



Conjugacy and entropy of piecewise mobius contact deformations  
by Scott Calvin Lewis

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of  
Philosophy in Mathematical Sciences  
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Abstract:

Random matrix products of  $2 \times 2$  matrices may be thought of in a dynamical system setting as iterations functions of mobius maps, with matrix multiplication replacing composition of functions. Sometimes branches of mobius maps may be restricted to form classical dynamical systems. One such family is the tent family about which much is known. Similar properties are investigated for a two parameter family of symmetric piecewise mobius maps (which include the tent family). Using kneading theory, symbolic dynamics, and other techniques, A parameter space is found which foliates into curves of constant kneading sequence on which maps may be pairwise conjugate depending on if the maps restricted to a (forward invariant) core interval are transitive. Investigations into iterated function systems given by the inverse branches of the symmetric family are made by defining a shift on two symbols (depending upon some interval of definition) that models the iterated function system. Continuous deformations of the interval are made and properties of entropy are found. In some cases entropy of the shift space is continuous as the interval is deformed, while in other cases there are discontinuities.

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A dissertation submitted in partial fulfillment  
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APPROVAL

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Scott Calvin Lewis

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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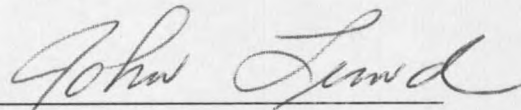


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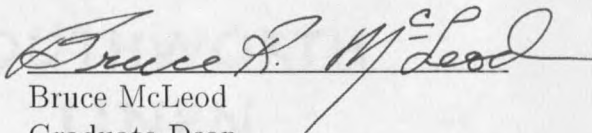


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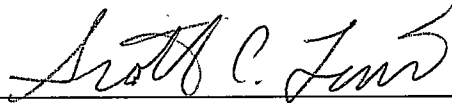
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## TABLE OF CONTENTS

	Page
<b>LIST OF TABLES</b> . . . . .	v
<b>LIST OF FIGURES</b> . . . . .	vi
<b>1. Introduction</b> . . . . .	1
The Setup . . . . .	2
The Results . . . . .	4
<b>2. Classical Theory</b> . . . . .	6
Transitive Maps . . . . .	9
Foliation from Conjugacy . . . . .	23
Finding and Approximating Entropy . . . . .	39
Measure Theory . . . . .	41
<b>3. Nonclassical Systems</b> . . . . .	45
The Contact Shift on Classical Systems . . . . .	47
Contact Shifts of Iterated Function Systems . . . . .	59
Extending the Contact Interval . . . . .	61
<b>REFERENCES CITED</b> . . . . .	77
<b>Appendix A: Numerical data for Parameters <math>\lambda</math> and <math>r</math></b> . . . . .	80

## LIST OF TABLES

Table		Page
1	Shift maximal kneading sequences and associated polynomials . . . . .	27
2	Contact States for the Period 4 Example . . . . .	48
3	Some Kneading Sequences of Periodic Parameters and Associated Inadmissible Blocks for Contact Shift $X_I$ . . . . .	55
4	Itinerary codings for $(RLLRC)^\infty$ -(S means R, L, or C is possible) . . . . .	58
5	Itinerary codings for $RLR^\infty$ -(S means R, L, or C is possible) . . . . .	58
6	Summary of Information for Entropy of Contact Shifts of Extensions of $I=[0,1]$ of the Relation $R$ . . . . .	62
7	Numerical Data for Period 5 - $RLLLC$ . . . . .	80
8	Numerical Data for Period 5 - $RLLRC$ . . . . .	80
9	Numerical Data for Period 5 - $RLRRC$ . . . . .	80
10	Numerical Data for Period 6 - $RLLLLC$ . . . . .	81
11	Numerical Data for Period 6 - $RLLLRC$ . . . . .	81
12	Numerical Data for Period 6 - $RLLRRC$ . . . . .	81

## LIST OF FIGURES

Figure		Page
1	Changes in the Family $f_{\lambda,r}$ as $r$ Varies on the Unit Interval . . . . .	7
2	Parameter Region $Q$ of Core Intervals . . . . .	9
3	Farey Map and Core Interval . . . . .	10
4	Rescaled Core Maps with Fixed $r$ and Different $\lambda$ . . . . .	11
5	Transitive Boundary Curve in Parameter Space . . . . .	12
6	Region $G$ . . . . .	14
7	Region $D$ . . . . .	19
8	Extensions of Transitivity . . . . .	20
9	Constant Kneading Curves of Low Periods . . . . .	28
10	The Possible Graphs of $f_{\lambda,r}^n(x)$ about $x = \frac{1}{2}$ . . . . .	36
11	Period 4 Tent Map and Transition Graph . . . . .	41
12	Period 4 Contact Shift Transition Graph . . . . .	49
13	Graph Splitting . . . . .	54
14	The Relation $\mathcal{R}$ . . . . .	60
15	Transition Graph for the Relation $\mathcal{R}$ . . . . .	61
16	Entropy vs. $\epsilon$ for Extensions of the Contact Shift on the Relation $\mathcal{R}$ . . . . .	63
17	Restricted Tent Map . . . . .	65
18	Extended Tent Map I.F.S. . . . .	67

**ABSTRACT**

Random matrix products of  $2 \times 2$  matrices may be thought of in a dynamical system setting as iterations functions of möbius maps, with matrix multiplication replacing composition of functions. Sometimes branches of möbius maps may be restricted to form classical dynamical systems. One such family is the tent family about which much is known. Similar properties are investigated for a two parameter family of symmetric piecewise möbius maps (which include the tent family). Using kneading theory, symbolic dynamics, and other techniques, A parameter space is found which foliates into curves of constant kneading sequence on which maps may be pairwise conjugate depending on if the maps restricted to a (forward invariant) core interval are transitive. Investigations into iterated function systems given by the inverse branches of the symmetric family are made by defining a shift on two symbols (depending upon some interval of definition) that models the iterated function system. Continuous deformations of the interval are made and properties of entropy are found. In some cases entropy of the shift space is continuous as the interval is deformed, while in other cases there are discontinuities.



## CHAPTER 1

## Introduction

Random matrix products arise in many areas of mathematics. Some of these include harmonic analysis, random walks, quantum mechanics, skew product flows, Conley index theory, and ergodic theory. Mathematicians ranging from Gauss to Erdős have worked in this area. It is a rich field having connections to many other areas of mathematics.

In the early 1940's, Bellman [3] studied random products in a statistical setting. He found expected values for the elements of products of two  $2 \times 2$  positive matrices and made conjectures as to the limiting behavior of the elements. In 1960, Furstenberg and Kesten [14] extended Bellman's work by solidifying results of the limiting process for most random products of positive matrices. A few years later Furstenberg [13] generalized the previous results for random products of matrices with elements in non-compact semi-simple Lie groups.

Later, in the early 1980's, Pelikan [25] studied random maps of  $[0,1]$  which represent dynamical systems on the square  $[0,1] \times [0,1]$ . He found sufficient conditions for a random map to have an absolutely continuous invariant measure. He also discussed the number of ergodic components of a random map. Recently there has been more work done in the area of iterated function systems (see for instance some of the latest work of [28]) .

In the late 1970's people began to study random products in a dynamical systems setting. Bowen and Series [4] found that for any finitely generated discrete subgroup  $\Gamma$  of  $SL(2, \mathbf{Z})$  which acts on  $\mathbf{R}$  with dense orbits, one can associate to  $\Gamma$  a

map  $f_T: \mathbf{R} \cup \{\infty\} \rightarrow \mathbf{R} \cup \{\infty\}$  that is orbit equivalent on  $\mathbf{S}^1$  and preserves a measure equivalent to Lebesgue.

More recently Fried [12] gave a coding of the geodesic flow on a surface defined through a finite index subgroup of a nonuniform hyperbolic triangle group in the same spirit as the coding of the group  $PSL(2, \mathbf{Z})$  by the continued fraction expansion. The invariant measures for the interval maps of the triangle groups were determined, generalizing the Gauss Measure for the continued fraction map.

Very recently Kwapisz [19], motivated by the theory of the cohomological Conley index, has done work on noninvertible random products (cocyclic subshifts). These properly generalize sofic systems and topological Markov chains. They admit a structure theory with a spectral decomposition into mixing components. Also, a zeta-like generating function for cocyclic subshifts gives practical tools for detecting chaos in general discrete dynamical systems.

## The Setup

Random products of matrices are formed from an alphabet by taking a collection  $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$  of  $n \times n$  matrices. Under matrix multiplication, one constructs words from the alphabet and investigates such products. Such investigations may be tailored to specific kinds of problems by restricting the collection  $\mathcal{A}$ , or more generally, the pool of admissible words.

In this treatment, random products will be used in a dynamical system setting first and then later in an iterated function system (or non-classical) setting. While both may be considered together, there are certain tools that the classical setting will allow that may not always be used in an iterated function setting. Our scope will be somewhat limited due to the complexity and variety of iterated systems. The main

focus of the paper is entropy of random products, but other related ideas are also explored.

For simplicity, this discussion will be limited to an invertible pair of  $2 \times 2$  matrices  $A$  and  $B$ . With these limitations, the matrices and associated random products may be taken . These may be considered to act on  $\mathbf{R}$ , since matrix multiplication may be replaced by composition of linear fractional maps.

**Definition 1.1** The pair  $\{A, B\}$  determines a *classical dynamical system* or *c.d.s.* if the linear fractional maps associated with  $A, B$  are inverse branches of an interval map  $f: I \rightarrow I$  for some closed invariant interval  $I \in \mathbf{R}$ .

**Definition 1.2** If the linear fractional forms of  $A$  and  $B$  are individually inverses of interval maps the pair  $\{A, B\}$ , each defined on a subinterval of some interval  $I$ , is said to determine an *iterated function system* or *i.f.s.*

Notice that under the loose definition of iterated function systems that a c.d.s. is an i.f.s.

Consider a two parameter pair of matrices  $\mathcal{A} = \{A, B\}$  with spectra  $\sigma(A) = \{r, \lambda\}$  and  $\sigma(B) = \{r, -\lambda\}$ . This pair indicates an extensive range of possibilities as shall be shown below. Since maps can be affinely rescaled, it is natural to fix an interval  $I$  (for instance  $I=[0,1]$ ) and select the pair  $\{A, B\}$  accordingly. We will primarily be interested in the pair

$$A_{\lambda,r} = \begin{pmatrix} r & 0 \\ 2r-1 & \lambda \end{pmatrix}, B_{\lambda,r} = \begin{pmatrix} r-1 & \lambda \\ 2r-1 & \lambda \end{pmatrix}. \quad (1.1)$$

In Chapter 2 classical dynamical systems will be formed from the linear fractional maps given from the inverses of the matrices 1.1. By careful restrictions of the maps a two parameter symmetric unimodal family  $f_{\lambda,r} : [0, 1] \rightarrow [0, 1]$  is created. Sufficient conditions on the parameters  $\lambda$  and  $r$  are found so that  $f_{\lambda,r}$  has a forward

invariant interval  $\mathcal{I}$  containing the critical point in its interior, and  $f_{\lambda,r}$  restricted to  $\mathcal{I}$  is topologically transitive. A region  $G$  in the parameter space is found that fits these conditions.

## The Results

The main results of Chapter 2 include showing that  $G$  foliates into arcs of constant kneading sequence in the parameter space, for which all maps  $f_{\lambda,r}$ , given by parameters on the arc, are pairwise conjugate when restricted to the forward invariant set  $\mathcal{I}$ . This allows the calculation of topological entropy by pushing down along the curves to the tent map family (for which entropy is well known).

Along the way, many results which are known for the tent family are extended to  $f_{\lambda,r}$ , for fixed  $r$ . Some of these properties include monotonicity of kneading sequences, and denseness of periodic parameters. The monotonicity proof is different than the known proof of the tent family (see [5]).

In the last subsection of Chapter 2 the measures of  $f_{\lambda,r}$ , with parameters in  $G$ , are found to have absolutely continuous invariant measures. The one exception to this is  $f_{1,1}$ , the full Farey map. It does not have an absolutely continuous invariant measure.

In Chapter 3 the ideas of symbolic dynamics in relationship to entropy is explored for i.f.s. A shift space is defined which will be called the contact shift. This shift may be applied to all i.f.s. The contact shift depends upon the square  $J \times J$ . The interval  $J$  is called the contact interval and limits the possible random compositions, which gives a natural process of defining admissibility in the contact shift. The contact shift is shown to be conjugate to the Markov shift of finite type when  $f_{\lambda,r}$  has a periodic parameter.

Also in Chapter 3, investigations into the behavior of the entropy of the contact shift as the contact interval is deformed continuously are made. An example is given of i.f.s. where entropy changes continuously on some intervals, but are discontinuous at certain intervals. One result is that for all i.f.s., found by extending the branches of the tent map,  $T_\lambda$ , with transcendental parameters  $\lambda \in [\frac{\sqrt{2}}{2}, 1]$ , the contact entropy continuously increases as the contact interval  $J$  is continuously deformed. It is conjectured that this behavior occurs for all  $\lambda \in [\frac{\sqrt{2}}{2}, 1]$ .

## CHAPTER 2

## Classical Theory

To begin, consider a two parameter pair of matrices given in equation 1.1. If the matrices are replaced by their linear fractional form then, under certain restrictions, they are inverse branches of a single map acting on  $\mathbf{R}$ . As  $\lambda$  and  $r$  vary, a symmetric family of continuous piecewise hyperbolic unimodal maps  $f_{\lambda,r}:\mathbf{R} \rightarrow \mathbf{R}$  is obtained. To insure that the family forms a continuous interval map, both  $\lambda$  and  $r$  must be positive and the domain must be divided so that there is no overlap of branches. This family may be described in the following manner:

$$f_{\lambda,r} = \begin{cases} \frac{\lambda x}{(1-2r)x+r} & \text{for } -\infty < x \leq \frac{1}{2} \\ \frac{\lambda(1-x)}{(2r-1)x+1-r} & \text{for } \frac{1}{2} \leq x < \infty \end{cases} \quad (2.1)$$

Here  $\lambda$  is the maximum value of the map and  $\frac{\lambda}{r}$  is the derivative at  $x = 0$ . For fixed  $\lambda$  the parameter  $r$  determines the convexity of the map (see Figure 1). Also  $x = \frac{1}{2}$  is the critical point and the axis of symmetry. Notice that if the domain is restricted to the unit interval  $I=[0,1]$  and  $r = \frac{1}{2}$  then  $f_{\lambda,r}$  is the well known tent family. When  $r = 1$ ,  $f_{\lambda,r}$  is a special family called the Farey family (This family will be discussed in more detail later). To form classical dynamical systems out of the family 2.1 the domain must be restricted to a closed invariant interval. It should be noted that a given interval  $I$  is invariant only for certain values of the parameters  $\lambda$ , and  $r$ . For instance, if  $\lambda$  is too large then  $I$  is not invariant under  $f_{\lambda,r}$ . In this discussion the interval  $I$  will be the unit interval. Thus the family 2.1 restricted to  $I$  becomes:

$$f_{\lambda,r} = \begin{cases} \frac{\lambda x}{(1-2r)x+r} & \text{for } 0 \leq x \leq \frac{1}{2} \\ \frac{\lambda(1-x)}{(2r-1)x+1-r} & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases} \quad (2.2)$$

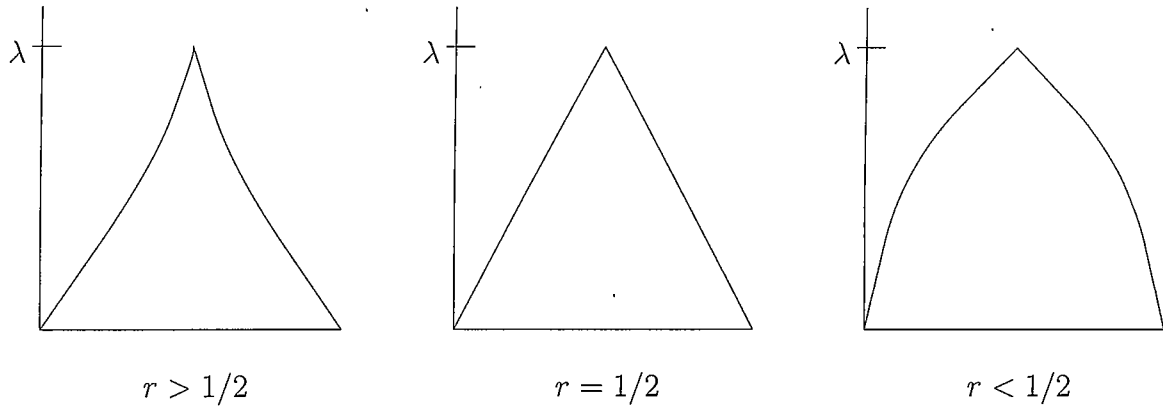


Figure 1: Changes in the Family  $f_{\lambda r}$  as  $r$  Varies on the Unit Interval

As mentioned before, not all the family 2.2 leave  $I$  invariant. Many of those that do, have subintervals contained in the unit interval where most of the interesting dynamics occur. For instance, the tent family ( $r = \frac{1}{2}$ ) with  $\frac{1}{2} < \lambda \leq 1$  has what is termed the *core* in [2]. Using the same terminology, we will define a similar interval.

**Definition 2.1** The *core interval*, denoted  $\mathcal{I}$ , for the map  $f_{\lambda, r}$  is  $[f_{\lambda, r}(\lambda), \lambda]$  if this interval is forward invariant and contains the critical point  $\frac{1}{2}$  in its interior. Otherwise, we will say no core interval exists.

The following theorem gives necessary and sufficient conditions on  $\lambda$  for the core interval to exist.

**Proposition 2.2** A core interval exists for  $f_{\lambda, r}$  in the family 2.2 if and only if the following conditions hold:

$$\frac{1}{2} < \lambda \leq 1 \tag{2.3}$$

$$f_{\lambda, r}(\lambda) < \frac{1}{2} \tag{2.4}$$

$$f_{\lambda, r}^2(\lambda) \geq f_{\lambda, r}(\lambda) \tag{2.5}$$

Proof:

Suppose the three conditions are satisfied. Equations 2.3 and 2.4 give  $f_{\lambda,r}(\lambda) < \frac{1}{2} < \lambda$ . To show that  $\mathcal{I}$  is invariant notice that the  $f_{\lambda,r}([\frac{1}{2}, \lambda]) = \mathcal{I}$ . Thus one need only show that  $f_{\lambda,r}([f_{\lambda,r}(\lambda), \frac{1}{2}]) \subseteq \mathcal{I}$ . Since  $f_{\lambda,r}$  restricted to  $[f_{\lambda,r}(\lambda), \frac{1}{2}]$  is orientation preserving, and by equation 2.5,

$$f_{\lambda,r}(\mathcal{I}) = [f_{\lambda,r}^2(\lambda), f_{\lambda,r}(\frac{1}{2})] = [f_{\lambda,r}^2(\lambda), \lambda] \subseteq [f_{\lambda,r}(\lambda), \lambda] = \mathcal{I}. \quad (2.6)$$

On the other hand, if one of the conditions is not satisfied then there is no core interval since  $[f_{\lambda,r}(\lambda), \lambda]$  is not invariant. To see this we need only relax each restriction one at a time and show that problems occur.

If  $\lambda > 1$ , then  $\mathcal{I} \not\subseteq [0,1]$ . However, while no core interval exists in  $I$  there is a forward invariant cantor set contained in  $[0,1]$ . For more details on such cases see [27], pp.26-36. Also, if  $\lambda \leq \frac{1}{2}$ , then  $f_{\lambda,r}(\mathcal{I}) \subseteq [0,1/2]$ . Thus  $\frac{1}{2}$  is not in the interior of  $\mathcal{I}$ . Hence there is no core interval. This takes care of the first condition. In the same manner if  $f_{\lambda,r}(\lambda) \geq \frac{1}{2}$  then the interval  $[f_{\lambda,r}(\lambda), \lambda] \subset [1/2,1]$ . The critical point  $c = \frac{1}{2}$  is not in the interior of  $[f_{\lambda,r}(\lambda), \lambda]$ . Thus no core interval exists. Therefore the second condition is necessary.

Finally, suppose  $f_{\lambda,r}^2(\lambda) < f_{\lambda,r}(\lambda)$ . This would imply  $[f_{\lambda,r}^2(\lambda), \lambda] \supset [f_{\lambda,r}(\lambda), \lambda]$ . But by equation 2.6,  $[f_{\lambda,r}^2(\lambda), \lambda] \subseteq [f_{\lambda,r}(\lambda), \lambda]$ . This leads to a contradiction. Thus the third condition is necessary.

□

The conditions on  $\lambda$  in Proposition 2.2 also place some constraints on what the parameter  $r$  may be. The first and third restrictions imply  $0 < r \leq 1$ . The first and second restrictions imply  $r > 1 - \lambda$ . Putting all conditions on  $\lambda$  and  $r$  gives the bounded region  $Q$  shown in Figure 2. For values of  $r$  and  $\lambda$  in the region  $Q$ , the



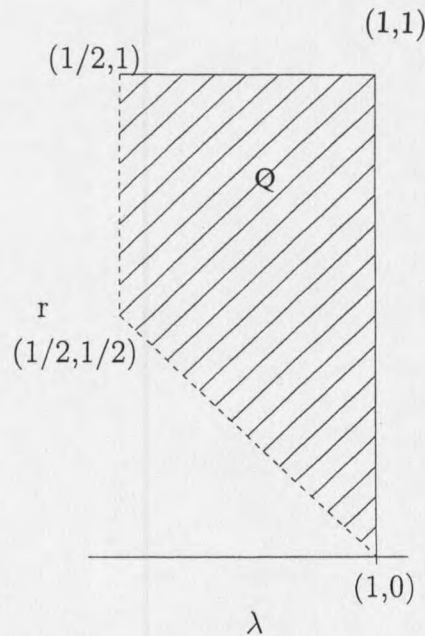


Figure 2: Parameter Region Q of Core Intervals

family 2.2 restricted to the core interval becomes:

$$f_{\lambda,r} = \begin{cases} \frac{\lambda x}{(1-2r)x+r} & \text{for } f_{\lambda,r}(\lambda) \leq x \leq \frac{1}{2} \\ \frac{\lambda(1-x)}{(2r-1)x+1-r} & \text{for } \frac{1}{2} \leq x \leq \lambda \end{cases} \quad (2.7)$$

The Farey family has a unique and interesting property with regard to the core interval. Each map  $f_{\lambda,1}$  has a symmetric core interval; that is,  $f_{\lambda,1}(\lambda) = 1 - \lambda$ . In fact, the point  $x = 1 - \lambda$  is a fixed point of  $f_{\lambda,1}$  for each  $\lambda \in [\frac{1}{2}, 1]$ . This means that each map in this family has a core interval on which it is affinely conjugate (shown in the section Foliation from Conjugacy) to one of the full ( $\lambda = 1$ ) maps in family 2.7. See Figure 3 for a graphical representation of one of these. No other such family in 2.7 with this property exists.

### Transitive Maps

An important property to consider in the family given in equation 2.7 is that of topological transitivity.

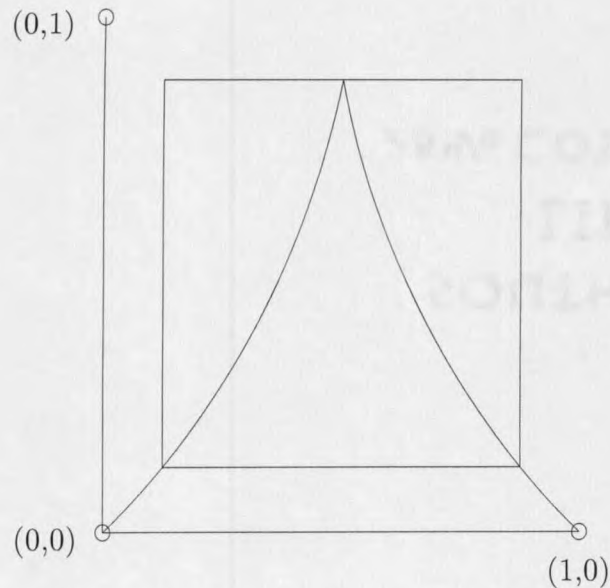


Figure 3: Farey Map and Core Interval

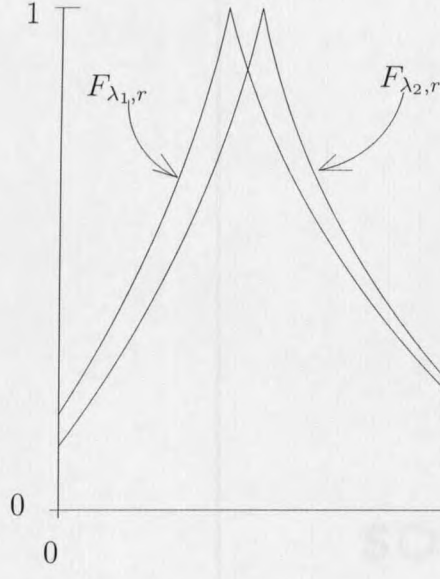
**Definition 2.3** A c.d.s.  $(I, F)$  is *topologically transitive* if there is a point  $y \in I$  such that its forward orbit,  $\mathcal{O}^+(y)$ , is dense in  $I$ .

