A rational interval of rotation numbers for periodic points in certain nonseparating plane continua
by Thor Hans Matison

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

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A RATIONAL INTERVAL OF ROTATION NUMBERS FOR PERIODIC POINTS IN CERTAIN NONSEPARATING PLANE CONTINUA

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A thesis submitted in partial fulfillment
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THOR HANS MATISON

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STATEMENT OF PERMISSION TO USE

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Early influences from my family played a great roll in my getting to this point in my education. Since Aunt Sonja received her Ph.D., I knew that I would work towards getting one, as well. My dad, Jerry first got me interested in the sciences; he showed me how to write with invisible ink. Both he and Lucy, my mom taught by example about the tenacity that one must have to finish a project such as this.

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ABSTRACT

Let F be an orientation preserving homeomorphism of the plane that has a fixed point that is contained in an invariant, nonseparating continuum Λ. If p/q is a reduced rational in the interior of the convex hull of the rotation set of Λ about the fixed point, then there exists a q-periodic point in Λ with rotation number p/q, provided that p/q is not the local rotation number about the fixed point and that Λ satisfies certain technical requirements. We also show that the local rotation number is a point in the closure of the rotation set of Λ and that if this rotation set is nondegenerate, then Λ is indecomposable.

CHAPTER 1

Introduction

In this chapter we briefly review some of the results related to those contained in this thesis. We provide some historical background and give rough statements of our results.

The study of celestial mechanics has been the impetus behind many advances in mathematics. Around the turn of the century the great French mathematician Henri Poincaré (1854-1912) made many of his discoveries while in pursuit of a better understanding of the motions in the solar system (see [Poi99]). His proof of the existence of periodic solutions in certain cases of the "three-body problem" was one such discovery.

From Newton's Law of Gravitation, we know that two point masses are attracted toward each other with a force proportional to the product of the two masses over the square of the distance between them. ($F = \frac{Gm_1m_2}{D^2}$ where G is the gravitational constant.) Suppose that there exist n point masses (bodies), each with a specified initial position and velocity. "Barring influence from any outside force, what is the equation of motion for each of the n point masses?" is the *n-body problem*. The problem was completely solved by Newton for $n = 2$, but for $n \geq 3$ the problem grows vastly more difficult. For such n it became of great importance to find which initial states yield a periodic solution, for the existence of a periodic solution manifests itself in that the initial state of the n bodies will reoccur after a length of time T, and again after another length T and so on.

The solution to the n-body problem for a fixed set of initial values has the form of a parameterized curve which lies on a manifold embedded in $\mathbb{R}^{6n}$. If one looks at all possible initial conditions, the union of their solution curves fills all of $\mathbb{R}^{6n}$ and (with their parameterizations) describes what is called a *flow* on $\mathbb{R}^{6n}$. Imagine taking a cross section of $\mathbb{R}^{6n}$ that is everywhere transverse to this flow, and suppose that the solution curve of every point in the cross section comes back in forward time and re-intersects the section. We can now define a (orientation preserving) homeomorphism of the cross section back into itself by sending every point to the unique point where its solution curve next re-intersects the cross section. This describes what is called a *first return map* or a *Poincaré map* after its originator. (More information about first return maps may be found in [GH83] or [Hc91], for instance.) Showing that there is a periodic point under this homeomorphism is equivalent to showing the existence of a closed (periodic) solution curve in the flow. This is one important reason why discrete transformations such as orientation preserving homeomorphisms of subsets of $\mathbb{R}^{6n}$ are studied.

Since the movement of the sun and planets takes place in a near vacuum, the frictional damping is often ignored when modeling their motion. Total energy and total momentum are then constant quantities and the system is said to be *conservative.* Therefore, when the friction is ignored, the flow for the n-body problem has the nice property that it preserves volume. That is, given any (Lesbegue) measurable subset U of $\mathbb{R}^{6n}$, the measure of U is the same as that of the set obtained by flowing U either forward or backward by any length of time. By carefully selecting the cross section, it may be possible to find a first return map that is also area preserving. (Incidentally, if the frictional force is taken into account, the system is said to be *dissipative* and the resulting flow is volume *contracting.*)

In 1909, Poincaré made a conjecture which, if proved to be true, would guarantee periodic solutions for a certain class of initial conditions in the three-body problem. In the following two years Poincaré was only able to verify the truth of his conjecture in special cases. Nearing his death, he wrote to the editor of a mathematics journal:

> "...at my age, I may not be able to solve it, and the results obtained, susceptible of putting researchers on a new and unexpected path, seem to me too full of promise, in spite of the deceptions they have caused me, that I should resign myself to sacrificing them..."[Bel37].

A short time after this conjecture was published [Poi12], George Birkhoff proved what has become known as "Poincaré's Last Geometric Theorem" or the "Poincaré-Birkhoff Theorem" (in [Bir13]). A rough statement of this theorem is as follows: Suppose that h is an orientation preserving homeomorphism of the closed annulus that advances one boundary component clockwise and the other counterclockwise. If h is area preserving, then the annulus possesses at least two points that are fixed under h. In fact, a corollary to this theorem guarantees an infinite number of periodic points, whose union of rotation numbers is all the rational points in an interval of real numbers. (A precise statement of the "advancing of the boundary components" requires passing to the universal covering space, since advancing by a clockwise angle α cannot otherwise be distinguished from a counterclockwise angular change of $2\pi-\alpha$. Precise definitions of terms pertinent to this thesis are in Chapter 2.)

Since its proof in 1913, generalizations of this theorem have been proven by Birkhoff in [Bir26] and recently by Franks in [Fra88]. (An excellent expository article giving a detailed proof based on Birkhoff's original may be found in [BN77], by Brown and Neumann.) Franks' result guarantees at least two fixed points for the open, closed, or half open annulus. Instead of requiring the homeomorphism to be area preserving, he employs the weaker hypothesis that every point be nonwandering, and instead of rotation of boundary components, he requires that there exist two open disks, of which one goes around the "hole" of the annulus clockwise and reintersects itself, while the other reintersects itself after going around counterclockwise. Additionally, theorems for other planar sets have been discovered that are closely related to the Poincaré-Birkhoff Theorem in that periodic orbits are shown to exist based on "something" rotating clockwise and "something" rotating counterclockwise. Typically, the planar set is a *continuum* (it is compact and connected) and is invariant under an orientation preserving homeomorphism of the plane. Under certain rotational conditions, the existence of periodic points has been shown by restricting attention to one of these invariant sets. Many of the planar continua that have been studied fall into one of two categories. Either the continuum separates the plane into exactly two components (as does an embedded circle), or it does not separate the plane at all (as in the case of a closed interval in the plane). We review some of the results for these kinds of continua.

Poincaré began the study of plane separating continua with his study of circle homeomorphisms. He defined the rotation number of a homeomorphism as a measurement of the average angular advancement and showed that for orientation preserving homeomorphisms, it is independent of the point on the circle from which it is calculated. Loosely speaking, for a given homeomorphism, every point on the circle must be moving around clockwise, every point must be moving counterclockwise, or every point must tend toward a fixed point on the circle. If a circle is embedded in the plane and is invariant under an orientation preserving homeomorphism h, one can calculate its rotation number. If that rotation number is rational, say p/q with p and q relatively prime, then there must be a periodic point of the circle with period equal to q. Furthermore, if there are other periodic points in this circle, they must also be of period q. If the rotation number happens to be irrational, then there are no periodic points. (These facts about circle maps with their proofs can be found in Devaney's book [Dev89].) An invariant circle is in some sense too simple to simultaneously have some points moving clockwise and others counterclockwise under a homeomorphism. To support this kind of phenomenon, the invariant set must be topologically more complicated. A strategy that has been useful in the study of complex invariant sets is to consider the rotation of the continuum as viewed from its complement, [Wal91].

Suppose that W is an open, connected and simply connected set in either the plane or its one point compactification, the 2-sphere ($\mathbb{R}^2 \cup \{\infty\} \approx S^2$). The set W must, therefore, be a topological open disk. Let h be a homeomorphism of $\mathbb{R}^2$ (or S^2) and suppose that W is invariant under h. If the boundary of W, ∂W is connected, it may be either a separating or nonseparating continuum. In either case, ∂W is also h-invariant. A point z of ∂W is said to be *accessible* from W if there is an arc lying in $W \cup \partial W$ such that the intersection of ∂W and the arc is the singleton $\{z\}$. The set of all points accessible from W, A is invariant. (If z is accessible, so must be $h(z)$.) The restriction of h to A induces a circle homeomorphism which has a well defined rotation number. This rotation number is called the *prime end rotation number* from W, as it may be obtained from Carathéodory's theory of prime ends [Car13]. (See [Mat82] or [AY92] for a modern reference to prime end theory, or see [BG91b] for another method of determining the prime end rotation number.) Using prime end theory, Cartwright and Littlewood showed in [CL51] that if a nonseparating continuum is invariant under an orientation preserving homeomorphism of the plane, then the continuum must contain a fixed point. O.H. Hamilton gave a short proof of this fact in [Ham54], and M. Brown has produced a one page proof using different methods in [Bro77]. A related theorem by Barge and Gillette states that every invariant continuum whose complement has exactly two components contains a fixed point if one of the prime end rotation numbers is zero [BG92]. In the same paper they sketch yet another proof of the Cartwright-Littlewood theorem.

Birkhoff began the investigation of certain topologically complex plane separating continua in with his "remarkable curve" paper of 1932. [Bir32] A plane separating continuum can be viewed as containing the boundaries of two invariant open topological disks, one of which contains the point at infinity. From this perspective, it is possible to obtain both an interior and an exterior (prime end) rotation number. While an invariant circle must have the same interior and exterior rotation number, Birkhoff's invariant remarkable curve has the property that they are different. Charpentier, in [Cha34], showed just how remarkable this continuum is, when he gave a proof that it was indecomposable. A continuum is *indecomposable* if it is not the union of two of its proper subcontinua. (See [HY61] or [Kur68] for some of the peculiar properties of indecomposable continua.)

A common requisite needed to prove the existence of certain periodic points is that the continuum be an attractor. (There are many possible ways to define *attractor*, one being that nearby points limit on the continuum.) In maps which arise from physical applications where friction is present, this is a property that one can often expect. Birkhoff's remarkable curve and the invariant sets derived from the van der Pol equations are examples (see [CL45], [CL51], [Lev49], [LeC88], and [GH83]). Under an assumption that the continuum is attracting, Alligood and Yorke obtain accessible periodic points based on prime end information. In contrast, Barge and Gillette in [BG91b] show that there is a rational interval (all rational numbers in a real interval) of rotation numbers for periodic points for irreducible plane separating continua without requiring that the continuum be attracting. ("Irreducible" in this context means that no proper subcontinuum separates the plane.) Nothing is said, however, of the accessibility of any of these periodic points. They also prove that the interior and exterior rotation numbers are in the convex hull of the rotation set of the continuum where the *rotation set* of the continuum is the union of all the rotation sets of all points contained in the continuum. If there are two different points in the rotation set, they show that the continuum is indecomposable (see also [BG91a]).

The results of this thesis are concerned with the rotation of certain invariant, nonseparating continua about a fixed point p that is in the continuum. Most closely related, perhaps, is Barge's paper [Bar], where he takes the compact closure of an immersed line, splits it open along the line, and transforms it into an irreducible plane separating continuum. Using this method he is, again, able to prove the existence of a rational interval of rotation numbers for periodic points. The continua that we consider include the nowhere dense, compact closure of an immersed line. We prove a similar result and show that if there is a point of the continuum that rotates clockwise about p and another point that rotates counterclockwise about this fixed point, then the continuum must contain a second fixed point provided that the local rotation about p is nonzero. These hypotheses actually insure an infinite number of periodic points and guarantee that the continuum be indecomposable. Many of the arguments presented in this thesis are modifications of those found in [BG91b], from the compact case of the closed annulus to the noncompact case of the punctured plane (or open annulus).

Chapter 2 contains precise definitions and some examples to illustrate some of the more technical concepts. In the third chapter we introduce the notion of a periodic disk chain and prove several lemmas that will be used to substantiate the main results. These results, their proofs, and more examples are in Chapter 4.

CHAPTER 2

Preliminaries

In this chapter we make the necessary definitions, including defining the planar sets that will be considered. We make precise the notion of a rotation set and prove some basic results for nonseparating plane continua.

Throughout this thesis, F will be an orientation preserving homeomorphism of the plane $\mathbb{R}^2$ onto itself. We will assume that Λ is a subset of $\mathbb{R}^2$ that is an invariant continuum. By *invariant*, we mean that $F(\Lambda) = \Lambda$, and by *continuum*, we mean a compact and connected set. ([Mun75] is a good basic topology reference.) We suppose that there is a point p in Λ that is fixed under F. That is, $F(p) = p$. Suppose further that Λ has the following four properties:

Standing Assumptions

1. Λ does not separate the plane. (The complement of Λ is connected.)
2. Λ is nowhere dense. (Λ contains no open disks.)
3. There exists a topological n-od Y and closed disk Δ such that $Y \subset \Delta \subset \mathbb{R}^2$, the vertex of Y is p, the endpoints of Y are in the boundary of Δ, and all non-endpoints of Y are contained in the interior of Δ. If C is the component of $\Delta \cap \Lambda$ which contains p, then $C \backslash \{p\}$ has exactly n components, with Y separating distinct components of $C \backslash \{p\}$ in Δ (see Figure 1).

Property 3 will sometimes be referred to as the "spokes of the wheel" property. It will be used in defining the local rotation about the fixed point p.

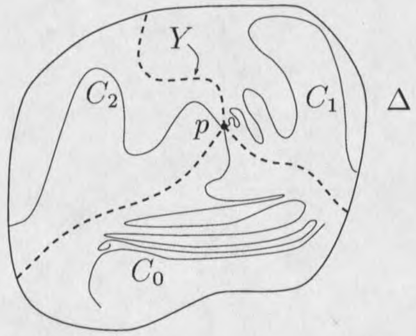

Figure 1: Y separates the $\mathbf{n} = 3$ components of $C\backslash\{p\}$

Fix a component C_0, of $C\backslash\{p\}$ in Δ, then label each of the other components C_1, C_2, ... , $C_{\mathbf{n}-1}$ in a counterclockwise order from C_0. There exists a universal covering map $\Pi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2\backslash\{p\}$ such that $\Pi(x+1, y) = \Pi(x, y)$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ and $\Pi^{-1}(C_j)$ is a subset of

$$\bigcup_{k\in\mathbb{Z}} \left\{ (x,y) \middle| \; k + \frac{j}{\mathbf{n}} < x < k + \frac{j+1}{\mathbf{n}} \text{ and } y \leq 1 \right\}$$

for each component C_j of $C\backslash\{p\}$ (see Figure 2). The universal covering space of the punctured plane $\mathbb{R}^2\backslash\{p\}$ is usually taken to be $\mathbb{R}^2$, but for convenience we use the homeomorphic open half plane $\mathbb{R} \times \mathbb{R}^+$. ([Mun75] is a good reference for covering spaces.)

Definition 2.1 For each $z \in \Lambda$, we define the *composant determined by* z, $\mathcal{C}_z$, to be the union of all proper subcontinua of Λ that contain z.

Fix the above covering map Π, and as one additional restriction on Λ, assume the following:

4. For every z in the composant determined by p, there is a subcontinuum Γ of Λ, containing both p and z, such that every component $\widetilde{\Gamma}$ of $\Pi^{-1}(\Gamma\backslash\{p\})$ is bounded (see Figure 3).

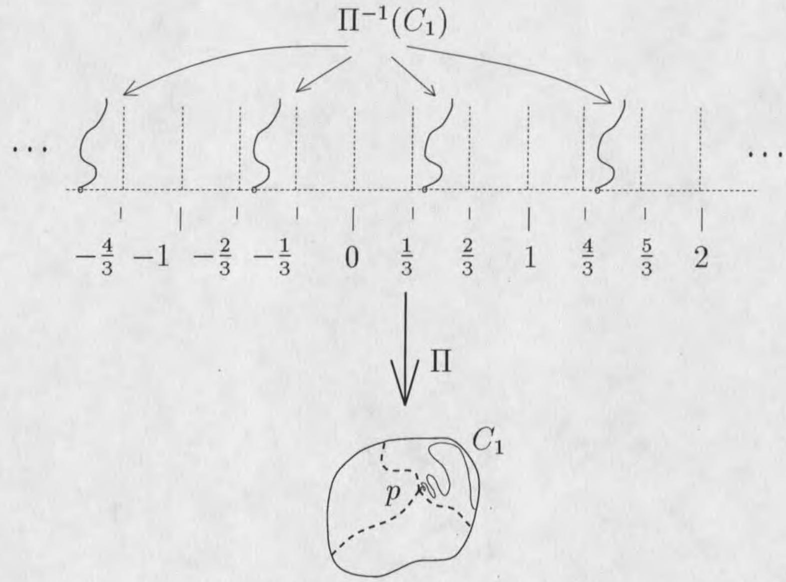

Figure 2: For example, $\Pi^{-1}(C_1)$ is contained in $\cup_{k\in\mathbb{Z}}\{(x,y)|k+\frac{1}{3}<x<k+\frac{2}{3}\}$

Definition 2.2 Given a subset Γ of $\mathbb{R}^2$ and U a subset of the complement of Γ, a point p of Γ is said to be *accessible from* U provided that there is an arc α in $U \cup \Gamma$ such that $\alpha \cap \Gamma = \{p\}$. If U is not specified and if p is accessible from $\mathbb{R}^2 \backslash \Gamma$, p is simply said to be *accessible.*

I believe that it is the case that if for each $z \in \mathcal{C}_p$, there is a subcontinuum $\Gamma = \Gamma(z)$ of Λ, containing both p and z, such that p is an accessible point of Γ, then there is a covering map Π, such that Property 4 is satisfied.

