Uniformly accelerated solutions of Einsteins Equations
by Timothy James Henline

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

Abstract:
By examining the effect of Cosgrove's transformation, Q4s' when acting on some simple space-times, the argument is put forward that the new solutions which are generated represent the space—times due to linearly accelerated axially symmetric sources. The procedure used is to look at the effect of the transformation on some simple space—times and compare the results with space-times which are known to represent accelerated sources. The effect of the transformation on static solutions is carefully examined. The author feels the transformation generates the metric of linearly accelerated sources.

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ABSTRACT

By examining the effect of Cosgrove's transformation, Q_{4s}, when acting on some simple space-times, the argument is put forward that the new solutions which are generated represent the space-times due to linearly accelerated axially symmetric sources. The procedure used is to look at the effect of the transformation on some simple space-times and compare the results with space-times which are known to represent accelerated sources. The effect of the transformation on static solutions is carefully examined. The author feels the transformation generates the metric of linearly accelerated sources.

CHAPTER 1

INTRODUCTION

In all theories of gravity, an attempt is made to predict the gravitational influence of an object on its surroundings. This influence is most often revealed by how the object affects the trajectories of test particles in its vicinity. Newton's theory of gravity says that an object sets up a gravitational field in its vicinity described by a scalar function $\phi(\underset{\sim}{x})$. In a region of space containing a mass density $\sigma(\underset{\sim}{x})$, the Newtonian gravitational potential obeys Poisson's equation in the form

$$\nabla_3^2\phi(\underset{\sim}{x}) = 4\pi G\sigma(\underset{\sim}{x}). \qquad (1.1)$$

For regions of space containing no mass, $\phi(\underset{\sim}{x})$ obeys Laplace's equation

$$\nabla_3^2\phi(\underset{\sim}{x}) = 0 \;. \qquad (1.2)$$

The force of gravity on a test particle of mass m is

$$\underset{\sim}{F} = -m\nabla\phi. \qquad (1.3)$$

A particular solution to Poisson's equation is expressed as an integral over the mass density

$$\phi(\underset{\sim}{x}) = -G\iiint\sigma(\underset{\sim}{x}')\,|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}\, d^3x'. \qquad (1.4)$$

For points $\underset{\sim}{x}$ outside the mass density, this solution obeys

Equation (1.2).

Einstein's General Theory of Relativity says that the gravitational influence of an object is described by the four-dimensional space-time geometry in the vicinity of the object. The mathematical function which gives information on the space-time geometry is the metric tensor. The components of the metric tensor are denoted by $g_{ij}(x^k)$. Two points in space-time which have an infinitesimal coordinate separation dx^i are said to have a space-time interval between them ds, where

$$(ds)^2 = g_{ij}dx^i dx^j \quad . \tag{1.5}$$

Equation (1.5) is the most common way of exhibiting the components of the metric tensor. In most cases when a space-time interval of the form of Equation (1.5) is given, one is said to have a metric with respect to a given set of coordinates x^i. The Einstein field equations are a set of non-linear, partial differential equations which the components of the metric tensor must obey. Once the components of the metric tensor are known, the trajectories of test particles through space-time may be determined (Ohanian,1976,p.203).

Before continuing, we need to explain some terminology. When a metric tensor satisfying the Einstein field equations is known in closed form one is said to possess an 'exact solution'. Until recently, the number of exact solutions was small. The reason was that the field equations were so complicated that straightforward solution was impossible for all but a few cases. Direct solution often depends on the ability to make certain simplifying assumptions about the metric tensor or the form of the space-time interval. This work will consider a group of exact solutions which describe the space-time exterior to an axially symmetric source of gravity. There is a second term which needs explanation. The phrase 'solution generation technique' refers to the creation of new exact solutions by methods other than the direct solution of the differential equations. Often an initial 'seed' solution is manipulated to give a new exact solution. The advantage in acquiring solutions in this way is that one avoids solving non-linear partial differential equations.

STATIONARY, AXIALLY SYMMETRIC SPACE-TIMES

The notation introduced here follows the concise introduction of Kinnersley (1977,pp.1529-1530) and the more detailed introduction of Kramer (1980,Ch.17) The space-times to be considered are stationary and axially symmetric. A stationary space-time is one for which there exists a time coordinate, $x^1 = t$, such that all g_{ij} are independent of t, that is, $g_{ij,t} = 0$ where the comma denotes the partial derivative. An axially symmetric space-time is one in which all components of the metric tensor are independent of an azimuthal angle coordinate $x^2 = \phi$ $(0 < \phi < 2\pi)$, $g_{ij,\phi} = 0$. Therefore, in a stationary, axially symmetric space-time the components of the metric tensor are functions only of the two remaining spatial coordinates x^3 and x^4.

$$g_{ij} = g_{ij}(x^3,x^4) \qquad i,j = 1,2,3,4 \tag{1.6}$$

The space-time is also assumed to possess a property called orthogonal transitivity. This means the space-time interval, ds^2, is unchanged when one makes the coordinate transformation $(\phi,t) \rightarrow (-\phi,-t)$. Physically, the space-time is unchanged when the sources undergo motion reversal. This restricts the space-time interval to take the form

$$\begin{aligned} ds^2 = {} & g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \\ & + g_{33}(dx^3)^2 + 2g_{34}(dx^3)(dx^4) \\ & + g_{44}(dx^4)^2 \end{aligned} \tag{1.7}$$

A transformation to a new set of (x^3,x^4) coordinates can always be found which allows the last three terms of Equation (1.7) to be written in the form

$$f(x^3,x^4)[(dx^3)^2 + (dx^4)^2]. \tag{1.8}$$

(Kramer,1980,p.195) When one elects to use coordinates such that the last three terms of Equation (1.7) can be written in the form of Equation (1.8), x^3 and x^4 are called isotropic coordinates. Therefore, in isotropic coordinates, stationary, axially symmetric space-times are specified by four metric tensor components.

The form of these four metric functions used by Lewis

(1932) allows the space-time interval to be written as

$$(ds)^2 = e^{2u}(dt - \omega d\phi)^2 - e^{2B-2u}d\phi^2 - e^{2(\Omega-u)}[(dx^3)^2 + (dx^4)^2] \qquad (1.9)$$

The functions u, ω, B, Ω depend only on the (x^3,x^4) coordinates. Appendix B shows the non-zero components of the Einstein tensor for the metric in Equation(1.9) written with respect to an orthonormal tetrad. Notice the V operator of Equation(B.10a) in Appendix B. Consider a two-dimensional (x^3,x^4) subspace of the metric in Equation (1.9). The interval in this subspace is found by setting $dt = d\phi = 0$ in Equation (1.9)

$$(ds_2)^2 = e^{2(\Omega-u)}[(dx^3)^2 + (dx^4)^2]. \qquad (1.10)$$

The gradient of a scalar f in this subspace is a 2-vector and has components

$$e^{2(\Omega-u)}[f_3,f_4], \qquad (1.11)$$

while the Laplacian has the form

$$e^{2(\Omega-u)}[f_{33} + f_{44}]. \qquad (1.12)$$

In Equations (1.11) and (1.12), the subscripts on the function f indicate partial derivatives. Except where necessary in the remainder of this work, subscripts on functions indicate partial derivatives. It turns out to be advantageous (Kinnersley 1977, p.1529) to define a tilde operator on the two-dimensional (x^3,x^4) subspace. For each 2-vector, $\underset{\sim}{A}$,

$$A = [A^3,A^4] \qquad \underset{\sim}{A}\cdot\underset{\sim}{A} = (A^3)^2 + (A^4)^2 \qquad (1.13)$$

there is a related vector, $\tilde{A}$

$$\tilde{A} = [A^4, -A^3] \tag{1.14}$$

Note that

$$\nabla \cdot \tilde{A} = 0 \quad \text{implies} \quad A = \tilde{\nabla} f \tag{1.15}$$

where f is some scalar function.

VACUUM FIELD EQUATIONS

Einstein's vacuum field equations are found by setting the Einstein tensor equal to zero. From Equations (1.9) and (B.6), one of these field equations is

$$e^{-B} \nabla \nabla e^B = 0. \tag{1.16}$$

Because of Equation (1.15), this implies

$$\nabla e^B = \tilde{\nabla} f. \tag{1.17}$$

One solution to Equation (1.17) is

$$e^B = x^3 \tag{1.18a}$$

$$f = x^4 \tag{1.18b}$$

In fact, without loss of generality one may choose x^3, x^4 in accordance with Equation (1.18), and then $\rho = x^3$, $z = x^4$ are called Weyl canonical coordinates (Kramer 1980, p. 195). In these coordinates, the remaining vacuum field equations are

$$\nabla\ [e^{2u} \nabla(\rho e^{-2u})] = \rho^{-1} e^{4u}\ \nabla\omega\ \nabla\omega \tag{1.19}$$

$$\nabla\ [\rho^{-1} e^{4u}\ \nabla\omega] = 0 \tag{1.20}$$

$$\nabla\ \nabla\Omega = \nabla\ \nabla u - \nabla u\ \nabla u + \rho^{-1} u_\rho + (1/4)\rho^{-2} e^{4u} \nabla\omega\ \nabla\omega \tag{1.21}$$

$$\Omega_\rho = \rho[(u_\rho)^2 - (u_z)^2] - (1/4)\rho^{-1} e^{4u}[(\omega_\rho)^2 - (\omega_z)^2] \tag{1.22}$$

$$\Omega_z = 2\rho u_\rho u_z - (1/2)\rho^{-1}e^{4u}\omega_\rho \omega_z \tag{1.23}$$

Equations (1.19) and (1.20) involve only the metric functions u and ω. Once these two functions are found, Ω is found by simple integrations of Equations (1.22) and (1.23). Equation (1.21) gives no new information as it can be derived from the other four equations. Therefore, Equations (1.19) and (1.20) are the key equations for describing the metric of stationary, axially symmetric space-times.

The metric function ω gives a measure of how fast the space-time is dragging inertial frames around the symmetry axis (Bardeen 1970). In other words, for $\omega \neq 0$ one expects the sources of the gravitation to be rotating. The space-time exterior to sources which possess axial symmetry, but which are not rotating, are described by Equations (1.19) to (1.23) with the function ω set equal to zero. Such space-times are said to be static. These static space-times were first studied by Weyl. The key equation (Lewis 1932, p. 182) for static space-times is Equation (1.19) with $\omega = 0$

$$\nabla [e^{2u}\nabla(\rho e^{-2u})] = 0, \tag{1.24}$$

or

$$u_{\rho\rho} + \rho^{-1}u_\rho + u_{zz} = 0. \tag{1.25}$$

Equation (1.25) is of the same form as Laplace's equation in cylindrical coordinates (z,ρ,∅) for an axially symmetric function u(ρ,z). From Equation (1.2), any axially symmetric Newtonian Φ can be used as a solution to the Einstein vacuum field Equation (1.25). This implies for any axially symmetric mass distribution σ one has, via Equation (1.4) a solution to the static, axially symmetric Einstein vacuum field equation. However, one must be cautious. In General Relativity, the Weyl canonical coordinates ρ and z do not necessarily retain their Euclidean meaning as cylindrical coordinates. So while one is free to select axially symmetric mass distributions for use in Equation (1.4), one is not able to immediately able to say the corresponding $u(\rho,z)$ represents the space-time external to that particular mass distribution. That is, for a given σ, ϕ and u do not necessarily represent the same physical situation. To determine what mass distribution is giving a particular general relativistic solution $u(\rho,z)$, one must examine where $u(\rho,z)$ exhibits zeroes and singularities.

SL(2,R) TENSOR NOTATION

The SL(2,R) tensor formulation of the vacuum stationary, axially symmetric field Equations is now introduced (Kinnersley 1977). Stationary axially symmetric space-times with orthogonal transitivity can be written

$$ds^2 = f_{AB}dx^A dx^B - e^{2(\Omega-u)}[(dx^3)^2 + (dx^4)^2] \tag{1.26}$$

The vacuum field equations can be written

$$\nabla\,[\rho^{-1}f_A{}^X\,\nabla f_{XB}] = 0\ . \tag{1.27}$$

$$\rho^2 = -\det(f_{AB}). \tag{1.28}$$

The indices (A,B = 1,2) are raised and lowered according to

$$h^A = \varepsilon^{AX} h_X \,, \quad h_A = \varepsilon_{XA} h^X \tag{1.29}$$

$$\varepsilon_{AB} = \varepsilon^{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.30}$$

Using Equation (1.15), the field Equation (1.27) implies the existence of a set of functions g_{AB} such that

$$\nabla g_{AB} = -\rho^{-1} f_A{}^X \tilde{\nabla} f_{XB} \tag{1.31}$$

which can be inverted to yield

$$\nabla f_{AB} = \rho^{-1} f_A{}^X \tilde{\nabla} g_{XB} \tag{1.32}$$

If one defines a tensor

$$H_{AB} = f_{AB} + i g_{AB} \tag{1.33}$$

Equations (1.31) and (1.32) can be written in the single complex equation

$$\nabla H_{AB} = -i\rho^{-1} f_A{}^X \tilde{\nabla} H_{XB}. \tag{1.34}$$

This complex equation is equivalent to the vacuum field equations.

Kinnersley and Chitre (1978), by examining the group properties of the field equations, discovered solution generation techniques which yield asymptotically flat new exact solutions. Their techniques are most easily expressed in terms of two generating functions $G_{AB}(s,t)$ and $F_{AB}(s)$ (Kinnersley and Chitre 1978b). These generating functions are defined such that

$$i\partial G_{AB}(s,t)/\partial t \,\big|_{t=0} = i\partial F_{AB}(t)/\partial t \,\big|_{t=0} = H_{AB}. \tag{1.35}$$

In other words, the solution generating techniques of Kinnersley and Chitre involve algebraic manipulation of these functions rather than the metric functions f_{AB}.

Working over a different route, Cosgrove (1979) discovered a set of infinitesimal transformations which preserve the form of the vacuum field equations. Some of these infinitesimal transformations were exponentiated to give finite transformations which yield new metric functions. Cosgrove has also shown how his transformations can be expressed in terms of the generating functions of Kinnersley and Chitre (Cosgrove 1980, pp.2422-2424).

