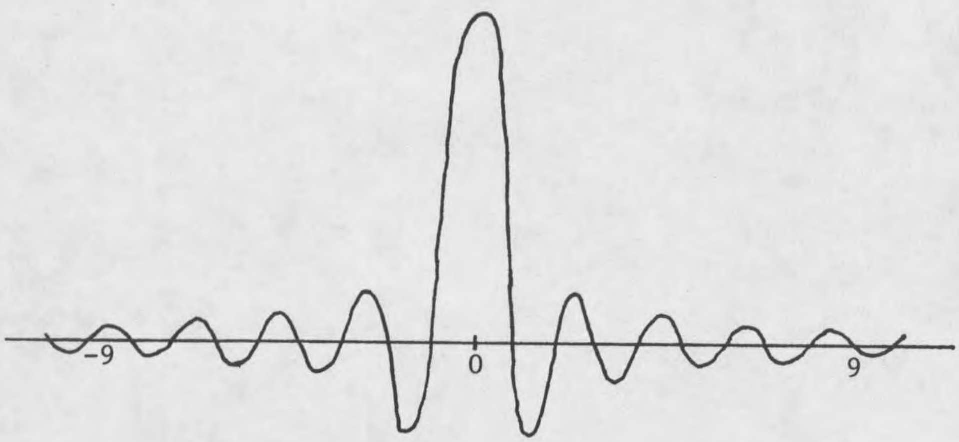

Figure 3: S(0,1)(x) = sinc(x) , $x \in \mathbf{R}$.

With regard to using (2.5) as an approximation tool, two important classes of functions must be identified. The first is the very restricted class for which (2.5) is exact. The second is the class of functions f for which the difference between f and C(f,h) is small. J.M. Whittaker [9] and McNamee, et.al. [10] accomplish this identification task by displaying a natural link between the Whittaker series and aspects of Fourier series and integrals.

The Fourier transform for a function g is

$$(2.6) \qquad \hat{g}(x) = \int_{-\infty}^{\infty} g(t)e^{ixt}dt \ .$$

A fundamental result of Fourier analysis is that if $g \in L^2(\mathbf{R})$ then $\hat{g} \in L^2(\mathbf{R})$ and g is recovered from $\hat{g}$ by the Fourier inversion integral

$$(2.7) \qquad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(x) e^{-ixt} dx \quad .$$

For a select set of functions, the Paley-Wiener Theorem shows that the inverse transform has compact support. Their result is

Theorem (2.8): If $g \in L^2(\mathbf{R})$, entire, and there exist positive constants A and C so that $|g(w)| \leq C\exp\{A|w|\}$ where $w \in \mathbb{C}$, then

$$(2.9) \qquad g(w) = \frac{1}{2\pi} \int_{-A}^{A} \hat{g}(x) e^{-ixw} dx \quad .$$

Showing that the sinc function satisfies the hypotheses of Theorem (2.8) with $A = \pi$ and $C = 1$ is straightforward. An elementary calculation gives

$$(2.10) \qquad \text{sinc}(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixw} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_{(-\pi,\pi)} e^{-ixw} dx \;;$$

hence, an immediate consequence of Theorem (2.8) is

$$(2.11) \qquad \widehat{\text{sinc}}(x) = \int_{-\infty}^{\infty} \text{sinc}(t) e^{ixt} dt = \chi_{(-\pi,\pi)}(x) \quad .$$

To accommodate the translated sinc function appearing in (2.4), a change of variables in (2.10) gives

$$(2.12) \qquad \widehat{S(k,h)}(x) = \int_{-\infty}^{\infty} \text{sinc}\left(\frac{t - \xi}{h}\right) e^{ixt} dt$$

$$= h\, e^{ix\xi}\, \chi_{(-\pi/h,\pi/h)}(x)$$

for fixed real ξ.

The support of the characteristic function in (2.12) prompts the definition of a class of functions, called the Paley-Wiener class, which is naturally associated with the Whittaker series.

Definition (2.13): Let B(h) be the set of functions g such that $g \in L^2(\mathbf{R})$, g is entire, and $|g(w)| \le C\exp\{\pi|w|/h\}$ where $w \in \mathbb{C}$.

Before the Whittaker function can be discussed, one result is vital.

Theorem (2.14): If $g \in B(h)$ then $g(w) = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\,\mathrm{sinc}\left(\frac{t-w}{h}\right)dt$.

A proof of Theorem (2.14) is found in [10]. For completeness the converse of Theorem (2.14) is

Theorem (2.15): If $g \in L^2(\mathbf{R})$ then $k(w) = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\,\mathrm{sinc}\left(\frac{t-w}{h}\right)dt$ is in B(h).

