Rotation sets of flows on higher dimensional tori
by Doreen Norma Dumonceaux

A dissertation submitted in partial fullillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Rotation sets of points under a flow measure the average displacement of orbits as time goes to infinity. In lower dimensions, it has been shown that there is a strong link between the properties of the rotation sets of a flow and the dynamics of that flow.

In this dissertation flows on the 3-torus are constructed with rotation sets that have 3-dimensional interior and periodic-point-free flows on the n-torus ($n \geq 4$) are constructed with rotation sets that have n-dimensional interior. A natural question is which sets can be rotation sets of flows. It is shown that: every polyhedron in R^3 with rational vertices that does not contain the origin is the rotation set for a flow on the 3-torus; every C^r curve in R^n is the rotation set of a flow on (n + l)-torus; and every compact 2-manifold that can be embedded in is the rotation set for a flow on the (n + 2)-torus. In assessing the box dimension of rotation sets of flows it is shown that: for any α □ [0,2] □ {3} there is a continuous flow on the 3-torus such that the rotation set of the flow has box dimension equal to α; and for any α □ [0,1] □ {2} □ {3} there is a smooth flow on the 3-torus such that the rotation set of the flow has box dimension equal to α.

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APPROVAL

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Doreen Norma Dumonceaux

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the college of Graduate Studies.

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STATEMENT OF PERMISSION TO USE

To my father, Dr. Robert Dumonceaux,
your example is the one I choose to follow.

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ABSTRACT

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In this dissertation flows on the 3-torus are constructed with rotation sets that have 3-dimensional interior and periodic-point-free flows on the n-torus ($n \geq 4$) are constructed with rotation sets that have n-dimensional interior. A natural question is which sets can be rotation sets of flows. It is shown that: every polyhedron in $\mathbb{R}^3$ with rational vertices that does not contain the origin is the rotation set for a flow on the 3-torus; every C^r curve in $\mathbb{R}^n$ is the rotation set of a flow on $(n+1)$-torus; and every compact 2-manifold that can be embedded in $\mathbb{R}^n$ is the rotation set for a flow on the $(n+2)$-torus. In assessing the box dimension of rotation sets of flows it is shown that: for any $\alpha \in [0,2] \cup \{3\}$ there is a continuous flow on the 3-torus such that the rotation set of the flow has box dimension equal to α; and for any $\alpha \in [0,1] \cup \{2\} \cup \{3\}$ there is a smooth flow on the 3-torus such that the rotation set of the flow has box dimension equal to α.

CHAPTER 1

HISTORY AND INTRODUCTION

History

The idea of a *rotation number* was first introduced by Henri Poincaré in the late nineteenth century ([17]). The concept was inspired by his study of the qualitative nature of the orbits of those flows on the torus which generate return maps that are circle homeomorphisms. The rotation number (also referred to as the *winding number*) measures the asymptotic rotation rate of iterates of these circle return maps. In particular, Poincaré proved that, in the setting where the return map from a circular cross-section of the torus back to itself is an orientation preserving homeomorphism of a circle, the rotation number exists and is independent of the point on the circle. He also proved that if a rotation number is rational, then that rotation number is realized by a periodic orbit; that is, there exists a periodic orbit of the homeomorphism with that rational as its rotation number ([17]). The rotation number has proven to be a useful invariant in the case of circle maps and much effort has been given to extend this concept to higher dimensional settings. Some relevant properties of rotation sets of homeomorphisms, maps, and flows on the circle, annulus, and n-dimensional torus are summarized in Tables 1, 2, and 3, respectively.

Unlike the circle homeomorphism case, rotation sets of individual orbits of circle endomorphisms are dependent upon the orbit under consideration. In 1979, the concept of rotation number was extended by Sheldon Newhouse, Jacob Palis, and Floris Takens to the *rotation set* of circle endomorphisms homotopic to the identity ([19]). Such rotation sets are the union of rotation sets of individual orbits and are invariant under conjugacy. The rotation set of an orbit of a circle endomorphism can be a singleton or a closed interval. Therefore, rotation sets of circle endomorphisms may have interior. Every rational contained in the full rotation set of the endomorphism is realized as the rotation number of a periodic orbit ([19]). In 1989, Ryuichi Ito proved that the full rotation set of a degree-one circle endomorphism is closed ([10]).

The emphasis of this dissertation is on rotation sets of flows on tori of various dimensions. By continuity, the rotation set of a flow is equal to the rotation set of the time-one map of that flow. In the one-dimensional case, the rotation set of a flow on the circle is always a single number and is independent of the orbit. We now turn to the annulus case.

The study of rotation sets has been further extended to homeomorphisms, maps, and flows on the annulus, $\mathbb{A} = \mathbb{S}^1 \times [0,1]$. In 1990, Michael Handel proved that if $f : \mathbb{A} \to \mathbb{A}$ is an orientation preserving, boundary component preserving homeomorphism, then the rotation set of f is closed ([8]). Furthermore, if f is also area-preserving, then the rotation set is a closed interval ([8]). John Franks showed that for each rational in the interior of the rotation set of f, there exists a periodic orbit with rotation number equal to that rational ([4]). It follows that, if the rotation set has interior, then the map must have periodic points. If $f : \mathbb{A} \to \mathbb{A}$ is map of the annulus, then it is an open question as to whether its rotation set is closed.

Again, because the rotation sets for flows of the annulus are the same as the time-one map, the rotation set of an annulus flow must be closed. A flow on the annulus which is the union of periodic orbits with varying periods has full rotation set with interior.

Now we consider the rotation sets of homeomorphisms, maps, and flows on n-dimensional tori, $\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n\text{- times}}$, $(n \geq 2)$. We will see that as n increases the link between rotation set structure and dynamical properties weakens. Michal Misiurewicz and Krystyna Ziemian developed an alternative definition of rotation set for homeomorphisms and maps on the n-dimensional torus and have established various properties of that rotation set ([15]). Consider a homeomorphism, $f : \mathbb{T}^2 \to \mathbb{T}^2$, which is homotopic to the identity. It follows from the work of Franks, as well of that of Misiurewicz and Ziemian, that the rotation set of f must contain its extreme points as well as the 2-dimensional interior of its closed convex hull ([5])([15]). Franks has also shown that every rational vector in the interior of the rotation set is the rotation vector for some periodic orbit ([5]). This followed the work of Handel who proved that periodic-point-free homeomorphisms of the 2-torus cannot have rotation sets with interior ([7]). Jaume Llibre and Robert MacKay proved related results ([14]). But, it is still unknown whether or not the rotation set of a homeomorphism, $f : \mathbb{T}^2 \to \mathbb{T}^2$, is closed.

A natural question is the following: Which subsets of the plane may be realized as rotation sets of n-dimensional toral homeomorphisms? Jaroslaw Kwapisz has shown that every convex polygon with rational vertices is realized as the rotation set for some homeomorphism of the 2-torus ([12]). Since rotation sets depend continuously on the map, this implies that some polygons with other than rational vertices can be rotation sets of some toral homeomorphisms ([16]). Kwapisz later constructed a smooth diffeomorphism on the 2-torus with a non-polygonal rotation set with interior ([13]).

Marcy Barge and Russell Walker provide examples of periodic-point-free endomorphisms on $\mathbb{T}^n$ which are C^∞, and C^∞ diffeomorphisms on the n-torus, ($n \geq 3$), with rotation sets which have interior ([1]). These examples illustrate that the link between rotation sets of maps with interior and the periodicity of orbits breaks down on n-tori when $n \geq 3$.

If $\varphi^t : \mathbb{T}^2 \to \mathbb{T}^2$ is a flow on the torus with lift, $\widetilde{\varphi}^t : \mathbb{R}^2 \to \mathbb{R}^2$, then again the nature of the rotation set of φ implies the existence of certain dynamics. Franks and Misiurewicz consider the more general Misiurewicz-Ziemian (M-Z) rotation set,

$\rho_{\text{M-Z}}(\varphi)$ ([15]). This is defined by $v \in \rho_{\text{M-Z}}(\varphi)$ if and only if there are sequences $x_i \in \mathbb{R}^2$ and $t_i \in \mathbb{R}^+$ with $\lim_{i\to\infty} t_i = \infty$ such that $\lim_{i\to\infty} \frac{\widetilde{\varphi}^{t_i}(x_i) - (x_i)}{t_i} = v$. Franks and Misiurewicz show that there are only three possibilities for the M-Z rotation set of a 2-torus flow.

1. The rotation set may be a single point, $v \in \mathbb{R}^2$.
2. The rotation set may be a closed segment contained in a line passing through 0 and another rational point in $\mathbb{R}^2$ (the segment need not contain 0).
3. The rotation set may be a closed line segment with one end at 0 and having irrational slope.

In particular, M-Z rotation sets of flows on the 2-torus must be closed and cannot have 2-dimensional interior ([6]). The M-Z rotation set contains the point rotation set which is the case of the M-Z rotation set when a fixed point in the domain is considered rather than a sequence of points. We use the point rotation set throughout this dissertation. In Chapter 6, we show that there is a flow on the 2-torus with (point) rotation set equal to a Cantor set in $\mathbb{R}^2$.

Now consider the case of homeomorphisms of $\mathbb{T}^n$ ($n \geq 3$). These rotation sets may have interior, but as previously discussed, Barge and Walker demonstrate that a rotation set with non-empty interior does not guarantee the existence of periodic points ([1]). Richard Swanson and Walker have constructed homeomorphisms on $\mathbb{T}^n$ with rotation sets that are not closed ([21]). Since the homeomorphisms constructed by Swanson and Walker are flowable (time-one maps of a flow), rotation sets for flows on the n-torus, ($n \geq 3$) need not be closed.

Table 1. Rotation Sets of Homeomorphisms

	Must be Singleton	Must be Closed	Can have Interior	Can Have Interior w/o Periodic Points
$\mathbb{S}^1$	Yes [17]	Yes [17]	No [17]	No [17]
$\mathbb{A}^2$	No	Yes [8]	Yes	No [4]
$\mathbb{T}^2$	No	Open	Yes [5][16]	No [5][14]
$\mathbb{T}^3$	No	No [21]	Yes	Yes [1]
$\mathbb{T}^n$	No	No [21]	Yes	Yes

Table 2. Rotation Sets of Maps

	Must be Singleton	Must be Closed	Can have Interior	Can have Interior w/o Periodic Points
$\mathbb{S}^1$	No [19]	Yes [10]	Yes [19]	No [19]
$\mathbb{A}^2$	No	Yes	Yes [2]	No
$\mathbb{T}^2$	No	Open	Yes	Yes [1]
$\mathbb{T}^3$	No	No [21]	Yes	Yes [1]
$\mathbb{T}^n$	No	No [21]	Yes	Yes [1]

Table 3. Rotation Sets of Flows

	Must be Singleton	Must be Closed	Can have Interior	Can Have Interior w/o Periodic Points
$\mathbb{S}^1$	Yes	Yes	No	No
$\mathbb{A}^2$	No	Yes	Yes	No
$\mathbb{T}^2$	No	Yes	No [6]	No
$\mathbb{T}^3$	No	No [21]	Yes Thm 2.1	Open
$\mathbb{T}^4$	No	No	Yes Thm 3.4	Yes Thm 3.4
$\mathbb{T}^n$	No	No	Yes Thm 3.4	Yes Thm 3.4

Results

In this dissertation, we prove several theorems about the rotation sets of flows on higher dimensional tori. These results further demonstrate the breakdown in the link between rotation set topology and flow dynamics on n-tori, as n increases. Several of these results also address the question as to which sets can be the rotation set of a flow on the n-torus.

We first use direct techniques to prove two results about flows with "thick" rotation sets:

THEOREM 2.1. There exists a C^∞ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, that has a rotation set with 3-dimensional interior.

THEOREM 3.4. For each $n \geq 4$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$, such that

(i) the rotation set ρ_φ, has n-dimensional interior, and

(ii) φ^t has no periodic points.

Next, we use more sophisticated but indirect techniques to extend Theorem 2.1.

THEOREM 4.2. Let $K \subset \mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ such that $(0,0,0) \notin K$. Then there exists a $\mathbb{C}^\infty$ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the rotation set $\rho_\varphi = K$.

Since the rotation set of a flow is equal to the rotation set of its time-one map, we have the following corollary to Theorem 4.2.

COROLLARY 4.5. Let $K \subset \mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ such that $(0,0,0) \notin K$. Then there exists a $\mathbb{C}^\infty$ homeomorphism, $f : \mathbb{T}^3 \to \mathbb{T}^3$ with lift $\widetilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$, such that the rotation set $\rho_f = K$.

Focusing on the question of which sets can be rotation sets, we prove the following:

THEOREM 5.1. For any C^r curve, $\gamma : [0,1] \to \mathbb{R}^n$, there exists a C^r flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, such that the rotation set $\rho_\varphi = \text{Image}(\gamma) \times \{0\} \subset \mathbb{R}^{n+1}$.

Using a Theorem due to Hans Hahn and Stefan Mazurkiewicz ([9]), we have the following corollary:

COROLLARY 5.2. For any $K \subset \mathbb{R}^n$, that is compact, connected, and locally connected, there exists a continuous flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, such that the rotation set $\rho_\varphi = K \times \{0\} \subset \mathbb{R}^{n+1}$.

For Theorem 5.1 and Corollary 5.2 the smoothness of the flow is restricted by the smoothness of γ. If the dimension of the image of γ is greater than 1, γ can be only continuous. But, if the dimension of the domain of the map is higher, more smoothness can be realized:

THEOREM 5.3. Let $H : [0,1] \times [0,1] \to \mathbb{R}^n$ be a C^∞ map such that $H(r,0) = H(r,1)$, for all $r \in [0,1]$, and $H(0,s) = H(1,s)$, for all $s \in [0,1]$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, such that the rotation set $\rho_\varphi = \text{Image(H)} \times \{0\} \times \{0\} \subset \mathbb{R}^{n+2}$.

Whether there exist endomorphisms of the 2-torus with circular rotation sets is an open question. We have the following corollary for a smooth flow on the 4-torus:

COROLLARY 5.4. Let D^2 be the unit disk contained in $\mathbb{R}^2$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^4 \to \mathbb{T}^4$, such that the rotation set $\rho_\varphi = D^2 \times \{0\} \times \{0\} \subset \mathbb{R}^4$.

Next we show that any compact 2-manifold, M, imbedded into $\mathbb{R}^n$ is the rotation set for a flow on the $(n+2)$-torus.

COROLLARY 5.6. Let M be a compact 2-manifold imbedded in $\mathbb{R}^n$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, such that the rotation set $\rho_\varphi = M \times \{0\} \times \{0\} \subset \mathbb{R}^{n+1}$.

In the last chapter we turn to the question of the box-counting dimension, $\dim_B$, of rotation sets of toral flows. We prove the following:

THEOREM 6.3. For any $0 \leq \alpha \leq 1$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^2 \to \mathbb{T}^2$, such that the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

The following corollary to Theorem 5.3 concludes that rotation sets for flows on the 3-torus may have fractional box dimension between 1 and 2.

COROLLARY 6.4. For any $1 < \alpha < 2$, there exists a continuous flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

To summarize the results of this dissertation concerning the dimension of rotation sets, we present the following two theorems.

THEOREM 6.5. For any $\alpha \in [0, 2] \cup \{3\}$, there exists a continuous flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

THEOREM 6.6. For any $\alpha \in [0, 1] \cup \{2\} \cup \{3\}$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

Definitions

We now establish some notation and definitions that will be used throughout this dissertation.

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the unit circle and let $\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}}$ be the n-dimensional torus. As our universal cover we use the map, $\Pi : \mathbb{R}^n \to \mathbb{T}^n$, given by:

$$\Pi(x_1, x_2, \ldots, x_n) = (\exp(2\pi i x_1), \exp(2\pi i x_2), \cdots, \exp(2\pi i x_n)) \tag{1.1}$$

where $x_j \in \mathbb{R}$ for each $j = 1, \ldots, n$.

We denote by $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ the *lift of the map*, $f : \mathbb{T}^n \to \mathbb{T}^n$, if $\widetilde{f}$ is continuous and $\Pi \circ \widetilde{f} = f \circ \Pi$. Note that for any given f, which is isotopic to the identity, with lift $\widetilde{f}$, $\widetilde{f}(\widetilde{p} + k) = \widetilde{f}(\widetilde{p}) + k$ for all $\widetilde{p} \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$.

Definition 1.1. *The* rotation set of $\widetilde{p}$ under the lift $\widetilde{f}:\mathbb{R}^n \to \mathbb{R}^n$ of $f:\mathbb{T}^n \to \mathbb{T}^n$ *is defined by*

$$\rho(\widetilde{f},\widetilde{p}) = \mathrm{LIM}\left\{ \frac{\widetilde{f}^n(\widetilde{p}) - \widetilde{p}}{n} \middle| n \in \mathbb{N} \right\}$$

where $\widetilde{p} \in \Pi^{-1}(p)$ *for* $p \in \mathbb{T}^n$.

Here, and throughout this dissertation, we use "LIM" to denote "the set of all limit points." Note that since the set of all limit points of any set is closed, then by definition the rotation set of $\widetilde{p}$ is closed.

Definition 1.2. *The* rotation set of $\widetilde{f}:\mathbb{R}^n \to \mathbb{R}^n$, *denoted by* $\rho(\widetilde{f})$, *is defined by*

$$\rho(\widetilde{f}) = \bigcup_{\widetilde{p}\in\mathbb{R}^n} (\rho(\widetilde{f},\widetilde{p}))$$

where $\widetilde{p} \in \Pi^{-1}(p)$ *for* $p \in \mathbb{T}^n$.

Note that the definition of rotation set used here is called the "point-wise rotation set" by Misiurewicz and Ziemian ([15]). Rotation sets of different lifts of a map, f, differ only by integer translates. We will refer to the rotation set of a map, $f:\mathbb{T}^n \to \mathbb{T}^n$, computed using a fixed lift for by ρ_f.

A Toroidal Horseshoe Map on the 2-Torus

One type of map that will be utilized in the construction of several examples in this dissertation will be the 3-symbol toroidal horseshoe homeomorphism. For our purposes, we list the properties of such a horseshoe which we will need. See Barge-Walker for a detailed construction ([1]).

The 3-symbol toroidal horseshoe, $f:\mathbb{T}^2 \to \mathbb{T}^2$, contains a rectangle $R \subset \mathbb{T}^2$, which in turn contains three disjoint sub-rectangles, I_0, I_1, I_2. f contracts R in one direction while stretching R in the perpendicular direction about both generators of the 2-torus so that $f(I_0)$, $f(I_1)$, and $f(I_2)$ meet R as shown in Figure 1. On $I_0 \cup I_1 \cup I_2$, f linearly contracts in one direction and linearly expands in the perpendicular direction.

DEFINITION 1.3. *Λ is an* invariant set *of a map $f : \mathbb{T}^n \to \mathbb{T}^n$, if $f(\Lambda) \subseteq \Lambda$.*

Furthermore, Λ_f is the non-wandering set of f ([18]).

For our 3-symbol toroidal horseshoe f, $\Lambda_f = \bigcap_{k \in \mathbb{Z}} \{\{f^k(R)\} \cup \{p_1\} \cup \{p_2\}\}$ where $p_1,\ p_2 \in \mathbb{T}^2 \backslash R$ are fixed points of f ([1]). $\Lambda_f \cap R$ is a Cantor set and any point $p \in \Lambda_f \cap R$ can be represented by an infinite 3-symbol sequence $\underline{\eta}(p) \in \{0,1,2\}^{\mathbb{Z}}$. The sequence $\underline{\eta}(p)$ keeps track of the forward and backward itinerary of each $p \in \Lambda_f$ in the following manner: if $\underline{\eta}(p) = \{\eta_i\}_{i=-\infty}^{\infty}$, then

$$\eta_i = \begin{cases} 0 & \text{if } f^i(p) \in I_0 \\ 1 & \text{if } f^i(p) \in I_1 \\ 2 & \text{if } f^i(p) \in I_2. \end{cases}$$

The map $\underline{\eta} : \Lambda_f \cap R \to \{0,1,2\}^{\mathbb{Z}}$ is one-to-one and onto. For simplicity, we denote by $p : \{0,1,2\}^{\mathbb{Z}} \to \Lambda_f$, the inverse of $\underline{\eta}$. We will also make use of finite forward 3-symbol sequences $\underline{\eta} \in \{0,1,2\}^k$ for some fixed $k \in \mathbb{N}$. If $\underline{\alpha} \in \{0,1,2\}^k$ and $\underline{\beta} \in \{0,1,2\}^l$, then the juxtaposition of two sequences, $\underline{\alpha\beta} \in \{0,1,2\}^{k+l}$, is defined to be the terms of $\underline{\alpha}$ concatenated with the terms of $\underline{\beta}$. $\underline{\beta}^n$ denotes $\underline{\beta}$ repeated n-times; $\underline{\beta}^\infty$ denotes the element of $\{0,1,2\}^{\mathbb{N}}$ where β is concatenated with itself infinitely often. Let $\sigma : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$ be the full shift map on 3-symbols such that $\sigma(\underline{\eta}) = \underline{\xi}$ where $\eta_i = \xi_{i+1}$. Notice that $\underline{\eta}(f(p)) = \sigma(\underline{\eta}(p))$ for all $p \in \Lambda_f$.

We now discuss the rotational properties of points, $p \in R$, which are such that their forward sequence representation is $\underline{\eta} \in \{0,1,2\}^{\mathbb{N}}$. We denote the set of all such points, p, by $p(\underline{\eta}) \in \Lambda_f$.