PURPOSE

Specifically, what this work will attempt to do is to suggest a physical interpretation to one of Cosgrove's transformations. First to be examined will be the new solutions generated when Cosgrove's technique is applied to static axially symmetric solutions. It will be argued that the physical effect of his transformation is to accelerate the initial solution. The effect of this transformation is illustrated in Figure 1. Finally, the action of the same transformation on stationary axially symmetric solutions will be examined.

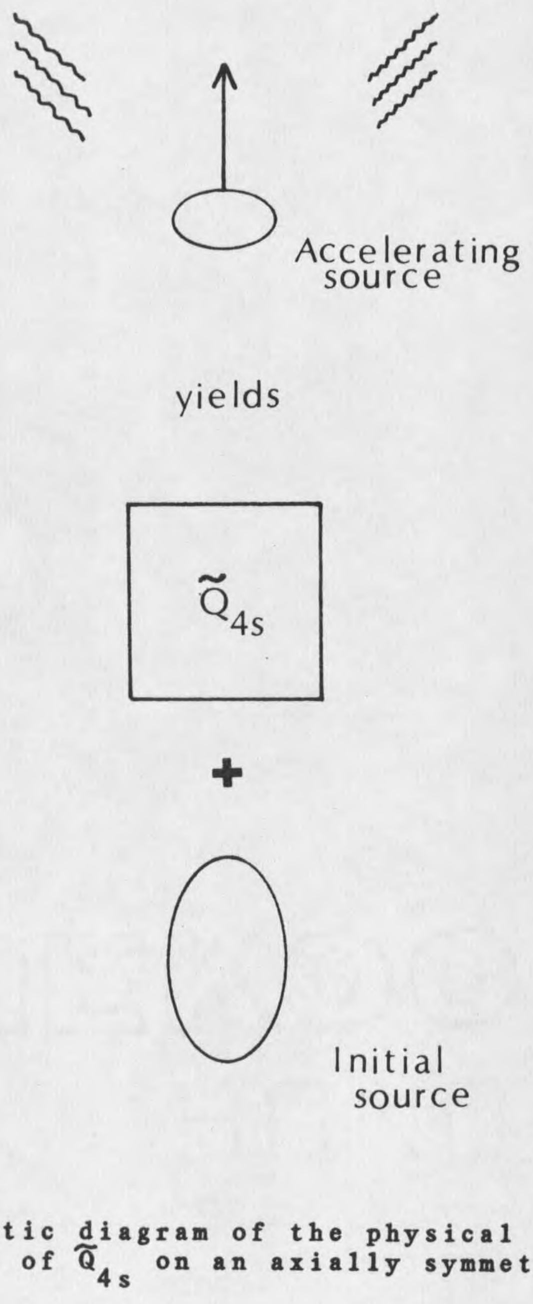

Figure 1: Schematic diagram of the physical effect of $\tilde{Q}_{4s}$ on an axially symmetric solution

CHAPTER 2

FLAT SPACE-TIME FOR ACCELERATED OBSERVERS

The simplest metric tensor is the metric tensor η_{ij} describing flat space-time. The space-time interval appears as

$$\begin{aligned} ds^2 &= \eta_{ij} dx^i dx^j \\ &= (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2, \end{aligned} \qquad (2.1)$$

where x^1 is a time coordinate and x^2, x^3, x^4 are three Cartesian spatial coordinates. Several other coordinate systems will prove useful. For example, $x^1 = T$, $x^2 = Z$, $x^3 = R\cos\emptyset$, $x^4 = R\sin\emptyset$ changes Equation (2.1) to the form

$$ds^2 = dT^2 - dZ^2 - dR^2 - R^2 d\emptyset^2. \qquad (2.2)$$

All that has been done is in Equation (2.2) is to transform from Cartesian spatial coordinates to cylindrical spatial coordinates. Even under this simple coordinate change, the metric tensor components change their functional form. All coordinate transformations may not yield such a direct interpretation. With

$$u = x^1 + x^2 \,, \; v = x^1 - x^2$$

$$\eta = x^3 + ix^4 \,, \; \overset{*}{\eta} = x^3 - ix^4$$

Equation (2.1) becomes

$$ds^2 = du dv - d\eta d\bar{\eta}.$$

The point is that flat space-time is easy to disguise by using non-Cartesian coordinate systems. Therefore, when simply looking at the components of a particular metric tensor g_{ij}, it may be impossible to tell if that particular metric tensor is representing flat space-time or if it represents a curved space-time with corresponding gravitational effects. However, there is an effective way to tell if the space-time is flat. With the functional form of the components of the metric tensor, the components of the Riemann curvature tensor R_{ijkl} may be computed. If all the components of this curvature tensor are zero, the space-time is flat. The drawback is that the computation of the components of R_{ijkl} is generally long and tedious (Misner et.al.1973,Ch.14). This chapter shows flat space-time displayed in ways which occur frequently in the remainder of this work.

A coordinate system of interest is one used by an observer moving with uniform, rectilinear acceleration. The notation here follows the presentation in (Misner et.al. 1973, Ch. 6) To get at this coordinate system, consider first an observer at rest in flat space-time. This observer specifies an event by giving its coordinates $\{x^1 = T, x^2 = Z, x^3 = X, x^4 = Y\}$ with respect to an orthonormal tetrad of basis vectors $\{\underset{\sim}{e}_i\}$ $(i=1,2,3,4)$. The observer specifies the position four-vector, $\underset{\sim}{x}$, of the event as a four-vector

$$\underset{\sim}{x} = T\underset{\sim}{e}_1 + Z\underset{\sim}{e}_2 + X\underset{\sim}{e}_3 + Y\underset{\sim}{e}_4.$$

This is all a mathematical way of saying how an observer at rest would measure events. A physical observer would use a set of clocks to measure the time T of an event and a set a meter sticks to measure the spatial location X,Y,Z of an event. One of the reasons for putting these physical acts in mathematical language (coordinates, basis vectors) is that for some observers it is not easy to intuitively imagine how they would go about actually measuring things. Mathematics allows one to sidestep questions about how a given observer would measure things, but at the same time it allows predictions as to what the observer would measure.

Assume another observer is moving in the positive $\underset{\sim}{e}_2$-direction such that he feels a constant acceleration of magnitude A. The position four-vector of this observer is:

$$\underset{\sim}{x} = [A^{-1}\sinh A\tau]\underset{\sim}{e}_1 + [A^{-1}\cosh A\tau + z_0 - A^{-1}]\underset{\sim}{e}_2, \qquad (2.3)$$

(Misner et.al. Ch.6) The symbol τ is the proper time along the trajectory of the accelerated observer. It may be viewed as representing time intervals measured by a clock riding along with the accelerated observer. The symbol z_0 is the position of the accelerated observer at time $\tau = 0$. Equation (2.3) claims $[Z(\tau) - z_0 - A^{-1}]^2 - [T(\tau)]^2 = A^{-2}$ = constant for all values of τ. On a T-Z graph, the trajectory of the accelerated observer is a hyperbola. For this reason, one-dimensional uniformly accelerated motion is termed hyperbolic motion. Figure 2 shows the trajectory of an observer in hyperbolic motion.

The reason for introducing an observer in hyperbolic motion is to look at flat space-time from this accelerated observer's point of view. One expects the accelerated observer to use his own set of coordinates $\{\bar{t},\bar{z},\bar{x},\bar{y}\}$ adopted to the orthonormal tetrad of basis vectors $\{\bar{\underset{\sim}{e}}_i\}$ he carries with him. By making precise the notion of 'carrying basis vectors', (Fermi-Walker transport, Misner et.al. 1973, p.173) it can be shown that the accelerated observer's coordinates are related to the stationary observer's coordinates by

$$T = (\bar{z} + A^{-1})\sinh A\bar{t} \tag{2.4a}$$

$$Z - z_0 - A^{-1} = (\bar{z} + A^{-1})\cosh A\bar{t} \tag{2.4b}$$

$$X = \bar{x} \tag{2.4c}$$

$$Y = \bar{y} \tag{2.4d}$$

In these accelerated coordinates, the flat space-time interval of Equation (2.1) becomes

$$ds^2 = (1 + A\bar{z})^2 d\bar{t}^2 - d\bar{z}^2 - d\bar{x}^2 - d\bar{y}^2 \tag{2.5}$$

Equation (2.5) displays the metric of flat space-time in terms of coordinates used by an accelerated observer. One property of equation (2.5) is of primary interest. If the coordinate transformation $\bar{x} = \bar{\rho}\cos\bar{\phi}$, $\bar{y} = \bar{\rho}\sin\bar{\phi}$ is performed

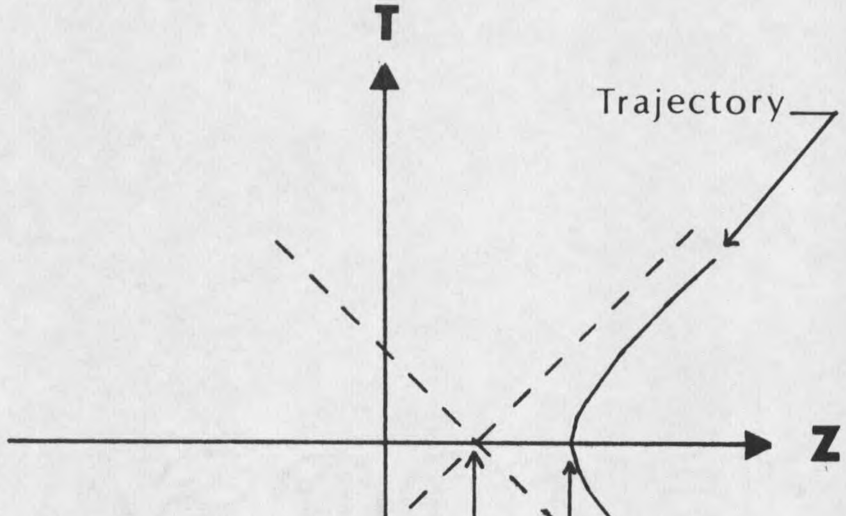

Figure 2: Trajectory of an observer in uniform rectilinear accelerated motion ('hyperbolic motion')

in Equation (2.5), the space-time interval becomes

$$ds^2 = (1 + A\bar{z})^2 d\bar{t}^2 - d\bar{z}^2 - d\bar{\rho}^2 - \bar{\rho}^2 d\bar{\phi}^2. \tag{2.6}$$

Comparison of this equation with Equation (1.9) (with $\omega = 0$), shows Equation (2.6) represents a static, axially symmetric space-time. As it stands, however, Equation (2.6) is not in the canonical form of Lewis (1932):

$$ds^2 = e^{2u}dt^2 - e^{-2u}\rho^2 d\phi^2 - e^{2(\Omega-u)}[d\rho^2 + dz^2]. \tag{2.7}$$

The necessary coordinate transformation is (Kramer 1980, p.202)

$$\bar{t} = t, \quad \bar{\phi} = \phi$$
$$(1 + A\bar{z}) = A^{1/2}(\tilde{r} + \tilde{z})^{1/2}$$
$$\bar{\rho} = A^{-1/2}(\tilde{r} - \tilde{z})^{1/2},$$

where

$$\tilde{r}^2 = \rho^2 + \tilde{z}^2$$
$$\tilde{z} = z + 1/(2A), \tag{2.8}$$

puts Equation (2.6) in the form

$$ds^2 = A(\tilde{r} + \tilde{z})dt^2 - (2A\tilde{r})^{-1}[d\rho^2 + dz^2] - A^{-1}(\tilde{r} + \tilde{z})^{-1}\rho^2 d\phi^2. \tag{2.9}$$

Comparison of Equation (2.9) with Equation (2.7) allows the metric functions of Lewis to be identified.

$$u = (1/2)\ln[A(\tilde{r} + \tilde{z})] \tag{2.10}$$

$$\Omega = (1/2)\ln[(\tilde{r} + \tilde{z})/2\tilde{r}]. \tag{2.11}$$

The reason for wanting the metric functions in Weyl canonical coordinates is that the solution generating techniques to be discussed are defined in terms of their action on Weyl coordinates. Whenever the metric function g_{tt} (in Weyl coordinates) has a term with the functional form $z - c + ((z-c)^2 + \rho^2)^{1/2}$ (c = constant), g_{tt} will be said to contain an 'acceleration factor'.

The space-time interval of Equation (2.9) is an example of the remarks at the beginning of the chapter. It is difficult to guess that Equation (2.9) represents flat space-time. Normally, flat space-time has $g_{tt} = +1$. In Equation (2.9), the metric function g_{tt} is a function of the (ρ,z) coordinates. The g_{tt} of Equation (2.9) violates the fundamental test for a physical space-time. A physical space-time (due to a localized gravity source) obeys the condition g_{tt} approaches unity for large values of the coordinates. In the case of Weyl canonical coordinates this would be the coordinate condition $[(\rho)^2 + (z^2)] \text{-->} \infty$. That is, physical space-times are asymptotically flat space-times. The asymptotic value of g_{tt} in Equation (2.9) is infinity. For any space-time, the surface defined by $g_{tt} = 0$ is an infinite red-shift surface (Ohanian 1976, p.308). The equation of the infinite red-shift surface for Equation (2.9) is $\rho = 0$, $z < 1/2A$. Equation (2.9) nonetheless, represents flat space-time.

CHAPTER 3

NEW SOLUTIONS VIA ACCELERATION FACTORS

As first mentioned in Chapter 1, the space-time interval for stationary, axially symmetric space-time is

$$ds^2 = e^{2u}(dt - \omega d\phi)^2 - e^{-2u}\rho^2 d\phi^2 - e^{2(\Omega - u)}[d\rho^2 + dz^2]. \quad (3.1)$$

Equation (3.1) is written using the metric parametrization of Lewis (1932) and assuming (ρ, z) as Weyl canonical coordinates. The Einstein vacuum field equations are written out in Equations (1.19)-(1.23). The Lewis (1932) parametrization and the selection of an orthonormal tetrad (Appendix B) allows the field equations to assume a fairly compact form. Weyl canonical coordinates allow the loose interpretation of z as a distance parallel to the symmetry axis and of ρ as a perpendicular distance away from the symmetry axis. This interpretation is strictly true only when $u = \omega = \Omega = 0$. For this case, Equation (3.1) reduces to the flat space-time interval written in cylindrical spatial coordinates. The advantage in Weyl coordinates is somewhat offset by the fact Weyl coordinates can give simple space-time a complicated functional form. The vacuum field equations for this metric are Equations(1.19)-(1.23) with $\omega=0$

$$\nabla [\rho \nabla u] = 0, \tag{3.2}$$

$$\Omega_\rho = \rho[(u_\rho)^2 - (u_z)^2], \tag{3.3}$$

$$\Omega_z = 2\rho u_\rho u_z \, . \tag{3.4}$$

Equations (3.3) and (3.4) are integrable, since $\Omega_{\rho z} = \Omega_{z\rho}$ follows from Equation (3.2) Once a function u is found which obeys Equation (3.2), the metric function Ω may be found by integration of Equations (3.3)-(3.4). This is possible because Equation (3.2) is the integrability condition on Equations (3.3)-(3.4). A solution set to these equations is denoted by $\langle u,\Omega\rangle$. Notice how static, axially symmetric solutions are completely characterized by the single metric function u. This is one reason why static space-times are easier to analyze than stationary space-times. The stationary space-times require the extra function ω for their complete specification. In this chapter, a very direct approach to generating static, axially symmetric solutions will be introduced.