An equivalent characterization of transitivity may be found in the following well known lemma.

**Lemma 2.4** *If  $F: I \rightarrow I$  is a continuous map of a locally compact separable metric space  $I$  into itself, then  $F$  is topologically transitive if and only if for every two nonempty open sets  $U, V \subset I$  there exists some integer  $N = N(U, V)$  such that  $F^N(U) \cap V \neq \emptyset$ .*

See [18] for a proof of Lemma 2.4.

There are several significant geometrical correspondences in terms of transitivity that can be made for the family of 2.7. The best way to see how the geometry changes as  $\lambda$  changes for a fixed  $r$  is to consider  $F_{\lambda, r}$ , the *rescaled core map* of  $f_{\lambda, r}$ . These rescaled core maps are found by rescaling the domain  $\mathcal{I}$  to  $[0, 1]$ . The rescaled

Figure 4: Rescaled Core Maps with Fixed  $r$  and Different  $\lambda$ 

core maps are given by

$$F_{\lambda, r} = \begin{cases} \frac{-r\lambda x + r + \lambda - r\lambda - 1}{(r\lambda - 2r\lambda^2 - 2r^2\lambda + 4r^2\lambda^2)x + \lambda^2 - \lambda + 3r\lambda - 2r\lambda^2 - r + r^2 - 2r^2\lambda} & \text{for } 0 \leq x \leq \frac{\lambda - 1 + r}{2r\lambda} \\ \frac{-r\lambda x + r\lambda}{(r\lambda - 2r\lambda^2 - 2r^2\lambda + 4r^2\lambda^2)x + 1 - 2\lambda - 2r + 5r\lambda + \lambda^2 + r^2 - 2r^2\lambda - 2r\lambda^2} & \text{for } \frac{\lambda - 1 + r}{2r\lambda} \leq x \leq 1 \end{cases} \quad (2.8)$$

Notice that the rescaled maps are also symmetric in  $\lambda$  and  $r$ . The importance of this symmetry will be addressed later in this discussion.

As  $\lambda$  increases for a fixed  $0 < r < 1$ ,  $F_{\lambda, r}(0)$  decreases continuously with  $F_{\lambda, r}(0) = 0$  when  $\lambda = 1$  (see Figure 4). There are different ways to show this. One can either show that  $\frac{\partial}{\partial \lambda} F_{\lambda, r}(0) < 0$  or show that the numerator of  $F_{\lambda, r}(0)$  is strictly decreasing while the denominator is nondecreasing in magnitude as  $\lambda$  increases from  $\frac{1}{2}$  to 1. We shall demonstrate the second method for  $\frac{1}{2} < r < 1$ . Notice that

$$\begin{aligned} F_{\lambda, r}(0) &= \frac{r + \lambda - r\lambda - 1}{\lambda^2 - \lambda + 3r\lambda - 2r\lambda^2 - r + r^2 - 2r^2\lambda} \\ &= \frac{(1 - \lambda)(r - 1)}{(1 - 2r)\lambda^2 + (3r - 2r^2 - 1)\lambda - r + r^2}. \end{aligned}$$

The numerator of  $F_{\lambda, r}(0)$  is negative and increasing monotonically to 0 as  $\lambda$  increases

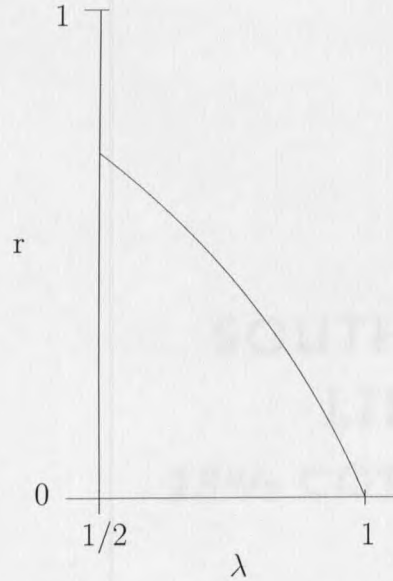


Figure 5: Transitive Boundary Curve in Parameter Space

to 1. The denominator is a quadratic in  $\lambda$  with roots

$$\lambda = \frac{r-1}{2} \pm \sqrt{\left(\frac{r-1}{2}\right)^2 - \frac{r(r-1)}{1-2r}}.$$

Both roots are negative since

$$\frac{r(r-1)}{1-2r} > 0 \text{ and } \frac{r-1}{2} < 0$$

while

$$\left(\frac{r-1}{2}\right)^2 > \frac{r(r-1)}{1-2r}$$

for  $\frac{1}{2} < r < 1$ . Since the leading coefficient,  $1-2r$ , is negative and both roots are negative for  $\frac{1}{2} < r < 1$ , the denominator is negative and decreasing monotonically as  $\lambda$  increases. Thus  $F_{\lambda,r}(0) > 0$  and is strictly decreasing.

In the same manner the critical point continuously moves across the interval too (see Figure 4). It approaches  $\frac{1}{2}$  from the left as  $\lambda$  approaches 1.

The geometry just discussed corresponds to transitivity in the following way: Transitivity changes occur as  $F_{\lambda,r}(0)$  moves across the fixed point  $p$  in the interval

$[\frac{\lambda-1+r}{2r\lambda}, 1]$  (or as  $f^2_{\lambda,r}(\lambda)$  moves across the fixed point  $p$  in the interval  $[\frac{1}{2}, 1]$ ). A curve in the parameter space (we will call it the transitive boundary since for any fixed  $r$ , maps given by parameters  $\lambda$  to the left of the curve are not transitive) can be found by setting  $F_{\lambda,r}(0)$  (or  $f^2_{\lambda,r}(\lambda)$ ) equal to the fixed point of the right branch. This gives the symmetric level curve

$$(2\lambda - 1)r^3 + (3\lambda^2 - 5\lambda + 2)r^2 + (2\lambda^3 - 5\lambda^2 + 4\lambda - 1)r - \lambda + 2\lambda^2 - \lambda^3 = 0. \quad (2.9)$$

Solving for  $r$  in terms of  $\lambda$  gives a symmetric curve in the parameter space (see Figure 5). We first show that for parameters such that  $F_{\lambda,r}(0) > p$  then  $F_{\lambda,r}$  is not transitive on  $[0,1]$ .

**Theorem 2.5** *Each map  $f_{\lambda,r}$ , given by parameters in  $Q$ , with  $f^2_{\lambda,r}(\lambda) > p$ , where  $p$  is the fixed point to the right of the critical point, is not transitive on the core interval  $[f_{\lambda,r}(\lambda), \lambda]$ .*

Proof:

Suppose  $f^2_{\lambda,r}(\lambda) > p$ . Consider the three intervals:  $\mathcal{I}_1 = (f_{\lambda,r}(\lambda), f^3_{\lambda,r}(\lambda))$ ,  $\mathcal{I}_2 = (f^3_{\lambda,r}(\lambda), f^2_{\lambda,r}(\lambda))$ ,  $\mathcal{I}_3 = (f^2_{\lambda,r}(\lambda), \lambda)$ . Here since  $f^2_{\lambda,r}(\lambda) > p$  then  $f^3_{\lambda,r}(\lambda) < p$ . Notice that  $f_{\lambda,r}(\mathcal{I}_3) = \mathcal{I}_1$  and  $f_{\lambda,r}(\mathcal{I}_1) \subseteq \mathcal{I}_3$ . To see this consider two cases. The first is where  $f^3_{\lambda,r}(\lambda) \leq \frac{1}{2}$ . Here  $f^4_{\lambda,r}(\lambda) \leq \lambda$  and so the results follow. The second case is where  $\frac{1}{2} < f^3_{\lambda,r}(\lambda) < p < f^2_{\lambda,r}(\lambda)$ . Here since  $f_{\lambda,r}$  is expanding we have  $|f^4_{\lambda,r}(\lambda) - p| > |f^3_{\lambda,r}(\lambda) - p| > |f^2_{\lambda,r}(\lambda) - p|$ . This implies that  $f^4_{\lambda,r}(\lambda) > f^2_{\lambda,r}(\lambda)$ , and so  $f_{\lambda,r}(\mathcal{I}_1) = \mathcal{I}_3$ . Thus no iterate of  $\mathcal{I}_3$  can meet  $\mathcal{I}_2$  and the result follows from lemma 2.4.

□

We define a region  $G$  in the parameter space (see Figure 6) that contains all parameters  $\lambda$  and  $r$  to the right of (and including) the transitive boundary (where  $f_{\lambda,r}^2(\lambda) = p$ ) and where  $|f_{\lambda,r}'(x)| \geq 1$  (except at the critical point). This region can be found by taking the derivative of one of the branches evaluated at the critical

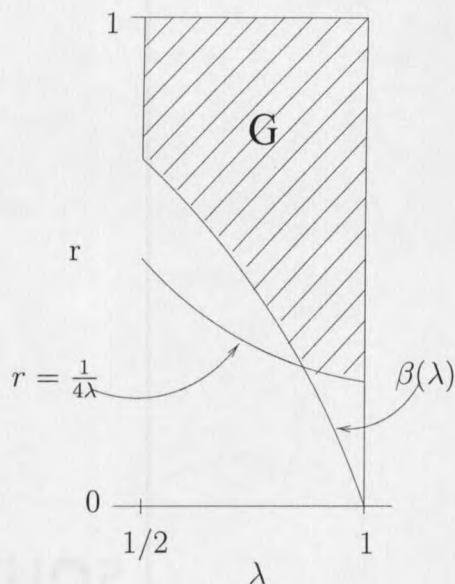


Figure 6: Region G

point for parameters  $r < \frac{1}{2}$ , or by taking the derivative of the right branch evaluated at  $\lambda$  for parameters  $r \geq \frac{1}{2}$ . For the parameters  $r \geq \frac{1}{2}$  all maps  $f_{\lambda,r}$  have the desired property for  $(\lambda, r) \in Q$ , but for the set of parameters  $r < \frac{1}{2}$  only the region

$$r(\lambda) \geq \frac{1}{4\lambda} \quad (2.10)$$

has the desired property. Putting the transitive curve together with this curve defines a region  $G$  of the parameter space (see Figure 6). Thus the region  $G$  may be defined as the region satisfying the equations  $\frac{1}{2} < \lambda \leq 1$ ,  $0 < r \leq 1$ ,  $r \geq \frac{1}{4\lambda}$  and  $r \geq \beta(\lambda)$ , where  $\beta(\lambda)$  is the transitive boundary curve shown in Figure 5.

**Definition 2.6** The itinerary  $,it(q)$ , of a point  $q$  is an infinite word (or sequence)  $W(R, L, C)$  formed by placing the letter  $R$  in the  $i + 1^{th}$  position of  $W(R, L, C)$  if  $f_{\lambda,r}^i(c) > c$ ,  $L$  if  $f_{\lambda,r}^i(c) < c$ , and  $C$  if the iterate lands on  $c$ .

Each core interval map,  $f_{\lambda,r}$  given by parameters in  $G$ , is unimodal with critical point  $c = \frac{1}{2}$ . Following the forward orbit of the critical point it is possible to code the point

$c$  by its itinerary.

**Definition 2.7** The itinerary  $it(f_{\lambda,r}(c))$  is called the *kneading sequence* of the map  $f_{\lambda,r}$ .

The kneading sequence will be denoted by  $\mathbf{k}_{f_{\lambda,r}}$ , or just  $\mathbf{k}$  if it is understood which map is being referred to, or if a general sequence is being discussed. We will denote the  $i^{\text{th}}$  term of the kneading sequence by  $\mathbf{k}(i)$ .

While the kneading sequence is the most important itinerary for each unimodal map, each point has its own individual itinerary which is different from that of any other point.

**Lemma 2.8** *Given  $x < y$ , both in the core interval of  $f_{\lambda,r}$ , then  $it(x) \neq it(y)$ .*

Proof:

Let  $x < y$  be two points in the core and assume  $it(x) = it(y)$ . If  $x < \frac{1}{2} \leq y$  (or  $x \leq \frac{1}{2} < y$ ) then we already have a contradiction to the assumption that  $it(x) = it(y)$ . Hence one may suppose that  $x$  and  $y$  lie on the same side of the critical point. Since  $|f'_{\lambda,r}| > 1$  almost everywhere on  $\mathcal{I}$  then for all subintervals  $J \subset \mathcal{I}$ , with  $J$  entirely on one side or the other of the critical point and the length of the interval  $l(J) \geq y - x$ , there is an  $\epsilon > 0$ , depending on  $x$  and  $y$ , such that  $l(f_{\lambda,r}(J)) > l(J) + \epsilon$ . Now by assumption  $f_{\lambda,r}^m(x)$  and  $f_{\lambda,r}^m(y)$  must lie on the same side of the critical point for all  $m \in \mathbf{Z}^+$ . By induction the  $m^{\text{th}}$  iterate of the interval  $[x, y]$  for  $m = 1, 2, 3, \dots$  must have at least measure  $y - x + m\epsilon$ . Thus for some  $m$  we have  $y - x + m\epsilon > \max\{\lambda - \frac{1}{2}, \frac{1}{2} - f_{\lambda,r}(\lambda)\}$ . This leads to a contradiction and implies that  $x$  and  $y$  cannot have the same itinerary. □

Maps  $f_{\lambda,r}$  given by parameters in  $G$  are transitive on their core interval. To show this the following definition is needed.

**Definition 2.9** The parameter  $\lambda$  is said to be *periodic of order  $n$*  at  $r$  if  $f_{\lambda,r}^n(\lambda) = \lambda$  and  $f_{\lambda,r}^m(\lambda) \neq \lambda$  for  $m < n$ .

When the term *periodic* is used for the parameter  $\lambda$ , it will be assumed that  $r$  is some fixed value. We first show transitivity for the periodic parameters. It will be helpful to consider

**Lemma 2.10** *Suppose  $\lambda$  is a periodic parameter of period  $n$  of  $f_{\lambda,r}$ , and let  $a < b$  be any two points in  $\mathcal{O}^+(\lambda)$ . Then  $\mathcal{I} = f_{\lambda,r}^i[a, b]$ , for some  $i \in \mathbf{Z}^+$ .*

Proof:

Since  $a$  and  $b$  are points in the periodic orbit there are positive integers  $k$  and  $j$  such that  $f_{\lambda,r}^k(a) = f_{\lambda,r}(\lambda)$  and  $f_{\lambda,r}^j(b) = \lambda$ . Also,  $f_{\lambda,r}^{kjn}(a) = f_{\lambda,r}(\lambda)$  and  $f_{\lambda,r}^{jkn}(b) = \lambda$ . Thus if  $i = jkn$ , then  $\mathcal{I} = f_{\lambda,r}^i[a, b]$ .

□

**Theorem 2.11** *If  $(\lambda, r)$  is on the transitive boundary or  $f_{\lambda,r}$  has a periodic parameter  $\lambda$  at  $r$ ,  $(\lambda, r) \in G$ , then  $f_{\lambda,r}$  is transitive on its core interval  $\mathcal{I}$ .*

Proof:

We first show that all maps  $f_{\lambda,r}$  given by parameters on the transitive boundary are transitive on their core intervals. To see this suppose  $f_{\lambda,r}^2(\lambda) = p$ , where  $p$  is the fixed point to the right of the critical point. Consider the intervals  $\mathcal{I}_1 = [f_{\lambda,r}(\lambda), p]$  and  $\mathcal{I}_2 = [p, \lambda]$ . Notice that  $f_{\lambda,r}(\mathcal{I}_2) = \mathcal{I}_1$  and  $f_{\lambda,r}(\mathcal{I}_1) \supseteq \mathcal{I}_2$ . Let  $U \subset \mathcal{I}$  be any open interval. Since  $|f'_{\lambda,r}| > 1$  (in the region  $G$ ), ( $x \neq \frac{1}{2}$ ) except at perhaps a point (two for  $f_{1,1}$ ) on the core, subsequent iterations  $U_i = f_{\lambda,r}^i(U)$  are expanding in length. We desire to show first that  $\lambda \in U_i$  for some  $i \in \mathbf{Z}^+$ . There are several different cases, but we shall consider only the worst case scenario with  $r \geq \frac{1}{2}$  since the rest have similar proofs. In fact, for most of this chapter (for simplicity of proofs) only the maps  $f_{\lambda,r}$ ,



with  $\frac{1}{2} < \lambda \leq 1$  and  $\frac{1}{2} \leq r \leq 1$ , will be discussed unless proofs are identical for maps with parameters  $r < \frac{1}{2}$ .

When  $r \geq \frac{1}{2}$  the graph of  $f_{\lambda,r}$  is convex. If  $|f'_{\lambda,r}(x)| = 1$  for some point  $x \in \mathcal{I}$  it will be at  $x = \lambda$ . Thus suppose  $m\{U\} = \delta > 0$ , where  $m$  denotes length. Let  $J = (\lambda - \delta, \lambda)$ . Notice that  $J$  is expanded least among intervals of length  $\delta$ . Thus

$$\begin{aligned} m\{J\} &< m\{f_{\lambda,r}(J)\} \\ &\leq m\{f_{\lambda,r}(U)\} \end{aligned}$$

for all  $U$ , with  $m\{U\} \geq \delta$ , not containing the critical point. Thus let  $m\{f_{\lambda,r}(J)\} = \epsilon_0 + \delta$ ,  $\epsilon_0 > 0$ . Then for any interval  $U$ , with  $m\{U\} = \delta$ , we have  $m\{U_1\} \geq \epsilon_0 + \delta$ . By induction,  $m(U_i) > i\epsilon_0 + \delta$  for all  $i \in \mathbf{Z}^+$ , unless  $\lambda \in U_j$ , for some  $j \in \mathbf{Z}^+$ . But then  $p \in U_{j+2}$ . Once  $p \in U_{j+2}$  it is only a matter of taking enough iterates to get all of  $\mathcal{I}_2$  in some  $U_{j+n+2}$ , for some  $n \in \mathbf{Z}^+$ . This implies the transitivity of  $f_{\lambda,r}$  by lemma 2.4.

We will use a similar approach for the rest of the maps given by periodic parameters in  $G$ . Let  $\lambda$  and  $r$  be parameters in  $G$ , and suppose  $\lambda$  is periodic. Let  $U$  be any open interval of  $\mathcal{I}$  of  $f_{\lambda,r}$ .

Claim: The set,  $\mathcal{P}$ , of points that fall into the critical orbit under iterations of  $f_{\lambda,r}$  is dense in  $U$ .

Proof (of claim): By way of contradiction assume  $\mathcal{P} \cap U$  is not dense in  $U$ . We may without loss of generality assume that  $\mathcal{P} \cap U = \emptyset$  since if not we can find some open subinterval  $U' \subset U$  that does have this property. Let  $a < b$  be points in  $U$  and let  $J = [a, b]$ . As above,  $m\{f_{\lambda,r}^i(J)\} \geq (b - a) + i\epsilon$ , for all  $i \in \mathbf{Z}^+$  and some  $\epsilon > 0$  since no point in the interval can land on  $\lambda$ . But for some  $i$ ,  $m\{f_{\lambda,r}^i[a, b]\} \geq \lambda - f_{\lambda,r}(\lambda)$ . This leads to a contradiction. Thus  $\mathcal{P} \cap U$  is dense in  $U$ .

To finish the proof let  $s < t$  be two points in  $U$  that land in the critical orbit. Suppose  $f_{\lambda,r}^k(s) = f_{\lambda,r}(\lambda)$  and  $f_{\lambda,r}^j(t) = \lambda$ . Then by lemma 2.10,  $f_{\lambda,r}$  is transitive on

its core interval.

□

**Definition 2.12** The parameter  $\lambda$  is said to be *eventually periodic* at  $r$  for  $f_{\lambda,r}$  if  $\mathcal{O}^+(\lambda)$  is finite.

**Definition 2.13** The parameter  $\lambda$  is said to be *prefixed* at  $r$  for  $f_{\lambda,r}$  if  $\mathcal{O}^+$  contains the fixed point to the right of the critical point.

The proof of Theorem 2.11 may be modified slightly to give the following

**Theorem 2.14** *The map  $f_{\lambda,r}$  is transitive on its core interval if  $(\lambda, r) \in G$  and  $\lambda$  is prefixed or eventually periodic.*

The proof of the next Theorem is similar to the standard method of showing transitivity in the tent family (see, for instance, [6]). The same techniques may also be used to prove Theorem 2.11 and Theorem 2.14. For historical interest and for motivation we shall include a discussion of this method as well as the new proof.

We begin with a discussion of the ideas used to show transitivity. Because of the technical nature and complexity of the process not all details will be given. However, the main ideas will be discussed. The idea is a simple one; If the map is expansive enough then repeated iterations of any open interval contained in the core interval map across the whole core interval.

Claim: Let  $U$  be any open set in  $\mathcal{I}$ , and suppose  $m\{U\} = \delta$ . If  $|f_{\lambda,r}^2| > 2$  then  $f_{\lambda,r}$  is transitive on  $\mathcal{I}$ .

Proof of Claim: If both  $U$  and  $f_{\lambda,r}(U)$  contain the critical point then  $f_{\lambda,r}^2(U)$  contains the fixed point  $p$ . If at most one of  $U$  and  $f_{\lambda,r}(U)$  contain the critical point then The Mean Value Theorem implies that  $m\{f_{\lambda,r}^2(U)\} > (2\delta)/2 = \delta$ . Thus future iterates of  $U$  under  $f_{\lambda,r}^2$  are expanding and must eventually contain the fixed point  $p$ . In

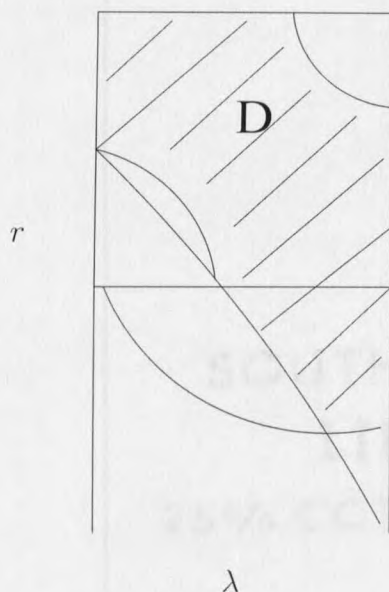


Figure 7: Region D

either circumstance, because future iterates always contain the fixed point and are expansive some iterate eventually contains  $\lambda$ . The next iterate contains the rest of the core interval. Hence  $f_{\lambda,r}$  is transitive on  $\mathcal{I}$ .

Notice that derivatives of  $f_{\lambda,r}$  are smallest in absolute value at the ends of the core interval. Thus  $|\frac{d}{dx}f_{\lambda,r}^2(x)|$  is smallest at  $x = \lambda$ . Figure 7 shows a rough graph of the part of  $G$  where  $|\frac{d}{dx}f_{\lambda,r}^2(\lambda)| > 2$ . There are two sections where this is not true. The upper section boundary has the smallest kneading sequence when  $\lambda = r = \frac{\sqrt{2}}{16}(1 + 5\sqrt{2} + \sqrt{-13 + 10\sqrt{2}})$ . The lower section boundary has the largest kneading sequence when  $\lambda = r = \frac{\sqrt{2}}{16}(1 + 5\sqrt{2} - \sqrt{-13 + 10\sqrt{2}})$ .

