Extensions of Cantor set maps to disk homeomorphisms

by Michael David Sanford

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

Abstract:
We show that the shift map on the inverse limit space of a surjective positive entropy map of the cross product of the Cantor Set and the interval cannot be tamely embedded into a near homeomorphism of the disk. We strengthen this result to cover the case of "temperate embeddings." When f is a "block permuting" Cantor set homeomorphism, we show there exists a continuously differentiable Kupka-Smale diffeomorphism F of the disk without sources or sinks and an F-invariant Cantor set A such that F restricted to Λ is conjugate to f. All k-symbol adding machines and finite products of such are "block permuting."

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Doctor of Philosophy

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Mathematics

MONTANA STATE UNIVERSITY
Bozeman, Montana

April 1995

APPROVAL

of a thesis submitted by

MICHAEL DAVID SANFORD

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

11 April 95

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ACKNOWLEDGEMENTS

I would like to thank my parents, Richard and Darlene Sanford, and my brother, Scott Sanford, without whose help and support this thesis would never have been possible.

I would like to thank Russell Walker, for his invaluable assistance and introducing me to Misiurewicz's lightning bolt which set this trek in motion.

I would like to thank Marcy Barge, Richard Swanson, and Richard Gillette for their help.

I would like to dedicate this to my cousin Tom, who I promised a copy. Unless he's looking over my shoulder from some other dimension he will never be able to read my thesis.

Thomas M. Thompson

1960-1991

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ABSTRACT

We show that the shift map on the inverse limit space of a surjective positive entropy map of the cross product of the Cantor Set and the interval cannot be tamely embedded into a near homeomorphism of the disk. We strengthen this result to cover the case of "temperate embeddings." When f is a "block permuting" Cantor set homeomorphism, we show there exists a continuously differentiable Kupka-Smale diffeomorphism F of the disk without sources or sinks and an F-invariant Cantor set Λ such that F restricted to Λ is conjugate to f. All k-symbol adding machines and finite products of such are "block permuting."

CHAPTER 1

Introduction

In 1990 M. Barge and J. Martin [BM90] proved the shift map on the inverse limit space (I, f) for any map $f : [0,1] \to [0,1]$ can be "realized as a global attractor in the plane." The main goal of this thesis is to show that analogous techniques for maps $F : C \times [0,1] \to C \times [0,1]$ where C is a Cantor set, $F(x,y) = (F_1(x), F_2(x,y))$ is a surjective map with positive topological entropy (Definition 1.4), and F_1 is a homeomorphism, do not work.

In Chapter 2 we show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map such that $F(x,y) = (F_1(x), F_2(x,y))$, F_1 is a homeomorphism and $F_2(x_0, \cdot) : [0,1] \to [0,1]$ is nonmonotone (Definition 1.2) for some x_0, then there exists no "embedding" of F into a near homeomorphism (Definition 1.1) of the disk. We will show this by assuming such a near homeomorphism does exist and then obtaining a contradiction using a result of S. Schwartz [Sch92] (Theorem 1.3) concerning nonmonotone maps. Unless otherwise specified X and Y are metric spaces.

Definition 1.1 A map $f : X \to Y$ is called a near homeomorphism provided there exists a sequence $\{f_k : X \to Y\}_{k=1}^{\infty}$ of homeomorphisms which uniformly converge to f.

Definition 1.2 A map $f : X \to Y$ is monotone provided $f^{-1}(V)$ is connected, whenever $V \subset Y$ is connected.

Theorem 1.3 (S. Schwartz [Sch92]) Suppose that X is a locally connected compact metric space. If $f : X \to X$ is a near homeomorphism then f is monotone.

In Chapter 3 we will show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map with positive topological entropy (Definition 1.4), which is tamely embedded in the disk (Definition 3.2,[Bin54]), then F cannot be extended to a near homeomorphism of the disk. The proof uses theorems of R. Bowen (Theorem 1.5) [Bow71] and M. Barge (Theorem 1.6) [Bar87]. Throughout π_1 and π_2 on $X \times Y$ are the first and second coordinate projection maps.

Definition 1.4 (Topological Entropy) Assume that X, Y are compact metric spaces. Suppose that $F : X \times Y \to X \times Y$ is a surjective map and has the form $F(x,y) = (F_1(x), F_2(x,y))$. Fix x_0 and let $\epsilon > 0$. A set $E \subset Y$ is (n,ϵ)–separated by $F|_{\pi_1^{-1}(x_0)}$ if for all $y_0, y_1 \in E$, $y_0 \neq y_1$, $d(\pi_2 F^k(x_0,y_0), \pi_2 F^k(x_0,y_1)) > \epsilon$ for some $k \in [0,n)$, where d is the Y–metric. Since Y is compact and $n < \infty$, card $E < \infty$. Let the maximum number of (n,ϵ)-separated orbits for each ϵ be

$$s(n,\epsilon) = \max\{\text{card } E \mid E \subset Y \text{ such that } E \text{ is } (n,\epsilon) - \text{separated by } F|_{\pi_1^{-1}(x_0)}\}.$$

Now, let the growth rate of $s(n,\epsilon)$ (or ϵ-topological entropy) be

$$h_{top}\left(F|_{\pi_1^{-1}(x_0)}, \epsilon\right) = \limsup_{n\to\infty} \frac{\log s(n,\epsilon)}{n}.$$

Lastly we let $\epsilon \to 0$ and define topological entropy for $F|_{\pi_1^{-1}(x_0)}$.

$$h_{top}\left(F|_{\pi_1^{-1}(x_0)}\right) = \lim_{\epsilon\to 0} h_{top}\left(F|_{\pi_1^{-1}(x_0)}, \epsilon\right).$$

The topological entropy $h_{top}(F_1)$ of the homeomorphism F_1 is defined similarly (See [Bow71]).

Theorem 1.5 (R. Bowen[Bow71]) If $F : X \times Y \to X \times Y$ has the form $F(x,y) = (F_1(x), F_2(x,y))$ then $h_{top}(F) \leq h_{top}(F_1) + \sup_{x\in X}\left\{h_{top}\left(F|_{\pi_1^{-1}(x)}\right)\right\}$. If $h_{top}(F_1) = 0$ then $h_{top}(F) = \sup_{x\in X}\left\{h_{top}\left(F|_{\pi_1^{-1}(x)}\right)\right\}$.

Theorem 1.6 (M. Barge [Bar87]) If $F : X \times [0,1] \to X \times [0,1]$ has the form $F(x,y) = (F_1(x), F_2(x,y))$, $F(x,\cdot) : [0,1] \to [0,1]$ is monotone for each x and $h_{top}(F_1) = 0$ then $h_{top}(F) = 0$.

In Chapter 4 we will prove a strengthened version of Theorem 3.1 from Chapter 3. We modify Theorem 3.1 by weakening the condition that the embedding of the Cantor set cross the interval is tame. Instead, we assume that each fiber of the embedded Cantor set cross the interval makes only "one crossing from near the bottom to near the top." We call such embeddings "temperate."

Chapter 5 addresses another aspect of homeomorphic extension onto disks. In 1962, S. Smale asked whether there exist C^r diffeomorphisms of the 2-sphere ($r \geq 1$) without periodic sources or sinks such that i) all periodic orbits are hyperbolic and ii) all intersections of stable and unstable manifolds are transverse ([Sma62]). Diffeomorphisms satisfying (i) and (ii) are now called "Kupka-Smale" and are known to be C^r generic on compact manifolds ([Kup63], [Sma63]). In 1976, R. Bowen and J. Franks answered Smale's question in the affirmative for $r = 1$ [BF76]. Their Kupka-Smale sphere diffeomorphism has a saddle fixed point and two saddle orbits of minimum period 2^k, for each k, but no periodic sources or sinks (see Figure 16). It has an attractor/repeller pair of Cantor sets which carry the dynamics of the 2-symbol adding machine. (See Chapter 5 Section 4 for the definition of the k-symbol adding machine.) J. Franks and L-S. Young subsequently constructed a C^2 model ([FY80]), and more recently J. Gambaudo, S. van Strien, and C. Tresser applied renormalization theory to demonstrate the existence of a C^∞ example ([JGT89]). In Chapter 5 we will exhibit a class of C^1 disk diffeomorphism that answers S. Smale's question in the affirmative. We first define a class of homeomorphisms of the Cantor set which we call "block permuting." All k-symbol adding machines and finite products of such are block permuting (Lemma 5.5).

E. Moise has shown that every homeomorphism between two Cantor sets in the disk can be extended to a homeomorphism of the whole disk ([Moi77]). However, there exist many homeomorphisms of the Cantor set which have infinite topological entropy. But by B. Kushnirenko's theorem ([Kus67]), C^1 disk maps have bounded topological entropy. Thus there exists no C^1 version of Moise's theorem even if one is free to choose how the Cantor set is embedded in the disk.

M. Barge and R. Walker have shown that no homeomorphism of the two disk, F, has an invariant set Λ homeomorphic to the Cantor set cross the interval, such that $F|_\Lambda$ is semi-conjugate to a 2-symbol adding machine ([BW93]). However, (2-symbol adding machine) $\times$ (Identity) on the C $\times$ C is a block permuting homeomorphism of the Cantor set. Therefore large diameter Cantor sets can be "shuttled" over a 2-symbol adding machine by C^1 disk diffeomorphism (See Corollary 5.6).

CHAPTER 2

Nonmonotone Maps of the Cantor Set Cross the Interval

Preliminaries

Let $C \subset [0,1]$ be a Cantor set. Let $\Sigma = C \times [0,1]$ and $\Sigma_\alpha = \{\alpha\} \times [0,1] \subset \mathbb{R}^2$ for $\alpha \in C$. The goal of this chapter is to prove Theorem 2.1 to follow. But first some preliminaries.

2.1.1 Assume $F : \Sigma \to \Sigma$ is a surjective map that has the form

$$F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$$

where $F_1 : C \to C$ is a homeomorphism. Furthermore for a given $\alpha_0 \in C$, $F_2(\alpha_0, y) = t(y)$ where $t : [0,1] \to [0,1]$ is a continuous nonmonotone map (see Figure 1 for an example). Let $\lambda_0 = F_1(\alpha_0)$.

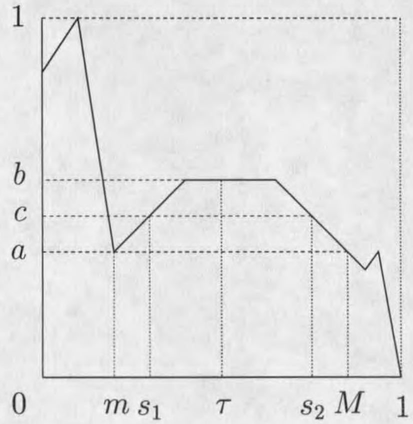

Figure 1: Example of a nonmonotone map

2.1.2 Since t is nonmonotone and continuous, there is an $a \in (0,1)$ such that $t^{-1}(a)$ is closed and not connected. Thus, there is an interval $(m, M) \subset [0,1] \setminus t^{-1}(a)$ such that $a = t(m) = t(M)$, and $t([m, M]) = [a, b]$ (or $[b, a]$) for some $b \neq a$. Without loss of generality we will assume that $a < b$. Let $\tau \in t^{-1}(b)$. By the intermediate value theorem, $t([m, M]) = [t(m), t(\tau)]$. Now choose $c = \frac{1}{2}(a + b)$. Since t is continuous there are $s_1 \in (m, \tau)$ and $s_2 \in (\tau, M)$ such that $c = t(s_1) = t(s_2)$ (See Figure 1).

2.1.3 By the continuity of F, for any $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that $F(x, y) \in \mathcal{B}_\epsilon(\lambda_0, t(y))$ when $d(\alpha_0, x) < \delta_1$ and $y \in [0,1]$. Suppose $K_1 = K_1(\epsilon) \in \mathbb{N}$ is such that $\frac{1}{K_1} < \delta_1$.

Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\}$. Now let $h_0 : \Sigma \to D$ be an arbitrary topological embedding. Then there is a homeomorphism $h_1 : D \to \mathrm{D}$ such that $(h_1 \circ h_0)(\alpha_0, y) = (\alpha_0, y)$ and $(h_1 \circ h_0)(\lambda_0, y) = (\lambda_0, y)$ for all $y \in [0,1]$. So h_1 "straightens out" $h_0(\alpha_0 \times I)$ and $h_0(\lambda_0 \times I)$ in a strong sense. Notice that $\Sigma \to D$.

2.1.4 By the uniform continuity of $h_1 \circ h_0$, for all $\epsilon > 0$ there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that for all $y \in [0,1]$, $h_1 \circ h_0(x, y) \in \mathcal{B}_\epsilon(\alpha_0, y)$ and $h_1 \circ h_0(x', y) \in \mathcal{B}_\epsilon(\lambda_0, y)$, for all $(x, y) \in \mathcal{B}_{\delta_2(\epsilon)}(\alpha_0, y)$ and $(x', y) \in \mathcal{B}_{\delta_2(\epsilon)}(\lambda_0, y)$. Let $K_2 = K_2(\epsilon) \in \mathbb{N}$ be such that $\frac{1}{K_2} < \delta_2$.

2.1.5 With a, b defined as in [2.1.2], let $\hat{d} = \min\{a, 1 - b, |a - b|\}$. For $0 < \epsilon_0 < \frac{\hat{d}}{100}$ choose $0 < \delta_0 \leq \min\left\{\delta_1(\epsilon_0), \delta_2(\epsilon_0), \frac{M-m}{100}\right\}$. So in particular [2.1.3] and [2.1.4] are satisfied. Note that $t([m, M]) \subset [\epsilon_0, 1 - \epsilon_0]$. Let $K_0 \geq \max\{K_1(\epsilon_0), K_2(\epsilon_0)\}$ be such that $\frac{1}{K_0} < \delta_0$. Then there is a sequence $\{\alpha_k\} \subset C$ such that $\alpha_k \to \alpha_0$ as $k \to \infty$, and $\Sigma_{\alpha_k} \subset \mathcal{N}_{\delta_0}(\Sigma_{\alpha_0})$, for all $k > K_0$. Let $\lambda_k = F_1(\alpha_k)$. It follows that $\lambda_k \to \lambda_0$ as $k \to \infty$ and $\Sigma_{\lambda_k} \subset \mathcal{N}_{\epsilon_0}(\Sigma_{\lambda_0})$ for all $k > K_0$. For a possibly larger K_0, also called K_0, and $o_k \in \mathcal{B}_{\epsilon_0}(\lambda_0, c)$, $k > K_0$, there exist $q_1(k)$ and $q_2(k)$ such that

$\{q_1(k), q_2(k)\} \subset F^{-1}(o_k)$, $q_1(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$ and $q_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2)$.

We now state our first theorem.

<u>Nonmonotone Nonextension Theorem</u>

Theorem 2.1 Let $F : \Sigma \to \Sigma$ be a map of the form $F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$ where $F_1 : C \to C$ is a homeomorphism. Furthermore, assume $F_2(\alpha_0, \cdot) : [0,1] \to [0,1]$ is surjective but not monotone for some α_0. Then there exists no extension of $h_0 \circ F \circ h_0^{-1}$ to a near homeomorphism of the disk D, for any topological embedding $h_0 : \Sigma \to D$.

Proof: Assume $h, K_0, \epsilon_0, \delta_0, \{\alpha_0\}, \{\lambda_k\}, q_1(k)$, and $q_2(k)$ are defined as in [2.1.1-5]. Suppose that $H_0 : D \to D$ is a near homeomorphism such that $H_0|_{h_0(\Sigma)} = h_0 \circ F \circ h_0^{-1}$. And let $H_1 : D \to D$ be given by $H_1 = h_1 \circ H_0 \circ h_1^{-1}$. Thus H_1 is also a near homeomorphism. So the diagram in Figure 2 commutes.

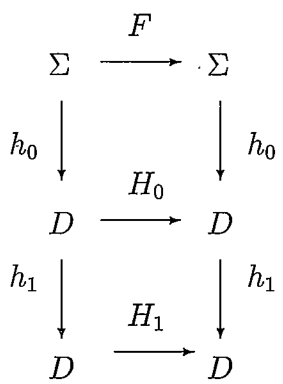

Figure 2: Commuting Diagram

<u>**2.1.6**</u> Let $\Lambda_\alpha = h_1 \circ h_0 (\Sigma_\alpha)$ for all $\alpha \in C$. By [2.1.3] $h_1 \circ h_0$ is a homeomorphism and if $\Sigma_\alpha \bigcap \Sigma_\lambda = \emptyset$ (when $\alpha \neq \lambda$), then $\Lambda_\alpha \bigcap \Lambda_\lambda = \emptyset$. Let ℓ_β be the horizontal line $\{y = \beta\}$. And let $\ell_\beta^\alpha(k) = \Lambda_{\alpha_k} \bigcap \ell_\beta$ and $\ell_\beta^\lambda(k) = \Lambda_{\lambda_k} \bigcap \ell_\beta$. Because $h_1 \circ h_0 (\alpha_k, 0) \in \mathcal{B}_{\delta_0}(\alpha_0, 0)$, $h_1 \circ h_0 (\alpha_k, 1) \in \mathcal{B}_{\delta_0}(\alpha_0, 1)$, and Λ_{α_k} is connected, then $\ell_\beta^\alpha(k) \neq \emptyset$ and all $k \geq K_0$ (See Figure 3 and [2.1.2]). Similarly $\ell_\beta^\lambda(k) \neq \emptyset$, for all $\beta \in [\epsilon, 1-\epsilon]$ and $k \geq K_0$. Note that

if $p \in \ell_\beta^\lambda(k)$ for given $k \geq K_0$ then $p \in \mathcal{B}_{\epsilon_0}(\lambda_0, \beta)$.

