Extremal properties characterizing weakly l-valent principal functions
by Dennis Evo Garoutte

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree DOCTOR OF PHILOSOPHY in Mathematics of
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Abstract:
Extremal properties, characterizing weakly λ-valent principal functions F0^ and F1^ defined on planar bordered Riemann surfaces, are developed. The λ-valent principal functions are constructed by using Sario's linear operator method. The extremal properties derived are analogous, in a rather natural way, to those of the univalent principal functions F0 and F1 of Sario. Such extremal properties characterize the λ-valent principal functions in a class much larger than the class of weakly λ-valent functions. The class consists of functions λ-valent near the border and near a fixed interior point, with the valence being arbitrary elsewhere.

The principal function F0^λ is shown to be the λ-th power of the univalent function F0. The function F0^λ then has the geometric property of being a weakly λ-valent radial slit disk mapping. The functional extremized by F1^λ is slightly different than the expected one. It does reduce to the expected one, however, when the class of competing functions is taken to be a smaller class consisting only of properly normalized λ-th powers of univalent functions.

129

EXTREMAL PROPERTIES CHARACTERIZING WEAKLY λ-VALENT PRINCIPAL FUNCTIONS

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DENNIS EVO GAROUTTE

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Head, Major Department

Chairman, Examining Committee

Graduate Dean

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TABLE OF CONTENTS

ABSTRACT

Extremal properties, characterizing weakly λ-valent principal functions F_0^λ and F_1^λ defined on planar bordered Riemann surfaces, are developed. The λ-valent principal functions are constructed by using Sario's linear operator method. The extremal properties derived are analogous, in a rather natural way, to those of the univalent principal functions F_0 and F_1 of Sario. Such extremal properties characterize the λ-valent principal functions in a class much larger than the class of weakly λ-valent functions. The class consists of functions λ-valent near the border and near a fixed interior point, with the valence being arbitrary elsewhere.

The principal function F_0^λ is shown to be the λ-th power of the univalent function F_0. The function F_0^λ then has the geometric property of being a weakly λ-valent radial slit disk mapping. The functional extremized by F_1^λ is slightly different than the expected one. It does reduce to the expected one, however, when the class of competing functions is taken to be a smaller class consisting only of properly normalized λ-th powers of univalent functions.

INTRODUCTION

L. Sario has used linear operators to establish the existence of certain univalent canonical mappings of planar bordered Riemann surfaces $\overline{W}$ onto slit disks [6]. These mappings F_0 and F_1, called principal analytic functions, are formed from principal harmonic functions, which themselves are constructed by applying the linear operator method of [5]. The singularity function s, which plays a central role in the linear operator method, is defined near the border γ as the flux function s_γ, which is constant on γ with flux 2π there, while near a fixed interior point ξ it is equal to $\log |z - \xi|$. By exhausting the planar bordered surface $\overline{W}$, one constructs the mappings F_0 and F_1 of $\overline{W}$ onto a slit plane disk, with radial or circular slits, possibly degenerate, removed.

In this investigation, we study the extremal properties of λ-th principal functions F_0^λ and F_1^λ which result when the singularity function s is taken to be s_γ^λ near γ and $\lambda \log |z - \xi|$ near ξ. Here s_γ^λ is constant on γ with a flux of $2\pi\lambda$. In a manner similar to [6] the canonical maps F_0^λ and F_1^λ are constructed. These maps were first shown to be λ-th powers by Nickel [4] and hence their geometric properties are completely understood. Namely, they are weakly λ-valent slit disk mappings. In a manner different from [4], it will be shown by using derived extremal properties that F_0^λ is the λ-th power of F_0. The similar property for F_1^λ is not proved here because it is needed to derive an extremal property for F_1^{λ}.

Chapter I is introductory by nature, and contains important theorems and definitions used in the sequel.

In Chapter II, we consider a planar bordered Riemann surface $\overline{W}$ of infinite connectivity. It is assumed that $\overline{W}$ has one isolated, compact, border component γ and a point $\xi \in \mathrm{Int}\,\overline{W}$ to be held fixed in our discussion. The surface $\overline{W}$ is approximated by an increasing sequence of compact bordered Riemann surfaces $\left\{\overline{W}_n\right\}$, each of finite connectivity and each containing γ and ξ. On W_n the analytic functions F_{on}^{λ} and F_{in}^{λ} are constructed. Among a class of normalized analytic functions F of W_n, F_{on}^{λ} maximizes the functional

$$2\pi\lambda \ \log r(F) + A(F)$$

and F_{in}^{λ} minimizes

$$2\pi\lambda \ \log r(F) - A(F) - 2\int_{\beta_n} \log |F_{in}^{\lambda}| \ d \arg F.$$

Here $r(F)$ represents the radius of the image of γ under F and $A(F)$ is the complementary logarithmic area.

In Chapter III it is then shown by using the Reduction Theorem (Sario [8]) that the extremal property holds in the limit for the limit function F_o^{λ}. A deviation formula is derived, from which it is established that F_o^{λ} is the λ-th power of the univalent function F_o of [6]. The geometric properties of F_o^{λ} may be inferred from those of F_o. In fact, F_o^{λ} is a weakly λ-valent radial slit disk mapping. Other extremal properties of F_o^{λ} are also derived.

It is shown in Chapter IV that the extremal property holds in the limit for the limit function F_1^λ . Sario's Reduction Theorem could not be applied and an argument based on the convergence of certain Dirichlet integrals is used. One additional extremal property is also established.

Such extremal properties characterize the λ-valent principal functions in a much larger class of functions. Namely, this class of functions consists of functions which are λ-valent near Υ and ξ . There are no further restrictions.

CHAPTER I

PRELIMINARIES

Important definitions and theorems used throughout this discourse are listed in this chapter.

§ 1. Basic properties of Riemann surfaces.

Definition. A Riemann surface (bordered Riemann surface) is a connected Hausdorff topological space W together with a covering V by open sets O and a family Φ of mappings φ satisfying the following properties:

1. Each $\varphi \in \Phi$ is a homeomorphism of an $O \in V$ onto an open set of the complex plane (closed upper half plane).