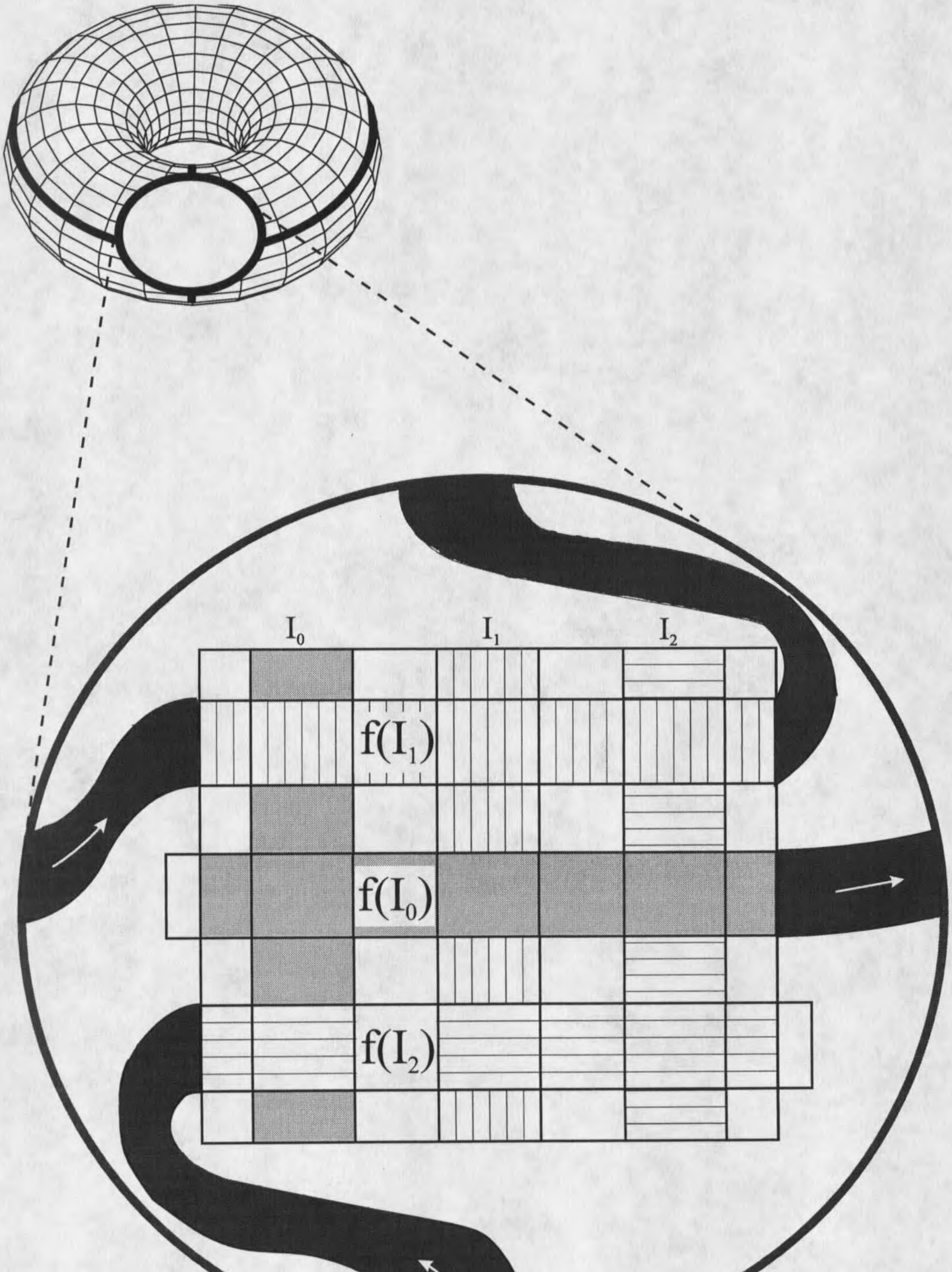


Figure 1. The toroidal horseshoe, f

For $k \in \{0,1,2\}$, define $\Gamma : \{0,1,2\} \to \mathbb{R}^2$ by

$$\Gamma(k) = \begin{cases} (0,0) & \text{if } k = 0 \\ (1,0) & \text{if } k = 1 \\ (1,1) & \text{if } k = 2. \end{cases} \tag{1.2}$$

PROPOSITION 1.4. *If f is the 3-symbol toroidal horseshoe as described above with lift $\widetilde{f}$ as in Figure 2, and $p \in \Lambda_f$ with forward 3-symbol sequence $\underline{\eta}(p)$, then*

$$\rho_f = \mathrm{LIM}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\Gamma(\eta_i) \;\middle|\; n \in \mathbb{N}\right\} = <(0,0),\ (0,1),\ (1,1)>$$

where $<x_1, \ldots, x_k>$ is the closed convex hull of $\{x_1, \ldots, x_k\}$.

Proof: Let $p \in \Lambda_f \cap R$ and $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the fixed lift of f via the projection $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$ as in Figure 2, $\widetilde{p} \in \Pi^{-1}(p)$, and $\widetilde{R}$ be the component of $\Pi^{-1}(R)$ which contains $\widetilde{p}$. The key observation in the construction of f which we make use of is that $\widetilde{f}(\widetilde{p}) \in \widetilde{R} + \Gamma(\eta_0)$, because $p \in I_{\eta_0}$, see Figure 2. Similarly, for $n \in \mathbb{N}$, $\widetilde{f}^n(\widetilde{p}) \in \widetilde{R} + \sum_{i=0}^{n-1}\Gamma(\eta_i)$ and thus, $\widetilde{f}^n(\widetilde{p}) - \sum_{i=0}^{n-1}\Gamma(\eta_i) \in \widetilde{R}$. Therefore,

$$\begin{aligned} \left\|(\widetilde{f}^n(\widetilde{p}) - \widetilde{p}) - \sum_{i=1}^{n-1}\Gamma(\eta_i)\right\| &= \left\|\left(\widetilde{f}^n(\widetilde{p}) - \sum_{i=1}^{n-1}\Gamma(\eta_i)\right) - \widetilde{p}\right\| \\ &\le \operatorname{diam}\widetilde{R}\,. \end{aligned}$$

Let $p = p_1$ or p_2. Then, p is fixed and $\rho_f(p) = (0,0)$ by our choice of lift, $\widetilde{f}$.

We can now easily compute the rotation set for all points $p \in \Lambda_f$ which have forward symbol sequence equal to $\underline{2}^\infty$.

$$\begin{aligned} \rho_f(p\ (\underline{2}^\infty)) &= \mathrm{LIM}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\Gamma(2) \;\middle|\; n \in \mathbb{N}\right\} \\ &= \mathrm{LIM}\left\{\frac{n(1,1)}{n} \;\middle|\; n \in \mathbb{N}\right\} \\ &= (1,1). \end{aligned}$$

It is left to the reader to check that:

$$\rho_f(p\ (\underline{0}^\infty)) = (0,0); \qquad \rho_f(p\ (\underline{1}^\infty)) = (1,0)$$

and that for $\underline{\eta} = \{(\eta_1\eta_2\dots) \mid \eta_i \in \{0,1,2\} \text{ for all } i > 0\}$,

$$\rho_f = \bigcup_{\underline{\eta}} \rho_f(p\ (\underline{\eta})) = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \text{ and } 0 \le x \le 1\right\}.$$

or $<(0,0),(1,0),(1,1)>$. ∎

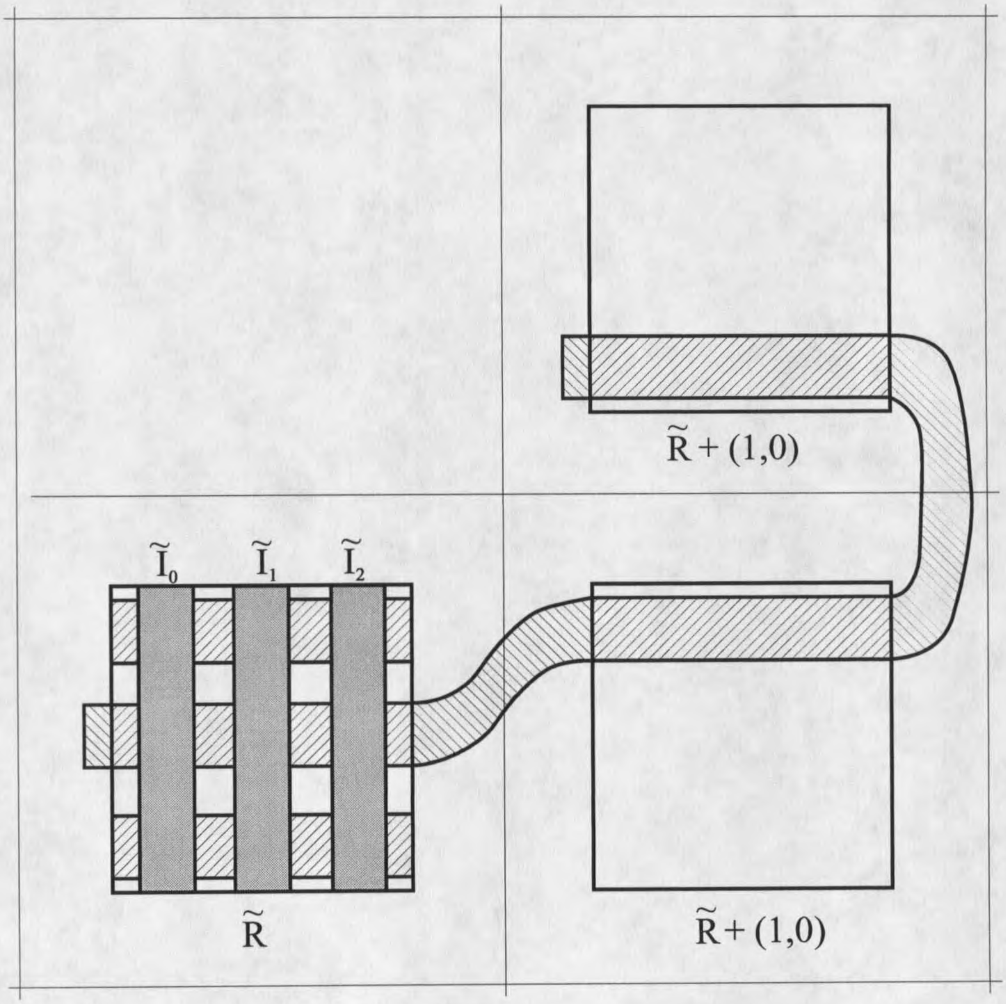


Figure 2. The lift of the toroidal horseshoe, $\widetilde{f}$

As mentioned previously, the focus of this dissertation is on the rotation sets of flows on the n-dimensional torus. In the transition from maps to flows, we shift our concern from the discrete-time dynamical system to the continuous-time dynamical system.

DEFINITION 1.5. *Let $\varphi : \mathbb{T}^n \times \mathbb{R} \to \mathbb{T}^n$ be a C^r function. Then φ is a C^r* flow *$\mathbb{T}^n$ if,*

(i) φ satisfies the group property, $\varphi^t \circ \varphi^s(p) = \varphi^{t+s}(p)$ for each $p \in \mathbb{T}^n$ and $t, s \in \mathbb{R}$ and,

(ii) for each fixed $t \in \mathbb{R}$, φ^t is a homeomorphism on $\mathbb{T}^n$.

For the rest of this dissertation $\varphi(p, t)$ will be denoted by $\varphi^t(p)$.

A flow can also be thought of as a family of homeomorphisms, $\varphi = \{\varphi^t : \mathbb{T}^n \to \mathbb{T}^n\}_{t\in\mathbb{R}}$ that satisfy the group property.

DEFINITION 1.6. *$\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$ is a* lift of the flow, *$\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$, if the following diagram commutes for each $t \in \mathbb{R}$.*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\widetilde{\varphi}^t} & \mathbb{R}^n \\ \downarrow \Pi & & \downarrow \Pi \\ \mathbb{T}^n & \xrightarrow{\varphi^t} & \mathbb{T}^n \end{array}$$

DEFINITION 1.7. *The* rotation set of $\widetilde{p}$, under the lift, *$\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$, of a flow, $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$, is defined by*

$$\rho(\widetilde{\varphi}, \widetilde{p}) = \underset{t\to\infty}{\mathrm{LIM}} \left(\frac{\widetilde{\varphi}^t(\widetilde{p}) - \widetilde{p}}{t} \right)$$

where $\widetilde{p} \in \Pi^{-1}(p)$ for $p \in \mathbb{T}^n$. Here "$\underset{t\to\infty}{\mathrm{LIM}}(f(t))$" means the set of all limits, $\lim_{j\to\infty}(f(t_j))$, for all infinite sequences, $\{t_1, t_2, \dots\}$ such that $\lim_{j\to\infty}(t_j) = \infty$.

Furthermore, the rotation set of $\widetilde{\varphi}$, *denoted by $\rho(\widetilde{\varphi})$, is defined by*

$$\rho(\widetilde{\varphi}) = \bigcup \rho(\widetilde{\varphi}, \widetilde{p})$$

where the union is taken over all $\widetilde{p} \in \mathbb{R}^n$.

Unlike the case of maps, there is only one lift of a flow φ. So, the rotation set of a flow is a uniquely defined set. We will refer to the rotation set of a flow by ρ_{φ}.

Suspension Flows

An interesting and useful flow construction that will be used throughout this dissertation is one which takes a C^r-diffeomorphism, $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$, which is isotopic to the identity, and creates a C^r*-suspension flow of* f, $\varphi_f^t : \mathbb{T}^n \to \mathbb{T}^n$, which we now describe. For any given C^r-diffeomorphism, $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$, that is isotopic to the identity, consider the space $X = \mathbb{T}^{n-1} \times \mathbb{R}$ under the equivalence relation, $(p, s+1) \sim (f(p), s)$ for all $p \in \mathbb{T}^{n-1}$ and $s \in \mathbb{R}$. Because f is isotopic to the identity, under this equivalence relation, the quotient space, $\hat{X} = X/_{\sim}$, is C^r-diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{S}^1$ ([18]). All points of $\hat{X}$, are represented by points (p, s) where $p \in \mathbb{T}^{n-1}$ and $0 \leq s < 1$. Consider the "vertical" vector field on X given by:

$$\dot{p} = 0$$

$$\dot{s} = 1.$$

This vector field induces a C^r flow on X which passes to the C^r flow, $\varphi_f^t : \hat{X} \to \hat{X}$, under the equivalence relation $\sim$. See Figure 3. Then φ_f^t is a *suspension flow of* f ([18]).

PROPOSITION 1.8. *If* $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ *has rotation set,* $\rho_f \subset \mathbb{R}^{n-1}$*, then there is a suspension flow of* f*,* $\varphi_f^t : \mathbb{T}^n \to \mathbb{T}^n$*, such that* $\rho_{\varphi_f} = \rho_f \times \{1\} \subset \mathbb{R}^n$.

Proof: The proposition follows from the observation that $\varphi_f^1(p, 0) = (f(p), 0)$ for all $(p, 0) \in \hat{X}$ and that $\mathbb{T}^{n-1} \times \{0\}$ is a cross-section of the flow, φ_f^t. ∎

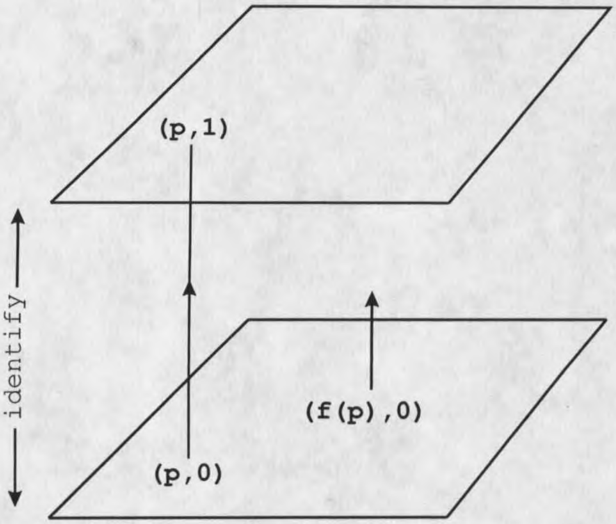


Figure 3. Suspension Flow of a map, f

λ-Scaled Suspension Flows

In a suspension flow of a map, all points move at constant speed one. A more general concept of a suspension flow allows points to move at variable speeds ([11]). Assume $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ is C^r $(r > 0)$ and is smoothly isotopic to the identity. Since the tangent space of $\mathbb{T}^n$, $\mathrm{T}(\mathbb{T}^n) \cong \mathbb{T}^n \times \mathbb{R}^n$, the C^r-suspension flow of f, φ_f^t, is generated by a vector field, $\mathbb{X}_f : \mathbb{T}^n \to \mathbb{R}^n$. We define the lift of $\mathbb{X}_f$, $\widetilde{\mathbb{X}}_f : \mathbb{R}^n \to \mathbb{R}^n$, to be the vector field, $\widetilde{\mathbb{X}}_f(\mathbf{x}) = \mathbb{X}_f(\Pi(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$. Then $\widetilde{\mathbb{X}}_f$ generates the flow, $\widetilde{\varphi}_f^t : \mathbb{R}^n \to \mathbb{R}^n$, which in turn is the lift of $\varphi_f^t : \mathbb{T}^n \to \mathbb{T}^n$. Now let $\lambda : \mathbb{R}^n \to \mathbb{R}^+$ be a C^r function such that $\lambda(\mathbf{x} + \mathbf{m}) = \lambda(\mathbf{x})$ for all $\mathbf{m} \in \mathbb{Z}^n$. Define $\widetilde{\mathbb{X}}_{f,\lambda} \equiv \lambda \widetilde{\mathbb{X}}_f$. The vector field, $\widetilde{\mathbb{X}}_{f,\lambda}$, generates a new C^r flow, $\widetilde{\varphi}_{f,\lambda}^t : \mathbb{R}^n \to \mathbb{R}^n$. $\widetilde{\varphi}_{f,\lambda}^t$ covers $\varphi_{f,\lambda}^t : \mathbb{T}^n \to \mathbb{T}^n$ because $\varphi_{f,\lambda}^t(p) = \Pi \circ \widetilde{\varphi}_{f,\lambda}^t(\widetilde{p})$ for $\widetilde{p} \in \Pi^{-1}(p)$ where $p \in \mathbb{T}^n$. We refer to $\varphi_{f,\lambda}^t$ as the *λ-scaled suspension flow of f*. For any C^r-function, $s : \mathbb{T}^{n-1} \to \mathbb{R}^+$, we can choose the scaling function, $\lambda_s : \mathbb{R}^n \to \mathbb{R}^+$, so that $\varphi_{f,\lambda_s}^{s(p)}$ restricted to the cross-section $T^{n-1} \times \{0\}$ is the Poincare´ (first-return) map of the flow, φ_{f,λ_s}^t.

So, $s(p)$ is the *transition time* for a point, $(p, 0)$, to return to the cross-section,

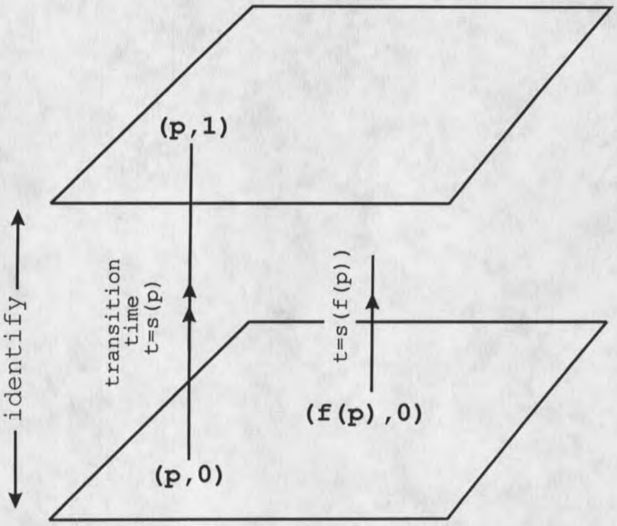


Figure 4. λ_s-scaled suspension flow of a map, f

$\mathbb{T}^{n-1} \times \{0\}$, under the flow, φ^t_{f,λ_s}, see Figure 4.

For standard suspension flows, an iterate of the time-one map corresponds to traversing the generator of the n-torus in the flow direction once. Such is no longer the case for λ-scaled suspension flows. So what makes computations for λ-scaled suspension flows difficult is that transition times back to the $\mathbb{T}^{n-1} \times \{0\}$ cross-section can vary dramatically. Furthermore, once a suspension flow is λ-scaled ($\lambda \neq 1$), the rotation set in the $\mathbb{T}^{n-1}$ directions is no longer that of the suspended map, f.

CHAPTER 2

A FLOW ON A 3-TORUS WITH 3-DIMENSIONAL ROTATION SET

The goal of this chapter is to explicitly exhibit a smooth flow on the 3-torus which has a rotation set with 3-dimensional interior. To that end, we construct a special λ-scaled suspension flow, $\varphi^t_{f,\lambda} : \mathbb{T}^3 \to \mathbb{T}^3$ of the 3-symbol toroidal horseshoe, $f : \mathbb{T}^2 \to \mathbb{T}^2$, as described in Chapter 1. The key to obtaining a rotation set with 3-dimensional interior is to scale the vector field of a suspension flow of f in such a way that the rotation set is "thickened" in the third coordinate. We take advantage of the periodic orbit structure of f to relate transition times to iterates of the map, f. This gives us a relationship between the displacement in the lift of the flow and time. By carefully assigning transition times to each point in Λ_f, the invariant set of f (described in Chapter 1), we obtain four non-coplanar points in the rotation set of the flow, $\varphi^t_{f,\lambda}$. Finally, by Lemma 2.7, an orbit is constructed which has rotation set that is the convex hull of these four non-coplanar points.

We state the main result of this chapter now and postpone the proof until we establish some notation and preliminary results.

THEOREM 2.1. *There exists a C^∞ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, that has a rotation set with 3-dimensional interior.*

We begin by defining a transition time function, s, which determines a λ_s-scaled suspension flow of f, $\varphi^t_{f,\lambda_s} : \mathbb{T}^3 \to \mathbb{T}^3$. It is this flow that has rotation set with 3-dimensional interior.

Throughout this chapter, rotation vectors are computed indirectly. That is, we use the symbol sequence of a point inherited from the toroidal horseshoe, f, rather than working directly with the lift. We show that this computation of the rotation set is equivalent to the definition.

Let $s : \mathbb{T}^2 \to \mathbb{R}^+$ be a C^∞ map such that for $p \in \Lambda_f$ and $\underline{\eta}(p) = (\eta_0, \eta_1, \ldots)$,

$$s(p) = \begin{cases} a & \text{if } \eta_0 = 0 \\ b & \text{if } \eta_0 = 1 \\ c & \text{if } \eta_0 = 2\,, \end{cases} \tag{2.1}$$

for fixed $a, b, c \in \mathbb{R}^+$ to be specified later. As described in Chapter 1, $\lambda_s : \mathbb{R}^3 \to \mathbb{R}^+$ scales the (vector field which generates) suspension flow of f in such a way that the transition times are precisely s. This scaled vector field generates the λ_s-scaled suspension flow of f, $\varphi^t_{f,\lambda_s} : \mathbb{T}^3 \to \mathbb{T}^3$, which has the property that for each $p \in I_k$ ($k = 0, 1, 2$), the first-return time of $(p, 0)$ under φ^t_{f,λ_s}, back to $\mathbb{T}^2 \times \{0\}$, equals the transition time, $s(p)$. For brevity, let $\varphi^t = \varphi^t_{f,\lambda_s}$.

For any $p \in \Lambda_f$ and $q = (p, 0) \in \mathbb{T}^3$, define $n(q, t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{N}$ by

$$n(q,t) = \operatorname{card}\left\{ \{\varphi^\tau(q) |\ \tau \in (0, t]\} \bigcap \left(\mathbb{T}^2 \times \{0\}\right) \right\}. \tag{2.2}$$

Here $n(q, t)$ equals the number of times the orbit of q intersects the cross-section, $\mathbb{T}^2 \times \{0\}$, as time ranges from 0 to t. This is also the number of times the orbit of q traverses the third generator of $\mathbb{T}^3$ in t time units.

Given any $q = (p, 0) \in \mathbb{T}^3$ as above, let $\{t_k\}_{k=1}^\infty$ be the strictly increasing sequence of times such that

$$t_k = \sum_{i=0}^{k-1} s(f^i(p)) \tag{2.3}$$

Then, t_k is the time of the k-th intersection of the orbit of q with the cross-section, $\mathbb{T}^2 \times \{0\}$. Equivalently, $n(q, t_k) = k$. The following proposition shows an alternative way to compute the rotation set of a point under a λ_s-scaled suspension flow.