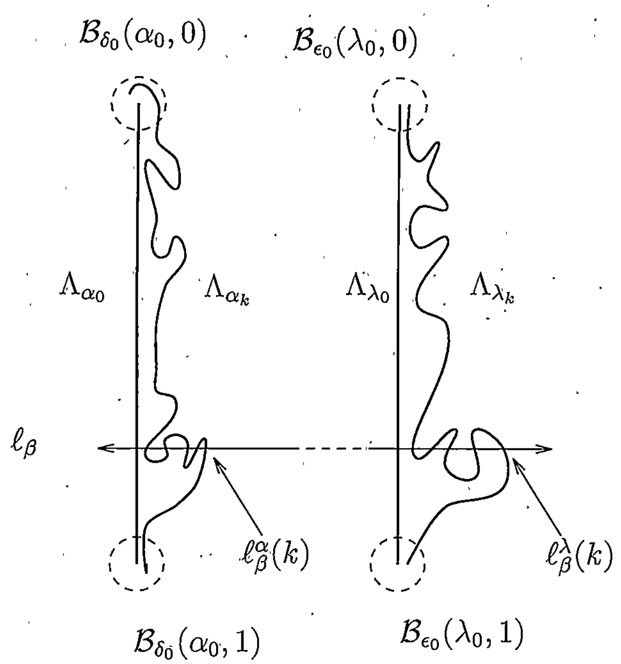

Figure 3: Intersection of Λ_{α_k} with ℓ_β

Denote by $\pi_i : \mathbb{R}^2 \to \mathbb{R}$, the ith coordinate projection map, $i = 1, 2$. Lemma [2.2] follows from the continuity of h_1, h_0, and π_1.

Lemma 2.2 Choose $p_k \in \ell_\beta^\alpha(k)$ for each k. Then $\pi_1 p_k \to \alpha_0$ as $k \to \infty$.

Notice that $\pi_1(h_1 \circ h_0)\left(\alpha_k, \frac{1}{2}\right) \neq \alpha_0$ for sufficently large k. So either

$$\text{card}\left\{k \middle| \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) > \alpha_0\right\} = \infty$$

or

$$\text{card}\left\{k \middle| \pi_1\left(h_1 \circ h_0\left(\alpha_k, \frac{1}{2}\right)\right) < \alpha_0\right\} = \infty.$$

2.1.7 So without loss of generality we may assume there exist distinct $\{k_n\}_{n=1}^{\infty}$ such that $k_n \to \infty$ as $n \to \infty$, and

$$\pi_1\left(h_1 \circ h_0\left(\alpha_{k_n}, \frac{1}{2}\right)\right) > \alpha_0.$$

2.1.8 For the sake of simplicity we denote α_{k_n} by α_n, $\Lambda_{\alpha_{k_n}}$ by Λ_{α_n}, $\Lambda_{\lambda_{k_n}}$ by Λ_{λ_n} and $\ell_\beta^{k_n}$ by ℓ_β^n.

Lemma 2.3 Let N_0 be such that $k_n \geq K_0$ for all $n \geq N_0$. Then

$$\Lambda_{\alpha_n} \bigcap \left\{ \left(x, \frac{1}{2}\right) \middle| x < \alpha_0 \right\} = \emptyset.$$

Proof: Fix $n \geq N_0$ and assume there exists $p_1 \in \Lambda_{\alpha_n} \bigcap \left\{ \left(x, \frac{1}{2}\right) \middle| x < \alpha_0 \right\}$, and let $p_2 = \left(h_1 \circ h_0\left(\alpha_{k_n}, \frac{1}{2}\right)\right)$. By [2.1.7] $\pi_1(p_2) > 0$. Let A be the arc in Λ_{α_n} with end points p_1 and p_2. By [2.1.6], $p_1, p_2 \in \mathcal{B}_{\epsilon_0}\left(\alpha_0, \frac{1}{2}\right)$. So by [2.1.5],

$$d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_2)) < \delta_0.$$

Since $\Lambda_{\alpha_n} \bigcap \Lambda_{\alpha_0} = \emptyset$, then using a Jordan Curve argument, it follows

$$A \bigcap \{(\alpha_0, y) | y > 1 \text{ or } y < 0\} \neq \emptyset.$$

Let $p_3 \in A \bigcap \{(0, y) | y > 1 \text{ or } y < 0\}$. So $d(p_1, p_3) > \frac{1}{2}$. But because $p_3 \in A$, either

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_1) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_2)$$

or

$$\pi_2 \circ (h_1 \circ h_0)^{-1}(p_2) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_1).$$

In either case $d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_3)) < \delta_0$. And so $d(p_1, p_3) < \epsilon_0$ which is a contradiction. □

2.1.9 Assume $n \geq N_0$. Let $g_n : [0,1] \to \Lambda_{\alpha_n}$ be the parameterization of Λ_{α_n} defined by $g_n(\beta) = h_1 \circ h_0(\alpha_n, \beta)$. Using τ, m from [2.1.2] $\Lambda_{\alpha_n} \cap \ell_\tau \neq \emptyset$ and $\Lambda_{\alpha_n} \cap \ell_m \neq \emptyset$ (see Figure 4) so by Lemma [2.3] and the connectivity of Λ_{α_n} there is a largest β, call it β_n^-, such that $g_n(\beta_n^-) \in \ell_m$. Let $a_n = g_n(\beta_n^-)$ (See Figure 4). Similarly there is a smallest β, call it β_n^+, such that $g_n(\beta_n^+) \in \ell_m$. Let $b_n = g_n(\beta_n^+)$.

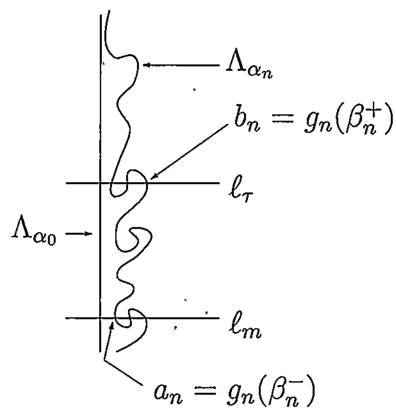

Figure 4: First and Last Intersections

2.1.10 If necessary, renumber the k_n's so that if $k_n < k_{n+1}$ then $\pi_1(a_n) > \pi_1(a_{n+1})$. It follows by an argument similar to that of Lemma [2.3] that $\pi_1(b_n) > \pi_1(b_{n+1})$.

Considering [2.1.10] and [2.1.7] and to simplify the notation assume

$$\text{card}\left\{k | \pi_1 (h_1 \circ h_0) \left(\alpha_k, \frac{1}{2}\right) > \alpha_0\right\} = \infty$$

and $\pi_1(a_k) > \pi_1(a_{k+1})$ for all k.

Using [2.1.9], for $k \geq N_0$ define the four curves $I(k,m)$, $I(k,\tau)$, J_{k-1}, and J_{k+1} in the following manner (see Figure 5). Let $I(k,m)$ be the line segment in ℓ_m between a_{k+1} and a_{k-1} and $I(k,\tau)$ be the line segment in ℓ_τ between b_{k+1} and b_{k-1}. Let

$$J_{k-1} = \left\{g_{k-1}(\beta) \left| \beta_{k-1}^- \leq \beta \leq \beta_{k-1}^+ \right.\right\}, \text{ and } J_{k+1} = \left\{g_{k+1}(\beta) \left| \beta_{k+1}^- \leq \beta \leq \beta_{k+1}^+ \right.\right\}.$$

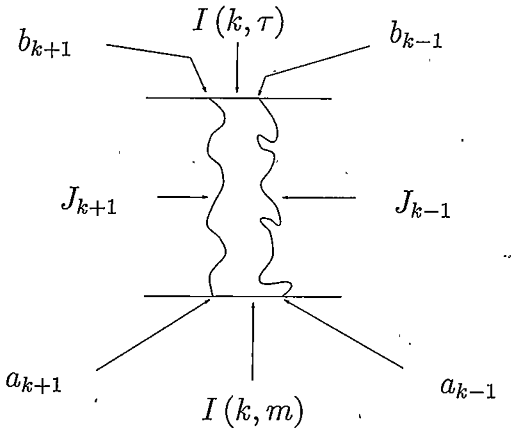

Figure 5: The Boundary of R_k.

Lemma 2.4 $I(k,\tau) \bigcup J_{k-1} \bigcup I(k,m) \bigcup J_{k+1}$ is a simple closed curve.

Proof: Since $\Lambda_{\alpha_{k-1}} \cap \Lambda_{\alpha_{k+1}} = \emptyset$ we have $J_{k-1} \cap J_{k+1} = \emptyset$. By [2.1.6] $I(k,m) \cap I(k,\tau) = \emptyset$. And by [2.1.9] we have that

$$a_{k-1} = J_{k-1} \bigcap I(k,m) \text{ and } a_{k+1} = J_{k+1} \bigcap I(k,m)$$

and

$$b_{k-1} = J_{k-1} \bigcap I(k,\tau) \text{ and } b_{k+1} = J_{k+1} \bigcap I(k,\tau).$$

And so the lemma follows. □

Let R_k be the closed and bounded set with boundary

$$I(k,m) \bigcup J_{k-1} \bigcup I(k,\tau) \bigcup J_{k+1}$$

(See Figure 5). Recall from [2.1.2] that $s_1 \in [m,\tau]$ and from [2.1.6] that $\ell^{\alpha}_{s_1}(k) = \Lambda_{\alpha_k} \cap \ell_{s_1}$ (See Figure 3).

Lemma 2.5 $R_k \cap \ell^{\alpha}_{s_1}(k) \neq \emptyset$.

Proof: Let γ_k be the arc $\left\{g_k(\beta) | 0 \leq \beta \leq \beta_k^+\right\}$. Let $S_k = R_k \cap \pi_2^{-1}[s_1,\tau]$ (See Figure 6). So $\partial S_k \supset I(k,\tau)$ and by [2.1.10] $b_k \in I(k,\tau)$. But b_k is not an endpoint of $I(k,\tau)$

because the endpoints of $I(k,\tau)$ are b_{k-1} and b_{k+1}. And so there is an $\eta > 0$ such that if $p \in \mathcal{B}_\eta(b_k)$ and $\pi_2(p) < \tau$ then $p \in \text{int } S_k$. Now, if $q \in \gamma_k \setminus \{b_k\}$ then $\pi_2(q) < \frac{1}{2}$. And since γ_k connects $h_1 \circ h_0(\alpha_k, 0)$ to b_k, we have that $(\gamma_k \cap \mathcal{B}_\eta(b_k)) \setminus \{b_k\} \neq \emptyset$. Thus there exists $p_0 \in \gamma_k \cap \mathcal{B}_\eta(b_k) \cap \text{int} S_k$. Let $A_k \subset \gamma_k$ be the arc with endpoints p_0 and $h_1 \circ h_0(\alpha_k, 0)$ (See Figure 6).

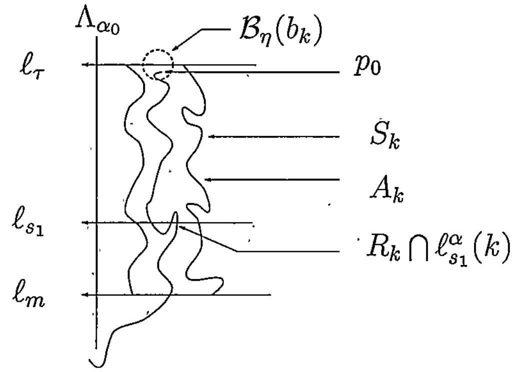

Figure 6: The Arc A_k

Because $p_0 \in \text{int} S_k$ and $h_1 \circ h_0(\alpha_k, 0) \notin S_k$ then $A_k \cap \partial S_k \neq \emptyset$. Since $A_k \cap \Lambda_{\alpha_{k-1}} = \emptyset$, $A_k \cap \Lambda_{\alpha_{k+1}} = \emptyset$, $A_k \cap I(n,\tau) = \emptyset$ and $\ell_{s_1} \cap R_k \subset \partial S_k$, we have that $A_k \cap \ell_{s_1} \cap R_k \neq \emptyset$, or $R_k \cap \ell^\alpha_{s_1}(k) \neq \emptyset$. □

<u>**2.1.11**</u> Note that since $\ell^\alpha_{s_1}(k) \cap \partial R_k = \emptyset$ then $\ell^\alpha_{s_1}(k) \subset \text{int} R_k$.

Lemma 2.6 *$\left[H_1^{-1}(h_1 \circ h_0(\lambda_k, y))\right] \cap \Lambda_{\alpha_l} = \emptyset$ for $k \neq l$.*

Proof: Suppose that $\rho \in H_1^{-1}(h_1 \circ h_0(\lambda_k, y)) \cap \Lambda_{\alpha_l}$ for $k \neq l$. Then $H_1(\rho) = h_1 \circ h_0(\lambda_k, y)$ But $H_1 \Lambda_{\alpha_l} = \Lambda_{\lambda_l}$. Thus $H_1(\rho) \in \Lambda_{\lambda_l}$ So $h_1 \circ h_0(\lambda_k, y) \in \Lambda_{\alpha_l}$ Or $(\lambda_k, y) \in C_{\lambda_l}$. Which contradicts, [2.1.6] since $k \neq l$. □

Proof of Theorem [2.1] continued. By Lemma [2.5] there exists $p_1(k) \in R_k \cap \ell^{\alpha}_{s_1}(k)$ for all $k \geq N_0$. By [2.1.5] $(h_1 \circ h_0)^{-1}(p_1(k)) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$. Using [2.1.5], let $o_k = F \circ (h_1 \circ h_0)^{-1}(p_1(k))$. So there exists $\{q_1(k), q_2(k)\} \subset F^{-1}(o_k)$ such that $q_1(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_1)$ and $q_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2)$. Choose $q_1(k)$ so that $p_1(k) = h_1 \circ h_0(q_1(k))$. And let $p_2(k) = h_1 \circ h_0(q_2(k))$ and $r_k = h_1 \circ h_0(o_k)$ (see Figure 7). Because $H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F$ then $\{p_1(k), p_2(k)\} \in H_1^{-1}(r_k)$. By the size of δ_0 chosen in [2.1.5], $p_2(k) \in \mathcal{B}_{\delta_0}(\alpha_0, s_2) \not\subset R_k$.

Recall that H_0 and H_1 are near homeomorphisms. Near homeomorphisms are monotone on locally connected compact metric spaces ([Sch92]). Thus pre-images of connected sets under H_1 are connected. So $H_1^{-1}(r_k)$ is a connected set which contains $p_2(k) \notin R_k$ and by [2.1.11] $p_1(k) \in \text{int}R_k$. Then $H_1^{-1}(r_k) \cap \partial R_k \neq \emptyset$. By Lemma [2.6] either $H_1^{-1}(r_k) \cap I(k, \tau) \neq \emptyset$ or $H_1^{-1}(r_k) \cap I(k, m) \neq \emptyset$. So there is an infinite sequence $\{\rho_{k_j}\}$ such that either $\rho_{k_j} \in I(k, \tau) \cap H_1^{-1}(r_{k_j})$ or $\rho_{k_j} \in I(k, m) \cap H_1^{-1}(r_{k_j})$ for all j (See Figure 7).

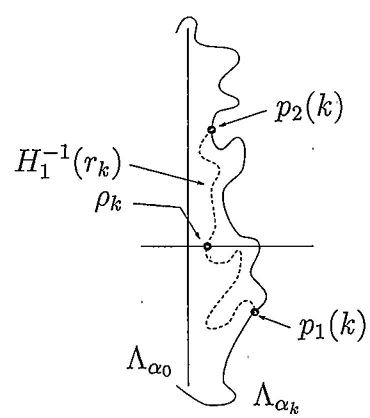

Figure 7: Subsequence and Pre-image

Now by Lemma [2.2] either $\rho_{k_j} \to h_1 \circ h_0(\alpha_0, \tau)$ or $\rho_{k_j} \to h_1 \circ h_0(\alpha_0, m)$ as $j \to \infty$. Since H_1 is continuous for all j, either

$$H_1 \rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, \tau) \quad \text{or} \quad H_1 \rho_{k_j} \to H_1 \circ h_1 \circ h_0(\alpha_0, m).$$

Because $H_1 \circ h_1 \circ h_0 = h_1 \circ h_0 \circ F$, then either

$$r_{k_j} \rightarrow h_1 \circ h_0 \left(\lambda_0, t(\tau)\right) \text{ or } r_{k_j} \rightarrow h_1 \circ h_0 \left(\lambda_0, t(m)\right).$$

Since $h_1 \circ h_0$ is a homeomorphism either

$$o_{k_j} \rightarrow (\lambda_0, b) \quad \text{or} \quad o_{k_j} \rightarrow (\lambda_0, a).$$

which is a contradiction since $\{o_{k_j}\} \subset \mathcal{B}_{\epsilon_0}(\lambda_0, c)$. □

CHAPTER 3

Positive Entropy Maps of Tame Embeddings of a Cantor Set Product the Interval

Introduction

Let $C \subset \mathbb{R}$ be a Cantor set and $I = [0,1] \subset \mathbb{R}$. In this chapter we use the results of Chapter 2 to prove the following:

Theorem 3.1 Let $F : C \times I \to C \times I$ be a surjective map such that $F(a,y) = (F_1(a), F_2(a,y))$, where $F_1 : C \to C$ is a homeomorphism. If $h_{top}(F) > 0$ then there exists no tame topological embedding (Definition 3.2) $h_0 : C \times I \to D \subset \mathbb{R}^2$ such that $h_0 \circ F \circ h_0^{-1}$ extends to a near homeomorphism of the disk D.

