The effect on final achievement in a beginning calculus course resulting from the use of programmed materials written to supplement regular classroom instruction

by William Albert Stannard

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF EDUCATION
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Abstract:
This investigation resulted from the belief that programmed materials written to supplement regular classroom instruction of beginning calculus would significantly improve the final achievement of students studying these materials in addition to their regular classroom instruction, The author consulted pertinent literature as a means of investigating the extent to which the general area of the problem had been explored. The literature revealed no evidence which supported or denied that programmed materials of the type specified would significantly effect the learning of beginning calculus. Hence an experiment was designed to measure the effect on final achievement in a beginning calculus course resulting from the study of programmed materials written to supplement regular class room instruction. The first step of the experiment was the preparation of a set of preliminary programmed materials written to cover the critical areas of beginning calculus which cause students the greatest learning problems. These preliminary materials were used in a pilot study to detect flaws in construction as well as to provide the author with experience in conducting the experiment. Following the pilot study the materials were revised to correct discovered inadequacies.

An experimental group was then selected to use the revised materials as a supplement to regular classroom instruction. The students of the experimental group were volunteers from the beginning calculus course and were chosen on the basis of their mathematical achievements in previous college mathematics courses. For one college quarter the programmed materials were administered to the experimental group in addition to their regular classroom instruction. At the end of this quarter the final examination scores earned by members of the experimental group were recorded as a measure of individual student achievement.

The control group was next selected. Standards for admission into this group were similar to those for the experimental group with the exception that the control group did not have access to the programmed materials. The measure of the final achievements of the members of the control group was by means of the same final achievement examination taken by members of the experimental group. The final achievement scores of the two groups were compared by appropriate statistical tests.

As a result of reviewing literature the author concluded: (1) Modern scientific-technological pressures have brought about many changes in the emphasis and content of current mathematics. Since these changes have influenced the teaching of calculus, a need has developed for more effective calculus teaching techniques. (2) The results of research done on the effectiveness of supplementing class-room teaching of mathematics by using programmed materials has been inconclusive. From his investigation the author concluded that the effect on the final achievement of students in a beginning calculus course resulting from the study of programmed materials prepared by the author was not significantly above the corresponding achievement of students not studying such materials.

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THE EFFECT ON FINAL ACHIEVEMENT IN A BEGINNING CALCULUS COURSE RESULTING FROM THE USE OF PROGRAMMED MATERIALS WRITTEN TO SUPPLEMENT REGULAR CLASSROOM INSTRUCTION

by

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of

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CHAPTER I

INTRODUCTION

There has been good reason to refer to the present age as a scientific-technological era. Never before in the history of man has the destiny of the future been so closely controlled by scientific technology. In the past twenty years single nations have come to possess, through scientific means, the machines of war capable of destroying entire civilizations. In fact the grasp of modern science has reached into the lives of people everywhere in the form of television, automobiles, satellites, electric dishwashers, medical drugs, and dozens of other areas of everyday life. Thus for the citizen of tomorrow to understand, appreciate, and properly use these tools he must be well qualified to judge them and must possess the technical ability to devise other efficient machines which will help him design a more compatible tomorrow.

However, the ability to properly judge, use, and design the complex scientific devices of this modern era has been found to be strongly dependent upon the degree of comprehension of the supporting science, mathematics. With the current shift to an increasingly automated world where tolerance for error has been found to be small, the society of today has turned to the truths of mathematics to guide the way to scientific understanding and utilization.

Naturally this heavy concentration on mathematics has had far-reaching effects. In fact, so much pressure has been brought to bear on the subject that the resulting vast changes in mathematics have been aptly described as a true revolution. This point was clearly stated in

1961 by mathematician G. Baley Price, Chairman of the Department of Mathematics at the University of Kansas, when he wrote:

> The changes in mathematics in progress at the present time are so extensive, so far-reaching in their implications, and so profound that they can be described only as a revolution. [1]

With such intense focus on mathematics by mathematicians and non-mathematicians throughout this country, the impact on education has been inescapable. In the past few years mathematicians and educators have found it necessary to work in close harmony in attempting to devise effective teaching programs capable of meeting modern needs. With the acceleration in scientific advancements, mathematics educators have been pressed to present the subject to the learner in the most effective way possible so that the subject matter presented will be in harmony with future expansion of mathematical knowledge. Numerous instructional improvement projects have been put into operation through federally sponsored institutes for teachers, in-service education courses, lecture series, educational television programs, and whatever other means have seemed appropriate.

Actually this revolution has repeatedly complicated itself because not only has each major advancement in mathematics implied

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[1] Price, G. Baley, "Progress in Mathematics and Its Implications for the Schools," in The Revolution in School Mathematics, A Report of Regional Orientation Conferences in Mathematics by the National Council of Teachers of Mathematics, p. 1.

a need for a curriculum revision at graduate level study, but each curriculum revision at the graduate level has in turn implied a need for curriculum revision at all prerequisite levels. This snowballing effect has severely challenged educational efforts to keep teaching up-to-date and in line with future needs.

Since the primary problem has become one of preparing more and more students with a broader mathematical background, it has become increasingly clear that ways need to be found to improve the transmission of mathematical knowledge from the instructor to the student at all levels. One critical level which has been found to demand such improved learning-teaching effectiveness is at the early college level where the beginning calculus is encountered.

Statement of the Problem

The problem of this study was to determine the effect on final achievement of students in a beginning calculus course who had worked through a series of lessons of programmed instruction which had been written to teach certain basic ideas in the calculus.

The existence of the problem investigated had its origin in the fact that the learning of beginning calculus by the student has been recognized to be largely governed by the teaching he receives in class, by the reading he does out of class, and by his homework. Since so many students have failed to perform satisfactorily in calculus, the learning in general has either been incomplete or the information learned has been incorrect. In view of this learning problem the question was posed: to what extent would programmed materials written to cover the areas of calculus which cause the greatest learning problems be a significant learning aid to the student? This question in turn led to the statement of the hypothesis (H_0) with which this investigation was concerned: there is no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such materials.

Procedures

In order to determine if programmed materials specifically written to cover the topics which cause learning problems for the student in a beginning calculus course would be a favorable supplementary learning aid for the student the following procedures were used:

1. Pertinent literature was examined to provide background information which would orient the study in terms of past and present practices and problems involved in teaching beginning calculus and to summarize other investigations which have dealt with the effects of programmed materials which have resulted from their uses.

2. An identification was made of the topics of beginning calculus which cause students learning problems.

3. A set of programmed materials was written which possibly would assist the student in learning these difficult topics.

4. An evaluation of the effectiveness of the preliminary programmed materials was made following the administration of these materials to a pilot group. This evaluation involved a rating system whereby items of the programmed materials were ranked according to their effectiveness.

5. Sections of the preliminary programmed materials which indicated lack of teaching effectiveness were rewritten.

6. The rewritten programmed materials were administered to a carefully chosen and regulated experimental group.

7. A study of the effect of the programmed materials on final achievement was made by statistically comparing the final achieved scores of an experimental group which used the prepared materials with the final achieved scores of a control group which did not use the prepared materials.

The control and experimental groups were initially of comparable background and ability.

In order to determine the areas of beginning calculus which cause students learning problems, a list of topics covered in a typical first quarter calculus sequence was prepared. This list was given to twelve people well qualified to judge the learning problems of students in a beginning calculus sequence. This same jury then ranked the topics which in their judgment caused the greatest learning problems.

The topics of the list were then tabulated according to the frequency of items marked. By considering possible available student time it was decided that no more than seventeen evening meetings could be expected of students desiring to work through the programmed materials which would be written. Thus the seventeen topics most frequently indicated were denoted and programmed materials were written to cover each of these areas.

The seventeen programmed lessons were administered to students requesting assistance in a beginning calculus course. These students kept a record of the items of the programmed materials with which they had difficulty. By using Holland's 10 percent permissible error ratio[2] those items of the materials which displayed over 10 percent error were rewritten to insure greater clarification of subject matter.

In order to test the hypothesis (H_0) an experimental group was next selected from the following quarter's beginning calculus sequence enrollment at Montana State College. On a weekly schedule during the quarter these students were provided with fifteen lessons of programmed materials written to correct the faults of the original seventeen lessons. At the end of the quarter the entire beginning calculus class was given the same comprehensive final examination for the course. Scores on this final examination were recorded and statistically studied to test the hypothesis (H_0) that there was no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. Teaching

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[2]Fry, E. G., Teaching Machines and Programmed Instruction, p. 4.

of all of the beginning calculus students was done by five experienced and well qualified instructors, and general supervision of all teaching was carefully carried out by one individual whose extensive qualifications included forty years of college mathematics teaching and fifteen years as head of a mathematics department.

Limitations

The following limitations were imposed on the investigation:

1. The number of students in the experimental and control groups was limited to twenty-one and twenty-two respectively. These sample sizes were judged large enough to give the statistical tests sufficient discrimination power.

2. The experimental portion of the investigation was limited in time to the duration of the winter quarter of the 1964-65 college academic year. This period of investigation was judged sufficient because during that quarter the experimental group had ample time to carefully study enough programmed materials to master those topics selected as being essential to success in beginning calculus.

3. The topics of beginning calculus which were programmed for use by the experimental group were fifteen in number. The choice of these particular fifteen topics was judged appropriate because a rating system was used to identify them as being critical to the learning of beginning calculus.

4. The regular classroom instruction of the experimental and

control group was limited to instruction by five experienced and competent members of the mathematics staff at Montana State College. Supervision of the instruction was kept uniform and of high quality by a competent and experienced supervisor.

5. The measurement of the final achievements by the members of the experimental and control groups was limited to one comprehensive examination. This limitation was necessary in order that the investication could be carried out within the structure of existing mathematics departmental policies at Montana State College.

6. The level of significance for the statistical tests was set at 5 percent. This level was judged appropriate in view of the nature of the investigation.

Under these limitations the investigation was made to discover the effects on final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. The first step of this investigation, a review of related literature, is reported next.

CHAPTER II

REVIEW OF LITERATURE

The purpose of the review of literature was to provide background information which would orient the study in terms of past and present practices and problems involved in teaching beginning calculus and to summarize other investigations which have dealt with the effects of programmed materials resulting from their application under similar circumstances and for purposes similar to those of this study. In organizing the results of the literature reviewed four major topics were selected and correspondingly developed through reference to pertinent literature: (1) recent changes in the content and spirit of beginning calculus, (2) learning problems associated with current beginning calculus, (3) the role of programmed instruction in learning current beginning calculus, and (4) a brief summary of investigations similar in manner and purpose to that of this study. The first topic is developed in the following section.

Recent Changes in the Content and Spirit of Beginning Calculus

In the 1950's serious changes began to take place in the mathematics curriculum which had been established in the United States since the 1890's.[3] One of the first major breaks with the traditional mathematics curriculum came in 1951 when the Mathematics Department

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[3]Dyer, H. S.; Kalin, Robert; and Lord, F. M.; Problems in Mathematical Education, pp. 18-19.

at the University of Illinois reacted to the poor mathematical preparation of students in the engineering curriculum by forming a committee to specifically look into college preparatory mathematics.[4] Typifying the reports of these investigations was the following observation:

> Few freshman students had any insight into the structure of mathematics; they regarded the subject as a collection of unrelated and often inconsistent rules. . . . Textbooks were often written as though mathematics were a completed thing, there to be learned, rather than a growing subject in whose development one might take part.[5]

In an attempt to correct this shortcoming the University of Illinois Committee on School Mathematics, usually referred to as the U. I. C. S. M., was formed with Max Beberman as its director. Soon this active group was busy writing texts and devising a curriculum to meet the modern mathematical needs of society. It was significant that the U. I. C. S. M. singled out the importance of student understanding through discovery of mathematical facts as a major factor in learning mathematics. The emphasis on this aspect of learning mathematics can be traced throughout the writings of this group in such places as the first sentence of the first page of their High School Mathematics where they state to the teacher: "We believe that students should be given an

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[4]Henderson, K. B., et al., Mathematical Needs of Prospective Students, p. 5.