Remark: The **Standing Assumptions**, Properties 1-4, will be denoted as **SA** throughout this thesis.

We give an example of an invariant set that has the properties of Λ. Suppose that p is a hyperbolic saddle of an area contracting, orientation preserving C^1-diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$. The unstable manifold $W^u(p) = \{z \in \mathbb{R}^2 | F^n(z) \to p \text{ as } n \to -\infty\}$ of p is an immersed line, the continuous one-to-one image of the reals.

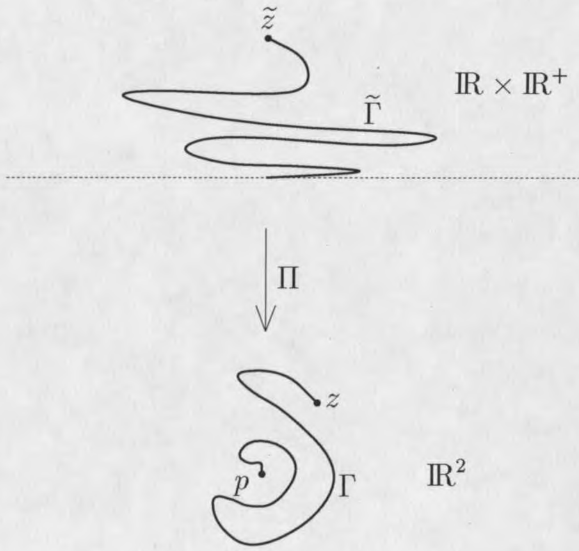

Figure 3: Property 4 of the SA: $\tilde{\Gamma}$ is bounded.

Suppose p is contained in the interior of a closed disk D and $F(D) \subset \text{int } D$. The closure of $W^u(p)$ is then compact. For $\Lambda = \text{cl } W^u(p)$, there is a disk Δ containing p such that the component C of $\Delta \cap \Lambda$ that contains p is homeomorphic to a closed interval and $C \backslash \{p\}$ is the disjoint union of $\mathbf{n} = 2$ half open intervals (see Figure 4).

This situation occurs in the horseshoe map. (See (Figure 5) and see [Sma62], [Sma63] or [GH83] for more on the Smale horseshoe.) The homeomorphism F of the plane can be constructed so that it is area contracting and the attracting set Λ, which contains the nonwandering Cantor set, is contained in a closed disk that is mapped into its interior. Λ has two fixed points, one accessible (the "endpoint") and one which is not. Let p be the inaccessible fixed point and let Λ be the closure of its unstable manifold. Then Λ satisfies the SA. Since F contracts area, Λ is nowhere dense and nonseparating. The local unstable manifold of p provides the $\mathbf{n} = 2$ spokes of the wheel, which are interchanged under each iteration of F. In this case, the composant determined by p is its unstable manifold. For any z in $W^u(p)$, the unique arc Γ which is contained in $W^u(p)$ and has p and z as its endpoints meets the requirements of

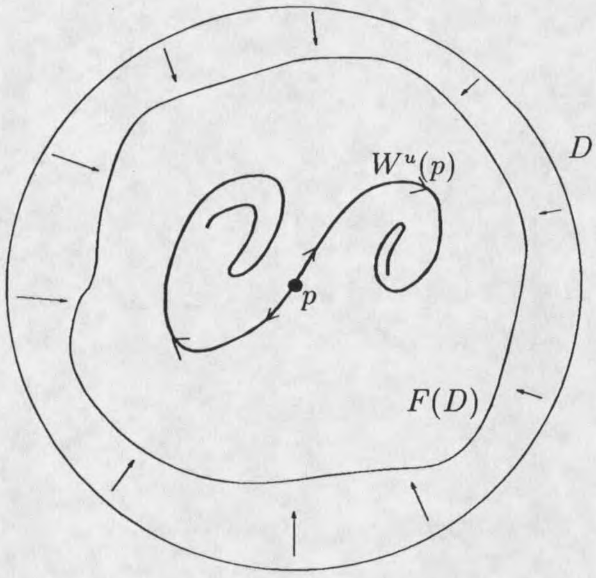

Figure 4: $\Lambda = \text{cl}\, W^u(p)$ satisfies the SA (Standing Assumptions).

Property 4.

Though it is not necessarily the case that given any diffeomorphism as above, the unstable manifold of p must equal the composant determined by p, this *is* the case in many interesting examples.

Next, we give examples of sets having any three of the defining properties of the class of plane continua containing Λ but not the fourth (see Figure 6).

Example 2.3 A depicts the union of a circle, its center p, and a path which begins at the centerpoint and spirals outward to limit on the circle. (By a path, in this instance, we mean a set homeomorphic to a half open interval.) This union separates the plane but has the last three properties of Λ. If p is a boundary point of a closed disk as in Example B, the disk has Properties 1, 3, and 4, but not 2 since it is not nowhere dense. Let C be the Cantor set and let $\Sigma = C \times (0, 1]$. Example C illustrates the one point compactification of Σ, $\Sigma \cup \{p\}$, a Cantor fan. Properties 1, 2, and 4 are satisfied but there is no disk and n-od as in Property 3. The fourth property

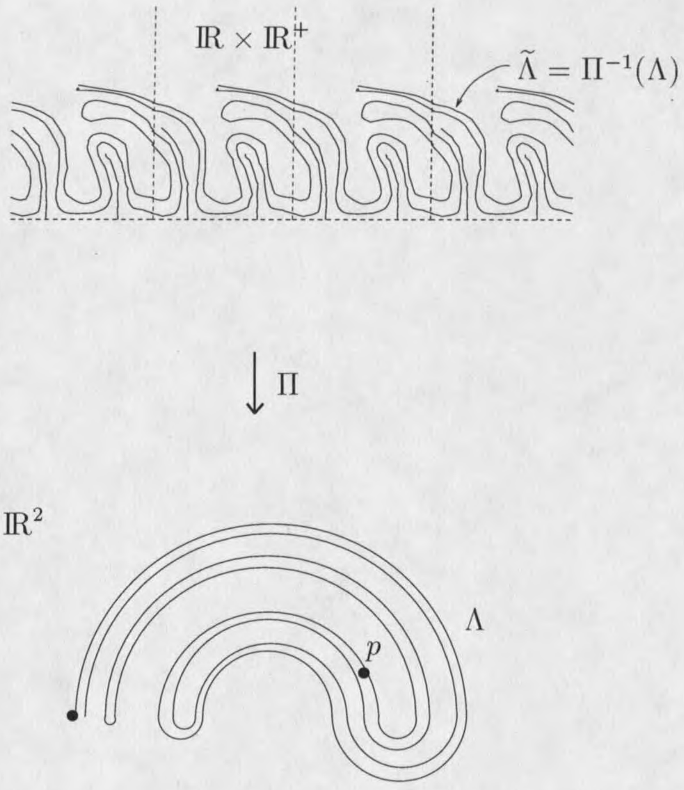

Figure 5: The attracting set of the horseshoe map

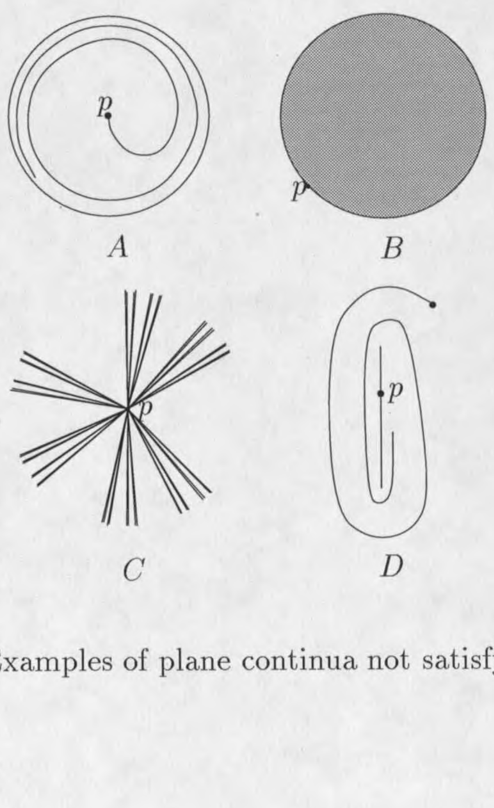

Figure 6: Examples of plane continua not satisfying the SA

is not satisfied by a set consisting of the union of a closed line segment containing p and a path (as in Example A) spiraling onto the line segment as in Example D. The first three properties, however, are satisfied. These examples demonstrate the independence of the four properties of the SA.

In the following claim and lemma we need only assume that Λ is any nonseparating plane continuum.

Claim 2.4 *The complement of* Λ *in* $\mathbb{R}^2$ *is homeomorphic to a punctured plane.*

Proof: Since Λ is closed and nonseparating, $U_e = \mathbb{R}^2 \backslash \Lambda$ is open and connected, and is therefore path connected. Let $\mathbb{R}^2 \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^2$ and γ be any simple closed curve in the path connected set $U_e \cup \{\infty\}$. From the Jordan Curve Theorem, the complement of γ in $\mathbb{R}^2 \cup \{\infty\}$ consists of two disjoint open disks A and B, having γ as their common boundary. Since Λ is connected, it must lie entirely in one of A or B. Assume then, that $\Lambda \subset A$. It must then be the case that $B \cup \gamma$ is a closed disk which is disjoint from Λ. It follows that γ is contractible to a point in $(B \cup \gamma) \subset U_e \cup \{\infty\}$ and since γ was arbitrary, every simple closed curve in $U_e \cup \{\infty\}$ must be contractible. Therefore $U_e \cup \{\infty\}$, being open, connected and simply connected, is a topological open disk. By removing the point ∞, we conclude that U_e a topological punctured plane. □

Lemma 2.5 *There exists a nested sequence* $\{D_n\}_{n=1}^{\infty}$ *of closed disks, each containing* Λ *in its interior, such that* $\Lambda = \cap_{n=1}^{\infty} D_n$.

Proof: From Claim 2.4 there is a homeomorphism $\psi : \mathbb{R}^2 \backslash \{0\} \to U_e$, where $U_e = \mathbb{R}^2 \backslash \Lambda$. We may assume that for every sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}^2 \backslash \{0\}$ with $|z_n| \to 0$, we have $d(\psi(z_n), \Lambda) \to 0$. For every positive integer n, let $A_n = \{z \in \mathbb{R}^2 \backslash \{0\} \mid |z| \leq \frac{1}{n}\})$ and let $D_n = \Lambda \cup \psi(A_n)$. Now, the interior of $\psi(A_n)$ is that part of U_e which is in the bounded component of the complement of the simple closed curve $\psi(\{z \mid |z| = \frac{1}{n}\})$, while $\psi(A_n)$ is the union of this subset of U_e with $\psi(\{z \mid |z| = \frac{1}{n}\})$. Therefore, $D_n = \Lambda \cup \psi(A_n)$ is the union of a simple closed curve and the bounded component of its complement in $\mathbb{R}^2$. That is, each D_n is a topological closed disk.

To see that D_{n+1} is contained in D_n for each $n \in \mathbb{N}$, let $z \in D_{n+1} = \Lambda \cup \psi(A_{n+1})$. If $z \in \Lambda$ then $z \in D_n = \Lambda \cup \psi(A_n)$. If $z \in \psi(A_{n+1})$, then $\psi^{-1}(z)$ is in A_{n+1} so $|\psi^{-1}(z)| \leq \frac{1}{n+1}$. Since $\frac{1}{n+1}$ is less than $\frac{1}{n}$, $\psi^{-1}(z)$ must also be an element of A_n, implying that z is in $\psi(A_n)$ which is contained in the interior of D_n. Since Λ is a subset of D_{n+1} and $D_{n+1} \subset \text{int}D_n$, we also have that Λ is contained in the interior of D_n.

Since $\Lambda \subset D_n$ for every n, Λ is contained in $\cap_{n=1}^{\infty} D_n$. In order to show that $\cap_{n=1}^{\infty} D_n$ is contained in Λ, let z be an element of $\cap_{n=1}^{\infty} D_n$. Then $z \in \Lambda \cup \psi(A_n)$ for every $n \in \mathbb{N}$. If z is not in Λ, z must be in $\psi(A_n)$ for every positive integer n. However, for n larger than $\frac{1}{|\psi^{-1}(z)|}$, $\psi^{-1}(z)$ cannot be in A_n since $|\psi^{-1}(z)| > \frac{1}{n}$. Consequently, z must have been in Λ, establishing that $\Lambda = \cap_{n=1}^{\infty} D_n$. □

In the remainder of this chapter, we assume that Λ is not only an invariant nonseparating continuum, but also contains the point p and satisfies Properties 1-4 of the SA.

There is a closed topological disk D containing Λ in its interior such that $\widetilde{D} = \Pi^{-1}(D) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ \, |y \leq \kappa\}$ for some $\kappa > 0$. Let $\widetilde{\Lambda} = \Pi^{-1}(\Lambda \backslash \{p\})$. Then $\widetilde{\Lambda}$ is contained in the interior of $\widetilde{D}$ as in Figure 7 and

Claim 2.6 $\widetilde{\Lambda}$ *has the following properties:*

1. $\widetilde{\Lambda}$ *is nowhere dense.*

2. $\widetilde{\Lambda}$ *is a closed subset of* $\mathbb{R} \times \mathbb{R}^+$.

3. *No closed subset of* $\widetilde{\Lambda}$ *separates* $\mathbb{R} \times \mathbb{R}^+$.

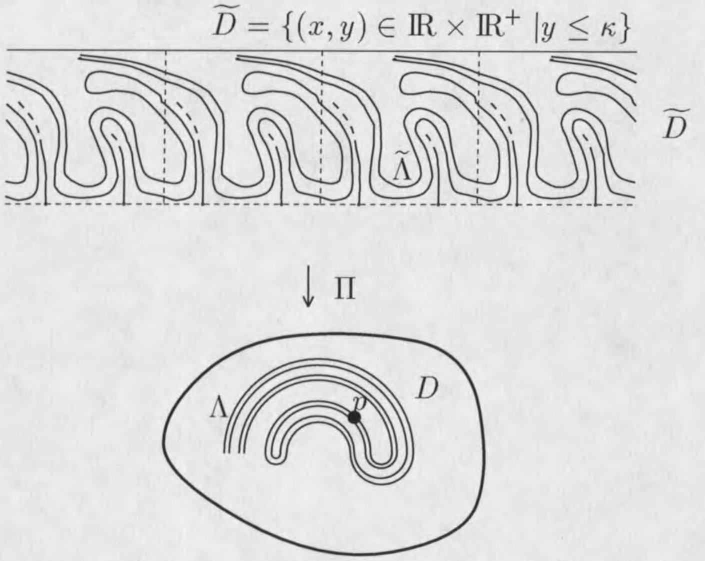

Figure 7: $\widetilde{\Lambda}$ is contained in the interior of $\widetilde{D}$.

4. If $(x, y) \in \widetilde{\Lambda}$, then $(x + n, y) \in \widetilde{\Lambda}$ for every $n \in \mathbb{Z}$.

Proof: If $\widetilde{\Lambda}$ were not nowhere dense, it would contain a set $\widetilde{U}$ which is open in $\mathbb{R}\times\mathbb{R}^+$. Since Π is a covering map, it is open (open sets map to open sets). Therefore, $\Pi(\widetilde{U})$ would be an open set in the plane that would be contained in Λ. Since Λ is nowhere dense, this cannot be the case.

In order to show that $\widetilde{\Lambda}$ is closed, we show that its complement is open. Let $\widetilde{z}$ be a point in $(\mathbb{R} \times \mathbb{R}^+)\backslash\widetilde{\Lambda}$. Since Π is a covering map, it is a local homeomorphism, so there exists an open set $\widetilde{U}$ containing $\widetilde{z}$ such that $\Pi|_{\widetilde{U}}$ is an embedding. Since $\mathbb{R}\times\mathbb{R}^+$ and $\mathbb{R}^2$ are both surfaces, $U = \Pi(\widetilde{U})$ is an open set in $\mathbb{R}^2$. Let V be an open set in $\mathbb{R}^2$ such that $z \in V$ and $V \cap \Lambda = \emptyset$. If we let $W = V \cap U$ and $\widetilde{W} = \Pi^{-1}(V) \cap \widetilde{U}$, then Π maps $\widetilde{W}$ homeomorphically onto the open set W so that $\widetilde{W}$ must be open in $\mathbb{R}\times\mathbb{R}^+$. Since z is in W and $W \cap \Lambda = \emptyset$, we know that $\widetilde{z} \in \widetilde{W}$ and $\widetilde{W} \cap \widetilde{\Lambda} = \emptyset$. The point $\widetilde{z}$ was arbitrary, so $(\mathbb{R} \times \mathbb{R}^+)\backslash\widetilde{\Lambda}$ is open, implying that $\widetilde{\Lambda}$ is a closed subset of $\mathbb{R} \times \mathbb{R}^+$.