2. Every $O \in V$ is the domain for some $\varphi \in \Phi$.

3. If O_1 and O_2 are the domains of φ_1 and φ_2 then $\varphi_2 \circ \varphi_1^{-1}$ is a directly conformal mapping of $\varphi_1 (O_1 \cap O_2)$.

We will denote a bordered Riemann surface by $\overline{W}$ and W will be its interior, which is itself a Riemann surface.

Definition. The border of a bordered Riemann surface $\overline{W}$ is that set of points belonging to $\overline{W}$ which is mapped into the real axis by some $\varphi \in \Phi$.

A bordered Riemann surface $\overline{W}$ is known to be orientable, and hence the border has a positive as well as a negative direction.

Definition. Let C be an arbitrary Jordan curve on a Riemann Surface W. The surface is called planar if whenever each p on C has a neighborhood N such that if N - C is composed of two components, then W - C is composed of at least two components. That is to say, if C locally separates W then it also separates W.

Definition. A real-valued (complex-valued) function f on W is called harmonic (analytic) if the functions $f \circ \varphi^{-1}$ are, where they are defined, harmonic (analytic) in the usual sense, for all $\varphi \in \Phi$.

Definition. An exhaustion of a Riemann surface W is a sequence $\{W_n\}$ of compact bordered Riemann surfaces satisfying:

1. $\overline{W}_n \subset W_{n+1}$

2. The boundary of $\overline{W}_n$ consists of a finite number of disjoint analytic Jordan curves.

3. Each component of $W - W_n$ is relatively non-compact.

4. $W = \cup W_n$

The sets W_n will be referred to as "approximating surfaces."

Let u and v be harmonic functions on a compact bordered Riemann surface $\overline{W}$. Then Green's formula may be expressed as

$$(1) \quad \iint_{\overline{W}} (u_x v_x + u_y v_y)\, dxdy = \int_{B(\overline{W})} u\, dv* = \int_{B(\overline{W})} v du*$$

where $B(\overline{W})$ is the border of $\overline{W}$ and $du*$ is the conjugate differential.

The left side of (1) is the mixed Dirichlet integral for the surface $\overline{W}$ and is denoted by $D_{\overline{W}}(u, v)$. Green's formula may also be expressed as

$$\int_{B(\overline{W})} udv* - vdu* = 0.$$

For functions p, harmonic on Int $\overline{W}$, but possibly not defined on $B(\overline{W})$, $\int_{\beta_i} pdp*$ is understood as $\lim_n \int_{\beta_i'}(W_n) pdp*$. Here β_i' denotes the border of an approximating surface W_n and is homologous to β_i.

We shall be concerned with an important solution of the Dirichlet boundary value problem for a Riemann surface. Such is called a <u>flux function</u> and is defined in the following manner. Let σ be a border component of a compact bordered Riemann surface $\overline{W}$. In addition, let σ' be a curve contained in the interior of $\overline{W}$ which is homologous to σ. Denote the region bounded by σ and σ' by $N(\sigma)$. In $N(\sigma)$ let w solve the following boundary value problem; namely, $w = 0$ on σ, $w = 1$ on σ', and w is harmonic in $N(\sigma)$. A <u>flux function</u> for a neighborhood of σ is defined as:

$$s_\sigma^\lambda = (2\pi\lambda / \int_\sigma dw*) \; w.$$

Evidently, s_σ^λ has flux $2\pi\lambda$, is constant on σ', 0 on σ, and harmonic in $N(\sigma)$.

§ 2. <u>Normal linear operators on a Riemann surface</u>.

<u>Definition</u>. G is a <u>regular region</u> of an open Riemann surface W if (1) G is relatively compact, (2) G and its exterior have a common boundary which is a 1-dimensional analytic submanifold, and (3) all

components of $W - G$ are non-compact.

Now let K be any compact set contained in W and let u be a function harmonic on its complement. It is possible to enclose K with a regular region G with a positively oriented boundary β. The flux of u will be denoted by $\int_\beta du*$. That this is independent of the choice of G is a consequence of Green's formula. As described above, β will be said to represent the ideal boundary of W. The border of a compact bordered Riemann surface $\overline{W}$ trivially represents the ideal boundary β of $\overline{W}$, while $\cup\beta_i$, where β_i are the border components of an exhausting W_n, represents the ideal boundary of the Riemann surface W.

Definition. W' is a regularly embedded open set of the open Riemann surface W if the interior and exterior of W' are distinguishable in the following sense: W' and its exterior have the same boundary and this boundary is a 1-dimensional submanifold.

Let α be the boundary of a regularly embedded open set W' with compact complement. We denote by L an operator whose domain consists of the set of continuous real-valued functions on α and whose range is the set of functions harmonic on W' and continuous on $\overline{W}'$. In addition, L has the following properties:

1. $L(f) = f$ on α.
2. $L(c_1 f_1 + c_2 f_2) = c_1 Lf_1 + c_2 Lf_2$.
3. $L1 = 1$.
4. If $f \geq 0$, then $Lf \geq 0$.
5. $\int_\beta d(Lf)* = 0$.

Here β represents the ideal boundary of W. Such an operator is called a <u>normal linear operator</u>.

Some important examples of normal linear operators are the principal operators L_0 and L_1 of [2]. For a given f on α, $L_0 f$ solves the boundary value problem in W' with a vanishing normal derivative on $\beta(W)$. To define L_1 we let Uf be the solution of the boundary value problem with boundary values f on α and 0 on β. Let w be the harmonic function which has boundary values 0 on α and 1 on β. Then $L_1 f$ is defined by the equation

$$L_1 f = Uf - \left(\int_\beta d(Uf)* \Big/ \int_\beta dw*\right) w.$$

Observe that $L_1 f$ equals f on α and is constant on β with vanishing flux there.

A third example is the operator L^*, referred to as the operator associated with a removable singularity. If Δ is a parametric disk centered at a point ξ of W and f is a continuous real-valued function defined on the boundary α of Δ, then L^*f is harmonic on Δ, continuous on $\overline{\Delta}$, and equal to f on α.

In general, W' need not be connected. We can therefore consider separating W' into a finite number of disjoint sets W'_i with boundary α_i. Let f_i be the restrictions of f to α_i and suppose that an operator L_i is defined on each W'_i. Then we can define a new operator L on W' by setting $Lf = L_i f_i$. Such an operator L is called the <u>direct sum</u> of the operators L_i.