[5]American Association for the Advancement of Science, Science Education News, Miscellaneous Publication No. 65-7, citing one of the members of the University of Illinois Committee on School Mathematics, p. 1.

opportunity to discover a great deal of mathematics which they are expected to learn."[6]

Shortly after the U. I. C. S. M. began its attempt to design a better mathematics curriculum other groups and agencies began to show interest in these same areas. In 1952 the Board of Governors of the Mathematical Association of America appointed a Committee on the Undergraduate Program to look at what the undergraduate program in mathematics should be like.[7] Among their recommendations which later became published in text form for the first year college calculus students was the unique feature of an intuitive approach to calculus to aid the student gain good insight into what was being presented. Secondly, they agreed on a carefully written formal development of the calculus through definitions, axioms, theorems, and mathematically complete proofs.[8]

These early efforts by the Committee on the Undergraduate Program in Mathematics and the U. I. C. S. M. set the stage for much of the mathematical spirit and curriculum content which was soon to blossom forth into what many writers referred to as a revolution in school mathematics. From this beginning the emphasis on understanding

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[6]University of Illinois Committee on School Mathematics, Unit I: Arithmetic of Real Numbers, High School Mathematics, Teacher's Edition, p. I.

[7]Begle, E. G.; MacLane, Sanders; Price, G. B.; Lectures on Experimental Programs in Collegiate Mathematics, p. 4.

[8]Ibid., p. 17.

through intuition and discovery and the careful attention paid to mathematical structure has continued to dominate the mathematical scene in curriculum revision up to the time this study was made.

A later group which was very influential in continuing and emphasizing this same spirit of modern mathematics was the School Mathematics Study Group. It was formed in 1958 by the president of the American Mathematical Society with the approval of the National Council of the Teachers of Mathematics and the Mathematical Association of America.[9] The School Mathematics Study Group singled out the importance of mathematical structure as one of their primary concerns. In fact structure was identified as being their major objective in the first sentence in the "Preface of the Teacher's Commentary" of the First Course in Algebra:

> The principal objective of this First Course in Algebra is to help the student develop an understanding and appreciation of some of the algebraic structure exhibited by the real number system, and the use of this structure as a basis for the technique of algebra.[10]

With this strong emphasis on structure and complete understanding at all academic levels as the spirit of modern mathematics the content of mathematics courses was changed from the problem oriented approach to a logical development with student awareness

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[9] Wagner, John, "The Objectives and Activities of the School Mathematics Study Group," The Mathematics Teacher 53:454, October, 1960.

[10] School Mathematics Study Group, First Course in Algebra, p. i.

approach. This was particularly true at the college calculus level. Agnew, Professor of Mathematics at Cornell University and well known calculus author, aptly portrayed the spirit and content of calculus prior to 1900 by stating: "Problems were the important things, and meaningful formulations of axioms, postulates, definitions, hypotheses, conclusions, and theorems either were not written or played minor roles."[11] Agnew went on to describe the transition which took place during the next fifty years as a direct contrast to the earlier attitudes:

> Through most of the first half of the twentieth century, elementary textbooks in our subject taught unexplained but "well motivated" intuitive ideas along with their problems. Enthusiasm for this approach to calculus waned when it was realized that students were not nourished by stews in which problems, motivations, fuzzy definitions, and fuzzy theorems all boiled together while something approached something else without ever quite getting there. About the middle of the twentieth century, precise formulations of basic concepts began to occupy minor but increasingly important roles.[12]

Finally Agnew indirectly characterized the current spirit and content of modern calculus by stating in his 1962 calculus text:

> Each student is expected to read the text and problems of each section as carefully as an alert physicist reads an account of a newly developed nuclear reaction, and to learn as much as he can. . . . The text, problems, and remarks frequently give students quite unusual opportunities and incentives to think and to become genuine authorities on developments of ideas, terminologies, notations, and theories.[13]

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[11]Agnew, R. P., Calculus, p. v.

[12]Ibid., p. v.

[13]Ibid., pp. v-vi.

As to the future of mathematical course content and spirit some extrapolations have been made. In 1963 under the financial support of the National Science Foundation twenty-five prominent and well-qualified mathematicians assembled at Cambridge, Massachusetts to assess the potential of mathematical learning and to set tentative goals towards which mathematical education should attempt to direct its efforts in the next thirty years. The goals of particular courses were specifically spelled out, and the role of structure permeated the entire system at all levels as was indicated by the statement:

> We hope to make each student in the early grades truly familiar with the structure of the real number system and the basic ideas of geometry. . . . Moreover, we want to make students familiar with part of the global structure of mathematics.[14]

Furthermore, the Cambridge group endorsed the method of teaching mathematics which results in discovery by the students:

> Even modestly endowed students can recreate larger parts of mathematics if they can remember just a few basic ideas. This fact repeatedly has been demonstrated in the classroom by the proponents of the so-called discovery method. The building of confidence in one's own analytical powers is another goal of mathematics education.[15]

Hence, a study of the literature pointed out that the spirit of contemporary mathematics at the level of beginning calculus as well as

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[14]Educational Services Incorporated, The Report of the Cambridge Conference on School Mathematics, Goals for School Mathematics, p. 8.

[15]Ibid., p. 9.

at all levels has been largely associated with complete mathematical understanding of the subject from a structural viewpoint. That is, the modern trend in mathematics has placed a premium on clear insight into the structure of mathematics rather than on mathematical ability without understanding. Student discovery and development of mathematical truths were frequently referred to as the best means of developing this structural awareness.

In the next section the current problems associated with learning mathematics with this emphasis on structure by means of student discovery are discussed.

Learning Problems Associated with Current Beginning Calculus

The literature reviewed clearly reflected a recent swing to emphasis on mathematical structure and student development of mathematical ideas. A considerable number of authors of the present widely used calculus texts have agreed to this principle.[16] However, not all of the students currently enrolled in beginning calculus have had a background in mathematics which emphasizes structure to the degree

[16] Randolph, J. E., Calculus and Analytic Geometry, pp. v-vii; Protter, M. H. and Morrey, C. B., College Calculus with Analytic Geometry, pp. v-vii; Agnew, op. cit., pp. v-vii; Apostol, T. M., Calculus, Vol. 1, pp. vii-ix; Goodman, A. W., Analytic Geometry and the Calculus, pp. v-vii; Thomas, G. B., Calculus and Analytic Geometry, pp. v-vi.

modern calculus courses demand. Hence, the student whose background is entirely traditional often has had difficulty catching the spirit of modern mathematics. This fact was well substantiated when Zant, in discussing problems facing beginning college mathematics students in Oklahoma, stated:

> Another problem. . . is that of the good student who has studied only traditional mathematics courses in high school. Somehow and as quickly as possible, these students must be taught the point of view towards mathematics and the basic concept involved in a modern program in secondary mathematics. If our college courses beginning at the analytics-calculus level have been modernized, then these students, even though they have studied traditional mathematics for three or more years in high school, are not prepared to enroll in such a course.[17]

Along with this emphasis on structure the major unifying ideas of mathematics have often been expressed using notation not commonly found in the traditional mathematics. For example, Randolph of the University of Rochester in his 1961 calculus text has stated:

> The introduction into elementary courses of the well-established notation for the set of elements, each satisfying a given condition is long overdue and rapidly finding favor. Even though some books state once and for all that 'the line $x = -2$' means 'a point (x, y) is on the line if and only if $x = -2$', some students will be confused about how x and y

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[17] Zant, J. H., "Effects of New Mathematics Programs in the Schools on College Mathematics Courses," The American Mathematical Monthly 63:200, February, 1963.

> can be both $x = -2$ and $y^2 = 4x$ in the same problem. After gradually becoming acquainted with the symbolism, then $\{(x, y) \mid x = -2\}$ and $\{(x, y) \mid y^2 = 4x\}$ forces a reading and quick comprehension where equivalent worded statements only annoy and muddle.[18]

The net result of this emphasis on structure when accompanied with corresponding modern notation has often compounded newness and difficulty thus resulting in learning problems for the student.

The wide spectrum of student needs and abilities were also reported as a learning problem. Evidence was plentiful which pointed out that not all students require the same type of calculus course. The mathematics majors need a firm understanding of the theory as well as a mastery of mechanical competency oriented always to their past and future training. On the other hand the engineer is required to master the application of mathematical fact to physical phenomenon which he will likely encounter in engineering problems. A third extreme is the student who takes calculus more for cultural improvement than as a prerequisite for future use. In this last case the calculus is not problem oriented but instead based on a developmental scheme via a rigorous sequence of definitions, axioms, theorems, and proofs.

The literature further revealed that many attempts have been made to meet this problem. In general the reported trend was toward special sections, honors courses, advanced placement, regular courses, and

[18] Randolph, op. cit., p. vi.

cultural courses geared to fit the needs and abilities of the students. However, even with such varied alternatives the problem of differing student needs still has caused concern. For example, Brown described as follows the beginning calculus problem at Dartmouth where in spite of alternative offerings of regular calculus classes, advanced placement, and honors mathematics, students' needs were not being met: "It seems to me that the first course . . . should be as broad as possible. Admittedly we have to steer between breadth and superficiality."[19]

Thus, since the instructor in the average class has been challenged to direct his teaching according to the class needs, treatment of a particular topic has often not been as complete for individual needs as would be desirable. That is, structural development of the calculus may have been handled too extensively for some students whereas it was found by others too shallow.

In this same area of thought, student abilities within a particular class have been known to range widely despite attempts to group homogeneously. Jones and Pingrey made this point clear by stating: "Students in any classroom, even in those where ability grouping is used, will differ widely in both general background and special abilities and backgrounds."[20] Hence, since ". . . it is still the individual classroom

[19] Brown, B. H., "Offerings for Freshmen," American Mathematical Monthly 68:262, March, 1961.

[20] Jones, R. S. and Pingry, R. E., "Instructional Arithmetic" in The Twenty-Fifth Yearbook of the National Council of the Teachers of Mathematics, 1960, p. 122.

teacher who must adapt instruction to the differences he finds among his pupils,"[21] the instructor has been challenged to develop the calculus in a way which is both inspirational to excellent students and within the grasp of poor students. Frequently in an attempt to satisfy one particular ability level the instructor has sacrificed the other extremes. Thus learning problems for the neglected students have resulted.

The role programmed instruction can play in meeting these learning problems is discussed next.

The Role of Programmed Instruction in Learning Current Beginning Calculus

A search of the literature revealed that learning problems in beginning calculus are in general fourfold: (1) unfamiliar notation has provided reason for low achievement, (2) new spirit of modern mathematics has been unfamiliar to many students, (3) individual student differences have caused teaching problems, and (4) differing student needs have been found to influence teaching effectiveness. The purpose of this section will be to show the role of programmed materials as associated with these needs.

Unfamiliar notation: Research has established that students can learn effectively by programmed instruction.[22] Hence, since the problem of unfamiliar mathematical notation has been shown to be a

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[21] Ibid., pp. 145-6.

[22] Schramm, Wilbur, The Research on Programed Instruction, p. 3.

hinderance to learning, it follows that students should be able to learn this troublesome notation and what it means by materials programmed for that specific purpose.

The spirit of modern mathematics: Although the question of being able to learn unfamiliar mathematical notation by study of properly written programmed materials was found to have been possibly answered, the question of being able to catch the proper spirit of modern mathematics was not conclusively answered. That is, the problem as to the ability of students to properly connect an understanding of mathematical structure with logical reasoning to produce new mathematical facts was found to be unanswered in the literature. Differing opinions and statements were abundant.

Relative to this question the Department of Audiovisual Instruction of the National Education Association and the National Society for Programmed Instruction in reference to research by Cartier stated: "Francis Cartier points out that a number of aspects of rational thinking (e.g. formal logic) can be taught by present programming techniques."[23] But in the next line the text read: "Formal logic accounts for only a part of the rational and creative processes we call thinking."[24]

A considerable amount of skepticism as to the value of programmed instruction in teaching mathematical thinking of the type desired in modern mathematics was found in May's Programed Learning and Mathe-

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[23] Ofiesh, G. D. and Meierhenry, W. C., editors, Trends in Programmed Instruction, Department of Audiovisual Instruction of the N. E. A. and the National Society for Programmed Instruction, p. 415.

[24] Ibid., p. 415.

matical Education. May stated emphatically: "Programed materials are incapable of eliciting the full range of behavior included in the objectives of college mathematics."[25] In particular May was concerned that programmed materials do not cause the student to "get out and grub" for mathematical truth; instead he contended that the student "is fed ideas intravenously drop by drop" which is not the spirit of modern mathematics."[26]

Buck was found to agree with May in this observation because he stated that programmed instruction does not develop creative ability but merely exercises the student in drill. In particular he said:

> In this I feel that a grave mistake has been made: to put it bluntly, creativity is the heart and soul of mathematics at all levels. The collection of special skills and techniques is only the raw material out of which the subject itself grows. To look at mathematics without the creative side of it, is to look at a black-and-white photograph of a Cezanne; the outlines may be there, but everything that matters is missing.[27]

Hence, the question of programmed instruction being able to satisfactorily cause the student to grasp the spirit of modern mathematics was found to be open to question.