In General Relativity, one generally expects the field equations governing the metric functions to be nonlinear. Therefore, the form of Equation (3.2) is a pleasant surprise. One advantage of this equation is discussed in Chapter 1 following Equation (1.25). Solutions to Equation (3.2) can be written as a Poisson integral (Equation (1.4)) of an axially symmetric mass distribution. Another advantage is that Equation (3.2) is a linear partial differential equation. If two functions u_1 and u_2 separately obey Equation (3.2), the function (u_1+u_2) also obeys Equation (3.2). This is solution generation at its simplest: the linear superposition of two known metric functions u_1 and u_2. If the gravitational fields described by the metric functions u are assumed to be produced by localized sources it is tempting to conclude that this technique yields the space-time due to the superposition of those sources. However, suppose one solution $\langle u_1, \Omega_1 \rangle$ is due to one source and $\langle u_2, \Omega_2 \rangle$ is due to another source. Equation (3.2) says that (u_1+u_2) is another solution, but the corresponding metric function Ω is not equal to $\Omega_1+\Omega_2$. That is, $\langle (u_1+u_2), (\Omega_1+\Omega_2) \rangle$ is not a solution because of the nonlinearity of Equations (3.3)-(3.4). To determine the function Ω corresponding to (u_1+u_2), assume

$$\Omega = \Omega_1 + \Omega_2 + u_2 - \beta. \tag{3.5}$$

This auxiliary function β obeys an equation of the form

$$\nabla\beta = (1 - 2u_{1,\rho})\nabla u_2 + 2\rho u_{1,z}\tilde{\nabla} u_2. \tag{3.6}$$

Equation (3.6) is the price for this simple superposition of solutions, and the ease of the technique depends on the ease of integrating Equation (3.6).

Suppose now that one of the solutions $\langle u_1, \Omega_1 \rangle$ is given by the metric functions of Equations (2.10)-(2.11)

$$u_1 = (1/2)\ln[A(\tilde{r} + \tilde{z})], \tag{3.7}$$

$$\Omega_1 = (1/2)\ln[(\tilde{r} + \tilde{z})/2\tilde{r}], \tag{3.8}$$

where $\tilde{r}$ and $\tilde{z}$ are defined as in Equation (2.8). The expressions in Equations (3.7)-(3.8) do satisfy the vacuum field equations, as they simply represent flat space-time from the point of view of an accelerated observer. When this solution $\langle u_1, \Omega_1 \rangle$ is superposed with another solution $\langle u_2, \Omega_2 \rangle = \langle u, \Omega \rangle$, Equation (3.6) for the auxiliary function β is

$$\nabla\beta = \tilde{r}^{-1}[\tilde{z}\nabla u + \rho\tilde{\nabla} u]. \tag{3.9}$$

The space-time interval for this new static solution has the form

$$ds^2 = A(\tilde{r} + \tilde{z})e^{2u}dt^2 - A^{-1}(\tilde{r} + \tilde{z})^{-1}e^{-2u}\rho^2 d\phi^2$$
$$- (2A\tilde{r})^{-1}e^{2(\Omega-\beta)}[d\rho^2 + dz^2]. \tag{3.10}$$

The solution generation procedure is simple. First, solve Equations (3.2)-(3.4). Second, solve for the auxiliary function β. Finally, the new metric is displayed in Equation (3.10). While the procedure is simple, it is not so easy to see a physical interpretation for the metric generated by this technique.

First, recall that the symbol 'A' in the above expressions (in particular (Equations (3.7)-(3.8)) represents the magnitude of an acceleration. This acceleration is associated with an observer (Chapter 2) who is moving through flat space-time. This observer describes flat space-time by means of the solution $\langle u_1, \Omega_1 \rangle$. What the above solution generation technique does is to add to this accelerated observer's view of flat space-time a solution $\langle u, \Omega \rangle$. If $\langle u, \Omega \rangle$ is asymptotically flat, it may be assumed to be the result of localized sources which are axially symmetric around the z axis. Therefore, the above generation technique takes flat space-time (as seen by an accelerated observer) and 'adds' the gravity field $\langle u, \Omega \rangle$ of localized sources. The metric generated represents the curved space-time around sources as seen by the accelerated observer: or equivalently, the new metric might be viewed as the space-time due to localized sources which are in uniform, rectilinear acceleration of magnitude 'A'. To find the metric of an accelerating source of gravity, simply find the gravity field $\langle u, \Omega \rangle$ for the sources at rest and 'add' an acceleration factor.

The argument above assumes that Equations (3.7)-(3.8) are the metric functions for flat space-time from the view of an accelerated observer. Fortunately, there is a way to confirm the assumption. Kinnersley and Walker (1970) in a thorough analysis, discuss a particular metric called the charged C-metric. Among many properties they show how the C-metric describes the space-time around a uniformly accelerating charged mass. The symbols e and m in their paper represent the charge and mass of the accelerating particle. The C-metric has the space-time interval:

$$ds^2 = A^{-2}(x + y)^{-2}[F(y)dt^2 - F^{-1}(y)dy^2 - G^{-1}(x)dx^2 - G(x)dz^2], \tag{3.11}$$

where

$$G(x) = 1 - x^2 - 2mAx^3 - e^2A^2x^4,$$

$$F(y) = -G(-y).$$

The flat space limit is when $e = m = 0$. When the C-metric (in this limit) is changed to the Weyl canonical form, the metric function u is of the form

$$u = (1/2)\ln[(z+1/2A)^2+((z+1/2A)^2+\rho^2)^{1/2}], \tag{3.12}$$

where (ρ, z) are Weyl canonical coordinates. Note how Equation (3.12) is of the same form as Equation (3.7). Since Equation (3.12) represents accelerated flat space-time, Equation (3.7) does also.

Equation (3.9) for the auxiliary function β is of interest for a couple of reasons. The equation is in terms of $\nabla\beta$, which means the equation is really two equations since ∇ is a two-dimensional gradient operator: $\nabla = [\partial_\rho, \partial_z]$. In order to find β given u, Equation (3.9) must obey the integrability condition $\nabla \tilde{\nabla}\beta = 0$. Taking $\tilde{\nabla}$ of both sides of Equation(3.9) gives

$$\nabla \tilde{\nabla}\beta = \tilde{r}^{-1}\nabla [\rho\nabla u]. \tag{3.13}$$

But, since u obeys Equation (3.2), the right hand side of Equation (3.13) is zero and the equation for the auxiliary function β is integrable.

The discussion following Equation (1.25) points out that solutions to Equation(3.2) can be written as an integral:

$$u(\rho,z) = -\iiint \sigma(\rho',z')|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}d^3x'. \tag{3.14}$$

$\sigma(\rho,z)$ can be thought of as an axially symmetric mass density in (ρ,z) Weyl canonical coordinate space. For points outside of the mass density, σ, Equation (3.14) is a solution to Equation (3.2). For points inside of the mass density $\sigma(\rho,z)$, Equation (3.14) is a solution of Poisson's equation

$$\rho^{-1}\nabla[\rho\nabla u] = 4\pi\sigma(\rho,z). \tag{3.15}$$

If the u in Equation (3.9) satisfies Equation (3.15) at all points in (ρ,z) space, the corresponding Poisson equation for the auxiliary function β is

$$\rho^{-1}\nabla\ [\rho\nabla(\tilde{r}^{-1}\beta)] = 4\pi\tilde{r}^{-2}\tilde{z}\sigma(\rho,z), \tag{3.16}$$

with a solution

$$\beta(\rho,z) = -\tilde{r}(\rho,z)\iiint \tilde{r}^{-2}(\rho',z')\langle z'+1/2A\rangle\sigma(\rho',z')\ \times$$
$$\times\ \ |\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}d^3x'. \tag{3.17}$$

β is determined by a modified mass density:

$$\tilde{r}^{-2}\tilde{z}\sigma(\rho,z). \tag{3.18}$$

The important point is that β may be written as a Poisson integral.

Recall, however, that selecting a mass density $\sigma(\rho,z)$ for an axially symmetric configuration of matter does not mean that going through Equations (3.14)-(3.17) will yield the space-time when that particular mass configuration is in rectilinear acceleration. The Weyl canonical coordinates are the problem: they only retain their flat space-time interpretation far away from the sources. The symbols ρ, z appearing in $\sigma(\rho, z)$ can not be interpreted as cylindrical coordinates for all values.

A second interesting thing about Equation (3.13) is the appearance of the factor $\tilde{r}$. In Weyl canonical coordinate (ρ, z) space, $\tilde{r}$ is the distance between the point $(0, -1/2A)$ and the point (ρ, z). This simple solution generation technique selects a preferred point, P_0, along the z-axis which depends on the acceleration A. As $A \to +\infty$, the preferred point slides up the symmetry axis toward the origin of the (ρ, z) coordinate system and for $A \to 0$, the preferred point slides out to negative infinity. This motion of P_0 in $\tilde{r}$ is potentially troublesome if the unaccelerated solution is due to a localized mass density $\sigma(\rho, z)$. For the moment, it is assumed the point $(0, -1/2A)$ lies on the z-axis below any sources $\sigma(\rho, z)$. However, the appearance of an expression which selects a preferred point along the z-axis is something which appears in the more elaborate solution generation schemes of other investigators (Kinnersley 1977, Cosgrove 1980). In their work, $\tilde{r}$ is replaced by a function S(s) defined as

$$S(s) = [(1-2sz)^2 + 4s^2\rho^2]^{1/2}. \tag{3.19}$$

This ubiquitous expression first appears in the group-theoretic analysis of Kinnersley and Chitre (1977b) when differential equations for the metric generating functions $F_{AB}(s)$ are derived. This gives the first clue that the simple solution generation technique here may be related to the techniques of other workers.

Another clue to a connection to other techniques is the appearance of the Equation (3.9) for the auxiliary function β. This equation results primarily from using Equations (3.3)-(3.4) to determine a metric function Ω which results from adding a general static solution to an accelerated flat space-time solution. Equation (3.9) corresponds very closely with Equation (2.17) of Hoenselaers et.al. (1979b). The only differences are a sign and a different parameter (t instead of A). The equation governing their β is

$$\nabla\beta = S^{-1}(t)[(1-2tz)\nabla u - 2t\rho\tilde{\nabla}u] \qquad (3.20)$$

Their β is used in solving for the generating functions of static, axially symmetric space-times. Their β comes about as a result of their definitions of the generating functions. There is no reference to acceleration or superposition of solutions. The advantage in their β is that the limit t-->0 is simpler in many cases. Figure 3 shows the preferred point and singularity of the acceleration factor expressed in Weyl canonical coordinates.

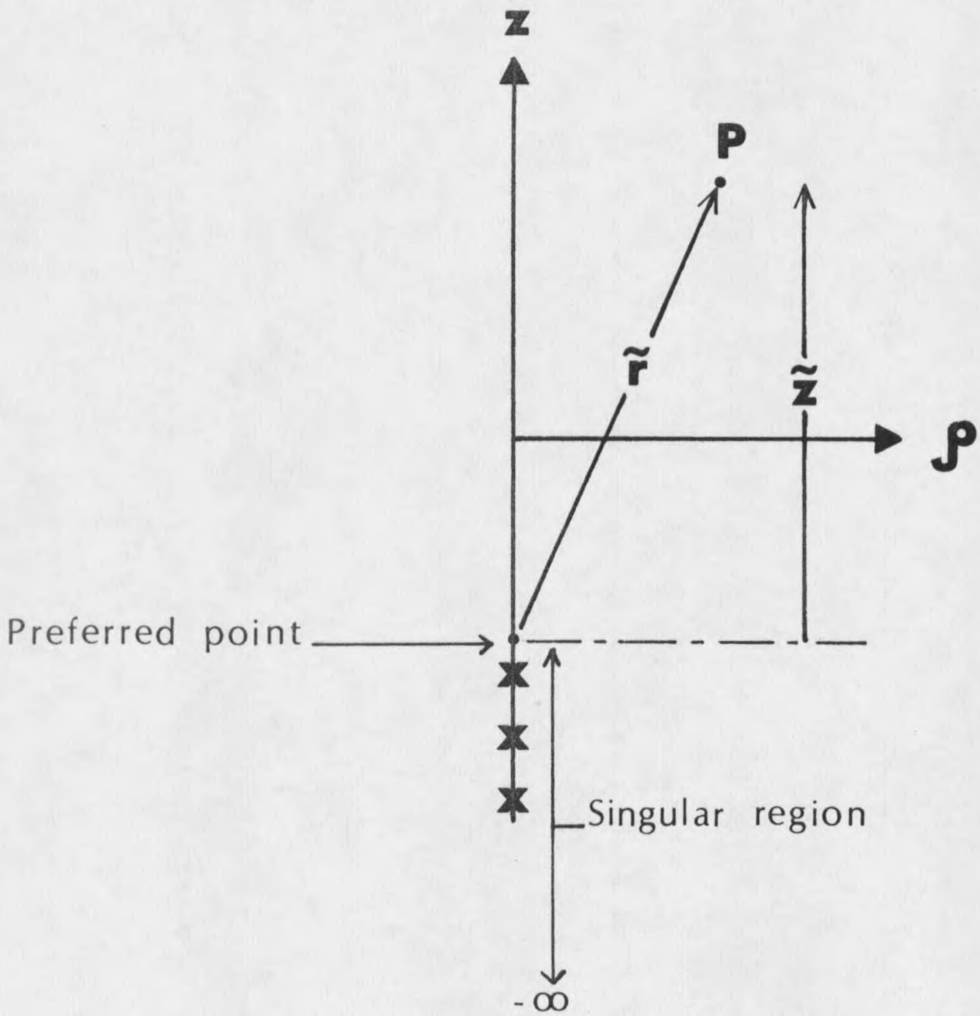

Figure 3: Preferred point and singularity of acceleration factor

CHAPTER 4

SIMPLE EXAMPLES

Too often, in many areas of General Relativity, an elaborate mathematical formalism is introduced accompanied by no specific calculations or examples. There are a couple of reasons for this. First, the notion of gravity being described as the curvature of a four-dimensional space-time allows the compact notation of Riemannian geometry to be employed. Sometimes the notation is too compact. The symbol for the metric tensor, g_{ij}, represents sixteen functions of the four space-time coordinates. The symbol for the curvature tensor, R_{ijkl}, can represent 256 different functions. Even though symmetries reduce the number of independent components, explicitly displaying the components of these tensors is tedious and time consuming. Yet, displaying components is necessary to examine the structure of a given space-time.