To extend the results as discussed in the claim consider the parameters in  $G$  that are not in  $D$  such that  $|\frac{d}{dx}f_{\lambda,r}^4(\lambda)| > 4$ . For the maps considered there are two sections that have significantly different kneading sequences. Those near the transitive boundary have sequences larger than  $RLRR\dots$ , while those near the top right hand corner in  $G$  begin  $RLLL\dots$ . As before, derivatives are smallest in magnitude nearest

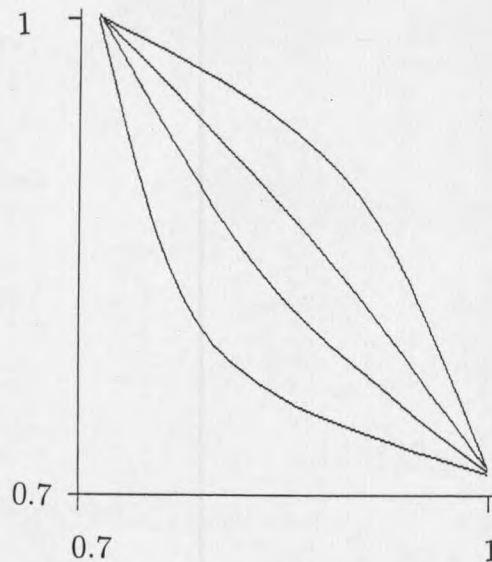


Figure 8: Extensions of Transitivity

the edges. Derivatives taken from the compositions  $f_r \circ f_l \circ f_l \circ f_r$  and  $f_l \circ f_l \circ f_l \circ f_r$  and evaluated at  $\lambda$  are smallest respectively for the lower and upper sections. Analysis shows that the results shown in Figure 7 are extended to more of  $G$ .

This same process may be continued. Considering where  $|\frac{d}{dx} f_{\lambda,r}^{2i}(\lambda)| > 2^i$ , for  $i = 1, 2, 3, \dots$  continues to extend the part of  $G$ . See figure 8 for progression of extensions in the upper right corner of  $G$ . Included are where  $i = 1$ , (which is the lowest of the four plots shown) through  $i = 4$  ( $i = 4$  gives the uppermost plot).

It is reasonable to conclude that as  $i$  increases that the region of  $G$  that is transitive keeps extending as demonstrated. It has been verified up through  $i = 8$  that this is the case.

It would be very difficult, but perhaps not impossible to use the above method for a convincing proof of the transitivity of all  $f_{\lambda,r}$  given by parameters in  $G$ . A somewhat less complicated argument is

**Theorem 2.15** *The map  $f_{\lambda,r}$  is transitive on its core interval if  $(\lambda, r) \in G$ .*

Proof:

Consider  $f_{\lambda,r}$  given by parameters in  $G$ , and let  $U$  be any open interval in the core of  $f_{\lambda,r}$ . It will be shown that for each  $V$  in  $\mathcal{I}$ , there is some  $n \in \mathbf{Z}^+$  such that  $f_{\lambda,r}^n(U) \cap V \neq \emptyset$ .

Define  $X_j = \bigcup_{i=0}^j \{f_{\lambda,r}^i(U)\}$ , for  $j \in \mathbf{Z}^+$ . Let  $a = f_{\lambda,r}(\lambda)$ , and consider the largest interval  $[a, c_j)$  contained in  $X_j$ . Define  $b$  to be the supremum of the  $c_j$ . If  $b = \lambda$  there is nothing to prove, so it may be assumed that  $b < \lambda$  (In fact we may assume  $b < \min\{f_{\lambda,r}^2(\lambda), \frac{1}{2}\}$ , since it is easily seen that otherwise the union of a finite number of iterates of  $[a, b)$  contains all of  $\mathcal{I}$ ).

Let  $I_0 = [a, b)$  and define  $I_j = f_{\lambda,r}^j(I_0)$ , for all  $j \in \mathbf{Z}^+$ . By Theorem 2.11 there is some smallest positive integer  $k$  such that  $f_{\lambda,r}^k(I_0)$  contains  $\frac{1}{2}$ . Notice that  $f_{\lambda,r}^{k+2}(I_0)$  contains  $a$ . It will be shown that  $l(f_{\lambda,r}^{k+2}(I_0))$ , the length of  $f_{\lambda,r}^{k+2}(I_0)$  is larger than  $l(I_0)$ . Thus we have a contradiction to  $b < \lambda$ .

First note that  $l(I_j) < l(I_{j+1})$ , for  $j + 1 < k$ . This is given by the Mean Value Theorem and the fact that  $|f_{\lambda,r}'| > 1$ , except perhaps at one point on the core interval where  $|f_{\lambda,r}'| = 1$ . In particular,  $l(I_1) > f_{\lambda,r}'(a)l(I_0)$ . By induction,  $l(I_k) > f_{\lambda,r}'(a)l(I_0)$ .

Next it should be noted that  $I_{k+1} \subset (f_l^{-1}(p), p)$ , where  $p$  is the fixed point of  $f_{\lambda,r}$ . If this were not the case then  $p$  would be in some  $I_j$  and hence (since  $|f_{\lambda,r}'| = 1$ ) future iterates would contain all of  $\mathcal{I}$  as in the transitive boundary case as stated in Theorem 2.11. It should also be noted that in the worst case  $\frac{1}{2}$  is in the exact center of  $I_k$  so  $f_{\lambda,r}$  maps  $I_k$  onto  $I_{k+1}$  twice. Thus

$$l(I_{k+1}) > \frac{1}{2}|f_{\lambda,r}'(p)|l(I_k) > \frac{1}{2}|f_{\lambda,r}'(p)||f_{\lambda,r}'(a)|l(I_0).$$

Finally, since  $|f_{\lambda,r}'|$  is smallest when  $x = \lambda$  (when  $r \geq \frac{1}{2}$ ) then  $l(I_{k+2})$  may be underestimated by  $l(I_{k+2}) > |f_{\lambda,r}'(\lambda)|l(I_{k+1})$ . Putting this together with the estimate

above for  $l(I_{k+1})$  we have

$$l(I_{k+2}) > \frac{1}{2} f_{\lambda,r}'(a) |f_{\lambda,r}'(p)| f_{\lambda,r}'(\lambda) l(I_0).$$

Thus, it will only be necessary to show that  $\frac{1}{2} f_{\lambda,r}'(a) |f_{\lambda,r}'(p)| f_{\lambda,r}'(\lambda) > 1$ . Since

$$f_{\lambda,r}'(a) = \frac{\lambda r (2\lambda r - \lambda + 1 - r)^2}{(\lambda - \lambda^2 - 3\lambda r + 2\lambda^2 r + 2\lambda r^2 + r - r^2)^2},$$

$$|f_{\lambda,r}'(p)| = f_{\lambda,r}'(f_r^{-1}(p)) = \frac{(\lambda - 1 + r + \sqrt{\lambda^2 - 2\lambda + 6\lambda r + 1 - 2r + r^2})^2}{4\lambda r},$$

and

$$f_{\lambda,r}'(\lambda) = \frac{\lambda r}{(2\lambda r - \lambda + 1 - r)^2},$$

then

$$\frac{1}{2} f_{\lambda,r}'(a) |f_{\lambda,r}'(p)| f_{\lambda,r}'(\lambda) = \frac{\lambda r (\lambda - 1 + r + \sqrt{\lambda^2 - 2\lambda + 6\lambda r + 1 - 2r + r^2})^2}{8(\lambda - \lambda^2 - 3\lambda r + 2\lambda^2 r + 2\lambda r^2 + r - r^2)^2}.$$

Contour plots of this quantity show that for all  $(\lambda, r) \in G$  it is larger than 1.

□

This proof is not very satisfactory in that it forces one to do very complex analysis or use technology. Thus we will outline a different proof that uses some simple ideas from Chapter 3.

Proof:(of Theorem 2.15)

The idea of the proof is to show that there are forward orbits that start arbitrarily close to  $\lambda$  and converge to the fixed point  $p$ .

Consider the map  $f_{\lambda,r}$ , with  $(\lambda, r) \in G$ . Now  $f_{\lambda,r}$  has points with itineraries between its kneading sequence  $\mathbf{k}$  and  $\sigma\mathbf{k}$  (see Lemmas 3.6 and 3.7). In particular,  $RLR^\infty$  is one of them, and corresponds to landing on the fixed point  $p$  after two iterates. Let  $it(\lambda_i)$  be a segment itinerary (of length  $i$ ) that is the same as  $\mathbf{k}$ . Choose  $i$  so that the last term is  $L$  and so that all points having itineraries that begin with  $it(\lambda_i)$  are within  $\epsilon > 0$  of  $\lambda$ . This can be done since  $f_{\lambda,r}$  is expanding. Augment

$it(\lambda_i)$  with  $R^m$ , for arbitrary  $m \in \mathbf{Z}^+$ . Denote this augmented itinerary by  $it(p_m)$ . Note that  $it(p_m)$  is an admissible itinerary segment (there is some point in the core interval of  $f_{\lambda,r}$  that begins with this itinerary segment) since  $\sigma \mathbf{k} < \sigma^j(it(p_m)) < \mathbf{k}$ , for all  $j \leq i + m - 1$ . Thus it may be concluded that there is a point arbitrarily close to  $\lambda$  that terminates close to  $p$ . Since this may be done for each  $m \in \mathbf{Z}^+$ , it may be assumed that the point lands arbitrarily close to  $p$ . Now since the preimages of  $\lambda$  are dense in the core interval, so are the preimages of neighborhoods of  $p$ . This implies that  $f_{\lambda,r}$  is transitive. □

Maps given by parameters in the region bounded by  $\lambda = 1$ ,  $r = \beta(\lambda)$ , and  $r = \frac{1}{4\lambda}$  do not have derivatives with modulus larger than or equal to one. However, it can be shown that  $|(f_{\lambda,r}^2)'| > 1$ , for maps given by parameters in this region. A similar argument to Theorem 2.11 shows  $f_{\lambda,r}^2$  is transitive on  $\mathcal{I}_2$ , for those maps given by parameters on the transitive boundary. Hence,  $f_{\lambda,r}$  is transitive on all of  $\mathcal{I}$ . This leads to consideration that the maps given by parameters in this tail end region may be transitive.

## Foliation from Conjugacy

We begin with a definition.

**Definition 2.16** Suppose  $X$  and  $Y$  are topological spaces. Two maps  $F : X \rightarrow X$  and  $G : Y \rightarrow Y$  are *topologically conjugate* if there is a homeomorphism  $H : X \rightarrow Y$  such that  $H \circ F(x) = G \circ H(x)$ .

Since the family  $f_{\lambda,r}$  is jointly continuous in  $(\lambda, r)$  for each map  $f_{\lambda_0, r_0}$  with an interval core there could be a map  $f_{\lambda_1, r_1}$  with  $r_1$  close to  $r_0$  and  $\lambda_1$  close to  $\lambda_0$  such that the two are topologically conjugate when restricted to their respective cores. One would hope that there exist curves in parameter space corresponding to maps

that are all topologically conjugate to one another. In this section we will show that this is the case; there are curves of conjugacy in the parameter space in region  $G$ . Thus the main goal of this section is to characterize these curves. The main result of Chapter 2 will be

**Theorem 2.17** *The region  $G$  admits a foliation whose leaves are algebraic curves  $C_\mu$ ,  $\frac{\sqrt{2}}{2} < \mu < 1$ . The curves  $C_\mu$  have the property that the maps  $f_{\lambda_1, r_1}$  and  $f_{\lambda_2, r_2}$  are conjugate if and only if  $(\lambda_1, r_1)$  and  $(\lambda_2, r_2)$  lie on the same curve  $C_\mu$ .*

The proof of Theorem 2.17 will be given later in the chapter after necessary material has been developed.

There is an ordering on kneading sequences and itineraries of unimodal maps called the *parity-lexicographic ordering*. Points in the core interval of  $f_{\lambda, r}$  are ordered exactly as their itineraries in this ordering. The ordering comes from the fact that the left and right branches of the map  $f_{\lambda, r}$  on the core are, respectively, orientation preserving and orientation reversing. It is defined as follows. First order  $R > C > L$ . Then if  $\underline{W} = w_1 w_2 w_3 \dots$  and  $\underline{V} = v_1 v_2 v_3 \dots$  are two distinct itineraries let  $m$  be the first index where they differ. If  $m = 1$ , then  $\underline{W} < \underline{V}$  if and only if  $w_1 < v_1$ . When  $m > 1$ , consider the number  $n$  of  $R$ 's in  $w_1 w_2 \dots w_{m-1}$ . If  $n$  is even then  $\underline{W} < \underline{V}$  if and only if  $w_m < v_m$ , and if  $n$  is odd then  $\underline{W} < \underline{V}$  if and only if  $v_m < w_m$ . We say that a finite sequence,  $\underline{W}$  is *even* if it contains an even number of  $R$ 's. Otherwise it is said to be *odd*.

**Definition 2.18** The shift map  $\sigma$  defined on itineraries is given by  $\sigma(W) = V$ , where  $v_m = w_{m+1}$  for all  $m \in \mathbf{Z}^+$ .

Since  $\lambda$  is the maximum value of  $\mathcal{I}$  for  $f_{\lambda, r}$  and  $\mathcal{I}$  is forward invariant, it follows that any shift  $\sigma^n \mathbf{k}$  gives an itinerary that is less than  $\mathbf{k}$  (but larger than or equal to the itinerary of  $f_{\lambda, r}(\lambda)$ ) in the parity-lexicographic ordering.



Recall that  $\lambda$  is periodic of order  $n$  if  $f_{\lambda,r}^n(\lambda) = \lambda$  and  $f_{\lambda,r}^m(\lambda) \neq \lambda$  for  $m < n$ .

An alternative definition is

**Definition 2.19** For a fixed  $r$ , the parameter  $\lambda$  is periodic if  $\sigma^{n-1}(\mathbf{k}) = it(\frac{1}{2})$  for some  $n \in \mathbf{Z}^+$ .

From this alternate definition it is easy to see that kneading sequences corresponding to periodic parameters are repeating and must be of the of the following form:

$$(RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^\infty. \quad (2.11)$$

Such kneading sequences are called *periodic*. Notice that since  $R$  or  $L$  may precede  $C$ ,  $n_j$  may be 0, but  $n_i \neq 0$ , for  $i < j$ .

**Definition 2.20** A kneading sequence is said to be *shift maximal* if  $\sigma^i(\mathbf{k}) < \mathbf{k}$ , for all  $i$  less than the period of the kneading sequence.

The idea (if not the name) of shift maximal kneading sequences was developed in relation to the tent family ( $f_{\lambda, \frac{1}{2}}$  as  $\lambda$  varies) in [10]. It will be seen that kneading sequences of the family  $f_{\lambda,r}$  are shift maximal (see the remark after Theorem 2.25). We will use the property that  $n_{2i+1} < n_1$ , for all  $i \in \mathbf{Z}^+$ .

At this point it should be mentioned that the curves,  $C_\mu$ , of mutually conjugate maps correspond to the locus of points  $(\lambda, r)$  in  $G$  which define maps having the same kneading sequence (see Theorem 2.25). A curve,  $C_\mu$ , of constant periodic kneading sequence may be found by composing left and right branches, respectively

$$f_l = \frac{\lambda x}{(1-2r)x+r} \quad (2.12)$$

and

$$f_r = \frac{\lambda(1-x)}{(2r-1)x+1-r}, \quad (2.13)$$

in the opposite order of the kneading sequence with  $f_l$  replacing  $L$  and  $f_r$  replacing  $R$ . Thus solutions  $(\lambda_0, r_0)$  to the composition

$$f_r^{n_j} \circ f_l^{n_{j-1}} \circ \dots \circ f_r^{n_2} \circ f_l^{n_1} \circ f_r(\lambda) = \frac{1}{2} \quad (2.14)$$

are candidates for points on the curve corresponding to the kneading sequence of 2.11. Notice that  $f_r(\frac{1}{2}) = f_l(\frac{1}{2}) = \lambda$ . Thus solutions  $(\lambda, r)$  to equation 2.14 give a periodic parameter pair for  $f_{\lambda,r}$ .

There is a trivial correspondence here between the inverses of  $f_r$  and  $f_l$  and periodic kneading sequences of  $f_{\lambda,r}$ . With a slight abuse of notation, define  $f_r^{-1} = R$  and  $f_l^{-1} = L$ . Then if the kneading sequence  $\mathbf{k}_{f_{\lambda,r}} = RL^{n_1}R^{n_2}\dots r^{n_j}C$  we have  $R \circ L^{n_1} \circ R^{n_2} \circ \dots \circ R^{n_j}(\frac{1}{2}) = \lambda$ . We mention this correspondence for several reasons. The first and foremost reason is that this correspondence inspired the ideas developed in Chapter 3. Another reason is that some arguments are much simpler using the inverses  $R$  and  $L$ , which are contractions.

Simplification of equation 2.14 leads to a polynomial equation

$$P_{\mathbf{k}}(\lambda, r) = 0. \quad (2.15)$$

Thus, most of the time, finding solutions requires numerical methods. Notice that the polynomial equation  $P_{\mathbf{k}}(\lambda, r) = 0$  may also be found using a composition of  $R$  and  $L$  as the order of the kneading sequence.

There are no guarantees that solutions are in the interval core region or that the  $C_\mu$  even exist at this point. However, if such curves do exist, they are symmetric in  $r$  and  $\lambda$  (in regions where that makes sense), that is, if  $(\lambda_1, r_1)$  is on the arc, then  $(r_1, \lambda_1)$  is also on the arc. This fact occurs because compositions of symmetric maps are symmetric. To understand why this is so for the  $f_{\lambda,r}$ , recall that the rescaled core maps,  $F_{\lambda,r}$ , are symmetric in  $r$  and  $\lambda$ . Notice also that the rescaled maps may be used

Table 1: Shift maximal kneading sequences and associated polynomials

$\mathbf{k}_f$	$P_{\mathbf{k}}(\lambda, r)$
<i>RLC</i>	$r^2 + (\lambda - 1)r + \lambda^2 - \lambda$
<i>RLLC</i>	$r^3 + (\lambda - 1)r^2 + (\lambda^2 - \lambda)r + \lambda^3 - \lambda^2$
<i>RLLLC</i>	$r^4 + (\lambda - 1)r^3 + (\lambda^2 - \lambda)r^2 + (\lambda^3 - \lambda^2)r + \lambda^4 - \lambda^3$
<i>RLLRC</i>	$r^4 + (3\lambda - 2)r^3 + (3\lambda^2 - 4\lambda + 1)r^2 + (3\lambda^3 - 4\lambda^2 + \lambda)r + \lambda^4 - 3\lambda^3 + \lambda^2$
<i>RLLRRC</i>	$r^4 + (5\lambda - 3)r^3 + (7\lambda^2 - 10\lambda + 3)r^2 + (5\lambda^3 - 10\lambda^2 + 6\lambda - 1)r + \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$
<i>RLLLLC</i>	$r^5 + (\lambda - 1)r^4 + (\lambda^2 - \lambda)r^3 + (\lambda^3 - \lambda^2)r^2 + (\lambda^4 - \lambda^3)r + \lambda^5 - \lambda^4$
<i>RLLLLRC</i>	$r^4 + (2\lambda - 2)r^3 + (\lambda^2 - 2\lambda + 1)r^2 + (2\lambda^3 - 2\lambda^2)r + \lambda^4 - 2\lambda^3 + \lambda^2$
<i>RLLRRC</i>	$r^5 + (5\lambda - 3)r^4 + (7\lambda^2 - 10\lambda + 3)r^3 + (7\lambda^3 - 12\lambda^2 + 6\lambda - 1)r^2 + (5\lambda^4 - 10\lambda^3 + 6\lambda^2 - \lambda)r + \lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2$

(similar to the  $f_{\lambda,r}$ ) to find the equations of 2.15. The same composition of rescaled branches is set equal to the new critical point  $c = \frac{\lambda-1+r}{2\lambda r}$ , which is symmetric in  $r$  and  $\lambda$  also. Since all the pieces of this equation are symmetric then so is the solution curve if it exists. See Figure 9 for numerical graphs of period 3 through period 6 constant kneading curves, and Table 1 of kneading sequences with their associated polynomials given by the above construction.

**Definition 2.21** The level curve of  $P_{\mathbf{k}}(\lambda, r)$  is  $C_{\mu}$ , where  $\mu$  is given by solving for  $\lambda$  in  $P_{\mathbf{k}}(\lambda, \frac{1}{2})$ .

Now that the method for finding the constant kneading curves,  $C_{\mu}$ , has been given, we need some machinery to prove the existence of such curves for shift maximal kneading sequences. The following theorems and lemmas will be important in developing the ideas needed to show the existence of curves of periodic kneading sequences in the desired parameter space region  $G$  for the periodic shift maximal kneading sequences.

Fix a shift maximal kneading pattern  $\mathbf{k} = (RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^{\infty}$ , of period  $n$  in (we are assuming  $RLLR^{\infty} < \mathbf{k} < RL^{\infty}$ ). Choose the product of  $f_r$

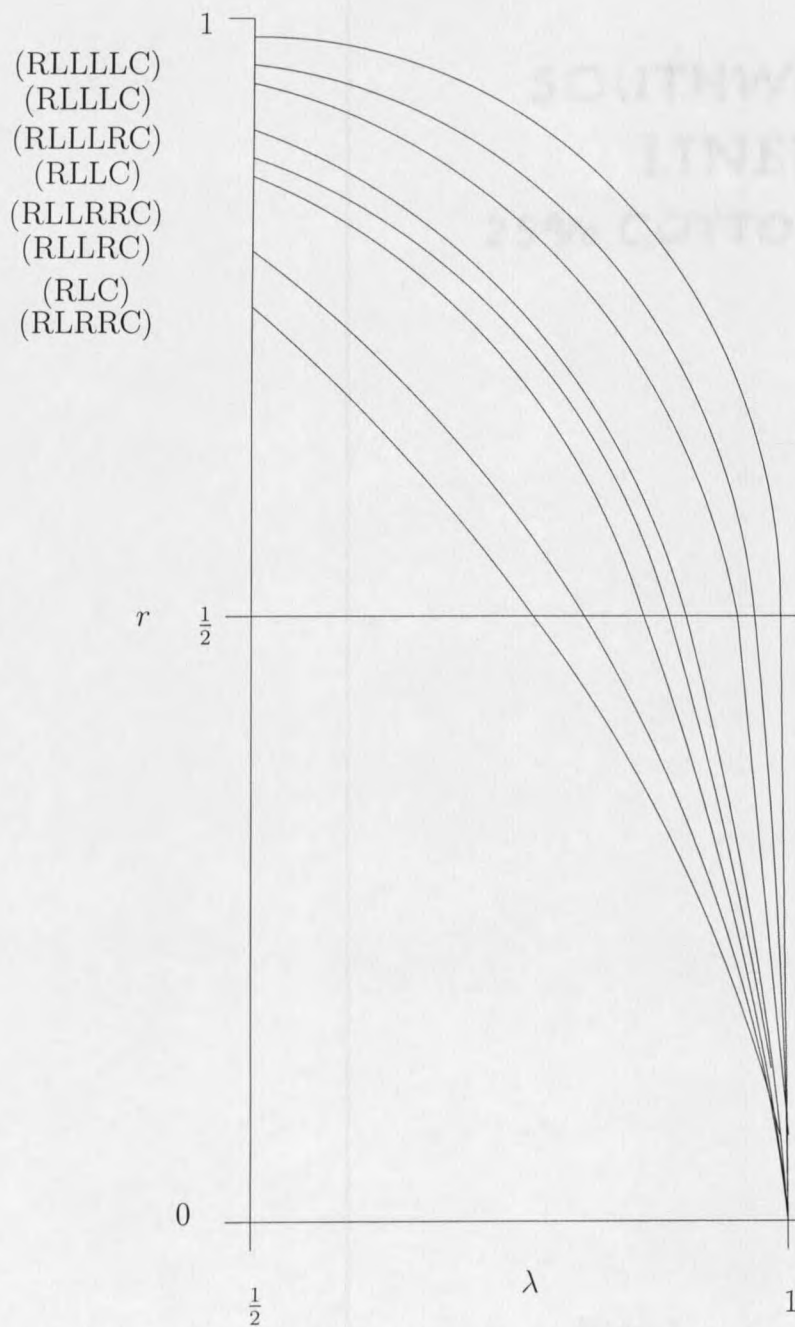


Figure 9: Constant Kneading Curves of Low Periods

and  $f_l$  as in equation 2.14 except instead of using  $(RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^\infty$  use  $C(RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^\infty$ . Define

$$\varphi_{\mathbf{k}}(\lambda, r) = f_r^{n_j} \circ f_l^{n_{j-1}} \circ \dots \circ f_r^{n_2} \circ f_l^{n_1} \circ f_r \circ f_l\left(\frac{1}{2}\right) - \frac{1}{2}. \quad (2.16)$$

Notice that this is a particular branch of the map  $f_{\lambda, r}^n\left(\frac{1}{2}\right)$ , (except subtraction by  $\frac{1}{2}$ ) and is a fixed rational function.

**Definition 2.22** Given a kneading sequence  $\mathbf{k} = (RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^\infty$  then  $f_{\lambda, r}^m(\lambda)$  following  $\mathbf{k}$  means the left and right branches of  $f_{\lambda, r}$  are composed according to  $\mathbf{k}$  not  $\mathbf{k}_{f_{\lambda, r}}$ . This will be denoted by  ${}_k f_{\lambda, r}^m(\lambda)$ .