To show that no closed subset of $\widetilde{\Lambda}$ separates $\mathbb{R} \times \mathbb{R}^+$, let $H \subset \widetilde{\Lambda}$ be closed and suppose to the contrary that H does separate $\mathbb{R} \times \mathbb{R}^+$. Let κ be large enough so that if $\widetilde{U} = \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid y > \kappa\}$, then $\widetilde{U} \cap \widetilde{\Lambda} = \emptyset$. The set $(\mathbb{R} \times \mathbb{R}^+)\backslash H$ is disconnected by hypothesis, so there exist distinct connected components $\widetilde{A}$ and $\widetilde{B}$ of $(\mathbb{R} \times \mathbb{R}^+)\backslash H$. Since H is closed, $\widetilde{A}$ and $\widetilde{B}$ are open in $\mathbb{R} \times \mathbb{R}^+$, and since Π must be an open map, $A = \Pi(\widetilde{A})$ and $B = \Pi(\widetilde{B})$ are open in $\mathbb{R}^2$. The set Λ is nowhere dense, hence there exist points $a \in A\backslash\Lambda$ and $b \in B\backslash\Lambda$. Claim 2.4 states that $\mathbb{R}^2\backslash\Lambda$ is a topological punctured plane so it is path connected. Therefore, there must exist arcs α and β from a to an arbitrary $z \in \Pi(\widetilde{U})$ and from b to this z, respectively, such that $\alpha \cap \Lambda = \emptyset = \beta \cap \Lambda$. For convenience, we may choose α and β to be disjoint. The arc α can be lifted to an arc $\widetilde{\alpha}$ with one endpoint in $\widetilde{A}$ and the other in $\widetilde{U}$, while β may be lifted to $\widetilde{\beta}$, an arc in $\mathbb{R} \times \mathbb{R}^+$ with an endpoint in $\widetilde{B}$ and one in $\widetilde{U}$. Since $\alpha \cap \beta = \emptyset$, it must be the case that $\widetilde{\alpha} \cap \widetilde{\beta} = \emptyset$. $\widetilde{U}$ is path connected, so $\widetilde{\alpha}$ and $\widetilde{\beta}$ may be joined with a third arc $\widetilde{\gamma}$, which connects the endpoint of $\widetilde{\alpha}$ in $\widetilde{U}$ with the endpoint of $\widetilde{\beta}$ in $\widetilde{U}$, such that $\widetilde{\alpha} \cap \widetilde{\gamma}$ and $\widetilde{\beta} \cap \widetilde{\gamma}$ are singletons. That is, $\widetilde{\alpha} \cup \widetilde{\gamma} \cup \widetilde{\beta}$ is an arc from $\widetilde{A}$ to $\widetilde{B}$ which is disjoint from H, contradicting $\widetilde{A}$ and $\widetilde{B}$ being distinct components of $(\mathbb{R} \times \mathbb{R}^+)\backslash H$. Therefore, $(\mathbb{R} \times \mathbb{R}^+)\backslash H$ must have exactly one component.

Finally, the fourth property in the claim follows immediately from the fact that $\Pi(x, y) = \Pi(x + 1, y)$ for every (x, y) in $\mathbb{R} \times \mathbb{R}^2$. □

The restriction of F to the plane punctured at p, $F|_{\mathbb{R}^2\backslash\{p\}}$, lifts to an orientation preserving homeomorphism $\widetilde{F} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ so that the diagram:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R}^+ & \xrightarrow{\widetilde{F}} & \mathbb{R} \times \mathbb{R}^+ \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathbb{R}^2\backslash\{p\} & \xrightarrow[F|_{\mathbb{R}^2\backslash\{p\}}]{} & \mathbb{R}^2\backslash\{p\} \end{array} \quad \text{commutes.}$$

That is to say $\Pi \circ \widetilde{F} = F|_{\mathbb{R}^2\backslash\{p\}} \circ \Pi$.

For any $z \in \Lambda\backslash\{p\}$ we have that following:

Definition 2.7 The *rotation set* $\rho(z,\widetilde{F})$ with respect to the chosen lift $\widetilde{F}$ of $F|_{\mathbb{R}^2\setminus\{p\}}$ is the closed interval

$$\rho(z,\widetilde{F}) = \left[\liminf_{n\to\infty} \frac{1}{n}\pi_1 \circ \widetilde{F}^n(\widetilde{z}), \limsup_{n\to\infty} \frac{1}{n}\pi_1 \circ \widetilde{F}^n(\widetilde{z})\right]$$

where $\widetilde{z}$ is any element of $\Pi^{-1}(z)$, π_1 is the first coordinate projection, and $\widetilde{F}^n(\widetilde{z}) = \underbrace{\widetilde{F} \circ \ldots \circ \widetilde{F}}_{n \text{ times}}(z)$. We define the *rotation set of* Λ with respect to $\widetilde{F}$ to be $\rho(\Lambda,\widetilde{F}) = \bigcup_{z\in\Lambda\setminus\{p\}} \rho(z,\widetilde{F})$. If $\rho(z,\widetilde{F})$ (or $\rho(\Lambda,\widetilde{F})$) degenerates to a singleton, $\rho(z,\widetilde{F})$ (or $\rho(\Lambda,\widetilde{F})$) will be refered to as the *rotation number of* z (or *of* Λ).

One can show that an equivalent definition of the rotation set of z is the set of all limit points of the sequence $\{\frac{1}{n}\pi_1 \circ \widetilde{F}^n(\widetilde{z})\}_{n=1}^{\infty}$. It is important to note that, though the rotation set $\rho(z,\widetilde{F})$ is independent of the choice of $\widetilde{z} \in \Pi^{-1}(z)$, it does depend on the choice of the lift $\widetilde{F}$. However, if $\widetilde{F}$ and $\widetilde{F}'$ are two lifts of $F|_{\mathbb{R}^2\setminus\{p\}}$, there is an integer j such that for any $z \in \Lambda\setminus\{p\}$, we have $\rho(z,\widetilde{F}') = \rho(z,\widetilde{F}) + j$.

Next, we define what is meant by the local rotation number about p. Let

$$t = \min_{1\leq j\leq \mathbf{n}}\left\{\max\left\{y\middle|(x,y)\in\Pi^{-1}(C_j)\right\}\right\} \text{ and let } \widetilde{W}_t = \left\{(x,y)\in\mathbb{R}\times\mathbb{R}^+\middle|y<t\right\}.$$

From the continuity of F and the properties of Π and $\widetilde{F}$, there exists an $s > 0$ such that if $\widetilde{W}_s = \{(x,y) \in \mathbb{R}\times\mathbb{R}^+|y < s\}$, then $\widetilde{F}(\widetilde{W}_s) \subset \widetilde{W}_t$. Let $k \in \mathbb{Z}$ and $j = k(\text{mod } \mathbf{n})$. Let $\widetilde{C}_k$ be the unique component of $\Pi^{-1}(C_j)$ which is contained in $\{(x,y)|\frac{k}{\mathbf{n}} < x < \frac{k+1}{\mathbf{n}}\}$ (see Figure 8). For a fixed lift $\widetilde{F}$ of F, there exists a unique integer μ such that $\widetilde{F}(\widetilde{C}_k \cap \widetilde{W}_s) \subset \widetilde{C}_{k+\mu}$ for every $k \in \mathbb{Z}$. Therefore, the natural ordering of the $\widetilde{C}_k$'s is preserved by $\widetilde{F}$ in the following way: If $\pi_1(\widetilde{C}_k) < \pi_1(\widetilde{C}_m)$, then $\widetilde{F}(\widetilde{C}_k \cap \widetilde{W}_s) \subseteq \widetilde{C}_{k+\mu}$, $\widetilde{C}_{m+\mu}$ contains $\widetilde{F}(\widetilde{C}_m \cap \widetilde{W}_s)$ and $\pi_1(C_{k+\mu}) < \pi_1(\widetilde{C}_{m+\mu})$. (By $\pi_1(\widetilde{C}_k) < \pi_1(\widetilde{C}_m)$, we mean that $\pi_1(\widetilde{z}) < \pi_1(\widetilde{w})$ for every $\widetilde{z} \in \widetilde{C}_k$ and every $\widetilde{w} \in \widetilde{C}_m$.)

Definition 2.8 The *local rotation number at* p *with respect to* $\widetilde{F}$, denoted by $\rho_l(p,\widetilde{F})$, will be defined to be the rational number $\frac{\mu}{\mathbf{n}}$.

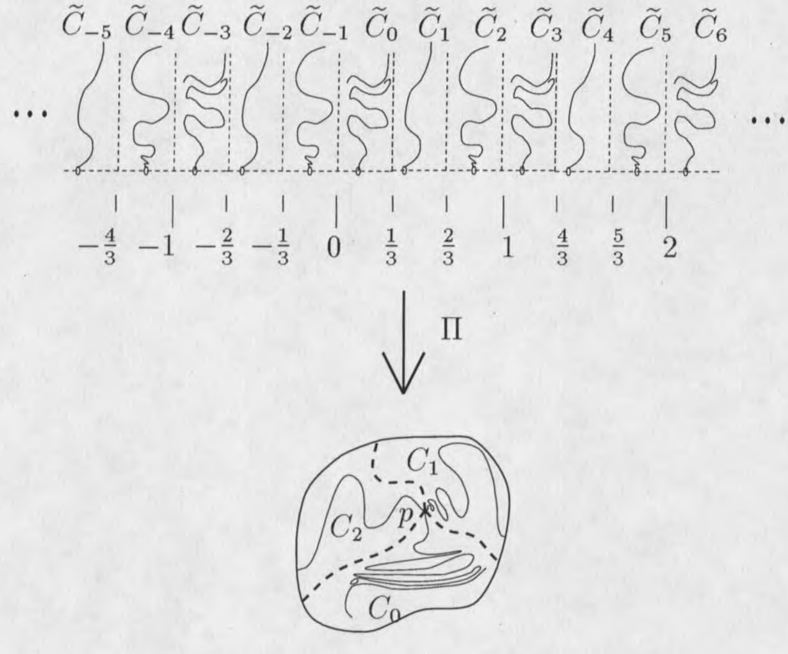

Figure 8: Lifting the spokes of the wheel

Loosely speaking, F rotates the "spokes of the wheel" about p. The local rotation number is a measurement of the number of spokes rotated through under F, relative to the total number of spokes emanating from p. It is nothing more than this, as the following example illustrates.

Example 2.9 Let Λ be the union of p and two half open arcs (each homeomorphic to a half open interval) which spiral onto p as in Figure 9. Let F be a homeomorphism of the plane that leaves Λ invariant, fixes p, and interchanges the two spiral arcs. F can be chosen so that as z approaches p, the direction of the vector $F(z) - z$ approaches that of a radial vector from p to z. One might suspect that $\rho_l(p; \tilde{F}) = 0$ for an appropriate lift $\tilde{F}$, but with our definition of the local rotation number, this is not the case. Since the arms of Λ are interchanged, $\rho_l(p; \tilde{F})$ must be $1/2$ (plus an integer). Note also, that in the figure, Π is not an appropriate covering map of $\mathbb{R}^2 \setminus \{p\}$, since if Γ is one of the spiral arms, components of $\Pi^{-1}(\Gamma)$ are not bounded as Property 4 of the SA requires. Π', however, does meet the requirements.

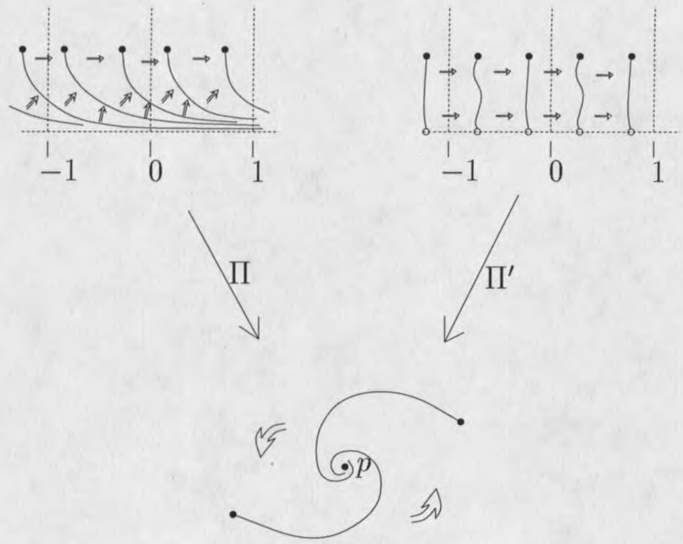

Figure 9: Two lifts of $\mathbb{R}^2\backslash\{p\}$

There is another way in which a rotation number in the neighborhood of a fixed point has been defined. Franks defines an *infinitesimal* rotation number for a fixed point p of a planar diffeomorphism in [Fra90]. This number is obtained by removing the fixed point and radially "blowing up" $\mathbb{R}^2\backslash\{p\}$ to $\mathbb{R}^2\backslash$ cl(D) (where cl(D) is a closed disk centered at p). The induced homeomorphism of $\mathbb{R}^2\backslash$ (cl(D)) extends to a homeomorphism of $\mathbb{R}^2\backslash D$ and the infinitesimal rotation number is defined to be the rotation number of homeomorphism restricted to the boundary circle ∂D. The infinitesimal rotation number is *not* the same as the local rotation number. Figure 10 provides an example of a nondifferentiable homeomorphism where this blowing up of the fixed point does not extend to the boundary circle (so no infinitesimal rotation number is defined.) In this figure, Λ consists of the fixed point p and two arms, each an embedded half open line segment, that limit on p. The first arm is contained in a radial line from p and the other contains an infinite number of bends as it limits on p. Choose F to be a homeomorphism of $\mathbb{R}^2$ such that $F(\Lambda)=\Lambda$, $F(p) = p$ and F interchanges the two arms. After blowing up p, the first arm limits at a single point on the circle ∂D, but the other arm limits on a nondegenerate arc on the circle. If the homeomorphism on $\mathbb{R}^2\setminus$ (cl(D)) were to extend to a homeomorphism of $\mathbb{R}^2\setminus D$, it would have to be the case that the point of ∂D on which the first arm limits would have to be mapped onto the entire arc of ∂D on which the other arm limits, an impossibility. The local rotation number is defined for this example, however. With the appropriate choice of a lift, $\rho_l(p; \widetilde{F}) = 1/2$. For any other lift, the local rotation number in an integer plus 1/2. We conclude this section with the following:

Conjecture 2.10 *If both the local and infinitesimal rotation numbers are defined, then they are equal.*

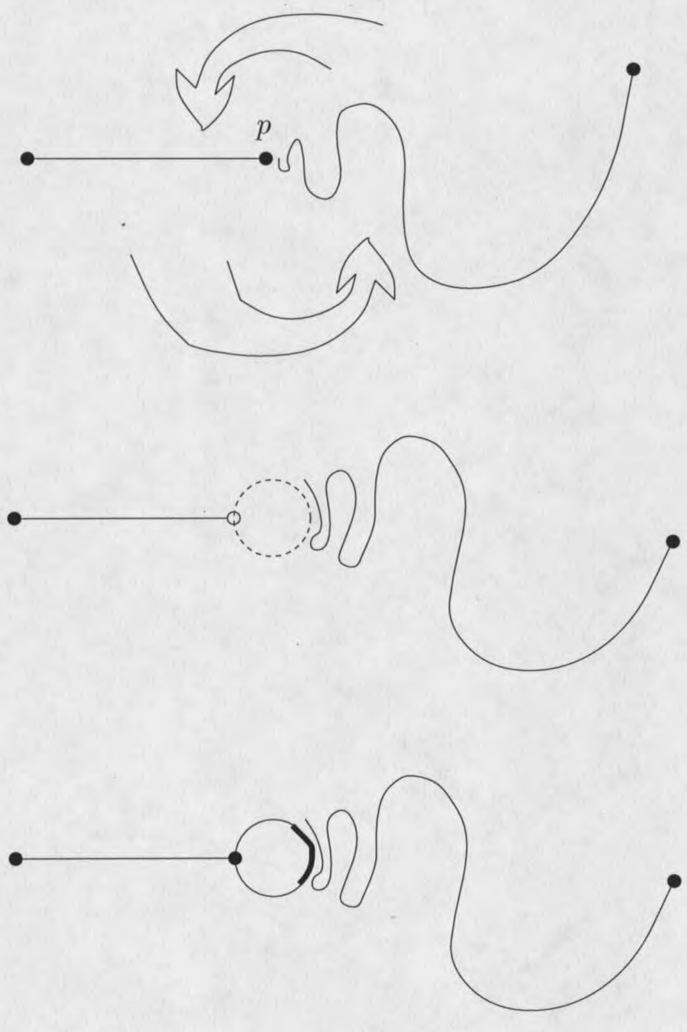

Figure 10: "Blowing up" the fixed point doesn't always extend to the boundary circle.

CHAPTER 3

Useful Tools

Periodic Disk Chains

The following definition is due to John Franks, [Fra88].

Definition 3.1 A *periodic disk chain* of a surface homeomorphism $G : M \to M$ is a finite collection $\{U_i\}_{i=1}^{k}$ of open topological disks $U_i \subset M$ with the following properties:

1. $G(U_i) \cap U_i = \emptyset$ for each $i \in \{1, 2, ..., k\}$,
2. $U_i \cap U_j = \emptyset$ if $i \neq j$, and
3. There exist positive integers m_i for each $i \in \{1, 2, ..., k\}$ such that $G^{m_i}(U_i) \cap U_{i+1} \neq \emptyset$ for $i \neq k$, and that $G^{m_k}(U_k) \cap U_1 \neq \emptyset$ (see Figure 11).

Proposition 1.3 of [Fra88] states that the existence of a periodic disk chain for an orientation preserving planar homeomorphism G implies that there is a fixed point for G. This is a generalization of Brouwer's theorem: Every orientation preserving homeomorphism of the plane that contains a periodic point must also have a fixed point (see [Bro12] and [Bro84]).