A second reason for not including specific examples occurs in solution generating techniques. Despite promises of not having to explicitly solve the Einstein field equations, often the method involves solving a difficult differential equation or acquiring an auxiliary function. Once various functions are found, there is the perpetual problem of giving a physical interpretation to all the mathematics.

This chapter shows specific examples of points raised in earlier chapters. Not every algebra step is shown, but there is enough so that any omitted steps are trivial.

One of the simplest metrics is the Schwarzchild metric (1916) which describes the space-time outside a point source of mass M

$$ds^2 = (1-2MR^{-1})dT^2 - R^2\sin^2\theta d\phi^2$$
$$- (1-2MR^{-1})^{-1}dR^2 - R^2 d\theta^2. \quad (4.1)$$

A spherically symmetric space-time possesses axial symmetry so it is possible to write Equation (4.1) in the form of Equation (3.1). The (T,ϕ) coordinates here are equated to the (t,ϕ) coordinates of Equation (3.1). Next, the terms in Equation (4.1) involving dR and $d\theta$ are cast in the isotropic form of Equation (1.8)

$$(1-2MR^{-1})^{-1}dR^2 + R^2 d\theta^2$$
$$= R^2[(R^2-2MR)^{-1}dR^2 + d\theta^2].$$

This part of the space-time interval is isotropic if a new coordinate α is introduced and defined as

$$d\alpha = (R^2-2MR)^{-1/2}dR$$
$$\alpha = \cosh^{-1}[(R-M)/M]$$
$$R = M[\cosh\alpha + 1]. \quad (4.2)$$

With the coordinate α, Equation (4.1) becomes

$$ds^2 = [\cosh\alpha - 1]/[\cosh\alpha + 1]dt^2$$
$$- M^2[\cosh\alpha + 1]^2[\sin^2\theta d\phi^2$$
$$- M^2[\cosh\alpha + 1]^2[d\alpha^2 + d\theta^2], \quad (4.3)$$

and by a comparison with Equation (3.1) allows the following functions and coordinates to be identified:

$$e^{2u} = [\cosh\alpha - 1]/[\cosh\alpha + 1] \quad (4.4)$$

$$\rho = M\sinh\alpha\sin\theta \quad (4.5)$$

$$e^{2\Omega} = e^{2u}[d\rho^2 + dz^2]^{-1}M^2[\cosh\alpha + 1]^2$$
$$\times [d\alpha^2 + d\theta^2]. \quad (4.6)$$

As the Weyl (ρ,z) coordinates and the (α,θ) coordinates are both isotropic, they are related by a conformal transformation. Hence, $\rho+iz$ and $\alpha+i\theta$ are analytic functions of one another and (ρ,z), (α,θ) each satisfy Laplace's equation in two dimensions, and are related by a set of Cauchy-Riemann equations. Therefore, z is the function conjugate to ρ. In this case, $z = M\cosh\alpha\cos\theta$ and Equation (4.6) reduces to

$$e^{2\Omega} = [\cosh^2\alpha - 1]/[\cosh^2\alpha - \cos^2\theta] \quad (4.7)$$

Writing the metric functions in terms of Weyl (ρ,z) coordinates is made easier by defining $x = \cosh\alpha$, $y = \cos\theta$:

$$\rho = M(x^2-1)^{1/2}(1-y^2)^{1/2}, \quad z = Mxy \quad (4.8)$$

$$e^{2u} = (x-1)/(x+1) = g_{tt} \quad (4.9)$$

$$e^{2\Omega} = (x^2-1)/(x^2-y^2) \quad (4.10)$$

Equation (4.8) is easily recognized as defining prolate spheroidal coordinates (Appendix E).

This calculation shows several things. Putting familiar metrics in Weyl canonical form destroys their familiar form. Equations (4.9) and (4.10) are difficult to recognize as representing the space-time outside of a point mass. In fact, there is a temptationto conclude something quite different. When u is written as a Poisson integral over a mass density in (ρ, z) space (Equation (3.16)) the mass density which would give the function in Equation (4.9) is that of a thin rod of linear mass density 1/2 and length 2M lying along the z-axis, and centered on $\rho = 0, z = 0$. The point source of the space-time of Equation (4.1) appears as a line source in Weyl canonical (ρ, z) coordinates. The source of the Schwarzchild is shown in Figure 4.

Equation (4.9) has an interesting limit. While keeping the density constant, if the top of the line source is placed at $\rho = 0$, $z = 0$, and the lower end is allowed to tend toward minus infinity, the metric function u in Equation (4.9) goes to

$$u \to (1/2)\ln[(4M)^{-1}[z + (\rho^2+z^2)^{1/2}] \qquad (4.11)$$

which is the same as an acceleration factor (Equation (2.10)). This suggests an acceleration factor in a metric function can be viewed as produced by an infinite line source with linear mass density 1/2. On one hand this is desirable because this infinite source of mass can be thought of as providing acceleration when added (Chapter 3) to a general static solution. That is, adding an acceleration factor is physically equivalent to adding an infinite line source to the space-time. On the other hand, an extended line source does cause uneasiness because physical sources of gravity are most often visualized as being of finite extent.

The metric function, u, for the Schwarzschild solution is a special case of a class of exact static solutions called Zipoy-Voorhees solutions (Zipoy 1966, Voorhees 1970). The metric functions for these solutions are

$$e^{2u} = [(x - 1)/(x + 1)]^{\delta} \tag{4.12}$$

$$e^{2\Omega} = (\delta^2/2)[(x^2 - 1)/(x^2 - y^2)] \tag{4.13}$$

The Schwarzchild solution is the $\delta = 1$ case of the Zipoy-Voorhees solution.

According to Chapter 3, if a new solution is found by adding an acceleration factor to a Zipoy-Voorhees solution, it will be necessary to solve for a function β (see Equation (3.11)) to completely specify the new space-time interval. From Equation (3.11), the two equations for β in (x,y) coordinates are

$$\beta_x = \tilde{r}^{-1}[\tilde{z}u_x + \rho(1-y^2)^{1/2}(x^2-1)^{-1/2}u_y] \tag{4.14}$$

$$\beta_y = \tilde{r}^{-1}[\tilde{z}u_y - \rho(x^2-1)^{1/2}(1-y^2)^{-1/2}u_y] \tag{4.15}$$

Using Equations (4.8) and (4.12), the function β is found to be

$$\beta = \delta \ \ln[(x + 2MAy - 2A\tilde{r})/(x^2 - 1)^{1/2}] \quad (4.16)$$

Another simple static axially symmetric metric is the Curzon solution (1924). The metric functions for this space-time are

$$u = -m/(\rho^2 + z^2)^{1/2} \quad (4.17)$$

$$\Omega = -(m\rho)^2/2(\rho^2 + z^2)^2 \quad (4.18)$$

Recall that the field equation for u is Laplace's equation in (ρ, z) coordinates. The function is the solution of Laplace's equation outside of a point source. Therefore, the metric function of Equation (4.17) stresses even better a point raised earlier. A novice theoretical physicist would claim that the simplest non-trivial solution to Laplace's equation

$$u_{\rho\rho} + \rho^{-1}u_{\rho} + u_{zz} = 0 \quad (4.19)$$

is given by the Equation (4.17) representing a point monopole source of strength m. Yet, as previously discussed, it is the considerably more complicated solution (Equation (4.9)) -- that of a thin rod source -- which eventually leads to the general relativistic space-time surrounding a point mass: the Schwarzschild solution. The physical interpretation of the Curzon solution is subtle -- some authors suggesting it represents the space-time exterior to a disk of radius 2m. At any rate, Weyl canonical coordinates truly compound the problem of giving a clear physical

physical interpretation to solutions expressed in (ρ, z) coordinates.

Finally, the function β (Equation (4.14)) related to the Curzon solution obeys the equations

$$\beta_\rho = m\rho[2z + 1/2A]/[\tilde{r}(\rho^2 + z^2)^{3/2}] \tag{4.20a}$$

$$\beta_z = m[z(z + 1/2A) - \rho^2]/[\tilde{r}(\rho^2 + z^2)^{3/2}] \tag{4.20b}$$

with solution

$$\beta = -2Am\tilde{r}/(\rho^2 + z^2)^{1/2}. \tag{4.20c}$$

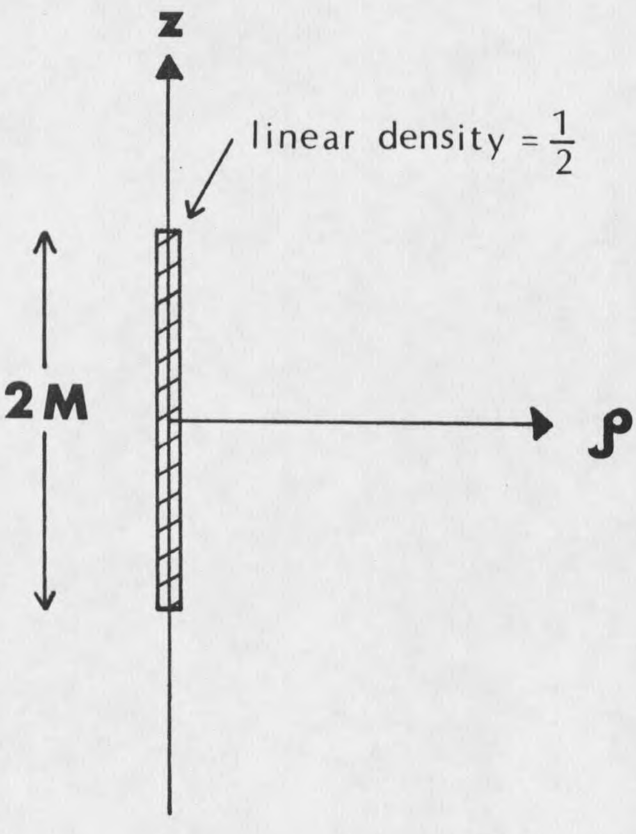

Figure 4: Source of the Schwarzchild solution in Weyl canonical coordinates

CHAPTER 5

SIMPLIFYING THE ACCELERATION FACTOR

Chapter 2 points out that flat space-time, when viewed by an accelerated observer can be written (Equations (2.9)-(2.11)):

$$ds^2 = e^{2u}dt^2 - e^{-2u}\rho^2 d\phi^2 - e^{2(\Omega-u)}[d\rho^2 + dz^2], \qquad (5.1a)$$

$$e^{2u} = A(\tilde{r} + \tilde{z}), \qquad e^{2\Omega} = (\tilde{r} + \tilde{z})/(2\tilde{r}) \qquad (5.1b)$$

$$\tilde{r}^2 = \rho^2 + \tilde{z}^2, \quad \tilde{z} = z + 1/2A \qquad (5.1c)$$

These functions were derived by performing coordinate transformations on flat space-time, so they are guaranteed to satisfy the Einstein vacuum field equations:

$$\nabla\,[\rho\nabla u] = 0 \qquad (5.2a)$$

$$\Omega_\rho = \rho[(u_\rho)^2-(u_z)^2] \qquad \Omega_z = 2\rho u_\rho u_z \qquad (5.2b)$$

The metric function g_{tt} has a line singularity at $\rho = 0$, $z = -1/2A$ and extending along the negative z-axis to minus infinity.

One of the reasons why Equations (5.1) look messy is that the coordinates (ρ, z) are centered on the point $\rho = 0$, $z = 0$, while the singularity in the metric function g_{tt} is displaced away from the origin down the z-axis. This chapter will show how an attempt to clean up this problem leads to some unexpected results.

The metric function g_{tt} in Equation (5.1b) depends on the two coordinates ρ and z. It would be simpler if g_{tt} depended on a single coordinate. Call this new coordinate ξ. The new coordinate is some function of the Weyl coordinates: $\xi = \xi(\rho, z)$. However, if g_{tt} is to only depend on ξ, then

$$\xi = \xi(y), \qquad y = \tilde{z} + \tilde{r}. \tag{5.3}$$

Assuming ξ is going to be part of a set of isotropic coordinates:

$$\xi_{\rho\rho} + \xi_{zz} = 0. \tag{5.4}$$

Using Equation (5.3), this equation becomes:

$$\xi_{yy}(y2y) + \xi_y = 0 \tag{5.5}$$

with solution:

$$\xi(y) = C_1(y)^{1/2} + C_2 \tag{5.6}$$

where C_1 and C_2 are constants of integration. To give Equation (5.1) the simplest appearance, select $C_1 = A^{-1/2}$ and $C_2 = 0$. Therefore:

$$\xi = A^{-1/2}(\tilde{r} + \tilde{z})^{1/2}. \tag{5.7}$$

The conjugate coordinate η is found by means of the Cauchy-Riemann conditions

$$\xi_\rho = -\eta_z \qquad \xi_z = \eta_\rho \tag{5.8}$$

The η coordinate is:

$$\eta = A^{-1/2}(\tilde{r} - \tilde{z})^{1/2} \tag{5.9}$$

Equations (5.7) and (5.9) can be inverted to yield:

$$2A^{-1}\tilde{z} = \xi^2 - \eta^2 \qquad A^{-1}\rho = \xi\eta \tag{5.10}$$

With this change of coordinates, the space-time interval in Equation (5.1) becomes:

$$ds^2 = A^2\xi^2 dt^2 - \eta^2 d\phi^2 - [d\xi^2 + d\eta^2]. \tag{5.11}$$

The coordinate transformation in Equation (5.10) shows that (ξ,η) are a set of parabolic cylindrical coordinates centered on the point $\rho = 0$, $z = -1/2A$. The constant coordinate curves for these coordinates are shown in Figure 5 They come about in this case by forcing g_{tt} to be a function of only one coordinate and forcing the new coordinates to maintain the isotropic form of the metric. However, this suggests changing the general static axially symmetric metric of Equation (5.1a) and the field equations (5.2) over to the (ξ,η) coordinates. Equation (5.1a) becomes:

$$ds^2 = e^{2u}dt^2 - e^{-2u}A^{-2}(\xi\eta)^2 d\phi^2 - e^{2(\Omega-u)}A^2(\eta^2 + \xi^2)[d\eta^2 + d\xi^2]. \tag{5.12}$$

and the field equations become:

$$u_{\eta\eta} + \eta^{-1}u_\eta + u_{\xi\xi} + \xi^{-1}u_\xi = 0. \tag{5.13a}$$

$$\Omega_\eta = (\eta^2 + \xi^2)^{-1}\eta\xi[\xi\langle(u_\eta)^2 - (u_\xi)^2\rangle + 2\eta u_\eta u_\xi] \tag{5.13b}$$

$$\Omega_\xi = (\eta^2 + \xi^2)^{-1}\eta\xi[-\eta\langle(u_\eta)^2 - (u_\xi)^2\rangle + 2\xi u_\eta u_\xi] \tag{5.13c}$$

In Chapter 7, Equations (5.12)-(5.13) will prove useful in a detailed analysis of radiation from new solutions.