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[25] May, K. O., Programed Learning and Mathematical Education, p. 5.

[26] Ibid., p. 7.

[27] Buck, R. C., "Teaching Machines and Mathematics Programs," American Mathematical Monthly 69:554, June-July, 1962.

Individual differences: A logical question often asked was: can programmed materials sufficiently solve the individual differences problem which results from instruction being beamed at class average hence causing the slower student to be unable to follow the materials being presented in class?

In many sources the reasoning has followed the logical line that programmed instruction "tends to level the differences in learning capacities among students; while all students exposed to the program may demonstrate achievement, the gain seems to be more conspicuous among the lower portion of the class distribution."[28] One of the first experimental efforts which verified this belief was reported in 1934 when Little published in The Journal of Experimental Education the "Results of Use of Machines for Testing and for Drill Upon Learning in Educational Psychology."[29] Then in 1959 Porter confirmed the belief by experimenting with spelling and programmed instruction.[30]

However, here again evidence was plentiful in the literature which indicated that individual differences are not significantly satisfied by a slow but sure pace due to programmed instruction. In summarizing the evidence May declared:

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[28] Lysaught, J. P. and Williams, C. M., A Guide to Programmed Instruction, p. 15.

[29] Fry, E. B., op. cit., p. 84.

[30] Porter, D., "Teaching Machines," Harvard Graduate School of Education Association Bulletin 3:1-5, March, 1958.

> Numerous studies have indicated that self-pacing is not as helpful as one might imagine. . . . It appears that pacing is not a very important issue and that self-pacing has no necessary or unique connection with programmed materials.[31]

As added substantiation of this point Schramm has pointed out seven studies which show there is no significant difference attributable to internal or external pacing.[32]

Different student needs: The final learning problem which was evident in the literature was differing student needs. Regarding mathematics and programmed learning May declares:

> The real possibilities for using programmed materials to cope with individual differences lie in different directions. One is the development of large libraries of brief units focused on narrow problems, beamed to specific student difficulties, and utilizing programing devices most appropriate to the audience and difficulty.[33]

Thus in summary it is clear that there has been much evidence to indicate the use of programmed materials can be beneficial in helping students learn. However, the degree to which programmed materials can be beneficial has been contested and the areas where it is of benefit has not been in total agreement.

In the next section research will be reviewed which has a bearing on learning beginning calculus by programmed materials.

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[31] May, op. cit., p. 6.

[32] Schramm, op. cit., p. 12.

[33] May, op. cit., pp. 6-7.

Survey of Investigations Similar in Manner and Purpose to This Study

A search of the literature revealed that no other experimental work had been carried out which measured the effect on achievement of beginning calculus students when their conventional classroom instruction was supplemented with programmed instruction. However, a number of studies had been completed which were related to the investigation with which this writer was concerned. One particular investigation reported by Lane which was of interest was concerned with different means of supplementing conventional classroom instruction of mathematics.

In 1962 Lane experimented with three means of supplementing instruction by closed circuit television of a basic mathematics course at the George Peabody College for Teachers. Lane used three types of supplementary instruction: film of assigned homework shown to students daily, daily classroom question-answer periods, and programmed instruction booklets prepared to cover the topic presented on television. The results of the study indicated that achievement of the group using programmed instruction was significantly better at the 5 percent level than the groups whose means of supplementing the televised instruction was by film or class question-answer periods.[34]

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[34] Lane, B. R., "An Experiment with Programmed Instruction as a Supplement to Teaching College Mathematics by Closed Circuit Television," The Mathematics Teacher 57: 395, October, 1964.

Numerous other investigations have been made into the effectiveness of teaching college mathematics and mathematics related courses by programmed instruction. In reviewing the literature the following investigations have been found to reveal results pertinent to this study.

Sharpe reported at the University of Buffalo on the effectiveness of using Skinner programmed materials in place of texts in pre-calculus courses:

> To date, the experiment as it has been conducted indicates a probability that programed materials may do an equivalent job, but presents no evidence that programed materials are superior. . . . Students in this experiment had programs backed up by good instructors, yet no records were broken.[35]

At the United States Air Force Academy Smith has studied the teaching of elementary statistics by the conventional classroom method versus the method of programmed instruction. The net result of the investigation has shown that learning achieved through study of statistics by a scrambled book program as compared to learning via conventional classroom teaching produced no significant difference in performance between experimental and control groups.[36]

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[35]May, op. cit., p. 5, citing Sharp, B., "Programed Mathematics, A Two Year Experiment 1962-1964," place and date unknown.

[36]Smith, H. N., "The Teaching of Elementary Statistics by the Conventional Classroom Method Versus the Method of Programed Instruction," Journal of Educational Research 55:417-20, 1962.

One of the first studies of the effectiveness of programmed learning in logic was done by Blyth at Hamilton College. Blyth reported the preliminary results of his investigation as encouraging. Contact hours with students were reduced by one third, level of subject matter mastery was raised, and the amount of material covered was increased.[37]

An important study was done during the fall of 1962 by Dobyns who reported on an experiment in the teaching of college algebra. Dobyns compared teaching certain topics of college algebra by conventional methods to teaching by programmed instruction. Among his conclusions were the following:

> There was no significant difference between the mean scores for the two teaching methods. . . . Significant gains made by the students using the programmed booklet covering inequalities, absolute value, coordinate systems, functions, and their graphical representations were lost after a time lapse.[38]

Kellems also reported on the effectiveness of programmed teaching in college algebra. His study was conducted during three semesters at Indiana State College during which time he administered programmed materials in three ways. For his control group Kellems

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[37]Blyth, J. W., "Teaching Machines and Logic," American Mathematical Monthly 67:285-7, March, 1960.

[38]Dobyns, R. A., "An Experiment in the Teaching of College Algebra," The Mathematics Teacher 57:86, February, 1964.

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Another related study was done by Heimer who conducted an experiment in which he measured the effects on achievement brought about by using different methods of instruction on college students in a beginning mathematics course. In particular he compared the effects of external and self-pacing when teaching was done by teaching machines, programmed textbooks, filmstrips, and conventional classroom teaching. No significant differences were revealed on final examination scores.[41]

In addition to this experiment by Heimer was one conducted by Lottes, Palmer, and Oakes who compared the effects on achievement due to differential rate of pacing programmed mathematics. No significant differences were observed in student performance or attitudes towards programmed learning when pacing was varied as much as 20 percent below and 10 percent above the average time required by a group of self-pacing students.[42]

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[41]Heimer, R. T., The Preparation of a Program in Contemporary Algebra and a Study of Its Effectiveness for Group Instruction Under Paced Conditions, Doctoral Dissertation, as reviewed in Dissertation Abstracts, Vol. XXIII, No. 10, April, 1963, pp. 3813-4.

[42]Schramm, Wilbur, op. cit., p. 32 citing Lottes, J. J., Palmer, G., and Oakes, H., "An Experimental Comparison of Differential Rates of Pacing Programed Math," published by the University Division of Instructional Services, the Pennsylvania State University, 1963.

Although numerous investigations have been carried out at the high school and junior high school level, the problem of motivation often makes the results of the investigations difficult to interpret.[43]

However, one such study at the high school level which had no mention of motivation problems and which was somewhat parallel to that of this study was done by Brown. Using the programmed booklets written by the U.I.C.S.M. Brown compared the test scores of an experimental group using programmed instructional materials with a control group not using programmed instructional materials. The experimental group in this study had conventional classroom teaching along with the programmed materials whereas the control group had only conventional classroom instruction. From this study the conclusion was reached that:

> The experimental group proved to be significantly superior to the control in a test of general ability, and about the same level of superiority was maintained in eight out of nine achievement tests given during the school term. The author concludes that 'no student was penalized in his level of mathematics achievement because of having used programmed material.'[44]

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[43] Randolph, P. H., "An Experiment in Programmed Instruction in Junior High School," The Mathematics Teacher 57:160-162, March, 1964; Jordy, J. L., "A Comparative Study of Methods of Teaching Plane Geometry," The Mathematics Teacher 57:472-8, November, 1964; Smith, W. I. and Moore, J. W., "Programed Materials in Mathematics for Superior Student in Rural Schools," published by Bucknell University, undated, p. 91.

[44] Schramm, op. cit., p. 26, citing Brown, O. Robert, "A Comparison of Test Scores of Students Using Programed Instructional Materials with Those of Students Not Using Programed Instructional Materials," mimeographed by the University of Illinois in 1962.

The review of literature has shown that there is evidence to indicate value of using programmed instruction in place of or as a supplement to conventional classroom instruction. However, there was also evidence which indicated the gains resulting from using such programmed materials are often not significant. Also, when differences do occur, they are sometimes in favor of the conventional method or disappear with a time lapse.

A summary of this entire chapter is presented next.

Summary

In summarizing the literature reviewed four points stand out: (1) beginning calculus has experienced a radical change in the past eight years from a problem-solution oriented attack to a careful structural attack achieved through student understanding and logical development, (2) many of the learning problems in beginning calculus have been brought about by the swing from a traditional type background to a modern approach thus leaving many students ill prepared for the degree of rigor required in modern mathematics, (3) the use of programmed instruction may reduce the imbalance in background experienced by traditionally prepared students through development of proper notation but may not entirely fill the gap in proper mathematical attitude such as is often required in mathematical reasoning, and (4) the investigations into the usefulness of programmed instruction do not consistently point to its unswerving value but instead the majority of comparisons of its value over conventional classroom instruction indicate the differences in final learning achieved through programmed instruction are not significantly greater than those achieved by conventional classroom teaching.

In the next chapter the procedures and results of the investigation will be presented.

CHAPTER III

INVESTIGATIONAL PROCEDURES AND STATISTICAL METHODS APPLIED

The research described in this chapter was conducted as a means of investigating the effect on final achievement in a beginning calculus course resulting from the use of a series of programmed instruction written to supplement the regular classroom instruction on the topics which ordinarily cause learning problems. In reporting the investigational procedures and statistical analysis the following topics were considered to be of major importance: (1) selection of topics to be programmed, (2) preliminary writing of the programmed materials and associated student participation, (3) final writing of the programmed materials and associated student participation, (4) means of instruction of the experimental and control groups, and (5) statistical analysis used to evaluate the results of the investigation. Each of these topics is discussed in the remainder of this chapter.

Selection of Topics to be Programmed

Prior to the fall quarter of 1964 a study was conducted by the writer to establish the conditions under which topics from beginning calculus would be selected and programmed for use by students in an experimental group. This investigation included three areas: identification of topics taught in a beginning calculus course, identification of those topics which cause student learning problems, and an estimation of a logical number of programmed learning sessions which would be convenient for the students to attend and yet would be extensive enough to guarantee an adequate coverage of material.

The first problem was to identify the topics taught in a beginning calculus course. This was done by comparing the development of the calculus in five widely used calculus texts.[45] In each of these texts the following topics appeared as introductory material: absolute value and inequalities; functions and functional notation; intersections of pairs of curves; limits; continuous functions; the derivative; simple differentiation formulas; differentiation of the product, quotient, and higher order derivatives; differentiation of the composite functions; implicit differentiation; application of the derivative to linear motion, motion under gravity, and related rates; sketching as related to increasing and decreasing functions; critical values, and extreme values of a function; Rolle's Theorem and the Mean Value Theorem; applications involving the theory of extremes; approximations by differentials; the indefinite integral; computation of area by integration; the definite integral; area as the limit of a sum; fundamental theorem of calculus; and mathematical induction.

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[45] Protter, M. H. and Morrey, C. B., College Calculus with Analytic Geometry; Taylor, Angus E., Calculus with Analytic Geometry; Thomas, George B., Calculus and Analytic Geometry; Johnson, R. E. and Kiokemeister, F. L., Calculus with Analytic Geometry; Goodman, A. W., Analytic Geometry and the Calculus.

In order to identify which of the above topics cause student learning problems a questionnaire was prepared which listed all of the above topics and briefly described the intent of the investigation.[46] The persons to whom the questionnaire was directed were asked to select seventeen areas which in their opinion cause beginning calculus students the greatest learning problems.