PROPOSITION 2.2. *Let φ^t be a λ_s-scaled suspension flow on $\mathbb{T}^3$ and t_k be defined as in (2.3). Then,*

$$\rho_\varphi(q) = \operatorname{LIM}\left\{ \frac{\widetilde{\varphi}^{t_k}(\widetilde{q}) - \widetilde{q}}{t_k} \,\middle|\, \widetilde{q} \in \Pi^{-1}(q) \textit{ and } k \in \mathbb{N} \right\}.$$

Proof: By the definition of $\rho_\varphi(q)$ in (1.7),

$$\mathrm{LIM}\left\{\frac{\widetilde{\varphi}^{t_k}(\widetilde{q})-\widetilde{q}}{t_k}\;\middle|\;\widetilde{q}\in\Pi^{-1}(q)\text{ and }k\in\mathbb{N}\right\}\subset\rho_\varphi(q).$$

Let $r\in\rho_\varphi(q)$. Then there exists a sequence of times $t_j\to\infty$ as $j\to\infty$ so that $\frac{\widetilde{\varphi}^{t_j}(\widetilde{q})-\widetilde{q}}{t_j}\to r$ as $j\to\infty$. Let $t_{k(j)}=\min\{t_k\mid t_k>t_j\}$. Since s is a continuous function on a compact set, $|t_{k(j)}-t_j|<b_0$ for some $b_0>0$.

Let $B_0=\operatorname{diam}\{\widetilde{\varphi}^t(\widetilde{q})-\widetilde{q}\,|\,t\in[0,b_0]\text{ and }\widetilde{q}\in\Pi^{-1}(q)\text{ where }q\in\mathbb{T}^3\}$. Then,

$$\begin{aligned}\lim_{j\to\infty}\frac{\widetilde{\varphi}^{t_{k(j)}}(\widetilde{q})-\widetilde{q}}{t_{k(j)}} &= \lim_{j\to\infty}\frac{\widetilde{\varphi}^{t_{k(j)}-t_j}(\widetilde{\varphi}^{t_j}(\widetilde{q}))-\widetilde{q}}{(t_{k(j)}-t_j)+t_j}\\ &= \lim_{j\to\infty}\frac{(\widetilde{\varphi}^{t_{k(j)}-t_j}(\widetilde{\varphi}^{t_j}(\widetilde{q}))-\widetilde{\varphi}^{t_j}(\widetilde{q}))+(\widetilde{\varphi}^{t_j}(\widetilde{q})-\widetilde{q})}{(t_{k(j)}-t_j)+t_j}\\ &= \lim_{j\to\infty}\left(\frac{B_0}{t_j+b_0}+\frac{(\widetilde{\varphi}^{t_j}(\widetilde{q})-\widetilde{q})}{t_j+b_0}\right)=r\,.\end{aligned}$$

Since r was chosen arbitrarily,

$$\mathrm{LIM}\left\{\frac{\widetilde{\varphi}^{t_k}(\widetilde{q})-\widetilde{q}}{t_k}\;\middle|\;\widetilde{q}\in\Pi^{-1}(q)\text{ and }k\in\mathbb{N}\right\}\supset\rho_\varphi(q).\qquad\square$$

For any point, $p\in\Lambda_f$, with forward symbol sequence , $\eta=\eta_0\eta_1\ldots$, the rotation set of the point $q=(p,0)$ under φ^t can be found by utilizing $\Gamma:\{0,1,2\}\to\mathbb{R}^2$, (as in (1.2)) and the following proposition:

Proposition 2.3. *Let f be the 3-symbol toroidal horseshoe, $p\in\Lambda_f$ with forward 3-symbol sequence $\underline{\eta}(p)$, $q=(p,0)$, $\varphi^t:\mathbb{T}^3\to\mathbb{T}^3$ be a λ_s-scaled suspension flow of f, and $\Gamma:\{0,1,2\}\to\mathbb{R}^2$ be as defined in (1.2). Then,*

$$\rho_\varphi(q)=\mathrm{LIM}\left\{\frac{1}{t_k}\left(\sum_{i=0}^{k-1}\Gamma(\eta_i),k\right)\;\middle|\;k\in\mathbb{N}\right\}.$$

Proof: Let $\widetilde{\varphi}^t:\mathbb{R}^3\to\mathbb{R}^3$ be the lift of φ^t. Recall $R\subset\mathbb{T}^2$ is the rectangle from the construction of f. Without loss of generality, choose $\widetilde{q}\in\Pi^{-1}(q)$ so that $\widetilde{q}\in\widetilde{R}\times\{0\}$.

The key observation in the construction of the λ_s-scaled suspension flow, φ^t, as described in Chapter 1, is that $\widetilde{\varphi}^{t_1}(\widetilde{q}) \in (\widetilde{R} + (\Gamma(\eta_0)), 1)$. This is because $p \in I_{\eta_0}$ and $\varphi^{t_1}(q)$ is the first-return to the cross-section, $\mathbb{T}^2 \times \{0\}$. Recall that $n(q, t_k) = k$. So,

$$\widetilde{\varphi}^{t_k}(\widetilde{q}) \in \left(\widetilde{R} + \left(\sum_{i=0}^{k-1} \Gamma(\eta_i) \right), k \right)$$

and $\widetilde{\varphi}^{t_k}(\widetilde{q}) - \left(\sum_{i=0}^{k-1} \Gamma(\eta_i), k \right) \in \widetilde{R} \times \{0\}$. Therefore,

$$\begin{aligned}
\left\| (\widetilde{\varphi}^{t_k}(\widetilde{q}) - \widetilde{q}) - \left(\sum_{i=0}^{k-1} \Gamma(\eta_i),\ k \right) \right\| &= \\
\left\| \widetilde{\varphi}^{t_k}(\widetilde{q}) - \left(\sum_{i=0}^{k-1} \Gamma(\eta_i),\ k \right) - \widetilde{q} \right\| &\leq \operatorname{diam}(\widetilde{R}).
\end{aligned}$$

By (2.2), the proposition follows. □

We now return to the task of finding four rotation vectors of $\widetilde{\varphi}^t$ that are non-coplanar. Recall that we use the notation $\underline{\omega}^\infty = \omega\omega\omega\ldots$ to represent the infinite sequence with ω concatenated with itself an infinite number of times. Let $p \in p(\underline{1}^\infty) \subset \mathbb{T}^2$, where $p(\underline{1}^\infty)$ is the set of all points with forward symbol sequence $\underline{1}^\infty$. By Proposition 2.3, the rotation set under φ^t of a point $(p, 0)$ is:

$$\begin{aligned}
r_1 = \rho_\varphi(p\ (\underline{1}^\infty), 0) &= \operatorname{LIM} \left\{ \frac{1}{t_k} \left(\sum_{i=0}^{k-1} \Gamma(1),\ k \right) \middle| \ k \in \mathbb{N} \right\} \\
&= \operatorname{LIM} \left\{ \frac{1}{t_k} \left(\sum_{i=0}^{k-1} (1,0,1) \right) \middle| \ k \in \mathbb{N} \right\} \\
&= \operatorname{LIM} \left\{ \left(\frac{k}{t_k}, 0, \frac{k}{t_k} \right) \middle| \ k \in \mathbb{N} \right\}.
\end{aligned}$$

By the definition of $s(p)$ in (2.1), and of t_k in 2.3, $t_k = kb$ when $p \in \mathbb{T}^2$ is such that its forward sequence is $\underline{1}^\infty$. So,

$$r_1 = \rho_\varphi(p\ (\underline{1}^\infty), 0) = \left(\frac{1}{b}, 0, \frac{1}{b} \right).$$

It is left to the reader to check that the rotation sets under φ^t_{f,λ_s} of the sets $(p\ (\underline{0}^\infty), 0)$, $(p\ (\underline{2}^\infty), 0)$, $(p\ (\underline{01}^\infty), 0)$, $(p\ (\underline{10}^\infty), 0)$ are:

$$r_0 \equiv \rho_\varphi(p\ (\underline{0}^\infty), 0) = (0, 0, \tfrac{1}{a});$$

$$r_2 \equiv \rho_\varphi(p\ (\underline{2}^\infty), 0) = (\tfrac{1}{c}, \tfrac{1}{c}, \tfrac{1}{c});$$

$$r_3 \equiv \rho_\varphi(p\ (\underline{01}^\infty), 0) = \rho_\varphi(p\ (\underline{10}^\infty), 0) = (\tfrac{1}{a+b}, 0, \tfrac{2}{a+b}).$$

Choose $a, b, c \in \mathbb{R}^+$ so that the convex hull of these rotation vectors, $< r_0, r_1, r_2, r_3 >$, has three-dimensional interior. By Definition 1.7, the rotation set of a single point is closed. Therefore, the proof of Theorem 2.1 is completed when we exhibit a point, $q = (p, 0) \in \mathbb{T}^2 \times \{0\} \subset \mathbb{T}^3$ with rotation set, $\rho_\varphi(q)$, that is dense in $< r_0, r_1, r_2, r_3 >$.

For the construction of such a point we need the following definitions and lemmas.

DEFINITION 2.4. *For any finite sequence,* $\underline{\beta} = \beta_1\beta_2\ldots\beta_n$, *the finite sequence,* $\underline{\beta}^* = \beta_1^*\beta_2^*\ldots\beta_m^*, m > n$, *is an* extension *of* $\underline{\beta}$ *if* $\beta_i^* = \beta_i$ *when* $1 \leq i \leq n$.

So, $\underline{\beta}$ can be *extended* to $\underline{\beta}^*$ by concatenating $\beta_{n+1}^*\beta_{n+2}^*\ldots\beta_m^*$ to the end of $\underline{\beta}$.

DEFINITION 2.5. *Let* $\widetilde{q} \in \Pi^{-1}(q)$ *for some* $q \in \mathbb{T}^n$ *and let* $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$ *be a flow with lift* $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$. *Define the* τ-rotation vector *of* p *under* $\widetilde{\varphi}^t$ *by,*

$$\rho_\varphi^\tau(q) = \frac{\widetilde{\varphi}^\tau(\widetilde{q}) - \widetilde{q}}{\tau}\ .$$

By the following lemma, given any finite sequence, β, and a set of rotation vectors of periodic points under a toroidal horseshoe, f, there exists $\tau > 0$ and an extension, β^* of β which has τ-rotation vector arbitrarily close to any given point in the convex hull of the set of rotation vectors at time, τ. In order to accomplish this β is extended so that it mimics for a while each of the periodic orbits. Since under a λ-scaled suspension flow points flow at different rates, we must also keep track of ratio: the time it takes the flow, φ^t, to mimic each of the periodic points, and the total time elapsed. We do this to ensure that the τ-rotation vector of β^* is as desired.

LEMMA 2.6. *Let f be a 3-symbol toroidal horseshoe on $\mathbb{T}^2$ and let $s : \mathbb{T}^2 \to \mathbb{R}^+$ be a C^∞ function. Let $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$ be a λ_s-scaled suspension flow of f and $\widetilde{\varphi}^t : \mathbb{R}^3 \to \mathbb{R}^3$ be the lift of φ^t. Let $\{p_1, p_2, \cdots, p_k\} \subset \Lambda_f$ be a set of f-periodic points. Denote by $r_i = \rho_\varphi(p_i, 0)$ and $K = < r_1, r_2, \cdots, r_k >$. Let $\underline{\beta} \in \{0,1,2\}^N$ for some finite $N > 0$. Then, for each $r \in K$ and $\varepsilon > 0$ there exists a finite extension, $\underline{\beta^*}$ of $\underline{\beta}$, and a finite time, $\tau_{\beta^*} > 0$, such that for all $q^* \in \{p(\underline{\beta^*}\underline{0}^\infty)\} \times \{0\}$,*

$$| \rho_\varphi^{\tau_{\beta^*}}(q^*) - r| < \varepsilon.$$

Proof: For brevity, let $p \in p\ (\underline{\beta 0}^\infty) \subset \Lambda_f \subset \mathbb{T}^2$ be a point with forward symbol sequence $\underline{\beta 0}^\infty \in \{0,1,2\}^{\mathbb{N}}$. Recall from the description of the toroidal horseshoe, f, in Chapter 1 that for any point, p, with symbol sequence, $\underline{\eta}$, $f^j(p) = p(\sigma^j(\underline{\eta}))$ where σ is the shift operator on 3 symbols.

Let $\tau_\beta = \sum_{j=0}^{N-1} s(f^j(p)) = \sum_{j=0}^{N-1} s(p(\sigma^j(\underline{\eta})))$. Notice that the τ_β-map of φ^t, $\varphi^{\tau_\beta} : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \{0\}$, is the N-th return map for the point, $(p, 0)$.

Let l_i be the period of p_i under f. Then, there exists a finite sequence, $\underline{x_i} = x_0\, x_1 \ldots x_{l_i - 1} \in \{0,1,2\}^{l_i}$ such that $\underline{\eta}(p_i) = \underline{x_i}^\infty$. Let $\tau_i = \sum_{j=0}^{l_i-1} s(p\ (\sigma^j(\underline{x_i})))$. Then τ_i is the time to flow through one period of p_i under φ^t, and the τ_i-map of φ^t, $\varphi^{\tau_i} : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \{0\}$ is the l_i-th return map for the point, $(p_i, 0)$.

Choose $n_0 \in \mathbb{N}$ so that $n_0 \tau_\beta \geq \sum_{i=1}^k \tau_i$. For brevity, we will abuse notation and redefine $\tau_\beta = n_0 \tau_\beta$ and $\underline{\beta} = \underline{\beta}^{n_0}$.

By hypothesis, all $r \in K$ can be written as $r = \sum_{i=1}^{k} \omega_i r_i$ for some ω_i such that $\sum_{i=1}^{k} \omega_i = 1$ and $0 \leq \omega_i \leq 1$.

Let $\varepsilon > 0$ be given and let

$$M = \left|\rho_\varphi^{\tau_\beta}(p, 0)\right| + k \cdot (\max_{r \in K} |r|)\ , \tag{2.4}$$

$$T > \frac{2\tau_\beta M}{\varepsilon} + \sum_{i=1}^{k} \tau_i, \tag{2.5}$$

and

$$n_i = \left\lfloor \frac{\omega_i T}{\tau_i} \right\rfloor \tag{2.6}$$

where $\lfloor y \rfloor \equiv \max\{n \in \mathbb{N} \mid n \leq y\}$ and $i = 1, \ldots, k$.

Set $\underline{\beta^*} = \underline{\beta}\ \underline{x_1}^{n_1} \underline{x_2}^{n_2} \ldots \underline{x_k}^{n_k}$ and let $p^* \in p\ (\underline{\beta}^* \underline{0}^\infty) \subset \Lambda_f \subset \mathbb{T}^2$ be a point with forward symbol sequence $\underline{\beta}^* \underline{0}^\infty \in \{0, 1, 2\}^{\mathbb{N}}$. Let

$$\tau_{\beta^*} = \tau_\beta + \sum_{i=1}^{k} n_i \tau_i \ .$$

Then the τ_{β^*}-map of φ^t, $\varphi^{\tau_{\beta^*}} : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \{0\}$, is the $(N + \sum_{i=1}^{k} n_i\ l_i)$-th return map for the point, $(p^*, 0)$.

Define $\Delta^t(q) = \widetilde{\varphi}^t(\widetilde{q}) - \widetilde{q}$ for all $\widetilde{q} \in \Pi^{-1}(q)$ where $q \in \mathbb{T}^3$. Δ^t satisfies the following property,

$$\Delta^{t_1+t_2}(q) = \Delta^{t_1}(q) + \Delta^{t_2}(\varphi^{t_1}(q)), \tag{2.7}$$

and the τ-rotation vector of q in terms of Δ^t is given by

$$\rho_\varphi^\tau(q) = \frac{\Delta^\tau(q)}{\tau} \ . \tag{2.8}$$

Then by (2.7),

$$\begin{aligned} \Delta^{\tau_{\beta^*}}(p^*, 0) \ = \ & \Delta^{\tau_\beta}(p^*, 0) + \Delta^{n_1\tau_1}(\varphi^{\tau_\beta}(p^*, 0)) + \\ & \Delta^{n_2\tau_2}(\varphi^{\tau_\beta + n_1\tau_1}(p^*, 0)) + \ldots + \\ & \Delta^{n_k\tau_k}(\varphi^{\tau_\beta + n_1\tau_1 + \ldots + n_{k-1}\tau_{k-1}}(p^*, 0)) \ . \end{aligned} \tag{2.9}$$

Since φ^t is a λ_s-scaled suspension flow of f,

$$\begin{aligned}\Delta^{\tau_{\beta^*}}(p^*,0) &= \Delta^{\tau_\beta}(p^*,0)+\Delta^{n_1\tau_1}(p\ (\sigma^N(\underline{\beta}^*\underline{0}^\infty)),0)+\\ &\quad \Delta^{n_2\tau_2}(p\ (\sigma^{N+n_1l_1}(\underline{\beta}^*\underline{0}^\infty)),0)+\ldots+\\ &\quad \Delta^{n_k\tau_k}(p\ (\sigma^{N+n_1l_1+\ldots+n_{k-1}l_{k-1}}(\underline{\beta}^*\underline{0}^\infty)),0)\ .\end{aligned}$$

From properties of the shift operator, σ,

$$\begin{aligned}\Delta^{\tau_{\beta^*}}(p^*,0) &= \Delta^{\tau_\beta}(p^*,0)+\Delta^{n_1\tau_1}(p\ (\underline{x_1}^{n_1}\underline{x_2}^{n_2}\ldots\underline{x_k}^{n_k}\underline{0}^\infty),0)+\\ &\quad \Delta^{n_2\tau_2}(p\ (\underline{x_2}^{n_2}\ldots\underline{x_k}^{n_k}\underline{0}^\infty),0)+\ldots+\\ &\quad \Delta^{n_k\tau_k}(p\ (\underline{x_k}^{n_k}\underline{0}^\infty),0)\ .\end{aligned}$$

Since $\varphi^{\tau_i}:\mathbb{T}^2\times\{0\}\to\mathbb{T}^2\times\{0\}$ is the l_i-th return map for the point, $(p_i,0)$,

$$\begin{aligned}\Delta^{\tau_{\beta^*}}(p^*,0) &= \Delta^{\tau_\beta}(p^*,0)+\Delta^{n_1\tau_1}(p_1,0)+\\ &\quad \Delta^{n_2\tau_2}(p_2,0)+\ldots+\Delta^{n_k\tau_k}(p_k,0)\ .\end{aligned}$$

And by the periodicity of each p_i we have,

$$\begin{aligned}\Delta^{\tau_{\beta^*}}(p^*,0) &= \Delta^{\tau_\beta}(p^*,0)+n_1\Delta^{\tau_1}(p_1,0)+\\ &\quad n_2\Delta^{\tau_2}(p_2,0)+\ldots+n_k\Delta^{\tau_k}(p_k,0).\end{aligned}$$

By (2.8) and since $\rho_\varphi^{\tau_i}(p_i,0)=\rho_\varphi(p_i,0)=r_i$,

$$\begin{aligned}\rho_\varphi^{\tau_{\beta^*}}(p^*,0)-r &= \frac{\Delta^{\tau_\beta}(p^*,0)}{\tau_{\beta^*}}+\sum_{i=1}^{k}\left(\frac{n_i\Delta^{\tau_i}(p_i,0)}{\tau_{\beta^*}}\right)-r\\ &= \frac{\tau_\beta}{\tau_{\beta^*}}\rho_\varphi^{\tau_\beta}(p^*,0)+\sum_{i=1}^{k}\left(\frac{n_i\tau_i}{\tau_\beta^*}\ \rho_\varphi^{\tau_i}(p_i,0)\right)-\sum_{i=1}^{k}\omega_i r_i\\ &= \frac{\tau_\beta}{\tau_{\beta^*}}\rho_\varphi^{\tau_\beta}(p^*,0)+\sum_{i=1}^{k}\left(\frac{n_i\tau_i}{\tau_\beta^*}-\omega_i\right)\ r_i \qquad (2.10)\end{aligned}$$

From (2.6) we have the following inequalities.

$$\frac{\omega_i T}{\tau_i}-1\le n_i\le\frac{\omega_i T}{\tau_i}\quad \text{for each } i\in 1,\ldots,k. \qquad (2.11)$$

Notice by (2.11) and (2.5)

$$\tau_{\beta^*} = \tau_\beta + \sum_{i=1}^{k} n_i \tau_i \geq \tau_\beta + T - \sum_{i=1}^{k} \tau_i > \frac{2\tau_\beta M}{\varepsilon}$$

so

$$\frac{\tau_\beta}{\tau_{\beta^*}} < \frac{\varepsilon}{2M}. \tag{2.12}$$

Also by (2.11)

$$\frac{n_i \tau_i}{\tau_\beta^*} - \omega_i \leq \omega_i \left(\frac{T}{\tau_{\beta^*}} - 1 \right) \leq \left(\frac{T}{\tau_{\beta^*}} - 1 \right).$$

By our choice of τ_β and (2.11), $\tau_{\beta^*} \geq T$. Thus by (2.12)

$$\left| \frac{T}{\tau_{\beta^*}} - 1 \right| = \left| \frac{\tau_{\beta^*} - T}{\tau_\beta^*} \right| \leq \left| \frac{(\tau_\beta + T) - T}{\tau_\beta^*} \right| = \left| \frac{\tau_\beta}{\tau_\beta^*} \right| < \frac{\varepsilon}{2M}. \tag{2.13}$$

Taking absolute values, applying the triangle inequality to (2.10) and substituting the bounds obtained in (2.12) and (2.13) we have,

$$\begin{aligned} \left| \rho_\varphi^{\tau_{\beta^*}}(p^*, 0) - r \right| &\leq \left| \frac{\tau_\beta}{\tau_{\beta^*}} \right| \left| \rho_\varphi^{\tau_\beta}(p^*, 0) \right| + \sum_{i=1}^{k} \left(\left| \frac{n_i \tau_i}{\tau_\beta^*} - \omega_i \right| |r_i| \right) \\ &< \frac{\varepsilon}{2M} \left(\left| \rho_\varphi^{\tau_\beta}(p^*, 0) \right| + \sum_{i=1}^{k} |r_i| \right). \end{aligned}$$

Since $\left| \rho_\varphi^{\tau_\beta}(p, 0) \right| = \left| \rho_\varphi^{\tau_\beta}(p^*, 0) \right|$ and by the choice of M in (2.4),

$$\left| \rho_\varphi^{\tau_{\beta^*}}(p^*, 0) - r \right| < \varepsilon. \qquad \square$$

The following lemma shows, for a collection of rotation vectors of *any* λ_s-scaled suspension flow of f, there exists a point with rotation set equal to the convex hull of the collection.