We note that if  $\mathbf{k}$  is not equal to  $\mathbf{k}_{f_{\lambda, r}}$  and the two sequences differ in the  $m^{\text{th}}$  position, then  ${}_k f_{\lambda, r}^m(\lambda) \geq \lambda$ .

**Lemma 2.23** *If  $\mathbf{k}_{f_{\lambda, r}} \neq \mathbf{k}$ , then some iterate  $f_{\lambda, r}^m(\lambda)$  following  $\mathbf{k}$  is to the right of the core interval of  $f_{\lambda, r}$ . Furthermore, the only way some iterate  ${}_k f_{\lambda, r}^{m+i}(\lambda)$  may return to the core interval is if  $f_r$  is applied to the first iterate outside the core then  $f_l$  is repeatedly applied.*

Proof:

Suppose  $\mathbf{k}$  and  $\mathbf{k}_{f_{\lambda, r}}$  differ in the  $m^{\text{th}}$  position, and this is the first position in which they do so. Thus  $f_{\lambda, r}^i(\lambda)$  following  $\mathbf{k}$  and  $f_{\lambda, r}^i(\lambda)$  are the same for all  $i < m$ . Without loss of generality it may be assumed that  $f_{\lambda, r}^{m-1}(\lambda) = p > \frac{1}{2}$ . Then  $f_l \circ f_{\lambda, r}^{m-1}(\lambda) = f_l(p)$  is the same as  $f_{\lambda, r}^m(\lambda)$  following  $\mathbf{k}$ , but  $f_l(p) > \lambda$  since  $f_l$  is increasing.

Because of similarity in the arguments, not all details are included here. Notice that 0 is a fixed point of  $f_l$  (as is  $\frac{\lambda-r}{1-2r}$  for maps given by parameters with  $r > \lambda$ ) and 1 maps to 0 under  $f_r$ . Using this fact along with the fact that  $f_l$  is increasing and  $f_r$  is decreasing on all of  $\mathbf{R}$ , once some iteration under  $\mathbf{k}$  is outside of  $[0, 1]$  (or

for  $r > \lambda$ ,  $[\frac{\lambda-r}{1-2r}, 1]$ ) no iterate can land back inside. Now let  $t > \lambda$ . Analysis shows that  $f_r(f_l(t)) < 0$  (or  $f_r(f_l(t)) < \frac{\lambda-r}{1-2r}$ , for  $r > \lambda$ ), for  $f_{\lambda,r}$  given by parameters in the region  $G$ . Thus once some iteration following  $\mathbf{k}$ ,  $t$ , is to the right of the core interval  $f_r$  must be applied next if some iterate will eventually land back inside the core. Similar analysis shows that  $f_r^3(t) < 0$ , for all maps  $f_{\lambda,r}$  given by parameters in the transitive region. Since  $f_l^i(f_r^2(t)) > f_r^2(t) > t$ , for  $i \in \mathbf{Z}^+$ , this forces  $f_l$  to be applied to  $f_r(t)$  if some iteration following  $\mathbf{k}$  will eventually land back inside the core.

□

**Proposition 2.24**  $\varphi_{\mathbf{k}}(\lambda_0, r_0) = 0$  only when  $f_{\lambda_0, r_0}$  has kneading sequence  $\mathbf{k}$ .

Proof:

Let  $\mathbf{k} = (RL^{n_1}R^{n_2} \dots L^{n_{j-1}}R^{n_j}C)^\infty$  be a shift maximal periodic kneading pattern. By way of contradiction assume there is some  $f_{\lambda_0, r_0}$  that does not have kneading sequence  $\mathbf{k}$  yet  $\varphi_{\mathbf{k}}(\lambda_0, r_0) = 0$ . We will show that this cannot occur. There are two main cases.

The first case is just a kneading sequence argument. Suppose first that  $\mathbf{k} < \mathbf{k}_{f_{\lambda_0, r_0}}$ . This implies that  $\sigma(\mathbf{k}) > \sigma(\mathbf{k}_{f_{\lambda_0, r_0}})$ . Let  $m$  be the last integer such that  ${}_k f_{\lambda_0, r_0}^m(\lambda_0) > \lambda_0$ . Let  $b = f_{\lambda_0, r_0}(\lambda_0)$  and  $a = f_r \circ {}_k f_{\lambda_0, r_0}^m(\lambda_0)$ . Then  $it(a) < it(b) < \sigma(\mathbf{k})$ , since  $a < b$ . This leads to a contradiction since no iteration of  $a$  following  $\mathbf{k}$  can land on  $\frac{1}{2}$ .

The second case is more complicated. Suppose that  $\mathbf{k} > \mathbf{k}_{f_{\lambda_0, r_0}}$ . Suppose also that  $\mathbf{k}$  and  $\mathbf{k}_{f_{\lambda_0, r_0}}$  differ first in the  $m^{\text{th}}$  position, that is,  $\mathbf{k}(m) \neq \mathbf{k}_{f_{\lambda_0, r_0}}(m)$ . There are three possibilities that need to be considered.

The first possibility of the second case is where  $m > n_1 + 1$ . Let  $z \in \mathbf{Z}^+$  be the last integer such that  ${}_k f_{\lambda_0, r_0}^z(\lambda_0) = t > \lambda_0$  before some iteration under  $\mathbf{k}$  lands on  $\frac{1}{2}$ . Then by lemma 2.23,  ${}_k f_{\lambda_0, r_0}^{z+1}(\lambda_0) = f_r \circ {}_k f_{\lambda_0, r_0}^z(\lambda_0)$ . Hence  $f_r(t) < f_r(\lambda_0)$ , but since  $n_1 > n_{2i+1}$ , for  $i = 1, 2, 3, \dots$ ,  $\sigma^{z+1}(\mathbf{k}) > \sigma(\mathbf{k})$ . Thus  $it(f_r(t)) < \sigma^{z+1}(\mathbf{k})$ ,

which forces  ${}_k f^{z+1+n_1}_{\lambda_0, r_0}(\lambda_0) > \lambda_0$  and a contradiction to that fact that  $z$  is the last integer where this occurs before some iteration under  $\mathbf{k}$  lands on  $\frac{1}{2}$ .

The second possibility to be considered is when  $m < n_1 + 1$ . Here  $\mathbf{k}(m) = L$  and  $\mathbf{k}(m+1) = L$  while  $\mathbf{k}_{f_{\lambda_0, r_0}}(m) = R$ . Thus  ${}_k f^m_{\lambda_0, r_0}(\lambda_0) = f_l \circ {}_k f^{m-1}_{\lambda_0, r_0}(\lambda_0) > \lambda_0$  and  ${}_k f^{m+1}_{\lambda_0, r_0}(\lambda_0) = f_l^2 \circ {}_k f^{m-1}_{\lambda_0, r_0}(\lambda_0)$  can't get back into the core as seen in lemma 2.23. Thus no iteration under  $\mathbf{k}$  may land on  $\frac{1}{2}$ .

The last possibility occurs when  $m = n_1 + 1$ . Here  $\mathbf{k}(m) = L$  and  $\mathbf{k}(m+1) = R$  while  $\mathbf{k}_{f_{\lambda_0, r_0}}(m) = R$ . Note that  $n_2 = 1$ , since otherwise by lemma 2.23 some iteration following  $\mathbf{k}$  would be outside the interval  $[0, 1]$  (Either  $f_r^3(t) < 0$  or  $f_r \circ f_l^i \circ f_r(\lambda_0) < 0$  for  $t > \lambda_0$ ). let  $s_1 = {}_k f_{\lambda_0, r_0}^{m-1}(\lambda_0)$  and  $s_1 - \frac{1}{2} = \delta$ . We want to show that further iterates under  $\mathbf{k}$  do not come within  $\delta$  of  $\frac{1}{2}$ . We will do this by showing that the best possibility fails to get as close.

Claim:  $f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) < \frac{1}{2}$  and  $(\frac{1}{2} - f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0)) > \delta$ .

Proof of Claim: We first note that  $f_l'(x) > 1$  and  $|f_r'(x)| > 1$  (unless  $x$  is outside the interval where some iteration following  $\mathbf{k}$  can get back into the core, in which case there is nothing to prove). By the Mean Value Theorem,  ${}_k f_{\lambda_0, r_0}^m(\lambda_0) - \lambda_0 > |f_l'(\frac{1}{2})|(s_1 - \frac{1}{2}) > \delta$ . Continuing this process,  $f_{\lambda_0, r_0}^{n_1}(\lambda_0) - f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) > \delta$ . Because  $f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) < f_{\lambda_0, r_0}^{n_1}(\lambda_0)$  then  $f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) < \frac{1}{2}$  else  $f_{\lambda_0, r_0}^{n_1}(\lambda_0) - f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) < \delta$  which cannot occur. For the other part of the claim we need to show that  $f_{\lambda_0, r_0}^{n_1}(\lambda_0) - f_l^{n_1-1} \circ f_r \circ f_l^{n_1} \circ f_r(\lambda_0) > 2\delta$ . To see this one uses the Mean Value Theorem estimate applied to  $f_r \circ f_l(\frac{1}{2})$ . Now

$$(f_r \circ f_l)'(x) = \frac{-\lambda_0^2 r_0^2}{(x - 3xr_0 + r_0 - \lambda_0 x + 2xr_0^2 - r_0^2 + 2x\lambda_0 r_0)^2}$$

Analysis shows that  $|f_r \circ f_l(\frac{1}{2})| > 2$  for all maps  $f_{\lambda, r}$  given by parameters in  $G$  with kneading sequence as above with  $n_1 \geq 2$ . This implies that  $S_1$  and  $\frac{1}{2}$  under two iterations following  $\mathbf{k}$  are separated by more than  $2\delta$ . More iterations only push them farther apart.

This claim and the fact that  $f_i$  is increasing show that

$$\left(\frac{1}{2} - {}_k f^{n_1+n_3+1}_{\lambda_0, r_0}(\lambda_0)\right) > \delta.$$

This also implies that

$${}_k f^{n_1+n_3+2}_{\lambda_0, r_0}(\lambda_0) > {}_k f^m_{\lambda_0, r_0}(\lambda_0).$$

Thus to finish off this proof one need only note that since  $n_{2i+1} > n_1$  for all  $i \in \mathbf{Z}^+$ , conditions are even worse. The restrictions on the kneading sequence along with those of lemma 2.23 will not allow any iteration under  $\mathbf{k}$  to be within  $\delta$  of  $\frac{1}{2}$ .  $\square$

**Theorem 2.25** *The zero set,  $\varphi_{\mathbf{k}}^{-1}(\{0\})$ , is a graph of a strictly increasing function of  $\lambda$ .*

Proof:

We assume that  $\frac{\partial \varphi_{\mathbf{k}}(\lambda, r)}{\partial \lambda} \neq 0$  and  $\frac{\partial \varphi_{\mathbf{k}}(\lambda, r)}{\partial r} \neq 0$ . (we will prove something much stronger in Lemma 2.28). Applying the implicit function, we have that  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  is locally a graph everywhere in  $G$ . In fact, since the partials of  $\varphi_{\mathbf{k}}(\lambda, r)$  are not zero,  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  must be a finite union of graphs. We claim that there is only one component in  $G$ . To see this, assume, by way of contradiction, that there are two components of  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  in  $G$ . Let  $\gamma(t)$  be a linear path from point  $\gamma(a)$  on one component to  $\gamma(b)$  on the other component. Since the gradient  $\Delta \varphi_{\mathbf{k}}(\lambda, r)$  is bounded away from 0, we have

$$\left| \int_a^b \Delta \varphi_{\mathbf{k}} \cdot \gamma'(t) dt \right| = |\varphi_{\mathbf{k}}(\gamma(b)) - \varphi_{\mathbf{k}}(\gamma(a))| > 0. \quad (2.17)$$

This gives a contradiction. Thus  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  must be a unique graph in  $G$ .  $\square$

**Remark:** Notice that Theorem 2.25 demonstrates that the only kneading sequences that need be considered are those that come from the tent family. Each periodic



parameter is associated with a graph in  $G$ . If the kneading sequence  $\mathbf{k}$  is not given by a parameter in the tent family then it is not possible to have a graph associated with it since it would not be defined at  $r = \frac{1}{2}$

**Corollary 2.26** *If  $\mathbf{k} < \mathbf{k}'$  are periodic kneading sequences given by parameters in the tent family then  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  lies below  $\varphi_{\mathbf{k}'}^{-1}(\{0\})$ .*

Proof:

Since kneading sequences of the tent family are monotone (see [5]), and  $\varphi_{\mathbf{k}}^{-1}(\{0\})$  cannot cross  $\varphi_{\mathbf{k}'}^{-1}(\{0\})$  (this would imply that some  $f_{\lambda,r}$  has two different kneading sequences) we have the desired result. □

This next lemma gives a bound on how fast  $f_{\lambda,r}^n(\frac{1}{2})$  changes as  $\lambda$  varies. It is instrumental in proving denseness of periodic parameters for  $f_{\lambda,r}$  (see Lemma 2.29). In this lemma the prime,  $'$ , denotes the derivative with respect to  $\lambda$ .

**Definition 2.27** *The skeleton map of  $f_{\lambda,r}$  is denoted by  $\varphi_n(\lambda) = f_{\lambda,r}^n(\frac{1}{2})$ .*

**Lemma 2.28** *Given a fixed  $r \in (\frac{1}{4}, 1)$ , for any closed interval  $J \subset (\lambda_r, 1]$  there are constants  $k_1, k_2 > 0$  and functions  $c_1(\lambda), c_2(\lambda) > 1$  such that for every  $\lambda \in J$ ,  $k_1(c_1(\lambda))^n \leq |\varphi_n'(\lambda)| \leq k_2(c_2(\lambda))^n$  holds whenever  $\varphi_n'(\lambda)$  exists.*

Proof:

Let  $J = [a, b]$  with  $\lambda_{r_0} < a < b \leq 1$ . We will assume that  $\frac{1}{2} \leq r \leq 1$  (a similar argument works for  $\frac{1}{4} \leq r < \frac{1}{2}$ ). First notice if  $\varphi_{n+1}'(\lambda)$  exists, then

$$\begin{aligned} \varphi_{n+1}'(\lambda) &= \frac{\partial}{\partial \lambda} (f_{\lambda,r}^{n+1}(\frac{1}{2})) \\ &= \frac{\varphi_n(\lambda)}{(1-2r)\varphi_n(\lambda) + r} + \frac{\lambda r \varphi_n'(\lambda)}{((1-2r)\varphi_n(\lambda) + r)^2}, \end{aligned} \tag{2.18}$$

when  $\mathbf{k}_{f_{\lambda,r}}(n) = L$ . Similarly,

$$\varphi'_{n+1}(\lambda) = \frac{1 - \varphi_n(\lambda)}{(2r - 1)\varphi_n(\lambda) + 1 - r} - \frac{\lambda r \varphi'_n(\lambda)}{((2r - 1)\varphi_n(\lambda) + 1 - r)^2}, \quad (2.19)$$

when  $\mathbf{k}_{f_{\lambda,r}}(n) = R$ . Next note that if  $\mathbf{k}_{f_{\lambda,r}}(n) = L$  then  $f_{\lambda,r_0}(\lambda) \leq \varphi_n(\lambda) \leq \frac{1}{2}$  and if  $\mathbf{k}_{f_{\lambda,r}}(n) = R$  then  $\frac{1}{2} \leq \varphi_n(\lambda) \leq \lambda$ . Thus maximizing equations 2.18 and 2.19 and using induction with  $\varphi'_1(\lambda) = 1$  we get

$$\begin{aligned} |\varphi'_{n+1}(\lambda)| &\leq 1 + |4\lambda r \varphi'_n(\lambda)| \\ &\leq 1 + 4\lambda r + |(4\lambda r)^2 \varphi'_{n-1}(\lambda)| \\ &\leq \frac{1 - (4\lambda r)^{n+1}}{1 - 4\lambda r} \\ &\leq \frac{(4\lambda r)^{n+1}}{4\lambda r - 1}. \end{aligned}$$

Let  $k_2 = \frac{1}{4ar-1}$  and  $c_2(\lambda) = 4\lambda r > 1$  to get

$$|\varphi'_n(\lambda)| \leq k_2 (c_2(\lambda))^n. \quad (2.20)$$

To get the lower estimate we need to minimize equations 2.18 and 2.19. For equation 2.18, notice that the first term

$$A_L = \frac{\varphi_n(\lambda)}{(1 - 2r)\varphi_n(\lambda) + r}$$

satisfies  $0 \leq A_L \leq 1$ . Also notice that the second term

$$B_L = \frac{\lambda r \varphi'_n(\lambda)}{((1 - 2r)\varphi_n(\lambda) + r)^2}$$

has the factor

$$\frac{\lambda r}{((1 - 2r)\varphi_n(\lambda) + r)^2} = f'_{\lambda,r}(\varphi_n(\lambda)).$$

Since  $f'_{\lambda,r}(x) = \frac{\lambda r}{((1 - 2r)x + r)^2} > 1$  for  $x \in [f_{\lambda,r}(\lambda), \frac{1}{2}]$ , then if  $\mathbf{k}_{f_{\lambda,r}}(n) = L$ , there is an  $\epsilon_0 > 0$  such that

$$|\varphi'_{n+1}(\lambda)| \geq (1 + \epsilon_0)|\varphi'_n(\lambda)| - 1.$$

We are assuming that  $|\varphi'_n(\lambda)| \geq 1$ . This follows from induction and the fact that  $\varphi'_1(\lambda) = 1$ .

For equation 2.19, notice that the first term

$$A_R = \frac{1 - \varphi_n(\lambda)}{(2r - 1)\varphi_n(\lambda) + 1 - r}$$

satisfies  $0 \leq A_R \leq 1$ , and that the second term

$$B_R = -\frac{\lambda r \varphi'_n(\lambda)}{((2r - 1)\varphi_n(\lambda) + 1 - r)^2}$$

has the factor

$$-\frac{\lambda r}{((2r - 1)\varphi_n(\lambda) + 1 - r)^2} = f'_{\lambda,r}(x)(\varphi_n(\lambda)).$$

As before,  $f'_{\lambda,r}(x) < -1$  for  $x \in [\frac{1}{2}, \lambda]$ . Thus if  $\mathbf{k}_{f_{\lambda,r}}(n) = R$  we have

$$|\varphi'_{n+1}(\lambda)| \geq (1 + \epsilon_0)|\varphi'_n(\lambda)| - 1.$$

This implies

$$\begin{aligned} |\varphi'_{n+2}(\lambda)| &\geq (1 + \epsilon_0)|\varphi'_{n+1}(\lambda)| - 1 \\ &\geq (1 + \epsilon_0)((1 + \epsilon_0)|\varphi'_n(\lambda)| - 1) - 1 \\ &\geq (1 + \epsilon_0)^2|\varphi'_n(\lambda)| - \epsilon_0 \\ &\geq (1 + \epsilon_0)^2|\varphi'_n(\lambda)| - 1. \end{aligned}$$

By induction  $|\varphi'_{n+k}(\lambda)| \geq (1 + \epsilon_0)^k|\varphi'_n(\lambda)| - 1$ . Thus letting  $c_1(\lambda) = 1 + \epsilon_0$  we may choose  $k_1$  such that

$$k_1(c_1(\lambda))^{k_1} \leq |\varphi'_n(\lambda)|. \quad (2.21)$$

□

For a fixed  $r \in (\frac{1}{4}, 1)$  define  $\lambda_r$  to be the value of  $\lambda$  given by the  $\max\{\frac{1}{2}, \frac{1}{4r}, \beta^{-1}(\lambda)\}$  on the boundary of the region  $G$ .

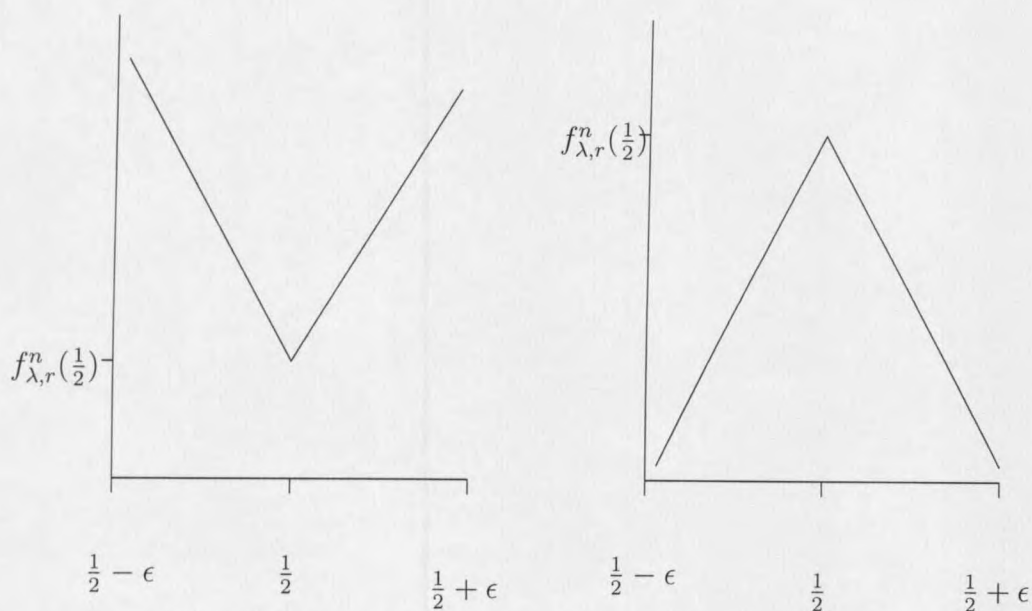


Figure 10: The Possible Graphs of  $f_{\lambda,r}^n(x)$  about  $x = \frac{1}{2}$

**Lemma 2.29** For each fixed  $r \in (\frac{1}{4}, 1)$  the family of core maps  $f_{\lambda,r}$  has a dense set of periodic parameters  $\lambda$  in the interval  $[\lambda_r, 1]$ .

Proof:

Fix  $r \in (\frac{1}{4}, 1)$  and let  $(\lambda_r, 1]$  be the interval in which transitivity occurs. Suppose there exists some interval  $U = [\alpha, \beta] \subset (\lambda_r, 1]$  containing no periodic parameters. Define  $\varphi_n(\lambda, r) = f_{\lambda,r}^n(\frac{1}{2})$ . Choose  $n \in \mathbf{N}$  such that

$$|\varphi'_n(\lambda, r)| > \frac{1}{2(\beta - \alpha)} \quad (2.22)$$

for all  $\lambda$  such that  $\varphi'_n(\lambda)$  exists. Note that  $\varphi'_n(\lambda, r)$  does not exist at parameters  $\lambda$  with period less than  $n$  (see [8] for the tent family case). The choice of  $n$  depends upon  $\alpha$ , the smallest  $\lambda$  value in  $[\alpha, \beta]$ , and the bound given in Lemma 2.28. If  $\varphi'_n(\lambda, r)$  does not exist for all  $\lambda \in [\alpha, \beta]$  then we have a periodic parameter in the interval of period less than  $n$ . Thus if  $\varphi'_n(\lambda, r)$  exists for all  $\lambda \in U$ , then by the Mean Value

Theorem there is some  $\lambda_0 \in U$  with

$$|\varphi'_n(\lambda_0, r)| = \left| \frac{\varphi_n(\beta, r) - \varphi_n(\alpha, r)}{\beta - \alpha} \right| \quad (2.23)$$

But then it must be that

$$|\varphi_n(\beta, r) - \varphi_n(\alpha, r)| \leq \frac{1}{2} \quad (2.24)$$

so we have

$$|\varphi'_n(\lambda_0, r)| \leq \frac{1}{2(\beta - \alpha)} \quad (2.25)$$

This gives a contradiction to equation 2.22. Thus there must exist  $\tilde{\lambda}$  such that  $\varphi'_n(\tilde{\lambda}, r)$  does not exist. This implies that  $f_{\tilde{\lambda}}^m(\frac{1}{2}) = \frac{1}{2}$  for some  $m < n$ , or  $\tilde{\lambda}$  is periodic.