For the remainder of this thesis we will assume that Λ is an invariant continuum that contains a fixed point p, and that Λ satisfies all the properties of the Standing Assumptions (SA). In the following lemma, however, only Property 1 (Λ does not separate the plane) of the SA is required. (Properties 2-4 are not needed.) As in the previous chapter let $\widetilde{\Lambda}$ be the lift of $\Lambda|_{\{p\}}$.

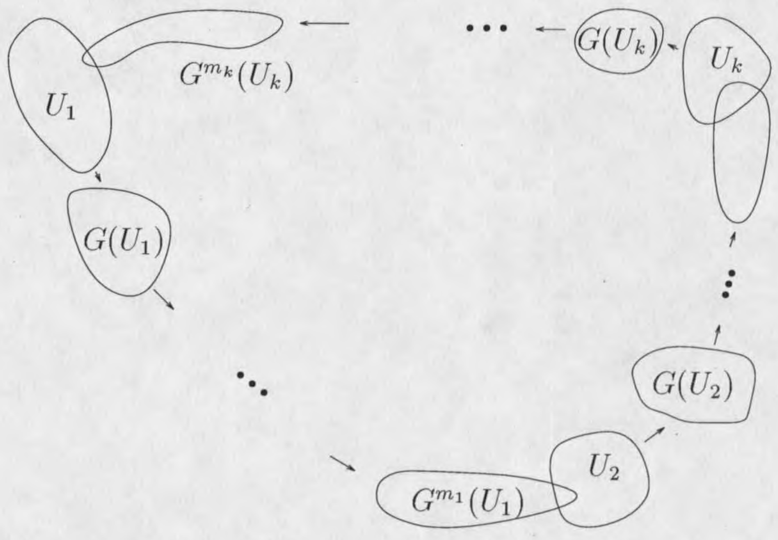

Figure 11: A periodic disk chain

Lemma 3.2 *Suppose that* $\widetilde{\Phi} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ *is an orientation preserving homeomorphism satisfying these hypotheses:*

1. $\widetilde{\Phi} \circ T = T \circ \widetilde{\Phi}$ *where* $T : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ *is the deck transformation* $T(x, y) = (x + 1, y)$. *(i.e.* $\widetilde{\Phi}$ *is a lift);*

2. $\widetilde{\Phi}(\widetilde{\Lambda}) = \widetilde{\Lambda}$*;*

3. *There exists an integer* m *and positive numbers* r *and* η*, such that if* $\widetilde{G} = \widetilde{\Phi}^m$*, then* $d(\widetilde{z}, \widetilde{G}(\widetilde{z})) > \eta$ *for every* $\widetilde{z}$ *with* $\pi_2(\widetilde{z}) < r$*; and*

4. $\widetilde{G}|_{\widetilde{\Lambda}}$ *has a chain recurrent point.*

Then there is a fixed point of $\widetilde{\Phi}$ *which is in* $\widetilde{\Lambda}$.

Proof: As in Lemma 2.5 let $\{D_n\}_{n=1}^{\infty}$ be a nested sequence of closed disks such that $\Lambda = \cap_{n=1}^{\infty} D_n$, where $\Lambda = \Pi(\widetilde{\Lambda})$ is contained in the interior of each D_n. Then (as in Figure 12), $\widetilde{\Lambda} = \cap_{n=1}^{\infty} \widetilde{D}_n$, the nested intersection of topological closed half-planes $\widetilde{D}_n = \Pi^{-1}(D_n)$, each containing $\widetilde{\Lambda}$ in its interior.

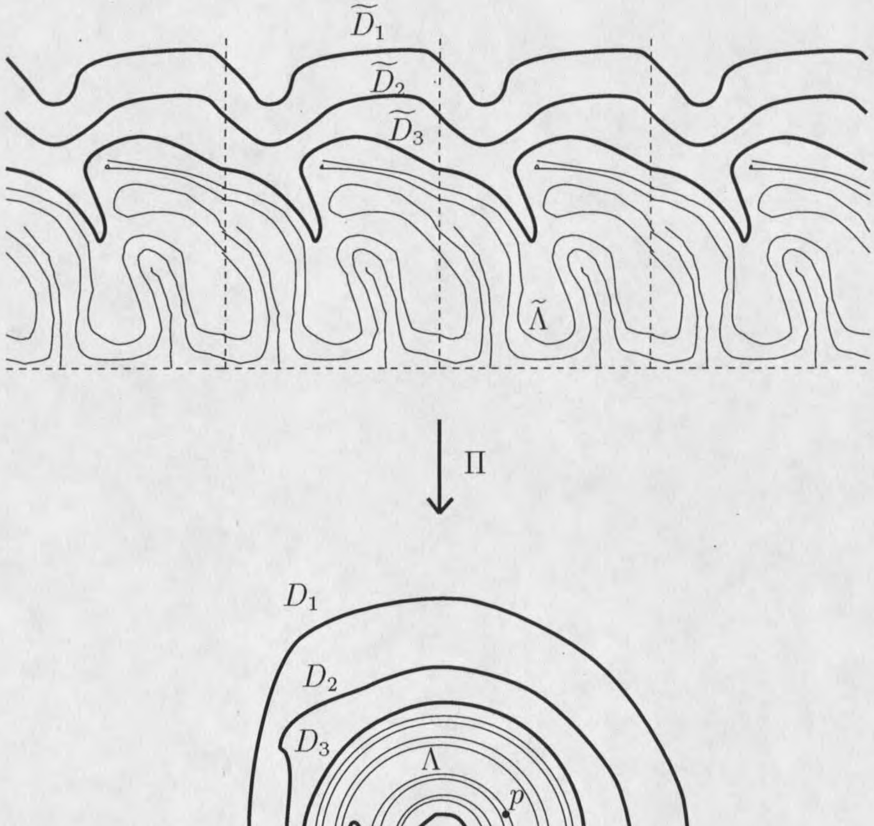

Figure 12: $\Lambda = \cap_{n=1}^{\infty} D_n$ and $\widetilde{\Lambda} = \cap_{n=1}^{\infty} \widetilde{D}_n$

Proof by contradiction: Let X be the fixed point set of $\widetilde{G}$ and suppose that for some n we have $\widetilde{D}_n \cap X = \emptyset$. Let V be the component of $(\mathbb{R} \times \mathbb{R}^+)\backslash X$ that contains $\widetilde{D}_n$ (so $\widetilde{G}(V) = V$) and let $P : \mathbb{R}^2 \to V$ be a universal cover. Since $\widetilde{D}_n$ is simply connected, $P^{-1}(\widetilde{D}_n)$ has a countable number of components (infinite if V is not simply connected or exactly one, otherwise) such that the restriction of P to any one of these components is an embedding. Let C be a component of $P^{-1}(\widetilde{D}_n)$ and let $\Lambda' = C \cap P^{-1}(\widetilde{\Lambda})$. There is an orientation preserving lift $g : \mathbb{R}^2 \to \mathbb{R}^2$ of $\widetilde{G}|_V$ which maps Λ' homeomorphically onto itself. We will first prove the result under the assumption that there is a periodic disk chain $\{\widetilde{U}_i\}_{i=1}^k$ for $\widetilde{G}$, with $\widetilde{U}_i \subset \widetilde{D}_n$ for $i \in \{1, 2, ..., k\}$ and then show that such a chain exists. For each i, let U_i be the unique component of $P^{-1}(\widetilde{U}_i)$ having nonempty intersection with C. Then $\{U_i\}_{i=1}^k$ is a periodic disk chain for the orientation preserving homeomorphism g of $\mathbb{R}^2$. By Franks' proposition, there is a fixed point z of g which implies that $P(z) \in V$ must be fixed by $\widetilde{G}$. This contradicts the fact that V is contained in the complement of the fixed point set. Thus, $\widetilde{G}$ must have a fixed point in every half-plane $\widetilde{D}_n$. In fact, by the first hypothesis, there must be a fixed point in $\{\tilde{z} | \; 0 \leq \pi_1(\tilde{z}) \leq 1\}$, while the third hypothesis implies that if $\tilde{z}$ is fixed, then $\pi_2(\tilde{z})$ must be greater than or equal to r. Therefore, for every positive integer n, there must be a fixed point $\tilde{z}_n$ in the compact set $\widetilde{D}_n \cap \{\tilde{z} | r \leq \pi_2(\tilde{z}) \leq M$ and $0 \leq \pi_1(\tilde{z}) \leq 1\}$ where $M = \max\{\pi_2(\tilde{z}) | \tilde{z} \in \widetilde{D}_1\}$. A limit point of $\{\tilde{z}_n\}_{n \in \mathbb{N}}$ is in $\widetilde{\Lambda} = \cap_{n=1}^{\infty} \widetilde{D}_n$ and is a fixed point of $\widetilde{G}$.

This fixed point of $\widetilde{G}$ in $\widetilde{\Lambda}$ is a periodic point of (not necessarily least) period m for $\widetilde{\Phi}$ since $\widetilde{G} = \widetilde{\Phi}^m$. At this point, note that if we had made the assumption that $\widetilde{G}$ have a periodic point in $\widetilde{\Lambda}$ instead of assuming that there is a chain recurrent point of $\widetilde{G}|_{\widetilde{\Lambda}}$, we could have lifted the periodic orbit to $C \subset \mathbb{R}^2$ instead of lifting the periodic disk chain. There would then exist a fixed point of $g : \mathbb{R}^2 \to \mathbb{R}^2$ by the Brouwer theorem. We could then proceed to arrive at the same conclusion: $\widetilde{G}$

has a fixed point in $\widetilde{D}_n$ for every $n \in \mathbb{N}$. Now, this is precisely the situation for $\widetilde{\Phi}$. That is, $\widetilde{\Phi}$ has a periodic orbit in $\widetilde{\Lambda}$ implying, by the above discussion, that there is a $\widetilde{\Phi}$-fixed point in every closed half-plane $\widetilde{D}_n$. Since each fixed point must also be fixed by $\widetilde{G}$, Properties 1 and 3 require that there is a sequence of $\widetilde{\Phi}$-fixed points in $\{\tilde{z}|r \leq \pi_2(\tilde{z}) \leq M \text{ and } 0 \leq \pi_1(\tilde{z}) \leq 1\}$. A limit point is in $\widetilde{\Lambda}$ and is fixed under $\widetilde{\Phi}$.

What remains to be shown is that $\widetilde{G}$ has a periodic disk chain $\{\widetilde{U}_i\}_{i=1}^k$ with $\widetilde{U}_i \subset \widetilde{D}_n$ for $i = 1, 2, \ldots, k$.

Claim: There exists a positive δ such that every set with diameter less than δ which intersects $\widetilde{\Lambda}$ is contained in $\widetilde{D}_n$.

The proof of this claim is a modification of a proof of a similar result given by Franks in [Fra88].

Proof of Claim: Let $t = \frac{1}{2}\min\{\pi_2(\tilde{z})|\tilde{z} \in \partial\widetilde{D}_n\}$, let $T = \max\{\pi_2(\tilde{z})|\tilde{z} \in \partial\widetilde{D}_n\}$ and let $\widetilde{S} = \{\tilde{z} \in \mathbb{R} \times \mathbb{R}^+|0 \leq \pi_1(\tilde{z}) \leq 1 \text{ and } t \leq \pi_2(\tilde{z}) \leq T\}$. Then $\widetilde{S}$ is compact so $\partial\widetilde{D}_n \cap \widetilde{S}$ and $\widetilde{\Lambda} \cap \widetilde{S}$ are compact as well. If δ is chosen to be one half the distance between $\partial\widetilde{D}_n \cap \widetilde{S}$ and $\widetilde{\Lambda} \cap \widetilde{S}$ if $\widetilde{\Lambda} \cap \widetilde{S}$ is nonempty or equal to t otherwise, then this δ has the desired property.

Now, having this δ , let $\{\tilde{z} = \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{k+1} = \tilde{z}\} \subset \widetilde{\Lambda}$ be an ϵ-chain from $\tilde{z}$ to itself, where $\epsilon = \frac{1}{3}\min\{\delta, \eta\}$. There exists a collection $\{\widetilde{U}_i\}_{i=1}^k$ of open topological disks such that $\tilde{z}_1$ and $\widetilde{G}(\tilde{z}_k)$ are in $\widetilde{U}_1$, and for each $i \in \{2, 3, \ldots, k\}$ we have $\tilde{z}_i$ and $\widetilde{G}(\tilde{z}_{i-1})$ in $\widetilde{U}_i$. It is possible to choose $\{\widetilde{U}_i\}_{i=1}^k$ so that $\widetilde{U}_i \cap \widetilde{U}_j = \emptyset$ if $i \neq j$ and the diameter of each $\widetilde{U}_i$ is less that 2ϵ (see [Oxt77]). $\{\widetilde{U}_i\}_{i=1}^k$ is then the required periodic disk chain. □

More Tools

Notation: For any set $S \subset \mathbb{R} \times \mathbb{R}^+$, let $\lceil S$ denote the infimum of $\pi_1(S)$ and let $S\rceil$ denote the supremum.

For each z in the composant determined by p, let $\Gamma = \Gamma(z)$ be a subcontinuum of Λ which contains both z and p and has the property that for every component $\widetilde{\Gamma}$ of $\Pi^{-1}(\Gamma \backslash \{p\})$, the infimum and supremum of $\{\pi_1(\tilde{z}) | \tilde{z} \in \widetilde{\Gamma}\}$, (ie. $\lceil\widetilde{\Gamma}$ and $\widetilde{\Gamma}\rceil$) are both finite. The fourth property of the SA guarantees the existence of such a subcontinuum.

Lemma 3.3 *If $\widetilde{H} \subset \widetilde{\Lambda}$ is bounded, then for every $\epsilon > 0$ there is an $L \in \mathbb{R}$ such that if $\widetilde{K} \subset \widetilde{\Lambda}$ is any continuum with $\lceil\widetilde{K} < \lceil\widetilde{H} - L$ and $\widetilde{H}\rceil + L < \widetilde{K}\rceil$ then $d(\tilde{z}, \widetilde{K}) < \epsilon$ for every $\tilde{z}$ in $\widetilde{H}$.*

Lemma 3.4 *For every $\epsilon > 0$, there exists an $L \in \mathbb{R}$ such that if $\widetilde{K} \subset \widetilde{\Lambda}$ is a continuum satisfying: $\widetilde{K}\rceil - \lceil\widetilde{K} > L$, then $\Pi(\widetilde{K})$ is ϵ-dense in Λ.*

The proofs of these lemmas will use the following two claims:

Claim 3.5 *Let $\kappa > 0$ and let $\widetilde{S} \subset \mathbb{R} \times \mathbb{R}^+$ be closed and connected. Suppose that there exist real numbers, M and m such that $\widetilde{S}\rceil < M$ and $m < \lceil\widetilde{S}$. If $\widetilde{S} \cap (\mathbb{R} \times \{\kappa\}) \neq \emptyset$ and $\inf\{y | (x, y) \in \widetilde{S}\} = 0$, then $\{(x, y) | 0 < y \leq \kappa\} \backslash \widetilde{S}$ has at least two distinct components, one of which contains $W^- = \{(x, y) | x \leq m$ and $y \leq \kappa\}$ and another of which contains $W^+ = \{(x, y) | x \geq M$ and $y \leq \kappa\}$.*

Proof: Let $l^- = \{(x, y) | x \leq m$ and $y = \kappa/2\}$ and let $l^+ = \{(x, y) | x \geq M$ and $y = \kappa/2\}$. Then $l^- \subset W^-$ and $l^+ \subset W^+$. If we suppose that W^- and W^+ are in the same component of $\{(x, y) | y \leq \kappa\} \backslash \widetilde{S}$, there exists an arc α from the endpoint of l^- to the endpoint of l^+ with

1. $\alpha \cap \widetilde{S} = \emptyset$,

2. $\alpha \cap (l^- \cup l^+) = \{(m, \kappa/2), (M, \kappa/2)\}$, and

3. $T = \sup\{y | (x, y) \in \alpha\} < \kappa$.