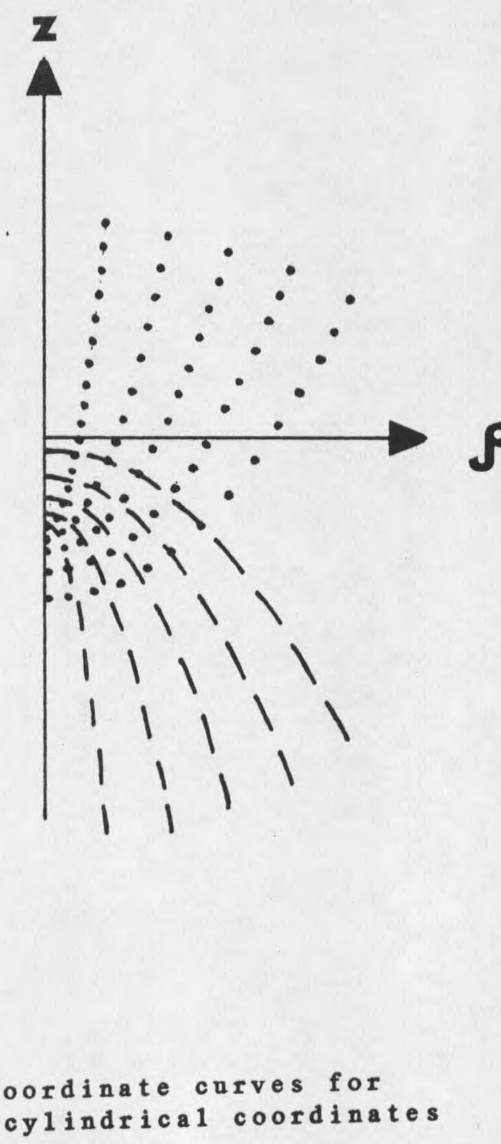

Figure 5: Constant coordinate curves for parabolic cylindrical coordinates

CHAPTER 6

THE ACTION OF $\mathfrak{Q}_{4s}$

It is well known in mathematical physics that many standard functions have representations in terms of generating functions. For example, Hermite polynomials, $H_n(x)$, can be completely defined in terms of their generating function, $F(x,t)$, as

$$F(x,t) = \exp(2xt-t^2) = \sum_{n=0}^{\infty} H_n(x)t^n/n! \qquad (6.1)$$

(Mathews and Walker (1970)). At first glance, it appears that the generating function buries all specific information about the Hermite polynomials inside an infinite summation. However, a short calculation shows how specific information about $H_n(x)$ is easily obtained.

$$F(x,t) = \exp(2xt-t^2) = \exp(x^2)\exp(-(t-x)^2)$$

$$= \exp(x^2) \sum_{n=0}^{\infty} \partial^n\{\exp(-(t-x)^2))\}/\partial t^n \; t^n/n!$$

$$= \exp(x^2) \sum_{n=0}^{\infty} (-1)^n d^n \{\exp(-x^2)\}/dx^n \; t^n/n! \quad (6.2)$$

When Equation (6.2) is compared to Equation (6.1), the functional form of $H_n(x)$ is found to be

$$H_n(x) = \exp(x^2)(-1)^n (d/dx)^n (\exp(-x^2)) \quad (6.3)$$

The main point is that a generating function contains most essential information about a set of functions.

In a series of papers Kinnersley and coworkers (Kinnersley-(1977), Kinnersley and Chitre-(1977),(1978a), (1978b) Hoenselaers, Kinnersley, and Xanthopoulos-(1979a),(1979b) discovered symmetry groups admitted by the stationary, axially symmetric Einstein-Maxwell field equations. One of the many consequences of their investigations is the fact that the symmetry transformations are most easily expressed in terms of suitably defined generating functions. The generating functions are SL(2,R) tensors and are constructed from a metric tensor in the following manner. A stationary, axially symmetric metric is expressed in the form of the SL(2,R) tensor, f_{AB}, as in Equation (1.26). Equations (1.31)-(1.33) are used to construct the complex tensor, H_{AB}. (For a concise summary of the notation of Kinnersley (1977), see Cosgrove (1980).) The tensor H_{AB} is used as the first of an infinite hierarchy of tensors.

$$H_{AB}^{(0)} = i\varepsilon_{AB},\ H_{AB}^{(1)} = H_{AB},\ H_{AB}^{(n)} = iN_{AB}^{(0,n)} \qquad (6.4\ a,b,c)$$

$$N_{AB}^{(m,n+1)} - N_{AB}^{(m+1,n)} = iN_{AX}^{(m,1)} H^{(n)X}{}_{B}$$

$$(m,n) \neq (-1,0),(0-1) \qquad (6.5)$$

$$N_{AB}^{(m,n)} - [N_{BA}^{(n,m)}]^{*} = [H_{XA}^{(m)}]^{*} H^{(n)X}{}_{B}$$

$$(m,n) \neq (0,0) \qquad (6.6)$$

$$\nabla(N_{AB}^{(m,n)}) = [H_{XA}^{(m)}]^{*} \nabla(H^{(n)X}{}_{B}), \qquad (6.7)$$

which are, in turn, employed in the definition of two generating functions

$$F_{AB}(t) = \sum_{n=0}^{\infty} H_{AB}^{(n)} t^{n} \qquad (6.8)$$

$$G_{AB}(s,t) = \sum_{n=0}^{\infty} N_{AB}^{(m,n)} s^{m} t^{n} . \qquad (6.9)$$

The two generating functions contain a dependence on the coordinates being used to describe the space-time, although only the dependence on the parameters, s,t, is shown in the above equations. If Weyl coordinates are used, the left side of Equation (6.8) is written $F_{AB}(\rho,z,t)$. It can be shown (Hoenselaers et.al. 1977) that the functional form of F_{AB} is governed by the differential equation

$$\nabla(F_{AB}) = itS^{-2}(t)[(1-2tz)\nabla(H_{AX}) - 2t\rho\tilde{\nabla}(H_{AX})]F^{X}{}_{B}. \qquad (6.10a)$$

$$S^{2}(t) = (1-2tz)^{2} + (2t\rho)^{2} \qquad (6.10b)$$

G_{AB} can always be expressed in terms of F_{AB}, so once F_{AB} is found, the function G_{AB} is automatically determined. The solution generation techniques to be examined are most cleanly expressed by formulas involving the generating functions above instead of formulas directly involving the metric tensor f_{AB}. Roughly speaking, the formulas are expressions of the form

$$(\text{New } F_{AB}) = (\text{Algebraic expression involving old } F_{AB}) \qquad (6.11)$$

Despite the fact that the generating functions involve an infinite number of tensors, it is the tensor f_{AB} which actually describes the space-time and it is this tensor which must eventually be recovered from any new F_{AB} produced. This is done using the definition of F_{AB} to give

$$f_{AB} = \mathrm{Re}\{\tfrac{d}{dt}F_{AB}(t)\}_{t=0} \qquad (6.12)$$

Cosgrove's method (1980) of solution generation stems from seeking infinitesimal transformations which do not change the form of the vacuum field equations. Cosgrove was able to exponentiate some of his infinitesimal transformations to finite transformations. The particular transformation to be looked at is denoted in Cosgrove by $\tilde{Q}_{4s}$. The effect of this transformation acting on a function is written

$$I' = \tilde{Q}_{4s}(I) \qquad (6.13)$$

The symbol s is a parameter which may vary continuously from minus infinity to plus infinity. When s = 0, Equation (6.13) becomes the identity transformation. The effect of $\tilde{Q}_{4s}$ on stationary, axially symmetric space-times is summarized by the expression

$$f'_{AB} = \tilde{Q}_{4s}(f_{AB}) = -S^{-1}(s)f_{XY}F^{X}{}_{A}(s)F^{Y}{}_{B}(s). \qquad (6.14)$$

The metric tensor f_{AB} is a known solution to the stationary, axially symmetric vacuum field equations and F_{AB} is the generating function constructed from the metric tensor f_{AB}. f'_{AB} is the new exact solution.

Until now, all expressions are written without an explicit coordinate dependence displayed. As it turns out, when $\tilde{Q}_{4s}$ acts on a function, not only is a new function produced, but $\tilde{Q}_{4s}$ also produces new coordinates. Weyl coordinates are changed according to

$$\{\rho',z'\} = \tilde{Q}_{4s}\{\rho,z\} = S^{-2}(\rho,z,s)\{\rho,z-2s(\rho^2+z^2)\} \qquad (6.15)$$

The coordinate dependence of the first part of Equation (6.14) should read

$$f'_{AB}(\rho',z',s) = \tilde{Q}_{4s}(f_{AB}(\rho,z)). \qquad (6.16)$$

In order to compare old solutions to new solutions, it is desirable to get rid of changes induced by $\tilde{Q}_{4s}$ acting on coordinates. This is accomplished by writing Equation (6.14) as

$$f'_{AB}(\rho,z,s) = -S^{-1}(\rho',z',s)f_{XY}(\rho',z')F^{X}{}_{A}(\rho',z',s) \times F^{Y}{}_{B}(\rho',z',s) \qquad (6.17)$$

where,

$$\{\rho',z'\} = \tilde{Q}_{-4s}\{\rho,z\} \qquad (6.18)$$

In the Lewis parametrization of stationary axially symmetric metrics (Equation (1.19)), the metric tensor f_{AB} has components

$$f_{11} = e^{2u}, \quad f_{12} = f_{21} = -\omega e^{2u},$$
$$f_{22} = \omega^2 e^{2u} - \rho^2 e^{-2u} \tag{6.19}$$

The (A=1,B=1) and (A=1,B=2) components of Equation (6.17) can be written in the form

$$f_{11} = (s^2 f_{11})^{-1}[(P_{11}T_1)^2 - (Q_{11}T_2)^2] \tag{6.20}$$

$$f_{12} = (s^2 f_{11})^{-1}[P_{11}P_{12}(T_1)^2 - Q_{11}Q_{12}(T_2)^2] \tag{6.21}$$

with

$$F_{AB} = P_{AB} + iQ_{AB} \tag{6.22}$$

$$2(T_1)^2 = (1-2sz) + S(\rho,z,s) \tag{6.23a}$$

$$2(T_2)^2 = -(1-2sz) + S(\rho,z,s) \tag{6.23b}$$

The auxiliary functions S, T_1, and T_2 do not depend on the original metric f_{AB} so the coordinate substitution, Equation (6.18), may be performed immediately. When this is done we get

$$S(\rho',z',s) = S^{-1}(\rho,z,-s) \tag{6.24}$$

$$T_1(\rho',z',s) = T_1(\rho,z,-s)S^{-1}(\rho,z,-s) \tag{6.25a}$$

$$T_2(\rho',z',s) = T_2(\rho,z,-s)S^{-1}(\rho,z,-s) \tag{6.25b}$$

The explicit coordinate dependence of Equations (6.20)-(6.21) is

$$f_{11}(\rho,z,s) = [s^2 f_{11}(\rho',z')S^2(\rho,z,-s)]^{-1}[\{P_{11}(\rho',z',s) \times T_1(\rho,z,-s)\}^2 - \{Q_{11}(\rho',z',s)T_2(\rho,z,-s)\}^2] \tag{6.26}$$

$$f_{12}(\rho,z,s) = [s^2 f_{11}(\rho',z') S^2(\rho,z,-s)]^{-1}$$
$$[P_{11}(\rho',z',s)P_{12}(\rho',z',s)(T_1(\rho,z,-s))^2$$
$$-Q_{11}(\rho',z',s)Q_{12}(\rho',z',s)(T_2(\rho,z,-s))^2] \qquad (6.27)$$

where the primed coordinates are defined by Equation (6.18).

Equations (6.26) and (6.27) show that whatever metric f_{AB} is acted on by $\tilde{Q}_{4s}$, the new metric is going to involve the three quantities S, T_1, and T_2. Notice that these functions can be written

$$S(\rho,z,-s) = 2s[(z+1/2s)^2 + \rho^2]^{1/2} \qquad (6.28)$$

$$2(T_1(\rho,z,-s))^2 = 2s[(z+1/2s) + (\rho^2+(z+1/2s)^2)^{1/2}] \qquad (6.29a)$$

$$2(T_2(\rho,z,-s))^2 = 2s[-(z+1/2s) + (\rho^2+(z+1/2s)^2)^{1/2}] \qquad (6.29b)$$

The expression for S in Equation (6.28) looks very much like the quantity $\tilde{r}$ defined in Equation (2.8) which came about by examining flat space-time from an accelerated observer. The symbol A in Equation (2.8) is here replaced by the symbol s. The expressions for T_1 and T_2 in Equations (6.29) look very much like the (ξ,η) parabolic cylindrical coordinates introduced in Equations (5.7) and (5.9). The zeroes of T_1 and T_2 are shown in Figure 6.