LEMMA 2.7. *Let f be a 3-symbol toroidal horseshoe on $\mathbb{T}^2$ and $\{p_1, p_2, \ldots, p_m\}$ be a set of f-periodic points. Let $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$ be a λ_s-scaled suspension flow of f with lift, $\widetilde{\varphi}^t : \mathbb{R}^3 \to \mathbb{R}^3$. Let $r_i = \rho_\varphi(p_i, 0)$ for each $i = 1 \ldots m$ and $< r_1, r_2, \ldots, r_m >$ be the convex hull of $\{r_1, r_2, \ldots, r_m\}$. Then there exists a point, $q \in \mathbb{T}^3$, such that the rotation set of q under $\widetilde{\varphi}^t$, $\rho_\varphi(q) = < r_1, r_2, \ldots, r_m >$.*

Proof: Let $K = \{k_1, k_2, \ldots\}$ be a countable and dense subset of $< r_1, r_2, \ldots, r_m >$. Remark: The desired point $q \in \mathbb{T}^3$, is such that $q = (p, 0)$ for some $p \in \Lambda_f$. We construct a sequence, $\underline{\eta}(p)$ corresponding to the forward symbol sequence of p, through an iterative process. At the n-th step, n-extensions are made to the finite sequence constructed at the $(n-1)$-st step. We keep track of what stage we are at in the process by subscripting the sequences. $\underline{\eta_{n,i}}$ is the i-th extension of the sequence at the n-th step where $i \in \{1, 2, \ldots, n\}$. For brevity, denote a point on the 2-torus with forward symbol sequence, $\underline{\eta_{n,i} 0^\infty}$, by $p_{n,i}$.

By Lemma 2.6 there exists a finite sequence $\underline{\eta_{1,1}} \in \{0, 1, 2\}^{N_{1,1}}$ for some $N_{1,1} > 0$ and $\tau_{1,1} > 0$ such that:

$$\left|\rho_\varphi^{\tau_{1,1}}(p_{1,1}, 0) - k_1\right| < 1.$$

Again by Lemma 2.6 we may extend $\underline{\eta_{1,1}}$ to the sequence $\underline{\eta_{2,1}} \in \{0, 1, 2\}^{N_{2,1}}$, where $N_{2,1} > N_{1,1}$, and there exists $\tau_{2,1} > \tau_{1,1}$ such that

$$\left|\rho_\varphi^{\tau_{2,1}}(p_{2,1}, 0) - k_1\right| < \frac{1}{2}.$$

Now extend $\underline{\eta_{2,1}}$ to the sequence $\underline{\eta_{2,2}} \in \{0, 1, 2\}^{N_{2,2}}$ where $N_{2,2} > N_{2,1}$ and let $\tau_{2,2} > \tau_{2,1}$ be such that

$$\left|\rho_\varphi^{\tau_{2,2}}(p_{2,2}, 0) - k_2\right| < \frac{1}{2}.$$

Continue this process so that at the nth step the sequence $\underline{\eta_{(n-1),(n-1)}}$ is extended to the sequence $\underline{\eta_{n,1}} \in \{0, 1, 2\}^{N_{n,1}}$ where $N_{n,1} > N_{(n-1),(n-1)}$, and let $\tau_{n,1} > \tau_{(n-1),(n-1)}$

be such that

$$\left|\rho_{\varphi}^{\tau_{n,1}}(p_{n,1},0) - k_1\right| < \frac{1}{n}.$$

Now for $i = 2 \ldots n$, extend the sequence $\underline{\eta_{n,(i-1)}}$ to the sequence $\underline{\eta_{n,i}} \in \{0,1,2\}^{N_{n,i}}$ where $N_{n,i} > N_{n,(i-1)}$, and let $\tau_{n,i} > \tau_{n,(i-1)}$ be such that

$$\left|\rho_{\varphi}^{\tau_{n,i}}(p_{n,i},0) - k_i\right| < \frac{1}{n}.$$

As n goes to ∞, the sequences $\underline{\eta_{n,i}0}^{\infty}$ limit on a sequence, $\underline{\eta} \in \{0,1,2\}^{\mathbb{N}}$. Under the lift of the flow, $\widetilde{\varphi}^t$, the rotation set of $q = (p(\underline{\eta}), 0)$ will contain each of the k_i's as they are limit points of $\left\{ \frac{\widetilde{\varphi}^t(\widetilde{q})-\widetilde{q}}{t} \;\middle|\; t \in \mathbb{R} \right\}$. Also, by construction, $\frac{\widetilde{\varphi}^t(\widetilde{q})-\widetilde{q}}{t} \in K$ for each t. So, since the closure of a set contains all of its limit points, $\rho_{\varphi} \subset Cl(K)$. Thus, the rotation set of q, $\rho_{\varphi}(q) = K$. □

Proof: (of THEOREM 2.1) By appropriate choices of a, b, c in the definition of the transition time function, s, there exists rotation vectors, r_0, r_1, r_2, r_3, of $\widetilde{\varphi}^t_{f,\lambda_s}$ that are not co-planar. By Lemma 2.7, there exists a point with rotation set equal to the convex hull of these rotation vectors. □

It is interesting to note that for choices of a, b, c which ensure that the rotation vectors are not co-planar, the projection of the convex hull of these rotation vectors onto $\mathbb{R}^2$ is not the rotation set of the toroidal horseshoe f. This projection is still triangular in shape but is not geometrically similar to the rotation set of f.

CHAPTER 3

PERIODIC-POINT-FREE FLOWS WHICH HAVE ROTATION SETS WITH INTERIOR

Barge-Walker have constructed a diffeomorphism $F : \mathbb{T}^n \to \mathbb{T}^n$ which is periodic-point-free and which has a rotation set with interior for $n \geq 3$ ([1]). The focus of this chapter is the construction of a periodic-point-free flow on the n-torus ($n \geq 4$) which has rotation set with non-empty interior. Their diffeomorphism, F, is essential in this construction although constructing the flow requires accounting for several scaling related issues simultaneously. We begin with a summary of the results of [1].

Let f be the fixed 3-symbol horseshoe as described in Chapter 1, and let $h = f^m$ for some $m = m(n)$. It suffices to let $m(n) = 2^{n-2}$. Then, $F : \mathbb{T}^n \to \mathbb{T}^n$ has a lift $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ of the form:

$$\tilde{F}(x, y, z_1, \ldots, z_{n-2}) = (\tilde{h}_1(x, y), \tilde{h}_2(x, y), z_1 + \tilde{g}_1(x, y), \ \ldots \ , z_{n-2} + \tilde{g}_{n-2}(x, y))$$

where $(\tilde{h}_1, \tilde{h}_2)$ is a C^∞ lift of h. Also, for $k = 1, 2, \ldots, n-2$, each $g_k : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ map for which $\tilde{g}_k(x, y) = \tilde{g}_k(x+1, y) = \tilde{g}_k(x, y+1)$. Notice that h has an invariant Cantor set, Λ_h, which supports a toroidal full shift on 3^m symbols. Depending on its 3^m symbol representation, each h-periodic orbit has a rotation vector in the closed triangle with vertices $(0, 0)$, $(0, m)$, (m, m). Each $\tilde{g}_k$ is designed to give (lifts of) each h-periodic orbit a net irrational push depending on its 3^m symbol representation. In this way, all h-periodic orbits are destroyed. The dependence of each $\tilde{g}_k$ on these 3^m symbol representations is designed so that the rotation set of $\widetilde{F}$, ρ_F, has n-dimensional interior.

To simplify the discussion, we construct only the flow on the 4-torus. It will then be clear how to make analogous constructions in higher dimensions. Let $h = f^2$. Then there exist 3^2 disjoint regions, $I_{(i,j)} \subset R$, where $I_{(i,j)} = I_i \times I_j$ for $i, j = 0, 1, 2$.

See Figure 5. The dynamics of h restricted to $\Lambda_h \cap R$ can be represented as symbol sequences of the 3^2-symbol space, namely, $\{0,1,2\} \times \{0,1,2\}$. For brevity, we denote the forward sequence space, $\left\{\{0,1,2\} \times \{0,1,2\}\right\}^{\mathbb{N}}$, by $\left\{\{0,1,2\}^2\right\}^{\mathbb{N}}$. For each $\underline{\eta} = \eta_0\eta_1\eta_2 \ldots \in \left\{\{0,1,2\}^2\right\}^{\mathbb{N}}$ there is a point, $p = p\left(\underline{\eta}\right) \in \Lambda_h$, such that $h^n(p) \in I_{\eta_n}$ for each $n = 0,1,2,\ldots$. Notice that if a point, $p \in \mathbb{T}^2$ has a sequence representation, $\underline{\eta}(p)$, under h, then the sequence representation of $h(p)$, $\underline{\eta}(h(p))$, is equal to the image of $\underline{\eta}(p)$ under the shift map, $\sigma : \left\{\{0,1,2\}^2\right\}^{\mathbb{N}} \to \left\{\{0,1,2\}^2\right\}^{\mathbb{N}}$ where $\sigma(\eta_0, \eta_1, \eta_2, \ldots) = (\eta_1, \eta_2, \ldots)$. So, $\eta(h(p)) = \sigma(\underline{\eta}(p))$.

The diffeomorphism, F, has a lift $\tilde{F} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form:

$$\tilde{F}(x,y,z) = (\tilde{h_1}(x,y), \tilde{h_2}(x,y), z + \tilde{g}(x,y)) \tag{3.1}$$

where the lift of h is denoted as $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)$. The function, $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$, gives (lifts of) each h-periodic orbit an irrational push in the z-direction depending on its 3^2 symbol representation. So, $\tilde{g}$ is defined as follows: Let $\underline{0}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta} \in \left\{\{0,1,2\}^2\right\}$ be defined by

$$\underline{0} \;=\; (0,0),\; \underline{\alpha} = (1,0),\; \underline{\beta} = (0,1),\; \underline{\gamma} \;=\; (2,0),\; \text{and } \underline{\delta} \;=\; (0,2)\;.$$

In order to "thicken" the rotation set of $\widetilde{F}$, i_0, i_1, and i_2 are chosen to be irrational numbers such that $-2 < i_0 < 2$, $i_1 < -4$, and $i_2 > 4$. We also require that $1, i_0, i_1,$ and i_2 are independent over $\mathbb{Q}$.

Then,

$$\tilde{g}(\tilde{p}) = \begin{cases} i_0 & \text{if } \eta_0 = \underline{0} \\ i_1 & \text{if } \eta_0 = \underline{\beta} \text{ or } \underline{\delta} \\ i_2 & \text{if } \eta_0 = \underline{\alpha} \text{ or } \underline{\gamma} \end{cases} \tag{3.2}$$

where $\tilde{p} \in \Pi^{-1}(p)$ and $p \in \Lambda_h$.

We now construct a λ_s-scaled suspension flow of F, $\varphi^t_{F,\lambda_s} : \mathbb{T}^4 \to \mathbb{T}^4$ which has the desired properties. Let $s : \mathbb{T}^2 \to \mathbb{R}^+$ be a smooth function such that:

$$s(p\left(\underline{\eta}\right)) = \begin{cases} a & \text{if } \eta_0 = \underline{\alpha} \\ b & \text{otherwise} \end{cases} \tag{3.3}$$

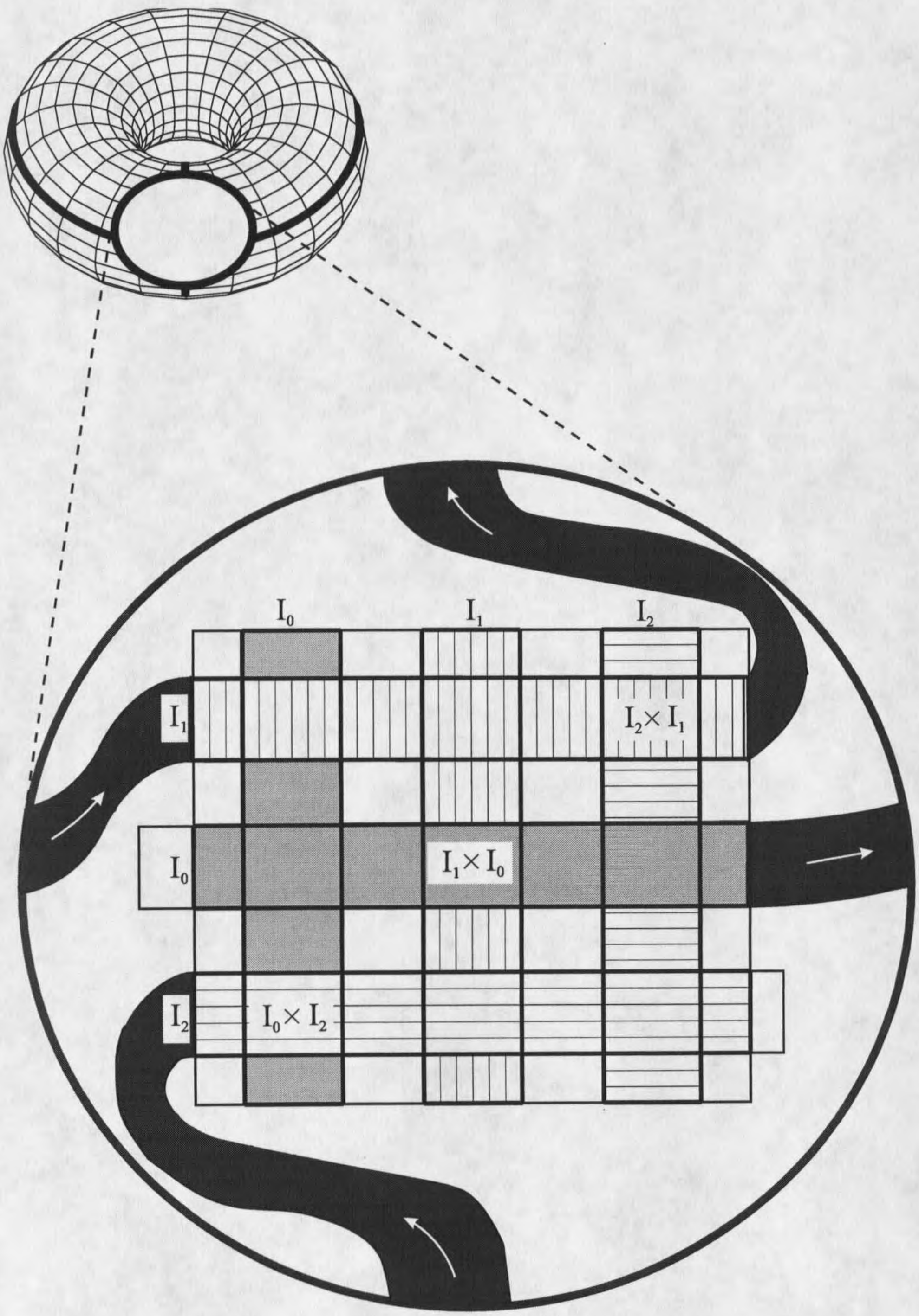


Figure 5. Toroidal horseshoe of the map, $h = f^2$

where $a, b \in \mathbb{R}^+$ and $a \neq b$. As described in Chapter 1, $\lambda_s : \mathbb{R}^4 \to \mathbb{R}^+$ scales the (vector field which generates) suspension flow of F in such a way that the transition times are precisely s. This scaled vector field generates the λ_s-scaled suspension flow of F, $\varphi^t_{F,\lambda_s} : \mathbb{T}^4 \to \mathbb{T}^4$. For the remainder of this chapter we denote $\varphi^t_{F,\lambda_s} : \mathbb{T}^4 \to \mathbb{T}^4$ by φ^t.

As in Chapter 2, for $p \in \Lambda_h$ and $q \in \{p\} \times \mathbb{S}^1 \times \{0\} \subset \mathbb{T}^4$, define $n(q,t) : \mathbb{T}^4 \times \mathbb{R} \to \mathbb{N}$ by

$$n(q,t) = \text{card}\left\{ \{\varphi^\tau(q) | \tau \in (0,t]\} \bigcap \left(\mathbb{T}^3 \times \{0\}\right) \right\}. \tag{3.4}$$

Here $n(q,t)$ determines the number of times the orbit of q intersects the cross-section, $\mathbb{T}^3 \times \{0\}$, for times in the range $(0,t]$. This is also the number of times the orbit of q traverses the fourth generator of $\mathbb{T}^4$ in t time units.

Given any $q \in \mathbb{T}^4$ as above, let $\{t_k\}_{k=1}^\infty$ be the strictly increasing sequence of times given by

$$t_k = \sum_{i=0}^{k-1} s(h^i(p)). \tag{3.5}$$

Then t_k is the time of the k-th intersection of the forward orbit of q with the cross-section, $\mathbb{T}^3 \times \{0\}$. Thus, $n(q, t_k) = k$. By an argument similar to that of Proposition 2.2, the rotation set of q under $\widetilde{\varphi}^t$ is

$$\rho_\varphi(q) = \text{LIM}\left\{ \frac{\widetilde{\varphi}^{t_k}(\widetilde{q}) - \widetilde{q}}{t_k} \,\middle|\, \widetilde{q} \in \Pi^{-1}(q) \text{ and } k \in \mathbb{N} \right\}. \tag{3.6}$$

Table 4 specifies $\Gamma : \left\{\{0,1,2\}^2\right\} \to \mathbb{R}^3$ for $\underline{0}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}$, where $\Gamma = (\Gamma_x, \Gamma_y, \Gamma_z)$. For any $p \in \Lambda_h$ with forward 3^2-symbol sequence $\underline{\eta}$, and for $\widetilde{q} \in \{p\} \times \mathbb{S}^1 \times \mathbb{S}^1$, where $\widetilde{q} = (x,y,z,u)$, Γ can be defined on all elements of $\left\{\{0,1,2\}^2\right\}$ so that $\widetilde{\varphi}^{t_1}(q) \in \left\{ \left(\widetilde{R} + (\Gamma_x(\eta_0), \Gamma_y(\eta_0))\, ,\; z + \Gamma_z(\eta_0),\; u+1 \right) \right\}$.

For any point, $p \in \Lambda_h$, with forward symbol sequence, $\underline{\eta}(p) \in \{\underline{0}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}\}^{\mathbb{N}}$, the rotation set of the point, $q \in \{p\} \times \mathbb{S}^1 \times \{0\}$, under φ^t can be found by utilizing Γ as defined in Table 4 and the following proposition:

Table 4. Γ values

	Γ_x	Γ_y	Γ_z
$\underline{0} = (0,0)$	0	0	i_0
$\underline{\alpha} = (1,0)$	1	0	i_2
$\underline{\beta} = (0,1)$	1	0	i_1
$\underline{\gamma} = (2,0)$	1	1	i_2
$\underline{\delta} = (0,2)$	1	1	i_1

PROPOSITION 3.1. *Let $F : \mathbb{T}^3 \to \mathbb{T}^3$ be a periodic-point-free diffeomorphism as described above, $p \in \Lambda_h$, with 3^2-symbol sequence, $\underline{\eta}(p)$, $q \in \{p\} \times \mathbb{S}^1 \times \{0\} \subset \mathbb{T}^4$, and let $\varphi^t : \mathbb{T}^4 \to \mathbb{T}^4$ be a λ_s-scaled suspension flow of F, $\widetilde{\varphi}^t : \mathbb{R}^4 \to \mathbb{R}^4$ be the lift of φ^t and $\{t_k\}$ as in (3.5). Then,*

$$\rho_\varphi(q) = \mathrm{LIM}\left\{ \frac{1}{t_k} \sum_{i=0}^{k-1} (\Gamma_x(\eta_i), \Gamma_y(\eta_i), \Gamma_z(\eta_i), 1) \,\middle|\, k \in \mathbb{N} \right\}.$$

Proof: By Proposition 2.3, we need only show that for some $B < \infty$

$$\left\| \left(\widetilde{\varphi}^{t_k}(\widetilde{q}) - \widetilde{q} \right)_z - \sum_{i=0}^{k-1} \Gamma_z(\eta_i) \right\| \leq B$$

where $\left(\widetilde{\varphi}^t(\cdot) - (\cdot) \right)_z$ denotes the third coordinate of $\widetilde{\varphi}^t(\cdot) - (\cdot)$. Since under the lift of the time-t_1 map, $\widetilde{\varphi}^{t_1}$, points are given a net irrational push in the third coordinate, $\left(\widetilde{\varphi}^{t_1}(\widetilde{q}) \right)_z = \Gamma_z(\eta_0)$. So, $\left(\widetilde{\varphi}^{t_k}(\widetilde{q}) \right)_z = \sum_{i=0}^{k-1} \Gamma_z(\eta_i)$. Thus, $B = 0$ suffices above. □

By Proposition 3.1, the rotation set of $q \in (p(\underline{\alpha}^\infty), 0, 0) \in \mathbb{T}^4$ under $\widetilde{\varphi}^t$ is:

$$\begin{aligned} r_1 \equiv \rho_\varphi(q) &= \mathrm{LIM}\left\{ \frac{1}{t_k} \sum_{i=0}^{k)-1} (\Gamma_x(\underline{\alpha}), \Gamma_y(\underline{\alpha}), \Gamma_z(\underline{\alpha}), 1) \,\middle|\, k \in \mathbb{N} \right\} \\ &= \mathrm{LIM}\left\{ \frac{1}{t_k} \sum_{i=0}^{k-1} (1, 0, i_2, 1) \,\middle|\, k \in \mathbb{N} \right\} \\ &= \mathrm{LIM}\left\{ \left(\frac{k}{t_k}, 0, \frac{k \cdot i_2}{t_k}, \frac{k}{t_k} \right) \,\middle|\, k \in \mathbb{N} \right\}. \end{aligned}$$

By definition of $s(p)$ in (3.3) and of $n(q,t)$ in (3.4), $t_k = ka$. So,

$$r_1 = \mathrm{LIM}\left\{ \left(\frac{k}{t_k}, 0, \frac{k \cdot i_2)}{t_k}, \frac{k}{t_k} \right) \middle| \, k \in \mathbb{N} \right\} \;=\; \left(\frac{1}{a}, 0, \frac{i_2}{a}, \frac{1}{a} \right).$$

It is left to the reader to check that the rotation sets under $\widetilde{\varphi}^t$ of the sets, $(p\ (\underline{0}^\infty), 0, 0)$, $(p\ (\underline{\beta}^\infty), 0, 0)$, $(p\ (\underline{\gamma}^\infty), 0, 0)$, $(p\ (\underline{\delta}^\infty), 0, 0)$ are:

$$r_0 \equiv \rho_\varphi(p\ (\underline{0}^\infty), 0, 0) = (0, 0, \tfrac{i_0}{b}, \tfrac{1}{b})\ ;$$

$$r_2 \equiv \rho_\varphi(p\ (\underline{\beta}^\infty), 0, 0) = (\tfrac{1}{b}, 0, \tfrac{i_1}{b}, \tfrac{1}{b})\ ;$$

$$r_3 \equiv \rho_\varphi(p\ (\underline{\gamma}^\infty), 0, 0) = (\tfrac{1}{b}, \tfrac{1}{b}, \tfrac{i_2}{b}, \tfrac{1}{b})\ ;$$

$$r_4 \equiv \rho_\varphi(p\ (\underline{\delta}^\infty), 0, 0) = (\tfrac{1}{b}, \tfrac{1}{b}, \tfrac{i_1}{b}, \tfrac{1}{b})\ .$$

The selection of s in (3.3) requires that $a \neq b$. Thus, the set of rotation vectors, $\{r_0, r_1, r_2, r_3, r_4\}$, is not contained in a 3-dimensional hyper-plane. So, $<r_0, r_1, r_2, r_3, r_4>$ has 4-dimensional interior. In order to show that ρ_φ has interior, it is enough to exhibit a point in $\mathbb{T}^4$ with a rotation set under $\widetilde{\varphi}^t$ that is a dense subset of $< r_0, r_1, r_2, r_3, r_4 >$. In order to do this we need the following lemmas which are analogous to Lemma 2.6 and Lemma 2.7.