□

**Remark:** The proof of Lemma 2.29 was inspired by [6].

Proof of Theorem 2.17:

From Theorem 2.25 there are curves for all periodic kneading sequences given by periodic parameters in the tent family. We desire to show that this is true for all non periodic kneading sequences given by parameters in the tent family. For each tent family parameter  $\mu$ , let  $\mathbf{k}_\mu$  denote the kneading sequence of  $T_\mu$ . Define  $\varphi_U = \inf_{\mathbf{k} > \mathbf{k}_\mu} \varphi_{\mathbf{k}}(\lambda, r)$ , with  $\mathbf{k}$  periodic. Similarly, define  $\varphi_L = \sup_{\mathbf{k} < \mathbf{k}_\mu} \varphi_{\mathbf{k}}(\lambda, r)$ , with  $\mathbf{k}$  periodic. Then  $\varphi_U$  and  $\varphi_L$  are continuous strictly monotone graphs. Suppose  $\varphi_U > \varphi_L$  for some fixed  $r$ . By construction there are no periodic parameters between  $\varphi_U$  and  $\varphi_L$ . But by Lemma 2.29 there must be one since the periodic parameters are dense. This leads to a contradiction. Therefore,  $\varphi_U = \varphi_L$ . This implies the desired result. To prove that the curves of constant kneading sequence also have the property that maps  $f_{\lambda, r}$  given by parameters on the curve are pairwise conjugate, we rely upon Theorem 3.2 of [9]. This theorem implies that if two unimodal maps have the same kneading sequence, no wandering intervals, no intervals of periodic points, and no periodic attractors

then they are conjugate. Hence, any two maps defined by different points on the curve of constant kneading sequence that are transitive (on the core) are conjugate, since transitivity rules out the possibility of wandering intervals, intervals of periodic points, and periodic attractors.

□

There are a couple of results that are immediate from Theorem 2.17 and the fact that these results are known to occur in the tent family.

**Corollary 2.30** *Kneading sequences of  $f_{\lambda,r}$ , for fixed  $r$  are monotonically increasing as  $\lambda$  increases in the region  $G$ .*

**Corollary 2.31** *The topological entropy of  $f_{\lambda,r}$ , for fixed  $r$  is monotonically increasing as  $\lambda$  increases in the region  $G$ .*

There will be no attempt to find most of the conjugating maps explicitly. However, in the case of transitive maps  $f_{a,b}$  and  $f_{b,a}$ ,  $a, b \in (\frac{1}{2}, 1)$  it is surprisingly easy to do so. It seems that the symmetry of the periodic constant kneading curves forces the rescaled core maps  $F_{a,b}$  and  $F_{b,a}$  to be identical. The maps  $f_{a,b}$  and  $f_{b,a}$  are the same shape but have different sized cores; One is just an affine rescaling of the other. Thus the affine map

$$H = \frac{2b-1}{2a-1}x + \frac{a-b}{2a-1}$$

is such that  $H(f_{a,b}(x)) = f_{b,a}(H(x))$  for  $x$  in the core of  $f_{a,b}$ .

This is also the case for the Farey family,  $f_{\lambda,1}$ , and the set of full maps,  $f_{1,r}$ , for  $\frac{1}{2} < r \leq 1$ . It was noticed earlier on in the discussion that  $f_{\lambda,1}(\lambda) = 1 - \lambda$  is a fixed point. Thus  $[1 - \lambda, \lambda]$  is the core interval for  $f_{\lambda,1}$  and the rescaled core map  $F_{\lambda,1}$  is a full map. To see which full map it is one matches up the derivative of  $F_{\lambda,1}$  at  $x = 0$  (or the derivative of  $f_{\lambda,1}$  at  $x = 1 - \lambda$ ) with the same point derivative of  $f_{1,r}$ . Now

$f'_{1,r}(0) = \frac{1}{r}$  and  $F_{\lambda,1}(0) = \frac{1}{\lambda}$ . Thus  $F_{a,1} = f_{1,a}$ . This means the conjugacy between  $f_{a,1}$  and  $f_{1,a}$  is just a rescaling as are the maps on the periodic constant kneading curves. The same affine map  $H$  may be used.

## Finding and Approximating Entropy

The curves of constant kneading sequence make it possible to find the topological entropy of maps defined by points in the region  $G$ . It is a well known fact that conjugate maps have the same entropy (see for instance [27]). Thus given a map  $f_{\lambda,r}$  if the map has a periodic parameter  $\lambda$  the entropy can be found by finding the value  $\lambda_o$  in the tent family with the same kneading sequence. The entropy is then  $\log(2\lambda_o)$ . In other words if  $P_k(\lambda, r)$  is the associated symmetric polynomial of the periodic kneading sequence of  $f_{\lambda,r}$  then

$$h(f_{\lambda,r}) = \log(2\{\text{Root Of}\{P_k(\lambda, \frac{1}{2})\}\}). \quad (2.26)$$

The root in equation 2.26 is the maximal root of  $P_k(\lambda, \frac{1}{2})$ . For some examples of this see appendix material.

If the kneading sequence of  $f_{\lambda,r}$  is not periodic then the entropy may still be estimated using the periodic curves. Since the periodic constant kneading curves are dense, a sequence of these limits on any shift maximal non-periodic kneading sequence. These periodic constant kneading curves each have an entropy given by the above formula 2.26. The sequence of entropy values limits to the entropy value of the non-periodic curve. Although this is not an exact method, one can get reasonably close with moderate computation. There are a couple of drawbacks to this approximation approach. There is always some roundoff error and there is no way to decide what size of finite kneading sequence is needed to get within a certain amount unless both upper and lower approximations are made for which twice the work is required.

There are other ways to calculate the entropy of maps given by periodic, eventually periodic, and prefixed parameters. Symbolic dynamics may be used to find entropy. This method will be demonstrated in the example below and will be used as an introduction to finding methods of calculating entropy of non-classical iterated function systems.

**Example 2.1:** Consider the tent map  $T_\lambda$  with periodic parameter  $\lambda \approx .9196433776$  of period 4 with kneading sequence  $\mathbf{k} = (RLLC)^\infty$ . The orbit  $\mathcal{O}^+(\lambda)$  partitions the core  $\mathcal{I}$  into three subintervals:  $I_1 = [2\lambda(1 - \lambda), (2\lambda)^2(1 - \lambda)]$ ,  $I_2 = [(2\lambda)^2(1 - \lambda), \frac{1}{2}]$ , and  $I_3 = [\frac{1}{2}, \lambda]$ . The Markov shift of finite type is given by  $X_F$ , with

$$F = \{(11), (21), (22), (13)\},$$

the set of finite blocks or words which cannot occur or are *inadmissible*. Notice that the block  $(i j)$  is inadmissible if  $I_j \not\subseteq T_\lambda(I_i)$ . This is equivalent to saying that there is no arrow from  $I_i$  to  $I_j$  in the transition graph (see Figure 11). The entropy of  $X_F$  is given by

$$h(X_F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(b_n(X_F)), \quad (2.27)$$

where  $b_n(X_F)$  is the number of admissible blocks of length  $n$  for  $X_F$ . This definition may be used under certain circumstances but most of the time it is too difficult to find a formula for  $b_n(X_F)$ . However, the above information may be used to compute the entropy of the shift by forming the transition matrix,  $A_t$ , whose  $ij^{\text{th}}$  entry is given by

$$A_t(i, j) = \begin{cases} 1 & \text{if } T_\lambda(I_i) \supseteq I_j \\ 0 & \text{if } T_\lambda(I_i) \not\supseteq I_j. \end{cases}$$

Using Perron-Frobenius theory, the entropy of the shift given by the possible infinite words formed from the indices of paths on the graph is the topological entropy of the map, and is found by calculating the largest eigenvalue of  $A_t$ , as long as  $A_t$  irreducible.



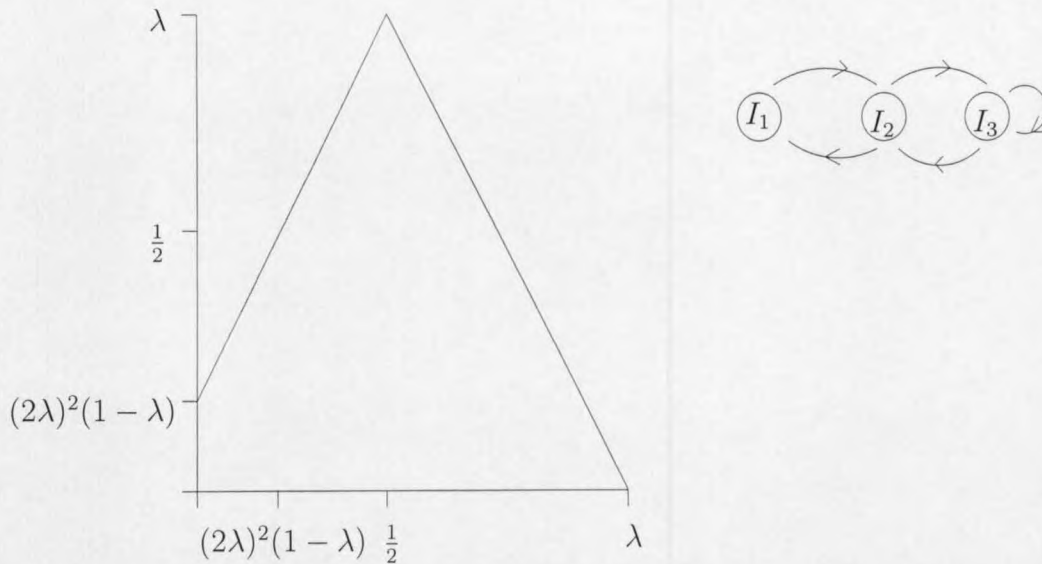


Figure 11: Period 4 Tent Map and Transition Graph

Recall that an  $n \times n$  matrix  $B$  is irreducible if for each  $1 \leq i, j \leq n$  there is some  $m(i, j) > 0$  such that  $B^m(ij) > 0$ . In this example

$$A_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

which has Perron eigenvalue  $2\lambda \approx 2(.9196433776)$  as expected.

## Measure Theory

**Theorem 2.32** *The only map  $F_{\lambda,r}$  from 2.7 that does not have an absolutely continuous invariant measure is the full Farey map  $F_{1,1}$ .*

Proof:

The family  $F_{\lambda,r} : [0, 1] \rightarrow [0, 1]$  has derivative  $|F'_{\lambda,r}| \geq 1$  everywhere except at  $x = \frac{1}{2}$ .

Equality occurs only at the endpoints 0 and 1 on  $F_{1,1}$ ;  $F'_{1,1}(1) = -1$  and  $F'_{1,1}(0) = 1$ .

Applying Theorem (Theorem 1 of [20]) for piecewise  $C^2$  functions we have that all the

maps except  $F_{1,1}$  have an absolutely continuous invariant measure on the core interval since  $|F'_{\lambda,r}| > 1$ , except at the critical point. The theorem is inconclusive for  $F_{1,1}$ , but it will be shown that it does not have an absolutely continuous invariant measure. For simplicity of notation for the remainder of the proof  $F$  will be denote  $F_{1,1}$ . To begin notice that  $F$  is a measurable nonsingular transformation. The term *Nonsingular* means that for any measurable set  $B$ ,  $m(F^{-1}(B)) = 0$  whenever  $m(B) = 0$  (Here  $m$  denotes Lebesgue measure). Define the Frobenius-Perron operator  $P_F : L_1 \rightarrow L_1$  by

$$P_F f(x) = \frac{d}{dx} \int_{F^{-1}([0,x])} f(s) ds.$$

There are two intervals given by  $F^{-1}([0, x])$ . Applying the Frobenius-Perron operator to both intervals we have

$$P_F f(x) = \frac{1}{(x+1)^2} f\left(\frac{x}{x+1}\right) + \frac{1}{(x+1)^2} f\left(\frac{1}{x+1}\right).$$

The first important step in the proof is to show that for  $f_0 \equiv 1$  the sequence  $g_n(x) = x f_n(x)$ , where  $f_n = P_F^n f_0$ , converges to a constant  $k_0$ . Since  $\frac{g_n(x)}{x} = f_n(x)$ ,

$$\begin{aligned} P_F\left(\frac{g_n(x)}{x}\right) &= \frac{1}{(x+1)^2} \frac{g_n\left(\frac{x}{x+1}\right)}{\frac{x}{x+1}} + \frac{1}{(x+1)^2} \frac{g_n\left(\frac{1}{x+1}\right)}{\frac{1}{x+1}} \\ &= \frac{1}{(x+1)} \frac{g_n\left(\frac{x}{x+1}\right)}{x} + \frac{1}{(x+1)^2} g_n\left(\frac{1}{x+1}\right). \end{aligned}$$

Multiplying through by  $x$  we get

$$x P_F\left(\frac{g_n}{x}\right) = \frac{1}{x+1} g_n\left(\frac{x}{x+1}\right) + \frac{x}{x+1} g_n\left(\frac{1}{x+1}\right).$$

Now since  $g_{n+1}(x) = x P_F\left(\frac{g_n}{x}\right)$  we have the recursive formula:

$$g_{n+1}(x) = \frac{1}{x+1} g_n\left(\frac{x}{x+1}\right) + \frac{x}{x+1} g_n\left(\frac{1}{x+1}\right), g_0 \equiv x. \quad (2.28)$$

Since  $g'_n(x) \geq 0$  for each  $n \in \mathbb{N}$  (by induction) all functions  $g_n$  are positive and

increasing. Thus according to equation 2.28,

$$\begin{aligned} g_{n+1}(1) &= \frac{1}{2}g_n\left(\frac{1}{2}\right) + \frac{1}{2}g_n\left(\frac{1}{2}\right) \\ &= g_n\left(\frac{1}{2}\right) \\ &\leq g_n(1). \end{aligned}$$

Since the  $g_n$  are non increasing we have  $\lim_{n \rightarrow \infty} g_n(1)$  exists. Therefore let  $\lim_{n \rightarrow \infty} g_n(1) \equiv k_o$ . Notice that  $\lim_{n \rightarrow \infty} g_n\left(\frac{1}{2}\right) = k_o$  also, since  $g_{n+1}(1) = g_n\left(\frac{1}{2}\right)$ . Now let  $z_0 = 1$  and  $z_{m+1} = \frac{z_m}{z_m+1}$ . Notice that  $z_m \leq z_{m-1}$ . Replacing  $x$  with  $z_m$  in 2.28 gives the equation

$$g_{n+1}(z_m) = \frac{1}{z_m+1}g_n(z_{m+1}) + \frac{z_m}{z_m+1}g_n(1-z_{m+1}). \quad (2.29)$$

Fix  $m$  and suppose the  $\lim_n g_n(x) = k_o$ , for  $z_m \leq x \leq 1$  (this is true at least for  $m = 0$ ). Since  $g_n$  is increasing equation 2.29 implies  $\lim_{n \rightarrow \infty} g_n(x) = k_o$ , for all  $x \in \left[\frac{1}{2}, 1\right]$  (since  $g_{n+1}(1) = g_n\left(\frac{1}{2}\right)$  and since  $g_n$  is increasing). we continue this process. Since

$$g_{n+1}\left(\frac{1}{2}\right) = \frac{2}{3}g_n\left(\frac{1}{3}\right) + \frac{1}{3}g_n\left(\frac{2}{3}\right), \quad (2.30)$$

we have  $\lim_{n \rightarrow \infty} g_n(x)$  exists for all  $x \in \left[\frac{1}{3}, 1\right]$ . Now  $1 - z_{m+1} \geq z_{m+1}$  for sufficiently large  $m$ , so  $\lim_n g_n(1 - z_{m+1}) = k_o$ . As  $n \rightarrow \infty$  in equation 2.29 it is seen that

$$k_o = \frac{1}{z_m+1} \lim_n g_n(z_{m+1}) + \frac{z_m}{z_m+1} k_o.$$

This forces  $\lim_n g_n(z_{m+1}) = k_o$ . Therefore since the  $g_n$  are increasing  $\lim_n g_n(x) = k_o$  uniformly for all  $x \in [z_{m+1}, 1]$ . Since  $\lim_m z_m = 0$  and since the above argument works for all  $m$  sufficiently large  $\lim_n g_n(x) = k_o$  for all  $x \in (0, 1]$ . This also implies that  $\lim_n f_n(x) = \frac{k_o}{x}$ .

The next step in the proof is to show that  $k_o = 0$ . To see this recall one of the properties of the Frobenius-Perron operator for  $f \in L_1$ :

$$\int_0^1 P_F f dm = \int_0^1 f dm.$$

This property implies that  $\|f_n\|_1 = 1$ , for all  $n \in \mathbb{N}$ , since  $\|f_o\|_1 = 1$ . Now suppose  $k_o \neq 0$ . Then  $\int_{\epsilon}^1 \frac{k_o}{x} dx = -k_o \ln \epsilon > 1$  for sufficiently small  $\epsilon > 0$ . But then  $\lim_n \int_{\epsilon}^1 f_n(x) dx = \int_{\epsilon}^1 \frac{k_o}{x} dx > 1$  for sufficiently small  $\epsilon > 0$  which is not possible since  $\int_{\epsilon}^1 f_n(x) dx < \|f_n\|_1 = 1$ . Thus  $k_o = 0$ .

The third step in the proof is to show that  $f_n \rightarrow 0$  uniformly. This can be achieved using induction to show that each of the functions  $f_n$  is decreasing. Thus  $f_n$  converges uniformly to 0 on any interval  $[\epsilon, 1]$  with  $\epsilon > 0$ . The last step is to show that the results in the above steps hold for an arbitrary sequence  $P_F^n f$  with  $f \in L_1$ . Define  $f^+ = \max\{0, f\}$  and  $f^- = \max\{0, -f\}$ . Given  $\epsilon > 0$  consider a constant  $c$  such that

$$\int_0^1 (f^- - c)^+ dm + \int_0^1 (f^+ - c)^+ dm \leq \epsilon.$$

Then

$$\begin{aligned} \int_{\epsilon}^1 |P_F^n f| dm &= \int_{\epsilon}^1 P_F^n f^+ dm + \int_{\epsilon}^1 P_F^n f^- dm \\ &\leq 2 \int_{\epsilon}^1 P_F^n c dm + \int_{\epsilon}^1 P_F^n (f^+ - c) dm + \int_{\epsilon}^1 P_F^n (f^- - c) dm \\ &\leq 2c \int_{\epsilon}^1 P_F^n 1 dm + \epsilon. \end{aligned}$$

Since  $P_F^n 1$  converges on  $[\epsilon, 1]$  uniformly to 0, then  $\lim_n \int_{\epsilon}^1 |P_F^n f| dm = 0$ . This implies that  $P_F^n f$  converges in measure to 0. Therefore  $P_F f = f$  has only  $f = 0$  as a solution. Thus there is no absolutely continuous non-trivial invariant measure.  $\square$

## CHAPTER 3

## Nonclassical Systems

In the second chapter the discussion centered around classical dynamical systems as related to the specific class of families in equation 2.7. Here the topic will be broadened to nonclassical systems. Although many i.f.s. are not classical dynamical systems, they are still of considerable interest. One such family is the Erdős family of maps with parameter  $\alpha \in (0, 1)$  given by

$$E_\alpha(x) = \begin{cases} \frac{x}{\alpha} & \text{for } 0 \leq x \leq \frac{\alpha^2}{1-\alpha} \\ \frac{x+\alpha-1}{\alpha} & \text{for } \alpha \leq x \leq \frac{\alpha}{1-\alpha} \end{cases} \quad (3.1)$$

With all the effort put into studying the family  $E_\alpha(x)$  there is still much that is not known. For instance, complete characterization of parameters  $\alpha$  for which  $E_\alpha(x)$  has an absolutely continuous invariant measure is still unknown. For more background information on this and related subjects see [11] and [15].

With some experimentation it becomes obvious that many of the techniques used in collecting information about classical dynamical systems no longer work for the majority of nonclassical families. Thus new techniques and ideas must be formulated in an effort to extract important information out of a nonclassical dynamical system. One tool that may still be used is symbolic dynamics and subshifts. Recall that a subshift of the one-sided full shift  $\Sigma_n$  on  $n$  symbols is a subset  $\sigma(F)$ , where  $F$  denotes a collection of forbidden blocks (or inadmissible subsequences) with elements in the set  $\{1, 2, 3, \dots, n\}$ . In an effort to use symbolic dynamics for a more extensive class of systems, a shift space  $X_I \subset \Sigma_2$ , similar in many aspect to the Markov shift of finite type (see Example 2.1), will be defined.

Consider a two branch map and define  $f$  and  $g$  to be the inverse branches of the i.f.s. Note that  $f$  and  $g$  are defined on  $\mathbf{R}$  (or perhaps subset of  $\mathbf{R}$  in some instances). Let  $W(g, f)$  be the set of all infinite sequences of  $f$ 's and  $g$ 's. For each finite subsequence,  $w^j$  (of length  $j$ ), of  $w \in W(g, f)$  we define an associated composition  $t^j$  given by composing  $f$  and  $g$  in the same order as they appear in  $w^j$ , that is if  $w^j = f^{n_1}g^{n_2}f^{n_3} \dots f^{n_k}$  then  $t^j = f^{n_1} \circ g^{n_2} \circ f^{n_3} \circ \dots \circ f^{n_k}$ .

It would be uninteresting to allow all of  $W(g, f)$  to be admissible in a shift space defined on the two symbols  $f$  and  $g$  (the shift space would be  $\Sigma_2$ ). Thus some rules of inadmissibility are in order.

**Definition 3.1** An infinite sequence,  $w = f^{j_1}g^{j_2}f^{j_3} \dots$ , with  $j_i \geq 0$ , is said to be *I-admissible*,  $I$  an interval, if for each  $w^j$  of  $w$ , the associated composition  $t^j$  has the property that  $m\{(t(I)) \cap I\} > 0$  ( $m$  denotes Lebesgue measure).

As far as we can ascertain, this is concept of *I-admissibility* is new.

For example, any of the family  $f_{\lambda,r}$  of Chapter 2 with kneading sequence  $\mathbf{k} = (RL^{n_1}R^{n_2} \dots C)^\infty$  and  $I = \mathcal{I}$  has an  $\mathcal{I}$ -admissible sequence  $g^\infty$ , while the sequence  $f^\infty$  is not  $\mathcal{I}$ -admissible (Here  $g$  is the inverse of  $f_r$  and  $f$  is the inverse of  $f_l$ ). This would imply that  $m\{g^n(\mathcal{I}) \cap \mathcal{I}\} \neq 0$  for any  $n \in \mathbf{N}$  and that  $m\{f^n(\mathcal{I}) \cap \mathcal{I}\} = 0$ , for some  $n \in \mathbf{Z}^+$ . While these facts may not be obvious at the moment they will become so later on. Notice that infinite words  $W(f, g)$  describe all possible random products.

We are now ready to define the subshift  $X_I$ .

**Definition 3.2** The *contact subshift*  $X_I = \{w \in W(g, f) : w \text{ is } I\text{-admissible}\}$ .

It may be verified, using the definition of subshift given in [21], that  $X_I$  is a subshift on two symbols. We will call  $I$  the contact interval. It should be obvious that *I-admissibility* depends upon  $I$  as well as the iterated function system from which  $f$  and  $g$  are taken. In this point of view, the two branches are regarded as primary

and the interval as secondary. This is somewhat similar to J. Kwapisz' view in [19], except that his treatment is for noninvertible matrices.

## The Contact Shift on Classical Systems

We begin the discussion with an example to show how entropy is calculated for contact shifts of finite type.

**Definition 3.3** A shift of *finite type* is one in which the minimal set,  $F$ , of forbidden or inadmissible blocks is finite.

We should also at this point introduce some ideas that will be used throughout the chapter.

**Definition 3.4** A block,  $w^j$ , that is not  $I$ -admissible is said to be *minimal* if every subblock of  $w^j$  is  $I$ -admissible.

**Example 3.1** Let  $f$  and  $g$  respectively be the left and right inverses of the tent map and reconsider the period 4 tent map  $(RLLC)^\infty$  regarded as a contact shift. This shift is of finite type with inadmissible block  $(fff)$ . Since all inadmissible blocks contain the subblock  $(fff)$  then the minimal set of forbidden blocks is  $F = \{(fff)\}$ . This shift is called *2-step* (or has *memory-2*) by [21]. It indicates the maximal number of positions (in any block) that must be considered in order to find which symbols may follow that block. For instance, the block  $gff$  may not be followed by  $f$  since  $fff$  is not  $\mathcal{I}$ -admissible (note:  $gfff$  is not  $\mathcal{I}$ -admissible since  $fff$  is not). The memory of a shift space also indicates the maximal length of blocks to be tested for admissibility (in this case three or one more than the memory). Thus, finite  $\mathcal{I}$ -admissible blocks of length three may be used to represent the *states* of the shift. These states are analogous to the subintervals of the Markov shift. In this example there are seven

Table 2: Contact States for the Period 4 Example

1	2	3	4	5	6	7
$(ffg)$	$(fgf)$	$(fgg)$	$(gff)$	$(gfg)$	$(ggf)$	$(ggg)$

states. For purposes of forming a transition graph the states will be labeled one through seven (see Table 2 and Figure 12).