Recall that $\pi_1 : C \times I \to C$ is the projection map onto the first coordinate. By work of R. Bowen [Bow71] we know that $h_{top}(F) \leq h_{top}(F_1) + \sup_{a \in C} \left\{ h_{top}\left(F|_{\pi_1^{-1}(a)}\right)\right\}$. It has been shown by M. Barge and R. Walker [BW93] that any near homeomorphism that extends $h_0 \circ F \circ h_0^{-1}$ to the disk must preserve a certain local order on the set of fibers $\{h_0(a \times I) | a \in C\}$. But we will show that if $h_{top}(F_1) > 0$ no such local order is preserved. So in fact $h_{top}(F_1) = 0$. Using [Bow71] and a result of M. Barge [Bar87], if $h_{top}(F) > 0$ then for some $a \in C$, $F_2(a, \cdot)$ is a nonmonotone map. We know that when h_0 is tame there is a fiber map that is nonmonotone, so by Theorem 2.1, $h_0 \circ F \circ h_0^{-1}$ cannot be extended to a near homeomorphism of the disk.

Proof of Theorem 3.1

Definition 3.2 (Tame Embedding) $h_0 : C \times I \to D \subset \mathbb{R}$ is a tame embedding provided there is a homeomorphism $h_1 : D \to D$ such that for all $a \in C$, $h_1 \circ h_0(\{a\} \times I) = (\{a'\} \times I)$ for some $\{a'\}$. Using E. Moise's theorem [Moi77] we can also require that $h_1 \circ h_0(\{a\}, \{i\}) = (\{a'\}, \{i\})$ for all a and $i = 0, 1$.

Proof of Theorem 3.1

3.2.1 Let h_1 be as in Definition 3.2. Denote by Λ the set $h_1 \circ h_0(C \times I)$ and by $\Lambda(a)$ the set $h_1 \circ h_0(a \times I)$. Note that $\pi_1(\Lambda(a)) = a'$ for some $a' \in \mathbb{R}$. Assume there is a near homeomorphism $H : D \to D$ such that on $C \times I$, $h_1 \circ h_0 \circ F = H \circ h_1 \circ h_0$. Before continuing with the proof, we stop to define a local ordering on $\Lambda = \{\Lambda(a) \,|a \in C\}$ and prove a lemma.

Order Definitions and Lemmas

Here we show that H preserves the local order of fibers as defined by M. Barge and R. Walker [BW93], which we will write as $<_{bw}$. And it will follow that $F_1 : C \to C$ is a "local order preserving homeomorphism."

Note: Since h_0 is tame, in this chapter, one could use the order of $\{\Lambda(a)|a \in C\}$ induced by π_1 instead of the $<_{bw}$. That is, $\Lambda(a) < \Lambda(b)$ if $\pi_1\Lambda(a) < \pi_1\Lambda(b)$. This is equivalent to the $<_{bw}$ order, and any extension of $h_1 \circ h_0 \circ F \circ (h_1 \circ h_0)^{-1}$ to D must preserve this order on $\Lambda(a)$. However, in the next chapter since the embedding is not tame we will need local preservation of the $<_{bw}$ order to the complete the proof of Theorem 4.8.

Barge-Walker Ordering For $a, b \in C$ suppose that γ_- and γ_+ are arcs in the plane with the properties:

γ_- has endpoints $h_1 \circ h_0\,(a, 0)$ and $h_1 \circ h_0\,(b, 0)$, and γ_- is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; γ_+ has endpoints $h_1 \circ h_0\,(a, 1)$ and $h_1 \circ h_0\,(b, 1)$ and γ_+ is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; and $(\gamma_- \cup \gamma_+) \cap \left([0, 2] \times \left\{\frac{1}{2}\right\}\right) = \emptyset$. Such arcs γ_- and γ_+ will be called <u>admissible arcs</u> joining $\Lambda(a)$ and $\Lambda(b)$. Here $\Lambda(a) = \{a'\} \times I$ and $\Lambda(b) = \{b'\} \times I$ and $\Lambda(a) <_{bw} \Lambda(b)$ if and only if $a' < b'$.

Definition 3.3 Given $a, b \in C$, $a \neq b$, then $\Lambda(a) <_{bw} \Lambda(b)$ if there are admissible arcs joining $\Lambda(a)$ and $\Lambda(b)$, as above, and the orientation $\gamma_- \to \Lambda(b) \to \gamma_+ \to \Lambda(a)$ is positive (counterclockwise) on the simple closed curve $\gamma_- \cup \Lambda(b) \cup \gamma_+ \cup \Lambda(a)$.

Definition 3.4 $<_X$ is <u>local ordering</u> on X if for all $x \in X$ there is a δ such that $<_X$ is an order relation on $\mathcal{B}_\delta(x)$. $(X, <_X)$ is a <u>locally ordered metric space</u>.

In [BW93] it is shown that if a and b are sufficiently close, $a \neq b$, then such admissible arcs exist. So either $\Lambda(a) <_{bw} \Lambda(b)$ or $\Lambda(b) <_{bw} \Lambda(a)$. Furthermore let $<_{bw}$ is a local ordering on $\Lambda = \{\Lambda(a)\,|a \in C\}$ where we use the metric $d(\Lambda(a), \Lambda(b)) = d(a, b)$.

Definition 3.5 Let $a, b \in C$. Then $a <_C b$ provided $\Lambda(a) <_{bw} \Lambda(b)$.

It follows from the proceeding remark $h_1 \circ h_0$ is uniformly continuous, that $<_C$ is a local ordering on C. Because $C \times I$ is tamely embedded, if a and b are sufficiently close then $a <_C b$ if and only if $a' < b'$ where $a' = \pi_1 \Lambda(a)$ and $b' = \pi_1 \Lambda(b)$.

Definition 3.6 Let $(X, <_X)$ and $(Y, <_Y)$ be locally ordered metric spaces. Let $G : (X, <_X) \to (Y, <_Y)$ be a homeomorphism. G is a <u>local order preserving homeomorphism</u>, if there is a $\delta > 0$ such that if $x_0, x_1 \in X$, $|x_0 - x_1| < \delta$, and $x_0 <_X x_1$, then $G(x_0) <_Y G(x_1)$.

Denote by $[x, y] = \{z \in C | x \leq_C z \leq_C y\}$. We next show $<_C$ on C is $\mathbb{R}$-like in the following sense.

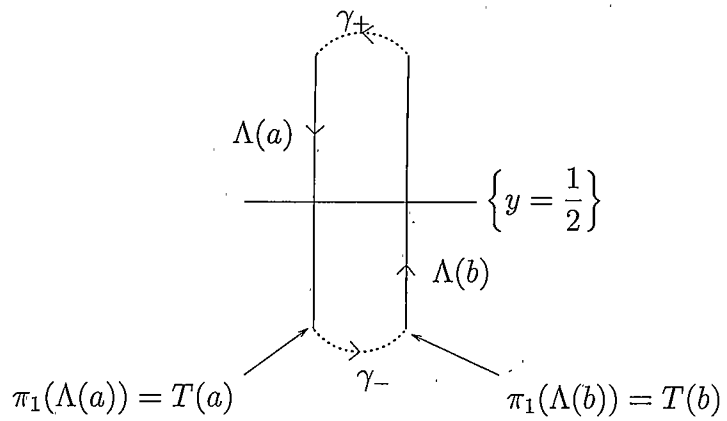

Figure 8: Barge–Walker Ordering on Cantor Fibers

Lemma 3.7 Given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in C$ and $|x - y| < \delta$, then for all $z \in [x, y]$, $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

Proof: Suppose throughout that $x, y, z \in C$ and $x <_C z <_C y$. By Definition 3.5 there are admissible arcs γ_1^+, γ_1^-, γ_2^+, and γ_2^- such that $\Lambda(z) \to \gamma_1^+ \to \Lambda(x) \to \gamma_1^-$ and $\Lambda(y) \to \gamma_2^+ \to \Lambda(z) \to \gamma_2^-$ have positive orientation.

Sublemma 3.8 For $\epsilon > 0$ there is a $\delta_1 > 0$ such that If $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ then $|x - z| < \epsilon$

Proof: By the continuity of $(h_1 \circ h_0)^{-1}$, if $\epsilon > 0$ there is a $\delta_1 > 0$ such that if $d(p, q) < \delta_1$ where $p, q \in \Lambda$ then $d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$. So if $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ there is $p \in \Lambda(x)$, $q \in \Lambda(z)$ such that $d(p, q) < \delta_1$. Thus $|x - z| = |\pi_1((h_1 \circ h_0)^{-1}(p)) - \pi_1((h_1 \circ h_0)^{-1}(q))| \leq d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$

□

Choose $\delta_1 > 0$ smaller so that if $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ and $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(y)) \neq \emptyset$ then $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

By the continuity of $(h_1 \circ h_0)$ there is $\delta > 0$ such that if $|x - y| < \delta$ then $\Lambda(x) \subset \mathcal{N}_{\delta_1}(\Lambda(y))$ and $\Lambda(y) \subset \mathcal{N}_{\delta_1}(\Lambda(x))$.

Suppose that $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \bigcap \mathcal{N}_{\delta_1}(\Lambda(y)) = \emptyset$. So either $\pi_1\Lambda(z) < \pi_1\Lambda(x)$ or $\pi_1\Lambda(y) < \pi_1\Lambda(z)$. Thus either $\Lambda(z) \to \gamma_1^+ \to \Lambda(x) \to \gamma_1^-$ has negative orientation or $\Lambda(z) \to \gamma_2^+ \to \Lambda(y) \to \gamma_2^-$ has positive orientation which contradicts $x <_C z <_C y$.

Thus $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset$ and $\Lambda(z) \bigcap \mathcal{N}_{\delta_1}(\Lambda(y)) \neq \emptyset$. So by the choice of δ then $|x - z| < \epsilon$ and $|y - z| < \epsilon$ as desired. □

Lemma 3.9 Let $f : (C, <_C) \to (C, <_C)$ be a local order preserving homeomorphism. Then there is a $\delta > 0$ such that if $|x - y| < \delta$ then $f([x, y]) = [f(x), f(y)]$

Proof: By Definition 3.6 there is an $\epsilon > 0$ such that for any $x, y \in C$ if $|x - y| < \epsilon$, and $x <_C y$ then $f(x) <_C f(y)$. By Lemma 3.7 there is a $\delta > 0$ such that if $x <_C z <_C y$ and $|x - y| < \delta$ then $|x - z| < \epsilon$ and $|y - z| < \epsilon$. Thus $f(x) <_C f(z)$ and $f(z) <_C f(y)$. □

The proof of the following lemma was suggested by M. Barge.

Lemma 3.10 Let $f : (C, <) \to (C, <)$ be a local order preserving homeomorphism. Then $h_{top}(f) = 0$.

Proof: Recall that $S \subset C$ is an <u>(n, ϵ)-spanning set</u>, for f if for all $x \in C$ there is a $y \in S$ such that $|f^k(x) - f^k(y)| < \epsilon$ for all $k = 0, 1, 2, \ldots n - 1$. Then $(h_{top})_\epsilon(f) = \limsup_{n\to\infty} \dfrac{\log \text{card } S(n, \epsilon)}{n}$, and $h_{top}(f) = \lim_{\epsilon \to 0} (h_{top})_\epsilon(f)$.

Choose δ as in Lemma 3.9 and suppose that $S \subset C$ is an (n, ϵ)-spanning set where $0 < \epsilon \leq \delta$ (δ from the lemma). Let X be a finite set of C that is ϵ-dense, say *card* $X = N$. Before proceeding with the proof of Lemma 3.10 we prove the following sublemma.

Sublemma 3.11 $S \bigcup f^{-n}(X)$ is an $(n+1, \epsilon)$-spanning set.

Proof: Let $x \in C$. Suppose that $y \in S$ is such that $|f^k(x) - f^k(y)| < \epsilon$ for $k = 0, 1, 2, \ldots n-1$. There is a $z \in X$ such that either $z \in [f^n(x), f^n(y)]$ or $z \in [f^n(y), f^n(x)]$, and such that $|f^n(x) - z| < \epsilon$. Then we have that $f^{-n}(z) \in S \cup f^{-n}(X)$ and z satisfies $|f^k(x) - f^k(z)| < \epsilon$ for $k = 0, 1, 2, \ldots n$ as desired. □

Continuing with proof of Lemma 3.10, it follows from Sublemma 3.11 that there exists a constant $K > 0$ such that for all n, card $S(n.\epsilon) \leq K + nN$. Thus,

$$\begin{aligned} h_{top}(f) &= \lim_{\epsilon \to 0} (h_{top})_\epsilon (f) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log card\ S(n, \epsilon)}{n} \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(K + nN)}{n} = 0 \end{aligned}$$

□

Lemma 3.12 Either H or H^2 locally preserves $<_{bw}$ on $\{\Lambda(a) | a \in C\}$.

Proof: By Lemma 2.1, $H|_{\Lambda(c)}$ is monotone for all $c \in C$. Fix $a_0 \in C$ and assume that $h_1 \circ h_0(\{a\} \times \{i\}) \subset \ell_i$ and $H \circ h_1 \circ h_0(\{a_0\} \times \{i\}) \subset \ell_i$ for $i = 0$ or 1. (The other cases are similar.) For all $a \neq a_0$ there exists an admissible arc, γ_a^- linking $h_1 \circ h_0(\{a_0\} \times \{0\})$ to $h_1 \circ h_0(\{a\} \times \{0\})$ and an admissible arc, γ_a^+ linking $h_1 \circ h_0(\{a_0\} \times \{1\})$ to $h_1 \circ h_0(\{a\} \times \{1\})$. Now H is monotone on the simple closed curve $\Gamma = \Lambda(a_0) \cup \gamma_a^- \cup \Lambda(a) \cup \gamma_a^+$. Thus H can be approximated by a homeomorphism $H' : D \to D$ such that $H'\Lambda(a_0) = H(\Lambda(a_0))$, $H'\Lambda(a) = H(\Lambda(a))$, $H\gamma_a^- = H(\gamma_a^-)$, and $H\gamma_a^+ = H(\gamma_a^+)$. So the orientation of $H(\Gamma)$ is identical to the orientation of $H'(\Gamma)$. For a sufficiently close to a_0 H' (or $(H')^2$) preserves $<_{bw}$ between $\Lambda(a_0)$ and $\Lambda(a)$

[BW93]. Thus H (or $(H)^2$) does as well. $\square$

Proof of Theorem 3.1 continued We now complete the proof of Theorem 3.1. We need only consider two cases. First suppose that F_1 does not locally preserve $<_C$. Then by Definition 3.5 H cannot locally preserve $<_{bw}$ on the fibers $\{\Lambda(a)|\ a \in C\}$, this contradicts Lemma 3.12 .

Next suppose F_1 does locally order preserve $<_C$. Then by Lemma 3.10 we have that $h_{top}(F_1) = 0$. And so by [Bow71] $h_{top}(F) = h_{top}(F_1) + \sup_{a\in C}\left\{h_{top}\left(F|_{\pi_1^{-1}(a)}\right)\right\} = \sup_{a\in C}\left\{h_{top}\left(F|_{\pi_1^{-1}(a)}\right)\right\}$. Thus if $h_{top}(F) > 0$ there is an $a_0 \in C$ such that $h_{top}\left(F|_{\pi_1^{-1}(a_0)}\right) > 0$. Then by Theorem 1.6 ([Bar87]) $F_2|_{a_0\times I}$ is not monotone. So by Theorem 2.1 no such near homeomorphism extension H of $h_1 \circ h_0 \circ F \circ (h_1 \circ h_0)^{-1}$ exists. $\square$

CHAPTER 4

Positive Entropy Maps of Temperate Embeddings of a Cantor Set Cross the Interval

The goal of this chapter is to prove Theorem 4.8 a strengthened version of Theorem 3.1 where the tame embedding hypothesis is weakened. First we need to classify non-topologically transverse intersections.

Topological Embeddings and Transversality

Let $D = \left\{(x,y) |\ x^2+y^2 \leq 4\right\}$. Also let $J = [-2,2] \times \left\{\frac{1}{2}\right\} \subset D$ and $\gamma : [0,1] \rightarrow D$ be a topological embedding.

Definition 4.1 Let $t_0 \in (0,1)$. γ is topologically transverse to J at $\gamma(t_0) \in J$, if there is an $\epsilon > 0$ such that $\pi_2 \circ \left(\gamma|_{(t_0-\epsilon, t_0+\epsilon)}\right)$ is one to one. γ is topologically transverse to J, or $\gamma \pitchfork J$, if γ is topologically transverse to J at all $p \in \gamma \bigcap J$.