The following linear operator theorem (Ahlfors and Sario [2]) is used to construct harmonic functions which will play a key role in this investigation.

Theorem. Let W be an open Riemann surface and L a normal linear operator defined with respect to a regularly embedded open set $W' \subset W$ where W' has compact complement. If s is harmonic on W' and continuous on $\overline{W}'$ with vanishing flux, then there exists a function p harmonic on W satisfying on W' $p - s = L(p - s)$. The function p is uniquely determined to within a constant.

That is to say, p modulo s has the properties of L on W'.

§ 3. The Reduction Theorem.

Let $\{W_n\}$ be an exhaustion of the open surface W, and $\{f_n\}$ be a sequence of functions such that f_n is harmonic (analytic) on W_n.

Definition. $\{f_n\}$ converges on compact sets of W if for every compact set $K \subset W$, there exists an N such that $\{ f_n ; n \geq N \}$ converges uniformly on K.

Definition. $\{f_n\}$ is a normal family on W when every sequence of $\{ f_n \}$ contains a subsequence which converges on compact sets of W.

A well known sufficient condition for a family of analytic functions $\{ f_n \}$ to be normal is the following. If for every compact set K, there exists a constant M and an integer N such that $|f_n(z)| < M$ for all

$n > N$ and all $z \in K$, then $\{f_n\}$ is a normal family.

Let $\{W_n\}$ be an exhaustion of W and let C and C_n for each n be classes of functions with domains W and W_n respectively. In addition, let m and m_n be functionals defined on W and W_n.

<u>Reduction</u> <u>Theorem</u> (Sario [8]). Suppose that the functionals m and m_n satisfy the following conditions with respect to the classes C and C_n:

1. $m(f) = \lim_n m_n(f)$.

2. If $W_m \subset W_n$ and if $f \in C_n$ then $f|_{W_m} \in C_m$. The same must be true if W_n is replaced by W and C_n by C.

3. If $\{f_k\}$ is a sequence of functions in C_n and if $\{f_k\}$ converges uniformly to $f \in C_n$ then $m_n(f_k)$ converges to $m_n(f)$.

4. There exists a function $f_n \in C_n$ such that f_n minimizes the functional m_n among all $f \in C_n$.

5. For $k < n$ and $f \in C_n$ $m_k(f) \leq m_n(f)$.

6. The family of minimizing functions mentioned in (4) is a normal family with the limiting functions belonging to C.

Then any limit function $f = \lim_n f_n$ minimizes m among all $f \in C$, and the value of the minimum is $m(f) = \lim_n m_n(f_n)$.

CHAPTER II

EXTREMAL PROPERTIES OF HARMONIC AND ANALYTIC FUNCTIONS ON APPROXIMATING SURFACES

Consider a planar bordered Riemann surface $\bar{W}$ of infinite connectivity and having one compact border component γ. In order to describe the remaining part of the boundary of W, we recall that such a surface can be embedded in a Riemann sphere S^2 (Ahlfors and Sario [2]). With respect to this embedding, it is assumed that no point of γ is a limit point of points belonging to any other boundary components. Such a border component is called "isolated". Also, it is assumed that the surface $\bar{W} - \gamma$ is open in S^2. We fix a point ξ belonging to the interior of $\bar{W} - \gamma$. Operations in $\bar{W}$ such as Int, boundary, etc., are referred to S^2 in the following.

It is known that such a surface $\bar{W}$ can be exhausted by a sequence of compact bordered surfaces $\{\bar{W}_n\}$, each of finite connectivity and each containing γ and ξ. Furthermore, $\bar{W}_n \subset W_{n+1}$ and $W = \bigcup_1^\infty W_n$. Evidently, γ is one border component of $\bar{W}_n$. Denote the remaining border components by $\beta_1, \beta_2, \ldots, \beta_{k(n)}$ and for convenience set $\bar{\beta}_n = \bigcup_1^k \beta_i$.

In this chapter we propose to develop extremal properties of certain harmonic and analytic functions defined on the approximating surfaces W_n.

§ 4 . Construction of harmonic functions on approximating surfaces.

Using Sario's linear operator method described in [5], the harmonic functions p_{on}^{λ} and p_{in}^{λ} will be constructed on each of the approximating regions W_n. The singularity function s will be taken to be $\lambda \log |z-\xi|$ in a neighborhood of ξ, identically zero in neighborhoods of the β_i, and equal to the singularity function s_γ^λ near γ. The singularity function s_γ^λ is that function which has flux equal to $2\pi\lambda$. Note that the total flux of s is zero.

The direct sum of the linear operators L_1 near γ, L_0 near each β_i, and L^* near ξ, with an application of the linear operator method, yields the λ-th harmonic function p_{on}^λ up to a constant. The function p_{on}^λ is normalized at ξ so as to get property ii) of $H_n(\lambda)$ satisfied. Here, L^* is the operator associated with the class of harmonic functions having a removable singularity at ξ. The λ-th harmonic function p_{1n}^λ is derived from the same direct sum of operators with the exception that L_1 is used near each β_i.

The functions p_{on}^λ and p_{1n}^λ possess the necessary properties required to belong to a class of harmonic functions $H_n(\lambda)$ which will be defined in the next section. In addition, the function p_{on}^λ has its normal derivative $\frac{\partial p_{on}^\lambda}{\partial n} = 0$ on each β_i, while p_{1n}^λ is constant on each β_i with flux $\int_{\beta_i} dp_{1n}^{\lambda *} = 0$. These functions will be seen to satisfy certain extremal properties.

§ 5 . Extremal properties of harmonic functions on approximating surfaces.

The extremal properties that the λ-th harmonic functions are to enjoy are with respect to classes of functions defined as follows:

Definition. $H_n(\lambda)$ is the set of functions $p(z)$ which are harmonic on $W_n - \xi$ while satisfying i.) $p(z) = \text{const.} = c(p)$ for $z \in \gamma$ and $\int_\gamma dp_* = 2\pi\lambda$, and ii.) the function $h(z) = p(z) - \lambda \log|z-\xi|$ has a harmonic continuation to ξ with $h(\xi) = 0$.