To form the transition graph each state block is first appended by  $g$  and  $f$  to form blocks of length four. Next admissibility is checked. Finally, arrows are placed from the original state blocks to the states that match up with the last three places in the appended blocks if the appended block is  $\mathcal{I}$ -admissible. In other words arrows are placed between states if the last two elements in the block match up with the first two of another block and the resulting concatenation is admissible. For example, the state  $(ffg)$  matches up with  $(fgg)$  since  $(ffgg)$  is admissible, but does not match up with  $(gff)$ .

An *adjacency* matrix  $\mathcal{A}$ , similar to the Markov transition graph, may be formed by letting the  $ij^{\text{th}}$  entry

$$\mathcal{A}(i, j) = \begin{cases} 1 & \text{if there is an arrow from state } i \text{ to state } j \\ 0 & \text{otherwise} \end{cases}$$

For the period 4 tent map the contact transition matrix is

$$\mathcal{A} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$



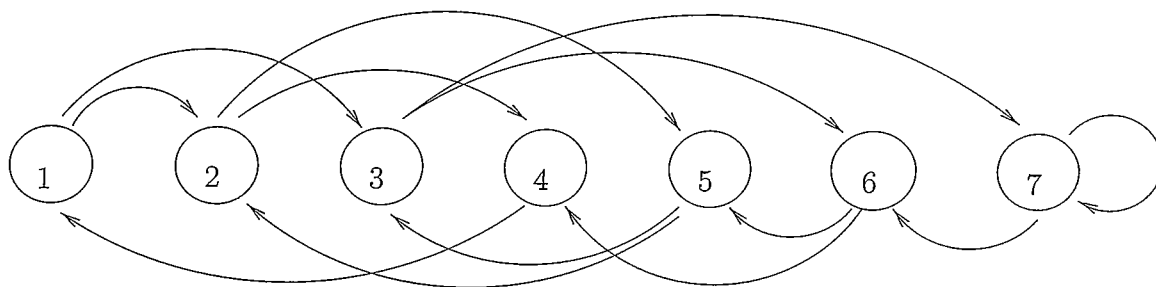


Figure 12: Period 4 Contact Shift Transition Graph

The entropy,  $h(X_{\mathcal{I}})$ , of the shift is found as in the Markov case. Again the Perron eigenvalue is  $2\lambda$ . The contact shift  $X_{\mathcal{I}}$  appears to be conjugate to a subset of  $\Sigma_7$ .

In the period 4 example it appears that the contact shift and the Markov shift are very different. The transition matrices are different, and the states in the graphs seem to be very different also. The entropy of both shifts, however, is the same. This does not necessarily mean that the two shifts are conjugate, but the following theorem guarantees it.

**Theorem 3.5** *Given a map  $f_{\lambda,r}$  (see equation 2.7) with core interval  $\mathcal{I}$  and periodic critical point then the Markov shift of finite type is conjugate to the contact shift  $X_{\mathcal{I}}$ .*

Before proving Theorem 3.5 there are some ideas that must be discussed. It will be assumed in the following lemmas that the kneading sequence of  $f_{\lambda,r}$  is of the general form

$$\mathbf{k}_{f_{\lambda,r}} = (RL^{n_1}R^{n_2}L^{n_3} \dots R^{n_k}C)^\infty.$$

An inadmissible block  $w^j$  is said to be minimal if all subblocks of  $w^j$  is admissible. In the interest of calculating entropy of shift spaces it is necessary to know what the set of minimal inadmissible blocks are. The following lemma shows a peculiarity of all minimal inadmissible blocks for contact shifts given by the maps  $f_{\lambda,r}$  of the family 2.7.

**Lemma 3.6** *Given  $f_{\lambda,r}$  on its core  $\mathcal{I}$  with a periodic parameter  $\lambda$ , all minimal inadmissible blocks of length  $j$  of the contact shift,  $X_{\mathcal{I}}$ , are given by compositions,  $t^j$  of  $f$  and  $g$ , such that every point of  $t^j(\mathcal{I})$  is less than or equal to the left endpoint of  $\mathcal{I}$ .*

Proof:

Given a minimal inadmissible block  $w^j(f, g)$  let  $t^j$  be the associated composition. Suppose every point of  $t^j(\mathcal{I})$  is larger than or equal to  $\lambda$ . For this to occur  $g$  must be applied last in the composition (or  $f$  must be applied to an interval far to the right of  $\mathcal{I}$  which would mean that the block is not minimal). Then every point of  $f_{\lambda,r} \circ t^j(\mathcal{I})$  is less than  $f_{\lambda,r}(\lambda)$ . This implies that the subblock  $w^{j-1}$  of length  $j-1$  formed by deleting the first element of  $w^j(f, g)$  is inadmissible since the composition form of  $w^{j-1}$ ,  $\tilde{t}^{j-1} = f_{\lambda,r}(t^j)$ , applied to  $\mathcal{I}$  has every point less than or equal to the left endpoint of  $\mathcal{I}$ .

□

Lemma 3.6 demonstrates that inadmissible sequences are those in which the associated composition 'loses contact' at the left side. This also means that no minimal inadmissible block begins with  $g$ .

There is a connection between the itineraries of the points in the core of  $f_{\lambda,r}$  (given by parameters in the region  $G$ ) and the admissible words in the contact shift. Each point  $q \in \mathcal{I}$  has a unique itinerary (see Lemma 2.8). The itinerary of  $q$  can be recoded by a sequence of  $f$ 's and  $g$ 's by replacing  $R$  with  $g$ ,  $L$  with  $f$ , and  $C$  with either  $f$  or  $g$ . This recoding is not one-to-one whenever  $C$  is in the itinerary of  $q$ . However, Lemma 3.7 shows that there is some choice in replacing the  $C$ 's such that the recoding is  $\mathcal{I}$ -admissible. Let  $w_q(f, g)$  represent the recoding of  $it(q)$ .

**Lemma 3.7** *Let  $f_{\lambda,r}$  be given by parameters in  $G$  and suppose  $q \in \mathcal{I}$ . Then there is a recoding  $w_q(f, g)$  of  $it(q)$  that is  $\mathcal{I}$ -admissible. Furthermore, no two points have the same associated admissible sequence.*

Proof:

It will first be shown that each point  $q \in \mathcal{I}$  has a sequence  $w_p(f, g)$  that is  $\mathcal{I}$ -admissible. The itinerary of  $q$  either contains  $C$ 's or does not depending on if  $\mathcal{O}^+(q) \supseteq \mathcal{O}^+(\lambda)$  or not. These two possibilities will be considered separately as follows:

Suppose that  $\mathcal{O}^+(q) \cap \mathcal{O}^+(\lambda) = \emptyset$ . Let  $it(q)$  be the itinerary of  $q$ . Notice that  $it(q)$  is a word in  $R$  and  $L$  only. Thus, consider the sequence  $w_q(f, g)$ , where  $f$  replaces  $L$  and  $g$  replaces  $R$ . This is the candidate for the corresponding admissible sequence. To check admissibility let  $w_q^j(f, g)$ ,  $j \in \mathbf{Z}^+$ , be the finite word given by the first  $j$  elements of  $w_q(f, g)$  and let  $t_q^j$  be the associated composition of inverse branches in the same order, that is, if  $w_q^2(f, g) = gf$  then  $t_q^2 = g \circ f$ . We will call the composition corresponding to the block its *associated composition*. Now  $U_j = t_q^j(\mathcal{I}) \cap \mathcal{I}$  is the closed interval of all points in  $\mathcal{I}$  that have itineraries that begin with the first  $j$  elements of the itinerary of  $q$ . Since  $q \in U_j$  and since  $q \neq \lambda$  and  $q \neq f_{\lambda, r}(\lambda)$ , the endpoints of the core interval, then  $q$  is in the interior of  $\mathcal{I}$ . Thus,  $U_j$  is not just a single point. If  $U_j$  were a single point this would force  $t_q^j(\mathcal{I}) = \{q\}$ , which is not possible. This implies that for all  $j$  the block  $w_q^j(f, g)$  is admissible. Therefore,  $w_q(f, g)$  is admissible.

Now suppose  $\mathcal{O}^+(q) \supseteq \mathcal{O}^+(\lambda)$ . Then  $q$  has itinerary of the form

$$it(q) = w^i(R, L)C(RL^{n_1}R^{n_2}L^{n_3} \dots R^{n_k}C)^\infty, \quad (3.2)$$

for some  $i \in \mathbf{N}$ . As above, form finite blocks  $w_q^j(f, g)$  by replacing  $R$  with  $g$ ,  $L$  by  $f$ , and  $C$  by  $f$  or  $g$ . The replacement codings are of the form

$$w^i(g, f)_g^f(gf^{n_1}g^{n_2}f^{n_3} \dots g^{n_k}f_g)^\infty. \quad (3.3)$$

Both  $f$  and  $g$  cannot always arbitrarily replace  $C$  due to inadmissibility problems. From Lemma 3.6 the inadmissible sequences of 3.3 are those in which some subblock

has an associated composition,  $t$ , such that every point in  $t(\mathcal{I}) \leq f_{\lambda,r}(\lambda)$ . Since

$$f^{n_1} \circ g^{n_2} \circ \dots \circ g^{n_k} \circ_g^f(\lambda) = f_{\lambda,r}(\lambda),$$

then if  $n_2 + n_4 + \dots + n_k$  is even

$$f^{n_1} \circ g^{n_2} \circ \dots \circ g^{n_k} \circ g(f_{\lambda,r}(\lambda)) > f_{\lambda,r}(\lambda)$$

and

$$f^{n_1} \circ g^{n_2} \circ \dots \circ g^{n_k} \circ f(f_{\lambda,r}(\lambda)) < f_{\lambda,r}(\lambda).$$

This implies that  $g f^{n_1} g^{n_2} f^{n_3} \dots g^{n_k} g$  is  $\mathcal{I}$ -admissible while  $g f^{n_1} g^{n_2} f^{n_3} \dots g^{n_k} f$  is not. Similarly, if  $n_2 + n_4 + \dots + n_k$  is odd then  $g f^{n_1} g^{n_2} f^{n_3} \dots g^{n_k} f$  is  $\mathcal{I}$ -admissible while  $g f^{n_1} g^{n_2} f^{n_3} \dots g^{n_k} g$  is not. This takes care of which choice of  $f$  or  $g$  is possible for the repeating portion of 3.3. Either  $f$  or  $g$  may be used after  $w^i(g, f)$  as long as  $w^i(g, f) \neq f^{n_1} g^{n_2} f^{n_3} \dots g^{n_k}$ . With the right choice of  $f$  or  $g$  as described, each  $U_j = t_q^j(\mathcal{I}) \cap \mathcal{I}$  is a nondegenerate interval. Hence,  $w_q(f, g)$  is admissible.

It will now be shown that if  $\tilde{q} \neq q$  then  $w_{\tilde{q}} \neq w_q$ . Assume by way of contradiction that the points  $\tilde{q} \neq q \in \mathcal{I}$  have the same associated  $\mathcal{I}$ -admissible sequence.  $w_q \in X_{\mathcal{I}}$ . By Lemma 2.8, the itineraries of  $\tilde{q}$  and  $q$  are different. This leads to a contradiction. Thus, at least one of the points must land in the orbit of the periodic parameter, since  $w_{\tilde{q}}$  and  $w_q$  would be different as they are direct replacements ( $g$  for  $R$  and  $f$  for  $L$ ). The other possibilities may also be ruled out by noticing that  $q \in t_q^j(\mathcal{I})$  and  $\tilde{q} \in t_{\tilde{q}}^j(\mathcal{I})$  for all  $j \in \mathbf{Z}^+$ . Then  $w_q = w_{\tilde{q}}$  implies that  $q = \tilde{q}$ .

□

The parity-lexicographic ordering on itineraries of points in the core may be used on associated  $\mathcal{I}$ -admissible words in  $X_{\mathcal{I}}$ . The ordering is given by using the rules outlined for  $R$  and  $L$  on  $g$  and  $f$  respectively.

**Lemma 3.8** *If  $p \neq q \in \mathcal{I}$  have parity-lexicographic ordering  $it(p) < it(q)$  then their associated admissible words  $w_p$  and  $w_q$  in  $X_{\mathcal{I}}$  have the same order, that is,  $w_p < w_q$ .*

Proof:

Let  $p \neq q \in \mathcal{I}$  and suppose  $it(p) < it(q)$  with the itineraries differing in the  $i^{th}$  position for some  $i \in \mathbf{Z}^+$ . Notice that the first  $i - 1$  positions of  $it(p)$  and  $it(q)$  do not contain a  $C$  since if that were the case then both itineraries would continue from that point with the kneading sequence  $k$ . Thus, they would never differ. There are two possibilities to consider. If  $C$  doesn't appear in the  $i^{th}$  position of either itineraries then the associated words in  $X_{\mathcal{I}}$  are direct replacements so  $w_p < w_q$ , using the rules  $R$  and  $L$  respectively for  $g$  and  $f$ . The other possibility is that one of the itineraries has  $C$  in its  $i^{th}$  position. Since the other cases are so similar we will consider only the case in which  $it(p)$  has  $C$  as its  $i^{th}$  component and an odd number of  $R$  in the first  $i - 1$  positions. This means that the  $i^{th}$  component of  $it(q)$  is  $f$ . Now in the corresponding exchange, as discussed in Lemma 3.7,  $C$  is replaced by either  $g$  or  $f$ . If  $f_{\lambda,r}^j(p)$  lands on the left endpoint for some  $j \in \mathbf{Z}^+$  with  $j < i$  then only  $g$  ensures admissibility. If  $C$  is replaced by  $g$ , then  $w_p < w_q$  as desired.

□

As we consider iterated function systems by extending the domain of each of the forward branches of tent maps, Lemma 3.8 no longer holds. There will be new admissible words that do not follow the parity-lexicographic ordering.

The idea of *state splitting* is crucial in the proof of Theorem 3.5. State splitting is a procedure for constructing new graphs from a given graph. Taking a partition of the *edges* (or arrows) of the graph, each state is split into a number of derived states. To describe the procedure in a precise manner, consider splitting a single state. Let  $H$  be a graph with  $\mathcal{V}$  the set of states and  $\mathcal{E}$  the set of edges. Fix a state  $V \in \mathcal{V}$  and let  $\mathcal{E}_V$  denote the set of edges in  $\mathcal{E}$  starting at  $V$ . Partition  $\mathcal{E}_V$  into two disjoint subsets  $\mathcal{E}_V^1$  and  $\mathcal{E}_V^2$  and construct a new graph  $\tilde{H}$ . The states of  $\tilde{H}$  are those of  $H$ , except  $V$  is replaced by two states  $V_1$  and  $V_2$ . For each edge  $E \in \mathcal{E}_V^i$ ,  $i=1$  or  $2$ , put

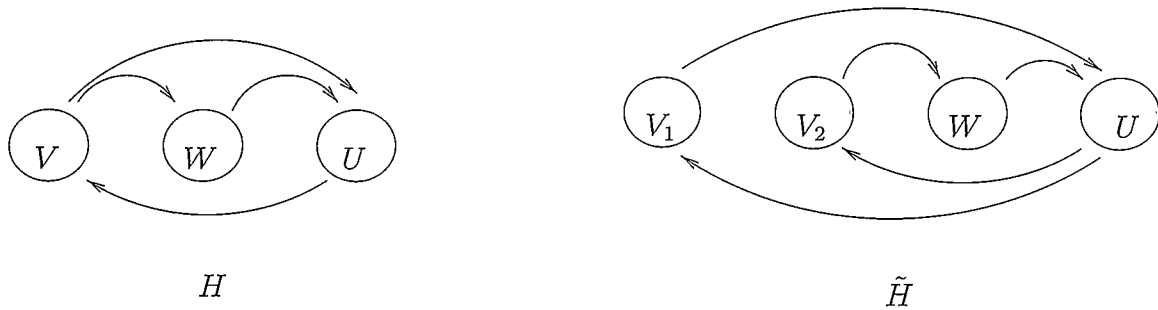


Figure 13: Graph Splitting

an edge from  $V_i$  to where it went in  $H$  (see Figure 12). For all edges in  $H$  ending at  $V$  place an edge in  $\tilde{H}$  from the original state to both  $V_1$  and  $V_2$ . All other edges in  $\tilde{H}$  should be placed as the edges were in  $H$ . In common terms the outgoing edges are split (they leave from either  $V_1$  or  $V_2$ ) and incoming edges are copied (go to both  $V_1$  and  $V_2$ ). While graphs given by state splittings look very different from the original graph they are not. This important property of splittings is formalized in Lemma 3.9. If  $H$  is a transition graph then  $X_H$  is the shift given by paths on the graph.

**Lemma 3.9** *If a graph  $\tilde{H}$  is a splitting of a graph  $H$  then the associated shifts  $X_H$  and  $X_{\tilde{H}}$  are conjugate.*

See [21], p.54 for a proof of Lemma 3.9.

Lemma 3.6 combined with the results of Lemmas 3.7, 3.8, and 3.9 give a process for finding the inadmissible set  $F$  for the contact shifts of the family 2.7 with periodic parameter  $\lambda$ . Simply consider the shift  $\sigma(it(\lambda))$ , which is the smallest itinerary of points on the core of  $f_{\lambda,r}$ . Find all itineraries of smaller points (one need only consider  $n - 1$  positions or less) and take the replacement sequences out to the position that determines lexicographic-parity size. These give the set  $F$ . For instance, consider the kneading sequence  $(RLLRC)^\infty$ . The shift is  $\sigma(RLLRC)^\infty = (LLRCR)^\infty$ . Smaller itineraries are ones that begin with  $LLL$  or  $LLRR$ . Thus, the inadmissible set for

Table 3: Some Kneading Sequences of Periodic Parameters and Associated Inadmissible Blocks for Contact Shift  $X_{\mathcal{I}}$

Period	$\mathbf{k}_f$	Inadmissible Set $F$
3	<i>RLC</i>	$\{(ff)\}$
4	<i>RLLC</i>	$\{(fff)\}$
5	<i>RLLLC</i>	$\{(ffff)\}$
5	<i>RLLRC</i>	$\{(fff), (ffgg)\}$
5	<i>RLRRC</i>	$\{(ff), (fggf)\}$
6	<i>RLLLLC</i>	$\{(fffff)\}$
6	<i>RLLLLRC</i>	$\{(ffff), (ffffgg)\}$
6	<i>RLLRRC</i>	$\{(fff), (ffggf)\}$
7	<i>RLLLLLC</i>	$\{(ffffff)\}$
7	<i>RLLLLRC</i>	$\{(ffffff), (ffffffgg)\}$
7	<i>RLLLRRC</i>	$\{(ffff), (ffffggf)\}$
7	<i>RLLRLC</i>	$\{(ffff), (ffffgg), (ffffgfg)\}$
7	<i>RLLRRLC</i>	$\{(fff), (ffggff)\}$
7	<i>RLLRRRC</i>	$\{(fff), (ffggf), (ffgggg)\}$
7	<i>RLLRLRC</i>	$\{(fff), (ffgg), (ffgfgf)\}$
7	<i>RLRRLC</i>	$\{(ff), (fggfgg)\}$
7	<i>RLRRRC</i>	$\{(ff), (fggf), (fggggf)\}$

the contact shift is  $\{(fff), (ffgg)\}$  (See Table 3 for more examples). One can see that the contact shift for this example has memory-3. This notion of memory size generalizes to all maps with periodic parameters in 2.7. If the parameter has period  $m$ , the contact shift has memory- $m$ .

Proof of Theorem 3.5:

We will show that the transition graph of the contact shift is a splitting of the transition graph of the Markov shift of finite type. Let  $G$  be the transition graph of the Markov shift of finite type, and let  $H$  be the transition graph of the contact shift. Let  $I_m$ , for  $m = 1, 2, 3, \dots, n-1$  be the states of the Markov shift of finite type. Find the set of possible itineraries for points in the interior of  $I_m$  for each  $m$ . Since the endpoints of  $I_m$  are periodic one need only consider  $n-1$  places. After the initial

$n - 1$  places  $R$ ,  $L$ , or  $C$  are all possibilities. Notice that in some of the initial places in the itineraries of  $I_m$ , more than one of  $R$ ,  $L$ , or  $C$  are possible. These are the states which will be split.

For each  $I_m$ , excluding endpoints, find the first position ( $i^{th}$ ) where  $R$  or  $L$  is possible. Split  $I_m$  into two subsets,  $I_{m_1}$ , having  $L$  in the  $i^{th}$  position, and  $I_{m_2}$ , having  $R$  in the  $i^{th}$  position. By the parity-lexicographic ordering on itineraries  $I_{m_1}$  and  $I_{m_2}$  are intervals having one point  $x$  in common. The point  $x$  has  $C$  in the  $i^{th}$  position of its itinerary. Continue the splitting process on  $I_{m_1}$  and  $I_{m_2}$  until  $I_m$  has been subdivided into intervals with the interiors of each having the same itinerary up to and including the  $n - 1^{th}$  position. By Lemmas 3.7 and 3.8 each of the new subinterval states of the Markov shift match up one to one ( $R$  for  $g$  and  $L$  for  $f$ ) with the contact states. The arrows in the splitting of  $G$  are found by taking the shift of the itinerary of each subinterval. Arrows are applied from each subinterval state go to each subinterval state whose first  $n - 2$  terms match the last  $n - 2$  terms. This is precisely the criteria for adjacency of contact states. Thus, the splitting of  $G$  is the same as  $H$ . By Lemma 3.9 the contact shift and the Markov shift of finite type are conjugate.

□

**Remark:** Another way to think of the contact shift is to find the partition given by the set of  $f_{\lambda,r}^{-(n-1)}(\{c\})$ . The contact shift is the Markov shift of this partition.

**Example 3.2:** To demonstrate the application of Theorem 3.5 consider the period 5 map  $f_{\lambda,r}$  with kneading sequence  $(RLLRC)^\infty$ . There are four Markov states:  $I_1 = [f_{\lambda,r}(\lambda), f_{\lambda,r}^2(\lambda)]$ ,  $I_2 = [f_{\lambda,r}^2(\lambda), \frac{1}{2}]$ ,  $I_3 = [\frac{1}{2}, f_{\lambda,r}^3(\lambda)]$ , and  $I_4 = [f_{\lambda,r}^3(\lambda), \lambda]$ . The state  $I_1 \mapsto I_2 \cup I_3$ ,  $I_2 \mapsto I_4$ ,  $I_3 \mapsto I_3 \cup I_4$ , and  $I_4 \mapsto I_1 \cup I_2$ . The transition matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$



The itinerary coding for the interiors of  $I_1$ ,  $I_2$ ,  $I_3$ , and  $I_4$ , and the associated contact states, are given in Table 4. The state  $I_1$  is split into two intervals or new states,  $I_{1a}$  corresponding to the interval whose interior has the possible itineraries  $LLRR\dots$  and  $I_{1b}$  corresponding to  $LRRR\dots$ . Before  $I_1$  was split it mapped onto  $I_2$  and  $I_3$ . Since the shift  $\sigma(LLR\dots) = LR\dots$  and  $\sigma(LRR\dots) = RR\dots$ , then  $I_{1a} \mapsto I_2$  and  $I_{1b} \mapsto I_3$ . Similarly, states  $I_3$  and  $I_4$  are each split into two states.

While Markov shifts of finite type given by maps with periodic parameters in 2.7 and contact shifts of those maps on their core interval are conjugate, the same cannot be said for all of the maps in 2.7, not even all those that have Markov shifts of finite type.