LEMMA 3.2. *Let $F : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ be a periodic-point-free diffeomorphism as above where $h : \mathbb{T}^2 \to \mathbb{T}^2$ has an invariant set which supports a full shift on 3^m-symbols. Let $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$ be a λ_s-scaled suspension flow of F with lift $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$. Let $\{p_1, p_2, \cdots, p_k\} \subset \mathbb{T}^2$ be a set of h-periodic points. Let $q_i = (p_i, \overbrace{0, \ldots, 0}^{n-2\ times}) \subset \mathbb{T}^n$. Define $r_i = \rho_\varphi(q_i)$, for $i = 1 \cdots k$. Let $\underline{\beta} \in \left\{\{0,1,2\}^m\right\}^N$ for $0 < N < \infty$. For each $r \in K =< r_1, r_2, \cdots, r_k >$ and $\varepsilon > 0$ there exists a finite time, $\tau_* > 0$, and an extension of $\underline{\beta}$, $\underline{\beta}^*$, such that for all $q^* \in \{p(\underline{\beta}^*\underline{0}^\infty)\} \times \overbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n-3\ times} \times \{0\}$,*

$$|\rho_\varphi^{\tau_*}(q^*) - r| < \varepsilon\ .$$

Proof: Recall from the construction of the λ_s-scaled suspension flow, φ^t, that λ_s is chosen so that $s : \mathbb{T}^2 \to \mathbb{R}$ determines the time for a point, $q \in \mathbb{T}^2 \times \overbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n-3 \text{ times}} \times \{0\}$ to flow around the n-th generator of $\mathbb{T}^n$. For brevity, denote $\overbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}}$ by $\{\mathbb{S}^1\}^n$.

We begin by defining some notation analogous to that in Lemma 2.6.

Let $\tau_\beta = \sum_{j=0}^{N-1} s(h^j(p\,(\underline{\beta}\,\underline{0}^\infty)))$. Notice that $\varphi^{\tau_\beta} : \mathbb{T}^{n-1} \times \{0\} \to \mathbb{T}^{n-1} \times \{0\}$ is the N-th return map for all $q \in \{p\,(\underline{\beta}\,\underline{0}^\infty)\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$.

By hypothesis, each p_i is h-periodic. Let l_i be the period of p_i under h. Then, there exists a sequence $\underline{x_i} = x_0\, x_1 \ldots x_{l_i-1} \in \left\{\{0,1,2\}^m\right\}^{l_i}$ such that $\underline{\eta}(p_i) = \underline{x_i}^\infty$.

Let $\tau_i = \sum_{j=0}^{l_i-1} s(h^j(p_i))$. Notice that for any $q \in \{p_i\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$, $\varphi^{\tau_i} : \mathbb{T}^{n-1} \times \{0\} \to \mathbb{T}^{n-1} \times \{0\}$ is the l_i-th return map.

Choose $n_0 \in \mathbb{N}$ so that $n_0\tau_\beta \geq \sum_{i=1}^{k} \tau_i$. For brevity, we will abuse notation and redefine $\tau_\beta = n_0\tau_\beta$ and $\underline{\beta} = \underline{\beta}^{n_0}$.

Let $q_\beta \in \{p(\underline{\beta}, \underline{0}^\infty)\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$ and $M = \left|\rho_\varphi^{\tau_\beta}(q_\beta)\right| + k\left(\max_{r\in K} |r|\right)$.

Let $\varepsilon > 0$ be given and T be such that $T > \frac{2\tau_\beta M}{\varepsilon} + \sum_{i=1}^{k} \tau_i$.

By hypothesis $r \in K$, so $r = \sum_{i=1}^{k} \omega_i r_i$ for some ω_i such that $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^{k} \omega_i = 1$.

So let $n_i = \left\lfloor \frac{\omega_i T}{\tau_i} \right\rfloor$ where $\lfloor y \rfloor = \max\{n \in \mathbb{N} \mid n \leq y\}$.

Set $\underline{\beta^*} = \underline{\beta}\,\underline{x_1}^{n_1}\,\underline{x_2}^{n_2} \ldots \underline{x_k}^{n_k}$. Let the time to flow through β^* be denoted by $\tau_* = \tau_\beta + \sum_{i=1}^{k} n_i \tau_i$. Notice that $\varphi^{\tau_*} : \mathbb{T}^{n-1} \times \{0\} \to \mathbb{T}^{n-1} \times \{0\}$ is the $\left(N + \sum_{i=1}^{m} n_i l_i\right)$-th return map for any $q^* \in \{p\,(\underline{\beta^*}\underline{0}^\infty)\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$.

Define $\Delta : \mathbb{T}^n \to \mathbb{R}^n$ by $\Delta^t(q) = \widetilde{\varphi}^t(\widetilde{q}) - \widetilde{q}$ for $\widetilde{q} \in \Pi^{-1}(q)$ where $q \in \mathbb{T}^n$.

Then the τ-rotation vector (see Defintion 2.5) is given by $\rho_\varphi^\tau(q) = \frac{\Delta^\tau(q)}{\tau}$, and $\Delta^{t_1+t_2}(q) = \Delta^{t_1}(q) + \Delta^{t_2}(\varphi^{t_1}(q))$. Notice that because $\widetilde{\varphi}^t$ gives (lifts of) each p in the invariant set of h, Λ_h, a net irrational push in the 3-rd through the $(n-1)$-st coordinates, the following property holds. For any $t > 0, p \in \Lambda_h$, $\widehat{q_1} = (p, 0, \ldots, 0)$ and $\widehat{q_2} = (p, z_3, \ldots, z_{n-1}, 0)$ where $z_i \in \mathbb{S}^1$, $i = 3, ..., n-1$, we have that $\Delta^t(\widehat{q_1}) = \Delta^t(\widehat{q_2})$. Let $q^* = (p\ (\underline{\beta}^* \underline{0}^\infty), 0, \ldots, 0) \in \mathbb{T}^n$.

Then as in the proof of Lemma 2.6 and (2.9)

$$\Delta^{\tau_*}(q^*) = \Delta^{\tau_\beta}(q^*) + n_1 \Delta^{\tau_1}(q_1) + n_2 \Delta^{\tau_2}(q_2) + \cdots + n_m \Delta^{\tau_k}(q_k) \ .$$

And similarly, as in the proof of Lemma 2.6 and (2.10),

$$\rho_\varphi^{\tau_*}(q^*) - r = \frac{\tau_\beta}{\tau_*} \rho_\varphi^{\tau_\beta}(q^*) + \sum_{i=1}^{k} \left(\frac{n_i \tau_i}{\tau_*} - \omega_i \right) r_i \ .$$

And lastly, by the proof of Lemma 2.6, and the choice of M, T, n_i, we have,

$$\left| \rho_{\tilde{\varphi}}^{\tau_*}(q^*) - r \right| \leq \varepsilon. \qquad \square$$

The following lemma shows that, for $n \geq 4$ and $F : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$, the periodic-point-free diffeomorphism as described above, and for any collection of rotation vectors of φ^t, the λ_s-scaled suspension flow of F, there exists a point in the n-torus with rotation set equal to the convex hull of the given collection. This is the n-dimensional version of Lemma 2.7. The proof uses a more sophisticated, infinite to one, indexing scheme than was used in the proof of Lemma 2.7. In actuality, the fact that F is periodic-point-free is not needed in the proof, but Lemma 3.2 is heavily used.

LEMMA 3.3. *Let $n \geq 4$, $F : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ be the periodic-point-free diffeomorphism as described above, and $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$ be a λ_s-scaled suspension flow of F with lift, $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$. Let $r_1, r_2, \ldots, r_k$ be any collection of rotation vectors of $\widetilde{\varphi}^t$. Then there exists a point $q \in \mathbb{T}^n$ for which $\rho_\varphi(q) = < r_1, r_2, \ldots, r_k >$.*

Proof: Remark: The desired point $q \in \mathbb{T}^n$, is such that $q \in \{p\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$ for some $p \in \Lambda_h$. We construct a sequence corresponding to p, through an iterative process. At the j-th step, the sequence constructed at the $(j-1)$-st step is extended. We keep track of what stage we are at in the process by subscripting the sequences. We let $\underline{\eta_j}$ denote the j-th extension of the sequence. For brevity, denote a point on the 2-torus with forward symbol sequence, $\underline{\eta_j 0^\infty}$, by p_j. We let $q_j \in \{p_j\} \times \{\mathbb{S}^1\}^{n-3} \times \{0\}$.

Let K be a countable dense subset of $<r_1, r_2, \ldots, r_k>$. As K is countable, there exists an onto map $\kappa : \mathbb{N} \to K$ such that for each $k \in K$, $card(\kappa^{-1}(k)) = \infty$. By Lemma 3.2 there exists a finite sequence $\underline{\eta_1} \in \left\{\{0,1,2\}^m\right\}^{N_1}$ for some $N_1 > 0$ and a time, t_1, such that

$$|\rho_\varphi^{t_1}(q_1) - \kappa(1)| < 1$$

Now extend $\underline{\eta_1}$ to the sequence $\underline{\eta_2} \in \left\{\{0,1,2\}^m\right\}^{N_2}$, where $N_2 > N_1$ and there is a time, $t_2 > t_1$, such that

$$|\rho_\varphi^{t_2}(q_2) - \kappa(2)| < \frac{1}{2}$$

Such an extension exists by Lemma 3.2.

Continue in this manner and at the j-th step, extend $\underline{\eta_{j-1}}$ to $\underline{\eta_j} \in \left\{\{0,1,2\}^m\right\}^{N_j}$ where $N_j > N_{j-1}$ and there is a time, $t_j > t_{j-1}$, such that

$$|\,\rho_\varphi^{t_j}(q_j) - \kappa(j)| < \frac{1}{j}\,. \tag{3.7}$$

The sequence, $\underline{\eta_j 0^\infty} \in \left\{\{0,1,2\}^m\right\}^\infty$, limits on the sequence, $\underline{\eta} \in \left\{\{0,1,2\}^m\right\}^{\mathbb{N}}$ as $j \to \infty$ which is the forward symbol sequence of a point, $p \in \Lambda_h$. Let $q \in \{p\} \times \mathbb{S}^1 \times \{0\}$. We now show that $\rho_\varphi(q) = <r_1, r_2, \ldots, r_k>$. Choose a $k \in K$. In order for $k \in \rho_\varphi(q)$, k must be a limit point of $\left\{\frac{\tilde{\varphi}^t(\tilde{q}) - \tilde{q}}{t}\middle| t \in \mathbb{R}\right\}$. We require a strictly increasing sequence of times, $\{\tau_i\}_{i=1}^\infty$, such that for $\varepsilon > 0$, $\frac{\tilde{\varphi}^{\tau_i}(\tilde{q}) - \tilde{q}}{\tau_i} \in \mathrm{B}_\varepsilon(k)$ for i. Here $\mathrm{B}_\varepsilon(k)$ is the ε-ball of k. We now show such a sequence of times exists.

For any $k \in K$, the $card\{\kappa^{-1}(k)\} = \infty$, there is an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ so that $\kappa(n_i) = k$ for all i. Let $m > 0$ be such that $\frac{1}{n_m} < \varepsilon$. Let $\tau_1 = t_{n_m+1}$, $\tau_2 = t_{n_{m+2}}$, and $\tau_l = t_{n_{m+l}}$ for each $l > 0$. So, $\tau_1 < \tau_2 < \ldots$.

By the construction of the sequence $\underline{\eta}$ and by 3.7, for each $l > 0$

$$\left| \rho_\varphi^{\tau_l}(q) - k \right| = \left| \rho_\varphi^{t_{n_{m+l}}}(q) - \kappa(n_{m+l}) \right| < \frac{1}{n_{m+l}} < \frac{1}{n_m} < \frac{1}{\varepsilon}.$$

Thus $\rho_\varphi^{\tau_l}(q) \in \mathrm{B}_\varepsilon(k)$. Hence k is a limit point of $\rho_\varphi(q)$. Since q is such that $\rho_\varphi^t(q) \in < r_1, r_2, \ldots, r_k >$ for each t, and since the set of limit points of a set is contained in the closure of that set, $\rho_\varphi^t(q)$ is contained in $< r_1, r_2, \ldots, r_k >$. Thus, $\rho_\varphi(q) = < r_1, r_2, \ldots, r_k >$. □

We now have the results needed to prove the following theorem:

THEOREM 3.4. *For each $n \geq 4$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$, such that*

(i) the rotation set ρ_φ, has n-dimensional interior, and

(ii) φ^t has no periodic points.

Proof: Since F has no periodic points, the λ_s-scaled suspension flow of F constructed as above cannot have periodic points.

By construction, there exists a collection of points on the n-torus which have rotation sets that contain points which are not co-hyper-planar. By Lemma 3.3 there exists a point, $q \in \mathbb{T}^n$, such that $\rho_\varphi(q)$ is equal to the convex hull of this collection and thus has n-dimensional interior. Since $\rho_\varphi(q) \subset \rho_\varphi$, ρ_φ has n-dimensional interior.

□

CHAPTER 4

ANY POLYHEDRON IS THE ROTATION SET FOR A FLOW

In [22], Ziemian expands upon a widely known method which uses the transition graph to compute the rotation set of a map on the torus which restricts to a subshift of finite type on an invariant Cantor set. In this chapter, we show that this method can be extended and used to compute rotation sets of λ-scaled suspension flows of maps which restrict to subshifts of finite type on invariant Cantor sets. Then, using a theorem of Kwapisz ([12]), we show that given any convex polyhedron with rational vertices contained in $\mathbb{R}^3$, there exists a flow on the 3-torus which has that polyhedron as its rotation set.

Rotation Sets of Subshifts of Finite Type

We begin by establishing notation and stating a relevant theorem of Ziemian. Following Ziemian, by "$f : \mathbb{T}^n \to \mathbb{T}^n$ is a subshift of finite type" we mean that f restricts to a subshift of finite type on an invariant Cantor set. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a transitive subshift of finite type which is homotopic to the identity with lift $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$. Let $R = \{R_i\}_{i=1}^m$ be a Markov partition of f ([18]). The $m \times m$ *transition matrix* of f, $B_f = (b_{ij})$ is defined by:

$$b_{ij} = \begin{cases} 1 & \text{if } \operatorname{int}(f(R_i)) \bigcap \operatorname{int}(R_j) \neq \emptyset \\ 0 & \text{if } \operatorname{int}(f(R_i)) \bigcap \operatorname{int}(R_j) = \emptyset \, . \end{cases}$$

The *shift space* for the transition matrix, B_f, is defined as:

$$\Sigma_f = \left\{ \underline{\eta} = \{\eta_i\}_{i=-\infty}^{\infty} \mid b_{\eta_i \, \eta_{i+1}} = 1 \text{ for all } i \right\} \, .$$

Remark: For many choices of B_f, the shift space is empty. We will focus on those cases where the shift space is non-empty, which is always the case when f is a transitive subshift of finite type.

Let $\sigma : \{1, 2, \dots m\}^{\mathbb{Z}} \to \{1, 2, \dots m\}^{\mathbb{Z}}$ be the shift map on m-symbols defined by $\sigma(\underline{\eta}) = \underline{\xi}$ where $\xi_i = \eta_{i+1}$.

The *transition graph*, $\mathcal{G}_f$, of the Markov partition for f is constructed from the transition matrix, B_f, as follows: Let the rectangles of the Markov partitions, $R_1, R_2, \dots, R_m$, be represented by vertices of $\mathcal{G}_f$, labelled $A_1, A_2, \dots, A_m$, respectively. There is an arrow (directed edge), (A_i, A_j), in the graph, $\mathcal{G}_f$, which begins at A_i and ends at A_j if and only if $b_{ij} = 1$. A *path* in $\mathcal{G}_f$, denoted by $P = (A_{\eta_0}, A_{\eta_1}, \dots, A_{\eta_n})$ is a collection of vertices such that there is an arrow from A_{η_i} to $A_{\eta_{i+1}}$ for $i \in \{0, 1, \dots, n-1\}$. Notice that the set of all possible infinite paths of $\mathcal{G}_f$ corresponds to the set of all allowable sequences, Σ_f. The *length* of a path, P, denoted by $|P|$, is the number of arrows in the path. So, if $P = (A_{\eta_0}, A_{\eta_1}, \dots, A_{\eta_n})$, then $|P| = n$. If a path, $P = (A_{\eta_0}, A_{\eta_1}, \dots, A_{\eta_n})$, is such that $A_{\eta_0} = A_{\eta_n}$, then we refer to P as a *loop* of $\mathcal{G}_f$. A loop is said to be *elementary* if it is *not* the concatenation of 2 shorter loops. That is, $P = (A_{\eta_0}, A_{\eta_1}, \dots, A_{\eta_n})$ is an elementary loop of $\mathcal{G}_f$ provided $A_{\eta_0} = A_{\eta_n}$ and if $i \neq j$, then $A_{\eta_i} \neq A_{\eta_j}$ for $i \in \{0, 1, \dots, n\}$ and $j \in \{1, 2, \dots, n-1\}$.

Define $\Delta_f : \mathbb{T}^n \to \mathbb{R}^n$ by $\Delta_f(p) = \widetilde{f}(\widetilde{p}) - \widetilde{p}$ where $\widetilde{p} \in \Pi^{-1}(p)$ for some $p \in \mathbb{T}^n$. Then Δ_f is the displacement map for $\widetilde{f}$. The expression, $\frac{\widetilde{f}^n(\widetilde{p}) - \widetilde{p}}{n}$, which is used in computing the rotation set of f, can be rewritten using the displacement map as $\frac{1}{n} \sum_{i=0}^{n-1} \Delta_f(f^i(p))$. This is the average displacement of points along part of the orbit of p. Since orbits can be represented as sequences of Σ_f, as well as paths of $\mathcal{G}_f$, if we record the displacement of f along each of the arrows of $\mathcal{G}_f$, all information needed to compute the rotation set of f would be stored in the graph, $\mathcal{G}_f$. To this end, we would like to represent Δ_f, the displacement by f, by a map on the arrows of $\mathcal{G}_f$. The desired map, which we denote by $D_f : \Sigma_f \to \mathbb{R}^n$, must be "constant along the arrows of $\mathcal{G}_f$" otherwise it is not well-defined. By "constant along the arrows of $\mathcal{G}_f$" we mean constant on cylinders of length 2 in the sequence space, Σ_f.

A *cylinder of length 2* is defined by $C_{ab} = \{\underline{\eta} \in \Sigma_f \,|\eta_0 = a \text{ and } \eta_1 = b\}$ where $a, b \in \{1, \ldots, m\}$. When f is a transitive subshift of finite type, Ziemian exhibits a map, $D_f : \Sigma_f \to \mathbb{R}^n$ that is constant on cylinders of length 2 ([22]). That is, there is a $M > 0$, so that for all $p \in \mathbb{T}^n$ with symbol sequence, $\underline{\eta}(p)$, and all $k \in \mathbb{N}$,

$$\|\sum_{i=0}^{k-1} \Delta_f(f^i(p)) - \sum_{i=0}^{k-1} D_f(\sigma^i(\underline{\eta}(p)))\| < M\,.$$

Therefore,

$$\lim_{k\to\infty} \left\| \frac{1}{k} \sum_{i=0}^{k-1} \Delta_f(f^i(p)) - \frac{1}{k} \sum_{i=0}^{k-1} D_f(\sigma^i(\underline{\eta}(p))) \right\| = 0\,. \tag{4.1}$$

Since D_f is constant on cylinders of length 2 and hence on arrows of $\mathcal{G}_f$, we assign to each arrow, (A_{η_0}, A_{η_1}), the value of $D_f(\underline{\eta})$ for any $\underline{\eta} \in C_{\eta_0 \eta_1}$. For each (finite) path, $P = (A_{\eta_0}, A_{\eta_1}, \ldots, A_{\eta_n})$ contained in $\mathcal{G}_f$, define the rotation vector of a path, P, by,

$$\rho_f(P) = \frac{1}{n} \sum_{i=0}^{n-1} D_f(\sigma^i(\underline{\eta}))\,. \tag{4.2}$$

Then, $\rho_f(P)$ is the average value of D_f on the arrows of P. Let $l_1, l_2, \ldots, l_k$ be the elementary loops of $\mathcal{G}_f$. Then, $\rho_f(l_1), \rho_f(l_2), \ldots, \rho_f(l_k)$ are the rotation vectors of these loops. As we see from the following theorem of Ziemian ([22]), these rotation vectors completely determine the rotation set of f.

THEOREM (ZIEMIAN) ([22]): *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a transitive subshift of finite type. Then, the (point) rotation set of f,*

$$\rho_f = < \rho_f(l_1), \rho_f(l_2), \ldots, \rho_f(l_k) >\,.$$

where $< \cdot >$ denotes the convex hull.

Remark: This method of computing the rotation set can also be extended to include non-transitive subshifts of finite type ([22], pg. 191) .

Rotation Sets of Scaled Suspension Flows of Subshifts of Finite Type

We now show how the result of Ziemian can be extended to include certain λ_s-scaled suspension flows of a C^∞ map, $f : \mathbb{T}^2 \to \mathbb{T}^2$, which is a subshift of finite type. The class of λ_s-scaled suspension flows we will consider are those that have a C^∞ transition time function, $s : \mathbb{T}^2 \to \mathbb{R}^+$, which is constant on each of the rectangles of the Markov partition of f. Such an s exists because the Markov partitions we will consider consist only of disjoint rectangles. In order to extend Ziemian's results to these λ_s-scaled suspension flows, we must keep track of the time it takes for points to flow around the third generator of the 3-torus. Since the transition time function is constant on each of the rectangles of the Markov partition of f, we are able to assign the value of the transition time to each of the arrows of the transition graph of φ^t. Then, using the information stored in the graph, we are able to compute the rotation set of the λ_s-scaled suspension flow.

We begin by establishing notation. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a subshift of finite type with Markov partition, $\{R_j\}_{j=1}^m$. Let $p \in \Lambda_f$, the invariant Cantor set of f. Then the forward orbit of p can be represented as a sequence, $\underline{\eta}(p)$, in the symbol space, $\{1, 2, \ldots, m\}^{\mathbb{N}}$, where $\eta_i = k$ if and only if $f^i(p) \in R_k$.

Assume the transition time function, $s : \mathbb{T}^2 \to \mathbb{R}^+$, is constant on each R_j. As described in Chapter 1, λ_s is the function which scales (the vector field which generates) a suspension flow of f. Let $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$ be the λ_s-scaled suspension flow of f and $\widetilde{\varphi}^t : \mathbb{R}^3 \to \mathbb{R}^3$ be the lift of φ^t. Since $\mathbb{T}^2 \times \{0\}$ is a cross-section to the flow, φ^t, we need only look at the displacements (under the lift) of points, $q = (p, 0) \in \mathbb{T}^3$, for all $p \in \mathbb{T}^2$. By uniform continuity of φ^t, we need only consider times when the forward orbit of such a q intersects the cross-section, $\mathbb{T}^2 \times \{0\}$.