The harmonic functions p_{on}^{λ} and p_{1n}^{λ} belong to the class $H_n(\lambda)$. A lemma that is essential in obtaining their extremal properties will first be stated and proved.

Lemma 1. If δ is the border of a parametric disk about ξ, with orientation induced by the surface W_n, then for every p and $q \in H_n(\lambda)$ we have $\int_\delta p\,dq_* - q\,dp_* = 0$.

Proof. Adding and subtracting the same term we have

$$(1) \quad \int_\delta p\,dq_* - q\,dp_* = \int_\delta (p-q)\,dq_* - q\,d(p-q)_* .$$

Since the function $p-q$ has a removable singularity at ξ, it has a single valued conjugate and, upon using an integration by parts, the right hand side of (1) may be written as

$$\int_\delta (p-q)\,dq_* + (p-q)_*\,dq .$$

But this is equal to $\text{Im}\int_\delta (P-Q)\,dQ$ where $P = p + ip_*$ and $Q = q + iq_*$. The functions P and Q are multiple valued but the function $P-Q$ is single valued. Using property ii.) of $H_n(\lambda)$ together with Cauchy's Integral Formula this can be seen to be zero.

The extremal property that p_{on}^{λ} possesses is stated in the following theorem.

Theorem 1. Among all $p \in H_n(\lambda)$ the function p_{on}^{λ} maximizes the functional $\Phi_n(p) = 2\pi\lambda c(p) - \int_{\bar{\beta}_n} p dp_*$. The value of the maximum is $2\pi\lambda\ c(p_{on}^{\lambda})$. The deviation of this functional from its maximum is $D_{\bar{W}_n}(p-p_{on}^{\lambda})$ and the maximizing function is unique.

Proof. By Green's formula we have

$$D_{\bar{W}_n}(p-p_{on}^{\lambda}) = \int_{\gamma\cup\bar{\beta}_n} (p-p_{on}^{\lambda})\ d\ (p-p_{on}^{\lambda})_* .$$

But since p and $p_{on}^{\lambda} \in H_n(\lambda)$, $\int_{\gamma} (p-p_{on}^{\lambda})\ d\ (p-p_{on}^{\lambda})_* = 0$.

Green's formula then becomes

(2) $$D_{\bar{W}_n}(p-p_{on}^{\lambda}) = \int_{\bar{\beta}_n} p dp_* + \int_{\bar{\beta}_n} p_{on}^{\lambda}\ d\ p_{on}^{\lambda}{}_* - \int_{\bar{\beta}_n} p_{on}^{\lambda}\ dp_* + p\ dp_{on*}^{\lambda} .$$

The second term of the right hand expression vanishes because the normal derivative $\frac{\partial p_{on}^{\lambda}}{\partial n} = 0$ on $\bar{\beta}_n$. For the same reason $\int_{\bar{\beta}_n} p dp_{on*}^{\lambda} = 0$. Therefore we change its sign and upon applying Green's formula obtain for the last term of (2) the expression

$$\int_{\gamma\cup\delta} p_{on}^{\lambda}\ dp_* - p dp_{on*}^{\lambda} .$$

Here δ is the border of a parametric disk Δ about ξ with orientation induced by $W_n - \Delta$. By lemma 1.) the contribution of δ to this integral is 0. Using property i.) of $H_n(\lambda)$ the part due to γ is $2\pi\lambda\ c_{\gamma}(p_{on}^{\lambda}) - 2\pi\lambda\ c(p)$. Collecting all of these contributions we obtain the formula

(3) $$2\pi\lambda\, c(p_{on}^{\lambda}) - D_{\bar{W}_n}(p-p_{on}^{\lambda}) = 2\pi\lambda\ c_{\gamma}(p) - \int_{\bar{\beta}_n} p dp_* .$$

Since $D_{\bar{W}_n}(p-p_{on}^{\lambda}) \geq 0$ we have the desired extremal property. The deviation is clearly $D_{\bar{W}_n}(p-p_{on}^{\lambda})$.

As for the uniqueness of the maximizing function, suppose another function p^1 maximizes the functional. Then the deviation $D_{\bar{W}_n}(p^1 - p_{on}^{\lambda}) = 0$ and hence $p^1 - p_{on}^{\lambda} = C$ (constant). But by the normalization at ξ, $C = 0$ and $p^1 = p_{on}^{\lambda}$.

The function p_{1n}^{λ} possesses an extremal property similar to that of p_{on}^{λ}, but the functional that it extremizes differs from the one originally anticipated, which was $2\pi\lambda\, c(p) + \int_{\beta_n} p\,dp*$.

<u>Theorem 2</u>. The function p_{1n}^{λ} minimizes the functional

$$\psi_n(p) = 2\pi\lambda\, c(p) + \int_{\beta_n} p\,dp* - 2\int_{\beta_n} p_{1n}^{\lambda}\, dp*.$$

among all $p \in H_n(\lambda)$. The minimum value is $2\pi\lambda\, c(p_{1n}^{\lambda})$. The deviation from this minimum is $D_{\bar{W}_n}(p - p_{1n}^{\lambda})$ and the minimizing function is unique.

<u>Proof</u>. We have by using Green's Theorem

$$D_{\bar{W}_n}(p - p_{1n}^{\lambda}) = \int_{\gamma \cup \beta_n} (p - p_{1n}^{\lambda})\, d\,(p - p_{1n}^{\lambda})*.$$

As in Theorem 1 the contribution of γ to the integral is 0, and we write

$$(4)\quad D_{\bar{W}_n}(p - p_{1n}^{\lambda}) = \int_{\beta_n} p\,dp* + \int_{\beta_n} p_{1n}^{\lambda}\, d\, p_{1n}^{\lambda}* - \int_{\beta_n} p\,dp_{1n}^{\lambda}* - p_{1n}^{\lambda}dp* - 2\int_{\beta_n} p_{1n}^{\lambda}\, dp*.$$

Since p_{1n}^{λ} is constant on each β_i and the flux $\int_{\beta_i} d\, p_{1n}^{\lambda}* = 0$ on each β_i the second term in (4) vanishes. Upon applying Green's Theorem again, the third term may be written as

$$\int_{\gamma \cup \delta} p\,dp_{1n}^{\lambda}* - p_{1n}^{\lambda}\, d\, p*.$$

By lemma 1. the contribution of δ to the above integral is 0 and, using property i.) of $H_n(\lambda)$, this reduces to $2\pi\lambda\, c(p) - 2\pi\lambda\, c(p_{1n}^{\lambda})$. Hence, (4) becomes

$$(5)\quad 2\pi\lambda\, c(p_{1n}^{\lambda}) + D_{\bar{W}_n}(p-p_{1n}^{\lambda}) = 2\pi\lambda\, c(p) + \int_{\beta_n} p\,dp^* - 2\int_{\beta_n} p_{1n}^{\lambda}dp^*.$$

Since $D_{\bar{W}_n}(p-p_{1n}^{\lambda}) \geqq 0$ we have the desired result. The uniqueness of the minimizing functional follows exactly as the argument used in Theorem 1.