Proof: Using Parseval's Theorem with the inner product $\langle f,g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt$ establishes the growth estimate for k. Application of Morera's Theorem proves entirety and the Cauchy-Schwarz inequality yields $k \in L^2(\mathbf{R})$.

The significance of all the preceeding work is in the subsequent elegant theorem.

Theorem (2.16): If $g \in B(h)$ then $g(w) = \sum_{k=-\infty}^{\infty} \alpha_k \, S(k,h)(w)$

where $$\alpha_k = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\,\mathrm{sinc}\left(\frac{t-kh}{h}\right)dt = g(kh).$$

Proof: The Paley-Wiener Theorem, the identity

$$e^{-ixw} = \frac{h}{\pi}\sin\left(\frac{\pi w}{h}\right)\sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{ikhx}}{w+kh}, \tag{2.17}$$

and the uniform convergence theorem justify the ensuing steps:

$$g(w) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \hat{g}(x)e^{-ixw}dx$$

$$= \frac{h}{2\pi^2}\sin\left(\frac{\pi w}{h}\right)\int_{-\pi/h}^{\pi/h} \hat{g}(x)\left\{\sum_{k=-\infty}^{\infty}(-1)^k\frac{e^{ikhx}}{w+kh}\right\}dx$$

$$= \frac{h}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin(\pi w/h)}{w+kh}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(x)e^{ikhx}dx\right\}$$

$$= \frac{h}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin(\pi w/h)}{w+kh}\,g(-kh)$$

$$= \sum_{k=-\infty}^{\infty} g(kh)\,\mathrm{sinc}\left(\frac{w-kh}{h}\right).$$

Hence for $f \in B(h)$, $f(w) = C(f,h)(w)$ for all $w \in \mathbb{C}$. Note that this is a stronger result than originally sought. The initial quest was for a class of functions such that the Whittaker series was exact on **R**. An even stronger statement is derived from Theorem (2.16) and the identity

$$(2.18) \qquad \frac{1}{h}\int_{-\infty}^{\infty} S(k,h)(t)S(\ell,h)(t)dt = \delta_{\ell k}^{(0)} = \begin{cases} 1, & \text{if } \ell = k \\ 0, & \text{if } \ell \neq k \end{cases} ;$$

that is,

Theorem (2.19): The set $\left\{\frac{1}{\sqrt{h}} S(k,h)\right\}_{k=-\infty}^{\infty}$ is a complete orthonormal set in $B(h)$ [10].

Unfortunately the set $B(h)$ is extremely restrictive and some relaxation is necessary if (2.5) is used as a practical interpolatory tool. McNamee, et. al. [10] identifies a set, here called $B^p(D_S)$, where $C(f,h)$ is not exact but its approximation to f is very good. In particular, the domain of analyticity for $B^p(D_S)$ is D_S.

Definition (2.20): $D_S = \{z: z = x + iy, |y| < d \in \mathbf{R}, d > 0\}$

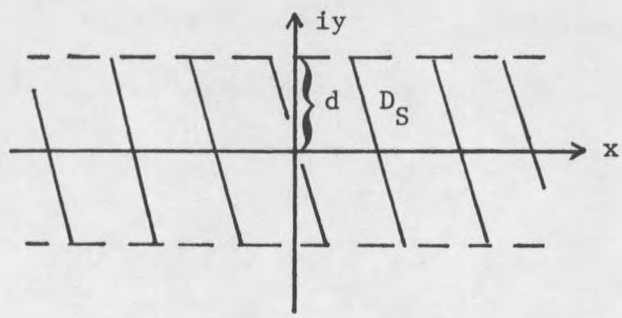

Figure 4: The Domain D_S.

The class $B^p(D_S)$ is specified by Definition (2.21).

Definition (2.21): $B^p(D_S)$ is the set of functions f such that

(2.22) f is analytic in D_S,

(2.23) $\int_{-d}^{d} |f(t+iy)|dy = O(|t|^a)$ as $t \to \pm\infty$ where $0 \le a < 1$,

and for p = 1 or 2

$$N^p(f,D_S) = \lim_{y \to d}\left[\left\{\int_{-\infty}^{\infty} |f(t+iy)|^p dt\right\}^{1/p} + \left\{\int_{-\infty}^{\infty} |f(t-iy)|^p dt\right\}^{1/p}\right] < \infty . \tag{2.24}$$

The exact form of the error is given by Theorem (2.25).