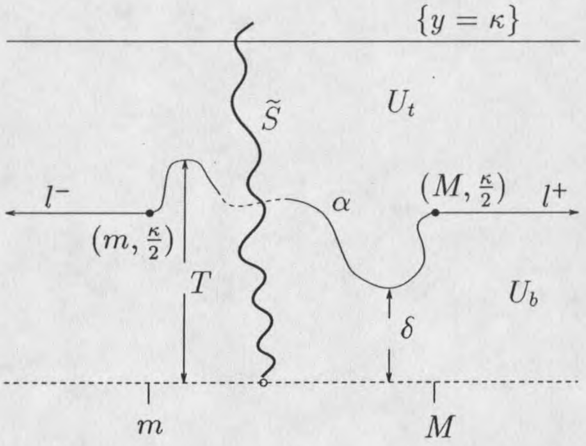

Figure 13: $l^- \cup \alpha \cup l^+$ must separate $\mathbb{R} \times \mathbb{R}^+$

Then $l^- \cup \alpha \cup l^+$ is a curve which separates $\mathbb{R} \times \mathbb{R}^+$ into two components as in Figure 13. Let U_t be the component containing $\{(x,y)|y = \kappa\}$ and let U_b be the component containing $\{(x,y)|y = \delta/2\}$ where $\delta = \min\{y|(x,y) \in \alpha\} > 0$. Since $\widetilde{S} \cap \{(x,y)|y = \kappa\} \neq \emptyset$ we have $\widetilde{S} \cap U_t \neq \emptyset$, and since $\inf\{y|(x,y) \in S\} = 0$ we have $\widetilde{S} \cap U_b \neq \emptyset$. Thus, $\{\widetilde{S} \cap U_t, \widetilde{S} \cap U_b\}$ is a separation of $\widetilde{S}$, contradicting $\widetilde{S}$ being connected. ∎

Claim 3.6 *Suppose that $\widetilde{V} \subset \mathbb{R} \times \mathbb{R}^+$ is an open set containing the point $\tilde{z}$ of $\widetilde{\Lambda}$. Then there exists a number L such that every continuum $\widetilde{K} \subset \widetilde{\Lambda}$ intersects $\widetilde{V}$ if $\lceil \widetilde{K} < \pi_1(\tilde{z}) - L$ and $\pi_1(\tilde{z}) + L < \widetilde{K} \rceil$.*

Proof: Let $\kappa > 0$ be such that $\widetilde{\Lambda} \subset \widetilde{D} = \{(x,y)|y < \kappa\}$ and let $\widetilde{B}$ be a closed disk with $\tilde{z} \in \widetilde{B}$ and $\widetilde{B} \subset \widetilde{V} \cap \widetilde{D}$. Let $B = \Pi(\widetilde{B})$, $z = \Pi(\tilde{z})$ and $D = \Pi(\widetilde{D})$. Since Λ is nowhere dense in $\mathbb{R}^2$, there exists a $w \in B \backslash \Lambda$, and since Λ doesn't separate the plane, there is an arc α from w to $\Pi\{(x,y)|y = \kappa\}$ (the boundary of D) with $\alpha \cap \Lambda = \emptyset$. Every composant of Λ is dense in Λ, so there exists a point $y \in B$ such that y is in the composant determined by p. Let C be a proper subcontinuum of Λ which contains both y and p, and let Γ' be the closure of the component of $C \backslash B$ that contains p.

If $\Gamma \subset \Gamma'$ is the continuum that irreducibly contains p and intersects B, then Γ is a subcontinuum of Λ with $\Gamma \cap \text{int}B = \emptyset$ and $\Gamma \cap B \neq \emptyset$. (See Theorem 2-16 of [HY61], the "Going to the Boundary Lemma".) Let $\widetilde{\Gamma}$ be a component of $\Pi^{-1}(\Gamma)$ such that $\widetilde{\Gamma} \cap \partial\widetilde{B} \neq \emptyset$ and $\widetilde{\Gamma} \cap \text{int}\widetilde{B} = \emptyset$, and let $\widetilde{\alpha}$ be a component of $\Pi^{-1}(\alpha)$ having nonempty intersection with $\widetilde{B}$. Finally, let $M = \max\{\ \widetilde{\Gamma}\rceil, \widetilde{B}\rceil, \widetilde{\alpha}\rceil\ \} + 1$ and let $m = \min\{\ \lceil\widetilde{\Gamma}, \lceil\widetilde{B}, \lceil\widetilde{\alpha}\ \} - 1$ (see Figure 14).

We have assumed (by Property 4 of the SA) that $\widetilde{\Gamma}$ is bounded. This, and the fact that $\widetilde{\alpha}$ and $\widetilde{B}$ are also bounded imply that m and M are finite, so the set $\widetilde{S} = \widetilde{\Gamma} \cup \widetilde{B} \cup \widetilde{\alpha}$ satisfies the hypothesis of Claim 3.5.

More Notation: Let $j = [\![(M-m)]\!]$ (where $[\![\ ]\!]$ is the greatest integer function) and for $S \subset \mathbb{R} \times \mathbb{R}^+$, let

$$S_+ = \{(x+j, y) \in \mathbb{R} \times \mathbb{R}^+ | (x,y) \in S\} \text{ and}$$
$$S_- = \{(x-j, y) \in \mathbb{R} \times \mathbb{R}^+ | (x,y) \in S\}.$$

In this notation, we have $\widetilde{\Gamma}_+$ and $\widetilde{\Gamma}_-$ contained in $\widetilde{\Lambda}$, since j is an integer and $\widetilde{\Gamma} \subset \widetilde{\Lambda}$. Also, note that both of $\widetilde{\Gamma}_+ \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+$ and $\widetilde{\Gamma}_- \cup \widetilde{B}_- \cup \widetilde{\alpha}_-$ are closed and connected sets which satisfy the requirements for the set $\widetilde{S}$ in Claim 3.5 (see Figure 15).

Let $L = 2(M-m)$ and $\widetilde{K} \subset \widetilde{\Lambda}$ be a continuum with $\lceil\widetilde{K} < \pi_1(\widetilde{z}) - L$ and $\pi_1(\widetilde{z}) + L < \widetilde{K}\rceil$. Since $\widetilde{K} \subset \widetilde{\Lambda}$, it must be the case that $\widetilde{K} \subset \widetilde{D}$. It is also true that $(\widetilde{\Gamma} \cup \widetilde{B} \cup \widetilde{\alpha})\rceil < \widetilde{K}\rceil$ and $\lceil\widetilde{K} < \lceil(\widetilde{\Gamma} \cup \widetilde{B} \cup \widetilde{\alpha})$, so by Claim 3.5, $\widetilde{K} \cap (\widetilde{\Gamma} \cup \widetilde{B} \cup \widetilde{\alpha}) \neq \emptyset$. Similarly, $(\widetilde{\Gamma}_- \cup \widetilde{B}_- \cup \widetilde{\alpha}_-)\rceil < \widetilde{K}\rceil$ and $\lceil\widetilde{K} < \lceil(\widetilde{\Gamma}_- \cup \widetilde{B}_- \cup \widetilde{\alpha}_-)$ imply that $\widetilde{K} \cap (\widetilde{\Gamma}_- \cup \widetilde{B}_- \cup \widetilde{\alpha}_-) \neq \emptyset$, and $(\widetilde{\Gamma}_+ \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+)\rceil < \widetilde{K}\rceil$ and $\lceil\widetilde{K} < \lceil(\widetilde{\Gamma}_+ \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+)$ imply that $\widetilde{K} \cap (\widetilde{\Gamma}_+ \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+) \neq \emptyset$. However, none of $\widetilde{\alpha}_-$, $\widetilde{\alpha}$, or $\widetilde{\alpha}_+$ meets $\widetilde{\Lambda}$, so each of $\widetilde{K} \cap (\widetilde{\Gamma}_- \cup \widetilde{B}_-)$, $\widetilde{K} \cap (\widetilde{\Gamma} \cup \widetilde{B})$, and $\widetilde{K} \cap (\widetilde{\Gamma}_+ \cup \widetilde{B}_+)$ is nonempty. Now, if $\widetilde{K} \cap \widetilde{B} \neq \emptyset$ we are done, so assume that $\widetilde{K} \cap \widetilde{B} = \emptyset$. This forces $\widetilde{K} \cap \widetilde{\Gamma}$ to be nonempty. $\widetilde{K} \cap \widetilde{\Gamma} \neq \emptyset$ requires that $\widetilde{K} \cap \widetilde{\Gamma}_- = \emptyset$ and $\widetilde{K} \cap \widetilde{\Gamma}_+ = \emptyset$ since, for example, if $\widetilde{K} \cap \widetilde{\Gamma}_-$ is not empty, then we would have

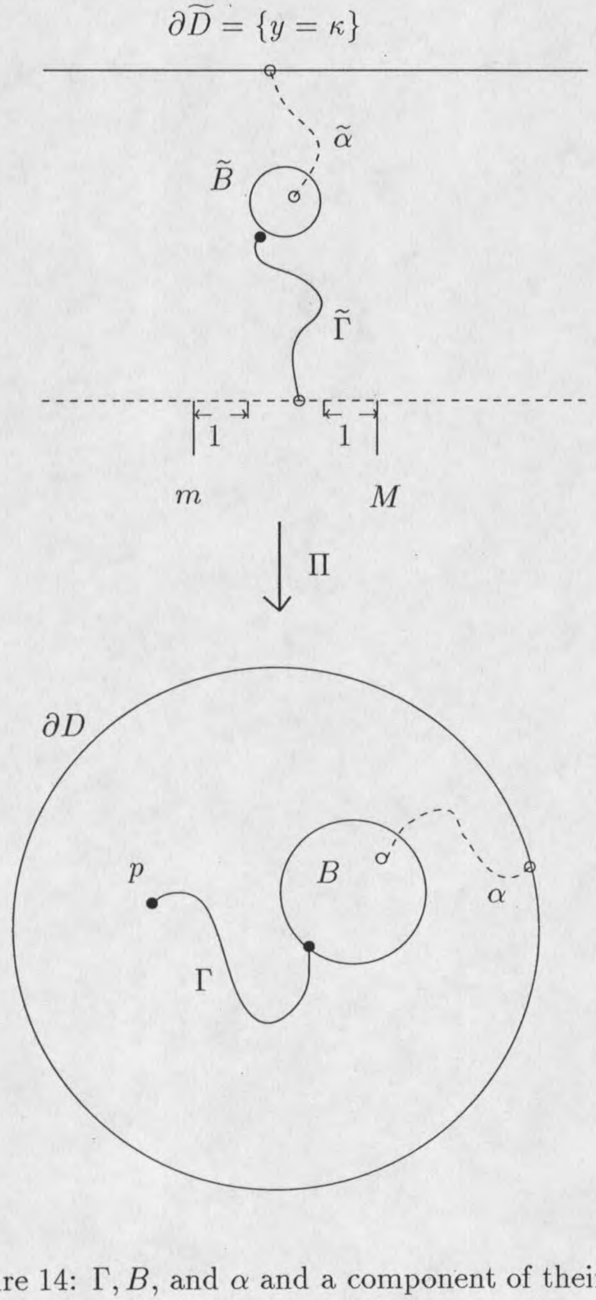

Figure 14: Γ, B, and α and a component of their lifts

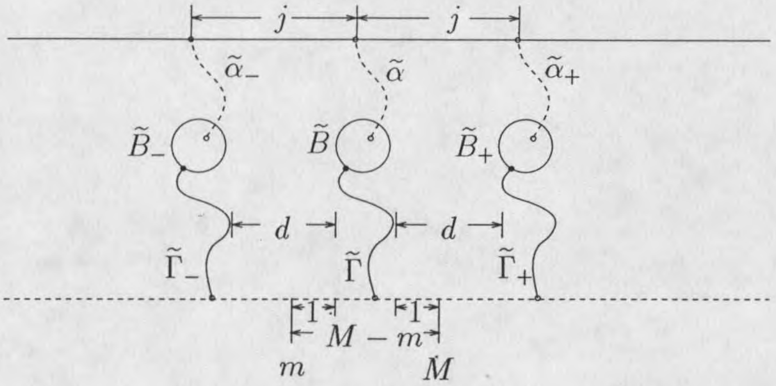

Figure 15: Three components of $\Pi^{-1}(\tilde{\Gamma} \cup \tilde{B} \cup \tilde{\alpha})$

$\tilde{\Gamma}_- \cup \widetilde{K} \cup \tilde{\Gamma}$ a closed subset of $\tilde{\Lambda}$ which separates $\mathbb{R} \times \mathbb{R}^+$. This cannot occur according to Property 3 of Claim 2.6.

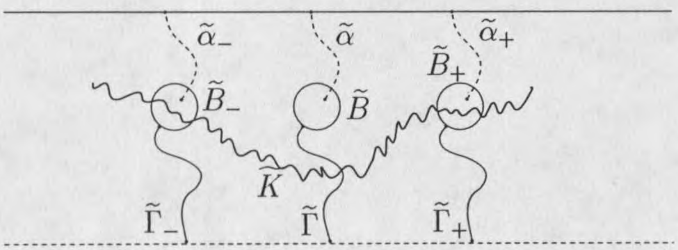

Figure 16: Such a $\widetilde{K} \subset \tilde{\Lambda}$ cannot exist.

Therefore, if $\widetilde{K} \cap \tilde{B}$ is empty, $\widetilde{K}$ must have nonempty intersection with each of $\tilde{B}_-$, $\tilde{\Gamma}$, and $\tilde{B}_+$ (see Figure 16). It will be shown that a $\widetilde{K}$ having these nonempty intersections does not exist, so that $\widetilde{K}$ must then have a point in common with $\tilde{B}$.

Subclaim: $\widetilde{H} = \tilde{\Gamma} \cup \widetilde{K} \cup \widetilde{K}_+ \cup \tilde{\Gamma}_+$ separates $\mathbb{R} \times \mathbb{R}^+$ (see Figure 17).

Proof of Subclaim: For $\delta = \min\{y|(x,y) \in \widetilde{K}\}$ and $M = \max\{\tilde{\Gamma}\rceil, \tilde{B}\rceil, \tilde{\alpha}\rceil\} + 1$, we'll show that the points $(M, \delta/2)$ and (M, κ) lie in distinct components of the complement of $\widetilde{H}$ (as in Figure 18). If not, there is an arc from $(M, \delta/2)$ to (M, κ) which is disjoint from $\widetilde{H}$. Let λ be the union of this arc with the half open arc $\{(x,y)|x = M$ and $0 < y \leq \delta/2\}$.

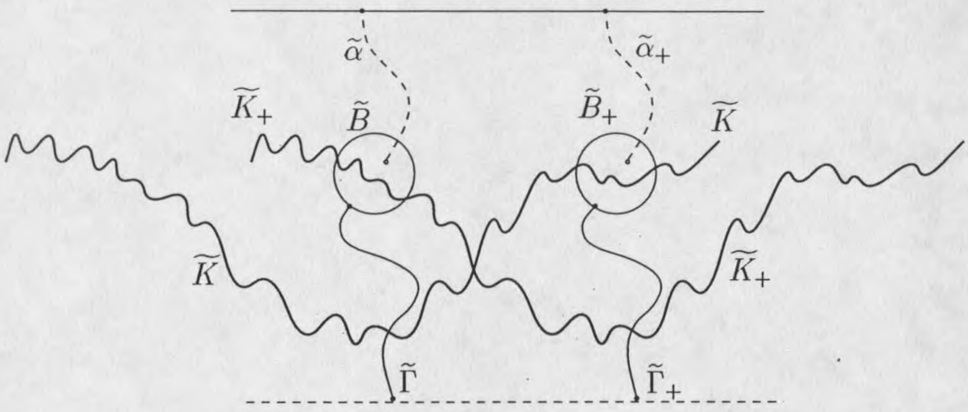

Figure 17: $\widetilde{H} = \widetilde{\Gamma} \cup \widetilde{K} \cup \widetilde{K}_+ \cup \widetilde{\Gamma}_+$ separates $\mathbb{R} \times \mathbb{R}^+$.

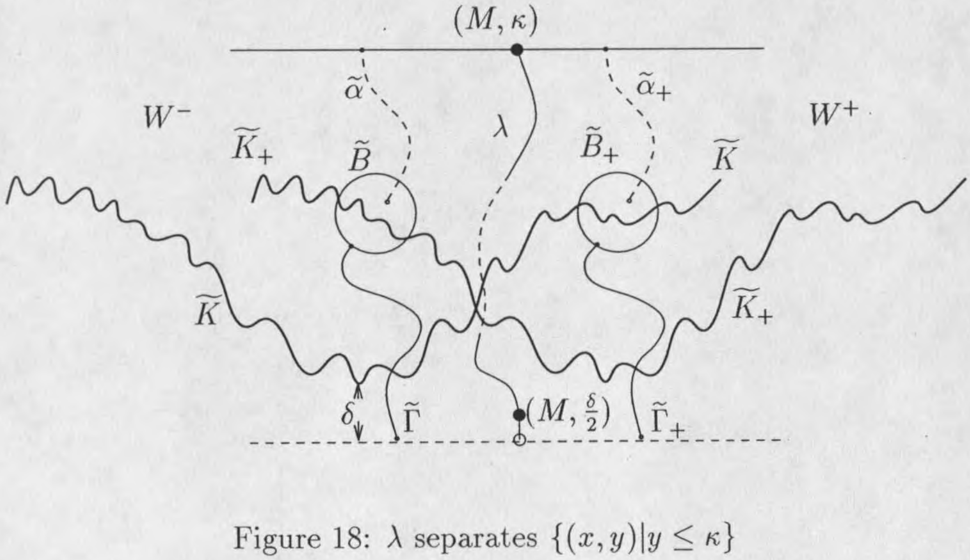

Figure 18: λ separates $\{(x, y) | y \leq \kappa\}$

Now, λ must separate $\{(x,y)|y \leq \kappa\}$ (Claim 3.5) in such a way that one component, W^-, of $\{(x,y)|y \leq \kappa\}\backslash\lambda$ contains $\widetilde{\Gamma}$ and another, W^+, contains $\widetilde{\Gamma}_+$. That is, $\widetilde{\Gamma} \subset W^-$ and $\widetilde{\Gamma}_+ \subset W^+$. Since $\widetilde{\Gamma} \cap \widetilde{K} \neq \emptyset$, $\widetilde{K}$ must also lie in W^-, and since $\widetilde{\Gamma}_+ \cap \widetilde{K}_+ \neq \emptyset$, $\widetilde{K}_+$ is contained in W^+. To arrive at a contradiction, we will show that $\widetilde{K} \cap \widetilde{K}_+ \neq \emptyset$, since $\widetilde{y} \in \widetilde{K} \cap \widetilde{K}_+$ will be in two supposed disjoint sets, W^- and W^+.