**Example 3.3** Consider the family of maps on the transitive boundary. The kneading sequence for these is  $RLR^\infty$  since the orbit  $\mathcal{O}^+(\lambda)$  lands on the fixed point,  $x_f$ , between  $\frac{1}{2}$  and  $\lambda$ . The family is termed *post critically finite* since  $\mathcal{O}^+(\lambda)$  falls into a periodic orbit. The Markov states are  $I_1 = [f_{\lambda,r}(\lambda), \frac{1}{2}]$ ,  $I_2 = [\frac{1}{2}, x_f]$ , and  $I_3 = [x_f, \lambda]$ . Here  $I_1 \mapsto I_3$ ,  $I_2 \mapsto I_3$ , and  $I_3 \mapsto I_1 \cup I_2$ . The transition matrix is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Using the indices of the states, one can see that there are five blocks forming the inadmissible sequences for the Markov shift of finite type, (11), (22), (12), (21), and (33). The contact shift is not of finite type but is *strictly sofic*, that is, all admissible words come from infinite paths on a graph, but the minimal inadmissible set  $F$  is not finite. Blocks of the form  $(fg^{2^j}f)$ ,  $j \in \mathbb{N}$  form the set of all inadmissible sequences for  $X_{\mathcal{I}}$ . Since shifts of finite type cannot be conjugate to strictly sofic shifts (see Thm.2.1.10, p.31, [21]), the Markov shift given by the forward orbit and the contact shift are not conjugate. To understand why the contact shift is not a splitting of the Markov shift of finite type follow the process of Theorem 3.5 (see Table 5 for itinerary

Table 4: Itinerary codings for  $(RLLRC)^\infty$ —(S means R, L, or C is possible)

State	Itineraries	Contact States
$I_1$	$LSRS \dots$	$\{(ffg), (fgg)\}$
$I_2$	$LRLS \dots$	$\{(fgf)\}$
$I_3$	$RRS \dots$	$\{(ggf), (ggg)\}$
$I_4$	$RLS \dots$	$\{(gff), (gfg)\}$

Table 5: Itinerary codings for  $RLR^\infty$ —(S means R, L, or C is possible)

State	Itineraries	Contact States
$I_1$	$LRSRSR \dots$	$\{(fg_f^g g_f^g g_f^g \dots)\}$
$I_2$	$RRSRSR \dots$	$\{(gg_f^g g_f^g \dots)\}$
$I_3$	$RSRSRS \dots$	$\{(g_f^g g_f^g \dots)\}$

information). In a sense, each Markov state would need to split a countably infinite number of times.

The following definitions concern transition matrices.

**Definition 3.10** A matrix  $A$  is *primitive* if it is irreducible and aperiodic.

For transition and adjacency matrices (that are nonnegative) primitivity is equivalent to the property that  $A^m > 0$  for all sufficiently large  $m$ .

**Definition 3.11** A matrix  $A$  is *eventually positive* if there is some  $n \in \mathbb{N}$  such that  $A^n > 0$ .

The question arises regarding the maps with eventually periodic parameters that have eventually positive transition matrices. In this case are the Markov shift of finite type and the contact shift conjugate? The answer must be an emphatic no, as the following example shows.

**Example 3.4** Consider the tent map  $T_\lambda$  with kneading sequence  $\mathbf{k}_{T_\lambda} = RL(LR)^\infty$ . In this example  $\lambda \approx .8478103848$  and  $\frac{1}{2} \rightarrow \lambda \rightarrow a \rightarrow b \rightarrow c \rightarrow b \dots$  under iterations of  $T_\lambda$ , where  $a \approx .25806$ ,  $b \approx .43756$ , and  $c \approx .74194$ . Using these values to partition the core interval we have  $I_1 = [a, b]$ ,  $I_2 = [b, \frac{1}{2}]$ ,  $I_3 = [\frac{1}{2}, c]$ , and  $I_4 = [c, \lambda]$ . Note that  $I_1 \rightarrow I_2 \cup I_3$ ,  $I_2 \rightarrow I_4$ ,  $I_3 \rightarrow I_2 \cup I_3 \cup I_4$ , and  $I_4 \rightarrow I_1$ . The transition matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

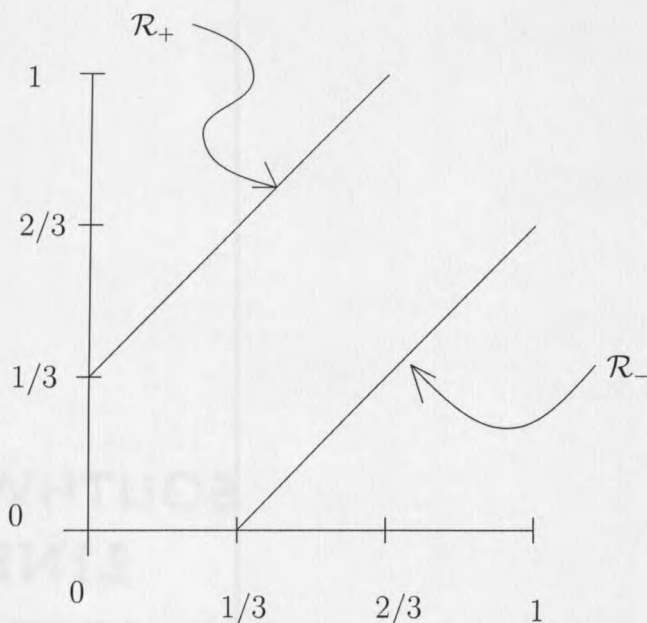
Since  $A^5 > 0$  then  $A$  is irreducible. This also implies that  $A$  is aperiodic. Thus,  $A$  is primitive. The inadmissible blocks for the Markov shift (using subscripts) are: (1 1), (1 4), (2 1), (2 2), (2 3), (3 1), (4 2), and (4 3). On the other hand, the inadmissible blocks for the contact shift contain the set  $\{(fff), (ff(gf)^{2n}), (ff(gf)^{2n}gg)\}$  since  $(RLLRC)^\infty > (RLLRLRC)^\infty > (RLLRLRLRC)^\infty \dots > (RLLR(LR)^nC)^\infty > \dots > (RL(LR)^\infty)$  in the lexicographic parity ordering. Thus, the contact shift is strictly sofic and cannot be conjugate to a shift of finite type.

### Contact Shifts of Iterated Function Systems

Thus far the contact shift has only been applied to maps from 2.7 on core intervals, which have all been classical dynamical systems. Such limitations are not necessary. The contact shift may be applied to many nonclassical settings. The beauty of the contact shift is that, unlike the set up in Chapter 2, domains do not need to be restricted so that a continuous map occurs, and overlapping domains cause no problems either.

Consider the following example. Let

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix}.$$

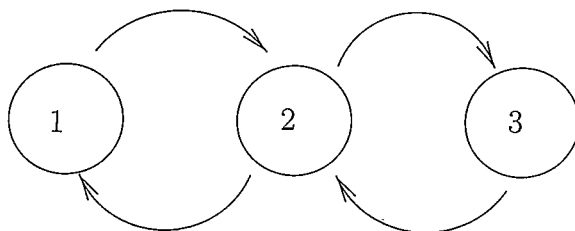
Figure 14: The Relation  $\mathcal{R}$ 

The inverses in linear fractional form are  $x_{-}^{\pm \frac{1}{3}}$ . Let  $I=[0,1]$  and restrict each domain of  $x_{-}^{\pm \frac{1}{3}}$  to the subinterval of  $I$  that remains in  $I$  after one iterate. This gives the relation

$$\mathcal{R} = \begin{cases} x + \frac{1}{3} & \text{for } 0 \leq x \leq \frac{2}{3} \\ x - \frac{1}{3} & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases} \quad (3.4)$$

For convenience let  $\mathcal{R}_- = x - \frac{1}{3}$  and  $\mathcal{R}_+ = x + \frac{1}{3}$  (see Figure 14).

The contact shift for this relation is what is known as a charge constrained shift since admissible blocks,  $w^j(\mathcal{R}_+, \mathcal{R}_-)$ , are those in which number of  $\mathcal{R}_+$  minus the number of  $\mathcal{R}_-$  in  $w^j(\mathcal{R}_+, \mathcal{R}_-)$  is in the set  $\{0, \pm 1, \pm 2\}$ . Notice that any admissible block  $w^j$  gives a composition  $t_w^j$  of  $\mathcal{R}_+$  and  $\mathcal{R}_-$  with  $t_w^j(I) \cap I$  being one of the intervals  $\{[0, \frac{1}{3}], [\frac{1}{3}, 1], [0, \frac{2}{3}], [\frac{2}{3}, 1]\}$ . If  $t_w^j(I) \cap I$  is the first interval in the set then  $w_j \mathcal{R}_+$  is admissible while  $w_j \mathcal{R}_-$  is not, and if  $t_w^j(I) \cap I$  is the last then  $w_j \mathcal{R}_-$  is admissible while  $w_j \mathcal{R}_+$  is not. Similarly if  $t_w^j(I) \cap I$  is one of the middle two intervals in the set then both  $w^j \mathcal{R}_{\pm}$  are admissible. Thus, admissible blocks may be lumped into three

Figure 15: Transition Graph for the Relation  $\mathcal{R}$ 

states with transition graph as in Figure 15, and transition matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, the entropy  $h(X_I)$  of the contact shift  $X_I$  for the relation  $\mathcal{R}$  is  $\log \sqrt{2}$ . (see section 4.4 of [21] for justification of the log of the perron eigenvalue being the entropy of the contact shift).

### Extending the Contact Interval

To introduce the next topic consider the relation  $\mathcal{R}$  and extend  $I$  to  $I_\epsilon = [-\epsilon, 1 + \epsilon]$ , for small  $\epsilon > 0$ . It can readily be seen that the new transition graph for  $X_{I_\epsilon}$  has five states and transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the entropy  $h(X_{I_\epsilon})$  is  $\log \sqrt{3}$ . Thus, as the interval is extended continuously the entropy for the contact shift does not always change continuously. The entropy remains constant as  $I_\epsilon$  varies until  $\epsilon$  is large enough. Then the entropy jumps to a different level. Each jump is given by a Toeplitz adjacency matrix of odd size of the same form as in the  $3 \times 3$  and  $5 \times 5$  examples. Table 6 gives approximate values of

Table 6: Summary of Information for Entropy of Contact Shifts of Extensions of  $I=[0,1]$  of the Relation  $R$

Matrix Size	Range of $\epsilon$	Approx. Growth Rate	$h(X_{I_\epsilon})$
1x1	$-\frac{1}{3} \leq \epsilon < 0$	1	0
3x3	$0 \leq \epsilon < \frac{1}{3}$	1.4142136	.3465736
5x5	$\frac{1}{3} \leq \epsilon < \frac{2}{3}$	1.732050808	.5493061
7x7	$\frac{2}{3} \leq \epsilon < 1$	1.847759065	.6139735
9x9	$1 \leq \epsilon < \frac{4}{3}$	1.902113033	.6429653
11x11	$\frac{4}{3} \leq \epsilon < \frac{5}{3}$	1.931851653	.6584789
13x13	$\frac{5}{3} \leq \epsilon < 2$	1.949855824	.6677554
15x15	$2 \leq \epsilon < \frac{7}{3}$	1.961570562	.6737456
25x25	$4 \leq \epsilon < \frac{13}{3}$	1.985417749	.6858293
35x35	$6 \leq \epsilon < \frac{19}{3}$	1.992389398	.6893345
45x45	$8 \leq \epsilon < \frac{25}{3}$	1.995337541	.6908132

same form as in the  $3 \times 3$  and  $5 \times 5$  examples. Table 6 gives approximate values of growth rates and entropy of the shift  $x_{I_\epsilon}$  for different ranges of  $\epsilon$ . Figure 16 gives a graphical representation of some the data listed. Notice that as  $\epsilon \rightarrow \infty$  the growth rate approaches 2, the maximum for a shift on two symbols.

Now reconsider the family of tent maps on the core interval, and suppose the core is extended continuously. For simplicity in presentation only the tent family will be considered, but similar arguments apply to much of the family 2.7. This next lemma applies to all iterated function systems.

**Lemma 3.12** *If  $I \subset J$  then  $h(X_I) \leq h(X_J)$ .*

Proof:

Consider the map  $T_\lambda$  with inverse branches  $f = \frac{x}{2\lambda}$  and  $g = 1 - \frac{x}{2\lambda}$ . Let  $I$  be an interval and suppose  $I \subset J$ . If the block  $w^j$  is admissible for  $X_I$ , then  $w^j$  is also admissible for  $X_J$ , since  $t_w^j(I) \cap I \neq \emptyset$  implies  $t_w^j(J) \cap I \neq \emptyset$ . Thus,  $t_w^j(J) \cap J \neq \emptyset$ . Now let  $b_j$  be the number of admissible blocks of length  $j$  for  $X_J$  and let  $a_j$  be the

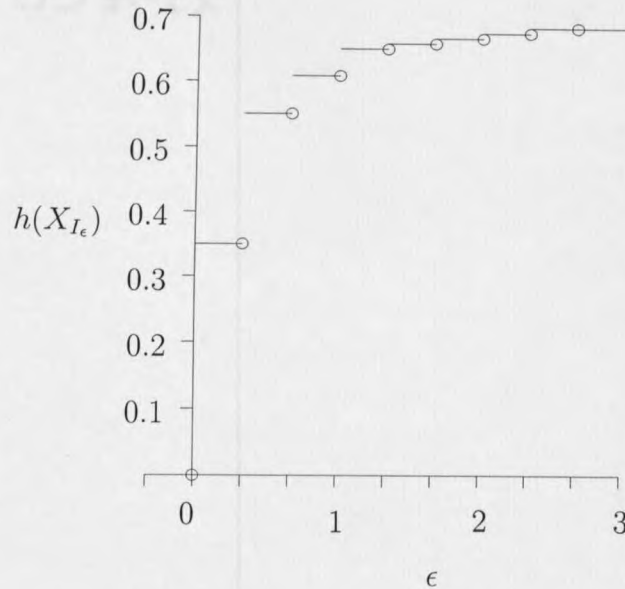


Figure 16: Entropy vs.  $\epsilon$  for Extensions of the Contact Shift on the Relation  $\mathcal{R}$

number of admissible blocks of length  $j$  for  $X_I$ . Since  $a_j \leq b_j$ , for all  $j \in \mathbf{N}$  then

$$\begin{aligned}
 h(X_I) &= \lim_{j \rightarrow \infty} \frac{1}{j} \log(a_j) \\
 &\leq \lim_{j \rightarrow \infty} \frac{1}{j} \log(b_j) \\
 &= h(X_J).
 \end{aligned}$$

□

There are an infinite number of different ways of expanding an interval so that the changes in the resulting contact shift may be studied. Most of the time, especially for extensions of the tent family to iterated function systems, the interval of interest,  $J = [a, b]$ , will be extended to  $J_\delta = [T_\lambda(b + \delta), b + \delta]$ .

**Definition 3.13** Extending or contracting the interval of interest of an iterated function system is called a *nonclassical deformation*.

With this extension of the branches to  $J_\delta$ , Lemma 3.6 still holds. In many ways this seems to be the most natural way of extending the interval. The following lemma shows why this might be.

**Lemma 3.14** *If the core  $\mathcal{I}$  of  $T_\lambda$  is expanded on the left only to  $J = [T_\lambda(\lambda) - \epsilon, \lambda]$  and  $T_\lambda(\lambda) - \epsilon \neq 0$  then  $h(X_{\mathcal{I}_\epsilon}) = h(X_{\mathcal{I}})$ .*

Proof:

Suppose the core interval  $\mathcal{I} = [2\lambda(1 - \lambda), \lambda]$ , for the map  $T_\lambda$ , is extended only on the left to  $J = [2\lambda(1 - \lambda) - \epsilon, \lambda]$ ,  $\epsilon > 0$ . Suppose also that there is some block  $w^j$  such that  $t_w^j(J) \cap J \neq \emptyset$  but  $t_w^j(J) \cap \mathcal{I} = \emptyset$ . This can only occur by applying  $f$  a certain number of times last in  $t_w^j$ . It is clear that  $gw^j$  is not  $J$ -admissible. Notice that for some  $m \in \mathbf{Z}^+$ ,  $f^m w^j$  is not admissible. For extensions where  $2\lambda(1 - \lambda) - \epsilon \geq 1 - \lambda$ , then  $f w^j$  is not admissible. In any case, once some composition  $t_w^j(\mathcal{I}_\epsilon)$  is contained in the extension  $[2\lambda(1 - \lambda) - \epsilon, 2\lambda(1 - \lambda)]$  neither  $f(t_w^j)(J)$  or  $g(t_w^j)(\mathcal{I}_\epsilon)$  intersects  $[2\lambda(1 - \lambda) - \epsilon, \lambda]$  except at perhaps one point. Thus, one may consider the points in the extension as a state which has transitions out of the state but not back into. The interval  $[2\lambda(1 - \lambda) - \epsilon, 2\lambda(1 - \lambda)]$  consists of points that under a finite number,  $m$ , of forward iterations are in the core interval. Therefore, the number of admissible blocks  $b_{j+m}$  of length  $m + j$  of the shift  $X_J$  is the same as  $m$  times the number of admissible blocks  $a_j$  of length  $j$  of the shift  $X_{\mathcal{I}}$ . Thus,

$$\begin{aligned} h(X_{\mathcal{I}}) &= \lim_{j \rightarrow \infty} \frac{1}{j} \log(a_j) \\ &= \lim_{j \rightarrow \infty} \frac{1}{j} \log(m a_j) \\ &= \lim_{j \rightarrow \infty} \frac{1}{j + m} \log(b_{j+m}) \\ &= h(X_{\mathcal{I}_\epsilon}). \end{aligned}$$

□



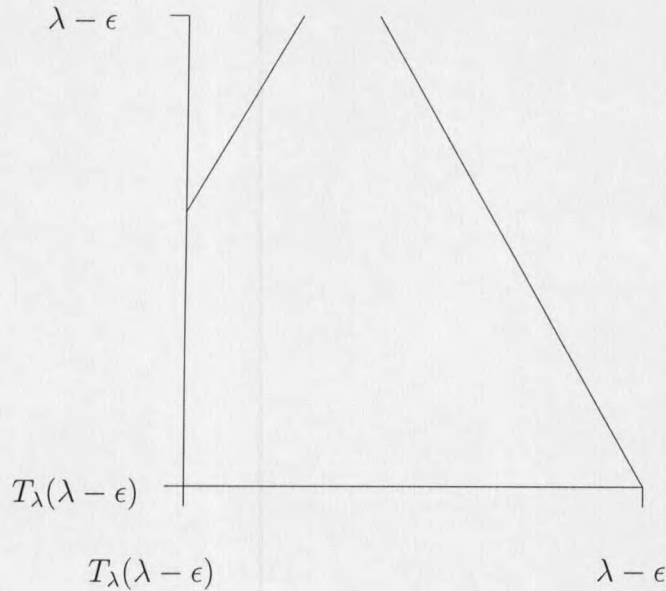


Figure 17: Restricted Tent Map

The argument of Lemma 3.14 above may be used for more than just the core interval. For an iterated function system given by the tent family on the interval  $J = [g^{-1}(b), b]$ ,  $b = \lambda + \epsilon$ , the same thing occurs. If  $J$  is extended only on the left then any new blocks,  $w^j$ , (those whose associated composition,  $t^j$ , applied to the extended interval intersect only in the extension) lead to no extra entropy since  $f^m w^j$  is inadmissible for some  $m \in \mathbf{Z}^+$  and  $g f^m w^j$  is inadmissible for all  $m \in \mathbf{N}$ . Thus, we have

**Theorem 3.15** *For an iterated function system (given by extending the forward branches of a tent map) defined on  $J = [g^{-1}(b), b]$ , if  $J$  is extended only on the left to  $K = [g^{-1}(b) - \epsilon, b]$  then  $h(X_K) = h(X_J)$ .*

It should be noticed that Lemma 3.14 and Theorem 3.15 apply to all of the family 2.7 in which the forward branches are expansive on  $[0, 1]$ . However, for simplicity of presentation we will only consider the tent family. We will assume in the

rest of the discussion that all extensions of the contact interval  $I = [a, b]$  of an i.f.s. will be of the form  $I_\epsilon = [g^{-1}(b + \epsilon), b + \epsilon]$ .

There is an important principle that should be understood about the entropy of these iterated function systems (given by extensions of the family 2.7) as the contact interval is non classically deformed to  $I = [0, 1]$ .

**Theorem 3.16** *The contact entropy  $h(x_J) \leq \log 2$  for all  $J \subseteq [0, 1]$ . Furthermore, when  $J = [0, 1]$ ,  $h(x_J) = \log 2$ , the maximum.*

Proof:

From Lemma 3.12 we have that  $h(x_J) \leq h(x_K)$  whenever  $J \subseteq K$ . Since  $f_r(1) = f_l(0) = 0$ , and 0 is a fixed point, any composition  $t$  of the inverses of  $f_r(x)$  and  $f_l(x)$  have the property that  $m(t(I) \cap I) \neq 0$ : Thus, all blocks of  $X_{[0,1]}$  are possible and  $h(X_{[0,1]}) = \log 2$ .

□

Theorem 3.16 raises an interesting question. As the contact intervals are continuously deformed does that mean that the contact entropy also changes continuously? The following is an investigation into that question.

**Theorem 3.17** *Given a continuous deformation  $\mathcal{I}_\epsilon = [g^{-1}(\lambda + \epsilon), \lambda + \epsilon]$ ,  $\epsilon > 0$ , of the core interval  $\mathcal{I}$  of  $T_\lambda$ , with contact shift  $X_{\mathcal{I}_\epsilon}$ , and periodic parameter  $\lambda$ , then  $\mathcal{I}_\epsilon \mapsto h(X_{\mathcal{I}_\epsilon})$  is continuous at  $\mathcal{I}$ .*

Proof:

We first show that the entropy is continuous from the left. Let  $\epsilon > 0$ . We constrict  $\mathcal{I}$  to a subinterval  $\mathcal{I}_{-\epsilon} = [g^{-1}(\lambda - \epsilon), \lambda - \epsilon]$  (see Figure 17) and consider  $\lim_{\epsilon \rightarrow 0} h(X_{\mathcal{I}_{-\epsilon}})$ . Theorem 1 of [26] shows that the limit is  $h(X_{\mathcal{I}})$ . The proof of continuity from the right involves expanding the core interval to  $\mathcal{I}_\epsilon = [T_\lambda(\lambda + \epsilon), \lambda + \epsilon]$ ,  $\epsilon > 0$ , and then

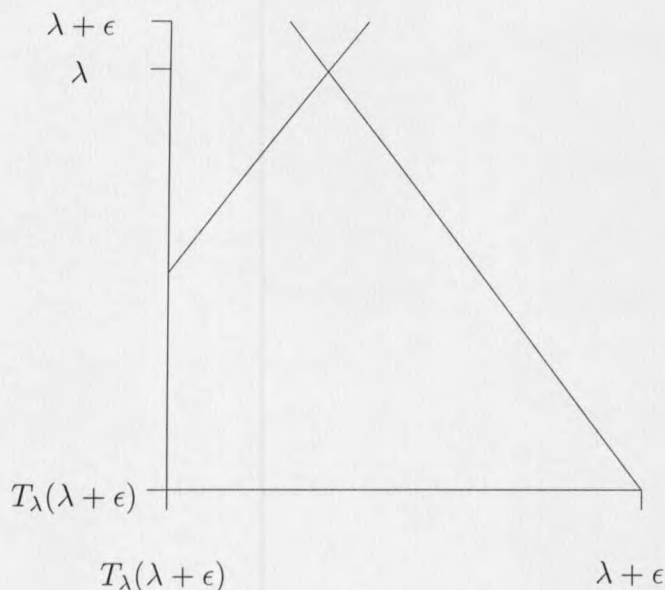


Figure 18: Extended Tent Map I.F.S.

showing that  $\lim_{\epsilon \rightarrow 0^+} h(X_{\mathcal{I}_\epsilon}) = h(X_{\mathcal{I}})$ . In this case the domain of the left and right branches is extended so that there is some overlap of definition (see Figure 18).

The idea for the proof is to create a nested sequence of shift spaces  $X_{I_1} \supseteq X_{I_2} \supseteq \dots$  with the intersection  $\bigcap_{i=1}^{\infty} X_{I_i} = Y$ , show that  $\lim_{i \rightarrow \infty} h(X_{I_{-i}}) = h(Y)$ , and then show that  $h(Y) = h(X_{\mathcal{I}})$ . Consider a sequence  $\{\epsilon_i\}_{i=1}^{\infty}$  of positive values with  $\epsilon_i > \epsilon_{i+1}$  for each  $i \in \mathbf{Z}^+$ . Suppose that  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ . Define  $X_{\mathcal{I}_i} = X_{\mathcal{I}_{\epsilon_i}}$ , where  $\mathcal{I}_{\epsilon_i} = [T_\lambda(\lambda + \epsilon_i), \lambda + \epsilon_i]$  (here  $T_\lambda(x) = 2\lambda(1 - x)$ ). Since  $\mathcal{I}_i \supset \mathcal{I}_{i+1}$  for every  $i \in \mathbf{Z}^+$ , then  $X_{\mathcal{I}_i} \supseteq X_{\mathcal{I}_{i+1}}$ . This defines the desired sequence.