Two factors necessitated limiting the number of areas to seventeen: (1) only a limited number of meetings could be expected of the volunteering students during the quarter, and (2) there was a potential difficulty of correlating the programmed materials with different class rates. The first factor which limited the areas to be covered to seventeen was judged essential since most of the students volunteering for the program were carrying full academic loads. Hence, it was unreasonable to expect them to meet more than twice weekly for the purposes of studying the prepared materials. The second factor was likewise judged necessary because the possibility existed that the topics selected by the jury of experienced teachers could be of the nature that course study of these topics would not occur at the beginning of the course but instead might be grouped timewise toward the end of the quarter. Hence, the decision was made to limit the potential topics selected to seventeen in number with the understanding that some modification of title and content might be necessary as the study progressed.

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[46]See Appendix A.

This questionnaire was then given to twelve college educators well-qualified to judge the learning problems of students in a beginning calculus class. Ten of these persons were prominent members of the mathematics department at Montana State College with extensive experience in teaching beginning calculus. Their experience was judged by their teaching assignments and duties as mathematicians at Montana State College. The group included one department head, three supervisors of teachers of elementary mathematics courses (one of whom was also a retired departmental head and supervisor of the beginning calculus sequence), an acting departmental head, a full professor, two associate professors, one assistant professor, and one instructor.

The two non-mathematics departmental members who filled in the questionnaire were chosen from the physics and chemistry departments. Their responsibilities in their areas indicated concern for and knowledge of learning problems of students in their fields as would be reflected by inability in different phases of calculus.

Upon receiving the results of the questionnaire a frequency tabulation was made to identify those areas of beginning calculus which cause the students the greatest learning problems. The seventeen topics which were most frequently identified were:

1. Inequalities and Absolute Value
2. Limits
3. Continuity
4. The Derivative
5. Composite Functions and the Chain Rule
6. Implicit Differentiation
7. Increasing and Decreasing Functions
8. Velocity and Related Rates
9. Rolle's Theorem and the Mean Value Theorem

10. Maximum and Minimum Problems
11. Differentials
12. Indefinite Integral
13. Definite Integral
14. Area by Integration
15. Mathematical Induction
16. Area as the Limit of a Sum
17. Fundamental Theorem of Integral Calculus

Preliminary Writing of the Programmed Materials and Associated Student Participation

Three factors influenced the scheduling of the meetings during which the students would study the programmed materials. First, on examining the seventeen areas selected by the experienced jury it was apparent that the topics for which the programmed materials had to be written occurred during the development of the course about two weeks after the quarter began. Hence, the first meeting with the volunteering students was not possible until at least two weeks of the quarter had elapsed. Second, since the amount of class time ordinarily devoted to some of the initial topics was comparatively longer than for later topics, a more liberal time schedule was anticipated at the start of the quarter than at the end of the quarter. Third, there was no guarantee that all of the instructors could hold closely to the day-by-day schedule of topics to be taught. These three factors suggested that a certain amount of flexibility was needed in the administration of the programmed materials. With these problems in mind a calendar was drawn up scheduling the programmed meetings on each available Tuesday and

Thursday of each week throughout fall quarter. By allowing for holidays and final examinations exactly seventeen such Tuesdays and Thursdays were found to be available.

Having determined the topics to be programmed and the number of meetings available, it was then possible to start writing the preliminary programmed materials. Preliminary to writing each section, the topic in question was studied carefully from the point of view of the author of the textbook from which the students would be required to read their daily assignments. Then the supervisor of the beginning calculus sequence was contacted to verify the teaching techniques to be used in presenting the topic. Finally, a close contact was maintained with individual instructors to be certain as to presentation of the topic in different classes. By combining these factors with the knowledge gained as a result of consulting literature about programming, the topics were then developed by writing appropriate programmed materials.

Each topic was developed in harmony with the theory and vocabulary used in class presentation. In most cases the topic had its foundation in a certain degree of mathematical theory. Consequently, a search for a clear, precise, intuitive presentation was made by the writer in various calculus texts. Upon examination of numerous leading calculus texts a harmonious development of the topic in conjunction with notation familiar to the students and in conjunction with a sufficient background of factual information evolved.

In cases where theory was presented an intuitive prelude was programmed into the materials which made the principle in question seem plausible. In other cases where procedure was the paramount idea the development of the procedure was gradually introduced to the student with frequent reference to the overall systematic approach. In the event variations of procedure were needed, each major variation was carefully identified and systematically developed.

During the preliminary writing of the materials care was exercised to insure a careful balance among available student time, proper presentation of the topic, and sufficient development of the idea to guarantee learning. In fact it was deemed appropriate to write materials which could be satisfactorily covered by the average student in an hour or less.

After the first writing of the materials for each topic the author proceeded to other duties. Later he returned to the first writing and systematically worked through these materials. As seemingly incomplete or inappropriate sections of the materials were encountered, these sections were rewritten to correct possible faults. Eventually, as the materials became quite polished they were typed up and dittoed for student use.

As a means of checking the quality of these preliminary writings, a pilot group of students was invited to use them on specified dates during the fall quarter of 1964. This pilot group consisted of all students from all sections of the beginning calculus sequence who desired additional help in calculus. In order to keep the number of volunteer participants in the pilot group of reasonable size no special efforts were made to encourage student participation. The availability of the materials was announced in class and from this brief invitation a sufficient number of student volunteers was obtained to accomplish the purpose of the pilot study. Because of the volunteer nature of the pilot investigation, the number of participants was not consistent throughout the quarter. However, the number was always above ten and below forty except for the last lesson when only five students completed the materials. In investigating the reason for the small interest in the last lesson it was found that the pressure of examinations at the end of the quarter and the fact that not all classes had yet covered this topic in class accounted for the poor attendance.

The purpose of the pilot study was to identify weaknesses in the preliminary writing of the programmed materials and to perfect the investigator's technique of writing programmed materials. The weaknesses in the preliminary writing were identified by instructing the student volunteers to denote by a check those items of the materials which they were unable to complete in accordance with the indicated response which followed each item. Also, a check was kept of time consumed by students in working the materials. Upon the completion of each lesson a check list was made of those items which showed 10 percent error ratio by student participants. The 10 percent figure was used in accordance to the tolerance specified by Holland.[47]

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[47] Fry, E. B., op. cit., p. 4.

In addition to a tabulation of error frequency a close contact was maintained with student participants to gain student opinions of particular phases of the materials. These opinions were noted and evaluated so that the final writing could reflect valid criticism.

Final Writing of the Programmed Materials and Associated Student Participation

During the winter quarter of 1965 at Montana State College the final writing and use of the programmed materials was carried out.

The final writing and administration of the materials were based on the experiences gained from the use of the preliminary materials. Modifications of the preliminary materials in the main consisted of three types: (1) modification of items which reflected a 10 percent error ratio; (2) modification of two lessons to facilitate efficient use of the materials; and (3) modification of the administration of the materials.

The first type of modification of the materials was one of content and was focused on individual items of the materials. That is, those items which reflected over a 10 percent error ratio were rewritten to insure greater clarity and thus reduce the error frequency. Sometimes this simply involved changes in wording or the addition of a sketch. In a few cases it was found desirable to redevelop the concept involved by introducing smaller steps and by including more explanatory passages. Considerable discussion was used to be certain the topics were clear and complete. To guard against possible incomplete reasoning and to insure a clear and favorable development in the rewritten portions, the sections when rewritten were presented to various members of the mathematics staff at Montana State College familiar with the program for comment and criticism. Upon agreement of a clear and consistent revision the rewritten portions were then included in the final version.

The second type of major modification of the materials also involved content. However, in this modification the focus was on efficient use of the materials from the standpoint of complete lessons. Three changes of this type were found necessary.

First, it was deemed advisable to use two meetings of the students for the first lesson because of the desirability of lengthening the time available for the introductory lesson. This was judged appropriate so as to better acquaint students with the purpose of the study and to better correlate the programmed materials with the contiguity of class presentation of corresponding topics.

Second, it was found necessary to eliminate the lesson "Fundamental Theorem of Integral Calculus" since class discussion of that lesson in the preceeding quarter was found to occur too late in the quarter to be satisfactorily supplemented by programmed instruction.

A third modification of the lessons was found necessary as the quarter progressed because class presentation fell behind the biweekly programmed materials sessions. Hence, the necessity presented itself during the quarter of dropping back one lesson. As a means of compensating for this the topic ratings by the panel of experts was reviewed and it was found that the indefinite integral was one of the topics least in need of supplementary work. Furthermore, the definite integral was a topic in great need of supplementary work. A careful examination of the preliminary writing showed that both of these goals could be best attained by altering the contents of the chapter on "Area by Integration" to include the definite integral and dropping the chapter on "The Indefinite and Definite Integral". With this final modification, which was accomplished without loss of continuity to the materials and without loss of purpose of the investigation, the writing of the materials was completed.[48]

The third major change in the final use of the materials over their preliminary use involved the administration of the materials. The most significant administrative change was that the materials were used by only a select group of students, the experimental group.

The selection of this experimental group took place at the beginning of winter quarter in 1965 after the instructors of the beginning calculus sequence at Montana State College were carefully briefed on the purpose of the forthcoming study into the effect on achievement of using programmed materials as a supplement to classroom instruction of beginning calculus. Upon agreement to the procedures of administration of the materials the instructors were asked to announce the purpose of the investigation to their classes

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[48] See Appendix B.

and to invite undergraduate students to volunteer for the program who had as their final mark in college algebra or college trigonometry grades of B, C, or D. Students were reminded that the nature of their participation was strictly voluntary and that no obligation to volunteer or to finish the program after starting existed. However, it was pointed out that of those starting the program the investigator desired as many to finish the program as could. Furthermore, it was noted that only students participating in the program would have access to the materials and application to start the program could not be accepted after the initial lessons had been completed.

Under these conditions students applied for acceptance into the experimental group. Screening of the candidates was carefully done on the basis of the criteria set up and an experimental group was chosen which consisted of all qualified students. The accepted students were notified of their acceptance and directed to report during the second week of classes for their orientation to the materials and their first lesson. At this first meeting a brief orientation was given as to how to use the materials and questions were answered regarding the program. An agreement was entered into whereby evening meetings would be held regularly twice weekly so that the scheduled lessons would all be worked. Also it was explained that each of these evening meetings would probably last about an hour and would be flexible enough to allow for different working rates of individual students. For a few students with conflicts on the scheduled evenings it was necessary to schedule a third night a week so that these students would also be able to complete all of the lessons.

From that time on regularly scheduled meetings were held, progress records were kept, and individual contacts were maintained to keep students up to date with their lessons. When students dropped behind in their lessons, reminders were sent them of their delinquency and they were encouraged to make up their missed lessons.[49]

In an effort to keep student advisors informed about the program and the progress of their advisees, explanations of the program and progress reports of advisees were sent to advisors at appropriate times.[50]

At no time during the quarter were the programmed materials allowed to be taken from the supervision of the investigator. In the event a student needed more time to make up a lesson or do extra work, special arrangements were made with the investigator to do the materials under his supervision. In this way the materials were carefully monitored and kept available only to the experimental group.

At the end of the quarter the students who had completed the entire program were noted and their names recorded so that their final records would be easily obtained for purposes of statistical comparisons.

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[49]See Appendix C.

[50]See Appendix D.

Also, at the end of the quarter the entire class roll of students completing the beginning calculus sequence was searched to identify members of the control group. That is, the backgrounds of all undergraduate students in the beginning calculus sequence were examined to determine those not in the experimental group who had taken as their last mathematics course college algebra or college trigonometry and had received a B, C, or D in that course. By restricting eligibility for selection into the control group to these standards a control group was obtained which was composed of students with backgrounds similar to those of the experimental group since that group was chosen by corresponding standards.

After carefully searching the student records all students meeting the above stated conditions were listed and were accepted as members of the control group.

Means of Instruction of the Experimental and Control Group

In order to be certain that instruction in the different beginning calculus classes was kept constant, the supervisor of instruction held weekly meetings with instructors to check on class progress and lesson presentation. In this way it was assured that all classes were quite uniform in the material presented and in the demands made of the student.

Because special mathematical notation facilitated the writing of the materials on various occasions, at each of these weekly meetings the investigator met with the instructors and supervisor to describe the progress of the study and to indicate notational usage in the materials which needed to be practiced in class to guarantee uniform familiarity.

In describing the progress of the study, the instructors and supervisor were notified of student participation by a progress chart which indicated the number of lessons each student had completed.[51] In addition to the progress sheets the instructors, supervisor, and department head were supplied weekly with the lessons currently being used by the experimental group.