§ 6. Extremal properties of analytic functions on approximating surfaces.

The classes of functions in which the λ-th analytic functions are to enjoy their extremal properties are defined as follows:

Definition. $A_n(\lambda)$ is the class of functions F analytic on W_n such that the following properties are satisfied:

i.) $|F(z)| = r(f) =$ const. for $z \in \gamma$ with $\int_{\gamma} d \arg F = 2\pi\lambda$.

ii.) F has a λ-th order zero at ξ.

iii.) $\lim_{z\to\xi} F(z)/(z-\xi)^{\lambda} = 1$.

Some interesting connections between the classes $H_n(\lambda)$ and $A_n(\lambda)$ are expressed in the following theorem.

Theorem 3. If $F \in A_n(\lambda)$ then $\log|F| \in H_n(\lambda)$. Furthermore, the λ-th analytic functions $F_{on}^{\lambda} = \exp(p_{on}^{\lambda} + i\, p_{on}^{\lambda}*)$ and $F_{1n}^{\lambda} = \exp(p_{1n}^{\lambda} + i\, p_{1n}^{\lambda}*)$ belong to $A_n(\lambda)$.

Proof. The function $\log|F|$ will be harmonic on $W_n - \xi$ provided $F \neq 0$ on $W_n - \xi$. An application of the argument principle gives the number of zeros Z_o on W_n as

$$Z_o = \frac{1}{2\pi}\int_{B(\bar{W}_n)} d \arg F = \frac{1}{2\pi}\left\{\int_{\gamma} d \arg F + \int_{\beta_n} d \arg F\right\} = \lambda.$$

Hence, the only place where $F = 0$ on W_n is at ξ.

Now F has a λ-th order zero at ξ so that near ξ, $F(z) = (z-\xi)^{\lambda} \cdot G(z)$ with $G(\xi) \neq 0$. By the normalization condition iii.) of $A_n(\lambda)$ it follows that $G(\xi) = 1$. Hence, the function h(z) mentioned in property ii.) of $H_n(\lambda)$ may be written as $h(z) = \log|F| - \lambda\log|z-\xi| = \log|G(z)|$. A sufficient condition for $h(z)$ to be harmonically extendable is that $\lim_{z\to\xi} (z-\xi)\, h(z) = 0$. This is satisfied and furthermore, by continuity considerations, the extension takes the proper value 0 since $\lim_{z\to\xi} h(z) = \log|G(\xi)| = 0$. Property ii.) of $H_n(\lambda)$ is therefore satisfied and property i.) is readily checked.

We prove the second part of Theorem 3 for F_{on}^{λ} and omit the analogous proof for F_{in}^{λ}. We must see that $F_{on}^{\lambda} = \exp(p_{on}^{\lambda} + i\, p_{on}^{\lambda}*)$ is defined as a function. The multiple valued conjugate function $p_{on}^{\lambda}*$ has periods $\int_{\gamma} d\, p_{on}^{\lambda}* = 2\pi\lambda$ and $\int_{\beta_i} d\, p_{on}^{\lambda}* = 0$ for each i. That is to say, it has a period of $2\pi\lambda$ and we have F_{on}^{λ} defined as a function. Here, F_{on}^{λ} is normalized by setting h*, the conjugate of the continuation for $p_{on}^{\lambda} - \lambda\log|z-\xi|$, equal to zero at ξ. The mapping F_{in}^{λ} is normalized in a similar manner. We have $|F_{on}^{\lambda}(z)| = \exp(p_{on}^{\lambda}(z)) = $ const. for $z \in \gamma$, and $\int_{\gamma} d \arg F_{on}^{\lambda} = \int_{\gamma} d\, p_{on}^{\lambda}* = 2\pi\lambda$. Using property ii.) of $H_n(\lambda)$, properties ii.) and iii.) of $A_n(\lambda)$ may be obtained.

Theorem 3 leads us to the following extremal properties for F_{on}^{λ} and F_{in}^{λ} stated below in Theorems 4 and 5.

Theorem 4. The function F_{on}^{λ} uniquely maximizes the functional

$$2\pi\lambda \log r(F) - \int_{\beta_n} \log |F| \, d \arg F$$

among $F \in A_n(\lambda)$. The maximum is $2\pi\lambda \log r(F_{on}^{\lambda})$ and the deviation from the maximum is $D_{\bar{W}_n}(\log |F/F_{on}^{\lambda}|)$.

Proof. We have $\log |F_{on}^{\lambda}| = p_{on}^{\lambda}$ and according to Theorem 1. this maximizes the functional $2\pi\lambda \, c(p) - \int_{\beta_n} p dp_*$. And by Theorem 3., if $F \in A_n(\lambda)$, then $\log |F| \in H_n(\lambda)$. Hence, F_{on}^{λ} maximizes the functional $2\pi\lambda \log r(F) - \int_{\beta_n} \log |F| \, d \arg F$ among all $F \in A_n(\lambda)$. The deviation is $D_{\bar{W}_n}(\log |F| - \log | F_{on}^{\lambda}|) = D_{\bar{W}_n}(\log |F/F_{on}^{\lambda}|)$.

Uniqueness is argued as in Theorem 1.

In an analogous manner the following theorem expressing the extremal property for F_{1n}^{λ} may be proved.