Theorem (2.25): If $f \in B^p(D_S)$ then $\varepsilon(f)(x) = f(x) - C(f,h)(x)$ where

$$\varepsilon(f)(x) = \frac{\sin(\pi x/h)}{2\pi i}\int_{-\infty}^{\infty}\left\{\frac{f(t-id)}{(t-x-id)\sin[\pi(t-id)/h]} - \frac{f(t+id)}{(t-x+id)\sin[\pi(t+id)/h]}\right\}dt . \tag{2.26}$$

Moreover, if $f \in B^1(D_S) \equiv B(D_S)$ then

$$\|\varepsilon(f)\|_\infty \le \frac{N^1(f,D_S)}{2\pi d \sinh(\pi d/h)} \equiv \frac{N(f,D_S)}{2\pi d \sinh(\pi d/h)} \tag{2.27}$$

while if $f \in B^2(D_S)$ then

$$(2.28) \qquad \|\varepsilon(f)\|_\infty \le \frac{N^2(f,D_S)}{2\sqrt{\pi d}\,\sinh(\pi d/h)}$$

and

$$(2.29) \qquad \|\varepsilon(f)\|_2 \le \frac{N^2(f,D_S)}{\sinh(\pi d/h)} \quad .$$

For a proof see Stenger [11]. Worth noting, is that the error statement of Theorem (2.25) is valid only on the real line. Theorem (2.16), recall, applies to the complex plane. However, the original goal was approximating on the real line and Theorem (2.25) certainly satisfies that.

Of far greater interest is the order statement derived from (2.27), (2.28), and (2.29). As $h \to 0$ $\sinh(\pi d/h) \to \infty$; hence, $\|\varepsilon(f)\|_{\infty,2} \to 0$ independent of whether $f \in B^1(D_S)$ or $B^2(D_S)$. Moreover, the rate of convergence is governed by $\sinh(\pi d/h)$; i.e., $1/\sinh(\pi d/h) = O(\exp(-\pi d/h))$ as $h \to 0$.

Although the exponential convergence rate is attractive, to be of practical importance it must be maintained when (2.5) is truncated. Denote the truncated Whittaker series for a function f by

$$(2.30) \qquad C_{M,N}(f,h)(x) \equiv \sum_{k=-M}^{N} f(kh)\,\mathrm{sinc}\left(\frac{x-kh}{h}\right)$$

where it is assumed $C(f,h)(x)$ converges.

Theorem (2.31): If $f \in B^p(D_S)$ for $p = 1$ or 2, $d > 0$ and there exist positive constants α and β such that

$$(2.32)\qquad |f(x)| \le L \begin{cases} e^{\alpha x} & \text{if } x \in (-\infty,0) \\ e^{-\beta x} & \text{if } x \in [0,\infty) \end{cases},$$

then choosing

$$(2.33)\qquad N = \left[\!\left[\frac{\alpha}{\beta} M + 1 \right]\!\right]$$

and

$$(2.34)\qquad h = (\pi d/(\alpha M))^{1/2}$$

gives

$$(2.35)\qquad \| f - C_{M,N}(f,h) \|_\infty \le C\, M^{1/2}\, e^{-(\pi d \alpha M)^{1/2}}$$

where C is a constant dependent on f.

Proof: From Theorem (2.25) there exists a constant L_1 such that $|f(x) - C(f,h)(x)| \le L_1 e^{-\pi d/h}$ for all $x \in \mathbb{R}$. Using the triangle inequality and (2.32),

$$\begin{aligned} |f(x) - C_{M,N}(f,h)(x)| &\le L_1 e^{-\pi d/h} + \sum_{k=M+1}^{\infty} |f(-kh)| + \sum_{k=N+1}^{\infty} |f(kh)| \\ &\le L_1 e^{-\pi d/h} + L \left\{ \sum_{k=M+1}^{\infty} e^{-\alpha k h} + \sum_{k=N+1}^{\infty} e^{-\beta k h} \right\} \\ &\le L_1 e^{-\pi d/h} + L \left\{ \frac{e^{-\alpha M h}}{\alpha h} + \frac{e^{-\beta N h}}{\beta h} \right\} . \end{aligned}$$

Now if N and h are defined by (2.33) and (2.34), then

$$\begin{aligned} |f(x) - C_{M,N}(f,h)(x)| &\le L_1 e^{-(\pi d \alpha M)^{1/2}} + L\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\sqrt{\frac{\alpha M}{\pi d}}\; e^{-(\pi d \alpha M)^{1/2}} \\ &\equiv C\, M^{1/2}\, e^{-(\pi d \alpha M)^{1/2}} . \end{aligned}$$

The choice of N and h is dictated by balancing asymptotic errors. The truncation errors for the lower and upper sums