Despite the coordinate transformations and definitions of various quantities, there is one property that stands out in the analysis of Chapters 2 and 3. When considering acceleration or metrics associated with acceleration, in

Weyl coordinates the point $\rho = 0$, $z = -1/2A$ becomes a special point. Even though the point has peculiar and even inconvenient properties, it seems very much associated with accelerations of magnitude A. Equations (6.26)-(6.29) show when $\tilde{Q}_{4s}$ acts on a metric, the point $\rho = 0$, $z = -1/2s$ is a preferred point in the new metric. Therefore, it appears that $\tilde{Q}_{4s}$ has the physical effect of giving an initial metric an acceleration with magnitude s. The advantage with $\tilde{Q}_{4s}$ is that it is not confined to act on static solutions but can also act on stationary solutions. Since stationary metrics are often set up by rotating sources, the effect of $\tilde{Q}_{4s}$ on such space-times raises the tantalizing possibility of easily generating the space-time exterior to a localized, rotating, <u>and</u> accelerating source.

To show the connection between $\tilde{Q}_{4s}$ and acceleration, the effect of the transformation on a general static, axially symmetric metric is examined. The form of the metric tensor, f_{AB} is

$$f_{AB} = \begin{bmatrix} e^{2u} & 0 \\ 0 & -\rho^2 e^{-2u} \end{bmatrix} \qquad (6.30)$$

As shown by Hoenselaers et.al. (1979a), the generating functions are of the form

$$F_{11} = sS^{-1}e^{u+\beta}\ ,\ F_{12} = iS^{-1}e^{u-\beta} \qquad (6.31a,b)$$

where

$$\nabla\beta = S^{-1}[(1-2sz)\nabla u - 2s\rho\tilde{\nabla}u] \qquad (6.32)$$

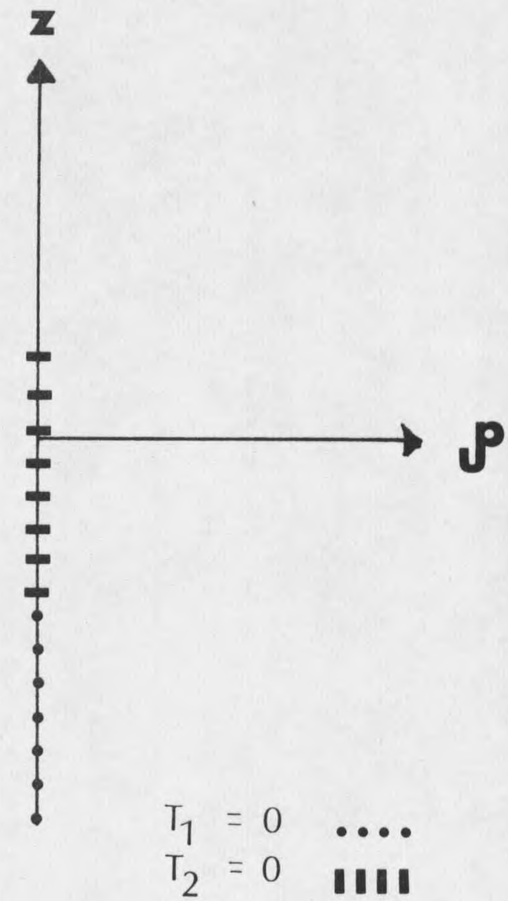

Figure 6: Zeroes of the functions in Equations (6.29a) and (6.29b)

Using these in Equations (6.26)-(6.27) gives the effect of $\tilde{Q}_{4s}$ on a general static solution.

$$u'(\rho,z,s) = \tilde{Q}_{4s}(u(\rho,z))$$
$$= (1/2)\ln[s(z+1/2s) + (\rho^2+(z+1/2s)^2)^{1/2}]$$
$$+ \beta(\rho',z',s) \qquad (6.33)$$

$$f_{12}(\rho,z,s) = \tilde{Q}_{4s}(f_{12}) = 0 \qquad (6.34)$$

Equation (6.34) says that $\tilde{Q}_{4s}$ preserves the static nature of the metric. Equation (6.33) shows that $\tilde{Q}_{4s}$ produces an acceleration factor similar to the one in Equation (2.10). This acceleration factor is added to a function β similar to the β defined in Equation (3.11). The only difference between $\tilde{Q}_{4s}$ and the analysis in Chapter 3 is that here the metric function containing an acceleration factor is added to β while in Chapter 3 the metric function containing an acceleration factor is added directly to the original metric function, u.

CHAPTER 7

RADIATION

Previous chapters have shown that the transformation of Cosgrove, $\mathfrak{Q}_{4s}$, acts on an exact solution of the Einstein vacuum field equation and produces an accelerated version of the original solution. The argument for $\mathfrak{Q}_{4s}$ producing acceleration thus far hinges on being able to compare the newly generated solution with already known solutions: flat space-time from an accelerated observer, or the C-metric. However, since the transformation $\mathfrak{Q}_{4s}$ allows new exact solutions to be easily produced, the number of new exact solutions quickly overruns the number of known solutions. For this reason, more general criteria are needed to test whether the exact solutions generated by the transformation $\mathfrak{Q}_{4s}$ really do describe the space-time geometry external to an accelerating gravitational source.

One of the most important predictions of Maxwell's electromagnetic theory that an accelerated point charge radiates energy (Jackson,1975,Chapter 14). Since electromagnetism is a linear theory, any finite size charged body (superposition of point charges) will radiate energy when the body is accelerated. This property is useful in that a distant observer can determine the attributes of the accelerating body without having to directly interact with the charged body. In particular, the radiation will identify the magnitude and direction of the acceleration.

General Relativity allows exact solutions which correspond to a source radiating energy in the form of gravitational waves. In contrast to electromagnetic theory which views the energy as being carried away by perturbations in the electric and magnetic fields, General Relativity claims the energy in gravitational radiation is carried away by perturbations in the space-time geometry around a source. In electromagnetic theory, when the electric and magnetic fields around a source decrease as:

$$(\text{fields}) \sim (\text{distance from source})^{-1}$$

the source is said to be radiating and somewhere its charges are accelerating. In General Relativity, the first partial derivatives of the metric tensor, g_{ij}, are analogous to the electric and magnetic fields. The temptation is to claim that if the first partial derivatives of the metric tensor decrease as the first inverse power of distance, then the source of gravitation is radiating. However, gravitational radiation shows up as perturbations in the tidal forces surrounding a source and the tidal forces are proportional to the second partial derivatives of the metric tensor. The quantity involving the second partial derivatives of the metric tensor is the Riemann tensor, R_{ijkl}. Therefore, roughly speaking, a source of gravitation is radiating if:

$$R_{ijkl} \sim (\text{Distance})^{-1}$$

The purpose of this chapter is to see if the above condition holds when applied to the solutions generated by the transformation $\tilde{Q}_{4s}$.

Radiation is most conveniently analyzed by looking at the space-time with respect to a complex null tetrad. Appendices C and D introduce the null tetrad notation of Newman and Penrose. In these appendices, the rotation coefficients, $\Gamma_{(a)(b)(c)}$, and the empty space Weyl tensor components, $\Psi_{(a)}$, are written out for a general stationary, axially symmetric space-time. There are a couple of reasons for using complex null tetrads to describe radiation. One reason can be seen by considering flat space-time (Equation (2.2)),

$$ds^2 = dT^2 - dZ^2 - dR^2 - R^2 d\phi^2 \qquad (7.1)$$

and a pulse of light moving down the +Z-axis. The contravariant components of the position vector of the pulse of light are,

$$X^i = [T, T, 0, 0]. \qquad (7.2)$$

The norm of this vector is

$$\underset{\sim}{X}\,\underset{\sim}{X} = X_i X^i = 0, \qquad (7.3)$$

showing that the position vector of the light pulse is a null vector. Null vectors point in the direction in which radiation will travel. Examining radiation in curved space-time may be handled by using the null directions defined by null tetrads. A second reason for using complex null tetrads is that since two of the tetrad vectors are complex conjugates of each other (see Equations (D.1c,d)), the complex conjugate of a tensor component projected onto the complex null tetrad frame is equivalent to interchanging tetrad indices (3) and (4). The number of independent tetrad components is cut in half.

Newman and Unti (1962) investigated empty space-times by examining the asymptotic behavior of Weyl tensor components when projected onto a complex null tetrad. In Appendix D, these components are given special symbols and are defined in Equations (C.3e). Newman and Unti (1962) and Janis and Newman (1965) show that these tensor components have Taylor series expansions in powers of the reciprocal of a suitably defined distance coordinate along a null direction,

$$\varphi_a = \sum_{n=0}^{\infty} C^n_{(a)}(\text{distance coordinate})^{-n-1} .$$

The coefficients in these expansions give information on whether or not a space-time is radiating energy and the shape of the gravitational radiation pattern. In this chapter the expansion coefficients for the new solutions of Chapter 3 and the transformation $\tilde{Q}_{4s}$ will now be computed for some cases and examined to see if the new solutions could represent a radiating source.

In Chapter 3, it was shown that when the metric function u ($g_{tt} = e^{2u}$) is a superposition of an acceleration factor (Equation (3.8)), and a solution to Laplace's equation (Equation(3.2)), the space-time interval can be written as Equation (3.10) with an auxiliary function β defined by Equation (3.9). Chapter 5 showed how to simplify solutions with an acceleration factor by defining two new isotropic coordinates (ξ,η). When written in terms of the (ξ,η) coordinates defined by Equations (5.7) and (5.9), the space-time interval of Equation (3.10) takes the form

$$ds^2 = A^2\xi^2 e^{2u} dt^2 - e^{-2u}\eta^2 d\phi^2 - e^{2(\Omega-\beta)}[d\xi^2 + d\eta^2]. \quad (7.4)$$

With respect to the null tetrad defined in Appendix D (Equations (D.1a)-(D.1d) with ω = 0), the components of the Weyl tensor for the metric displayed in Equation (7.4) have the form

$$\Psi_0 = 2^{-1}e^{-2\Delta}[(u_{\xi\xi}-u_{\eta\eta}-2\Delta_\xi u_\xi+2\Delta_\eta u_\eta+\xi^{-1}u_\xi-\eta^{-1}u_\eta-\xi^{-1}\Delta_\xi -\eta^{-1}\Delta_\eta$$
$$+i(2u_{\xi\eta}-2\Delta_\xi u_\eta-2\Delta_\eta u_\xi+\xi^{-1}u_\eta+\eta^{-1}u_\xi+\eta^{-1}\Delta_\xi-\xi^{-1}\Delta_\eta)] \quad (7.5a)$$

$$\Psi_2 = 2^{-1}e^{-2\Delta}[-(u_\xi)^2-(u_\eta)^2-\xi^{-1}u_\xi+\eta^{-1}u_\eta] \quad (7.5b)$$

$$\Psi_4 = -(\Psi_0)^* \quad (7.5c)$$

$$\Psi_1 = \Psi_3 = 0, \tag{7.5d}$$

where

$$\Lambda = \Omega - \beta.$$

Later it becomes necessary to consider the weak-limit limit, or linearization with respect to the function u of the above expressions. Equations (3.3) and (3.4) indicate that the function Ω is quadratic in the function u while Equation (3.9) indicates that the function β is linear in the function u. The linear approximation to Equations (7.5) is (going back to Weyl coordinates)

$$\Psi_0 \sim A[(-\tilde{z}(u_{\rho\rho}-u_{zz}) + 2\rho u_{\rho z} + 3u_z + i(\rho(u_{\rho\rho}-u_{zz}) + 2\tilde{z}u_{\rho z} + 3u_\rho)] \tag{7.6a}$$

$$\Psi_2 \sim A[-u_z + \rho^{-1}\tilde{z}u_\rho] \tag{7.6b}$$

$$\Psi_4 = -(\Psi_0)^* \tag{7.6c}$$

$$\Psi_1 = \Psi_3 = 0 \tag{7.6d}$$

As expected, Equations (7.5) and (7.6) vanish when $u = \Omega = \beta = 0$. For this case,

$$ds^2 = A^2\xi^2 dt^2 - \eta^2 d\phi^2 - d\xi^2 - d\eta^2 \tag{7.7}$$

and the coordinate transformation found by combining Equations (2.4),(5.7), and (5.9),

$$T = \xi \sinh At \tag{7.8a}$$

$$Z - z_0 - A^{-1} = \xi \cosh At \tag{7.8b}$$

$$R = \eta \tag{7.8c}$$

$$\phi = \phi \tag{7.8d}$$

puts Equation (7.7) into cylindrical spatial coordinates

(Equation (7.1)),

$$ds^2 = dT^2 - R^2 d\phi^2 - dZ^2 - dR^2.$$

With the further coordinate transformation,

$$T = U + r \tag{7.9a}$$

$$Z - z_0 - A^{-1} = r\cos\theta \tag{7.9b}$$

$$R = r\sin\theta \tag{7.9c}$$

$$\phi = \phi \tag{7.9d}$$

this flat space-time interval becomes

$$ds^2 = dU^2 + 2dUdr - r^2(d\theta)^2 - r^2\sin^2\theta d\phi^2 \tag{7.10}$$

The coordinate U represents a retarded time while the three coordinates (r, θ, ϕ) represent spherical spatial coordinates. A pulse of light traveling along a θ = constant, ϕ = constant trajectory has an equation of motion given by U = constant. Therefore, a complex null tetrad for the flat space-time in Equation (7.10) is

$$\underset{\sim}{L} = \partial/\partial r \tag{7.11a}$$

$$\underset{\sim}{N} = \partial/\partial U - 2^{-1}\partial/\partial r \tag{7.11b}$$

$$\underset{\sim}{M} = 2^{-1/2}r^{-1}[\partial/\partial\theta + i\csc\theta\partial/\partial\phi] \tag{7.11c}$$

$$\overset{*}{\underset{\sim}{M}} = 2^{-1/2}r^{-1}[\partial/\partial\theta - i\csc\theta\partial/\partial\phi] \tag{7.11d}$$

where the tetrad $\{\underset{\sim}{L}, \underset{\sim}{N}, \underset{\sim}{M}, \overset{*}{\underset{\sim}{M}}\} = \{\underset{\sim}{E}_{(a)}\}$ is expressed in the form of Equation (A.2). The tetrad vector $\underset{\sim}{L}$ points outward along a null direction and the coordinate r is a measure of spatial distance along this null direction. According to Newman and Unti, this is exactly the sort of null tetrad and coordinates to be used when examining the asymptotic behavior of the Weyl tensor to see if a space-time represents gravitational radiation. In particular, they found the leading terms of the Weyl tensor components should behave as

$$\Psi_0 = O(r^{-5}) \quad (7.12a)$$

$$\Psi_1 = O(r^{-4}) \quad (7.12b)$$

$$\Psi_2 = O(r^{-3}) \quad (7.12c)$$

$$\Psi_3 = O(r^{-2}) \quad (7.12d)$$

$$\Psi_4 = O(r^{-1}) \quad (7.12e)$$

The Weyl components expressed in Equations (7.5) or (7.6) are the components with respect to a complex null tetrad which in the flat space-time limit reduces to

$$\underset{\sim}{l} = 2^{-1/2}[A^{-1}\xi^{-1}\partial/\partial t + \eta^{-1}\partial/\partial\phi] \quad (7.13a)$$

$$\underset{\sim}{n} = 2^{-1/2}[A^{-1}\xi^{-1}\partial/\partial t - \eta^{-1}\partial/\partial\phi] \quad (7.13b)$$

$$\underset{\sim}{m} = 2^{-1/2}[\partial/\partial\xi + i\partial/\partial\eta] \quad (7.13c)$$

$$\underset{\sim}{\overset{*}{m}} = 2^{-1/2}[\partial/\partial\xi - i\partial/\partial\eta] \quad (7.13d)$$

which is, of course, a null tetrad for Equation (7.7). Such a tetrad is very convenient for explicit calculation of spin coefficients and Weyl tensor components.