Definition 4.2 (Nontransverse Intersection Types) Assume that $\gamma(t_0) \in J$ for $t_0 \in (0,1)$ and γ is not topologically transverse to J at $\gamma(t_0)$.

1. γ has type I intersection with J at $\gamma(t_0)$ if there exists $\epsilon_0 > 0$ such that either $\gamma^{-1}(\gamma[t_0-\epsilon_0, t_0] \bigcap J)$ or $\gamma^{-1}(\gamma[t_0, t_0+\epsilon_0] \bigcap J)$ is an interval.

2. γ has type II intersection with J at $\gamma(t_0)$ if there is an $\epsilon > 0$ such that

$$\gamma[t_0-\epsilon, t_0+\epsilon] \bigcap J = \{\gamma(t_0)\}.$$

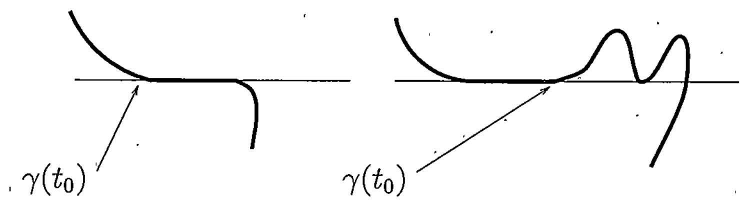

Figure 9: Type I intersections

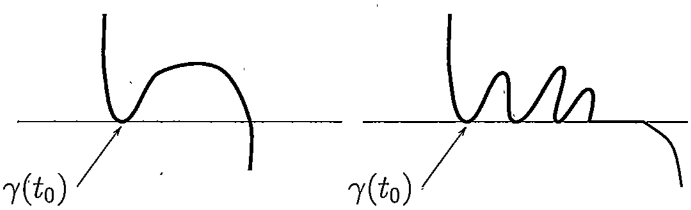

Figure 10: Type II intersections

3. γ has <u>type III intersection</u> with J at $\gamma(t_0)$ if for all $\epsilon > 0$,

$$\text{card } \left\{ \gamma^{-1}(\gamma([t_0 - \epsilon, t_0 + \epsilon]) \right\} > 1$$

and there exists $\epsilon_0 > 0$ such that $\mathcal{B}_\epsilon \gamma(t_0)$ does not contain any type I or II intersections.

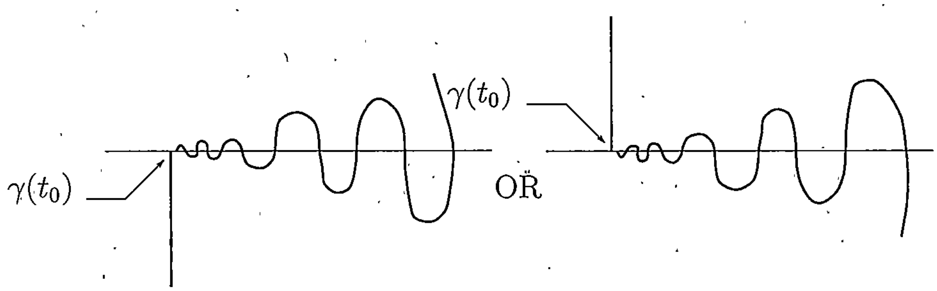

Figure 11: Type III intersections

4. γ has <u>type IV intersection</u> provided $\gamma(t_0)$ is not type I,II, or III

The next lemma is automatic.

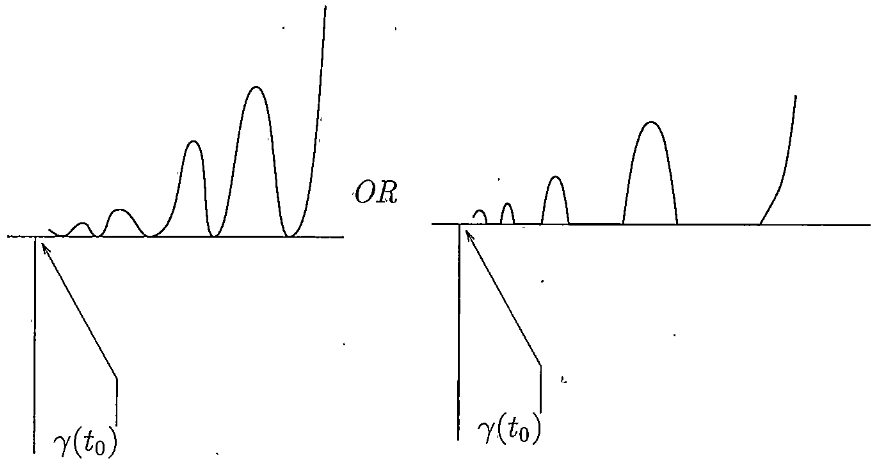

Figure 12: type IV intersection

Lemma 4.3 If $\gamma(t_0) \in J$ then either γ is topologically transverse to J at $\gamma(t_0)$ or has type I,II, III, or IV intersection with J at $\gamma(t_0)$.

4.1.1 Again let $C \subset [0,1]$ be a Cantor set with order $<_C$ inherited form $\mathbb{R}$. Let $h_0 : C \times I \to C \times I$ be a topological embedding. Again $\ell_0 = \{(x,y)|y = 0\}$ and $\ell_1 = \{(x,y)|y = 1\}$. By E. Moise ([Moi77]), there is a surjective homeomorphism $h_1 : D \to D$ such that $h_1 \circ h_0(a,0) \in \ell_0$ and $h_1 \circ h_0(a,1) \in \ell_1$ for all $a \in C$. Let $\Lambda = h_1 \circ h_0(C \times I)$ and $\Lambda\,(a) = h_1 \circ h_0(a \times I)$. Also let $\lambda_a : [0,1] \to \Lambda\,(a)$ be given by $\lambda_a(t) = h_1 \circ h_0(a,t)$. Lastly let

$$\mathcal{S} = \{a \in C | \Lambda\,(a) \text{ has a type I, II, or IV intersection with } J \text{ at some } p \in J\}.$$

Lemma 4.4 $\mathcal{S}$ is countable.

Proof: Here $\pi_1, \pi_2; D \to [-2,2]$ are the coordinate projection map. If $a \neq b$ then $\Lambda\,(a) \bigcap \Lambda\,(b) = \emptyset$. If $\Lambda\,(a)$ has a type I intersection with J then $\Lambda\,(a) \bigcap J$ contains an interval with nonzero length. J contains only a countable number of intervals with nonzero length [Roy88]. So the set of $a \in C$ such that $\Lambda\,(a)$ has type I intersection with J is countable.

Assume that $\Lambda(a_0)$ has type II intersection with J at $p = \lambda_{a_0}(t_0)$ and assume that $\epsilon_0 > 0$ such that $\lambda_{a_0}(t_0 - \epsilon_0, t_0 + \epsilon_0) \bigcap J = p$. Then either $\pi_2(\lambda_{a_0}(t_0 - \epsilon, t_0))$ and $\pi_2(\lambda_{a_0}(t_0, t_0 + \epsilon)) \subset \left[0, \frac{1}{2}\right)$ or both are contained in $\left(\frac{1}{2}, 1\right]$. Otherwise $\Lambda(a_0)$ is transverse to J at p. Assume the first. The other case is similar. Let $\ell_n = [-2, 2] \times \left\{\frac{1}{2} - \frac{1}{n}\right\}$. Then there exist n_0 such that $\ell_{n_0} \bigcap \lambda_{a_0}(t_0 - \epsilon, t_0) \neq \emptyset$ and $\ell_{n_0} \bigcap \lambda_{a_0}(t_0, t_0 + \epsilon) \neq \emptyset$. So there exists $t_0^-(n_0)$ and $t_0^+(n_0)$ such that $\lambda_{a_0}(t_0^-(n_0))$, $\lambda_{a_0}(t_0^+(n_0)) \in \ell_{n_0}$ but for all $t \in (t_0^-(n_0), t_0) \bigcup (t_0, t_0^+(n_0))$, $\lambda_a(t) \notin \ell_{n_0}$. We call $t^-(n_0)$ (resp. $t^+(n_0)$) the "previous" (resp. "next") intersection of $\Lambda(a)$ with ℓ_{n_0}. Now let $\ell_{n_0}(a_0) \subset \ell_{n_0}$ be the line segment with nonzero length which has endpoints $\lambda_{a_0}(t_0^-(n_0))$ and $\lambda_{a_0}(t_0^+(n_0))$. So the type II intersection of $\Lambda(a_0)$ with J has determined the nonzero line segment $\ell_{n_0}(a_0)$ in ℓ_{n_0}.

We now argue that this assignment is unique in the following sense: Assume for $a_1 \neq a_0$, $\Lambda(a_1)$ has type II intersection with J at $\lambda_{a_1}(t_1)$ with corresponding $\epsilon_1 > 0$. And assume $\ell_{n_0} \bigcap \lambda_{a_1}[(t_1 - \epsilon_1, t_1)] \neq \emptyset$ and $\ell_{n_0} \bigcap \lambda_{a_1}[(t_1, t_1 + \epsilon_1)] \neq \emptyset$. Then there exists $\ell_{n_0}(a_1) \subset \ell_{n_0}$ with the endpoints $\lambda_{a_1}(t_1^-(n_0))$ and $\lambda_{a_1}(t_1^+(n_0))$, as before. That $\ell_{n_0}(a_0) \bigcap \ell_{n_0}(a_1) = \emptyset$ follows from the fact that $\Lambda(a_0) \bigcap \Lambda(a_1) = \emptyset$, the choice of previous and next intersections, and the Jordan Curve Theorem [Mun75].

Let $A_n \subset C$ be the set of $a \in C$ such that $\ell_n(a)$ has been determined by the above procedure. Since each $\ell_n(a)$ has nonzero length(because $h_1 \circ h_0$ is an embedding). A_n is countable for each n. But as noted, if $\Lambda(a)$ has type II intersection with J then $a \in A_n$ for some n. Since arbitrarily near all type IV intersections there exist type I and II intersections of $\Lambda(a)$ with J, the proof follows. □

Temperate Embedding

The following conjecture is apparently open.

Conjecture 4.5 All topological embeddings, $h_0 : C \times [0,1] \to D$, are tame (Definition 3.2).

However, In this chapter we prove a stronger version of Theorem 3.1 for so-called temperate embeddings of $C \times I$ into D. All tame embeddings of $C \times I$ are "temperate."

Definition 4.6 If for a given $a \in C$ there exist unique $t_{bottom}(a)$ and $t_{top}(a)$ such that $\pi_2(\lambda_a(t_{top}(a))) = 1$, $\pi_2(\lambda_a(t_{bottom}(a))) = 0$, and $\pi_2(\lambda_a(t)) \in \pi_2^{-1}(0,1)$ for all $t_{bottom}(a) < t < t_{top}(a)$, then $\Lambda(a)$ makes one crossing from near the bottom to near the top.

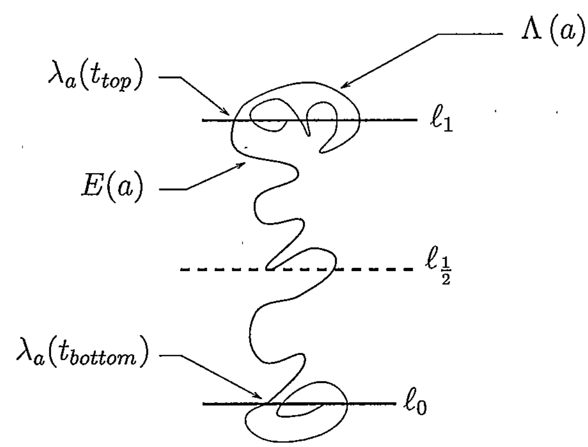

Figure 13: Temperate Embedding

Definition 4.7 A topological embedding, $h_0 : C \times I \to D$ is a temperate embedding if there exists $h_1 : D \to D$ such that for all $a \in C$, $h_1 \circ h_0(a,0) \in \ell_0$, $h_1 \circ h_0(a,1) \in \ell_1$ and each $\Lambda(a)$ makes one crossing from near the bottom to near the top.

Theorem 4.8 Let $F : \Sigma \times I \to \Sigma \times I$ be a surjective map such that $F(a,y) = (F_1(a), F_2(a,y))$ and $F_1 : C \to C$ is a homeomorphism. If $h_{top}(F) > 0$ then there exists no temperate embedding $h_0 : C \times I \to D$ such that $h_0 \circ F \circ h_0^{-1}$ extends to a near homeomorphism of the disk D.

Proof: Assume h_0 is a temperate embedding. That is (as in Definition 4.7), let $h_1 : D \to D$ be such that $h_1 \circ h_0 (C \times \{0\}) \subset \ell_0$, $h_1 \circ h_0 (C \times \{1\}) \subset \ell_1$, and each $\Lambda(a) = h_1 \circ h_0 (a \times I)$ makes one crossing form near the bottom to near the top. Assume there exists a near homeomorphism such that $H_1|_{h_1 \circ h_0(C \times I)} = h_1 \circ h_0 \circ F \circ h_0^{-1} \circ h_1^{-1}$.

We need to show that the $<_{bw}$ order on fibers $\{\Lambda(a)\}$ is "$\mathbb{R}$-like" in the same sense as Lemma 3.7. To do this we will produce an arc $\gamma : [0,1] \to D$ which crosses the fibers one at a time (although γ may follow along a countable number of fibers). More discussion of this scheme follows Lemma 4.12

Let $E(a) \subset \Lambda(a)$ be given by $E(a) = \{\lambda_a(t(a)) | \, t_{bottom}(a) < t < t_{top}(a)\}$. Again let $J = [-2, 2] \times \left\{\frac{1}{2}\right\}$. We will need closed sets $J_n \subseteq J$, where each J_n is J_{n-1} with a certain open interval removed. We will use the notation $K_n(a) = \min \pi_1(E(a) \bigcap J_n)$ which always exists. And $L_n(a) = \min \pi_1(E(a) \pitchfork J_n)$, if it exists. (Note: $\pitchfork$ denotes topologically transverse intersection.) For each a, let $\varepsilon_a : [0,1] \to E(a)$ be given by $\varepsilon_a(t) = \lambda_a((t_{top}(a) - t_{bottom}(a))t + t_{bottom}(a))$. So $\varepsilon_a(0) \in \ell_0$ and $\varepsilon_a(1) \in \ell_1$. Let $t_0(a,n)$ be given such that $\varepsilon_a(t_0(a,n)) = (K_n(a), \frac{1}{2})$. Now define $t_1(a,n)$ as follows:

4.1.3 If $K_n(a) \neq L_n(a)$, choose $t_1(a,n)$ such that

$$\varepsilon_a(t_1(a,n)) = \left(\max \pi_1(E(a) \bigcap J_n), \frac{1}{2}\right).$$

4.1.4 If $K_n(a) = L_n(a)$, then $t_1(a,n) = t_0(a,n)$.

Now we will recursively define $\{J_n\}$ and arcs $\{\gamma_k : [0,1] \to D\}$ and ultimately show that $\{\gamma_k\}$ converges uniformly. Let $\gamma_0 : [0,1] \to J$ be given by $\gamma_0(\tau) = \left(4\tau - 2, \frac{1}{2}\right)$.

4.1.5 Then there are $\tau_0, \tau_1 : C \to [0,1]$ such that $\gamma_0(\tau_0(a)) = \varepsilon_a(t_0(a,1)) = \left(K_1(a), \frac{1}{2}\right)$ and $\gamma_0(\tau_1(a)) = \varepsilon_a(t_1(a,1)) = \left(L_1(a), \frac{1}{2}\right)$. Notice $\tau_0(a) \leq \tau_1(a))$ and $\pi_1(\gamma_0(\tau_0(a))) \leq \pi_1(p)$ for all $p \in E(a) \bigcap J$ (see Figure 14).