Theorem 5. The function F_{1n}^{λ} uniquely minimizes the functional

$$2\pi\lambda \log r(F) + \int_{\beta_n} \log |F| \, d \arg F - 2\int_{\beta_n} \log |F_{1n}^{\lambda}| d \arg F$$

among $F \in A_n(\lambda)$. The minimum is $2\pi\lambda \log r(F_{1n}^{\lambda})$ and the deviation from the minimum is $D_{\bar{W}_n}(\log |F/F_{1n}^{\lambda}|)$.

CHAPTER III

EXTREMAL PROPERTIES OF THE FIRST λ-TH PRINCIPAL HARMONIC AND ANALYTIC FUNCTIONS

Extremal properties for surfaces of infinite connectivity which generalize the results of Chapter II will be derived in this chapter, with the main tool used being Sario's Reduction Theorem [8] :

§ 7 . Extremal properties of the first λ-th principal harmonic function.

Let $\overline{W}$ be a planar bordered Riemann surface and $\{\overline{W}_n\}$ an exhausting set as described in Chapter II.

Lemma 1. The family $\{p_{on}^{\lambda}\}$ is a normal family on the surface $W - \xi$.

Proof. Because $p_{on}^{\lambda} = \log |F_{on}^{\lambda}|$, it suffices to show that the family $\{F_{on}^{\lambda}\}$ is normal. A well known sufficient condition for normality of a family $\{f_n\}$ of analytic functions on a region is that the functions, for sufficiently large n, be uniformly bounded on every compact subset of the region.

Let K be an arbitrary compact set contained in W and choose N large enough such that $K \subset W_N$. Denote by $A(F_{on}^{\lambda}, W_N)$ the quantity $-\int_{\beta_N} \log |F_{on}^{\lambda}|\, d \arg F_{on}^{\lambda}$ for $n \geqq N$. This is a non-negative quantity, for consider the Dirichlet integral over the surface $\overline{W}_n - W_N$, whose orientation is induced by W. Then

$$(1)\quad 0 \leqq D_{\overline{W}_n - W_N}(\log |F_{on}^{\lambda}|) = \int_{\beta_n} \log |F_{on}^{\lambda}|\, d \arg F_{on}^{\lambda} + A(F_{on}^{\lambda}, W_N).$$

The first integral of (1) is zero and $A(F_{on}^{\lambda}, W_N) \geq 0$. Hence we have

(2) $2\pi\lambda \log r(F_{on}^{\lambda}) \leq 2\pi\lambda \log r(F_{on}^{\lambda}) + A(F_{on}^{\lambda}, W_N) \leq 2\pi\lambda \log r(F_{oN}^{\lambda})$.

The last inequality results from the extremal theorem for the function F_{oN}^{λ} on W_N. It must be noted that the expression in the middle is the functional given in the extremal theorem (Theorem 4. of Chapter II). Thus the $r(F_{on}^{\lambda})$ are bounded for all $n \geq N$. An application of the argument principle yields $|F_{on}^{\lambda}(z)| \leq r(F_{on}^{\lambda})$ for $z \in W_n$. To see this, suppose there is a $z_o \in W_n$ with $w_o = F_{on}^{\lambda}(z_o)$ such that $|w_o| > r(F_{on}^{\lambda})$. Then, there is no change in the argument of the function $F_{on}^{\lambda}(z) - w_o$ around each of the cycles γ and β_i for every i. Therefore, by the argument principle, the function $F_{on}^{\lambda}(z) - w_o$ has no zeros on W_n, i.e., the value w_o is assumed by F_{on}^{λ} for no $z \in W_n$. Hence, $\{F_{on}^{\lambda}\}$ for $n \geq N$ is uniformly bounded on every compact subset and is therefore a normal family on W.

We let p_o^{λ} be some limit of the normal family $\{p_{on}^{\lambda}\}$. Later, we will see that there is exactly one such p_o^{λ}. The function p_o^{λ} is referred to as the "first" λ-th principal harmonic function to distinguish it from the function p_1^{λ} which has an analogous development.

The Reduction Theorem will be used to develop an extremal property for p_o^{λ} in the class of functions defined as follows:

<u>Definition</u>. $H(\lambda)$ is the class of functions p which are harmonic on $W - \xi$ and satisfy the properties:

i.) $p(z) = c(p) = \text{const.}$ for $z \in \gamma$ with $\int_\gamma dp* = 2\pi\lambda$.

ii.) $h(z) = p(z) - \lambda \log |z - \xi|$ has a harmonic continuation with $h(\xi) = 0$.

The function p_o^λ does belong to $H(\lambda)$. A lemma that is needed to insure the existence of certain integrals is stated and proved below.

Lemma 2. If $m<n$ the inequality $\int_{\bar\beta_m} pdp* \leq \int_{\bar\beta_n} pdp*$ holds for all $p \in H_n(\lambda)$.

Proof. Consider the surface $\bar W_n - W_m$, with orientation induced by W. The Dirichlet integral of p on this surface is non-negative and we have

$$0 \leq D_{\bar W_n - W_m}(p) = \int_{\bar\beta_n - \bar\beta_m} pdp* = \int_{\bar\beta_n} pdp* - \int_{\bar\beta_m} pdp*.$$

In the following, $\int_{\bar\beta} pdp*$ is to be understood as $\lim_n \int_{\bar\beta_n} pdp*$, the existence of the limit being guaranteed by the monotonicity of such integrals as shown in lemma 2. This definition is independent of the exhaustion $\{W_n\}$ as can be seen by applying Green's Theorem.

Theorem 1. The function p_o^λ uniquely maximizes the functional $\Phi(p) = 2\pi\lambda\, c(p) - \int_\beta pdp*$ among all $p \in H(\lambda)$. The maximum is $2\pi\lambda\, c(p_o^\lambda)$ and the deviation from this maximum is given by $D_{\bar W}(p - p_o^\lambda)$.

Proof. It will be shown via the Reduction Theorem that p_o^λ minimizes the functional $-\Phi(p) = -2\pi\lambda\, c(p) + \int_\beta pdp*$. Then it will follow that p_o^λ maximizes the functional Φ.