Through: (1) the careful monitoring of the teaching by the supervisor, (2) the competency of the instructors, (3) the close cooperation among the investigator, instructors, supervisor, and department head, and (4) the fact that the members of the experimental and control group were scattered at random throughout all sections of the beginning calculus sequence, it was assumed that uniform instruction for both experimental group and control group resulted.

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[51] See Appendix E.

Measurement of the Final Achievements of the Experimental and Control Groups

The measurement of the final achievement of the experimental and control groups was accomplished by the same comprehensive final examination taken by all students who had completed the beginning calculus course the quarter during which this study was made. This single test was used to measure the final achievements for two reasons. First, the testing of the control group had to be accomplished within the structure of existing practices at Montana State College. Second, it was judged possible to design a valid and reliable test which would measure the final achievements of students completing the beginning calculus course that quarter.

In order to carefully control the design of this examination an experienced and well-qualified committee of calculus instructors was appointed by the supervisor of instruction to write an appropriate test which would measure the final achievement of the students completing the course. The examination[52] was then prepared and presented to the entire group of instructors of the beginning calculus sequence for their inspection and approval. With final approval announced by the supervisor of instruction, who was particularly well qualified to judge the validity and reliability of the test (having had over forty years of college mathematics teaching experience which

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[52]See Appendix F.

included extensive administrative responsibilities), the examination was accepted as an accurate measure of the final achievement of members of the experimental and control groups.

As a means of holding the scoring of this examination uniform, a committee was appointed by the supervisor of the beginning calculus sequence and was directed to prepare a sheet on criteria for grading each problem of the examination. This criteria for grading sheet was then given to each instructor of the beginning calculus class and he or she used it according to its purpose.

Statistical Analysis Used to Evaluate the Effect Resulting from the Use of the Programmed Materials

The problem of measuring the effect on the final achievement resulting from the use of programmed materials which were written to supplement classroom instruction involving topics which usually cause learning problems in beginning calculus was carried out by first carefully stating the hypothesis to be tested. Then appropriate statistical methods were used to test the hypothesis by comparison of final examination scores at the end of the quarter.

Statement of the hypothesis. As a prelude to actual statistical comparison of the achievement by the experimental and control groups the null hypothesis, H_O, was stated. This hypothesis was: there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. The alternative hypothesis, H_1, was: there is a significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. To test these hypotheses appropriate statistical methods were selected and applied.

Selection of statistical methods. In order to test the null hypothesis, H_0, the question of comparability of the experimental and control groups had to be first answered. Hence, in the final analysis two major questions presented themselves: (1) were the experimental and control groups comparable as far as mathematical and intellectual backgrounds concerned? and (2) was the achievement in beginning calculus of the experimental group superior to that of the control group in view of the possible differences in mathematical and intellectual backgrounds?

In order to answer the first question the backgrounds of the students in both the experimental and control groups had to be compared. In searching for such a comparison it was found that Montana State College had devised a criterion for placement of students in the beginning calculus sequence according to their mathematical and intellectual backgrounds. This criterion was a predicted grade in beginning calculus and was a composite score based upon the following sources: the score of the American Council of Education Psychological Examination, the score of the Minnesota Scholastic Aptitude Test (a short form of the Ohio Psychological Examination), the score on the Mechanical Comprehension Test - Form CC, a set of scores on examinations in arithmetic, algebra I, and algebra II (examinations prepared by the Mathematics Department at Montana State College), and the high school grade point average.[53] The fact that this scheme of predicting grades in mathematics had been used at Montana State College for over 30 years with some modification but without major revision attested to its accepted value.[54]

Hence by using these predicted grades in the beginning calculus sequence as calculated by the formula adopted by Montana State College, it was possible to assign a numerical score to each member of the experimental and control group. For the purposes of this investigation this score was considered to be a comprehensive measure of the mathematical and intellectual backgrounds of the students. Thus only a statistical means was needed whereby the mathematical and intellectual backgrounds as indicated by the predicted scores could be compared.

Both of the major problems to be investigated were kept in mind in the selection of this statistical test. Namely: (1) were the

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[53] Suvak, Albert, (unpublished research at the Montana State College Testing and Counseling Service).

[54] Brookhart, M. E., (unpublished statement by M. E. Brookhart, Director of the Montana State College Testing and Counseling Service).

experimental and control groups comparable as far as mathematical and intellectual backgrounds concerned? and (2) was the final achievement of the experimental group above that of the control group? Because both the question of background and the question of final achievement involved similar comparisons between the same two populations, a common statistical treatment of both questions was judged appropriate.

A decision was made to use the parametric t-test on both questions rather than non-parametric methods. This decision was based on the fact that the t-test is stronger than the alternative non-parametric methods in that the t-test is more likely to reject H_0 when H_0 is false.[55] Furthermore, the conditions required of the populations to be compared met the standards necessary for application of the t-test. These standards consisted mainly of being able to assume that the variances of the two populations being compared were equal.[56] Other standards of secondary importance such as being able to assume that the sampling was independent[57] presented no problem and were systematically accepted. Preliminary statistical investigations were necessary, however, in order to assume equality of variances of the two populations both in the case of comparing mathematical and intellectual backgrounds as well as comparing final achievement.

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[55] Siegel, Sidney, Nonparametric Statistics for the Behavorial Sciences, p. 19.

[56] Ibid., p. 19.

[57] Ibid., p. 19.

Such statistical investigation did show that one could not reject the hypothesis that the variances of the experimental and control populations in the case of comparing mathematical and intellectual backgrounds differed significantly.[58] Similarly, the same statistical procedure revealed that one could not reject the hypothesis that the variances of the experimental and control populations in the case of comparing final achievements differed significantly.[59]

A condition found listed as being essential to application of the t-test was that the populations being compared must both be normally distributed.[60] However, statistician Cochran in investigating this problem stated:

> The consensus from these investigations is that no serious error is introduced by non-normality in the significance levels of the F-test or the two-tailed t-test . . . . . If a guess may be made about the limits of error, the true probability corresponding to the tabular 5 percent significance level may lie between 4 and 7 percent.[61]

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[58]See p. 63

[59]See p. 65

[60]Siegel, op. cit., p. 19.

[61]Cochran, W. G., "Some Consequences When the Assumptions for the Analysis of Variance Are Not Satisfied," Biometrics Bulletin, Vol. III, No. 1, December, 1946, p. 24.

Although the case is not quite so solid for the one-tailed t-test, the permissiveness of non-normality is much the same. Kendall in summarizing the results of other statisticians who had studied this problem stated that:

> Their results may be broadly summarized by the statement that. . . tests on population means (i.e. "Students" t-tests for the mean of a normal population and for the difference between the means of two normal populations with the same variance) are rather insensitive to departures from normality. . . .[62]

Both arguments that non-normality is not essential for one or two-tailed t-testing stem from the fact that the t-test is based on calculations involving sample means and one would be concerned not so much with the normality of the population, but instead with the normality of the distribution of these sample means. But by the Central Limit Theorem, these sample means are themselves approximately normally distributed.[63]

Hence, the question of normality was not considered of major importance. Furthermore, the data tended to cause the investigator to believe the two populations to be normally distributed. Therefore, the t-test was judged appropriate for investigating the two questions: (1) were the experimental and control groups of comparable mathematical and intellectual backgrounds? and (2) was the final achievement of the experimental group significantly above from that of the control group?

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[62] Kendall, Maurice, The Advanced Theory of Statistics, Vol. II, p. 465.

[63] Freund, John E., Modern Elementary Statistics, p. 205.

Application of statistical methods. In order to best test the question of comparability of the experimental and control groups a null hypothesis, H_0, was formulated: there is no significant difference in mathematical and intellectual background of the experimental group as compared to the control group. The significance level was set at 5 percent and the test, because of the nature of the null hypothesis, was a two-tailed test.

The null hypothesis was tested by application of the t-test which involved calculation of the t statistic using the predicted scores in beginning calculus as measures of the mathematical and intellectual backgrounds of members of the experimental and control groups. Since the number of students in the experimental group was 21 and the number of students of the control group was 22, the t-distribution table was consulted at the 5 percent level with $t(.025, 41) = 2.0195$.

For the calculated t less than 2.0195 or greater than -2.0195 the null hypothesis that there was no significant difference in mathematical and intellectual background of the experimental as compared to the control group could not be rejected.

Next, the second major question was investigated. Namely, a statistical study was made to investigate the relative final achievement of the experimental group as compared to the control group.

Upon receiving the scores of the students of the experimental and control groups earned on the final examination at the end of the quarter the statistical problem of comparison of the final achievement of the experimental and control group was investigated. In recalling the null hypothesis, H_O: there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied from such programmed materials, it was clear that there too the Student t-test could be used. However, this time the t-test was one-tailed.

Again the level of significance was set at 5 percent and $t(.05, 41) = 1.6829$ was the critical value. That is, for the calculated t less than 1.6829 the null hypothesis could not be rejected that there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied from such programmed materials.

Summary

In order to investigate the effect on final achievement in a beginning calculus course resulting from the use of a series of programmed instruction written to supplement the regular classroom instruction on the topics which ordinarily cause learning problems a poll was made of well-qualified mathematicians and scientists familiar with the content and learning problems of a beginning calculus course.

This survey was in turn tabulated to identify the areas of beginning calculus which cause learning problems to students.

The identification of areas causing learning problems led to the preparation of a series of lessons of programmed instruction written to supplement the regular classroom instruction of the topics identified as causing learning problems. These lessons were made available to students enrolled in the beginning calculus class at Montana State College during the fall quarter of 1964.

The students were instructed to note passages of the programmed materials which they could not satisfactorily complete. Using a permissible error ratio of 10 percent the passages causing trouble to student participants were identified.

Prior to the winter quarter of 1965 decisions were made as to appropriate modifications of the preliminary writing. These modifications included: (1) rewriting passages and supplying necessary sketches to reduce the error ratio on troublesome passages below 10 percent, (2) alterations of the lessons by lengthening the time available for the introductory lesson by dropping a lesson which came too late in the quarter to be of benefit, and by condensing the content of two chapters into one, and (3) revising the administration of the materials such that a close control could be assured over the use of the materials by the experimental group.

At the beginning of winter quarter an experimental group was chosen from students volunteering for the investigation. The criteria for selection of the experimental group involved selection of students of demonstrated moderate achievement. Extreme levels of achievement were not allowed because of the possible skewing of final achievement results.

During the quarter as the students of the experimental group met twice weekly to complete the prepared programmed materials, a constant contact was maintained with instructors of the students and others concerned to assure uniform classroom instruction and notational usage.

At the end of the quarter a search was made of the records of all students completing the course to determine which students were qualified as members of a control group under the same criteria imposed on the admission into the experimental group. All such qualifying students were selected and the control group was thereby formed.

Also, at the end of the quarter a final examination meeting the approval of all instructors of the beginning calculus sequence was written and administered to all members of both the control and experimental group. This examination was scored in a uniform manner according to grading standards previously set down by a committee appointed by the supervisor of the beginning calculus sequence.

As a means of comparing the mathematical and intellectual backgrounds of students in the control group to those in the experimental group, the placement figures obtained by the Testing and Counseling Service at Montana State College were used. Since Montana State College had adopted a predictor of success in beginning calculus based on previous mathematical and intellectual background, this predictor was used as a means of comparing the backgrounds of the experimental and control group. The statistical test used to compare the two groups was the Student t-test. This test was chosen due to its applicability to the data and its strength.

Similarly at the end of the quarter the null hypothesis, H_O: there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students not studying from such programmed materials, was tested by the Student t-test. The raw data for this test was supplied by the scores of the final examination given to all members of the experimental and control group.

The results of the investigation are presented in the next chapter.

CHAPTER IV

EFFECT ON FINAL ACHIEVEMENT IN BEGINNING CALCULUS RESULTING FROM THE USE OF PROGRAMMED MATERIALS

In order to present the results of the investigation of the effect on final achievement by students in a beginning calculus course resulting from use of programmed materials administered as a supplement to classroom instruction of topics which ordinarily cause learning problems three areas were considered. These areas were: (1) selection of the experimental and control groups, (2) comparability of the experimental and control groups, and (3) the effect on final achievement resulting from the use of the programmed materials.