A summary at this point is in order. To examine the possible radiation in space-time (and therefore the acceleration of the sources), Newman and Unti express the asymptotic limit of the Weyl tensor components, Ψ_a, expressed in a set of coordinates (U, r, θ, ϕ) and defined with respect to a complex null tetrad ($\underset{\sim}{E}_{(a)}$) which reduces to Equation (7.11) in the flat space-time limit. However, what is available in Equations (7.5) are the Weyl tensor components, $\varphi_{(a)}$, expressed in a set of coordinates (t,ϕ,ξ,η) and defined with respect to a complex null tetrad ($\underset{\sim}{e}_{(a)}$) which reduces to Equations (7.13) in the limit of flat space-time. Therefore, what must be done to consider possible radiation is to determine relations of the form,

$$\bar{\varphi}_a = \bar{\varphi}_a(\varphi_b)$$

and to express the relations in terms of (U,r,θ,ϕ) coordinates. When this is done, the Weyl components may be expanded in powers of r^{-1} to compare the asymptotic behavior with the predicted behaviors of Newman and Unti for these components.

The calculation is simplified by assuming that the metric functions are weak enough so that the linear approximation may be used. Physically this means that the accelerating sources are creating weak gravitational fields which are 'painted' on to a background of flat space-time. From a computational point of view, this means that the problem of relating the two sets of Weyl tensors reduces to expressing the null tetrad of Equations (7.11) in terms of the null tetrad of Equations (7.13). The coordinate transformations in Equations (7.8) and (7.9) give the tetrad relations to be

$$2^{1/2}\underset{\sim}{L} = (-\xi^{-1}U\cos\theta)(\underset{\sim}{l} + \underset{\sim}{n}) + (-\xi^{-1}(U+r\sin^2\theta))(\underset{\sim}{m} + \overset{*}{\underset{\sim}{m}}) + (-i\sin\theta)(\underset{\sim}{m} - \overset{*}{\underset{\sim}{m}}) \tag{7.14a}$$

$$2^{3/2}\underset{\sim}{N} = (\xi^{-1}\cos\theta(U+2r))(\underset{\sim}{l} + \underset{\sim}{n}) + (-\xi^{-1}(U + r + r\cos^2\theta))(\underset{\sim}{m} + \overset{*}{\underset{\sim}{m}}) + i\sin\theta(\underset{\sim}{m} - \overset{*}{\underset{\sim}{m}}) \tag{7.14b}$$

$$2\underset{\sim}{M} = (\xi^{-1}\sin\theta(U+r))(\underset{\sim}{l} + \underset{\sim}{n}) + (-\xi^{-1}r\sin\theta\cos\theta)(\underset{\sim}{m} + \overset{*}{\underset{\sim}{m}}) + i(\underset{\sim}{l} - \underset{\sim}{n}) - i\cos\theta(\underset{\sim}{m} - \overset{*}{\underset{\sim}{m}}) \tag{7.14c}$$

where (from Equations (7.8a) and (7.8b)),

$$\xi^2 = (Z - z_0 - A^{-1})^2 - T^2 \tag{7.15}$$

Using Equations (7.14) and the general definition of the Weyl tensor components with respect to a complex null tetrad (Equations(C.3)) a long and tedious calculation shows that

$$\begin{aligned}\bar{\varphi}_0 = {} & [2\sin^2\theta\varphi_0] \\ & + [4U\cos\theta\varphi_1]r^{-1} \\ & + [U^2\cos\theta\csc^2\theta(\cos\theta\varphi_0-4\varphi_1+3\cos\theta\varphi_2)]r^{-2} \\ & + [U^3\cos\theta\csc^4\theta(-2\cos\theta\varphi_0+(7-3\sin^2\theta)\varphi_1-6\cos\theta\varphi_2 \\ & \quad +\cos^2\theta\varphi_3)]r^{-3} \\ & + [U^4 8^{-1}\csc^6\theta((7\sin^4\theta-42\sin^2\theta+43)\varphi_{(0)} \\ & \quad +\cos\theta(72\sin^2\theta-104)\varphi_1 \\ & \quad +(24\sin^4\theta-80\sin^2\theta+68)\varphi_2 \\ & \quad +24\cos^3\theta\varphi_3+(1-2\sin^2\theta+\sin^4\theta)\varphi_4)]r^{-4}\end{aligned} \tag{7.16a}$$

$$
\begin{aligned}
\bar{\varphi}_1 = {} & [2^{1/2}\sin\theta(\cos\theta\varphi_0-\varphi_1)] \\
& + [U\cos\theta(2^{1/2}\sin\theta)^{-1})(-\varphi_0+4\cos\theta\varphi_1-3\varphi_2)]r^{-1} \\
& + [U^2\cos\theta(2^{1/2}\sin\theta)^{-3}((2+2\cos^2\theta)\varphi_0 \\
& \quad -12\cos\theta\varphi_1+(6+6\cos^2\theta)\varphi_2-3\cos\theta\varphi_3)]r^{-2} \\
& + [U^3\cos\theta(2^{1/2}\sin\theta)^{-5}((11\sin^2\theta-15)\varphi_0 \\
& \quad +12\cos\theta(4-\sin^2\theta)\varphi_1+12(3\sin^2\theta-4)\varphi_2 \\
& \quad +4(\cos\theta)(4-\sin^2\theta)\varphi_3-\cos^2\theta\varphi_4)]r^{-3}
\end{aligned} \tag{7.16b}
$$

$$
\begin{aligned}
\bar{\varphi}_2 = {} & [\cos^2\theta\varphi_0-2\cos\theta\varphi_1+\varphi_2] \\
& + [U\sec\theta\csc^2\theta(-2\cos\theta\varphi_0 \\
& \quad +(2+2\cos^2\theta)\varphi_1-6\cos\theta\varphi_2+2\varphi_3]r^{-1} \\
& + [U^2\cos\theta 4^{-1}\csc^4\theta(\cos\theta(2\cos^2\theta+5)\varphi_0 \\
& \quad -2(2+9\cos^2\theta)\varphi_1+6\cos\theta(\cos^2\theta+3)\varphi_2 \\
& \quad -4(1+\cos^2\theta)\varphi_3+\cos\theta\varphi_4)]r^{-2}
\end{aligned} \tag{7.16c}
$$

$$
\begin{aligned}
\bar{\varphi}_3 = {} & [(2^{1/2}\sin\theta)^{-1}\cos^3\theta\varphi_0-3\cos^2\theta\varphi_1 \\
& \quad +3\cos\theta\varphi_2-\varphi_3] \\
& + [U\cos\theta(2^{1/2}\sin\theta)^{-3}(-3\cos^2\theta\varphi_0 \\
& \quad +2\cos\theta(2\cos^2\theta+3)\varphi_1-3(1+3\cos^2\theta\varphi_2 \\
& \quad +6\cos\theta\varphi_3-\varphi_4)]r^{-1}
\end{aligned} \tag{7.16d}
$$

$$
\begin{aligned}
\bar{\varphi}_4 = {} & [2^{-1}\csc^2\theta 9(\cos^4\theta\varphi_0-4\cos^3\theta\varphi_1 \\
& \quad +6\cos^2\theta\varphi_2-4\cos\theta\varphi_3+\varphi_4)]
\end{aligned} \tag{7.16e}
$$

The expressions above are written as expansions in the inverse power of the r coordinate. The reason for carrying out the expansion to different orders will become apparent later. The expressions above simplify somewhat, since from Equation(7.6d), $\varphi_1 = \varphi_3 = 0$. To further analyze the expressions in Equations (7.16), a specific metric function u--a solution to the field equation (3.2)-- is needed, to compute the functional form of the remaining φ_a via Equation (7.6a)-(7.6c).

A solution to the field equation (3.2) can be written as a multipole expansion:

$$u(\rho, z) = \sum_{n=0}^{\infty} m_n (r)^{-n-1} P_n(x). \tag{7.17}$$

$$r^2 = \rho^2 + z^2, \quad x = z/r$$

The symbol m_n represents the 2^n-pole moment and $P_n(x)$ is a Legendre polynomial. The metric function u in Equation (7.17) is axially symmetric and written in Weyl canonical coordinates and is axially symmetric. When this metric function is put into Equations (7.6a)-(7.6b), the results are

$$\varphi_a = \sum_{n=0}^{\infty} \varphi_a^{(n)} \tag{7.18}$$

where

$$\varphi_0^{(n)} = Am_n n(r)^{-n-2}\csc^2\theta$$
$$[-e^{-i\theta}[P_n(\cos\theta)[8\cos^4\theta(n+2)+2\cos^2\theta(-5n-11)+(2n+5)]$$
$$+\cos\theta P_n(\cos\theta)[\cos^2\theta(-4n-6)+(4n+7)]]$$
$$-i2e^{-i\theta}\sin\theta[\cos\theta P_n(\cos\theta)[4\cos^2\theta(n+2)+(-3n-7)]$$
$$+P_{n-1}(\cos\theta)[\cos^2\theta(-2n-3)+(n+2)]]$$
$$-i3e^{i\theta}\sin\theta P_{n-1}(\cos\theta)$$
$$+i3\sin\theta e^{i2\theta}P_n(\cos\theta)]$$
$$+(3/2)m_n e^{i2\theta}P_n(\cos\theta)r^{-n-3}$$
$$-(1/2)m_n n r^{-n-3}\csc^2\theta$$
$$[P_n(\cos\theta)[8\cos^4\theta(n+2)+2\cos^2\theta(-5n-11)+(2n+5)]$$
$$+\cos\theta P_{n-1}(\cos\theta)[\cos^2\theta(-4n-6)+(4n+7)]$$
$$+i2\sin\theta[\cos\theta P_n(\cos\theta)[4\cos^2\theta(n+2)+(-3n-7)]$$
$$+P_{n-1}(\cos\theta)[\cos^2\theta(-2n-3)+(n+2)]] \qquad (7.19a)$$

$$\varphi_2^{(n)} = m_n(2r^{n+3}\sin^2\theta)^{-1}$$
$$[P_n(\cos\theta)[-\sin^2\theta+2n\cos^2\theta-n]-n\cos\theta P_{n-1}(\cos\theta)$$
$$+2Anr[\cos\theta P_n(\cos\theta)-P_{n-1}(\cos\theta)]] \qquad (7.19b)$$

$$\varphi_4^{(n)} = -(\varphi_0^{(n)})^* \qquad (7.19c)$$

When Equations (7.19) are combined with the coordinate relations between Weyl coordinates and the $(U,r,\theta,\emptyset)$ coordinates (combining Equations (7.8) and (7.9)) the following expansions are found:

$$\rho \sim (-iA\sin^2\theta)r^2+(-iAU)r+(2^{-1}iAU^2\operatorname{ctn}^2\theta) \qquad (7.20a)$$

$$z \sim (-A\sin^2\theta)r^2+(-AU)r+2^{-1}(-AU^2-1/A) \qquad (7.20b)$$

The φ_a have the following behavior with respect to the r coordinate.:

$$\varphi_0 = \sum A_0^{(n)}(U,\theta,\phi)r^{-5} \quad (7.21a)$$

$$\varphi_2 = \sum A_2^{(n)}(U,\theta,\phi)r^{-3} \quad (7.21b)$$

$$\varphi_4 = \sum A_4^{(n)}(U,\theta,\phi)r^{-1} \quad (7.21c)$$

where the expansion coefficients $A_a^{(n)}$ are only functions of the retarded time coordinate U and the angular coordinates. Appendix F lists these coefficients out to n = 2 for the summations in Equations (7.21). The important point about these expressions is the r-dependent factors outside the summation. The r behavior for these expressions may be summarized as,

$$\begin{aligned} \varphi_a &= O(r^{-5+a}) \qquad a=0,2,4 \\ &= 0 \qquad a=1,3 \end{aligned} \quad (7.22)$$