The next objective is to show that  $\lim_{i \rightarrow \infty} h(X_{I_i}) = h(Y)$ . Begin by fixing  $\epsilon > 0$ . Since  $Y \subseteq X_{I_k}$ , then  $h(Y) \leq h(X_{I_k})$  for all  $k \in \mathbf{Z}^+$ . Define  $b_n(Y)$  to be the number of admissible blocks of length  $n$  for  $Y$ . Since  $\lim_{n \rightarrow \infty} \frac{1}{n} \log(b_n(Y)) = h(Y)$  there is an  $N > 1$  such that  $\frac{1}{N} \log(b_N(Y)) < h(Y) + \epsilon$ . There is also a  $K > 1$  such that  $b_N(X_{I_k}) = b_N(Y)$  for all  $k \geq K$ , since if this were not the case, there is a point in  $\cap X_{I_k}$  that is not in

$Y$ . Since  $h(X_{I_k}) = \inf_{n \geq 1} \frac{1}{n} \log(b_n(X_{I_k}))$  (see prop.4.1.8 of [21]) we have

$$\begin{aligned} h(X_{I_k}) &\leq \frac{1}{N} \log(b_N(X_{I_k})) \\ &= \frac{1}{N} \log(b_N(Y)) \\ &\leq h(Y) + \epsilon. \end{aligned}$$

Thus,  $h(Y) \leq h(X_{I_k}) < h(Y) + \epsilon$ . Since this occurs for each  $\epsilon > 0$  (with  $K$  increasing as  $\epsilon$  decreases) this forces  $\lim_{k \rightarrow \infty} h(X_{I_k}) = h(Y)$ .

We now demonstrate that  $h(Y) = h(X_{\mathcal{I}})$ . This will be done by showing that the growth rate of the extra blocks that are admissible for  $Y$ , but not  $\mathcal{I}$ -admissible is smaller than the growth rate of the  $\mathcal{I}$ -admissible blocks. This is done by characterizing all the new blocks of every length admissible for  $Y$ . There are several possibilities that must be considered.

First, suppose there is some block  $w^j$  with associated composition  $t^j$  such that such that  $t^j(\mathcal{I}) \cap \mathcal{I} = \emptyset$ . We will show that no such block is admissible for  $Y$  but not  $\mathcal{I}$ -admissible. To see this suppose that  $w^j$  is admissible for  $Y$  (hence for each  $X_{I_i}$ ) but not for  $X_{\mathcal{I}}$ . Also suppose that  $w_j$  is such that the associated composition has the property that  $t^j(\mathcal{I}) \cap \mathcal{I} = \emptyset$ . Without loss of generality assume that the associated composition  $t^j$  is orientation preserving. Then  $t^j(\lambda) = 2\lambda(1 - \lambda) - \delta$  for some  $\delta > 0$ . By continuity of  $t^j$ ,  $\lim_{\epsilon \rightarrow 0} t^j(\lambda + \epsilon) = 2\lambda(1 - \lambda) - \delta$  and  $\lim_{\epsilon \rightarrow 0} T_\lambda(\lambda + \epsilon) = 2\lambda(1 - \lambda)$ . Thus, for sufficiently small  $\epsilon$ ,  $t^j(\lambda + \epsilon) < T_\lambda(\lambda + \epsilon)$  This leads to a contradiction since  $w^j$  is supposed to be admissible for  $Y$ .

Next consider minimal blocks,  $w^j$ , with associated composition,  $t^j$ , such that either  $t^j(\mathcal{I}) \cap \mathcal{I} = \{2\lambda(1 - \lambda)\}$ , with  $t^j$  orientation preserving, or  $t^j(\mathcal{I}) \cap \mathcal{I} = \{2\lambda(1 - \lambda)\}$ , with  $t^j$  orientation reversing (other possibilities are  $t^j(\mathcal{I}) \cap \mathcal{I} = \{\lambda\}$ , which are just the previous case with  $g$  applied on the front, hence not minimal). These are  $I_i$ -admissible for all  $i$ , but not  $\mathcal{I}$ -admissible.

**Definition 3.18** Blocks that are  $I_i$ -admissible for all  $i$ , but not  $\mathcal{I}$ -admissible are *bounding blocks*.

Associated compositions,  $t$ , of bounding blocks are such that  $t(\mathcal{I}) \cap \mathcal{I}$  is a single endpoint of  $\mathcal{I}$ . If the kneading sequence of the map on  $\mathcal{I}$  is  $(RL^{n_1}R^{n_2} \dots R^{n_k}C)^\infty$  then blocks of the form  $f^{n_1}g^{n_2} \dots g^{n_k}f$  and  $f^{n_1}g^{n_2} \dots g^{n_k}f g$  with  $f$  and  $g$  chosen in the appropriate location to assure the correct orientation (otherwise the blocks are  $\mathcal{I}$ -admissible) are bounding blocks. Since the contact shift  $X_{\mathcal{I}}$  is conjugate to the Markov shift of finite type, then all bounding blocks must start with  $f^{n_1}g^{n_2} \dots g^{n_k}$ , and following blocks must be admissible. Thus, all bounding blocks must be of the form  $bb_1^i = f^{n_1}g^{n_2} \dots g^{n_k}f(gf^{n_1}g^{n_2} \dots g^{n_k}f)^i$  in the orientation preserving case ( $f$  and  $g$  must be chosen in the appropriate places so that the associated composition is orientation preserving; otherwise, the block will be admissible for  $X_{\mathcal{I}}$  as well as for  $Y$ ), and  $bb_2^i(f^{n_1}g^{n_2} \dots g^{n_k}f g)^m$  in the orientation reversing case, with  $m \in \mathbf{N}$ . For instance, consider the period 5 tent map,  $T$ , with kneading sequence  $\mathbf{k} = (RLLRC)^\infty$ . Then blocks  $ffg_g^f(gffg_g^f)^k$ ,  $k \in \mathbf{Z}^+$  are the possibilities whose associated compositions are orientation preserving and  $(ffg_g^f g)^k$  are the possible orientation reversing ones.

It should be noticed that all new blocks, admissible for  $Y$  but not  $\mathcal{I}$ -admissible, are a rather specialized mixture of blocks,  $w$ ,  $\mathcal{I}$ -admissible and the bounding blocks  $bb_2^m$ ,  $i = 1, 2$ . Notice that mixtures of bounding blocks are given by  $bb_2bb_1$ ,  $bb_1gbb_2 = bb_2^2$ ,  $bb_1gbb_1 = bb_2bb_1$ , and  $bb_2bb_2$ . In a sense then all bounding blocks are of the form  $bb_2^m$  or  $bb_2^{m-1}bb_1$ . The number of bounding blocks of any length is easy to calculate. The number at most doubles each time the length of block is a multiple of the period of the kneading sequence. More specifically, if  $\mathbf{k} = (RL^{n_1}R^{n_2} \dots R^{n_k}C)^\infty$  is of period  $n$ , then there are at most  $2^m$  blocks of length  $mn$  for each  $m \in \mathbf{N}$ .

We will overestimate the new blocks by assuming that all  $\mathcal{I}$ -admissible blocks ending in  $g$  (but not  $f$ ) may precede a bounding block. Notice that any  $\mathcal{I}$ -admissible

blocks that follow bounding blocks must have associated compositions that when applied to  $\mathcal{I}$  contain  $\lambda$  or  $2\lambda(1 - \lambda)$ . We will overestimate these blocks by assuming that all  $\mathcal{I}$ -admissible blocks may be used. Observe that once a bounding block has been followed by an  $\mathcal{I}$ -admissible block then no bounding block may follow since admissibility is lost except for those given in the bounding block overestimation.

We now estimate the number of new blocks of length  $nm$ ,  $m \in \mathbf{Z}^+$ , allowed by including the bounding blocks. Let  $n$  be the period of the periodic parameter. To find the number of new blocks we need to consider the position of the bounding block in the new block as well as the length of the bounding block. To maximize the number of possible new blocks, we assume for each  $j \in \mathbf{Z}^+$ , the number of new blocks of length  $nj$  containing  $bb_2^{j-1}bb_1$  or  $bb_2^j$  is at most  $2^{j+1}$ . Thus, new blocks of length  $nm$  containing  $bb_2^{j-1}bb_1$  or  $bb_2^j$  in a fixed location may be estimated by  $2^{j+1}(2\lambda)^{mn-nj}$  (Here the growth rate of the  $\mathcal{I}$ -admissible blocks,  $2\lambda$ , is used. If needed, a larger value could be used but will make no difference when estimating  $h(Y)$ ). Since there are at most  $nm - (nj - 1)$  different positions to place the bounding block  $bb_2^{j-1}bb_1$  or  $bb_2^j$  in a block of length  $nm$ , then the approximate number of new blocks of length  $nm$  is

$$\sum_{j=1}^m 2^{j+1}(n(m-j) + 1)(2\lambda)^{n(m-j)} \quad (3.5)$$

The sum 3.5 may be overestimated by

$$\sum_{j=1}^m 2^{j+1} 2nm (2\lambda)^{nm}.$$

To finish the proof we only need to show that  $h(Y) = h(X_{\mathcal{I}})$ . Let  $n$  be the period of the kneading sequence as given above given by the periodic parameter  $\lambda$ . Also let  $b_{nm}$  be the number of admissible blocks of length  $nm$ , and  $E_{nm}$  be the number

of new blocks admissible for  $Y$ . Then

$$\begin{aligned}
 h(Y) - h(X_{\mathcal{I}}) &= \lim_{m \rightarrow \infty} \frac{1}{nm} (\log(b_{nm} + E_{nm}) - \log(b_{nm})) \\
 &\leq \lim_{m \rightarrow \infty} \frac{1}{nm} \log \left( (2\lambda)^{nm} + \sum_{j=1}^m 2^{j+2} nm (2\lambda)^{nm} - (2\lambda)^{nm} \right) \\
 &\leq \lim_{m \rightarrow \infty} \frac{1}{nm} \log \left( 1 + \sum_{j=1}^m 2^{j+2} nm \right) \\
 &= 0.
 \end{aligned}$$

□

**Remark:** It is interesting that the new blocks formed have a growth rate that is exponential on the order of  $2^{\frac{1}{n}}$ . However, this is not enough to change entropy.

We will now extend some of the results of Theorem 3.17 to iterated function systems given by contact deformations of tent maps containing the core (see Figure 18).

**Lemma 3.19** *Given an inadmissible block  $w^j$  (with associated composition  $t^j$ ) for  $X_{\mathcal{I}}$  of a tent map,  $T_{\lambda}$ , there is a unique  $\epsilon \geq 0$  such that  $w^j$  is a bounding block for the contact shift,  $X_J$ ,  $J = [2\lambda(1 - (\lambda + \epsilon)), \lambda + \epsilon]$  of the iterated function system given by extending the branches of  $T_{\lambda}$ .*

Proof:

Without loss of generality assume  $t^j$  is orientation preserving. We first show uniqueness. Assume by way of contradiction that  $\epsilon_1 < \epsilon_2$  are two extensions of the desired type. Let  $b_1 = \lambda + \epsilon_1$ ,  $b_2 = \lambda + \epsilon_2$ ,  $a_1 = 2\lambda(1 - b_1)$ , and  $a_2 = 2\lambda(1 - b_2)$ . Notice that  $t^j(\lambda)a_2 < a_1$ . Thus,

$$a_2 - a_1 = 2\lambda(1 - b_2) - (2\lambda(1 - b_1)) = 2\lambda(\epsilon_1 - \epsilon_2). \quad (3.6)$$

But we also have  $|t^j(\lambda) - a_i| = \frac{\epsilon_i}{(2\lambda)^j}$  which implies

$$a_2 - a_1 = \frac{\epsilon_1}{(2\lambda)^j} - \frac{\epsilon_2}{(2\lambda)^j} = \frac{(\epsilon_1 - \epsilon_2)}{(2\lambda)^j}. \quad (3.7)$$

Setting the right hand sides of equations 3.6 and 3.7 we get

$$2\lambda(\epsilon_1 - \epsilon_2) = \frac{(\epsilon_1 - \epsilon_2)}{(2\lambda)^j}.$$

This forces  $\epsilon_1 = \epsilon_2$  as desired.

The existence of the desired  $\epsilon$  is given by the continuity of  $t^j$  and the fact that  $t^j(\lambda) \leq 2\lambda(1 - \lambda)$ . If  $t^j(\lambda) = 2\lambda(1 - \lambda)$ , then  $w^j$  is a bounding block for  $X_{\mathcal{I}}$ . If  $t^j(\lambda) < 2\lambda(1 - \lambda)$  then let  $U = \{\epsilon > 0 : t^j(\lambda + \epsilon) < 2\lambda(1 - \lambda - \epsilon)\}$  and  $D = \{\epsilon > 0 : t^j(\lambda + \epsilon) > 2\lambda(1 - \lambda - \epsilon)\}$ . These two sets are open and nonempty. For example, if  $\epsilon_1 \in D$  then so are all  $\epsilon > \epsilon_1$ . Also since  $t^j(\lambda + \epsilon_1) > 2\lambda(1 - \lambda - \epsilon_1)$  then by continuity of  $t^j(x)$  and  $2\lambda(1 - (\lambda + x))$  all  $\epsilon < \epsilon_1$  sufficiently close to  $\epsilon_1$  are in  $D$ . The closures  $\bar{U}$  and  $\bar{D}$  are closed and  $\bar{U} \cup \bar{D} \supset [0, \infty)$ . Hence  $\bar{U} \cap \bar{D} \neq \emptyset$ , which implies that there is some  $\epsilon$  such that  $t^j(\lambda + \epsilon) > 2\lambda(1 - \lambda - \epsilon)$ .

□

**Remark:** Formulas can be obtained for such  $\epsilon$  in low periods.

Before proving Theorem 3.21, which is an extension of Theorem 3.17, we need some terminology.

**Definition 3.20** A number  $\zeta$  is *algebraic* if there exists a polynomial  $p(x)$  with coefficients in  $\mathbf{Z}$  such that  $p(\zeta) = 0$ . If  $\zeta$  is not algebraic it is said to be *transcendental*.

Since the algebraic numbers are countable there is a dense set of transcendental values in  $[\frac{\sqrt{(2)}}{2}, 1]$ . Thus, for a dense set of the parameter  $\lambda$ , we have

**Theorem 3.21** *The map  $I_\epsilon \mapsto h(X_{I_\epsilon})$  is continuous from the right at all  $I = [T_\lambda(\lambda + \delta), \lambda + \delta]$ ,  $\delta > 0$ , for iterated function systems given by contact deformations of the tent family with transcendental parameter  $\lambda$ .*

Proof:

This proof uses the framework developed in Theorem 3.17. Let  $I = [a, b] \supset \mathcal{I}$  be



the contact interval of the contact deformation. Here  $b = \lambda + \delta$  for some  $\delta > 0$  and  $a = 2\lambda(1 - b)$ .

We must show that  $\lim_{\epsilon \rightarrow 0} h(X_{I_\epsilon}) = h(X_I)$ . Notice that no block that is admissible for the system on the core interval is inadmissible for any of the extended systems. Thus, inadmissible blocks must be smaller (in the lexicographic-parity ordering) than the bounding blocks of the core interval. Let  $X_{I_i} = X_{I_{\epsilon_i}}$  and notice that  $X_{I_1} \supset X_{I_2} \supset X_{I_3} \dots$ . Let  $Y = \bigcap_{i=1}^{\infty} X_{I_i}$ . As in Theorem 3.17, the only blocks that are admissible for  $Y$  that are not for  $X_I$  are blocks  $w^j$  such that the associated composition  $t^j$  has the property  $t^j(I) \cap I = a$ . If  $t^j$  is orientation preserving then  $t^j(b) = a$  and if  $t^j$  is orientation reversing then  $t^j(a) = a$ . These blocks are similar to the bounding blocks of  $X_I$ . Since both cases are very similar we will mainly concern ourselves with the orientation preserving blocks.

We want to show that for each  $X_I$ ,  $I = [a, b]$ , there is only a finite number of blocks  $w^j$  with  $t^j(b) = a$ . Suppose  $t^j$  is orientation preserving. Since  $b = \lambda + \epsilon$  then  $t^j(b) = 2\lambda(1 - \lambda) - 2\lambda\epsilon$ . Also, we have  $t^j(b) - t^j(\lambda) = \frac{\epsilon}{(2\lambda)^j}$ . Hence,

$$t^j(\lambda) = 2\lambda(1 - \lambda) - 2\lambda\epsilon - \frac{\epsilon}{(2\lambda)^j} \quad (3.8)$$

$$= \frac{(2\lambda)^{j+1}(1 - \lambda) - \epsilon(2\lambda)^{j+1} + \epsilon}{(2\lambda)^j}. \quad (3.9)$$

Since  $t^j$  is a composition of  $f(x) = \frac{x}{2\lambda}$  and  $g(x) = \frac{2\lambda - x}{2\lambda}$  then

$$t^j(\lambda) = \frac{\pm\lambda + \sum_{i=1}^j a_i (2\lambda)^i}{(2\lambda)^j}, \quad (3.10)$$

where  $a_i = 0, \pm 1$  (the last nonzero  $a_i = 1$ ) and depend upon the order and quantity of the  $f$ 's and  $g$ 's. Setting equations 3.9 and 3.10 equal and solving for  $\epsilon$  gives

$$\epsilon = \frac{\pm\lambda + \sum_{i=1}^j a_i (2\lambda)^i + (\lambda - 1)(2\lambda)^{j+1}}{-(1 + (2\lambda)^{j+1})}. \quad (3.11)$$

Now suppose there is another orientation preserving composition,  $s^l$ , such that

$s^l(b) = a$ . As in equation 3.11 we have

$$\epsilon = \frac{\pm\lambda + \sum_{i=1}^l b_i(2\lambda)^i + (\lambda - 1)(2\lambda)^{l+1}}{-(1 + (2\lambda)^{l+1})}. \quad (3.12)$$

If  $l = j$ , then setting equations 3.11 and 3.12 equal and simplifying give

$$\sum_{i=1}^j (b_i - a_i)(2\lambda)^i = 0. \quad (3.13)$$

Since  $2\lambda$  is transcendental then each  $a_i = b_i$ . Hence  $t^j$  and  $s^l$  are the same composition. If  $l > j$ , notice that  $g \circ t^j(b) = b$ . This is also true for  $g \circ s^l$ . In order to compare we need to take compositions  $t^j(g \circ t^j)^n$  and  $s^l(g \circ s^l)^m$  of the same size and both orientation preserving. We need both  $n$  and  $m$  to be even and satisfy

$$n(j + 1) + j = m(l + 1) + l. \quad (3.14)$$

Simplifying we get

$$\frac{j + 1}{k} = \frac{m + 1}{n - m} \quad (3.15)$$

Let  $l = j + k$ . Then equation 3.14 has a solution  $m = j$  and  $n = l$  if both  $j$  and  $l$  are even. Replacing the left hand side of equation 3.15 by  $\frac{2(j+1)}{2k}$  and solving gives  $m = j + 1$  and  $n = 2k + m$  when  $j$  is odd and  $l$  is even or odd. A similar process gives solutions to equation 3.14 when  $j$  is even and  $l$  is odd. In each instance with the two orientation preserving compositions  $t^j(g \circ t^j)^n(b) = a$  and  $s^l(g \circ s^l)^m(b) = a$  we have the same setup as in equation 3.13. This implies that  $t^j(g \circ t^j)^n = s^l(g \circ s^l)^m$ . Thus, there is only one unique bounding block of length  $n(j + 1) + j$ , and its associated composition is just repeated compositions of  $t^j$  and  $g$  as shown. In the same manner as the preceding argument there can be at most one block that is orientation reversing whose associated composition,  $s^j$ , satisfies  $m\{s^j(I) \cap I\} = \emptyset$ . Note that  $s^j(a) = a$ . Hence it cannot be applied as the orientation preserving blocks can.

The rest of the proof is to find the number of extra blocks given by including the bounding blocks as in Theorem 3.17. Then we show that the extra blocks are

not enough to change the contact entropy. To do this there are several important observations to make.

Let  $w_1$  and  $w_2$  be (respectively orientation preserving and orientation reversing) the extra admissible blocks of  $Y$  that are not admissible for  $X_I$ . We will make the assumption that all admissible blocks for  $X_I$  followed by  $g$  may be predecessors for  $w_1$  and  $w_2$ . (This is not the case but it will give us a good enough estimate. Actually the possible preceding blocks are those whose associated compositions keep  $b$  in  $I$ ). The blocks that follow  $w_1$  and  $w_2$  are very different. We will take each case separately.

The possible blocks that may follow  $w_1$  are those whose associated compositions,  $h^n$ , are such that  $b \in h^n(I) \cap I$ . These include  $w_1$  itself but not  $w_2$ . If  $w_1$  is followed by some other block  $v$  and then again by itself then  $v$  is some  $w_1(gw_1)^n$  else admissibility is lost. Since  $w_1(gw_1)^n$  does not create many new blocks, the only way one could hope to increase entropy with these new blocks is for there to be long follower blocks that are not of this form. It may be assumed that all admissible blocks may follow. The rest of the proof is as Theorem 3.17.

□

Extending Theorem 3.21 to all  $\lambda \in [\frac{\sqrt{(2)}}{2}, 1]$  presents a problem. For the extension not to be continuous from the right, it would have to be the case that the growth rate of new blocks given by introducing bounding blocks is larger than the growth rate of the contact shift. There would have to be many bounding blocks of sufficiently small size. While this is possible it is extremely unlikely. It would be very difficult to find conditions such that Conjecture 3.22 were not true. Thus it is reasonable to conjecture

**Conjecture 3.22** *Theorem 3.21 applies to all  $\lambda \in [\frac{\sqrt{(2)}}{2}, 1]$*

Other theorems extending Theorem 3.21 generally to i.f.s. might depend upon

each forward branch being expansive, and having some interval where the maximum possible entropy is achieved. This last property suggests fixed points or eventually fixed points for all maps in the i.f.s. defined at the ends of the interval. Future work will involve continuity from the left (or lack thereof) as well as from the right in contact deformations.

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APPENDIX A

Numerical data for Parameters  $\lambda$  and  $r$

Table 7: Numerical Data for Period 5 - *RLLLC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.9584	.68	.8919	.84	.7555
.54	.9525	.70	.8798	.86	.7293
.56	.9460	.72	.8666	.88	.6996
.58	.9388	.74	.8521	.90	.6655
.60	.9310	.76	.8364	.92	.6254
.62	.9225	.78	.8190	.94	.5768
.64	.9131	.80	.8000	.96	.5143
.66	.9029	.82	.7789	.98	.4228

Table 8: Numerical Data for Period 5 - *RLLRC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.8488	.68	.7251	.84	.5338
.54	.8359	.70	.7058	.86	.5017
.56	.8224	.72	.6854	.88	.4668
.58	.8082	.74	.6638	.90	.4282
.60	.7932	.76	.6411	.92	.3849
.62	.7774	.78	.6168	.94	.3351
.64	.7609	.80	.5910	.96	.2753
.66	.7435	.82	.5634	.98	.1961

Table 9: Numerical Data for Period 5 - *RLRRC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.7398	.68	.5873	.84	.3891
.54	.7226	.70	.5655	.86	.3593
.56	.7050	.72	.5430	.88	.3277
.58	.6868	.74	.5197	.90	.2940
.60	.6681	.76	.4956	.92	.2576
.62	.6488	.78	.4706	.94	.2175
.64	.6289	.80	.4446	.96	.1719
.66	.6084	.82	.4175	.98	.1158



Table 10: Numerical Data for Period 6 - *RLLLLC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.9798	.68	.9348	.84	.8265
.54	.9762	.70	.9259	.86	.8041
.56	.9721	.72	.9160	.88	.7784
.58	.9675	.74	.9049	.90	.7482
.60	.9623	.76	.8926	.92	.7121
.62	.9565	.78	.8788	.94	.6673
.64	.9501	.80	.8634	.96	.6082
.66	.9429	.82	.8461	.98	.5187

Table 11: Numerical Data for Period 6 - *RLLLRC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.9344	.68	.8513	.84	.6959
.54	.9265	.70	.8369	.86	.6671
.56	.9181	.72	.8214	.88	.6349
.58	.9089	.74	.8047	.90	.5982
.60	.8991	.76	.7865	.92	.5556
.62	.8884	.78	.7668	.94	.5046
.64	.8770	.80	.7453	.96	.4399
.66	.8646	.82	.7218	.98	.3476

Table 12: Numerical Data for Period 6 - *RLLRRC*

$\lambda$	$r$	$\lambda$	$r$	$\lambda$	$r$
.52	.8856	.68	.7745	.84	.5946
.54	.8743	.70	.7567	.86	.5639
.56	.8624	.72	.7378	.88	.5301
.58	.8497	.74	.7177	.90	.4925
.60	.8363	.76	.6964	.92	.4501
.62	.8221	.78	.6736	.94	.4008
.64	.8072	.80	.6492	.96	.3406
.66	.7913	.82	.6230	.98	.2590

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