For $q = (p, 0) \in \mathbb{T}^3$ where $p \in \mathbb{T}^2$ with symbol sequence, $\underline{\eta}(p)$, let

$$t_k(q) = \sum_{j=0}^{k-1} s(f^j(p)) = \sum_{j=0}^{k-1} s(p(\sigma^j(\underline{\eta}(p)))) \; . \tag{4.3}$$

Note, $t_k(q)$ is precisely the time of the k-th intersection of the forward orbit of q with the cross-section, $\mathbb{T}^2 \times \{0\}$.

Having assumed the transition time map is constant on the arrows of the transition graph of φ^t, we now seek an approximation of the displacement by $\widetilde{\varphi}^t$ which is also constant on arrows of the graph. Let $\Delta_\varphi : \mathbb{T}^2 \times \{0\} \to \mathbb{R}^3$ be the displacement by $\widetilde{\varphi}^t$ of (the lift of) a point, $q \in \mathbb{T}^2 \times \{0\}$, at $t_1(q)$. That is, for any $q = (p, 0) \in \mathbb{T}^3$ where $p \in \mathbb{T}^2$,

$$\Delta_\varphi(q) = \widetilde{\varphi}^{t_1(q)}(\widetilde{q}) - \widetilde{q} \, .$$

Since $t_k(q)$ is as in (4.3), and since φ^t is a λ_s-scaled suspension flow of f,

$$\Delta_\varphi(q) = \big((\widetilde{f}(\widetilde{p}) - \widetilde{p}), \; 1\big) \; . \tag{4.4}$$

By hypothesis, f is a subshift of finite type. So, by [22], there exists a map, $D_f : \Sigma_f \to \mathbb{R}^{n-1}$, such that D_f is constant on cylinders of length 2 and D_f satisfies (4.1). Define $D_\varphi : \Sigma_f \to \mathbb{R}^3$ by

$$D_\varphi(\underline{\eta}) = (D_f(\underline{\eta}), 1) \; . \tag{4.5}$$

It follows that D_φ is also constant on cylinders of length 2. So by (4.1), for $q = (p, 0)$ where $p \in \mathbb{T}^2$ with symbol sequence, $\underline{\eta}(p)$,

$$\lim_{k\to\infty} \left\| \frac{1}{t_k(q)} \sum_{i=0}^{k-1} \Delta_\varphi(\varphi^{t_i(q)}(q)) - \frac{1}{t_k(q)} \sum_{i=0}^{k-1} D_\varphi(\sigma^i(\underline{\eta}(p))) \right\| = 0 \; . \tag{4.6}$$

The itinerary of orbits are represented as sequences of Σ_f, as well as paths in $\mathcal{G}_f$. Let the graph of φ, $\mathcal{G}_\varphi$, contain all vertices and arrows of $\mathcal{G}_f$. Assign to each arrow, $(A_{\eta_0}, A_{\eta_1}) \in \mathcal{G}_\varphi$, the displacement, $D_\varphi(\underline{\eta})$, for any $\underline{\eta} = (\eta_0\, \eta_1 \ldots) \in \Sigma_f$. Also assign to (A_{η_0}, A_{η_1}) the transition time, $\tau(A_{\eta_0}, A_{\eta_1}) = \tau_1(q)$, for any $q = (p, 0) \in \mathbb{T}^3$ such that the symbol sequence of p is $\underline{\eta} = (\eta_0, \eta_1, \dots) \in \Sigma_f$. Note that D_φ is well-defined because, by [22], D_f is constant on arrows of $\mathcal{G}_f$. Now, all necessary information for computing the rotation set of the λ_s-scaled suspension flow, φ, is stored in the graph, $\mathcal{G}_\varphi$.

For any (finite) path, $P = (A_{\eta_0}, A_{\eta_1}, \dots, A_{\eta_n})$ contained in $\mathcal{G}_\varphi$, define the rotation vector of a path, P, by,

$$\rho_\varphi(P) = \frac{\sum_{i=0}^{n-1} D_\varphi(\sigma^i(\underline{\eta}))}{\sum_{i=0}^{n-1} s(p(\sigma^i(\underline{\eta})))} = \frac{\sum_{i=0}^{n-1} D_\varphi(\sigma^i(\underline{\eta}))}{\sum_{i=0}^{n-1} \tau(A_{\eta_i}, A_{\eta_{i+1}})} . \tag{4.7}$$

Let $l_1, l_2, \dots, l_k$ be the elementary loops of $\mathcal{G}_f$. We denote the rotation vectors of these loops by $\rho_f(l_1), \rho_f(l_2), \dots, \rho_f(l_k)$. Thus, we have the following proposition:

PROPOSITION 4.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a subshift of finite type and $s : \mathbb{T}^2 \to \mathbb{R}^+$ be constant on each rectangle of the Markov partition of f. Let $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$ be the λ_s-scaled suspension flow of f with lift, $\widetilde{\varphi}^t : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathcal{G}_\varphi$ be the transition graph of φ^t with elementary loops, $l_1, l_2, ..l_k$. Then, the rotation set of $\widetilde{\varphi}^t$,*

$$\rho_\varphi = <\rho_\varphi(l_1), \rho_\varphi(l_2), \dots, \rho_\varphi(l_k)> .$$

Proof: The proposition follows from (2.2), (4.7), and Theorem (Ziemian). □

Polyhedrons as Rotation Sets

A natural question concerning rotation sets is: What subsets of $\mathbb{R}^2$ can be rotation sets of toral maps? As a partial answer, Kwapisz has proved the following theorem:

Theorem (Kwapisz) ([12]) *Let $K \subset \mathbb{R}^2$ be a convex polygon with vertices at rational points of $\mathbb{R}^2$. Then there exists a diffeomorphism, $g : \mathbb{T}^2 \to \mathbb{T}^2$, which is homotopic to the identity, such that $\rho_g = K$.*

The analogous question for flows is: What subsets of $\mathbb{R}^3$ can be the rotation sets for flows on the 3-torus? We adapt the above result to give a partial answer which is summarized in the following:

THEOREM 4.2. *Let $K \subset \mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ such that $(0,0,0) \notin K$. Then there exists a $\mathbb{C}^\infty$ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the rotation set $\rho_\varphi = K$.*

In order to obtain the desired polyhedron as the rotation set of the flow, one might naively attempt the following: Project the vertices of the polyhedron onto $\mathbb{R}^2$. Use the result of Kwapisz to find a map on the 2-torus with rotation set, the convex hull of the projected vertices. Then simply suspend this map. However, as is evident in (4.7), all components of the rotation vectors are scaled by time. Moreover, this scaling is not uniform and may even destroy the geometry of the rotation set of the Kwapisz map in the $\mathbb{R}^2$ coordinates. That is, the rotation set of the λ_s-scaled suspension flow projected onto $\mathbb{R}^2$ will most likely *not* be the rotation set of the map we are suspending because the scaling in the flow direction distorts the polygon in the non-flow direction. Therefore, we must a priori account for the result of this time scaling. That is to say, given the desired rotation set, we must anticipate the scaling effects of time and "counter-scale" vertices. Just as the transition time depends upon the point on the torus, our "counter-scaling" depends upon the vector in $\mathbb{R}^3$.

Since Proposition 4.4 below provides polyhedron rotation sets contained in $\mathbb{R}^2 \times \mathbb{R}^+$, we first prove Lemma 4.3, which is similar to a lemma in [12] and which shows how we can linearly scale the rotation set of a given flow. Given a flow, φ^t on $\mathbb{T}^n$, and a linear isomorphism L, Lemma 4.3 provides a new flow, ψ^t, with rotation set, $\rho_\psi = L^{-1}\rho_\varphi$.

LEMMA 4.3. *Let L be a linear isomorphism on $\mathbb{R}^n$ such that $L(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$. If $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$, is the lift of a flow on $\mathbb{T}^n$, let $\widetilde{\psi}^t = L^{-1}\widetilde{\varphi}^t L$. Then*

(a) $\widetilde{\psi}^t(\mathbf{x}+\mathbf{v}) = \widetilde{\psi}^t(\mathbf{x}) + \mathbf{v}$ *for all* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{v} \in L^{-1}(\mathbb{Z}^n)$, *and* $t \in \mathbb{R}^n$.

(b) $\widetilde{\psi}^t$ *is the lift of a flow on* $\mathbb{T}^n$, *and*

(c) $\rho_\psi = L^{-1}(\rho_\varphi)$.

Proof of a) : Let $\mathbf{v} = L^{-1}(\mathbf{y})$ for some $\mathbf{y} \in \mathbb{Z}^n$ then,

$$\begin{aligned}
\widetilde{\psi}^t(\mathbf{x}+\mathbf{v}) &= L^{-1}\widetilde{\varphi}^t L(\mathbf{x} + L^{-1}(\mathbf{y})) \\
&= L^{-1}\widetilde{\varphi}^t(L\mathbf{x} + \mathbf{y}) \\
&= L^{-1}(\widetilde{\varphi}^t L\mathbf{x} + \mathbf{y}) \text{ as } \mathbf{y} \in \mathbb{Z}^n \text{ and } \widetilde{\varphi}^t \text{ is the lift of } \varphi^t \\
&= \widetilde{\psi}^t(\mathbf{x}) + L^{-1}(\mathbf{y}) \\
&= \widetilde{\psi}^t(\mathbf{x}) + \mathbf{v} \,.
\end{aligned}$$

Proof of b) : We show that $\widetilde{\psi}^t$ satisfies the group property.

$$\begin{aligned}
\widetilde{\psi}^{t+s}(\mathbf{x}) &= L^{-1}\widetilde{\varphi}^{t+s}L(\mathbf{x}) \\
&= L^{-1}\widetilde{\varphi}^t\widetilde{\varphi}^s L(\mathbf{x}) \\
&= L^{-1}\widetilde{\varphi}^t L L^{-1}\widetilde{\varphi}^s L(\mathbf{x}) \\
&= \widetilde{\psi}^t \circ \widetilde{\psi}^s(\mathbf{x}) \,.
\end{aligned}$$

Furthermore, since $L(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$, and $\widetilde{\psi}^t$ is a family of homeomorphisms on $\mathbb{R}^n$ which satisfies the group property. Thus, $\widetilde{\psi}^t$ is the lift of a flow on $\mathbb{T}^n$.

Proof of c) : The rotation set of the flow, ψ^t, with lift, $\widetilde{\psi}^t$ is such that:

$$\begin{aligned}
\rho_\psi &= \operatorname*{LIM}_{t\to\infty}\left(\frac{\widetilde{\psi}^t(\mathbf{x}) - \mathbf{x}}{t}\middle|\ \mathbf{x} \in \mathbb{R}^n\right) \\
&= \operatorname*{LIM}_{t\to\infty}\left(\frac{L^{-1}\widetilde{\varphi}^t L(\mathbf{x}) - \mathbf{x}}{t}\middle|\ \mathbf{x} \in \mathbb{R}^n\right) \\
&= \operatorname*{LIM}_{t\to\infty}\left(\frac{L^{-1}(\widetilde{\varphi}^t L\mathbf{x} - L\mathbf{x})}{t}\middle|\ \mathbf{x} \in \mathbb{R}^n\right) \,.
\end{aligned}$$

Let $\mathbf{y} = L\mathbf{x}$, then

$$\rho_\psi = \underset{t\to\infty}{\mathrm{LIM}} \left(\frac{L^{-1}(\widetilde{\varphi}^t\mathbf{y} - \mathbf{y})}{t} \Big| \, \mathbf{y} \in L(\mathbb{R}^n) = \mathbb{R}^n \right) .$$

Then since L^{-1} is a linear isomorphism,

$$\begin{aligned} \rho_\psi &= L^{-1} \left(\underset{t\to\infty}{\mathrm{LIM}} \left(\frac{(\widetilde{\varphi}^t\mathbf{y} - \mathbf{y})}{t} \Big| \, y \in \mathbb{R}^n \right) \right) \\ &= L^{-1}(\rho_\varphi) . \qquad \square \end{aligned}$$

(Note: Throughout the discussion to follow n will no longer be used to denote the torus dimension as it was in Lemma 4.3.

The proof of Theorem 4.2 is a consequence of Proposition 4.4 and Lemma 4.3. The proof of Proposition 4.4 relies upon Theorem (Kwapisz). So, before we begin the proof, we discuss the relevant properties of the diffeomorphism constructed by Kwapisz.

In the proof of Theorem (Kwapisz), given any finite set of points contained in $\mathbb{Q}^2$, $\{w_1, w_2, \ldots, w_n\}$, a diffeomorphism, g on $\mathbb{T}^2$ is constructed with rotation set, $\rho_g = < w_1, w_2, \ldots, w_n >$. This g is a subshift of finite type together with a finite number of sources and sinks. We denote the set of all sources and sinks of g by $\mathcal{S}_g$. The subshift of finite type of g has a Markov partition, $\{R_i\}_{i=0}^m$, such that $R_i \bigcap R_j \neq \emptyset$ if and only if $i = j$. As before, there is a $D_g : \Sigma_g \to \mathbb{R}^2$ which approximates the displacement of points under (the lift of) g, which is constant on cylinders of length 2 and satisfies (4.1). The transition graph of the subshift of finite type of g, which we denote by, $\mathcal{G}_g$, contains vertices, $A_1, \ldots, A_m$. We denote the elementary loops of $\mathcal{G}_g$ by $l_1, l_2, \ldots, l_k$. Note that for the Kwapisz diffeomorphisms the number of elementary loops, k, is greater than n, the number of vertices in the desired polyhedron. Each l_i is a loop which visits a subset of the set of vertices, $\{A_1, \ldots, A_m\}$. We give each vertex another label(s) which corresponds to the loop(s) to which it belongs. If we consider the j-th vertex on loop, l_i, we label it A_j^i. Thus, each $l_i = (A_0^i, A_1^i, \ldots, A_{N(i)}^i)$

where $A_0^i = A_{N(i)}^i$ for some $N(i)$ determined in ([12]). Since the displacement on an arrow, (A_j^i, A_{j+1}^i), under (lifts of) g is approximated by $D_g(\underline{\eta})$ for some $\underline{\eta} \in \Sigma_g$ where $A_{\eta_0} = A_j^i$ and $A_{\eta_1} = A_{j+1}^i$; we will abuse notation and denote $D_g(\underline{\eta})$ by $D_g(A_j^i, A_{j+1}^i)$.

The elementary loops of $\mathcal{G}_g$, $l_1, l_2, \dots, l_k$, can be characterized as these types.

Type I: For $i, j = 1, \dots, n$, $\rho_g(l_i) = w_i$, $|l_i| > 1$, and if $i \neq j$, l_i and l_j share no vertex and hence share no arrows.

Type II: For $i = n+1, \dots, n+m$, $|l_i| = 1$; that is, l_i is a loop consisting of a single arrow from a vertex back to itself. So, there is exactly one Type II loop for each vertex of the graph.

Type III: Let E denote set of all arrows contained in loops, $l_1, \dots, l_{n+m}$. For $i = n+m+1, \dots, k$, l_i contains at least one arrow that is not in E. Furthermore, if l_i contains 2 or more arrows that are not in E, these arrows are not consecutive in this loop. That is, every arrow in the compliment of E begins at a vertex in loop, l_r, and ends at a vertex in loop, l_ν, $r \neq \nu$ and $r, \nu = 1, \dots, n$. See Figure 6.

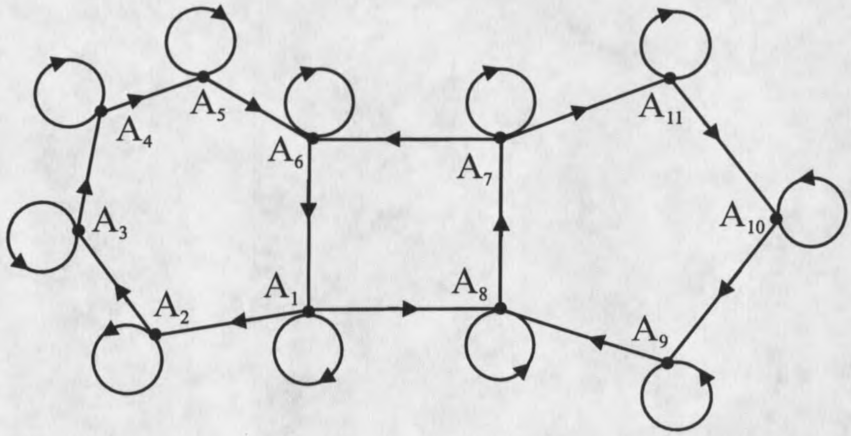
From Theorem (Ziemian) and the proof of Theorem (Kwapisz), we have that

$$\rho_g = <\rho_g(l_1), \rho_g(l_2), \dots, \rho_g(l_k)> = <\rho_g(l_1), \rho_g(l_2), \dots, \rho_g(l_n)> \quad . \qquad (4.8)$$

Now, we are ready to prove:

PROPOSITION 4.4. *Let $K \subset \mathbb{R}^2 \times \mathbb{R}^+$ be any 3-dimensional convex polyhedron with vertices, $v_1, v_2, \dots v_n$, where $v_i \in \mathbb{Z}^3$. Then, there exists a flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that $\rho_\varphi = K$.*

Proof: To construct a flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, with the desired rotation set, K we begin by scaling the vertices, $v_1, v_2, \dots v_n$, and then projecting them onto $\mathbb{R}^2$. Next, we use the result of Kwapisz ([12]) to produce a map, $g : \mathbb{T}^2 \to \mathbb{T}^2$, with rotation set equal to the convex hull of the projected vertices. Lastly, we carefully assign transition times



Type I Loops: $(A_1, A_2, A_3, A_4, A_5, A_6, A_1)$; $(A_7, A_{11}, A_{10}, A_9, A_8, A_7)$

Type II Loops: (A_i, A_i) for $i = 1, \ldots, 11$

Type III Loops: $(A_1, A_8, A_7, A_6, A_1)$

Figure 6. Example Type I, Type II, and Type III loops

to the arrows of the graph of g in such a way that the rotation set of the λ-scaled suspension flow of g is precisely $< v_1, v_2, \ldots v_n >= K$ as desired.

Let $v_i = (x_i, y_i, z_i) \in \mathbb{Q}^2 \times \mathbb{Q}^+$. Define $w_i \in \mathbb{Q}^2$ by

$$w_i = \left(\frac{x_i}{z_i}, \frac{y_i}{z_i} \right) . \tag{4.9}$$

Notice that by hypothesis, $(0,0) \in < w_1, w_2, \ldots, w_n >$. By [12] there exists $g : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\rho_g =< w_1, w_2, \ldots, w_n >$. Let $\mathcal{G}_g$ denote the graph of g and let $l_1, l_2, \ldots, l_n$ be the elementary loops of Type I. Denote each loop, l_i, by $l_i = (A_0^i, A_1^i, \ldots, A_{N(i)}^i)$ for $i = 1, \ldots, n$ where $A_0^i = A_{N(i)}^i$ for some $N(i)$ which is specified from the construction of g.

Let $D_g : \Sigma_g \to \mathbb{R}^2$ be the approximate displacement function of (the lift of) g

which satisfies (4.1). Then by (4.2) and the construction of g, for each $i = 1, \ldots, n$

$$\rho_g(l_i) = \frac{\sum_{j=0}^{N(i)-1} D_g(A^i_j, A^i_{j+1})}{N(i)} = w_i \,. \tag{4.10}$$

We now augment D_g on each arrow, (A^i_j, A^i_{j+1}) to obtain the displacement as a point flows around the third generator of $\mathbb{T}^3$. We denote this augmentation by $D_\varphi : \Sigma_g \to \mathbb{R}^3$, and it is defined by,

$$D_\varphi(A^i_j, A^i_{j+1}) = (D_g(A^i_j, A^i_{j+1}), 1) \,. \tag{4.11}$$

Rectangles of the Markov partition of g can be thought of as subsets of $\mathbb{T}^2 \times \{0\}$, a cross-section to the a λ-scaled suspension flow of g . So, $D_\varphi(A^i_j, A^i_{j+1})$ will be the approximated displacement of a point in the rectangle which corresponds to vertex A^i_j as it flows to the rectangle which corresponds to vertex A^i_{j+1} .

Now for loops, $l_1, \ldots, l_n$, assign to each arrow (A^i_j, A^i_{j+1}) of l_i a "transition time", $\tau(A^i_j, A^i_{j+1})$ given by

$$\tau(A^i_j, A^i_{j+1}) = \tau_i = \frac{1}{z_i} \,. \tag{4.12}$$

Recall z_i is the 3-rd coordinate of v_i.

At this point we have done the main work in determining a transition time function $s : \mathbb{T}^2 \to \mathbb{R}^+$. In fact, we will later show that the rotation set of the λ_s-scaled suspension flow of g with this particular s, is precisely the convex hull of the rotation vectors of the Type I loops. For any arrow that remains we must now carefully assign times so that the rotation vector of any loop lies within the convex hull of the rotation vectors of the Type I loops. We must also assign times similarly to points in $\mathcal{S}_g$.

For arrows that are not in a Type I loop assign times as follows: If (A^r_j, A^r_{j+1}) is the arrow in a Type II loop, l_r, then $A^r_j = A^r_{j+1}$ and A^r_j is a vertex in some Type I loop, l_i, for some $i = 1 \ldots n$. Assign to this arrow the transition time, $\tau(A^r_j, A^r_{j+1}) = \tau_i$. If (A^r_j, A^r_{j+1}) is an arrow in a Type III loop, l_r, then A^r_j is a vertex in l_i and A^r_{j+1} is a vertex in l_ν for some $i \neq \nu$, $i, \nu = 1, \ldots, n$. To this arrow we assign transition time, $\tau(A^r_j, A^r_{j+1}) = \frac{1}{2}\tau_i + \frac{1}{2}\tau_\nu$.