The set $\widetilde{\Gamma} \cup \widetilde{K} \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+$ also separates $\{(x,y)|y \leq \kappa\}$ as in Claim 3.5. Let U^- be the component of $\{(x,y)|y \leq \kappa\}\backslash(\widetilde{\Gamma} \cup \widetilde{K} \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+)$ containing $\{(x,y)|y = \kappa$ and $x > \widetilde{\alpha}_+\rceil\}$. Note that $\widetilde{K}_+ \cap U^+ \neq \emptyset$ since $\widetilde{K}_+\rceil > (\widetilde{\Gamma} \cup \widetilde{K} \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+)\rceil$. Also, note that $\widetilde{K}_+ \cap U^- \neq \emptyset$ since there is an arc in the interior of $\widetilde{B}$, except for its endpoints, connecting any point of $\widetilde{K}_+ \cap \widetilde{B}$ with $\widetilde{\alpha}$. Since $\widetilde{\alpha} \cap \{(x,y)|\ y = \kappa$ and $x < \lceil\alpha_+\}$ is nonempty and is in the complement of $\widetilde{\Gamma} \cup \widetilde{K} \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+$, $\widetilde{\alpha}$ must be contained in U^-. Therefore, $\widetilde{K}_+ \cap U^- \neq \emptyset$ so we must then have $\widetilde{K} \cap \widetilde{K}_+ \neq \emptyset$, since $\widetilde{K}_+ \cap (\widetilde{\Gamma} \cup \widetilde{B}_+ \cup \widetilde{\alpha}_+) = \emptyset$ and $\widetilde{K}_+$ is connected. This establishes the subclaim and completes the proof of the claim. □

Finally, we are in a position to prove the first two lemmas.

Proof of Lemma 3.3: Let $\{\widetilde{V}_i\}_{i=1}^n$ be a finite cover of $\widetilde{H}$ by open sets in $\mathbb{R} \times \mathbb{R}^+$ such that $\widetilde{z}_i \in \widetilde{V}_i \cap \widetilde{H}$ and the diameter of $\widetilde{V}_i$ is less that ϵ for each $i \in \{1, 2, ..., n\}$. Let L_i be the number from Claim 3.6 so that any continuum $\widetilde{K} \subset \widetilde{\Lambda}$ with $\lceil\widetilde{K} < \pi_1(\widetilde{z}_i) - L_i$ and $\pi_1(\widetilde{z}_i) + L_i < \widetilde{K}\rceil$ intersects $\widetilde{V}_i$. Then, $L = \max\{L_i\}_{i=1}^n + \widetilde{H}\rceil - \lceil\widetilde{H}$ meets the requirements of the lemma. □

Proof of Lemma 3.4: Let $\{\widetilde{V}_i\}_{i=1}^n$ be a finite cover of Λ by sets open in $\mathbb{R}^2$, each with diameter less that ϵ. For each $i \in \{1, 2, ..., n\}$, we may assume that $V_i \cap \Lambda \neq \emptyset$ so that there exists a $z_i \in V_i \cap \Lambda\backslash\{p\}$. Let $\widetilde{V}_i = \Pi^{-1}(V_i)$ and z_i be a point fixed in $\Pi^{-1}(z_i)$ for every i. Let $L_i > 0$ be as in Claim 3.6 and $L = \max\{L_i\}_{i=1}^n + 1$. Then, for any subcontinuum $\widetilde{K}$ of $\widetilde{\Lambda}$ with $\widetilde{K}\rceil$ - $\lceil\widetilde{K} > L$, we have $\widetilde{K} \cap \widetilde{V}_i \neq \emptyset$ for each i. Therefore, $\Pi(K) \cap V_i \neq \emptyset$ for every $i \in \{1, 2, ..., n\}$, so that the distance from w to

$\Pi(K)$ is less than ϵ for every $w \in \Lambda$. □

The following lemma will be used to show that if the rotation set is nondegenerate, then the continuum must be indecomposable.

Lemma 3.7 *No proper subcontinuum of Λ can contain two periodic points that have different rotation numbers. Furthermore, there is no proper subcontinuum containing both p and a periodic point with rotation number different from the local rotation number.*

Proof: Let z and w be periodic points of $\Lambda \backslash \{p\}$ with $\rho(z, \widetilde{F}) \neq \rho(w, \widetilde{F})$. Suppose that there is a proper subcontinuum $H \subset \Lambda$ which contains both z and w. We have two cases: Either p is an element of H or it is not.

<u>Case 1</u>: $p \notin H$. We will show that the existence of such an H implies that Λ must contain a plane separating subcontinuum, an impossibility. Since $p \notin H$, $\Pi^{-1}(H)$ is a countable union of connected components, each homeomorphic to H. Choose one such component to be $\widetilde{H}$. Then, since T commutes with Π, we have $\Pi^{-1}(H) = \bigcup_{n \in \mathbb{Z}} T^n(\widetilde{H})$, where $T^m(\widetilde{H}) \cap T^n(\widetilde{H}) = \emptyset$ for $m \neq n$. A key observation to make is that if $\widetilde{K}$ is a continuum that contains a point of $T^m(\widetilde{H})$ and a point of $T^n(\widetilde{H})$ for some $n \neq m$, then $\Pi(\widetilde{K}) \cup H$ must be plane separating.

Let $\widetilde{z}$ and $\widetilde{w}$ be the respective points of $\Pi^{-1}(z)$ and $\Pi^{-1}(w)$ that are contained in $\widetilde{H}$. Since z and w are periodic, there are integers p, q, r,and s so that $\rho(z, \widetilde{F}) = p/q$ and $\rho(w, \widetilde{F}) = r/s$. Therefore, $\widetilde{F}^q(\widetilde{z}) = T^p(\widetilde{z})$ and $\widetilde{F}^s(\widetilde{w}) = T^r(\widetilde{w})$. Consider the continuum $\widetilde{F}^{qs}(\widetilde{H})$ and notice that it meets $T^{ps}(\widetilde{H})$ since $\widetilde{F}^{qs}(\widetilde{z})$ is a member of $\widetilde{F}^{qs}(\widetilde{H}) \cap T^{ps}(\widetilde{H})$. It also meets $T^{qr}(\widetilde{H})$ since $\widetilde{F}^{qs}(\widetilde{w})$ is a point of $\widetilde{F}^{qs}(\widetilde{H}) \cap T^{qr}(\widetilde{H})$. Since z and w have different rotation numbers, we have $ps \neq rq$, so $\Pi(\widetilde{F}^{qs}(\widetilde{H})) \cup H$ is a plane separating subcontinuum of Λ, which cannot exist (see Figure 19).

<u>Case 2</u>: $p \in H$. We show that in this case there must be a closed subset of $\widetilde{\Lambda}$ that separates $\mathbb{R} \times \mathbb{R}^+$. This cannot occur according to Claim 2.6. At least one of z or

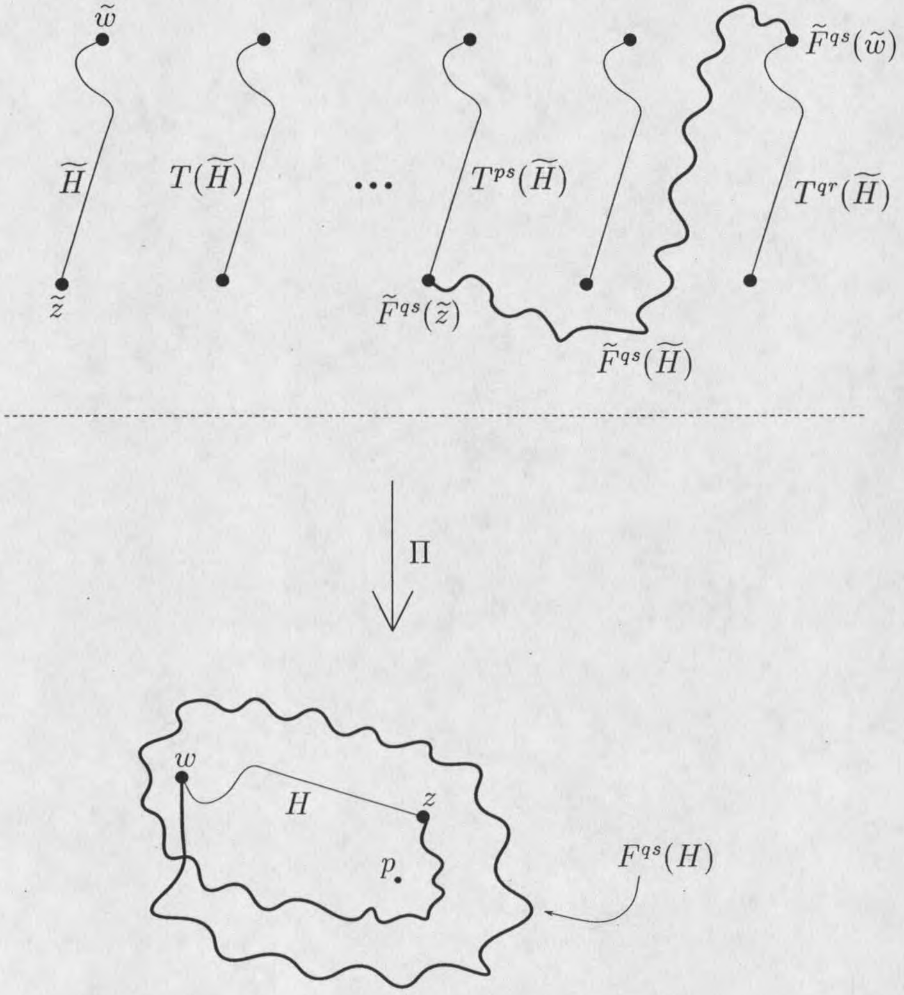

Figure 19: $F^{qs}(H) \cup H$ separates the plane

w does not have rotation number equal to $\mu/\mathbf{n} = \rho_l(p, \widetilde{F})$, the local rotation number. Suppose that z has this property. There must then exist integers p and q so that $\rho(z, \widetilde{F}) = p/q \neq \mu/\mathbf{n}$. Assume that $\mu/\mathbf{n} < p/q$. (The proof for $\mu/\mathbf{n} > p/q$ is similar.) Since z is in H and H is a *proper* subcontinuum of Λ, z must be in $\mathcal{C}_p$, the composant determined by p. Therefore, there exists a proper subcontinuum Γ of Λ that contains both p and z so that every component of $\Pi^{-1}(\Gamma)$ is bounded. (This is Property 4 of the SA.) Let $\widetilde{\Gamma}$ be one such component and $\widetilde{z}$ the point of $\Pi^{-1}(z)$ that is contained in $\widetilde{\Gamma}$. Let r and s be integers so that $\mu/\mathbf{n} < r/s < p/q$. We suppose without loss of generality that the denominators $\mathbf{n}, s$, and q are positive integers so that $ps - rq$ and $r\mathbf{n} - \mu s$ are also positive integers. Define $\widetilde{G} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ by $\widetilde{G} = T^{-r} \circ \widetilde{F}^s$. Note that $\rho_l(p, \widetilde{G}) < 0 < \rho(z, \widetilde{G})$ and that $\widetilde{G}^q(\widetilde{z}) = T^{ps-rq}(\widetilde{z})$, a positive translation of $\widetilde{z}$. Choose $k \in \mathbb{N}$ large enough so that $\widetilde{\Gamma}\rceil + 1 < T^{k(ps-rq)}(\widetilde{\Gamma})$ and let $\widetilde{\Gamma}_* = T^{k(ps-rq)}(\widetilde{\Gamma})$. $\widetilde{\Gamma}_*$ is a positive translation of $\widetilde{\Gamma}$ that contains $G^{kq}(\widetilde{z})$ and has the property that the first coordinate projection of any point it contains is at least one greater than the first coordinate projection of every point in $\widetilde{\Gamma}$.

<u>Claim</u>: $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ is closed, connected and separates the covering space, $\mathbb{R} \times \mathbb{R}^+$ (refer to Figures 20 and 21). To see that $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ is closed and connected, note that $\widetilde{\Gamma}_*$ and $\widetilde{G}^{kq}(\widetilde{\Gamma})$ are each closed and connected. Since $\widetilde{G}^{kq}(\widetilde{z})$ is an element of both $\widetilde{G}^{kq}(\widetilde{\Gamma})$ and of $\widetilde{\Gamma}_*$, the union of $\widetilde{G}^{kq}(\widetilde{\Gamma})$ and $\widetilde{\Gamma}_*$ must be closed and connected. To show that $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ separates $\mathbb{R} \times \mathbb{R}^+$, we use the fact that both $\widetilde{\Gamma}_*$ and $\widetilde{G}^{kq}(\widetilde{\Gamma})$ have 0 as the infima of their second coordinate projection. (They "limit at the bottom" of $\mathbb{R} \times \mathbb{R}^+$.) Since $\rho_l(p, \widetilde{G}) < 0$, the local rotation number for $\widetilde{G}^{kq}$; that is, $\rho_l(p, \widetilde{G}^{kq})$, is also strictly negative, so there exists an $\eta > 0$ so that $\pi_1(\widetilde{G}^{kq}(\widetilde{w})) < \pi_1(\widetilde{w})$ for every $\widetilde{w}$ with $\pi_2(\widetilde{w}) < \eta$. Let $\widetilde{K} = \{\widetilde{w} \in \widetilde{\Gamma} | \pi_2(\widetilde{w}) \geq \eta\}$. $\widetilde{K}$ is compact so $\delta = \min\{\pi_2(\widetilde{G}^{kq}(\widetilde{w}) | \widetilde{w} \in \widetilde{K}\}$ is greater than 0. Therefore, if x_0 is any number between $\widetilde{\Gamma}\rceil$ and $\lceil\widetilde{\Gamma}_*$, then the point $(x_0, \delta/2)$ is not in $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$. $(x_0, \delta/2)$ is not in $\widetilde{\Gamma}_*$ because

$x_0 < \lceil\widetilde{\Gamma}_*$, and $(x_0, \delta/2)$ is not in $\widetilde{G}^{kq}(\widetilde{\Gamma})$ since $\pi_1(\widetilde{G}^{kq}(\widetilde{w})) < \widetilde{\Gamma}\rceil < x_0$ for every $\widetilde{w} \in \widetilde{\Gamma}$ with $\pi_2(\widetilde{w}) < \eta$, and $\delta/2 < \pi_2(\widetilde{G}^{kq}(\widetilde{w}))$ for every $\widetilde{w} \in \widetilde{\Gamma}$ with $\pi_2(\widetilde{w}) \geq \eta$. Suppose that $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ does not separate. Then its complement in $\mathbb{R} \times \mathbb{R}^+$ is path connected. Let $\kappa > 0$ be such that $\pi_2(\widetilde{w}) < \kappa$ for every $\widetilde{w} \in \widetilde{\Lambda}$. Let α be a half open arc which limits at the bottom of $\mathbb{R} \times \mathbb{R}^+$, goes through $(x_0, \delta/2)$, has its endpoint at a point (M, κ), and does not meet $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$. This half open arc separates the set of all points in $\mathbb{R} \times \mathbb{R}^+$ with second coordinate projection less than κ into two open sets, $\widetilde{U}_-$ (on the left) and $\widetilde{U}_+$ (on the right). Since $\widetilde{G}^{kq}(\widetilde{\Gamma})$ and $\widetilde{\Gamma}_*$ limit at the bottom of $\mathbb{R} \times \mathbb{R}^+$, we can see that $\widetilde{\Gamma} \cap \widetilde{U}_-$ and $\widetilde{\Gamma} \cap \widetilde{U}_+$ are nonempty, contradicting $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ being a connected set and establishing the claim.

Therefore if $p \in H$, then $\widetilde{\Lambda}$ contains a closed subset that separates $\mathbb{R} \times \mathbb{R}^+$ which contradicts the third result in Claim 2.6 and completes the proof. ∎

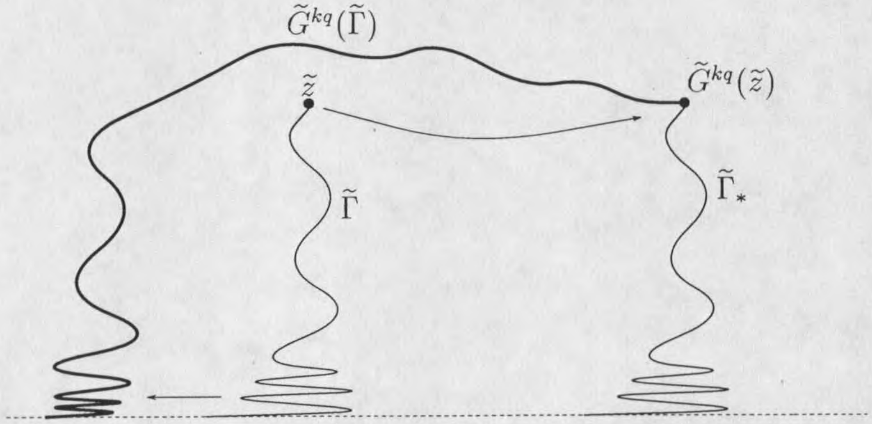

Figure 20: $\widetilde{G}^{kq}(\widetilde{\Gamma}) \cup \widetilde{\Gamma}_*$ separates $\mathbb{R} \times \mathbb{R}^2$.