Selection of the Experimental and Control Groups

The experimental group was selected from student volunteers from the beginning calculus class at Montana State College during the winter quarter of 1965.[64] Fifty-five students made application for admittance into the experimental group. Forty-seven of these applicants were selected under the adopted criteria[65] as participants in the programmed instruction investigation. Participation in the program was strictly voluntary. Twenty-one of these forty-seven students remained with the program throughout the quarter and systematically worked all of the programmed materials. Statistical

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[64]See p. 43

[65]See p. 42 and 43

measurements involving the effect of programmed materials were based on the final achievements of the twenty-one students who completed the entire sequence of programmed lessons.

The entire roll of students who completed the beginning calculus course was surveyed at the end of the quarter to identify those students eligible for the control group. Criteria for selection of the control group[66] reduced the potential membership to twenty-two students. Since all twenty-two of these students qualified as members of the control group, all were selected as control group members.

Thus the comparison of the final achievement of the experimental and control groups involved forty-three students. The twenty-one students in the experimental group had completed the beginning calculus course and in addition had completed all of the programmed materials. The twenty-two students in the control group had studied the same calculus course but their classroom instruction was not supplemented with programmed materials.

It was recognized by the author that the voluntary nature of the experimental group might introduce two biases which could effect the results of the investigation. First, it was foreseen that student volunteers may have applied for acceptance into the experimental group with the intent of learning calculus while not exerting a normal effort. Second, it was foreseen that the student volunteers who applied for acceptance into the experimental group might be comparatively

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[66] See p. 45

more enthusiastic students with more than the average desire to learn the subject matter.

Both of these biases were beyond the control of the experimenter. However, it was hoped that the number and duration of intensive study sessions would reduce the number of students that anticipated learning calculus with little or no effort. Since twenty-six students dropped the program, there was reason to believe that those students seeking learning without effort were substantially eliminated from the experimental group.

If the second bias were present, it was not evident in the results of the investigation.

Comparability of the Experimental and Control Groups

In order to study the effect resulting from the use of programmed materials on final achievement of the experimental and control groups it was first necessary to determine the comparability of the two groups. This comparability was based on a study of composite scores which indicated the mathematical and intellectual backgrounds of the students. Nearly all of these scores had been previously calculated by the Testing and Counseling Service at Montana State College and appeared in the Montana State College Placement Test Scores and Indices, September, 1964.[67] The student scores needed for the study which

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[67] Brookhart, M. E. and Suvak, Albert, Montana State College Placement Test Scores and Indices, September, 1964, 52 pp.

did not appear in the Montana State College test index booklet were calculated using the same formula as was regularly used to calculate these scores.[68]

The appropriateness of the t-test was investigated as a means of comparing these scores[69] which were measures of the mathematical and intellectual backgrounds of students in the experimental and control groups. Establishment of the appropriateness of the t-test was dependent upon permissiveness of assuming equality of variances of the two sample populations.[70] Hence, the question of assuming equality of variances was investigated by testing the hypothesis: the variances of the scores measuring the mathematical and intellectual backgrounds of the experimental and control populations were the same at the 5 percent level.

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[68]Predicted Mathematics 121 score = .00416 (total score on A. C. E.) - .013151 (Minn. Schol. Apt. Test) +.00012 (M. S. C. Arithmetic Test) +.02755 (M. S. C. Algebra I Test) +.02061 (M. S. C. Algebra II Test) +.01250 (Mech. Comp. Test) +.06899 (High School G. P. A.) -1.16826. (This formula was developed by Albert Suvak of the Montana State College Testing and Counseling Service to predict achievement of students enrolling in Mathematics 121.)

[69]See Appendix G , Table 1.

[70]Siegel, op. cit., p. 19

The test applied to this hypothesis was the two-tailed F-test.[71] The calculated F statistic was .65516. Since this value was between $F(.975, 20, 21) = .40816$ and $F(.025, 20, 21) = 2.42$, it was not possible to reject the hypothesis that the variances were equal at the 5 percent level. Thus it was concluded that the variances of scores measuring the mathematical and intellectual backgrounds of the experimental and control populations do not differ significantly.

The conclusion that the variances of the two populations did not differ significantly enabled the investigator to contend that the t-test was appropriate for testing the hypothesis: there was no significant difference in the mathematical and intellectual backgrounds of the experimental and control groups. This test was a two-tailed test and was accomplished by comparing the means of the sample popu-

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[71] The steps used in computing the two-tailed F statistic at the 5 percent level are as follows:

1. Determine the number of cases in groups I and II being compared. Denote them n_1 and 2_2 respectively.
2. Denote the observations in groups I and II by y_{1i} and y_{2j} respectively where $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.
3. Calculate: $\overline{Y_1} = \sum_{i=1}^{n_1} y_{1i}/n_1$ and $\overline{Y_2} = \sum_{j=1}^{n_2} y_{2j}/n_2$
4. Calculate:
$$F = \sum_{i=1}^{n_1} (y_{1i}-\overline{Y}_1)^2/(n_1-1) \div \sum_{j=1}^{n_2} (y_{2j}-\overline{Y}_2)^2/(n_2-1)$$
5. The F statistic of (4) is distributed as an F with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom.

6. Accept the null hypothesis if the calculated F in (4), F_o, is such that: $F(.975, v_1, v_2) < F_o < F(.025, v_1, v_2)$, otherwise reject the null hypothesis. (Ostle, Bernard, Statistics in Research, pp. 91-93.)

lations according to the procedure of calculating the t statistic.[72]

It was found that the t statistic calculated from the data listed in Table A was -.69982. Since this value was smaller than t(.025, 41) = +2.0195 and larger than -t(.025, 41) = -2.0195, the hypothesis was accepted. That is, the investigator expected that 95 percent of all samples compared (the samples each consisting of an experimental and a control group) would have no significant difference in their mathematical and intellectual backgrounds.

The Effect on Final Achievement Resulting From the Use of Programmed Materials

After establishing the comparability of the experimental and

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[72] The steps used in computing the two-tailed t statistic at the 5 percent level were as follows:

1. Determine the number of cases in groups I and II being compared. Denote them n_1 and n_2 respectively.
2. Denote the observations in groups I and II by y_{1i} and y_{2j} respectively where $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.
3. Calculate: $\overline{Y_1} = \sum_{i=1}^{n_1} y_{1i}/n_1$ and $\overline{Y_2} = \sum_{j=1}^{n_2} y_{2j}/n_2$.
4. Calculate: $\sqrt{\frac{n_1\, n_2}{n_1+n_2}}\,(\overline{Y}_1-\overline{Y}_2) \Big/ \sqrt{\frac{\sum_{i=1}^{n_1}(y_{1i}-\overline{Y}_1)^2 + \sum_{j=1}^{n_2}(y_{2j}-\overline{Y}_2)^2}{n_1+n_2-2}}$
5. The t statistic of (4) is distributed as a t with n_1+n_2-2 degrees of freedom.
6. If the calculated t statistic, t_o, is such that: $-t(.025, n_1+n_2-2) < t_o < t(.025, n_1+n_2-2)$, accept the null hypothesis. Otherwise reject the null hypothesis. (Mood, A. M. and Graybill, F. A., Introduction to the Theory of Statistics, pp. 306-307.)

control group by statistical means it was then possible to begin investigation of the main hypothesis: there is no significant difference in the achievement of students in a beginning calculus course who have studied programmed learning as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials.

Again it was desirable to use the parametric t-test in comparing the final achievements of the control and experimental groups. Again the appropriateness of the test had to be justified by demonstrating reason for assuming equality of variance for the final achievements of the control and experimental populations. This assumption was justified as a result of studying the achieved scores of the two sample populations.[73] That is, in order to show reason for assuming equality of variances for the two populations a test was made of the hypothesis: there is no significant difference between the variances of the final achievement scores in beginning calculus for the experimental population and the control population. The test was a two-tailed F-test and resulted in the F statistic being calculated as 1.267. Since this value was between $F(.975, 20, 21) = .40816$ and $F(.025, 20, 21) = 2.42$, it was not possible to reject the hypothesis that the variances were equal at the 5 percent level. Thus it was assumed that the variances of the final achievement scores of the experimental and control populations did not differ significantly.

[73] See Appendix G, Table 2.

By assuming that the variances of the two populations were not significantly different the investigator was then able to contend that the t-test was appropriate for testing the main hypothesis: there is no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials.

The t-test was one-tailed because of the nature of the hypothesis. On this basis the t statistic was calculated[74] from the data of Table B and was found to be $t = .60602$. Since this value was less than $t(.05, 41) = 1.6829$, the hypothesis was accepted. That is, the investigator expected that in 95 percent of all samples studied (the samples each consisted of one experimental and one control group) the experimental group would not have a final achievement mean greater than the mean of the final achievement of the control group.

Summary

Experimental and control groups were selected from the population of students in a beginning calculus course to compare the effect on final achievement in beginning calculus due to the use of programmed

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[74] The calculation of the one-tailed t statistic followed the procedure of calculating the two-tailed t statistic. (Ostle, op. cit., pp. 98-102.)

materials. Twenty-one of forty-seven students in the experimental group completed the programmed materials. The final achievement of these students was compared to the final achievement of the control group, consisting of twenty-two students, in order to provide reason for stating conclusions about the experimental and control populations.

In order to compare the final achievements of the experimental and control groups the comparability of the two groups had to be first established. This was done by application of the two-tailed t-test to the predicted achievement scores of the individuals in the control and experimental groups. These scores were calculated by using the formula developed and approved by the Montana State College Testing and Counseling Service. The appropriateness of the test was previously established by demonstrating permissiveness of assuming equality of variances for the two populations by F-test.

After establishing the comparability of the two groups the major hypothesis was considered: there is no significant difference in the achievement of students in a beginning calculus course who have studied programmed learning as a supplement to conventional class-room instruction of topics which ordinarily cause learning problems over students not studying such programmed materials.

This time the one-tailed t-test was needed for testing the hypothesis and hence was shown appropriate by establishing permissiveness of assuming equality of variance for the two groups by F-test. Since the one-tailed t-test was appropriate, it was applied and subsequently demonstrated the acceptability of the hypothesis.

CHAPTER V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

The purpose of this investigation was to compare the results obtained from the use of programmed materials as a supplement to classroom instruction of topics in a beginning calculus course which ordinarily cause learning problems with results obtained in classes not using programmed materials.

The need for such a study was in part due to the changes in mathematical emphasis which have accompanied the transition into a modern scientific-technological era. These same changes in mathematical emphasis have seriously effected the need for more effective means of increasing mathematical learning-teaching effectiveness. As one means of increasing the effectiveness, this investigation concentrated on the problems associated with teaching beginning calculus at the early college level. It was believed by the author that appropriate programmed materials could be used effectively by students in a beginning calculus course to supplement their regular classroom instruction and hence improve their final achievement. The purpose of this study was to investigate this belief.

Summary

Four major points became clear to the author from the review of the literature which was carried out as a preliminary step in this investigation.

1. Beginning calculus has experienced a radical change in the past eight years from a development depending heavily on solving particular problems to a careful structural development achieved through student understanding and logical reasoning.

2. Many of the learning problems in beginning calculus have been brought about by a recent change from traditional mathematics to modern mathematics. This change has been accompanied by revolutionary teaching methods and emphasis. Thus many students whose preparation has been traditionally oriented have been unsuccessful in making a transition to the type of thinking and rigor expected in modern mathematics.

3. The use of programmed instruction possibly can facilitate correction of the imbalance in background experienced by traditionally prepared students through development of proper notation, but evidence indicated considerable disagreement as to whether programmed instruction could entirely fill the gap in developing proper mathematical attitude such as is required in mathematical reasoning.

4. The reports of the investigations into the learning value of programmed instruction have not consistently described it as being infallible. Instead, a number of comparisons of its value over conventional classroom instruction indicated that the differences in final learning achieved through programmed instruction were not significantly greater than those differences achieved by conventional classroom teaching.

In the course of reviewing the literature no reports of investigations were found which described efforts to study the effect on final achievement of students in a beginning calculus course which had supplemented their conventional classroom instruction with programmed instruction. Hence, the author proceeded to investigate this problem. The first step in the investigation was the identification of those areas in beginning calculus which ordinarily cause learning problems. This was done by polling twelve college instructors familiar with the learning problems of students in calculus. Then a schedule of lesson involving supplementary programmed materials was drawn up to cover the topics which were identified as being troublesome.

The author consulted various standard calculus texts in order to assure a complete and effective presentation of each topic. Then he developed each topic by carefully writing a preliminary set of programmed materials. The preliminary materials were then tested for effectiveness by conducting a pilot study. This study pointed out specific needs for improved writing of certain passages, improved administrative techniques, and improvements regarding arrangements of topics to be covered. With these factors in mind the final writing of the materials was accomplished.