Let $s : \mathbb{T}^2 \to \mathbb{R}^+$ be the transition time function defined as follows: If $p \in \mathbb{T}^2$ has symbol sequence, $\underline{\eta} \in \Sigma_g$, then $s(p)$ is the transition time assigned to the arrow (A_{η_0}, A_{η_1}). Since the Markov partition of g is such that $R_i \bigcap R_j \neq \emptyset$ if and only if $i = j$, and since $\mathcal{S}_g$ is finite and does not intersect any rectangle of the Markov partition, s can be chosen as smooth as desired. Let $\lambda_s : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ scaling of the vector field which generates the suspension flow of g as described in Chapter 1. This scaled vector field generates the λ_s-scaled suspension flow of g, $\varphi^t_{g,\lambda_s} : \mathbb{T}^3 \to \mathbb{T}^3$. For the rest of this chapter we will denote φ^t_{g,λ_s} by φ^t. By Proposition 4.1, the rotation set of φ^t is the convex hull of rotation vectors of the elementary loops of the graph, $\mathcal{G}_\varphi$. For any Type I loop, l_i, where $i = 1, \ldots, n$, by (4.7),

$$\rho_\varphi(l_i) \quad = \quad \frac{\displaystyle\sum_{k=0}^{N(i)-1} D_\varphi(A^i_k, A^i_{k+1})}{\displaystyle\sum_{k=0}^{N(i)-1} \tau_i}.$$

Then by (4.11),

$$\rho_\varphi(l_i) \quad = \quad \frac{\displaystyle\sum_{k=0}^{N(i)-1} (D_g(A^i_k, A^i_{k+1})\ ,\ 1)}{N(i)\,\tau_i}.$$

Since l_i is a Type I loop, and by (4.10) and (4.12),

$$\rho_\varphi(l_i) \quad = \quad \frac{N(i)(w_i, 1)}{\frac{N(i)}{z_i}}.$$

Then by (4.9),

$$\rho_\varphi(l_i) \;=\; z_i\left(\frac{x_i}{z_i},\frac{y_i}{z_i},1\right) = (x_i,y_i,z_i) = v_i\ .$$

Now we must consider the rotation vectors of Type II and Type III elementary loops. By [12], for any Type II or Type III loop, l_r, $\rho_g(l_r) \in <\rho_g(l_1),\ldots,\rho_g(l_n)>$. Transition times were assigned to arrows in a Type II or Type III loop, l_r, so that the rotation vector of that loop,

$\rho_\varphi(l_r) \in <\rho_\varphi(l_1),\ldots,\rho_\varphi(l_n)>=<v_1,\ldots v_n>= K$. So, by Proposition 4.1,

$$\rho_\varphi =<\rho_\varphi(l_1),\ldots\rho_\varphi(l_k)>=<v_1,\ldots v_n>= K\ . \qquad \square$$

We now show how Theorem 4.2 follows from Proposition 4.4 and Lemma 4.3. Let K be any given polyhedron in $\mathbb{R}^3$ with rational vertices, $(v_1,\ldots,v_n)$ and $(0,0,0)\notin K$. Then there exists $b\in\mathbb{N}$ such that $bv_i\in\mathbb{Z}^3$, for each $i=1,\ldots,n$. Let $bK=<bv_1,bv_2,\ldots,bv_n>$. Then, bK satisfies the hypothesis of Proposition 4.4 and there is a flow, $\varphi_1^t:\mathbb{T}^3\to\mathbb{T}^3$, with lift, $\widetilde{\varphi}_1^t:\mathbb{R}^3\to\mathbb{R}^3$, such that the rotation set of $\widetilde{\varphi}_1^t$, $\rho_{\varphi_1}=bK$. Now, let $\widetilde{\varphi}^t=L^{-1}\widetilde{\varphi}_1^tL$ where $L(\mathbf{x})=b\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^3$. Then by Lemma 4.3 c), $\rho_\varphi=L^{-1}(\rho_{\varphi_1})=\frac{1}{b}(bK)=K$. $\square$

Since φ^t is a diffeomorphism and since the rotation set of a flow is equal to the rotation set of its time-one map, we have the following corollary to Theorem 4.2.

COROLLARY 4.5. *Let $K\subset\mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ and $(0,0,0)\notin K$. Then there exists a C^∞ homeomorphism, $f:\mathbb{T}^3\to\mathbb{T}^3$, with lift, $\widetilde{f}:\mathbb{R}^3\to\mathbb{R}^3$ such that the rotation set $\rho_f=K$.*

CHAPTER 5

ROTATION SETS WHICH ARE COMPACT 2-MANIFOLDS AND IMAGES OF CURVES

A natural question that we address in this chapter is which sets can be rotation sets of flows. Our goal is to expand the class of sets which are known to be rotation sets of flows. For any given closed curve γ in $\mathbb{R}^n$, we construct a continuous flow on $\mathbb{T}^{n+1}$ which has Image(γ) $\times \{0\}$ as its rotation set. And for any given compact 2-manifold, M embedded in $\mathbb{R}^n$, we construct a smooth flow on $\mathbb{T}^{n+2}$ with rotation set equal to $M \times \{0\} \times \{0\}$.

THEOREM 5.1. *For any C^r curve, $\gamma : [0,1] \to \mathbb{R}^n$, there exists a C^r flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, such that the rotation set of $\widetilde{\varphi}^t$,*

$$\rho_\varphi = \text{Image}(\gamma) \times \{0\} \subset \mathbb{R}^{n+1}.$$

Proof: There exists a C^r re-parameterization of γ, $\widehat{\gamma}$, so that $\widehat{\gamma}$ is C^r-flat at 0 and 1, $\widehat{\gamma}(0) = \gamma(0) = \widehat{\gamma}(1)$, and $\widehat{\gamma}(\frac{1}{2}) = \gamma(1)$. So, in fact, Image($\gamma$) = Image($\widehat{\gamma}$) .
Let $\widehat{\gamma}(s) = (\widehat{\gamma}_1(s), \widehat{\gamma}_2(s), \ldots, \widehat{\gamma}_n(s)) \in \mathbb{R}^n$. Define a vector field $\widetilde{\mathbb{X}}$ on $\mathbb{R}^{n+1}$ by:

$$\widetilde{\mathbb{X}}(\mathbf{x}, \overline{s}) = (\widehat{\gamma}_1(\overline{s}), \widehat{\gamma}_2(\overline{s}), \ldots, \widehat{\gamma}_n(\overline{s}), 0)$$

where $\mathbf{x} \in \mathbb{R}^n$, $s \in \mathbb{R}$, and $\overline{s} = s \bmod 1$. This vector field is constant on each of the n-dimensional hyper-planes of the form $\mathbb{R}^n \times \{s\}$ contained in $\mathbb{R}^{n+1}$. See Figure 7. Since $\widetilde{\mathbb{X}}(\mathbf{x}, s) = \widetilde{\mathbb{X}}(\mathbf{x}, s+1)$ for all $s \in \mathbb{R}$ and $\widetilde{\mathbb{X}}$ is constant in $\mathbf{x}$, $\widetilde{\mathbb{X}}$ covers a C^r vector field on the $(n+1)$-torus, $\mathbb{T}^{n+1}$.

$\widetilde{\mathbb{X}}$ induces a continuous flow, $\widetilde{\varphi}^t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, of the form:

$$\widetilde{\varphi}^t(\mathbf{x}, s) = (\widehat{\gamma}_1(\overline{s}) \cdot t,\ \widehat{\gamma}_2(\overline{s}) \cdot t, \ldots, \widehat{\gamma}_n(\overline{s}) \cdot t, s)$$

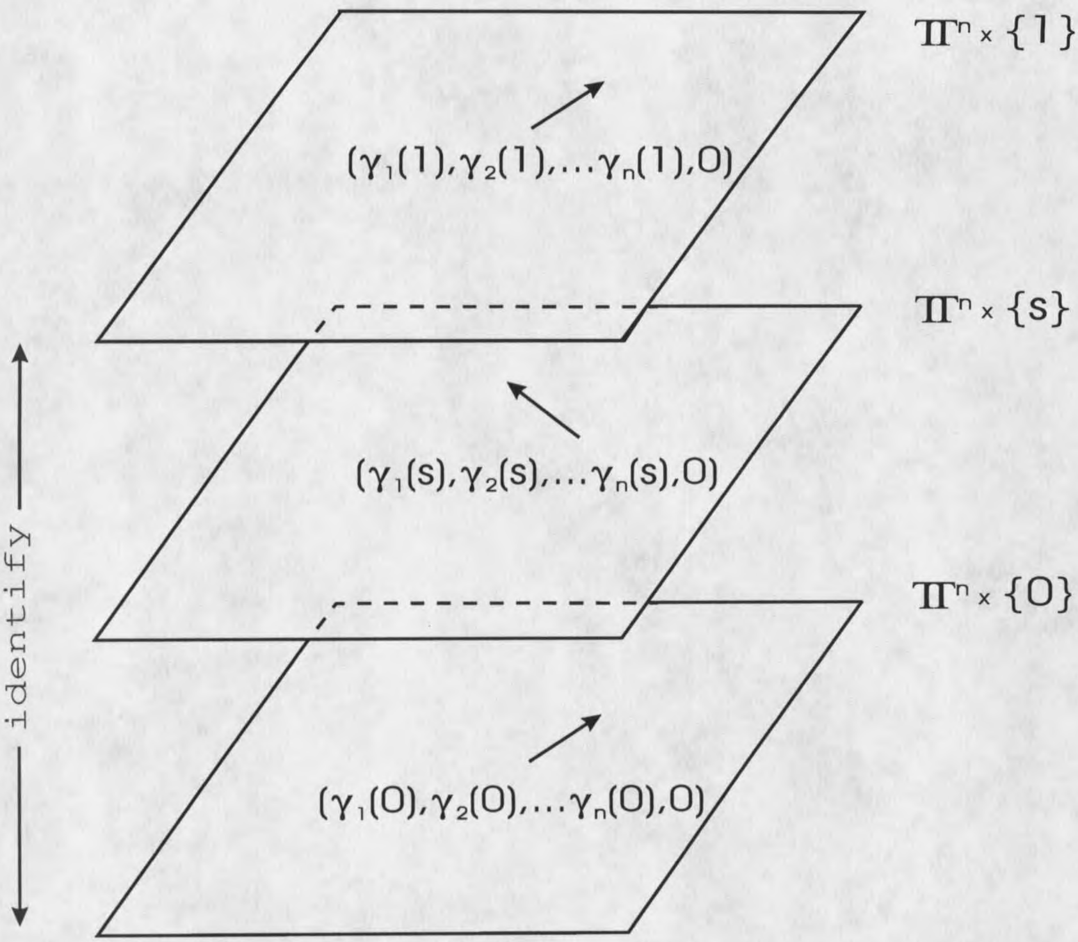


Figure 7. The vector field $\mathbb{X}$

which covers a flow, φ^t on $\mathbb{T}^{n+1}$. Since φ^t restricted to $T^n \times \{s\}$ depends linearly upon t, the rotation set for any point, $(p,s) \in \mathbb{T}^n \times \{s\}$ is the singleton, $(\widehat{\gamma}_1(s), \widehat{\gamma}_2(s), \ldots, \widehat{\gamma}_n(s), 0)$. The union of all such singletons is $\text{Image}(\gamma) \times \{0\}$. Thus,

$$\rho_\varphi = \bigcup_{(p,s)\in \mathbb{T}^{n+1}} \rho_\varphi(p,s) = \bigcup_{s\in[0,1]} (\widehat{\gamma}_1(s), \widehat{\gamma}_2(s), \ldots, \widehat{\gamma}_n(s), 0) = \text{Image}(\gamma) \times \{0\}. \quad \blacksquare$$

Corollary 5.2. *For any $K \subset \mathbb{R}^n$, that is compact, connected, and locally connected, there exists a continuous flow on $\mathbb{T}^{n+1}$ with $K \times \{0\}$ as its rotation set.*

Proof: Let K be a compact, connected, and locally connected subset of $\mathbb{R}^n$. Since $K \subset \mathbb{R}^n$, K is a metrizable Hausdorff space. By the following theorem of Hahn and Mazurkiewicz, K may be filled by a continuous curve ([9]).

Hahn-Mazurkiewicz Theorem: *A nonempty Hausdorff topological space can be completely filled by a continuous curve if and only if the space is compact, connected, locally connected and metrizable.*

Let $\gamma : [0,1] \to K$ be the curve which exists by the Hahn-Mazurkiewicz Theorem. There exists a continuous flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, with rotation set, $\rho_\varphi = K \times \{0\}$ by Theorem 5.1. □

Remark: The curves which fill these higher-dimensional spaces are not differentiable, which in turn destroys the differentiability of the flow. Therefore, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$ can be only C^0.

Theorem 5.3. *Let $H : [0,1] \times [0,1] \to \mathbb{R}^n$ be a C^∞ map such that $H(r,0) = H(r,1)$, for all $r \in [0,1]$, and $H(0,s) = H(1,s)$, for all $s \in [0,1]$. Then, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, such that $\rho_\varphi = \mathrm{Image(H)} \times \{0\} \times \{0\} \subset \mathbb{R}^{n+2}$.*

Proof: Let $H = (H_1, H_2, \ldots, H_n)$ where $H_i : [0,1] \times [0,1] \to \mathbb{R}$ for $i = 1 \ldots n$. Let $\mathbf{x} \in \mathbb{R}^n$ and define the vector field, $\widetilde{\mathbb{X}}$, on $\mathbb{R}^{n+2}$ by

$$\widetilde{\mathbb{X}}(\mathbf{x}, z, w) \quad := \quad (H_1(\overline{z}, \overline{w}), H_2(\overline{z}, \overline{w}), \ldots, H_n(\overline{z}, \overline{w}), 0, 0)$$

where $\overline{z} = z \bmod 1$ and $\overline{w} = w \bmod 1$. We denote the components of $\widetilde{\mathbb{X}}$ by $(\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{n+2})$. Since $\widetilde{\mathbb{X}}$ does not depend on $\mathbf{x}$ and since each H_i commutes with integer lattice by assumption, $\widetilde{\mathbb{X}}$ commutes with the integer lattice as well. Thus,

$\widetilde{\mathbb{X}}$ covers a vector field, $\mathbb{X}$, on $\mathbb{T}^{n+2}$. Let $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$ be the flow generated by $\mathbb{X}$. Since $\widetilde{X}_{n+1} = \widetilde{X}_{n+2} = 0$, φ^t leaves n-tori of the form $\overbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}} \times \{z_0\} \times \{w_0\}$ invariant, for fixed z_0 and w_0. Furthermore, $H_1, \ldots, H_n$ are constant on every n-torus of the same form. Thus, the rotation vector for all points $(\mathbf{x}, z_0, w_0) \in \mathbb{T}^{n+2}$,

$$\begin{aligned}
\rho_\varphi(\mathbf{x}, z_0, w_0) &= \operatorname*{LIM}_{t\to\infty} \left\{ \frac{\tilde{\varphi}^t(\mathbf{x}, z_0, w_0) - \mathbf{x}, z_0, w_0}{t} \right\} \\
&= \operatorname{LIM} \left(\frac{t(H_1(z_0, w_0), H_2(z_0, w_0), \ldots, H_n(z_0, w_0), 0, 0)}{t} \right) \\
&= \{(H_1(z_0, w_0), H_2(z_0, w_0), \ldots, H_n(z_0, w_0), 0, 0)\}
\end{aligned}$$

which is a singleton. Then it follows that the rotation set for φ^t is

$$\begin{aligned}
\rho_\varphi &= \bigcup_{\mathbf{x},z,w\in\mathbb{T}^{n+2}} \rho_\varphi(\mathbf{x}, z, w) \\
&= \bigcup_{z,w\in[0,1]} (H_1(z, w), H_2(z, w), \ldots, H_n(z, w), 0, 0) \\
&= \operatorname{Image}(H) \times \{0\} \times \{0\} \qquad \square
\end{aligned}$$

Using Theorem 5.3, we can obtain a variety of path-connected sets as the rotation sets of flows. In [13], Kwapisz constructs a 2-torus diffeomorphism with rotation set having an infinite number of vertices, but as yet there have been no constructions of 2-torus homeomorphisms with a round rotation set. By Corollary 5.2, there exists a continuous flow on the 3-torus with rotation set equal to the 2-dimensional disk, D^2. We now use this theorem to construct a *smooth* flow on the 4-torus with rotation set that is a two dimensional disk embedded into $\mathbb{R}^4$.

Corollary 5.4. *Let D^2 be the unit disk contained in $\mathbb{R}^2$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^4 \to \mathbb{T}^4$, such that $\rho_\varphi = D^2 \times \{0\} \times \{0\} \subset \mathbb{R}^4$.*

Proof: Let $H : [0,1] \times [0,1] \to R^2$ be defined by:

$$H(r, s) = \left(\frac{1}{2} + \frac{1}{2}\cos(2\pi r)\right) \left(\cos(2\pi s), \sin(2\pi s)\right).$$

H describes a C^∞ family of circles with radii ranging from 0 to 1. Notice that $H(r,0) = H(r,1)$ for all $r \in [0,1]$ and $H(0,s) = H(1,s)$ for all $s \in [0,1]$. Furthermore, the Image(H) = D^2. Thus, by Theorem 5.3 the corollary is proved. □

PROPOSITION 5.5. *Let M be any smooth compact 2-manifold. Then, there exists a C^∞ surjective map, $H : [0,1] \times [0,1] \to M$, such that*

$$H(r,0) = H(r,1) \text{ for all } r \in [0,1] \text{ and } H(0,s) = H(1,s) \text{ for all } s \in [0,1]\,. \tag{5.1}$$

Remark: In the proof of the proposition, we first show that we can find such an H for the torus, $\mathbb{T}^2$, and the projective plane, P^2. Then we show that, if there is such an H for a compact 2-manifold, M, we can find such an H for $M\# \, \mathbb{T}^2$ and $M\# \, P^2$ where # denotes the connected sum of two manifolds. A connected sum of two manifolds is the space obtained by deleting a small open disc from each of the manifolds and pasting together their boundaries. Finally we show that such an H exists for the 2-sphere, $\mathbb{S}^2$. The following well-known theorem then finishes the proof of the proposition.

THEOREM (DEHN AND HEEGARD) ([20]) *Every compact 2-manifold is homeomorphic to one of the following:*

$$S^2; \qquad X_1, X_2, \ldots; \text{ or } Y_1, Y_2 \ldots .$$

where $X_k = \overbrace{\mathbb{T}^2\# \ldots \#\mathbb{T}^2}^{k \text{ times}}$, *and* $Y_k = \overbrace{P^2\# \ldots \#P^2}^{k \text{ times}}$.

Proof: (of Proposition 5.5) Let $I^2 = [0,1] \times [0,1]$. It is well known that $\mathbb{T}^2$ can be constructed as the quotient space of I^2 with the identifications indicated in Figure 8 (a)

Let J be the quotient map from I^2 to $\mathbb{T}^2$. Then J satisfies (5.1) and can be made C^∞.

The projective plane, P^2, is defined as the space obtained by identifying each point on S^2 with its antipodal point. Equivalently, it is the quotient space obtained

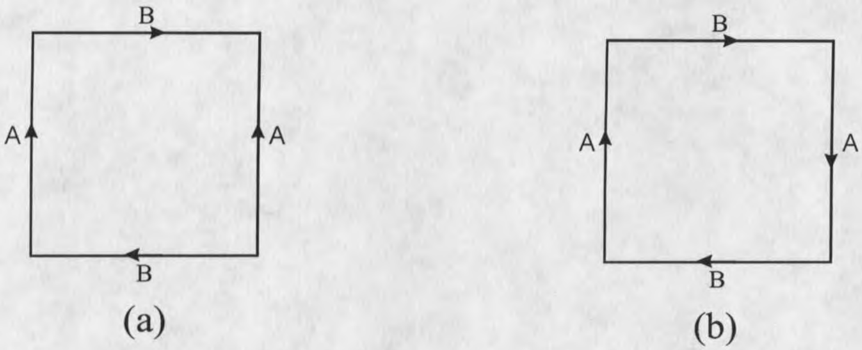


Figure 8. Standard identifications of I^2 for (a) T^2 (b) P^2

from I^2 by making the identifications pictured in Figure 8 (b). Clearly, this map can be made C^∞. Let $L : I^2 \to P^2$ be this quotient map. Observe that L does not satisfy (5.1). We want to make appropriate identifications on I^2 so that the image of the induced quotient map with these identifications is homeomorphic to P^2 and satisfies (5.1) . Define a decomposition of I^2 into the following sub-rectangles:

$$
\begin{aligned}
I_1 &= \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \qquad I_2 = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \\
I_3 &= \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \qquad I_4 = \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]
\end{aligned} \tag{5.2}
$$

Let $H : I^2 \to P^2$ be a map such that H restricted to each I_j is a scaled version of L as indicated in Figure 9.. Clearly, H is a quotient map which satisfies (5.1) and can be made as smooth as the function L.

Now let M be any compact 2-manifold. Assume there is a quotient C^∞ map $G : I^2 \to M$ satisfying (5.1). Let $M\# \, \mathbb{T}^2$ be the connected sum of M with $\mathbb{T}^2$. That is, $M\# \, \mathbb{T}^2$ is defined as the space obtained by deleting a small open disc, D from M and D' from $\mathbb{T}^2$, and pasting together their boundaries, C and C'. We denote the identified boundary by C, see Figure 10.

Let $Z : I^2 \to M\# \, \mathbb{T}^2$ be such that

$$
Z(x) = \begin{cases} G(x) & \text{for } x \in M \backslash D \\ J(x) & \text{for } x \in \mathbb{T}^2 \backslash D' \, . \end{cases}
$$

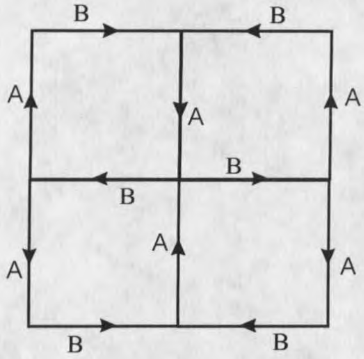


Figure 9. Identification of I^2 with underlying space, P^2

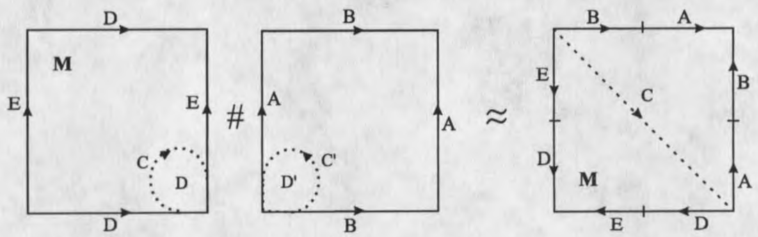


Figure 10. Connected sum of M and $\mathbb{T}^2$

Notice that maps, G and J, can be modified in a neighborhood of C in such a way that H is as smooth as G and J. Decompose I^2 as in (5.2). Let $H : I^2 \to M\#\,\mathbb{T}^2$ be a map such that H restricted to I_j is a scaled version of Z as indicated in Figure 11. Clearly, H is a quotient map which satisfies (5.1) and can be made as smooth as Z.

In a similar manner, for any compact 2-manifold, M, assume there is a map, $G : I^2 \to M$ satisfying (5.1). Let $M\#\,P^2$ be the connected sum of M with P^2, see Figure 12.