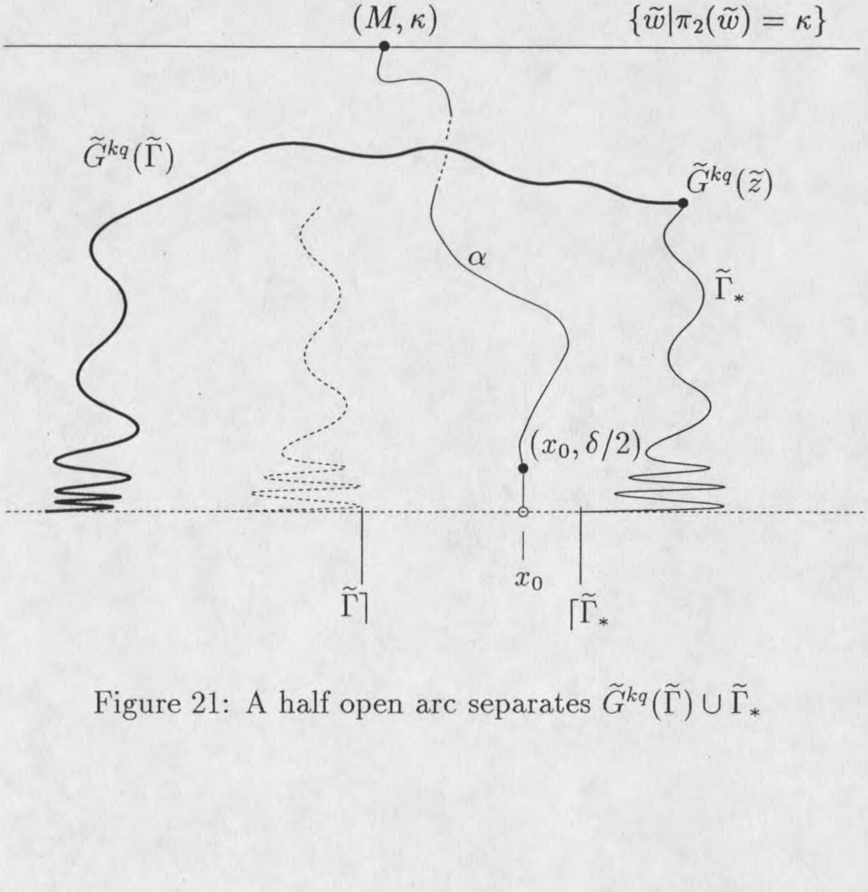

Figure 21: A half open arc separates $\tilde{G}^{kq}(\tilde{\Gamma}) \cup \tilde{\Gamma}_*$

CHAPTER 4

Theorems, Examples and Proofs

We first state the main theorem of this thesis.

Theorem 4.1 *Suppose that F is an orientation preserving homeomorphism of the plane which has a fixed point p in an invariant set Λ satisfying the Standing Hypotheses (SA) of Chapter 2. For a lift $\widetilde{F}$ of $F|_{\mathbb{R}^2\setminus\{p\}}$, suppose that $p/q \neq \rho_l(p, \widetilde{F})$ is a reduced rational number with $p/q \in int\langle\rho(\Lambda, \widetilde{F})\rangle$, the interior of the convex hull of $\rho(\Lambda, \widetilde{F})$. Then there exists a periodic point $z \in \Lambda$ with least period q and rotation number $\rho(z, \widetilde{F}) = p/q$.*

Note that the existence of z_1 and z_2 in Λ such that $\min \rho(z_1, \widetilde{F}) < p/q < \max \rho(z_2, \widetilde{F})$ is equivalent to $p/q \in \text{int}\langle\rho(\Lambda, \widetilde{F})\rangle$.

Before proving the theorem, we revisit two of the the previous examples.

Example 4.2 Figure 22 illustrates two plane continua which are invariant under an orientation preserving homeomorphism, contain a fixed point p, but have a rotation set about p which consists of only two points. The left example of the figure fails to meet the SA because it is plane separating and the right example fails Property 4. The planar homeomorphism which leaves the left example invariant is one in which the outer circle is rotated through by π-radians and the endpoint of the spiraling half-open arc is the fixed point p. All other points on this half-open arc move along the arc toward the fixed point. Under the appropriate lift of this homeomorphism, the rotation set about p is $\{1/2, 0\}$ since every point on the circle has rotation number $1/2$ and the rotation number of points on the spiral is 0 (they tend toward p along this arc). In the right diagram of Figure 22, the homeomorphism is to have two fixed points; an interior point of the central arc p, and the endpoint of the half-open arc which spirals onto the central arc. This homeomorphism may be constructed so that every other point of the central arc is a period 2 while every point on the spiral tends towards the fixed point on the spiral. Again, under appropriate lift, we have the rotation set equal to $\{1/2, 0\}$.

It is clear from these examples that Theorem 4.1 is false if either the first or the fourth property of the SA is eliminated. I do not know of an example where the theorem doesn't hold if only the nowhere dense (number 2) restriction is eliminated. The "spokes of the wheel" assumption (number 3) is needed for the definition of the local rotation number in the statement of the theorem.

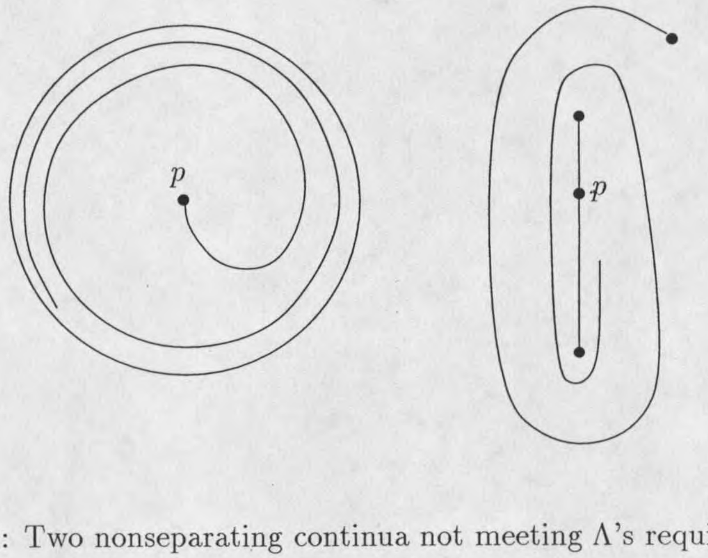

Figure 22: Two nonseparating continua not meeting Λ's requirements

Proof of Theorem 4.1: First, define $\widetilde{\Phi} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ to be $\widetilde{\Phi} = T^{-p} \circ \widetilde{F}^q$. Recall that $T(x, y) = (x + 1, y)$ and note that $\widetilde{\Phi}$ will have a fixed point in $\widetilde{\Lambda}$ if and only if F has a periodic point with least period q and rotation number equal to p/q.

Let $\widetilde{G} = \widetilde{\Phi}^3$. Since $p/q \in \text{int}\langle\rho(\Lambda, \widetilde{F})\rangle$, the interior of the convex hull of the rotation set of Λ with respect to $\widetilde{\Phi}$ must contain 0. Therefore, 0 is in $\text{int}\langle\rho(\Lambda, \widetilde{G})\rangle$.

Claim 1: There exist positive r, ω and η such that $\eta < \text{d}(\widetilde{z}, \widetilde{G}(\widetilde{z})) < \omega$ for any $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$.

Proof of Claim: Let $\{\widetilde{C}_j\}_{j\in\mathbb{Z}}, \mathrm{n}, s, t,$ and $\widetilde{W}_s$ be as in the paragraph preceding Definition 2.8. Since $p/q \neq \rho_l(p, \widetilde{F})$, there is a unique $\mu \in \mathbb{Z}\backslash\{0\}$ so that for every $\widetilde{C}_j$, $\widetilde{\Phi}(\widetilde{C}_j \cap \widetilde{W}_s) \subset \widetilde{C}_{j+\mu}$. Let $0 < r < t$ be such that for $\widetilde{W}_r = \{(x,y)|y \leq r\}$, we have $\widetilde{G}(\widetilde{W}_r)$ contained in $\widetilde{W}_t$. Since $\widetilde{G} = \widetilde{\Phi}^3$, $\widetilde{G}(\widetilde{C}_j \cap \widetilde{W}_r)$ is contained in $\widetilde{C}_{j+3\mu}$ for every j, but $\widetilde{G}(\widetilde{C}_j \cap \widetilde{W}_r) \cap \widetilde{C}_i = \emptyset$ if $i \neq j + 3\mu$. Let $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$ be given. Then there exists exactly one integer k such that $\widetilde{z} \in \{(x,y)|\frac{k}{\mathrm{n}} \leq x < \frac{k+1}{\mathrm{n}}\}$. By Claim 3.5, there are at least three components of the complement of $\widetilde{C}_{k-1} \cup \widetilde{C}_{k+1}$ in the closure of $\widetilde{W}_r$. Two of these components are unbounded, while one that is bounded, $\widetilde{V}$, must contain $\{(x,y)|\frac{k}{\mathrm{n}} \leq x < \frac{k+1}{\mathrm{n}}$ and $y \leq r\}$ and therefore $\widetilde{z}$. Since

$$\widetilde{G}(\widetilde{C}_{k-1} \cap \widetilde{W}_r) \subset \widetilde{C}_{k-1+3\mu} \subset \{(x,y)|\tfrac{k-1+3\mu}{\mathrm{n}} < x < \tfrac{k+3\mu}{\mathrm{n}}\},$$
$$\widetilde{G}(\widetilde{C}_{k+1} \cap \widetilde{W}_r) \subset \widetilde{C}_{k+1+3\mu} \subset \{(x,y)|\tfrac{k+1+3\mu}{\mathrm{n}} < x < \tfrac{k+2+3\mu}{\mathrm{n}}\},$$

and $\widetilde{G}(\widetilde{V})$ is in a bounded component of the complement of $\widetilde{G}(\widetilde{C}_{k-1} \cup \widetilde{C}_{k+1})$ in $\widetilde{G}(\widetilde{W}_r)$, it must be that $\frac{k-1+3\mu}{\mathrm{n}} < \pi_1(\widetilde{G}(\widetilde{z})) < \frac{k+2+3\mu}{\mathrm{n}}$. Since $|\mu| \geq 1$, the distance between $\pi_1(\widetilde{z})$ and $\pi_1(\widetilde{G}(\widetilde{z}))$ greater than $\frac{1}{\mathrm{n}}$, but less than $\frac{5}{\mathrm{n}}$. Therefore, for $\eta = \frac{1}{\mathrm{n}}$ and $\omega = \frac{5}{\mathrm{n}}$, we have $\eta < d(\widetilde{z}, \widetilde{G}(\widetilde{z})) < \omega$ for every $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$.

This establishes the claim which is the third property required of $\widetilde{\Phi}$ in Lemma 3.2. We must yet show the existence of a chain recurrent point of $\widetilde{G}|_{\widetilde{\Lambda}}$ (Property 4) to apply the lemma.

Suppose that for some $\widetilde{w} \in \widetilde{\Lambda}$, $\{\widetilde{G}^n(\widetilde{w})\}_{n=0}^{\infty}$ has a convergent subsequence. Then the limit of this convergent subsequence is in $\widetilde{\Lambda}$ and is chain recurrent. By Lemma 3.2, $\widetilde{G}$ has a fixed point in $\widetilde{\Lambda}$ so we are done. Assume then, that for every $\widetilde{w}$ in $\widetilde{\Lambda}$, $\{\widetilde{G}^n(\widetilde{w})\}_{n=0}^{\infty}$ has no convergent subsequence.

Let

$$R^+ = \{\tilde{z} \in \tilde{\Lambda} | \lim_{n\to\infty} \pi_1 \circ \tilde{G}^n(\tilde{z}) = +\infty\} \text{ and } R^- = \{\tilde{z} \in \tilde{\Lambda} | \lim_{n\to\infty} \pi_1 \circ \tilde{G}^n(\tilde{z}) = -\infty\}.$$

Claim 2: $\tilde{\Lambda}$ is contained in $R^+ \cup R^-$.

Proof of Claim: Let κ be such that $\tilde{\Lambda} \subset \tilde{D} = \{(x,y)|y \leq \kappa\}$. Since $d(\tilde{z}, \tilde{G}(\tilde{z})) < \omega$ for $\tilde{z}$ with $\pi_2(\tilde{z}) < r$, the distance between $\tilde{z}$ and $\tilde{G}(\tilde{z})$ is bounded from above by some $b > 0$ for $\tilde{z} \in \tilde{D}$. For every integer k, let $B_k = \{(x,y)|\ kb \leq x \leq (k+1)b \text{ and } y \leq \kappa\}$ and suppose that $\mu \geq 1$. (The argument for $\mu \leq 1$ is similar.) Consider the sequence $\{\tilde{G}^n(\tilde{z})\}_{n=0}^{\infty}$ for some $\tilde{z} \in \tilde{\Lambda}$. Since $\pi_1(\tilde{G}(\tilde{z})) > \pi_1(\tilde{z}) + \eta$ for any $\tilde{z}$ with $\pi_2(\tilde{z}) \leq r$, there must be some integer M_0 so that $\tilde{G}^n(\tilde{z}) \notin B_0$ for each $n > M_0$, for otherwise $\{\tilde{G}^n(\tilde{z})\}_{n=0}^{\infty}$ would have a convergent subsequence in the compact set $B_0 \cap \{(x,y)|y \geq r\}$. Then either $\pi_1 \circ \tilde{G}^n(\tilde{z}) > b$ for all $n > M_0$ or $\pi_1 \circ \tilde{G}^n(\tilde{z}) < 0$ for $n > M_0$. If $\pi_1 \circ \tilde{G}^n(\tilde{z}) > b$, there is an M_1 so that $n > M_1 > M_0$ implies that $\tilde{G}^n(\tilde{z}) \notin B_1$. In this case, we see that for each $k \in \mathbb{N}$, there is an $M_k > M_{k-1}$ such that if $n > M_k$, then $\tilde{G}^n(\tilde{z}) \notin B_k$, so that $\tilde{z} \in R^+$. Similarly, if $\pi_1 \circ \tilde{G}^n(\tilde{z}) < 0$ for $n > M_0$, then $\tilde{z} \in R^-$.

Claim 3: Both of R^- and R^+ are nonempty.

Proof of Claim: Suppose, for example, that $R^- = \emptyset$ so that by Claim 2, $\tilde{\Lambda}$ is contained in R^+. That is to say, $\lim_{n\to\infty} \pi_1 \circ \tilde{G}^n(\tilde{z}) = +\infty$ for every $\tilde{z}$ in $\tilde{\Lambda}$, so that for every $\tilde{z} \in \tilde{\Lambda}$ there exists an $N = N(\tilde{z})$ such that $\pi_1 \circ \tilde{G}^n(\tilde{z}) > 0$ for every $n > N$. It follows that no subsequence of $\{\frac{1}{n}\pi_1 \circ \tilde{G}^n(\tilde{z})\}_{n\in\mathbb{N}}$ can converge to a negative number. Therefore, $\inf\rho(\tilde{z}, \tilde{G}) \geq 0$ so it must be the case that the infimum of $\rho(\Lambda, \tilde{G})$ is greater than or equal to 0 , in contradiction to 0 being in the interior of the convex hull of $\rho(\Lambda, \tilde{G})$.

We have three cases to consider.

Case 1: Suppose that there exist composants $\mathcal{C}^-$ and $\mathcal{C}^+$ of Λ, such that $\Pi^{-1}(\mathcal{C}^-) \subset R^-$ and $\Pi^{-1}(\mathcal{C}^+) \subset R^+$. Every composant is dense in Λ, so R^- and R^+ must both be dense in $\tilde{\Lambda}$. Roughly, the idea here is to construct bounded ϵ-pseudo orbits by choosing points alternating between R^- and R^+. These pseudo orbits will be chosen in such a way so that each has a limit point in a common compact set. The bounded collection of these limit points will have a limit point which is chain recurrent.

Let $\widetilde{w}_0 \in R^-$ and choose $n_1 > 0$ so that $\pi_1 \circ \widetilde{G}^{n_1}(\widetilde{w}_0) < \epsilon$. Let $\widetilde{w}_i = \widetilde{G}^i(\widetilde{w}_0)$ for $0 < i < n_1$ and let $\widetilde{w}_{n_1} \in R^+$ with $d(\widetilde{w}_{n_1}, \widetilde{G}^{n_1}(\widetilde{w}_0)) < \epsilon$. As in Claim 2, let $b = \max\{\mathrm{d}(\widetilde{w}, \widetilde{G}(\widetilde{w})) | \widetilde{w} \in \widetilde{D}\}$ and choose $n_2 > n_1$ such that $\pi_1 \circ \widetilde{G}^{n_2-n_1}(\widetilde{w}_{n_1}) > b + \epsilon$. Let $\widetilde{w}_{n_1+1} = \widetilde{G}(\widetilde{w}_{n_1}), \widetilde{w}_{n_1+2} = \widetilde{G}^2(\widetilde{w}_{n_1}), \ldots, \widetilde{w}_{n_2-1} = \widetilde{G}^{n_2-n_1-1}(\widetilde{w}_{n_1})$ and let $\widetilde{w}_{n_2} \in R^-$ with $d(\widetilde{w}_{n_2}, \widetilde{G}^{n_2-n_1}(\widetilde{w}_1)) < \epsilon$. Continuing in this manner, we create an ϵ-pseudo orbit $\{\widetilde{w}_n\}_{n=0}^{\infty}$ which must contain an infinite subsequence in the compact $\{\widetilde{w} \in \widetilde{\Lambda} | 0 \leq \pi_1(\widetilde{w}) \leq b \text{ and } \pi_2(\widetilde{w}) \geq r\}$. That there are an infinite number of the $\widetilde{w}_n$ for which $\pi_2(\widetilde{w}_n) \geq r$ follows from the fact that either $\pi_1 \circ \widetilde{G}(\widetilde{w}) > \pi_1(\widetilde{w})$ for any $\widetilde{w}$ with $\pi_2(\widetilde{w}) < r$ or $\pi_1 \circ \widetilde{G}(\widetilde{w}) < \pi_1(\widetilde{w})$ every $\widetilde{w}$ with $\pi_2(\widetilde{w}) < r$. Therefore, there is a convergent subsequence of $\{\widetilde{w}_n\}_{n=0}^{\infty}$, converging to some $\widetilde{w}(\epsilon) \in \widetilde{\Lambda}$ with $\pi_2(\widetilde{w}(\epsilon)) \geq r$ and $0 \leq \pi_1(\widetilde{w}(\epsilon) \leq b$. Also, since $\widetilde{\Lambda}$ is closed, $\widetilde{w}(\epsilon)$ must be in $\widetilde{\Lambda}$. Now $\widetilde{w}(\epsilon)$ has the property that there is a 2ϵ-chain from $\widetilde{w}(\epsilon)$ to itself, so that if $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence converging to zero, a limit point $\widetilde{z}$ of $\{\widetilde{w}(\epsilon_n)\}_{n=0}^{\infty}$ in $\{\widetilde{w} \in \widetilde{\Lambda} | 0 \leq \pi_1(\widetilde{w}) \leq b$ and $\pi_2(\widetilde{w}) \geq \eta\}$ has the property that there is an ϵ-chain from $\widetilde{z}$ to itself for any $\epsilon > 0$. That is, $\widetilde{z}$ is a chain recurrent point of $\widetilde{G}|_{\widetilde{\Lambda}}$.