By Lemma 4.4 $\mathcal{S}$ is countable so let, $\mathcal{S} = \{a_k\}_{k=0}^{\infty}$. First set $J_1 = J$. Assume there exists a first $a \in \mathcal{S}$ call it a_1 such that $K_1(a_1) \neq L_1(a_1)$ (otherwise let $\gamma_k = \gamma_0$ for all k). We now modify γ_0 by following $E(a_1)$ as follows. Let $\gamma_1 : [0,1] \to \mathbb{R}^2$ be defined as follows:

$$\gamma_1(\tau) = \begin{cases} \varepsilon_{a_1}\left(\frac{t_1(a_1,1)-t_0(a_1,1)}{\tau_1(a_1)-\tau_0(a_1)}(\tau - \tau_0(a_1)) + t_0(a_1,1)\right) & ; \tau \in [\tau_0(a_1), \tau_1(a_1)] \\ \gamma_0(\tau) & ; \text{elsewhere} \end{cases}$$

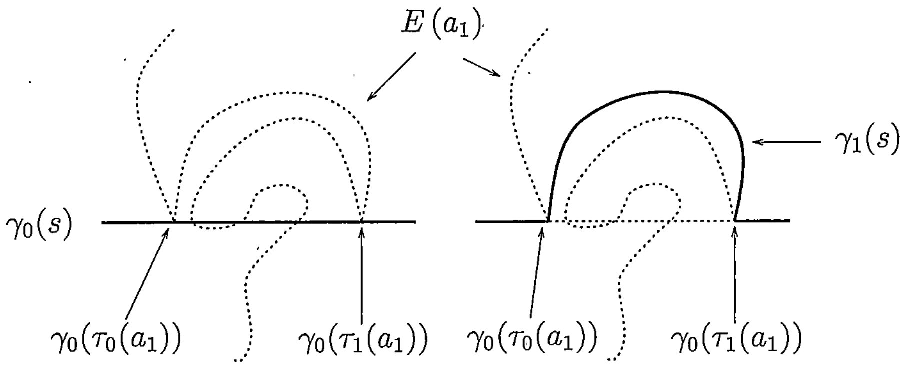

Figure 14: Creating γ_1

Let $I_1 \subset J_1$ be the open line segment from $\gamma_0(\tau_0(a_1))$ to $\gamma_0(\tau_1(a_1))$. Now set $J_2 = J_1 \setminus I_1$. Assume there exists a next $a \in \mathcal{S}$ call it a_2 such that $K_2(a_2) \neq L_2(a_2)$, otherwise $\gamma_k = \gamma_1$ for all $k \geq 2$. Adjust τ_0 and τ_1 so that $\tau_0(a_2)$ and $\tau_1(a_2)$ are such that $\gamma_0(\tau_0(a_2)) = \varepsilon_{a_2}(t_0(a_2,2)) = \left(K_2(a_2), \frac{1}{2}\right)$ and $\gamma_0(\tau_1(a_2)) = \varepsilon_{a_2}(t_1(a_2,2)) = \left(L_2(a_2), \frac{1}{2}\right)$. (Note: $\tau_0(a_2) < \tau_1(a_2)$). Now we modify γ_1 by following along $E(a_2)$ as follows. Let $\gamma_2 : [0,1] \to \mathbb{R}^2$ be defined as follows:

$$\gamma_2(\tau) = \begin{cases} \varepsilon_{a_2}\left(\frac{t_1(a_2,2)-t_0(a_2,2)}{\tau_1(a_2)-\tau_0(a_2)}(\tau - \tau_0(a_2)) + t_0(a_2,2)\right) & ; \tau \in [\tau_0(a_2), \tau_1(a_2)] \\ \gamma_1(\tau) & ; \text{elsewhere} \end{cases}$$

4.1.6 If necessary, recursively define γ_k as follows: Let $I_{k-1} \subset J_{k-1}$ be the open line segment from $\gamma_0(\tau_0(a_{k-1}))$ to $\gamma_0(\tau_1(a_{k-1}))$. (Note that $I_i \cap I_j = \emptyset$ if $i \neq j$.) Let $J_k = J_{k-1} \setminus I_{k-1}$. Assume there exists another $a \in \mathcal{S}$ call it a_k such that $K_k(a_k) \neq L_k(a_k)$, (otherwise set $\gamma_n = \gamma_{k-1}$ for $n \geq k-1$). Then adjust τ_0 and τ_1 such that

$\gamma_0(\tau_0(a_k)) = \varepsilon_{a_k}(t_0(a_k,k))$ and $\gamma_0(\tau_1(a_k)) = \varepsilon_{a_k}(t_1(a_k,k))$. (Note: $\tau_0(a_i) < \tau_1(a_i)$ for all $i \leq k$). Let $\gamma_k : [0,1] \to \mathbb{R}^2$ be defined as follows:

$$\gamma_k(\tau) = \begin{cases} \varepsilon_{a_k}\left(\frac{t_1(a_k,k)-t_0(a_k,k)}{\tau_1(a_k)-\tau_0(a_k)}(\tau - \tau_0(a_k)) + t_0(a_k,k)\right) & \tau \in [\tau_0(a_k), \tau_1(a_k)] \\ \gamma_{k-1}(\tau) & ;\text{ elsewhere} \end{cases}$$

So we may assume [4.1.4] holds for each a such that $a \notin \{a_k\}_{k=1}^{\infty}$. Next we show that $\{\gamma_k\}$ uniformly converge to a topological embedding γ. Each γ_k has been formed by removing a segment from γ_{k-1}, then linking the two end points via an arc from $E(a_k)$. Thus, the following lemma is clear.

Lemma 4.9 For all k, $\gamma_k : [0,1] \to \mathbb{R}^2$ is a topological embedding.

Lemma 4.10 The sequence $\{\gamma_k(t)\}$ is uniformly convergent.

Proof: Let $\epsilon > 0$ be given. See Lemma 4.4 for the construction of $\{a_k\}_{k=1}^{\infty}$. Assume $a \leq n < m$ By construction if $n < m$ then $\gamma_n(\tau) = \gamma_m(\tau)$ for $\tau \in I \setminus \left(\bigcup_{k=n+1}^{m} [\tau_0(a_k), \tau_1(a_k)]\right)$. Because $\{[\tau_0(a_k), \tau_1((a_k)]\}_{k=1}^{\infty}$ are disjoint and $\sum_{k=1}^{\infty} |\tau_0(a_k) - \tau_1(a_k)| \leq 4$ then $|\tau_0(a_k) - \tau_1(a_k)| \to 0$ as $k \to \infty$. Thus for all δ there exists K such that $|\tau_0(a_k) - \tau_1(a_k)| < \delta$ for all $k > K$. Assume thoughout that $n > K$.

We need only to consider $\tau \in [\tau_0(a_k), \tau_1(a_k)]$ for some $n+1 \leq k \leq m$. Then by construction $\gamma_m(\tau) \in E(a_k)$ (recall that since $m \geq n$, $\gamma_m(\tau_0(a_k)), \gamma_m(\tau_1(a_k)) \in E(a_k)$). And since $h_1 \circ h_0$ is uniformly continuous [Mun75] then for all $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that if $|\tau_0(a_k) - \tau_1(a_k)| < \delta_1$ then $\|\gamma_m(\tau_0(a_k)) - \gamma_m(\tau)\| < \frac{\epsilon}{2}$. Also by [4.1.6] $\gamma_n(\tau) \in J$ and so $\pi_1(\gamma_n(\tau_0(a_k))) \leq \pi_1(\gamma_n(\tau)) \leq \pi_1(\gamma_n(\tau_1(a_k)))$. Thus there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that if $|\tau_0(a_k) - \tau_1(a_k)| < \delta_2$ then $\|\gamma_n(\tau) - \gamma_n(\tau_0(a_k))\| < \frac{\epsilon}{2}$.

For $\epsilon > 0$ choose $\delta = \delta(\epsilon) < \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$ and K large enough that for $n > K$ and $n+1 \leq k \leq m$ then $|\tau_0(a_k) - \tau_1(a_k)| < \delta$. Thus since $\gamma_m(\tau_0(a_k)) = \gamma_n(\tau_0(a_k))$, we have

$$\|\gamma_m(\tau) - \gamma_n(\tau)\| = \|\gamma_m(\tau) - \gamma_m(\tau_0(a_k)) + \gamma_m(\tau_0(a_k)) - \gamma_n(\tau)\|$$

$$= \|\gamma_m(\tau) - \gamma_m(\tau_0(a_k)) + \gamma_n(\tau_0(a_k)) - \gamma_n(\tau)\| \leq$$

$$\|\gamma_m(\tau) - \gamma_m(\tau_0(a_k))\| + \|\gamma_m(\tau_0(a_k)) - \gamma_n(\tau)\| \leq \epsilon$$

And so $\{\gamma_k\}$ is uniformly Cauchy. □

Let $\gamma = \lim_{k\to\infty} \gamma_k$.

Lemma 4.11 For all $\tau \in [0,1]$, there is a K such that $\gamma(\tau) = \gamma_k(\tau)$ for all $k > K$.

Proof: Let τ be fixed. Either for all k, $\gamma_k(\tau) \in J$ and so $\gamma_k(\tau) = \gamma_0(\tau)$, and $\gamma(\tau) = \gamma_0(\tau)$. Or there is a $K > 0$ such that $\gamma_K(\tau) \in E(a_K) \setminus J$. In such case $\tau \in [\tau_0(a_K), \tau_1(a_K)]$. So assume $\tau \in [\tau_0(a_k), \tau_0(a_k)]$. But then by construction if $k > K$ then $\gamma_k(\tau) = \gamma_K(\tau)$ for $\tau \in I \setminus \left(\bigcup_{i=K+1}^{k} [\tau_0(a_i), \tau_1(a_i)] \right)$. Since $\tau \notin \bigcup_{i=K+1}^{k} [\tau_0(a_i), \tau_1(a_i)]$, $\gamma_k(\tau) = \gamma_K(\tau)$ for all $k > K$. □

Lemma 4.12 γ is a topological embedding.

Proof: By Lemma 4.10 $\{\gamma_k\}$ uniformly converges to γ. Thus γ is continuous. Suppose that there is $\tau_0 \neq \tau_1$ and $\gamma(\tau_0) = \gamma(\tau_1)$. By Lemma 4.11 there are $K_0 = K_0(\tau_0)$ and $K_1 = K_1(\tau_1)$, such that $\gamma(\tau_0) = \gamma_k(\tau_0)$ and $\gamma(\tau_1) = \gamma_k(\tau_1)$, if $k = \max\{K_0, K_1\}$. Thus $\gamma_k(\tau_0) = \gamma_k(\tau_1) = \gamma(\tau_0) = \gamma(\tau_1)$ and so γ_k is not one-to-one, which contradicts Lemma 4.9 □

Lemma 4.13 If $E(b_0) \pitchfork J$ at $\gamma_0\left(\left(\tau_0(b_0), \frac{1}{2}\right)\right)$ then $\tau_0(b_0)$ is continuous at b_0.

Proof: Let $q_0 = \Lambda(a) \bigcap \ell_{\frac{1}{2}} = \Lambda(a) \pitchfork \ell_{\frac{1}{2}}$ and $p_0 = (h_1 \circ h_0)^{-1}(q)$. Define the following sequences $\{p_n^+\}$, $\{p_n^-\}$, $\{p_{n,k}^+\}$ and $\{p_{n,k}^-\}$ as follows:

$$p_n^+ \to p_0 \text{ such that } p_n^+ \in (h_1 \circ h_0)^{-1}\Lambda(a) \text{ and } \pi_2 p_n^+ > \pi_2 p_0,$$

$$p_n^- \to p_0 \text{ such that } p_n^- \in (h_1 \circ h_0)^{-1}\Lambda(a) \text{ and } \pi_2 p_n^- < \pi_2 p_0,$$

$$p_{n,k}^+ \to p_n^+ \text{ such that } p_{n,k}^+ \in (h_1 \circ h_0)^{-1}\Lambda(a_k), \text{ and}$$

$$p_{n,k}^- \to p_n^- \text{ such that } p_{n,k}^- \in (h_1 \circ h_0)^{-1}\Lambda(a_k)$$

Let $q_{n,k}^+ = (h_1 \circ h_0)(p_{n,k}^+)$ and $q_{n,k}^- = (h_1 \circ h_0)(p_{n,k}^-)$. For large enough k and n, $\pi_2 q_{n,k}^+ > \frac{1}{2}$ and $q_{n,k}^- < \frac{1}{2}$. For any $\epsilon > 0$ there is large enough k and n such that $|q_{n,k}^+ - q_0| < \epsilon$ and $|q_{n,k}^- - q_0| < \epsilon$. Thus there are subsequences $\{q_{n_j,k_j}^+ = q_j^+\}, \{q_{n_j,k_j}^- = q_j^-\}$ such that $q_j^+ \to q_0$ and $q_j^- \to q_0$. Thus for large enough k and n, $|q_j^+ - q_j^-|\epsilon$.

Since $\pi_2 q_j^+ > \dfrac{1}{2}$ and $\pi_2 q_j^- < \dfrac{1}{2}$ there is point $\alpha_j \in \Lambda(a_{k_j}) \bigcap \ell_{\frac{1}{2}}$ and between q_j^+ and q_j^- on $\Lambda(a_{k_j})$. Now by the continuity of the inverse

$$\pi_2 q_j^- < \pi_2 (h_1 \circ h_0)^{-1}(\alpha_j) < \pi_2 q_j^+$$

and thus $\alpha_j \in \mathcal{B}_\epsilon q_0$ for all j.

Let $p_0 = \left(\tau_0, \frac{1}{2}\right)$. It follows from Definition 4.1 that for all $\epsilon > 0$, both

$$\mathcal{B}_\epsilon(p_0) \bigcap \left[\pi_2^{-1}\left(\left(\frac{1}{2}, 1\right)\right)\right] \bigcap E\left(b_0\right) \neq \emptyset$$

and

$$\mathcal{B}_\epsilon(p_0) \bigcap \left[\pi_2^{-1}\left(\left(0, \frac{1}{2}\right)\right)\right] \bigcap E\left(b_0\right) \neq \emptyset.$$

By the uniform continuity of $h_1 \circ h_0$ and the connectivity of each $E\left(b\right)$, if $b_k \to b_0$ then there exists $p_k \in E\left(b_k\right) \cap J$ such that $p_k \to p_0$. So $\pi_1(p_k) \to \pi_1(p_0)$ and by [4.1.5] $\pi_1 \circ \gamma(\tau_0(b_k)) \leq \pi_1(p_k)$ for all k.

Assume there is $\epsilon_0 > 0$ and that $\tau_0(b_k) < \tau_0(b_0) - \epsilon_0$ for k sufficiently large. By passing to a subsequence, if necessary, assume $\{\pi_1 \circ \gamma(\tau_0(b_k))\}$ converges to x_0 (or that $\left(\pi_1 \circ \gamma(\tau_0(b_k)), \frac{1}{2}\right) \to \left(x_0, \frac{1}{2}\right)$) where $x_0 \leq \pi_1 \circ \gamma(\tau_0(b_0)) - \epsilon_0$. Define $B_n = \mathcal{B}_{\frac{1}{n}}\left(x_0, \frac{1}{2}\right)$ $n \geq 1$. Then for sufficiently large k, $E\left(b_k\right) \bigcap B_n \neq \emptyset$. But by uniform continuity of $h_1 \circ h_0$ and that $a_k \to b_0$, we have that $E\left(b_0\right) \bigcap B_n \neq \emptyset$ for all $n \geq 1$. Let $q_n \in E\left(b_0\right) \bigcap B_n$. Then $q_n \to \left(x_0, \frac{1}{2}\right)$. But $E\left(b_0\right)$ is closed so

$\left(x_0, \frac{1}{2}\right) \in E(b_0)$ where $x_0 < \tau_0(b_0) - \epsilon_0$ contradicting the choice of $\tau_0(b_0)$. Similarly assuming $\tau_0(b_k) > \tau_0(b_0) + \epsilon_0$ for large k leads to a contradiction. □

Lemma 4.14 Assume $E(b_0)$ has type III intersection with J at $\left(\tau_0(b_0), \frac{1}{2}\right)$ and $E(b_0)$. If $\{b_k\} \to b_0$, then there exists $\left(x_k, \frac{1}{2}\right) \in E(b_k) \bigcap J$ such that $x_k \to \tau_0(b_0)$.

Proof: Let $x_0 = \tau_0(b_0)$. So there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $p \in E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right)$ and $p \neq \left(x_0, \frac{1}{2}\right)$, $E(a_0)$ is topologically transverse to J at p. For otherwise either $\left(x_0, \frac{1}{2}\right)$ is isolated (thus not type III) in $E(b_0) \bigcap J$ or $E(a_0)$ has a type I or II intersection with J.

Each $E(b_k) \bigcap J \neq \emptyset$. Therefore it is enough to show for all $\epsilon > 0$, there is $K = K(\epsilon)$ such that $E(b_k) \bigcap J \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset$. By our type III assumption there exists $p_\epsilon \in E(b_0) \bigcap J \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right)$ such that $E(b_0)$ is topologically transverse to J at p_ϵ. Then

$$\pi_2^{-1}\left[0, \frac{1}{2}\right) \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset$$

and

$$\pi_2^{-1}\left(\frac{1}{2}, 1\right] \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset.$$

Furthermore, there exists $t_\epsilon^+, t_\epsilon^- \in [0,1]$ such that

$$p^+ = \varepsilon_{b_0}(t^+) \in \pi_2^{-1}\left[0, \frac{1}{2}\right) \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset,$$

$$p^- = \varepsilon_{b_0}(t^-) \in \pi_2^{-1}\left(\frac{1}{2}, 1\right] \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset,$$

and $p_\epsilon \in \varepsilon_{b_0}[t_\epsilon^+, t_\epsilon^-] \subset \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right)$. Therefore, by the uniform continuity of $h_1 \circ h_0$ for k sufficent large, there is a point

$$p_k^+ \in \pi_2^{-1}\left[0, \frac{1}{2}\right) \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset$$

and

$$p_k^- \in \pi_2^{-1}\left(\frac{1}{2}, 1\right] \bigcap E(b_0) \bigcap \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right) \neq \emptyset.$$

Let $p_k^- = \varepsilon_{b_k}(t_k^-)$ and $p_k^+ = \varepsilon_{b_k}(t_k^+)$. Since $\varepsilon_{b_0}[t_k^+, t_k^-] \subset \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right)$, for k larger, $\varepsilon_{b_k}[t_k^+, t_k^-] \subset \mathcal{B}_\epsilon\left(x_0, \frac{1}{2}\right)$. But each $E(b_k)$ is path connected. So

$$\left(x_k.\frac{1}{2}\right) = \varepsilon_{b_k}\left(t_k^+, t_k^-\right) \bigcap J \bigcap \mathcal{B}_\epsilon x_0, \frac{a}{2} \neq \emptyset$$

has the desired property. □

Proof of Theorem 4.8 continued Next we define two special functions g and T. The function g is used to "crush out" the sections of γ, which follows along fibers of $\{E(a_k)\}$, to a single points. The function T will be a "local order preserving" topological embedding of $(C, <_C)$ (Definition 3.5) into $([0,1], <)$. Let $g : [0,1] \to [0,1]$ be an onto, continuous function that is constant on each $[\tau_0(a_k), \tau_1(a_k)]$, for all k, and strictly increasing elsewhere. Define $T : C \to \mathbb{R}$ by $T(a) = g(\tau_0(a))$.