Let $Z : I^2 \to M\#\,P^2$ be such that

$$Z(x) = \begin{cases} G(x) & \text{for } x \in M \backslash D \\ L(x) & \text{for } x \in P^2 \backslash D'. \end{cases}$$

As before, decompose I^2 as in (5.2). Let $H : I^2 \to M\#\,P^2$ be a map such that

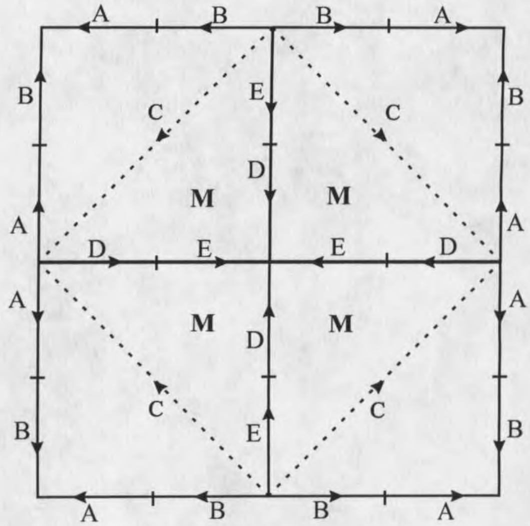


Figure 11. Identifications of I^2 under $H : I^2 \to M \# \mathbb{T}^2$

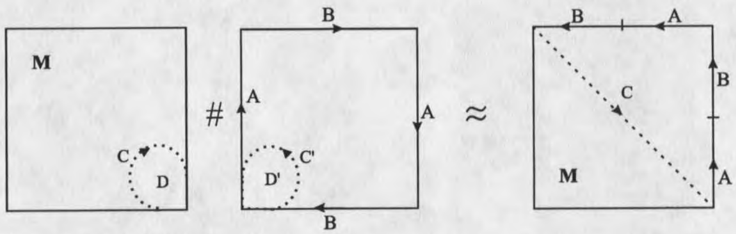


Figure 12. Connected sum of M and P^2

H restricted to I_j is a scaled version of Z as indicated in Figure 13. Clearly, H is a quotient map which satisfies (5.1) and can be made as smooth as Z.

Finally, let $\mathbb{S}^2$ be the 2-sphere contained in $\mathbb{R}^3$. Let $H : I^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ be defined by:

$$H(r, s) = \Big(\sin(2\pi r)\cos(2\pi s),\ \sin(2\pi r)\sin(2\pi s),\ \cos(2\pi r) \Big).$$

Then, H is C^∞, surjective and satisfies (5.1). ∎

Now, we have the following corollary to Theorem 5.3.

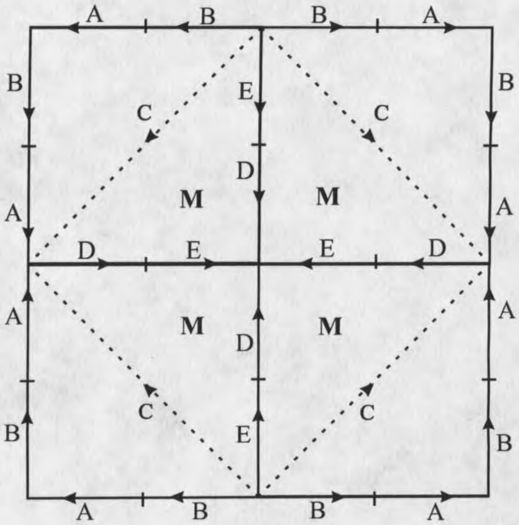


Figure 13. Identification of I^2 under $H : I^2 \to M\# P^2$

COROLLARY 5.6. *Let M be a 2-manifold imbedded in $\mathbb{R}^n$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, with lift, $\widetilde{\varphi}^t : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, such that the rotation set $\rho_\varphi = M \times \{0\} \times \{0\} \subset \mathbb{R}^{n+2}$.*

Proof: The proof follows from Proposition 5.5 and Theorem 5.3. ∎

CHAPTER 6

THE BOX DIMENSION OF ROTATION SETS FOR FLOWS ON THE N-TORUS

In this chapter we introduce the *box dimension* of a set and try to answer the question: What are the possible box dimensions of the rotation sets of toral flows? We find a relationship between the smoothness of a flow and which dimensions we can attain as the dimension of a rotation set of a flow on the torus. We begin with a brief introduction to the concept of box dimension.

The box-counting or box dimension is one of the most widely used dimensions in mathematics. The ease of calculation accounts for a large part of its popularity. It has been referred to as: entropy dimension, capacity dimension, metric dimension, logarithmic density, and information dimension ([3]). Let A be a non-empty bounded subset of $\mathbb{R}^n$. Partition $\mathbb{R}^n$ into boxes with sides of length ε in the following manner. For $\varepsilon > 0$ and $\mathbf{j} = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ let

$$R_{(j_1,j_2,\ldots,j_n)} = \{(x_1, x_2, \ldots, x_n) \mid j_i\varepsilon \leq x_i < (j_i + 1)\varepsilon \text{ for } 1 \leq i \leq n\}.$$

A box of this kind is said to be a *box from the ε-grid.* Let $N(\varepsilon, A)$ be the number of boxes, $R_{\mathbf{j}}$, among all $\mathbf{j} \in \mathbb{Z}^n$ such that $A \bigcap R_{\mathbf{j}} \neq \emptyset$.

DEFINITION 6.1. *For a compact set $A \subset \mathbb{R}^n$, the* lower *and* upper box dimension *of A respectively are defined by*

$$\begin{aligned} \underline{\dim}_{\mathrm{B}}(A) &= \liminf_{\varepsilon\to 0} \frac{\log(N(\varepsilon, A))}{\log(\varepsilon^{-1})} \\ \overline{\dim}_{\mathrm{B}}(A) &= \limsup_{\varepsilon\to 0} \frac{\log(N(\varepsilon, A))}{\log(\varepsilon^{-1})} \end{aligned}$$

If these are equal we refer to the common value as the box dimension *of A.*

$$\dim_{\mathrm{B}}(A) = \lim_{\varepsilon\to 0} \frac{\log(N(\varepsilon, A))}{\log(\varepsilon^{-1})}$$

Remark: A can also be covered, without fixing the ε-grid, by a finite set of closed cubes of length ε on a side and sides parallel to the axes, or by ε-balls. Let $N'(\varepsilon, A)$ be the minimum number of such cubes needed to cover A and let $N''(\varepsilon, A)$ be the minimum number of ε-balls needed to cover A. Then the box dimension of A can be computed using these coverings as follows (provided the limit exists).

$$\dim_{\mathrm{B}}(A) = \lim_{\varepsilon \to 0} \frac{\log(N'(\varepsilon, A))}{\log(\varepsilon^{-1})} = \lim_{\varepsilon \to 0} \frac{\log(N''(\varepsilon, A))}{\log(\varepsilon^{-1})}$$

<u>Middle-α Cantor Rotation Sets</u>

Unlike the topological dimension of a set which only takes on integer values, the box dimension of a set may not be integer-valued. The middle-α Cantor set, C, for $\alpha \in (0,1)$ is an example of a set which has fractional box dimension. The following proposition is established in ([18]).

PROPOSITION 6.2. ([18]) *For all $\alpha \in (0,1)$ there exists a Cantor set, $C \subset \mathbb{R}$, such that $\dim_{\mathrm{B}}(C) = \alpha$.*

Proof: Let C be a middle-α Cantor set in the real line. Let $\beta = \frac{1-\alpha}{2}$. The construction of the middle-α Cantor set requires that at the j^{th} step, there are 2^j intervals of length β^j which cover C. So $N'(\beta^j, C) = 2^j$ for each j. Since $\beta < 1$, $\beta^j \to 0$ as $j \to \infty$. Therefore,

$$\begin{aligned}
\dim_{\mathrm{B}}(C) &= \lim_{j \to \infty} \frac{\log(N'(\beta^j, C))}{\log((\beta^j)^{-1})} \\
&= \lim_{j \to \infty} \frac{\log(2^j)}{j\log(\beta^{-1})} \\
&= \lim_{j \to \infty} \frac{j\log(2)}{j\log(\beta^{-1})} \\
&= \frac{\log(2)}{\log(\beta^{-1})}.
\end{aligned}$$

Since $0 < \beta < \frac{1}{2}$, $0 < \dim_{\mathrm{B}}(C) < 1$. Thus the middle-$\alpha$ Cantor sets have non-integral box dimension dependent on α. By an appropriate choice of α, any number between 0 and 1 can be realized as the box dimension. □

THEOREM 6.3. *For any $0 \leq \alpha \leq 1$ there exists a C^∞ flow, φ^t, on $\mathbb{T}^2$ such that $\dim_{\mathrm{B}}(\rho_\varphi) = \alpha$.*

Proof: The cases when $\alpha = 0$ and $\alpha = 1$ are trivial. By Proposition 6.2, there exists a set, $C \subset (\frac{1}{3}, \frac{2}{3})$, such that $\dim_{\mathrm{B}}(C) = \alpha$. Let $X_1 : [0,1] \to (0,1]$ be a C^∞ map such that

$$X_1(y) = \begin{cases} y & \text{for } y \in [\frac{1}{3}, \frac{2}{3}] \\ > 0 & \text{otherwise} \end{cases}$$

and $X_1^{(k)}(0) = X_1^{(k)}(1)$ for all $0 \leq k \leq r$. Here $f^{(k)}(y)$ denotes the k-th derivative of f at y. Now let $X_2 : [0,1] \to [0,1]$ be a C^∞ map such that

$$X_2(y) = \begin{cases} 0 & \text{for } y \in C \\ > 0 & \text{for } y \in (0,1)\backslash C \end{cases}$$

and such that $X_2^{(k)}(0) = X_2^{(k)}(1)$ for all $0 \leq k \leq r$. Let $\mathbb{S}^1 = [0,1]/_\sim$ where "$\sim$" identifies 0 with 1. Since $X_1(0) = X_1(1)$ and $X_2(0) = X_2(1)$ both X_1 and X_2 project onto to C^∞ maps from $\mathbb{S}^1$ to $[0,1]$. We abuse the notation and refer to these projections as X_1 and X_2, respectively. Then $\mathbb{X}(x,y) = (X_1(y), X_2(y))$ defines a C^∞ vector field on $\mathbb{R}^2$. Since by the definition of X_1 and X_2, $X_1(0) = X_1(1)$ and $X_2(0) = X_2(1)$, $\mathbb{X}$ induces a vector field on the 2-torus which generates the C^∞ flow, $\varphi^t : \mathbb{T}^2 \to \mathbb{T}^2$. See Figure 14.

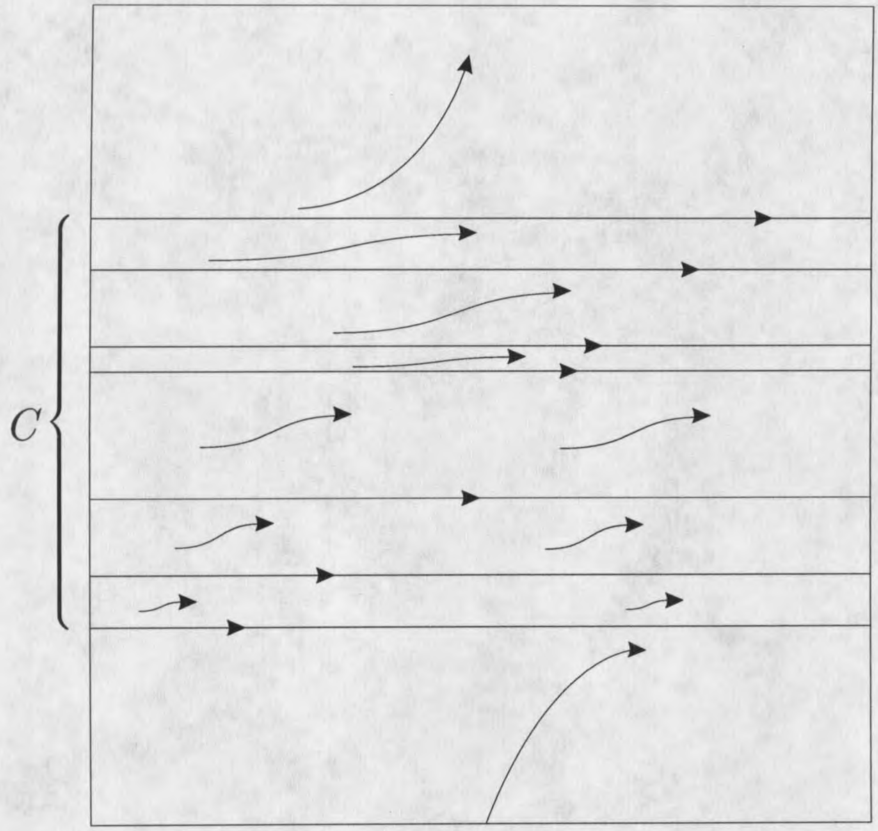


Figure 14. The flow on $\mathbb{T}^2$ generated by $\mathbb{X}$

We can now compute the rotation set of a point, $(\widetilde{x}, \widetilde{c}) \in \Pi^{-1}(x, c)$ where $x \in \mathbb{S}^1$ $c \in C$ and $\widetilde{\varphi}^t(\widetilde{x}, \widetilde{c}) = (\widetilde{x} + tc, \widetilde{c})$. So,

$$\begin{aligned} \rho_\varphi(x, c) &= \operatorname*{LIM}_{t \to \infty} \left(\frac{\widetilde{\varphi}^t(\tilde{x}, \widetilde{c}) - (\widetilde{x}, \widetilde{c})}{t} \right) \\ &= \operatorname*{LIM}_{t \to \infty} \left(\frac{(\widetilde{x} + tc, \widetilde{c}) - (\widetilde{x}, \widetilde{c})}{t} \right) \\ &= \operatorname*{LIM}_{t \to \infty} \left(\frac{(tc, 0)}{t} \right) \\ &= (c, 0). \end{aligned} \tag{6.1}$$

Let $p_0 = (x_0, y_0) \in \mathbb{T}^2$, for $y_0 \in \mathbb{S}^1 \backslash C$. Let $\widetilde{c}_{y0} = \min\left\{\widetilde{c} \in \widetilde{C} | \widetilde{c} > \widetilde{y}_0\right\}$. Here we use "$\sim$" to mean the lifted point(set) in $\mathbb{R}^2$. Since $\widetilde{X_2}(\widetilde{\varphi}^t(\widetilde{p_o})) > 0$ and $\widetilde{X_2}(\widetilde{\varphi}^t(0, \widetilde{c}_{y_0})) = 0$, there exists a sequence of times, t_k, approaching infinity such that $\lim_{k\to\infty} d(\widetilde{\varphi}^{t_k}(\widetilde{p_0}), \widetilde{q}) = 0$ for all $\widetilde{q}$ in the orbit of $(0, \widetilde{c}_{y_0})$.

We would like to show for any such p_0, the rotation set, $\rho_\varphi(p_0) \in C \times \{0\}$. We proceed with a proof by contradiction. Choose a point $r \in \rho_\varphi(p_0)$, Assume $|r - \rho_\varphi(0, c_{y_0})| = \delta > 0$. Let $(\cdot)_x$ be the x-component of a vector and $(\cdot)_y$ be the y-component of a vector. Since the ω-limit set of p_0, $\omega(\widetilde{p_0})$, is the orbit of $(0, \widetilde{c}_{y_0})$, there is a $T > 0$ such that $d(\widetilde{\varphi}^T(\widetilde{p_0}), \widetilde{q}) < \frac{\delta}{2}$ for some $\widetilde{q}$ in the orbit of $(0, \widetilde{c}_{y_0})$. By definition, $\left(\widetilde{\varphi}^t(\widetilde{p_0})\right)_y$ is strictly increasing for all $t > T$ and bounded above by $\widetilde{c}_{y_0}$. Thus, $\left(\rho_\varphi(p_0)\right)_y = \left(\rho_\varphi(0, c_{y_0})\right)_y = (r)_y = 0$. We now consider the x-component. Since X_1 is the identity on $[\frac{1}{3}, \frac{2}{3}]$ and thus strictly increasing, then for all $t > T$, $\left(\widetilde{\varphi}^t(\widetilde{p_0})\right)_x < \left(\rho_\varphi(p_0)\right)_x = (r)_x < \left(\rho_\varphi(0, \widetilde{c}_{y_0})\right)_x$. Thus, $c_{y_o} - \frac{\delta}{2} < \left(\rho_\varphi(p_0)\right)_x < c_{y_o}$. But, $\rho_\varphi(0, \widetilde{c}_{y_0}) = (c_{y_0}, 0)$. So, $\rho_\varphi(p_o) = r$ and $|\rho_\varphi(p_o) - \rho_\varphi(0, c_{y_0})| < \frac{\delta}{2}$ and we have a contradiction.

Thus, the rotation set of the flow, $\rho_\varphi = C \times \{0\}$. Since C was chosen such that $\dim_B(C) = \alpha$, $\dim_B(\rho_\varphi) = \dim_B(C \times \{0\}) = \alpha$ □

Rotation Sets with Box Dimension between 1 and 2

It is possible for a continuous function to be irregular enough that the box dimension of its graph is strictly greater than 1. For example, the graph of any space filling curve has box dimension equal to the dimension of the space it fills. The well-known Weierstrass function, is another example of a function whose graph has fractional box dimension. $W(t) : \mathbb{R} \to \mathbb{R}$ is the defined by

$$W(t) = \sum_{k=1}^{\infty} \beta^{(s-2)k} \sin(\beta^k t) \tag{6.2}$$

where $1 < s < 2$ and $\beta > 1$. Then $W(t)$ has the properties that it is a nowhere differentiable continuous function and provided β is chosen large enough, has box dimension equal to s ([3]). Figure 15 and Figure 16 show graphs of the Weierstrass function when $\beta = 1.5$ and $s = 1.3$ and 1.7 respectively.

COROLLARY 6.4. *For any $1 < \alpha < 2$, there exists a continuous flow, φ^t, on $\mathbb{T}^3$ such that* $\dim_B(\rho_\varphi) = \alpha$.

Proof: Let $W(t)$ be the Weierstrass function as defined in (6.2). Let $s = \alpha$ and $\beta > 1$. Define $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(s) : (s, W(s))$. By Theorem 5.1, there exists a flow, φ^t on $\mathbb{T}^3$ such that the rotation set, $\rho_\varphi = \text{Image}(\gamma) \times \{0\}$. Then, by definition of W,

$$\dim_B(\text{Image}(\gamma) \times \{0\}) = \alpha \,. \qquad \square$$

We summarize the results concerning the box dimension of rotation sets of flows in the following two theorems.

THEOREM 6.5. *If $\alpha \in [0,2] \cup \{3\}$, there exists a continuous flow, φ^t, on $\mathbb{T}^3$ such that* $\dim_B(\rho_\varphi) = \alpha$.

Proof: For $\alpha \in [0,1]$, apply Theorem 6.3.

For $\alpha \in (1,2)$, apply Corollary 6.4.

For $\alpha = 2$, let K be the unit disk. Since K is a compact, connected, and locally connected set by Corollary 5.2, K is the rotation set of a flow on the 3-torus.

For $\alpha = 3$, follows directly from Theorem 2.1. $\square$

THEOREM 6.6. *If $\alpha \in [0,1] \cup \{2\} \cup \{3\}$, there exists a C^∞ flow, φ^t, on $\mathbb{T}^3$ such that* $\dim_B(\rho_\varphi) = \alpha$.

Proof: For $\alpha \in [0,1]$, apply Theorem 6.3.

For $\alpha = 2$, let $\varphi_f^t : \mathbb{T}^3 \to \mathbb{T}^3$ be a suspension of a diffeomorphism, $f : \mathbb{T}^2 \to \mathbb{T}^2$, for which $\dim_{\mathrm{B}}(\rho_f) = 2$. Then by Proposition 1.8, $\rho_{\varphi_f} = \rho_f \times \{1\}$. Thus, $\dim_{\mathrm{B}}(\rho_{\varphi_f}) = 2$.

For $\alpha = 3$, follows directly from Theorem 2.1. □

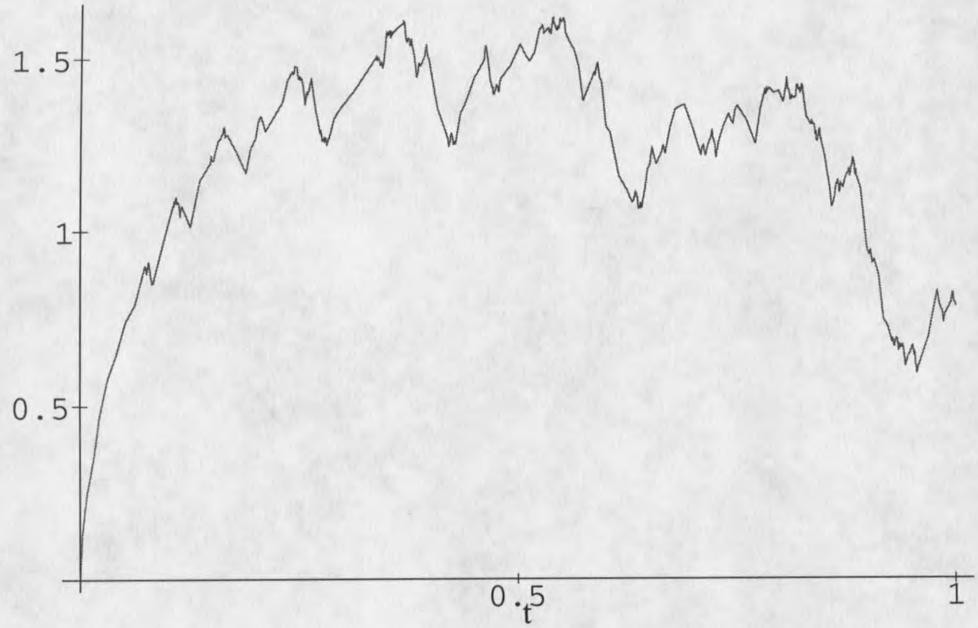


Figure 15. The Weierstrass function with $\beta = 1.5$ and $s = 1.3$

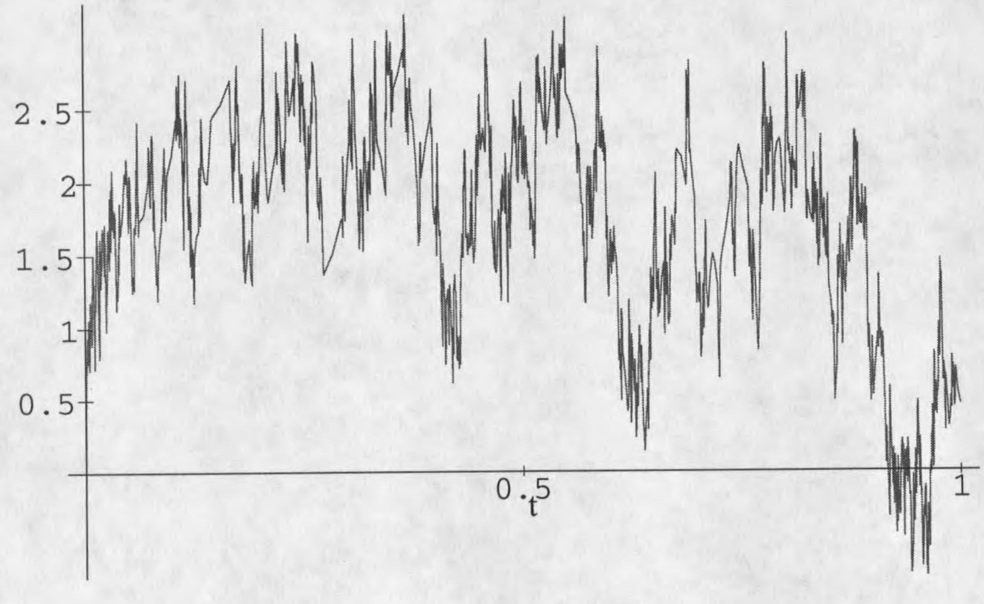


Figure 16. The Weierstrass function with $\beta = 1.5$ and $s = 1.7$

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