The functional $-\Phi$ satisfies $-\Phi(p) = \lim_n \left[-\Phi_n(p)\right]$, the existence of the limit being insured by the monotonicity of the integrals $\int_{\beta_n} p dp*$, as proved in lemma 2. The restrictions of functions defined on surfaces W_m belong to the class $H_n(\lambda)$ for $n \leq m$. The same may be said of functions defined on W. Condition 3.) is satisfied because of standard theorems on interchange of limits. The extremal theorem for p_{on}^{λ} fulfills the requirements of 4.). Lemma 2.) provides us with the monotone property of the functionals $-\Phi_n$, while the requirements of 6.) are met by lemma 1. Thus the functional Φ is maximized by p_o^{λ} with the maximum being $\lim_n \Phi_n(p_{on}^{\lambda}) = 2\pi\lambda\, c(p_o^{\lambda})$.

As for the deviation formula, consider the function $p_\epsilon = p_o^{\lambda} + \epsilon(p-p_o^{\lambda})$ which belongs to $H(\lambda)$ for arbitrary ϵ. The value it gives the functional Φ_n is

$$(3)\quad \Phi_n(p_\epsilon) = 2\pi\, c(p_o^{\lambda}) - \int_{\beta_n} p_o^{\lambda}\, dp_o^{\lambda}* + a_n\epsilon - \epsilon^2\int_{\beta_n}(p-p_o^{\lambda})\, d(p-p_o^{\lambda})*,$$

which is a quadratic polynomial in ϵ with a_n the coefficient of the ϵ term. The integral in the last term is precisely $D_{\overline{W}_n}(p-p_o^{\lambda})$.

Now suppose $D_{\overline{W}}(p-p_o^{\lambda}) = \lim_n D_{\overline{W}_n}(p-p_o^{\lambda})$ is finite. The finiteness of $D_{\overline{W}}(p-p_o^{\lambda})$ determines the finiteness of the integrals $\int_\beta p dp*$ and $\int_\beta p_o^{\lambda}\, d\, p_o^{\lambda}*$ as can be seen by using the triangle inequality for Dirichlet integrals on surfaces $\overline{W}-\Delta$, where Δ is a parametric disk about ξ. Upon setting $\epsilon = 1$, consider taking limits on both sides of (3.). In view of the preceding remarks, each term with the possible exception of a_n possesses a finite limit. Therefore so does a_n. Denote this limit by a. Taking limits of (3) we obtain

(4) $\Phi(\tilde{p}_\epsilon) = \Phi(p_o^\lambda) + a\epsilon - \epsilon^2 D_{\overline{W}}(p-p_o^\lambda)$.

But according to the first part of this theorem $\Phi(p_\epsilon)$ has a maximum when $\epsilon = 0$. Therefore its derivative with respect to ϵ evaluated at $\epsilon = 0$ must vanish, i.e., $a = 0$. The desired deviation formula results when $\epsilon = 1$ is substituted into (4).

If $D_{\overline{W}}(p-p_o^\lambda)$ is infinite, then the triangle inequality for Dirichlet integrals on surfaces $W - \tilde{\Delta}$ can be used to show that $\Phi(p)$ is infinite as well. The deviation formula holds in the sense that both sides are infinite.

The uniqueness of the maximizing function follows exactly as in the proof of the extremal theorem in the finite case. A consequence of the uniqueness of the maximizing function is the fact that p_o^λ is the only limit of the normal family $\{p_{on}^\lambda\}$. This completes the proof of Theorem 1.

An analytic function F_o^λ will be defined in an analogous manner to that of F_{on}^λ, i.e., $F_o^\lambda = \exp(p_o^\lambda + i\, p_{o*}^\lambda)$. The function F_o^λ will be referred to as the "first" λ-th principal analytic function to distinguish it from another principal function F_1^λ.

§ 8. Extremal Properties of the First λ-th Principal Analytic Function

The extremal property of the function p_o^λ leads immediately to the extremal property for the λ-th principal analytic function F_o^λ.

A class of competing analytic functions is defined on W as follows:

Definition. A (λ) is the class of functions F analytic on $\overline{W}$ and satisfying the properties:

i.) $|F(z)| = r(F) = \text{const.}$ for $z \in \gamma$ with $\int_\gamma d\arg F = 0$.

ii.) F has a λ-th order zero at ξ.

iii.) $\lim_{z \to \xi} F(z) / (z - \xi)^\lambda = 1$.

The function F_0^λ, already defined above, belongs to the class $A(\lambda)$. The quantity $-\int_\beta \log |F| \, d\arg F$, denoted by $A(F)$, is understood as

$$\lim_n - \int_{\beta_n} \log |F| \; d \arg F.$$

Theorem 2. Among $F \in A(\lambda)$ the functional $2\pi\lambda \log r(F) + A(F)$ is uniquely maximized by the function F_0^λ. The maximum is $2\pi\lambda \log r(F_0^\lambda)$ and the deviation from the maximum is $D_{\overline{W}}(\log |F/F_0^\lambda|)$.

Proof. If $F \in A(\lambda)$ then $\log |F| \in H(\lambda)$. The argument used in Theorem 4 of the preceding chapter yields the result.

An important corollary characterizing F_0^λ as a λ-th power of the univalent function F_0 of [6], i.e., the principal function corresponding to $\lambda = 1$, will now be drawn from Theorem 2.

Corollary 1. The function F_0^λ is the λ-th power of the univalent function F_0.

Proof. It is first observed that $[F_0(z)]^\lambda$ belongs to $A(\lambda)$. Because of the uniqueness of the maximizing function in Theorem 2, it only remains to be seen that the deviation $D_{\overline{W}}(\log |(F_0(z))^\lambda / F_0^\lambda(z)|) = 0$. For convenience, denote by $f(z)$ the analytic function $(F_0(z))^\lambda / F_0^\lambda(z)$ with a removable singularity at ξ. By Green's Theorem

$$D_{\hat{\overline{W}}}(\log|f|) = \int_{\gamma\cup\beta} \log|f| \, d\arg f.$$

The contibution of γ to the integral is 0. Now

(5) $0 \leq D_{\hat{\overline{W}}}(\log|f|) = \int_{\beta} \log|f| \, d\arg f = \lim_{n} \int_{\beta_n} \log|f| \, d\arg f.$

Let $f_m(z) = (F_{om}(z))^{\lambda} / F_{om}^{\lambda}(z)$. The family $\{f_m\}$ is a normal family with a limit being f. Therefore, interchanging limits and integration, the right hand side of (5) may be written, in the sense of iterated limits, as $\lim_{n} \lim_{m \geqslant n} \int_{\beta_n} \log|f_m| \, d\arg f_m$, and we have

(6) $D_{\hat{\overline{W}}}(\log|f|) = \lim_{n} \lim_{m\geqslant n} \int_{\beta_n} \log|f_m| \, d\arg f_m \geqq 0.$

Considering the Dirichlet integral of $\log|f_m|$ over the surface $\overline{W}_m - W_n$, we obtain

(7) $0 \leq D_{\hat{\overline{W}}_m - W_n}(\log|f_m|) = \int_{\beta_m} \log|f_m| \, d\arg f_m - \int_{\beta_n} \log|f_m| \, d\arg f_m.$

The first integral is 0 and we have

(8) $\int_{\beta_n} \log|f_m| \, d\arg f_m \leq 0$ for every $m \geqq n$.