Lemma 4.15 $T : C \to \mathbb{R}$ is a topological embedding.

Proof: All that is necessary is to show that T is continuous and injective. First we will show that T is continuous. Let $\epsilon > 0$ be given. By the continuity of g there is a δ_1 such that if $|\tau_1(a) - \tau_0(b)| < \delta_1$ or $|\tau_0(a) - \tau_1(b)| < \delta_1$, then $|g(\tau_1(a)) - g(\tau_0(b))| < \epsilon$ or $|g(\tau_0(a)) - g(\tau_1(b))| < \epsilon$. Also by the continuity of γ there is a δ_2 such that if $|\gamma(\tau_1(a)) - \gamma(\tau_0(b))| < \delta_2$ or $|\gamma(\tau_0(a)) - \gamma(\tau_1(b))| < \delta_2$ then $|\tau_1(a) - \tau_0(b)| < \delta_1$ or $|\tau_0(a) - \tau_1(b)| < \delta_1$. There is a δ_3 such that $d(a,b) < \delta_3$ then $E(a) \subset \mathcal{N}_{\delta_2}(E(b))$. By 4.1.3-4] and $\gamma(\tau_i(a)) = \varepsilon_a(t_i(a))$ for $i = 0, 1$ we have that $|\gamma(\tau_1(a)) - \gamma(\tau_0(b))| < \delta_2$ or $|\gamma(\tau_0(a)) - \gamma(\tau_1(b))| < \delta_2$. And so if $d(a,b) < \delta_3$ then $|g(\tau_1(a)) - g(\tau_0(b))| < \epsilon$ or $|g(\tau_0(a)) - g(\tau_1(b))| < \epsilon$. Since $g(\tau_1(a)) = g(\tau_0(a))$ for all a, then $|g(\tau_0(a)) - g(\tau_0(b))| < \epsilon$. Thus $|T(a) - T(b)| < \epsilon$. And T is continuous.

Next we will show that T is one-to-one. Suppose that $T(c) = T(d)$ then $g(\tau_0(c)) = g(\tau_0(d))$. Then $\tau_0(d) \in [\tau_0(c), \tau_1(c)]$. Then either we have a contradiction or by definition of γ, $\varepsilon_d(t_0(d)) \in E(c)$. But $\varepsilon_d(t_0(d)) \in E(d)$. And so $E(c) = E(d)$ or $c = d$. □

Order

Next it is necessary to show that T preserves $<_C$ which is induced by the Barge–Walker order of fibers (See Definition 3.5).

Lemma 4.16 If $\pi_1\gamma(\tau_0(a)) < \pi_1\gamma(\tau_0(b))$ then $g(\tau_0(a)) < g(\tau_0(b))$.

Proof: Using the final choice of $\tau(a)$ we know that $\gamma_0(\tau_0(a)) = \gamma(\tau_0(a))$ then $\pi_1\gamma_0(\tau_0(a)) < \pi_1\gamma_0(\tau_0(b))$. Since $\pi_1 \circ \gamma_0$ is increasing then $\tau_0(a) < \tau_0(b)$. Because g is nondecreasing and $\tau_0(b) \not\leq \tau_0(a)$ then $g(\tau_0(a)) < g(\tau_0(b))$. □

Lemma 4.17 $T : (C, <_C) \to (\mathbb{R}, <)$ is a local order preserving embedding.

Proof: Suppose that $a <_C b$. Then by Definition 3.5, $E(a) <_{bw} E(b)$. So as in Definition 3.3, there exist admissible arcs γ_l and γ_u such that the simple closed curve $E(b) \to \gamma_u \to E(a) \to \gamma_l$ is positively oriented (counterclockwise).

For brevity, let the projection of the "first' and "last" intersections of $E(a)$ and $E(b)$ with J be $m = \pi_1\varepsilon_a(t_0(a))$, $\tilde{m} = \pi_1\varepsilon_b(t_0(b))$, $M = \pi_1\varepsilon_a(t_1(a))$, and $\tilde{M} = \pi_1\varepsilon_b(t_1(b))$.

To apply Lemma 4.16 we need to show that $E(a) <_{bw} E(b)$ implies $m < \tilde{m}$. We will do this by considering all possible real line orderings of four points. But, because $m \leq M$ and $\tilde{m} \leq \tilde{M}$ we need only consider six possibilities for the order of $m, M, \tilde{m}$, and $\tilde{M}$. The conclusion is automatic for either $m \leq M < \tilde{m} \leq \tilde{M}$ or $m < \tilde{m} < M \leq \tilde{M}$.

Next let us consider possibilities that may occur. First assume either $\tilde{m} \leq \tilde{M} < m \leq M$ or $\tilde{m} < m \leq \tilde{M} < M$. In both cases the path $\gamma_l \to E(b) \to \gamma_u \to E(a)$ has negative orientation. Thus $E(b) <_{bw} E(a)$ which is not allowed.

Lastly, assume or $\tilde{m} < m \leq M < \tilde{M}$. Here we make use of our assumption one crossing from near the bottom to near the top of Definition 4.6. Let H^- and H^+ be sets such that $H^- \bigcup H^+ = \pi_2^{-1}[0,1]$ and $H^- \bigcap H^+ = E(b)$. (See Figure 15.)

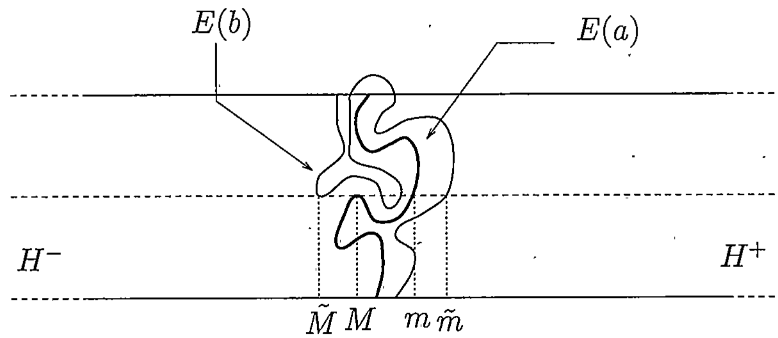

Figure 15: H^+ and H^-

Since $m > \tilde{m}$ then $\varepsilon_a(t_0(a)) \in H^-$. And since $\tilde{M} < M$ then $\varepsilon_a(t_1(a)) \in H^+$. By a Jordan curve agument and the connectivity of $E(a)$, we know $E(a) \bigcap \partial H^- \neq \emptyset$. Then there is a $t_2(a)$ such that $0 < t_0(a) < t_2(a) < t_1(a) < 1$ and the possibilities $\varepsilon_a(t_2(a)) \bigcap \ell_1 \neq \emptyset$, $\varepsilon_a(t_2(a)) \bigcap \ell_0 \neq \emptyset$, or $\varepsilon_a(t_2(a)) \bigcap E(b) \neq \emptyset$. This is a contradiction, since $E(b) \bigcap \ell_1 = \varepsilon_a(1)$ and $E(b) \bigcap \ell_0 = \varepsilon_a(0)$ and $E(a) \bigcap E(b) = \emptyset$. So we have that $m < \tilde{m}$.

By Lemma 4.16 and the fact that $m = \pi_1 \circ \varepsilon_a(t_0(a)) = \pi_1(\gamma(\tau_0(a))$, $\tilde{m} = \pi_1 \circ \varepsilon_b(t_0(b)) = \pi_1(\gamma(\tau_0(b))$,we conclude that $T(a) = g(\tau_0(a)) < g(\tau_0(b)) = T(b)$. □

<u>**4.2.5**</u> Recall that $F : \Sigma \times I \to \Sigma \times I$ is a homeomorphism such that $F(a,y) = (F_1(a), F_2(a,y))$. Let $f : T(C) \to T(C)$ be defined by $f = T \circ F_1 \circ T^{-1}$.

The proof of the next lemma follows similarly that of Lemma 3.12 using the observation that homeomorphism $H_1 : D \to D$ preserves $<_{bw}$ on $\{E(a)|a \in C\}$.

Lemma 4.18 Either H_1 or $(H_1)^2$ locally preserves $<_{bw}$ on $\{E(a)|a \in C\}$.

<u>**Proof of Theorem 4.8**</u>

This proof now follows similarly the proof of Theorem 3.1.

If F_1 does not locally preserves $<_C$, then by Definition 3.5, H_1 does not preserve the $<_{bw}$ order on fibers $\{E(a)\}$ this contradicts Lemma 4.18. And, thus F cannot be extended to a near homeomorphism of D.

So assume that F_1 is a local order preserving homeomorphism. Then by Lemma 4.17 $f : \mathbb{R} \to \mathbb{R}$ is a local order preserving homeomorphism. By Lemma 4.15, f and F_1 are conjugate. To apply 3.10 we need 3.7. Notice that it is necessary for the proof of 3.7 that $E(x)$, $E(y)$, and $E(z)$ are "straight;" it is not necessary that Λ be tamely embedded. This can be done composing with a homeomorphism $h_2 : D \to D$ such that $h_2(E(x)) = x' \times I$, $h_2(E(y)) = y' \times I$, and $h_2(E(z)) = z' \times I$, then applying the needed theorems. Now, by Lemma 3.10 $h_{top}(F_1) = h_{top}(f) = 0$. And by [Bow71] $0 < h_{top}(F) \leq \sup_a \left\{h_{top}\left(F|_{\pi_1^{-1}(a)}\right)\right\}$. Thus by [Bar87] there exists an a_0 such that $F_2|_{\pi_1^{-1}(a_0)}$ is not monotone. By Theorem 2.1 we cannot extend $h_0 \circ F \circ h_0^{-1}$ to a near homeomorphism of the disk. □

CHAPTER 5

Kupka-Smale Extensions of Cantor Set Homeomorphisms without Sources or Sinks

Introduction

In this chapter we consider the Cantor set C and a function, $F : C \to C$ which is a type of "generalized adding machine." It is then shown that F can be extended to a Kupka–Smale homeomorphism of the disk without periodic source or sinks. To do this we first divide the unit disk into a center disk and a finite number of annuli. In each annulus we create a new collection of disks so that when an iterate of F is applied to the each disk the inner disks are permuted; after enough applications of F each inner disk returns to where it started. We then recursively define similar behavior inside each of these inner disks At the intersection of these nested disks lies a Cantor set that supports a "generalized adding machine." Next we adjust F so that this Cantor set is an attracting set. This we do by forcing each inner disk to be an attracting disk. So to create these attracting disks it is necessary to have a periodic saddle point. The main result of Chapter 5 is the following:

Theorem 5.1 Let f be a block permuting homeomorphism of the Cantor set without periodic orbits. Then there exists a C^1 Kupka-Smale diffeomorphism F of the 2-disk without periodic sources or sinks, and an invariant Cantor set Λ such that $F|_\Lambda$ is conjugate to f.

This result is a partial generalization of the result of R. Bowen and J. Franks ([BF76]), which answers the question, posed by S. Smale ([Sma62]), as to whether C^r diffeomorphisms of the 2-sphere, $(r \geq 1)$ must have periodic sources or sinks.

Later J. Franks and L. Young built a C^2 version the Bowen–Franks model. For specific block permuting homeomorphisms, Franks and Young's techniques can be modified to create analogous C^2 models. However there is no reasonable modification of Franks and Young's scheme suitable for arbitrary block permuting homeomorphisms. Regardless of f, each of the disk diffeomorphism we construct must have at least one invariant Cantor set supporting a 2-symbol adding machine.

Conjecture 5.2 Every C^1 Kupka-Smale diffeomorphism of the 2-disk without periodic sources or sinks has an invariant Cantor set which supports a 2-symbol adding machine.

The author would like to acknowledge the valuable assistance Charles Pugh provided which made the generality of Theorem 5.1 possible. The author would also like to thank John Franks for suggesting Conjecture 5.2.

Preliminaries

Terminology

For dynamical systems terminology the reader should consult [Rob95],[GH83] or another basic text.

5.1.1 Let C be a Cantor set. Throughout we will assume that there exist collections $\mathcal{C}_n = \left\{ \mathrm{C}_n^k \right\}_{k=1}^{\mathrm{N}(n)}$, $n = 1, 2, 3, \ldots$, called the n^{th} level of blocks with these properties:

i.) $\mathrm{C} = \bigcup_{k=1}^{N(n)} \mathrm{C}_n^k$ for each n where $N(n) < \infty$.

ii.) C_n^k is closed for each n and $1 \leq k \leq N(n)$.

iii.) $C_n^i \bigcap C_n^j = \emptyset$; $i \neq j$ where $1 \leq i, j \leq N(n)$.

iv.) For each n and $1 \leq k \leq N(n)$ there exists $j = j\,(k, n)$ such that $C_n^k \subset C_{n-1}^j$.

v.) $\max\limits_{1 \leq k \leq N(n)} \left\{ \text{diam } C_n^k \right\} \to 0$ as $n \to \infty$.

Let $\mathcal{K}_{n-1}^j = \left\{ k : C_n^k \subset C_{n-1}^j \right\}$, and $N(n-1, j) = card\ \mathcal{K}_{n-1}^j$.

<u>**5.1.2**</u> For each Cantor set with blocks $\{\mathcal{C}_n\}_{n=1}^{\infty}$ call $f : C \to C$ <u>block permuting</u> provided for each $n-1$ and $1 \leq j \leq N(n-1)$ there is a least positive integer $m = m\,(n-1, j)$ such that

i.) $f^m C_{n-1}^j = C_{n-1}^j$, and

there exists a permutation $\sigma : \mathcal{K}_{n-1}^j \to \mathcal{K}_{n-1}^j$ such that

ii.) if $C_n^k \subset C_{n-1}^j$ then

$$f^m C_n^k = C_n^{\sigma(k)} \subset C_{n-1}^j.$$

<u>**5.1.3**</u> It follows that there exists a permutation

$$\sigma_n : \{1, 2, \ldots, N(n)\} \to \{1, 2, \ldots, N(n)\}$$

such that $f C_n^k = C_n^{\sigma_n(k)}$. Consequently for all $n-1$ and j, if $C_n^k \subset C_{n-1}^j$, then $f^m C_n^k = C_n^{\sigma_n^m(k)} \subset C_{n-1}^j$ where $m = m(n-1, j)$. That is, $\sigma_n^m \left(\mathcal{K}_{n-1}^j \right) = \mathcal{K}_{n-1}^j$. Write $\sigma = \sigma_n^m$ as the product of cycles, $\sigma = s_{n-1}^j(1) s_{n-1}^j(2) \ldots s_{n-1}^j(\nu_{n-1}^j)$. So ν_{n-1}^j is the number of cycles in σ_n^m. Denote the length of each cycle by $\left| s_{n-1}^j(i) \right|$, $1 \leq i \leq \nu_{n-1}^j$. Always assume $\left| s_{n-1}^j(1) \right| \leq \left| s_{n-1}^j(2) \right| \leq \ldots \leq \left| s_{n-1}^j(\nu_{n-1}^j) \right|$.

<u>**5.1.4**</u> Notice we have the following relationship between $m(n, k)$ and $m(n-1, j)$. If $C_n^k \subset C_{n-1}^j$ then for some $1 \leq i \leq \nu_{n-1}^j$, $m\,(n, k) = \left| s_{n-1}^j(i) \right| \cdot m\,(n-1, j)$.

Example

Before proceeding with the proof of the theorem, we now describe the construction at an intuitive level by way of example. First we describe a particular block permuting homeomorphism f. Then we use this description to start the construction of a diffeomorphism $F : \mathrm{D}^2 \to \mathrm{D}^2$ as required in Theorem 5.1.

The 0^{th} level is $\mathrm{C}_0^1 = \mathrm{C}$. Assume that C has five blocks at the first level $\mathcal{C}_1 = \{\mathrm{C}_1^1, \mathrm{C}_1^2, \ldots, \mathrm{C}_1^5\}$, and that f permutes these blocks according to a top level permutation $\sigma_1 = (12)(345)$. So the number of cycles of σ_1 is $\nu_0^1 = 2$. These are $s_0^1(1) = (12)$ and $s_0^1(2) = (345)$. The lengths of these cycles are $|s_0^1(1)| = 2$ and $|s_0^1(2)| = 3$. And so $f^{m(1,1)}\mathrm{C}_1^1 = \mathrm{C}_1^1, \ldots, f^{m(1,5)}\mathrm{C}_1^5 = \mathrm{C}_1^5$ where $m(1,1) = m(1,2) = 2$ and $m(1,3) = m(1,4) = m(1,5) = 3$.