The author selected an experimental group during the winter quarter of 1965 at Montana State College on the basis of their previous mathematical achievement. A schedule of meetings was drawn up, and the experimental group began using the programmed materials as a supplement to regular classroom instruction. At the conclusion of the quarter a control group was chosen using criteria similar to those used to select the experimental group. Then a final achievement examination was administered to both the control and experimental groups. Appropriate statistical procedures were applied to the test scores.

The statistical comparisons indicated that although the experimental and control populations were of comparable mathematical and intellectual backgrounds, the final achievements of students in a beginning calculus course studying programmed materials administered as a supplement to a conventional classroom instruction of topics which ordinarily cause learning problems were not significantly above the final achievements of students not studying such materials.

Conclusions

The following conclusions regarding the teaching of a beginning calculus course were evident to the investigator as a result of reviewing the literature:

1. Because scientific-technological advancements have placed a greater emphasis on the mathematical preparation of students and this emphasis has in turn influenced the manner of teaching mathematics, students in the beginning calculus courses are having trouble learning the concepts of mathematics. Hence, a need exists to improve the teaching of the basic concepts of beginning calculus.

2. The research done on the effectiveness of supplementing classroom teaching of mathematics by the use of programmed materials is inconclusive and additional research needs to be done.

The author concluded as a result of his investigation that statistically there is very little reason to believe that the effect on the final achievements of students in a beginning calculus course resulting from the study of programmed materials administered as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems would be significantly above the corresponding achievements by students not studying such materials.

Recommendations

As a result of this investigation four recommendations were formulated:

1. Because a greater number of students fail to satisfactorily learn the material in a beginning calculus course, more research is needed to identify the factors contributing to this lack of learning. In particular, research needs to be done in this area with full understanding of the changing mathematical background of the students. That is, the background of the entering college students is currently changing from being oriented in traditional mathematics to an orientation involving modern mathematics.

2. Research needs to be done to find ways of improving the teaching of beginning calculus. The average beginning calculus student was found to be interested in learning the material but in many cases the learning by these same students was found to be incomplete. Thus, research into the improvement of teaching calculus and hence into the improvement of mathematical communication at the level of beginning calculus was found desirable.

3. First, evidence was found in the literature to indicate that students can learn some things by studying appropriate programmed materials. Research needs to be carried out which will identify the areas of beginning calculus which can be effectively learned by studying programmed materials.

Second, once the areas of beginning calculus which can effectively be taught through the use of programmed materials have been identified, research needs to be done to discover the most efficient way to use programmed materials so that maximum learning of beginning calculus will result.

4. The background of students in the beginning calculus course at the time of this study was in general traditionally oriented. With the current emphasis on a modern structural approach to mathematics at all levels of education, learning problems of beginning calculus will likely change as students' background reflect the modern emphasis involving notation, the structure of mathematics, and the patterns of thought essential in successful reasoning in mathematics. Hence, the investigator makes the recommendation that this investigation be

replicated at a future date when experimental and control groups can be chosen from a population of students whose background is molded by modern mathematical training.

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APPENDIX

Appendix A

Instructions to the Panel of Judges which Selected the Topics to be Programmed

Instructions to the Panel of Judges Which Selected the Topics to be Programmed

Certainly there are topics which students enrolled in the beginning calculus sequence are expected to master. Obviously an inappropriate amount of difficulty is experienced by many students in attaining these goals. This situation has perpetuated itself despite attempts to match student ability with course content and to improve classroom teaching. However, despite attempted solutions the problem of too many unsuccessful students still exists. Hence it is clear that further investigation into ways to improve student learning and thus success are in order. I have chosen such an investigation as a problem for my doctoral thesis and request your opinion on one important point.

You see, as a means of aiding student learning, I intend to prepare a set of programmed materials which will be focused on the critical areas of beginning calculus which display the greatest learning problems. It is intended that such supplementary materials will do much to clear up the major concepts which stand before understanding in these vital areas.

Since student time is limited, so too must the selection of topics prepared be limited. Hence, the following list of topics studied in the beginning calculus sequence is presented for your inspection. In order that the most vital areas can be concentrated on, it is requested that you select the seventeen areas which you feel deserve the greatest emphasis. In addition to you, other individuals well qualified to judge the learning problems of students in calculus will be similarly polled. Then those topics which indicate the heaviest need for concentration will be selected as the areas to be developed.

Please check the seventeen topics which you feel are most difficult for students to master:

1. The locus problem.
   What is the graph of an equation and what is the equation of a graph; slope; equations of straight lines, parallel lines, perpendicular lines; distance formula and circle; parabola; ellipse; hyperbola; related problems.

2. Functions and functional notation.
   Cartesian cross product; relation; function; associated notation; related problems.

3. Intersections of pairs of curves.
   Sets of points satisfying equations individually and simultaneously; introduction of extraneous solution points; related problems.

4. Inequalities and absolute value.
   Definition of inequality symbols; basic theorems of inequalities; definition of absolute value; triangular inequality; related graphing; related problems.

5. Limits.
   Algebraic interpretation; geometric interpretation; basic theorems of limits; related problems.

6. Continuous functions.
   Definition of continuity at a point; definition of continuity over an interval; meaningful examples of continuous and discontinuous functions; related problems.

7. The derivative.
   Average rates of change; instantaneous rates of change; slope of a curve at a point; average and exact velocity; definition of the derivative; mechanics of differentiation when proceding from the definition; related problems.

8. Simple differentiation formulas.

   $D_x c$; $D_x cx^n$; $D_x (u+v)$; $D_x uv$; $D_x u/v$; related problems.

9. Differentiation of the product, quotient, and higher order derivatives.
   Development of product and quotient formulas; higher order derivatives; related problems.

10. Differentiation of composite functions.
   Proof of chain rule; use of chain rule; related problems.

11. Implicit differentiation.
   Discussion of implicit functions; related problems.

12. Application of the derivative to linear motion, motion under gravity, and related rates.
   Linear motion; motion under gravity; related rate discussion; related problems.

13. Sketching as related to increasing and decreasing functions, critical values, and extreme values of a function.
   Relation of the derivative to increasing and decreasing functions; use of the first derivative to isolate critical points; the use of the second derivative test; curve plotting using the derivatives; related problems.

14. Rolle's Theorem and Mean Value Theorem.
   Statement and proof of Rolle's Theorem; statement and proof of Mean Value Theorem; application of Mean Value Theorem to prove other basic theorems; related problems.

15. Applications involving the theory of extremes.
   Max - min problem types; problems.

16. Approximation by differentials.
   Definition of differentials; geometric interpretation of differentials; approximate change and differentials; related problems.

17. The indefinite integral.
   Meaning of indefinite integral; integration and differentiation formulas; constant of integration; related problems.

18. Computation of area by integration.
   Concept of area; relation of the integral to area under a curve; computation of areas bounded by more complicated curves; related problems.

19. The definite integral.
   Meaning of the definite integral; geometric interpretations; important properties of the definite integral; computation of the definite integral; related problems.

20. Area as the limit of a sum.
    Proof that area is the limit of a sum; related problems.

21. Fundamental theorem of calculus.
    Proof of the fundamental theorem; related problems.

22. Mathematical induction.
    Discussion of nature of mathematical induction; examples of mathematical induction properly and improperly applied; related proofs by mathematical induction.

Appendix B

Programmed Materials Studied by the Experimental Group

Lesson I

INEQUALITIES

1. Frequently you will find it necessary to use and understand the symbols: $>$, $<$, $\geqslant$, and $\leqslant$ in connection with the real numbers (the numbers which signify the distances to the right or left of 0 on the number line). The first symbol, $>$, is read: "is greater than". The second symbol, $<$, is read: "is less than". How would "$a > b$" be read?

- -

"a is greater than b".

---

2. How would "$a < b$" be read?

- -

"a is less than b".

---

3. The symbol $\geqslant$ is read: "is greater than or equal to". Correspondingly $\leqslant$ is read: "is less than or equal to. How would "$a \geqslant b$" be read?

- -

"a is greater than or equal to b".

---

4. How would "$a \leqslant b$" be read?

- -

"a is less than or equal to b"

---

5. Translate the following true statements into symbolic representation:

(1) 4 is greater than 1.
(2) 4 is less than 5.
(3) 6 is greater than or equal to 5.
(4) 6 is greater than or equal to 6.
(5) 5 is less than or equal to 7.
(6) 5 is less than or equal to 5.

- -

$4 > 1$
$4 < 5$
$6 \geqslant 5$
$6 \geqslant 6$
$5 \leqslant 7$
$5 \leqslant 5$

---

6. Translate the following symbolic representation into true statements:

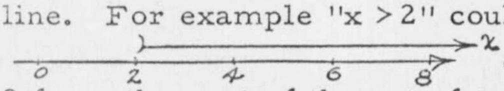
(1) $7 > 4$ (4) $2 \leq 3$
(2) $6 < 8$ (5) $3 \geq 3$
(3) $3 \geq 2$ (6) $2 \leq 2$

- -

"7 is greater than 4".
"6 is less than 8".
"3 is greater than or equal to 2".
"2 is less than or equal to 3"
"3 is greater than or equal to 3".
"2 is less than or equal to 2".

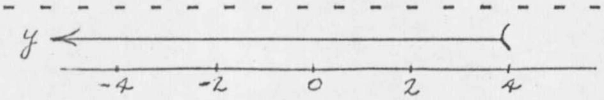
---

7. Often such symbols are used in conjunction with variables and the value of the variable can be indicated graphically on the number line. For example "$x > 2$" could be sketched as follows:



The symbol) separates the point 2 from the rest of the set of points to the right of 2. Indicate graphically $y < 4$.

- -



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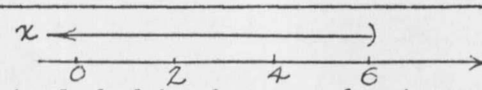
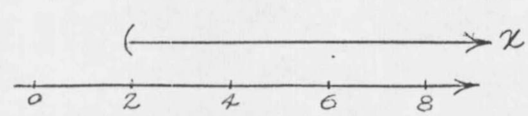
8. Similarly one could sketch $x \leq 6$ as:



Here the symbol) indicates that 6 is included in the set of points. Indicate graphically $x \geq 2$.

- -



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9. Sometimes in writing mathematical statements it saves time to group two statements involving inequality into one statement. For example, $6 < x$ and $x \leq 7$ could be written: $6 < x \leq 7$. Then reading from left to right: "6 is less than x is less than or equal to 7". Or to indicate that both conditions are fulfilled often this statement is read: "6 is less than x and x is less than or equal to 7", or simply: "x is greater than 6 but less than or equal to 7". What would "$3 \leq x < 5$" mean?

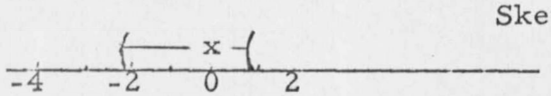
- -

x is greater than or equal to 3 but less than 5.

---

10. Now combining both symbols in one statement the sketch of $-2 \leq x < 1$ would appear:

Sketch: $1 < x \leq 3$.



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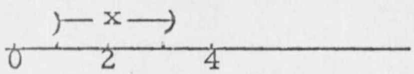

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11. What would you conclude about x if: $y \leq x \leq y$?

- -

$x = y$

---

12. Could it be true that $x < y \leq x$? Why?

- -

No. This would state that x is less than itself which is not possible for real numbers.

---

13. For purposes of mathematical proof it will be convenient to take as a definition that $x < y$ and $y > x$ are equivalent statements and that both mean that $y - x$ is positive or $y - x > 0$. What does $c < d$ mean?

- -

$d - c > 0$.

---

14. From this simple definition and the basic properties of real numbers (the sum of two positive reals is again a positive real, the associative laws for real numbers, the commutative laws for real numbers, etc.) many facts can be proven. We shall prove that: "if $a > b$ and $c > d$, then $a+c > b+d$".

Why is the argument that $3 < 4$ and $5 < 6$ and surely $3+5 < 4+6$ not a valid proof of the stated theorem?

- -

This argument is only good for one special case - not for every case as is desired.

---

15. In order to prove the theorem, start as follows: if $a > b$, then $a-b > 0$. Also, if $c > d$, then $c-d > 0$. Why is each statement true?

- -

According to definition (13)

---

16. But from (15): $(a-b) + (c-d) > 0$. Why?

- -

The sum of 2 positive real numbers is again a positive real number.