Thus, taking iterated limits of (8), we obtain

(9) $\lim_{n} \lim_{m\geqslant n} \int_{\beta_n} \log|f_m| \, d\arg f_m \leq 0.$

Combining (6) and (9) the desired result $D_{\hat{\overline{W}}}(\log|f|) = 0$. This completes the proof of Corollary 1.

A consequence of this corollary is the fact that F_o^{λ} is a weakly λ-valent radial slit mapping of $\overline{W}$ into a disc in the following sense:

<u>Definition</u>: The mapping F(z) is called <u>weakly</u> <u>λ-valent</u> if for

each $w \in F(\overline{W})$, the set $F^{-1}(w)$ consists of at most λ points $z \in \overline{W}$, and for some $w \in F(\overline{W})$, the set $F^{-1}(w)$ consists of exactly λ points. A weakly λ-valent mapping $F(z)$ of $\overline{W}$ into the point set S is called a radial slit mapping of $\overline{W}$ into S if each component of the set $\{ w \in S; F^{-1}(w)$ contains at most $\lambda-1$ points $z \in \overline{W} \}$ is a radial slit or point.

Two corollaries stating other extremal properties of F_o^λ will be drawn. Define a class of functions $\mathfrak{F}(\lambda)$ which is a subclass of $A(\lambda)$ and differing from $A(\lambda)$ in that $A(F) \geq 0$ for each $F \in \mathfrak{F}(\lambda)$.

Corollary 2. The function F_o^λ has the property

$$\max_{F \in \mathfrak{F}(\lambda)} r(F) = r(F_o^\lambda).$$

Proof. Let $F \in \mathfrak{F}(\lambda)$. The $A(F) \geq 0$ and

$$2\pi\lambda \log r(F) \leq 2\pi\lambda \log r(F) + A(F) \leq 2\pi\lambda \log r(F_o^\lambda).$$

The last inequality is the extremal property of F_o^λ. Hence the result.

Define a class of functions $\mathfrak{G}(\lambda)$ defined on W and differing from $\mathfrak{F}(\lambda)$ only in the condition $|F(z)| =$ const. on γ is dropped.

Corollary 3. The function F_o^λ has the extremal property

$$\max_{G \in \mathfrak{G}(\lambda)} \min_{z \in \gamma} |G(z)| = r(F_o^\lambda).$$

Proof. Suppose there is a function $G \in \mathfrak{G}(\lambda)$ with $\min_{z \in \gamma} |G(z)| = r > r(F_o^\lambda)$. Let W' be the simply connected region enclosed by $G(\gamma)$. Choose a function φ belonging to the class of univalent functions $A(1)$ defined for W' and consider the disk of radius ρ which is the image of W' under φ. Let C be the circle, contained entirely in W', having radius r and center at 0. Let M be the maximum value of $|\varphi(z)|$ for $z \in C$. By Cauchy's estimate for the derivative of an analytic function we have

$|\varphi'(o)| \leq M/r$. The normalization condition iii.) of A(1) implies that $\varphi'(o) = 1$ and we have $r \leq M$. Because $M \leq \rho$ we have $\rho > r(F_o^{\lambda})$. The function $\varphi \circ G$ belongs to $\mathfrak{F}(\lambda)$ and we have a contradiction to Corollary 1.

CHAPTER IV

EXTREMAL PROPERTIES OF THE SECOND λ-TH PRINCIPAL HARMONIC AND ANALYTIC FUNCTIONS

In this chapter extremal properties of certain λ-th principal functions on W will be derived.

§ 9. <u>The extremal property of the second λ-th principal harmonic function.</u>

First, a few properties of the functions p_{on}^{λ} are developed which are needed to obtain the normality of the family $\left\{ p_{1n}^{\lambda} \right\}$.

<u>Lemma 1</u>. If $m \geqq n$, then $\int_{\beta_n} p_{om}^{\lambda} \, d\, p_{om}^{\lambda *} \leqq 0$.

<u>Proof</u>. By lemma 2 of Chapter 3

$$\int_{\beta_n} p_{om}^{\lambda} \, d\, p_{om}^{\lambda *} \leqq \int_{\beta_m} p_{om}^{\lambda} \, d\, p_{om}^{\lambda *} = 0.$$

<u>Lemma 2</u>. The sequence $\left\{ c(p_{on}^{\lambda}) \right\}$ is a decreasing sequence.

<u>Proof</u>. Let $m \geqq n$. Using the extremal property for p_{on}^{λ} we have

$$2\pi\lambda\, c(p_{on}^{\lambda}) \geqq 2\pi\lambda\, c(p_{om}^{\lambda}) - \int_{\beta_n} p_{om}^{\lambda} \, d\, p_{om}^{\lambda *} \geqq 2\pi\lambda\, c(p_{om}^{\lambda}).$$

The last inequality follows from lemma 1.

<u>Lemma 3</u>. The family $\left\{ p_{1n}^{\lambda} \right\}$ is a normal family on $W - \xi$.

<u>Proof</u>. It suffices to show the family $\left\{ F_{1n}^{\lambda} \right\}$ normal because $p_{1n}^{\lambda} = \log \left| F_{1n}^{\lambda} \right|$. For an arbitrary compact set K contained in W chose N large enough such that $K \subset W_N$. Using the extremal property on W_n for F_{on}^{λ} we write for $n \geqq N$