At the second level let's assume we have twenty-two blocks,

$$\mathcal{C}_2 = \left\{\mathrm{C}_2^1, \mathrm{C}_2^2, \ldots, \mathrm{C}_2^{22}\right\},$$

and that f permutes these blocks according to the second level permutation,

$$\sigma_2 = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22).$$

Furthermore, $\mathcal{K}_1^1 = \{1,3,5,7,9\}$, $\mathcal{K}_1^2 = \{2,4,6,8,10\}$, $\mathcal{K}_1^3 = \{11,14,17,20\}$, $\mathcal{K}_1^4 = \{12,15,18,21\}$, and $\mathcal{K}_1^5 = \{13,16,19,22\}$. These sets indicate which blocks of level two are contained in each block of level one; that is, $\mathrm{C}_2^k \subset \mathrm{C}_1^j$ for $k \in \mathcal{K}_1^j$. Notice that $f^m\mathrm{C}_2^k = \mathrm{C}^{\sigma_2^m(k)}$ where $m = m(1,j)$, and furthermore $\sigma_2^{m(1,j)}\mathcal{K}_1^j = \mathcal{K}_1^j$ for $j = 1,2,3,4,5$. Each $\sigma_2^{m(i,j)}|_{\mathcal{K}_1^j}$ is the product of cycles:

$$\begin{aligned}
\sigma_2^{m(1,1)} &= s_1^1(1)s_1^1(2) = (1\ 3)(5\ 7\ 9) \\
\sigma_2^{m(1,2)} &= s_1^2(1)s_1^2(2) = (2\ 4)(6\ 8\ 10) \\
\sigma_2^{m(1,3)} &= s_1^3(1) = (11\ 14\ 17\ 20) \\
\sigma_2^{m(1,4)} &= s_1^4(1) = (12\ 15\ 18\ 21)
\end{aligned}$$

$$\sigma_2^{m(1,5)} = s_1^5(1) = (13\ 16\ 19\ 22)$$

Now, by definition, $m(2,k)$ is the least positive integer m such that $f^m \mathrm{C}_2^k = \mathrm{C}^{\sigma_2^m}$. (We will write $k \in s_1^j(i)$ when $s_1^j(i) = (k_1 k_2 \ldots k_s)$ and $k = k_i$ for some $1 \leq i \leq s$.) If $k \in \mathcal{K}_1^j$ then $\mathrm{C}_2^k \subset \mathrm{C}_1^j$ and $f^{m(1,j)}\mathrm{C}_2^k \subset \mathrm{C}_1^j$. But if also $k \in s_1^j(i)$, then $f^{m(2,k)}\mathrm{C}_2^k = \mathrm{C}_2^k$ where $m(2,k) = \left|s_1^j(i)\right| \cdot m(1,j)$.

We now describe construction of a C^1 Kupka-Smale diffeomorphism F of the unit disk $\mathrm{D}^2 \subset \mathbb{R}^2$, without sources or sinks, which permutes levels of blocks of a Cantor set $\Lambda \subset \mathrm{D}^2$ in the same way f permutes the blocks of C. Figure 16 depicts how this is done for f as in the current example.

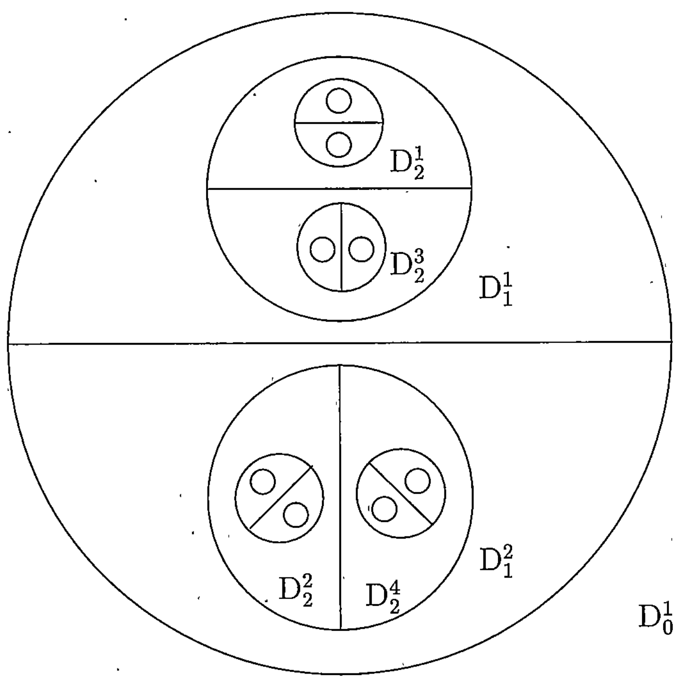

Figure 16: Bowen-Franks Model [BF76]

F fixes $\partial\mathrm{D}^2$, rotates an outer annulus by $\frac{2}{3}\pi$, and an inner annulus by π. At the center of D^2 lies a fixed copy of the Bowen-Franks model (See Figure 17).

All rotations are bumped off across collar neighborhoods of these annuli. There

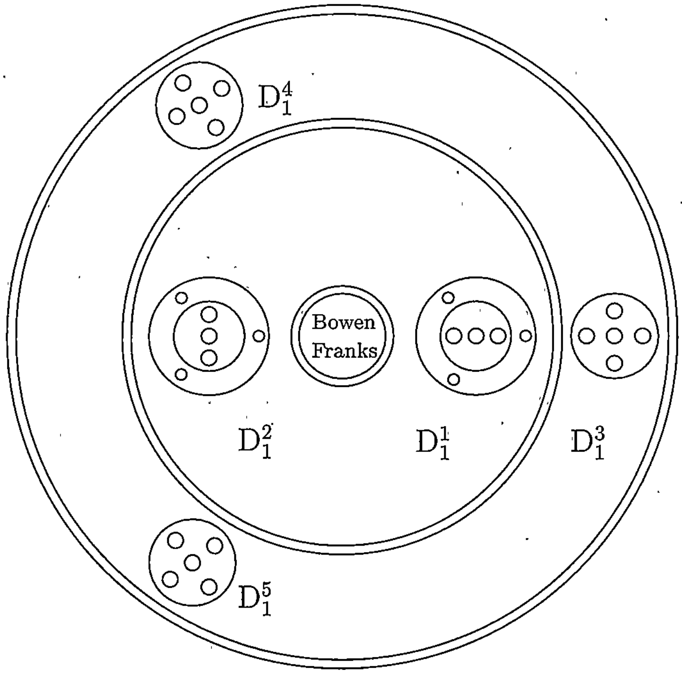

Figure 17: The Example

are five level one subdisks. F cyclically permutes $\{D_1^1, D_1^2\}$ and also $\{D_1^3, D_1^4, D_1^5\}$. On $D^2 - \bigcap_{j=1}^{5} D_1^j$, F is Morse-Smale (See Figure 17). D_1^3, for example, contains four second level disks $\{D_2^{11}, D_2^{14}, D_2^{17}, D_2^{20}\}$. $F^3 D_1^3 = D_1^3$ but F^3 rotates an annulus containing $\{D_2^{11}, D_2^{14}, D_2^{17}, D_2^{20}\}$ by $\frac{\pi}{2}$ thereby cyclicly permuting these. D_1^1 contains five second level disks $\{D_2^1, D_2^3, D_2^5, D_2^7, D_2^9\}$. For $j \in \mathcal{K}_1^1 = \{1, 3, 5, 7, 9\}$, one has $F^2 D_2^j = D_2^{\sigma(j)}$ where $\sigma = \sigma_2^2|_{\mathcal{K}_1^1} = (1\ 3)(5\ 7\ 9)$. These two cyclic permutations are accomplished by bumped off rotation of D_1^1 by F^2. F^2 has a fixed copy of the Bowen-Franks model at the centers of D_1^1 and D_1^2. Similarly F^3 has fixed copies of the Bowen-Franks model in $D_1^3, D_1^4,$ and D_1^5.

Proof of Theorem 5.1

Creating the "Disks" of $\mathbf{D}^2$

We now proceed with the construction of F. We assume a block permuting homeomorphism f and all permutations $\{\sigma_n\}$, cycles $\left\{s^j_{n-1}(i)\right\}$, and return iterations $\{m(n-1,j)\}$, are given as described in (5.1.1-4). Let Δ be the unit disk centered at $(0,0)$. First construct a set of reference diffeomorphisms $T_\alpha = T_\alpha(n-1,j,i) : \Delta \to \Delta$ for each $n-1$, j and $1 \le i \le \nu^j_{n-1}$. The exact form of $T_\alpha(n-1,j,i)$ depends on the lengths of the cycles, $\left\{\left|s^j_{n-1}(i)\right|\right\}$.

Lemma 5.3 Assume $2\pi \ge \frac{2\pi}{|s(1)|} \ge \frac{2\pi}{|s(2)|} \ge \ldots \ge \frac{2\pi}{|s(\nu)|} \ge 0$, where $\nu = \nu^j_{n-1}$. Then there exists $0 = r_0 < \frac{1}{8} = R_0 < r_1 < R_1 < \ldots < r_\nu < R_\nu < \frac{1}{2}$ and $\beta = \beta^j_{n-1} : [0,1] \to [0, 2\pi]$ such that

1. $\beta\,[r_i, R_i] = \frac{2\pi}{|s(i)|}$, $\beta\,[r_0, R_0] = 2\pi$ and $\beta\left[\frac{1}{2}, 1\right] = 0$.
2. β is strictly decreasing on $\left[0, \dfrac{1}{2}\right] - \displaystyle\bigcup_{i=0}^{\nu} [r_i, R_i]$.
3. $|\beta'(x)| \le 18$, for all x.

Proof: First let $\beta_0 : [0,1] \to [0, 2\pi]$ be a C^1-function such that

$$\beta_0\left[0, \frac{1}{8}\right] = 2\pi,$$
$$\beta_0\left[\frac{1}{2}, 1\right] = 0,$$

β_0 is strictly decreasing otherwise, and $|\beta_0'(x)| \le 17$. For each $\frac{2\pi}{|s(i)|}$, let $r_i = \beta_0^{-1}\left(\frac{2\pi}{|s(i)|}\right)$. The key observation is the following: if for each i, R_i is sufficiently close to r_i then there exist β^j_{n-1} as desired. This is because β_0 can be modified near each r_i in such away that the new map, β^j_{n-1}, is constant on $[r_i, R_i]$, but $\left|\left(\beta^j_{n-1}\right)'(x)\right| < |\beta_0'(x)| + 1$, for all x. □

<u>**5.2.1**</u> Let $T_\alpha = T_\alpha(n-1, j, i)$ be the time-α map of the flow on Δ generated by the system in polar coordinates,

$$\dot{\rho} = 0$$
$$\dot{\theta} = \beta^j_{n-1}(\rho)$$

So, in particular, $T_t\begin{pmatrix}\rho\\ \theta\end{pmatrix} = \begin{pmatrix}\rho\\ \theta + t\beta^j_{n-1}(\rho)\end{pmatrix}$. Write

$$(T_t - Id)\begin{pmatrix}\rho\\ \theta\end{pmatrix} = \begin{pmatrix}0\\ t\beta^j_{n-1}(\rho)\end{pmatrix} = \begin{pmatrix}\rho_t(\rho,\theta)\\ \theta_t(\rho,\theta)\end{pmatrix}.$$

Let $||\cdot||_{C^1}$ denote the C^1 norm on the space $Diff^1(\mathrm{D}^2)$ given (in this case) by $||T_t - Id||_{C^1} = ||\rho_t|| + ||\theta_t||$. Here we use $||g|| = \sup_{0<\rho\leq 1}|g(\rho)| + \sup_{0<\rho\leq 1}|g'(\rho)|$. Because each β^j_{n-1} is constant for $\rho \in (0, \frac{1}{8}]$, our $||\cdot||_{C^1}$ is equivalent to the usual C^1 norms on D^2. By Lemma 3.1 $||\rho_t(\rho,\theta)|| = 0$ and $||\theta_t(\rho,\theta)|| < 25\,|t|$.

<u>**5.2.2**</u> Thus we have these useful estimates

$$||T_\alpha(n-1,j,i) - \mathrm{Id}||_{C^1} < \kappa\cdot\alpha$$

and similarly

$$\left|\left|T_\alpha^{-1}(n-1,j,i) - \mathrm{Id}\right|\right|_{C^1} < \kappa\cdot\alpha$$

for all $1 \leq i \leq \nu^j_{n-1}$ when $\kappa = 25$.

For each $0 < r, R$, and s define the <u>reference sector</u>

$$\mathcal{S}(r,R,s) = \left\{(\rho,\theta) \;:\; \frac{-\pi}{s} \leq \theta \leq \frac{\pi}{s},\; r \leq \rho \leq R\right\}$$

<u>**5.2.3**</u> By (5.2.1) $T_{|s(i)|} = Id$ on $\mathcal{S}(r_i, R_i, |s(i)|)$ where $|s(i)| = \left|s^j_{n-1}(i)\right|$.

5.2.4 Our F-invariant Cantor set is $\Lambda = \bigcap_{n\geq 1} \bigcup_{k=1}^{N(n)} D_n^k$, where each D_n^k is a round disk yet to be specified. We now construct these disks. For each $1 \leq i \leq \nu_{n-1}^j$ specify a round reference sink disk $\Delta^i \subset int\ \mathcal{S}(r_i, R_i, s_i) \subset \left\{\rho < \frac{1}{2}\right\}$ with center $(x_i, 0)$, $r_i < x_i < R_i$. By (5.2.3) $T_{|s(i)|}\Delta^i = \Delta^i$.

5.2.5 We will find the following terminology useful: Let $\mathrm{D} \subset \mathrm{D}^2$ be any closed round disk. A round disk orbit of the diffeomorphism g on D^2 is a collection $\mathcal{O} = \{g^q(\mathrm{D}) \mid q \in \mathbb{Z}\}$ such that for some $p \in \mathbb{N}$,

i.) $\{g^q : 0 \leq q < p\}$ are pairwise disjoint.

ii.) $g^p(\mathrm{D}) = \mathrm{D}$.

We will call p the <u>period</u> of $\mathcal{O}$.

5.2.6 Call $\mathcal{O}$ a <u>round disk T_α orbit</u> if also

iii.) there are affine diffeomorphisms a, a' such that Figure 18 commutes.

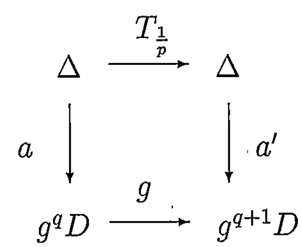

Figure 18: $T_{\frac{1}{p}} \circ a' = a \circ g$

So in particular $g^p = Id$ on $\partial\mathrm{D}$.

From (5.2.5,6) and the fact that T_α leaves $\partial\Delta$ pointwise fixed, we have that $g^p|_{\mathrm{D}}$ is a <u>replica</u> of T_1 for all q; that is $g^p|_{\mathrm{D}}$ is conjugate to T_1 via an affine conjugacy.

<u>**5.2.7**</u> Now assume $\mathcal{O} = \left\{G^q(\mathrm{D}^j_{n-1})\right\}$ has been defined and is a round disk T_α orbit of period $m = m(n-1, j)$ for the diffeomorphism G, where $T_\alpha = T_\alpha(n-1, j, i)$ is the reference diffeomorphism associated with a permutation $\sigma = \sigma_n^m$. We now proceed to define <u>subordinate n^{th} level round disk orbits</u>. For $\nu = \nu^j_{n-1}$ embed subordinate orbits $\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^\nu$ as follows: For each $\mathrm{D} = G^q\mathrm{D}^j_{n-1}$ fix an affine diffeomorphism $a : \Delta \to \mathrm{D}$. Each cycle $s^j_{n-1}(i)$ has the form $\left(k_1 k_2 \ldots k_{|s(i)|}\right)$. Let $\eta(i) = \eta^j_{n-1}(i)$ be the smallest such k.

<u>**5.2.8**</u> Label the disks of $\mathcal{O}^i$ by $\mathrm{D}_n^{\sigma^q\eta(i)} = aT_q\Delta_i$ (again where $\sigma = \sigma_n^m$, $m = m(n-1, j)$, and $0 \le q < \left|s^j_{n-1}(i)\right|$). So by (5.2.6–7) $G^m\mathrm{D}_n^{\eta(i)} = \mathrm{D}_n^{\sigma\eta(i)}$ and thus $G^{m\cdot q}\mathrm{D}_n^{\eta(i)} = \mathrm{D}_n^{\sigma^q\eta(i)}$. It follows then that $G^m\mathrm{D}_n^k = \mathrm{D}_n^{\sigma_n(k)}$ and $G^{m\cdot q}\mathrm{D}_n^k = \mathrm{D}_n^{\sigma_n^q(k)}$ for all $1 \le k \le N(n)$, and all $q \ge 0$.

Each $\mathcal{O}^i$ is a <u>round disk orbit</u> of period $m(i) = m(n-1, j) \cdot \left|s^j_{n-1}(i)\right|$ for some i (See 5.1.4). We now proceed to make these round disk T-orbits where $T = T_{\frac{1}{m(i)}}(n-1, j, i)$.

<u>**5.2.9**</u> For each q choose surjective affine diffeomorphisms $a^q_{n-1} : \Delta \to G^q\mathrm{D}^j_{n-1}$. Replace the rigid map $G|_{\mathrm{D}_n^k}$ by a replica T' of T, followed by G. The replacement is the map $\hat{G} = G \circ T'$ which makes Figure 19 commute.