---

17. Hence from (16): $(a+c) - (b+d) > 0$. Why?

- -

$(a-b) + (c-d)$ is the same as $(a+c) - (b+d)$ by permissible algebraic operations on real numbers. (Actually this involves associative and commutative laws and the properties of -.)

---

18. But (17) contends that: $a+c > b+d$. Why?

- -

The definition (13) contends that $y-x > 0$ means $y > x$.

---

19. Hence you have verified the theorem you set out to prove. A similar theorem is: "if $a > b$ and c is any real number, than $a+c > b+c$". The statements for the proof of this theorem follow. Supply the reasons.

Statements	Reasons
1. $a > b$ implies $a-b > 0$	1.
2. $+c-c = 0$	2.
3. $a-b+c-c > 0$	3.
4. $(a+c)-(b+c) > 0$	4.
5. $a+c > b+c$	5.

- -

1. Definition of >
2. Properties of real numbers
3. Properties of real numbers
4. Properties of real numbers
5. Definition of >

---

20. Suppose you were asked to prove: "if $a > b$ and $c > 0$ where a, b, and c are real numbers, $ac > bc$", a good place to start would be with the given fact $a > b$. What can be concluded from: $a > b$?

- -

$a > b$ means $a-b > 0$.

---

21. How would you reason from the facts: $a-b > 0$ and $c > 0$ to get $c(a-b) > 0$?

- -

$c(a-b) > 0$ because two positive real numbers have a positive product

---

22. Why is it true that $c(a-b) = ac-bc$?

- -

Permissible algebraic operations on real numbers are used. (The reason really depends on the distributive and commutative laws for real numbers.)

---

23. Now using (21) and (22) one can say: $ac > bc$. Why?

- -

Definition 13 says that $y-x > 0$ means $y > x$.

---

24. An interesting twist to the theorem stated in (20) is achieved by setting c negative instead of positive. What would happen to the relationship between ac and bc if $a = 4$, $b = 3$, and $c = -1$?

- -

$ac < bc$

---

25. On the basis of step (24) complete the theorem: "if $a > b$ and $c < 0$, ____________."

- -

$ac < bc$

---

26. The proof of this theorem could be made to closely follow that of theorem (20) by one small step: if $c < 0$, then $-c > 0$ by the properties of real numbers. Using this innovation prove: if $a > b$ and $c < 0$, $ac < bc$.

- -

1. $a > b$ implies $a-b > 0$	1. Definition of >
2. $c < 0$ implies $-c > 0$	2. Properties of real numbers - negative of a negative is positive
3. $\therefore -c(a-b) > 0$	3. Product of two positives is positive
4. $\therefore -ac + bc > 0$ or $bc - ac > 0$	4. Properties of real numbers - distributive and commutative laws for reals
5. $\therefore bc > ac$ or $ac < bc$	5. Definition of >

---

27. Now you should prove one alone to be sure you have the technique down solid. Prove: if $a < b$ and $b < c$, then $a < c$.

- -

1. $a < b$ means $b-a > 0$; $b < c$ means $c-b > 0$	1. Definition of $<$
2. $\therefore (b-a) + (c-b) > 0$	2. Sum of 2 positives is positive
3. $\therefore c-a > 0$	3. Simplification of step 2 - actually involves the idea of an additive inverse
4. $\therefore c > a$ or $a < c$	4. Definition of $>$

---

28. From the three theorems:
(a) if $a > b$ and c are real numbers, $a+c > b+c$;
(b) if $a > b$ and $c > 0$, $ac > bc$;
(c) if $a > b$ and $c < 0$, $ac < bc$;
many inequalities of a single variable can be solved. Use theorem (a) to solve: $x-3 > 4$.

- -

$x > 7$

---

29. Solve by using theorem (b): $3x < 5$.

- -

$x < 5/3$

---

30. Solve by using theorem (c): $-3x < 5$.

- -

$x > -5/3$

---

31. Solve: $-2x + 7 \geqslant 4x - 5$.

- -

$x \leqslant 2$

---

32. Of course there are many proofs involving inequalities which await investigation by the curious student. Let's look at another general type.

Suppose you were asked to prove: "if a and b are positive numbers, $a^2 + b^2 \geqslant -2ab$". You might start by writing $(a+b)^2 \geqslant 0$. Why would this be true?

- -

The square of any real number is positive or zero.

---

33. Of course this is equivalent to stating $a^2+2ab+b^2 \geqslant 0$. But from this statement one can easily conclude $a^2+b^2 \geqslant -2ab$. Why?

- -

$x-y \geqslant 0$ implies $x \geqslant y$.

---

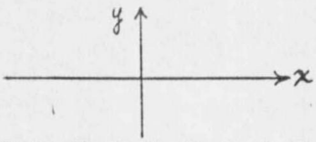
34. Now you try one. Prove: $a^2 + b^2 \geq 2ab$.

- -

1. $(a-b)^2 \geq 0$	1. Square of any real is positive
2. $\therefore a^2 + b^2 \geq 2ab$	2. $x - y \geq 0$ means $x \geq y$.

---

35. By now you should have the general notion as to how inequality proofs proceed. So let's turn our attention now to a two dimensional geometric interpretation of inequalities. For example consider $y > x$. One might wonder if such a condition could be geometrically represented - and the answer is yes. First, sketch $y = x$.



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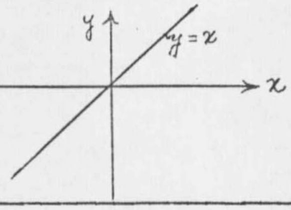

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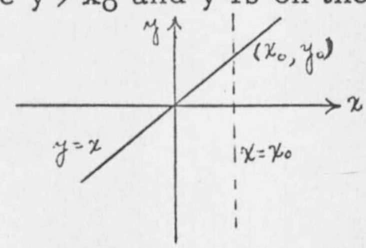
36. Now look at the sketch. Every point on that line is located such that the ordinate is exactly the same as the abscissa. On the following sketch consider in particular the point indicated on $y = x$ at $x = x_0$. On the line $x = x_0$ shade in those points where the ordinates are greater than x_0. (i.e. Shade in those points where $y > x_0$ and y is on the line $x = x_0$.)



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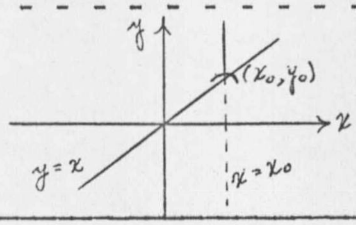

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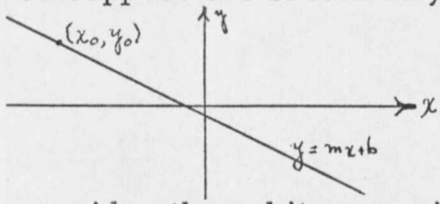
37. Of course there is nothing special about the particular position of (x_o, y_o) on the line - it could be anywhere on the line. But what is important is the relative position of the points where $y > x$! Where would all such points be?

- -

Above the line $y = x$.

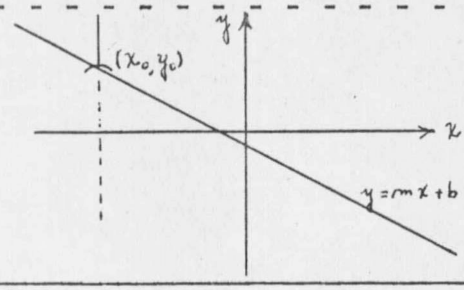
---

38. Let's get a little more variety in our analysis. Consider any straight line on the coordinate system - except the vertical lines. It is a well known fact from high school mathematics that every such line can be represented with an equation of the form $y = mx + b$. Our goal is to discover those points where $y > mx + b$. To do so suppose the sketch of $y = mx + b$ is as follows:



Again consider the arbitrary point (x_o, y_o) on the line and sketch in those points where $y > y_o = mx_o - b$. (Again consider y to be on the line $x = x_o$.)

- -



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39. Now go one step further and answer the general question: where are the points which cause the inequality $y > mx + b$ to be true?

- -

Above the line $y = mx + b$.

---

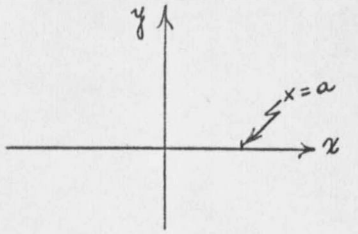
40. Correspondingly you should be able to answer the question: where are the points where $y < mx + b$? Do so.

- -

Below the line $y = mx + b$.

---

41. Although we have neglected the case of the vertical line in our consideration, it comes as a bonus to our one-dimensional work. That is, whereas in one dimension when we wanted all points such that $x > a$, we considered the points to the right of the real number line. Now we want a two dimensional consideration of the phenomenon $x > a$. Shade the area where $x > a$.



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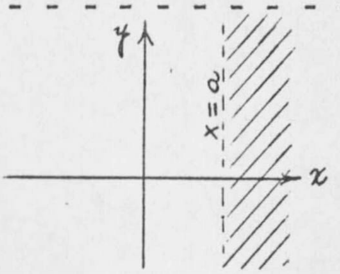

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42. Also, you should be able to conclude the position of the points where $x \leqslant a$. Where would they be?

- -

On the line $x = a$ and to the left of the line $x = a$.

---

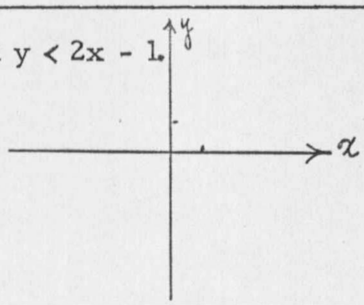
43. You see! Any straight line $y = mx + b$ separates the plane into two halves where the points on one side satisfy the inequality $y > mx + b$. What inequality do the points on the other side satisfy?

- -

$y < mx + b$.

---

44. Sketch $y < 2x - 1$.



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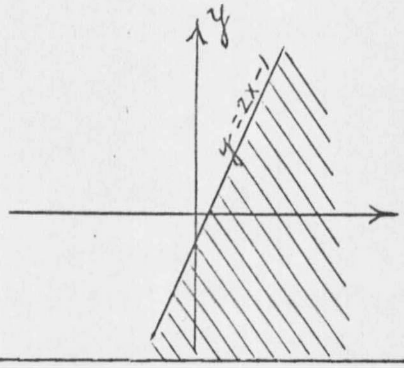
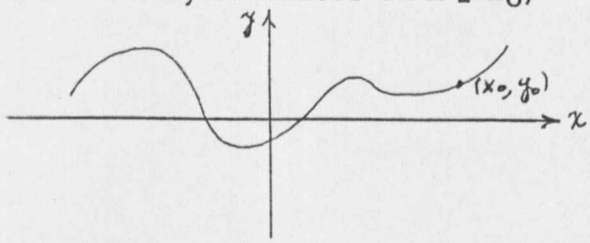


45. But what are we to do with other inequalities like $y > x^2$? To simplify our work let's just consider simple polynomial curves: $y = ax^n + bx^{n-1} + cx^{n-2} + \ldots + k$. Suppose we had sketched $y = ax^n + bx^{n-1} + \cdots + k$. At the point (x_o, y_o) on the curve where would $y > ax_o^n + bx_o^{n-1} + \ldots + k$ be? Sketch it in. (Consider only ordinates on $x = x_o$)



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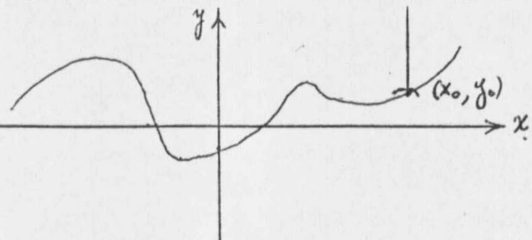
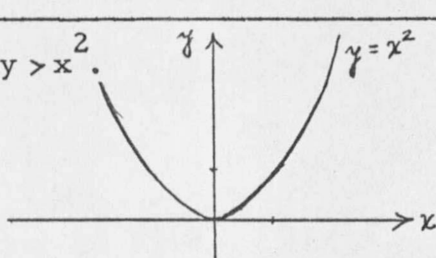


46. Now be a bit more general and specify where all points $y > ax^n + bx^{n-1} + \ldots + k$ would be.

- -

Above the curve $y = ax^n + bx^{n-1} + \ldots + k$.

47. Sketch: $y > x^2$.



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