Case 2: Suppose that there is a continuum H, contained in $\Lambda \backslash \{p\}$, such that $\Pi^{-1}(H) \cap R^-$ and $\Pi^{-1}(H) \cap R^+$ are nonempty. Since $p \notin H$, there exists a continuum $\widetilde{H} \subset \Pi^{-1}(H)$ which contains points $\widetilde{z}^- \in R^-$ and $\widetilde{z}^+ \in R^+$. It follows from Lemma 3.4 that for any $\epsilon > 0$ there must be a positive integer $N = N(\epsilon)$ so that $d(\widetilde{z}, \widetilde{G}^N(\widetilde{H})) < \epsilon$ for every $\widetilde{z} \in \widetilde{H}$ (see Figure 23).

We construct a nested sequence of compact sets $\{\widetilde{K}_n\}_{n=0}^{\infty}$ by letting $\widetilde{K}_0 = \widetilde{H}$ and $\widetilde{K}_{n+1} = \{\widetilde{w} \in \widetilde{H} | d(\widetilde{G}^N(\widetilde{w}), \widetilde{K}_n) \leq \epsilon\}$ for every $n \geq 0$. Let $\widetilde{w}_0 \in \widetilde{K} = \cap_{n \geq 0} \widetilde{K}_n$.

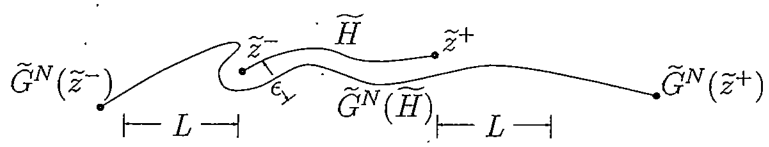

Figure 23: $\mathrm{d}(\widetilde{z}, \widetilde{G}^N(\widetilde{H})) < \epsilon$ for every $\widetilde{z}$ in $\widetilde{H}$.

Since $\widetilde{w}_0$ is in every $\widetilde{K}_n$, we have $d(\widetilde{G}^N(\widetilde{w}_0), \widetilde{K}_n) \leq \epsilon$ for each n. Therefore, there exists a $\widetilde{w}_N \in \widetilde{K}$ such that $d(\widetilde{G}^N(\widetilde{w}_0), \widetilde{w}_N) \leq \epsilon$. Let $\widetilde{w}_i = \widetilde{G}^i(\widetilde{w}_0)$ for $0 < i < N$ and for each $j \geq 2$, inductively choose $\widetilde{w}_{Nj} \in \widetilde{K}$ with $d(\widetilde{G}^N(\widetilde{w}_{(j-1)N}, \widetilde{w}_{jN})$ and let $\widetilde{w}_i = \widetilde{G}^{i-(j-1)N}(\widetilde{w}_{(j-1)N})$ for $(j-1)N < i < jN$. $\{\widetilde{w}_i\}_{i=0}^{\infty}$ is then an ϵ-pseudo orbit with $\widetilde{w}_{jN} \in \widetilde{H}$ for every positive integer j. Since $\widetilde{H}$ is compact, there is a subsequence of the subsequence $\{\widetilde{w}_{jN}\}_{j=0}^{\infty}$ converging to some point $\widetilde{w}(\epsilon)$ in $\widetilde{H}$. This limit is 2ϵ chain recurrent. As in Case 1, the exists a limit point of $\{\widetilde{w}(\epsilon_n) | \ \epsilon_n > 0 \text{ and } \epsilon_n \to 0\}$ which is chain recurrent. Lemma 3.2 guarantees a fixed point in $\widetilde{\Lambda}$.

Case 3: If neither of the first two cases occurs, it must be that if $\mathcal{C}$ is the composant of Λ which is the union of all proper subcontinua of Λ that contain p, then $\Pi^{-1}(\mathcal{C} \backslash \{ p\})$ contains both points of R^- and points of R^+. Furthermore, every continuum contained in $\Pi^{-1}(\mathcal{C} \backslash \{ p\})$ must be a subset of one of R^- or R^+. As in the proof of Claim 1, there are positive r and η such that either $\pi_1 \circ \widetilde{G}(\widetilde{z}) > \pi_1(\widetilde{z}) + \eta$ for every $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$, or $\pi_1 \circ \widetilde{G}(\widetilde{z}) < \pi_1(\widetilde{z}) - \eta$ for every $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$. Assume for $\widetilde{z}$ with $\pi_2(\widetilde{z}) < r$, that $\pi_1 \circ \widetilde{G}(\widetilde{z}) > \pi_1(\widetilde{z}) + \eta$. The proof for the other situation is similar. Let $\widetilde{z}^-$ be a point in $R^- \cap \Pi^{-1}(\mathcal{C} \backslash \{ p\})$ and let Γ be a subcontinuum of Λ which contains p and $\Pi(\widetilde{z}^-)$ such that (as in the SA - Property 4) if $\widetilde{\Gamma}$ is the component of $\Pi^{-1}(\Gamma \backslash \{ p\})$ which contains $\widetilde{z}^-$, then the infimum and supremum of $\{\pi_1(\widetilde{z}) | \widetilde{z} \in \widetilde{\Gamma}\}$ are finite. Since $\{\pi_2(\widetilde{z}) | \widetilde{z} \in \widetilde{\Gamma}\}$ is also bounded, $\widetilde{\Gamma}$ is bounded. Therefore, for each $\epsilon > 0$ there is an $L = L(\epsilon, \widetilde{\Gamma})$ as in Lemma 3.3. Since $\lim_{n\to\infty} \pi_1 \circ \widetilde{G}^n(\widetilde{z}) = -\infty$, there exists an M such that $\pi_1 \circ \widetilde{G}^N(\widetilde{z}^-) < \inf\{\pi_1(\widetilde{z}) | \widetilde{z} \in \widetilde{\Gamma}\} - L$ for every $N \geq M$.

Using an argument very similar to that in the proof of Claim 1, one can show that for any $N \in \mathbb{N}$ there exists an $r' > 0$ such that for any $\tilde{z}$ with $\pi_2(\tilde{z})$ less that r', we have $\pi_1 \circ \tilde{G}^N(\tilde{z}) > \pi_1(\tilde{z}) + N\eta$. It is therefore possible to choose $N \geq M$ large enough so that $\pi_1 \circ \tilde{G}^N(\tilde{z}) > \sup\{\pi_1(\tilde{z}) | \tilde{z} \in \tilde{\Gamma}\} + L$ for some $\tilde{z} \in \tilde{\Gamma}$. Let $\widetilde{H} \subset \tilde{\Gamma}$ be a continuum containing $\tilde{z}^-$ and a point $\tilde{z}^+$ with $\pi_2(\tilde{z}^+) < r'$. Then $\sup\{\pi_1 \circ \tilde{G}^N(\tilde{z}) | \tilde{z} \in \widetilde{H}\} > \sup\{\pi_1(\tilde{z}) | \tilde{z} \in \tilde{\Gamma}\} + L \geq \sup\{\pi_1(\tilde{z}) | \tilde{z} \in \widetilde{H}\} + L$ (see Figure 24). By Lemma 3.3, $d(\tilde{z}, \tilde{G}^N(\widetilde{H}))$ is less that ϵ for every $\tilde{z} \in \widetilde{H}$ so that we may proceed as in Case 2.

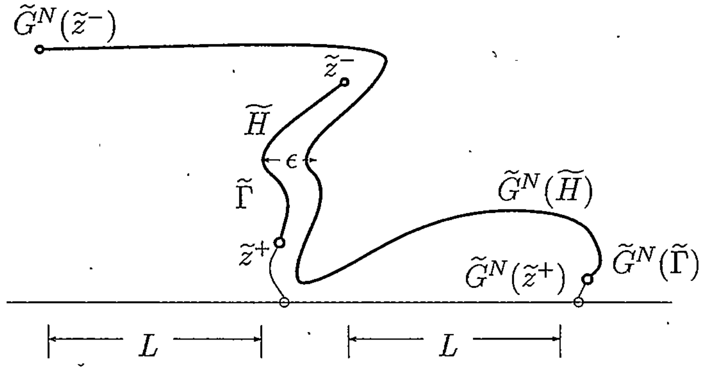

Figure 24: $d(\tilde{z}, \tilde{G}^N(\widetilde{H})) < \epsilon$ for every $\tilde{z} \in \widetilde{H}$

□

Theorem 4.3 *The local rotation number is in the closure of the rotation set of* Λ. *ie.* $\rho_l(p; \tilde{F}) \in cl(\rho(\tilde{\Lambda}; \tilde{F}))$.

Proof: Suppose that $\rho_l(p; \tilde{F}) \notin cl(\rho(\tilde{\Lambda}; \tilde{F}))$. Theorem 4.1 gives us that either $\rho_l(p; \tilde{F}) > t$ for every $t \in \rho(\tilde{\Lambda}; \tilde{F})$ or $t > \rho_l(p; \tilde{F})$ for every $t \in \rho(\tilde{\Lambda}; \tilde{F})$. Suppose that $\rho_l(p; \tilde{F}) > t$ for every $t \in \rho(\tilde{\Lambda}; \tilde{F})$, the other case being similar. Let $\mu/\mathbf{n}$ be as in the definition of $\rho_l(p; \tilde{F})$ and let r and s be integers such that $r/s < \mu/\mathbf{n}$, and r/s is strictly greater than t for each $t \in \rho(p; \tilde{F})$. If we define $\tilde{G} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ to be $\tilde{G} = T^{-r} \circ \tilde{F}^s$, then $\rho_l(p; \tilde{G})$ is greater than 0 and $t < 0$ for every $t \in \rho(\tilde{\Lambda}; \tilde{G})$.

That is, the local rotation number for $\widetilde{G}$ is strictly positive, while the rotation set of Λ under $\widetilde{G}$ contains only points that are strictly negative. Let $\widetilde{\Gamma}$ be any closed and bounded subset of $\widetilde{\Lambda}$ as in Property 4 of the SA. Our situation is now identical to that of Case 3 of Theorem 4.1 and Figure 24. The "bottom" of $\widetilde{\Gamma}$ "moves to the right" under iterations of $\widetilde{G}$, but every $\widetilde{z} \in \widetilde{\Gamma}$ has a negative rotation set so moves to the "left". As before, we can constuct a chain recurrent point in $\widetilde{\Gamma}$ which implies the existence of a fixed point for $\widetilde{G}$ in $\widetilde{\Lambda}$. This fixed point must have rotation number r/s under $\widetilde{F}$, which contradicts $r/s > t$ for all $t \in \rho(\widetilde{\Lambda}; \widetilde{F})$. □

Theorem 4.4 *Suppose that $\rho(\Lambda; \widetilde{F}) \neq \rho_l(p; \widetilde{F})$. Then Λ is indecomposable.*

Proof: Suppose that $\rho(\Lambda; \widetilde{F}) \neq \rho_l(p; \widetilde{F})$ but Λ is decomposable. Then there exist proper subcontinua, A and B of Λ, such that $A \cup B = \Lambda$. Since $\rho_l(p; \widetilde{F})$ is a point in the closure $\rho(\widetilde{\Lambda}; \widetilde{F})$ (from Theorem 4.3), $\rho(\Lambda, \widetilde{F})$ is nondegenerate. Theorem 4.1 gives us that Λ must contain an infinite number of periodic points with different rotation numbers. From Lemma 3.7 we have that neither A nor B can contain any two of these periodic points. □

Corollary 4.5 *If the rotation set $\rho(\Lambda; \widetilde{F})$ is nondegenerate (that is, contains more than a single point), then Λ is indecomposable.*

CHAPTER 5

Conclusion

In the main theorem (Theorem 4.1) of this dissertation we have shown that if p/q is in the interior of $\langle\rho(\Lambda; \widetilde{F})\rangle$ with p and q relatively prime, then there exists a periodic orbit in Λ with period q and rotation number p/q, provided that p/q is not the local rotation number about the fixed point. That is, there is an interval I, such that every rational number in I, except possibly for $\rho_l(p; \widetilde{F})$, is realized as the rotation number of a periodic orbit in Λ. We have also shown that $\rho_l(p; \widetilde{F})$ is in the closure of $\rho(\Lambda; \widetilde{F})$ and that Λ must be indecomposable if $\rho(\Lambda; \widetilde{F})$ is nondegenerate (Theorems 4.3 and 4.4, respectively).

There are a number of questions that are of immediate interest. Does the main theorem remain true if $p/q \in int\ \rho(\Lambda; \widetilde{F})$ *is* equal to the local rotation number? That is, must there be a periodic point in Λ with rotation number $\rho_l(p; \widetilde{F})$? What about the irrationals in the interval I? Must each irrational in I be the rotation number of some point in Λ? Other unanswered questions include

- Must $\rho(\Lambda; \widetilde{F})$ be closed?
- If $p/q \in \mathbb{Q}$ is an endpoint of $\rho(\Lambda; \widetilde{F})$, must p/q be the rotation number of a periodic point in Λ?
- Is the prime end rotation number $\rho_e(\Lambda; \widetilde{F})$ in $\rho(\Lambda; \widetilde{F})$?
- Does the main theorem hold if Property 2 of the Standing Assumptions is relaxed so that Λ is allowed to be somewhere dense?

- Are the hypotheses of the main theorem enough to guarantee more than one periodic orbit of a given rotation number?

I would conjecture that the answer to each of these questions is "yes".

Finally, two examples of invariant, nonseparating plane continua are given that contain no periodic points other than the fixed point. These examples are not directly related to the results of this dissertation, but may be useful in constructing counterexamples to other conjectures. The right diagram of Figure 25 illustrates two topological closed disks having exactly one point in common. Consider a homeomorphism of the plane for which the common point is fixed and which each of the two disks is an invariant portion of an elliptic sector of the the fixed point. Let F be the homeomorphism obtained by the composition of this initial homeomorphism and a rotation about the fixed point that interchanges the two disks. Under the appropriate lift, we have $\rho_l(p; \widetilde{F}) = 1/2 \; = \; \rho(\Lambda; \widetilde{F})$ though there are no period 2 points in the invariant continuum. In the left diagram of Figure 25, we have a similar situation occurring in a nowhere dense continuum. The homeomorphism for this example is the composition of a Π - radian rotation and a homeomorphism that advances each of the infinite number of spokes centered at the fixed point to the next counterclockwise spoke. Since there are not a finite number of such spokes, we cannot define a local rotation number. However, under any reasonable definition, this number must be 1/2 (for the proper lift choice). Again, we see that $\rho(\Lambda; \widetilde{F}) \; = \; 1/2$, yet there are no period 2 points in the continuum.

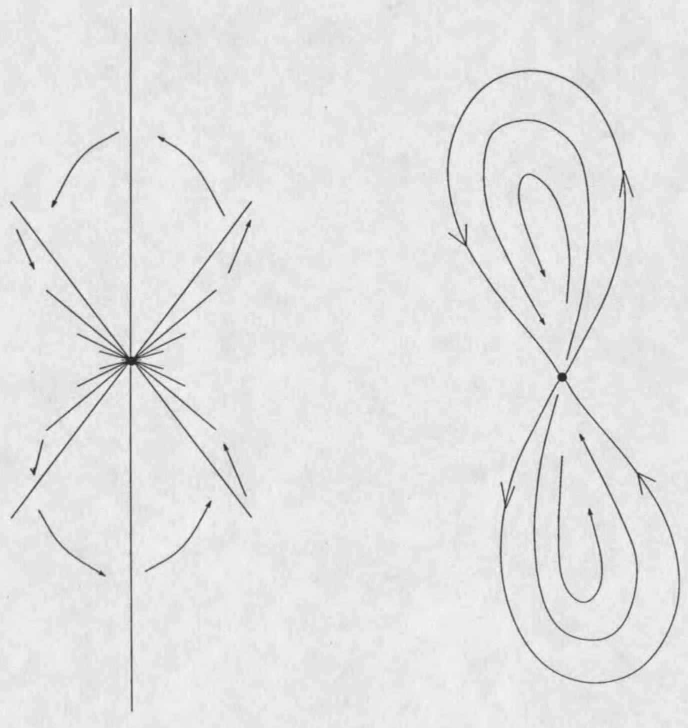

Figure 25: Two nonseparating continua having no non-fixed periodic points

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