For each i and q, T' leaves a neighborhood of $\partial G^q\mathrm{D}_n^k$ pointwise fixed so $\hat{G} = G \circ T'$ glues smoothly to G along the boundary circle. $\hat{G}$ is the composition of C^1 diffeomorphisms and has round disk T orbits $\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^\nu$ for $\nu = \nu^j_{n-1}$ as desired.

<u>**5.2.10**</u> By equation (5.2.2) and the fact that C^1 scales replicas correctly (See the Norms Rescaling Lemma of ([JGT89]) we have

$$\left\|G - \hat{G}\right\|_{C^1} < \frac{\kappa}{m(i)} \text{ and } \left\|G^{-1} - \hat{G}^{-1}\right\|_{C^1} < \frac{\kappa}{m(i)}.$$

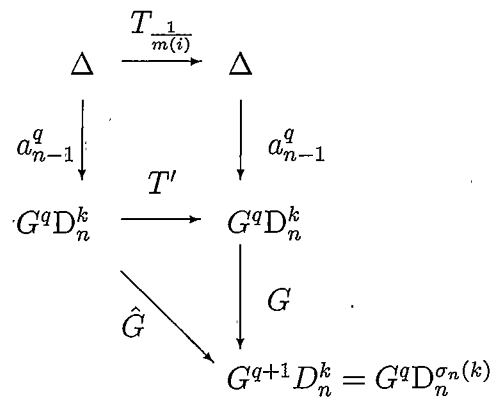

Figure 19: Commuting Diagram

5.2.11 By (5.2.9) for each $D^k_n \subset D^j_{n-1}$ we have that $\hat{G}^{m\cdot q}D^k_n = D^{\sigma^q(k)}_n$ for $1 \le i \le \nu^j_{n-1}$, $m = m(n-1, j)$ and $\sigma = \sigma^m_n$. Thus it follows that $\hat{G}^q D^k_n = D^{\sigma_n(k)}_n$.

Now start with $\hat{G}_0 = T_1(0, 1, i)$ where $1 \le i \le \nu^1_0$ and iterate the preceding construction. This gives a sequence of diffeomorphisms

$$\left\{\hat{G}_\mu : \Delta \to \Delta \mid \mu = 1, 2, 3, \ldots\right\}.$$

By (5.2.10), $\left\{\hat{G}_\mu\right\}$ converge uniformly to a C^1 diffeomorphism $\bar{G} : D^2 \to D^2$ as $\mu \to \infty$.

5.2.12 Clearly each round disk T_α-orbit of $\hat{G}_\mu$ is a round disk T_α-orbit for G. So for each n, k, $\left\{\bar{G}^q D^k_n\right\}$ is a round disk twist orbit. Thus $\bar{G}^m D^j_{n-1} = D^j_{n-1}$ and $\bar{G}^m D^k_n = D^{\sigma(k)}_n$ for $\sigma = \sigma^m_n$ where $m = m(n-1, j)$. It follows that for all n, and k, $\bar{G}D^k_n = D^{\sigma_n(k)}_n$.

Kupka-Smale Modification

Let $\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N(n)} D^k_n$. Then in fact $\bar{G}|_\Lambda$ is conjugate to f. But before exhibiting this conjugacy we first modify $\bar{G}$ away from Λ creating a Kupka-Smale diffeomorphism F which has no sources or sinks.

For each $n-1$ and j define a vector field $X = X(n-1, j)$ on Δ that has these properties on each sector $\mathcal{S} = \mathcal{S}(r_i, R_i, \left|s^j_{n-1}(i)\right|) \subset \Delta$. (See Figure 20) (Here again $|s(i)| = \left|s^j_{n-1}(i)\right|$ are generated by f as in Section 2.) On a collar neighborhood of $\partial\Delta^i$, X is purely radial (toward Δ^i) except on $\partial\Delta^i$ where $X \equiv 0$. On a collar neighborhood of $\partial\mathcal{S}$, X is purely radial (toward (0,0)) and $|X| \equiv 1$. Lastly the X-flow is Morse -Smale and has only the dynamics indicated in Figure 20.

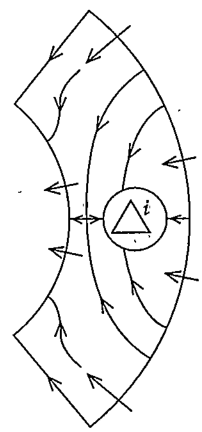

Figure 20: $\mathcal{S}\,(r_i, R_i, |s(i)|)$

Now extend X to the rest of Δ by first requiring that X be invariant under $DT_q(n-1, j, i)$ for each $q \geq 0$ and $1 \leq i \leq \nu^j_{n-1}$. Then require that X be purely radial (toward (0,0)) on a collar neighborhood of $\partial\Delta$ but that $X \equiv 0$ only on $\partial\Delta$. Also require $X \equiv 0$ on the subdisk $\{\rho \leq \frac{1}{8}\}$. Elsewhere (between the annuli) X is purely radial (toward (0,0)) and $X \neq 0$.

For all $n-1$ and j use the tangent map of each affine map $a^j_{n-1} : \Delta \to \mathrm{D}^j_{n-1}$ to push $X(n-1, j)$ onto D^j_{n-1}. This creates a C^1 vector field Y on D^2. Let ϕ_t be the Y-flow . Notice that $\bar{G} \circ \phi_t = \phi_t \circ \bar{G}$ for all t. Let $\Phi_0 = \bar{G} \circ \phi_1$. Φ_0 is not Kupka-Smale yet but we will make it so.

Let $E^j_{n-1} = a^j_{n-1}\left(\left\{\rho \leq \frac{1}{8}\right\}\right)$. Then $\Phi_0^{m(n-1,j)}$ fixes E^j_{n-1} since $Y \equiv 0$ there.

Furthermore Φ_0 is translation followed by rotation by $\frac{2\pi}{m(n-1,j)}$ on each E^j_{n-1}. R. Bowen and J. Franks [BF76] built a C^1 diffeomorphism $F_0 : \Delta \to \Delta$ which is Kupka-Smale without sources or sinks and fixes $\partial\Delta$ (See Figure 20). We now glue a replica of F_0 into each E^j_{n-1}. The new diffeomorphism $\Phi = \Phi_0$ off $\bigcup E^j_{n-1}$. But on E^j_{n-1}, $\Phi = \Phi_0 \circ a^j_{n-1} \circ F_0 \circ \left(a^j_{n-1}\right)^{-1}$. Thus $\Phi^{m(n-1,j)} = a^j_{n-1} \circ F_0^{m(n-1,j)} \circ \left(a^j_{n-1}\right)^{-1}$.

Let $\Psi : \mathrm{D}^2 \to \mathrm{D}^2$ be a diffeomorphism which fixes $\partial\mathrm{D}^2$ but pushes radially inward on a small collar neighborhood of each $\partial\mathrm{D}^k_n$ and each ∂E^k_n. If $H = \Psi \circ \Phi$ then H and Φ have identical dynamics except near each $\partial\mathrm{D}^k_n$ and each ∂E^k_n where H pushes radially inward. Furthermore, H is a C^1 diffeomorphism and except for saddles and separatrices, the α-limit sets lie in $\partial\Delta$ and the ω-limit sets in $\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N(n)} \mathrm{D}^k_n$ or in one of the disks supporting a Bowen Franks model(which are Kupka-Smale without sources or sinks). Finally apply the Kupka-Smale theorem [GH83] in the noncompact setting, $\mathrm{D}^2 - \Lambda$, to break all saddle connections without significantly changing the dynamics on Λ. Call the perturbed diffeomorphism F. Since F has no periodic orbits on Λ we have the following:

5.3.1 F is Kupka-Smale without sources or sinks and $F|_\Lambda = \tilde{G}|_\Lambda$.

Conjugacy between $F|_\Lambda$ and f

Next we need to show that $F|_\Lambda \approx f$. Let $p \in \mathrm{C}$ then $p \in \bigcap_{n=1}^{\infty} \mathrm{C}^{k_n}_n$ for some sequence $\{k_n\}$. By (5.1.1.*iv*) and (5.1.1 .*v*) $\mathrm{C}^{k_{n+1}}_{n+1} \subset \mathrm{C}^{k_n}_n$, and diam $\mathrm{C}^k_n \to 0$ as $n \to \infty$. Thus $\bigcap_{n=1}^{\infty} \mathrm{C}^{k_n}_n$ is the singleton p. So by (5.1.1.*iii*) for each p there exists a unique sequence $\{k_n\} = \{k_n(p)\}$ such that $p = \bigcap_{n=1}^{\infty} \mathrm{C}^{k_n}_n$ on E^j_{n-1}.

Let $x \in \Lambda$. Then by (5.2.4) there exists a sequence $\{l_n\}$ such that $x \in \bigcap_{n=1}^{\infty} \left(\mathrm{D}^{l_n}_n \bigcap \Lambda\right)$. By (5.2.4) and (5.2.8) , $\mathrm{D}^{l_{n+1}}_{n+1} \subset \mathrm{D}^{l_n}_n$, and diam $\mathrm{D}^{l_{n+1}}_{n+1} < \frac{1}{2}$diam $\mathrm{D}^{l_n}_n < \ldots < \frac{1}{2^{n+1}}$, so diam $D_n^{l_n} \to 0$ as $n \to \infty$. Thus $x = \left(\bigcap_{n=1}^{\infty} D_n^{l_n}\right) \bigcap \Lambda = \bigcap_{n=1}^{\infty} D_n^{l_n}$. And since $D_n^j \bigcap D_n^i = \emptyset$ when $i \neq j$, there exists a unique sequence $\{l_n\} = \{l_n(x)\}$ such that $x = \bigcap_{n=1}^{\infty} D_n^{l_n}$.

<u>5.4.1</u> Define $h : C \to \Lambda$ by $h(p) = \bigcap_{n=1}^{\infty} D_n^{k_n(p)}$. We have shown that h is well defined and is a bijection. We next show that h is a homeomorphism and $F|_\Lambda \circ h = h \circ f$.

Let d be the metric on C defined by $d(p,q) = \sum_{n=1}^{\infty} \frac{|k_n(p) - k_n(q)|}{2^{N(n)}}$ where $N(n)$ is as in (4.1.1.i). The proof of the following lemma is routine and is left to the reader.

Lemma 5.4 Let $p, q \in C$, and let $\epsilon > 0$ and K be such that $\epsilon < \frac{1}{2^{N(K)}}$. If $d(p,q) < \epsilon$ then $p, q \in C_K^{k_K}$.

To show the continuity of h, let $\epsilon > 0$, and choose K large enough that $\frac{1}{2^K} < \epsilon$. Choose δ such that $\delta < \frac{1}{2^{N(K)}}$. So if $d(p,q) < \delta$ then by Lemma 3.2, $p, q \in C_K^{k_K}$. Thus $h(p), h(q) \in D_K^{k_K}$ and $d(h(p), h(q)) \leq \text{diam } D_K^{k_K} \leq \frac{1}{2^K} < \epsilon$ by (4.2.4). Since C and Λ are compact Hausdorff spaces, h is a homeomorphism.

Lastly we need to verify that $h \circ f = F|_\Lambda \circ h$. From (4.1.3) $fC_n^{k_n} = C_n^{\sigma_n(k_n)}$. So $f(p) = f\left(\bigcap_{n=1}^{\infty} C_n^{k_n}\right) = \bigcap_{n=1}^{\infty} C_n^{\sigma_n(k_n)}$. Let $\Lambda(k_n) = D_n^{k_n} \bigcap \Lambda$. By (4.4.1) $hC_n^{k_n} = \Lambda(k_n)$. By (4.2.12) $\bar{G}D_n^{k_n} = D_n^{\sigma_n(k_n)}$ and by (4.3.1) $F|_\Lambda = \bar{G}|_\Lambda$. Thus $F|_\Lambda(\Lambda(k_n)) = \Lambda(\sigma_n(k_n))$. Since $\{F|_\Lambda(\Lambda(k_n))\}$ are pairwise disjoint, we have $h \circ f(p) = h \circ f \bigcap_{n=1}^{\infty} C_n^{k_n} = h \bigcap_{n=1}^{\infty} C_n^{\sigma_n(k_n)} = \bigcap_{n=1}^{\infty} \Lambda(\sigma_n(k_n)) = F|_\Lambda \bigcap_{n=1}^{\infty} \Lambda(k_n) = F|_\Lambda \circ h \bigcap_{n=1}^{\infty} C_n^{k_n} = F|_\Lambda \circ h(p)$.

<u>Finite Products of Adding Machines</u>

In this section we prove that finite products of "k-symbol adding machines" are block permuting. First we have the following useful lemma.

Lemma 5.5 Let $f, g : C \to C$ be block permuting homeomorphisms. Then $f \times g : C \times C \to C \times C$ is block permuting.

Proof: Let C^1 and C^2 be Cantor sets. C^1 has blocks $\{\mathcal{C}_n^1\}_{n=1}^\infty$ and C^2 has blocks $\{\mathcal{C}_n^2\}_{n=1}^\infty$ as in (4.1.1). For each n, form

$$\left\{\mathcal{C}_n^1 \times \mathcal{C}_n^2\right\}_{n=1}^\infty = \left\{C_n^{k(1)} \times C_n^{k(2)} : C_n^{k(1)} \in \mathcal{C}_n^1, C_n^{k(2)} \in \mathcal{C}_n^2\right\}.$$

Clearly $\{\mathcal{C}_n^1 \times \mathcal{C}_n^2\}_{n=1}^\infty$ satisfies (4.1.1). Let $C_{n-1}^{j(1)} \times C_{n-1}^{j(2)} \in \mathcal{C}_{n-1}^1 \times \mathcal{C}_{n-1}^2$. There are least positive integers $m(1)$ and $m(2)$ such that $f^{m(1)}C_{n-1}^{j(1)} = C_{n-1}^{j(1)}$ and $g^{m(2)}C_{n-1}^{j(2)} = C_{n-1}^{j(2)}$. Let $m = \operatorname{lcm}(m(1), m(2))$. Then

$$(f \times g)^m \left(C_{n-1}^{j(1)} \times C_{n-1}^{j(2)}\right) = f^m C_{n-1}^{j(1)} \times g^m C_{n-1}^{j(2)} = C_{n-1}^{j(1)} \times C_{n-1}^{j(2)}.$$

Also $m(n-1, j)$ must be a multiple of $m(1)$ and of $m(2)$. Thus $m(n-1, j) = \operatorname{lcm}(m(1), m(2))$.

It remains to verify (4.2.5.*ii*). Let $C_n^{k(1)} \times C_n^{k(2)} \subset C_{n-1}^{j(1)} \times C_{n-1}^{j(2)}$ and let $m = m(n-1, j) = a_1 m(1) = a_2 m(2)$ where $a_1, a_2 \in \mathbb{N}$. Then

$$\begin{aligned}(f \times g)^m \left(C_n^{k(1)} \times C_n^{k(2)}\right) &= f^m C_n^{k(1)} \times g^m C_n^{k(2)} \\ &= f^{a_1 m(1)} C_n^{k(1)} \times g^{a_2 m(2)} C_n^{k(2)} \\ &= C_n^{(\alpha^1)^{a_1}(k_{(1)})} \times C_n^{(\alpha^2)^{a_2}(k_{(2)})},\end{aligned}$$

where $\alpha^1 = (\sigma^1)_n^{m(1)} : \mathcal{K}_{n-1}^{j(1)} \to \mathcal{K}_{n-1}^{j(1)}$ and where $\alpha^2 = (\sigma^2)_n^{m(2)} : \mathcal{K}_{n-1}^{j(2)} \to \mathcal{K}_{n-1}^{j(2)}$. So $\alpha^1 \times \alpha^2$ on $\mathcal{K}_{n-1}^{j(1)} \times \mathcal{K}_{n-1}^{j(2)}$ is the desired permutation. □

Next we define what is meant by a "k-symbol adding machine". Let $\Sigma_k = \{0, 1, \ldots, k-1\}^{\mathbb{N}}$ and let $\Sigma_k^n = \{0, 1, \ldots, k-1\}^n$ for $k \geq 2$. The <u>k-symbol adding machine</u> $A_k : \Sigma_k \to \Sigma_k$ is defined as follows: for each $n \geq 1$ let $A_k^n : \Sigma_k^n \to \Sigma_k^n$ by $A_k^n(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ where

$$\begin{aligned} b_1 &= a_1 + 1 \bmod k \quad \text{and when } i > 1 \\ b_i &= \begin{cases} a_i + 1 \bmod k & \text{if } a_{i-1} = k-1 \text{ and } b_{i-1} = 0. \\ a_i & \text{otherwise} \end{cases}\end{aligned}$$

Then $A_k(\underline{a}) = \underline{b}$ where $(b_1, b_2, \ldots, b_n) = A_k^n(a_1, a_2, \ldots, a_n)$ for all n.

That all adding machines are block permuting is routine and left to the reader. Using Lemma 4.1 we have the following corollary to Theorem 5.1.

Corollary 5.6 Suppose that $f : \mathrm{C} \to \mathrm{C}$, where C is a Cantor set, is a homeomorphism conjugate to a finite product of adding machines. Then there exists a C^1 Kupka-Smale diffeomorphism F of the 2-disk without periodic sources or sinks, and an invariant Cantor set Λ such that $F|_\Lambda$ is conjugate to f.

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