Equations (7.22) explain why Equations (7.16) were carried out to different powers of r^{-1}. When the equations (7.22) are put in Equations (7.16) the results are

$$\bar{\varphi}_a = O(r^{-5+a}) \qquad a=0,1,2,3,4 \quad (7.23)$$

or more specifically,

r^{-5} contribution to $\bar{\varphi}_0$:

$$\begin{aligned} &[2\sin^2\theta]\varphi_0 \\ &+[3U^2 \mathrm{ctn}^2\theta]\varphi_2 r^{-2} \\ &+[8^{-1}U^4 \mathrm{ctn}^4\theta \csc^2\theta]\varphi_4 r^{-4} \end{aligned} \quad (7.23a)$$

r^{-4} contribution to $\bar{\Psi}_1$:

$$[-3U2^{-1/2}\mathrm{ctn}\theta]\Psi_2 r^{-1} + [-2^{-5/2}U^3\mathrm{ctn}^3\theta\csc^2\theta]\Psi_4 r^{-3} \tag{7.23b}$$

r^{-3} contribution to $\bar{\Psi}_2$:

$$\Psi_2 + [4^{-1}U^2\mathrm{ctn}^2\theta\csc^2\theta]\Psi_4 r^{-2} \tag{7.23c}$$

r^{-2} contribution to $\bar{\Psi}_3$:

$$[-2^{-3/2}U\mathrm{ctn}\theta\csc^2\theta]\Psi_4 r^{-1} \tag{7.23d}$$

r^{-1} contribution to $\bar{\Psi}_4$:

$$[2^{-1}\csc^2\theta]\Psi_4 \tag{7.23e}$$

Equations (7.23a-e) are the Weyl tensor components for a static, axially symmetric space-time which is generated by adding an acceleration factor to a multipole solution of the field equation (Equation (3.2)). With respect to the radial dependence of these components, the components behave in the fashion predicted by Newman and Unti (see Equations (7.12a-e)), (also see Appendix G) for radiating systems. Combining Equations (7.21) and (7.23), the expressions in Equations (7.23) can be written:

r^{-5} contribution to $\bar{\Psi}_0$:

$$2\sin^2\theta\sum A_0^{(n)} + 3U^2\mathrm{ctn}^2\theta\sum A_2^{(n)} + 8^{-1}U^4\mathrm{ctn}^4\theta\csc^2\theta\sum A_4^{(n)} \tag{7.24a}$$

r^{-4} contribution to $\bar{\varphi}_1$:

$$- 3U2^{-1/2}\text{ctn}\theta\sum A_2^{(n)}$$
$$- 2^{-5/2}U^3\text{ctn}^3\theta\csc^2\theta\sum A_4^{(n)} \qquad (7.24b)$$

r^{-3} contribution to $\bar{\varphi}_2$:

$$\sum A_2^{(n)} + 4^{-1}U^2\text{ctn}^2\theta\csc^2\theta\sum A_4^{(n)} \qquad (7.24c)$$

r^{-2} contribution to $\bar{\varphi}_3$:

$$- 2^{-3/2}U\text{ctn}\theta\csc^2\theta\sum A_4^{(n)} \qquad (7.24d)$$

r^{-1} contribution to $\bar{\varphi}_4$:

$$2^{-1}\csc^2\theta\sum A_4^{(n)} \qquad (7.24e)$$

In order to argue that the leading terms of the Weyl tensor components in Equations (7.24) represent a space-time containing radiation, one would like to show that Equations (7.24) can be expressed in the form of Equations (G.1)-(G.5) in Appendix G. If only one term in the summations in Equations (7.24) is used, from Appendix F one has:

$$A_0^{(0)} = -3m_0(1 + A^2U^2\csc^2\theta)^2/(8A^7U^5) \qquad (7.25a)$$
$$A_2^{(0)} = m_0/(2A^3U^3) \qquad (7.25b)$$
$$A_4^{(0)} = 3m_0\sin^4\theta/(2A^3U^5) \qquad (7.25c)$$

and putting these expressions in Equations (7.24) gives

r^{-5} contribution to $\bar{\varphi}_0$:

$$(3m_0/2A^3U)[-\sin^2\theta(1 + A^2U^2\csc^2\theta)^2/(2A^4U^4)$$
$$+ \text{ctn}^2\theta(1 + \cos^2\theta/8) \qquad (7.26a)$$

r^{-4} contribution to $\bar{\varphi}_1$:

$$-3m_0 \operatorname{ctn}\theta(1 + \cos^2\theta)/(2^{3/2}A^3U^2) \tag{7.26b}$$

r^{-3} contribution to $\bar{\varphi}_2$:

$$m_0(4 + 3\cos^2\theta)/(8A^3U^3) \tag{7.26c}$$

r^{-2} contribution to $\bar{\varphi}_3$:

$$-3m_0 \sin\theta\cos\theta/(2^{5/2}A^3U^4) \tag{7.26d}$$

r^{-1} contribution to $\bar{\varphi}_4$:

$$3m_0 \sin^2\theta/(4A^3U^5) \tag{7.26e}$$

Comparing Equations (7.26) with the expressions in Appendix G shows that it is not possible to select a set of multipole moments (a_N in Appendix G), which will yield the expressions in Equations (7.26). Chapter 9 will give more discussion on this last issue.

CHAPTER 8

ACCELERATED ROTATING SOLUTIONS

Chapter 7 performs a radiation analysis of the space-time produced when the transformation $\mathfrak{Q}_{4s}$ acts on a static space-time. This chapter points out some differences that result when the transformation $\mathfrak{Q}_{4s}$ acts on a stationary space-time.

As mentioned following Equations (1.19)-(1.21), stationary space-times are described by two metric functions u, and ω. Instead of the static case (ω = 0) in which there is only one field equation to solve (Equation (1.19)), stationary space-times require the solution to two field equations (Equations (1.19)-(1.20. Chapter 3 showed how new solutions could be generated by adding an acceleration factor to an initial metric function -- the price being the need to solve for an auxiliary function β (Equation (3.5) and (3.9)). A similar function appears when $\mathfrak{Q}_{4s}$ acts on a general static solution (Equation (6.33)).

A key result of Chapter 3 was the fact that the auxiliary function β can be written as a Poisson integral. Since the static metric function u obeys Laplace's equation, it can be written as a Poisson integral over an axially symmetric distribution function σ. The auxiliary function β can be written as a similar Poisson integral as in Equation (3.17). However, the distribution function which appears in the Poisson integral of β is a modified distribution function σ':

$$\sigma' = \tilde{r}^{-2}\tilde{z}\sigma(\rho, z) \quad (8.1)$$

$$\tilde{z} = z + 1/(2A)$$

$$\tilde{r}^2 = \rho^2 + \tilde{z}^2.$$

This is a pleasing result for a couple of reasons. First, it means that in 'adding' an acceleration factor to generate new solutions (Chapter 3) or in letting $\mathfrak{Q}_{4s}$ act on a static solution (Chapter 6), the new solution depends in a fairly direct manner on the original solution. If the methods of Chapters 3 and 6 are giving a static solution a linear acceleration, one would expect the distributions setting up the gravity fields to appear modified when undergoing acceleration. Secondly, Equation (8.1) suggests a possible extension to the case of $\mathfrak{Q}_{4s}$ acting on a stationary solution.

Ideally, what one would like to see is that when $\mathfrak{Q}_{4s}$ acts on a stationary space-time, the new space-time is determined by two auxiliary functions β_1, and β_2, which may, in turn be written as Poisson integrals over two separate modified distribution functions. These modified distribution functions would be related to distribution functions σ_1, and σ_2, which determine the original metric functions u, and ω. Unfortunately, such an optimistic is quickly tempered. The field equations governing the metric functions u and ω are a coupled set of nonlinear equations. It is impossible to write these original functions in terms of Poisson integrals over two separate distribution functions σ_1 and σ_2, let alone hope that the new solutions is governed by two separate modified mass densities.

The action of $\mathfrak{Q}_{4s}$ on stationary solutions is further confused by the fact that the new solutions will always contain a dependence on two functions T_1 and T_2 as defined in Equations (6.28a) and (6.28b). The function T_1 is of the functional form of an acceleration factor--an infinite line singularity at $\rho = 0$, $z < -1/(2A)$. The function T_2 is also a line singularity at $\rho = 0$, $z > -1/(2A)$. This additional factor compounds the trouble in analyzing the new solutions generated by $\mathfrak{Q}_{4s}$.

Nonetheless, the transformation $\mathfrak{Q}_{4s}$ can be applied to stationary solutions. If the initial space-time is being created by rotating sources, $\mathfrak{Q}_{4s}$ should give the resulting space-time when the sources are in uniform acceleration with acceleration magnitude s. The Kerr solution (1963) represents the space-time external to a point mass with angular momentum. The result of $\mathfrak{Q}_{4s}$ acting on a Kerr solution is presented in Appendix H (Equations (H.3a,b,c)). The parameter q in those expressions represents the amount of rotation in the original Kerr solution--q = 0 represents no angular momentum (Schwarzchild solution (1916)), while q = 1 represents 'extreme Kerr'. Any radiation analysis of this accelerated Kerr solution (as in Chapter 7) is immediately hindered by the fact that the new solution is conveniently expressed in both parabolic coordinates (T_1, T_2) and in prolate spheroidal (x,y) coordinates.

However, it is possible to determine the infinite red-shift surface ($g_{tt} = 0$), for the solution in Equation (H.3a), in the limit of small rotation ($q \ll 1$). The equation for this surface is

$$x = 1 + q^2[(1-y^2)/2(1+2smy)^2]$$

where (x,y) are prolate spheroidal coordinates in Weyl canonical (ρ,z) space centered on the two points

$$\rho = 0, \qquad z = m(1-2sm)^{-1}$$

$$\rho = 0, \qquad z = -m(1+2sm)^{-1}$$

Figure 7 illustrates this red-shift surface for s = 0, and for a finite value of s. Notice how the surface changes shape for different values of the parameter s: a parameter which is directly proportional to the magnitude of the acceleration.

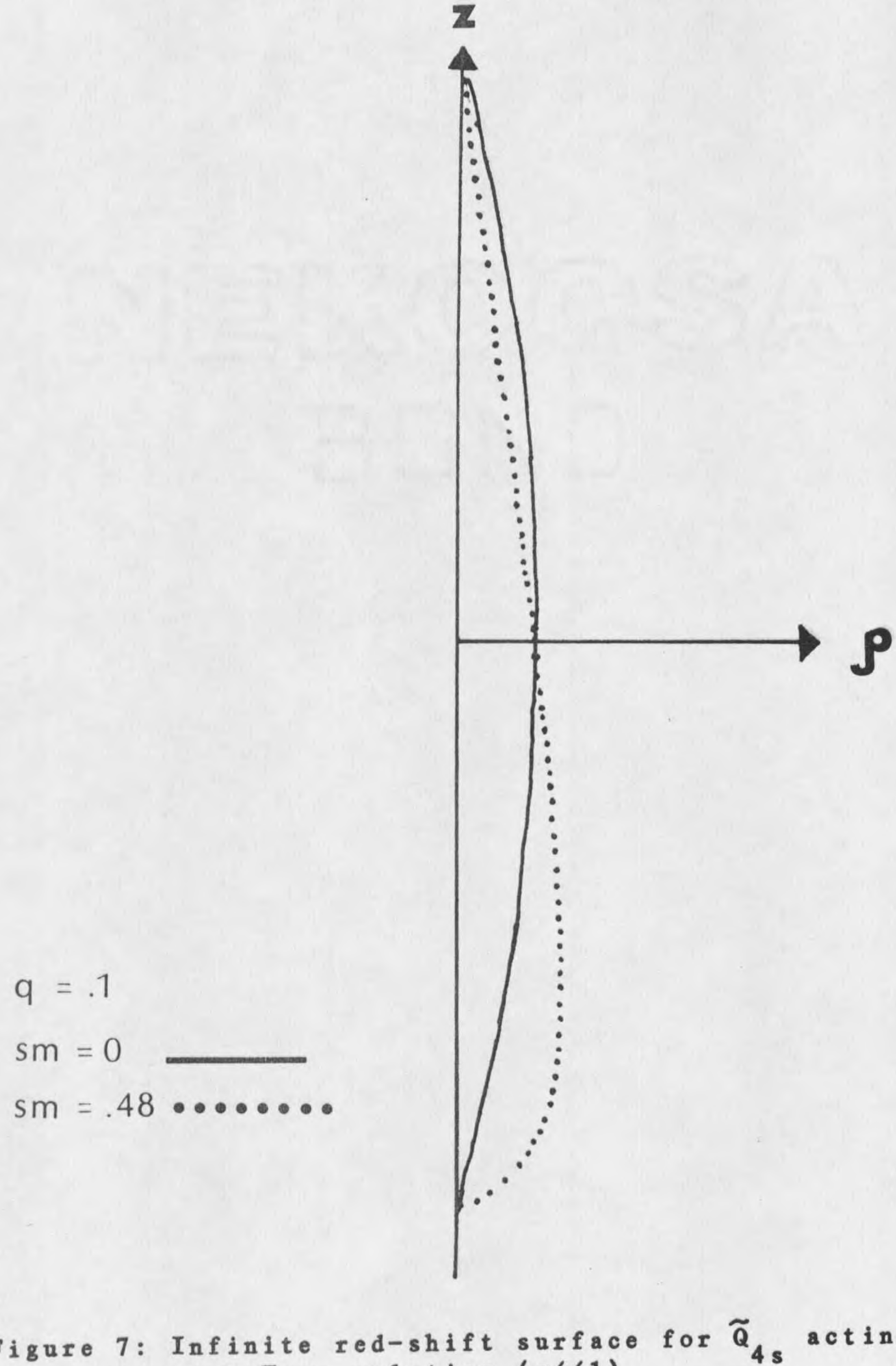

Figure 7: Infinite red-shift surface for $\tilde{Q}_{4s}$ acting on a Kerr solution ($q \lesssim 1$)

CHAPTER 9

SUMMARY AND CONCLUSIONS

In this work we have attempted to determine if one of the solution generating techniques for creating new stationary, axially symmetric space-times has an interesting and straightforward physical interpretation. The attempt has been to show, in the specific case of static, axially symmetric space-time, the transformation of Cosgrove, $\tilde{Q}_{4s}$, has the physical effect of taking an initial static, axially symmetric metric and giving it an acceleration along the symmetry axis. The new metric is an accelerated version of the original metric. This physical interpretation was surmised by examining flat space-time from the view of a uniformly accelerated observer (Chapter (2)) and comparing this with the action of $\tilde{Q}_{4s}$ on flat space-time (Chapters (3),(4),(5),(6)). In both cases, when the result is expressed in Weyl canonical coordinates, an acceleration factor appears in the metric (Equation (3.7)). This factor appears in the simplistic approach to generating solutions outlined in Chapter (3) as well as the more sophisticated treatments of Kinnersley, Cosgrove, and other workers.

The hypothesis is made that if (in Weyl canonical space) an acceleration factor represents flat space-time for an observer in linear acceleration, then an acceleration factor plus any static solution represents the space-time of that static solution when in uniform, rectilinear acceleration. Since such an accelerated system is expected to radiate gravitational energy, Chapter 7 makes an effort to see if, in fact, the new static space-times generated by the action of $\mathfrak{A}_{4s}$ represent radiation and therefore accelerated sources.

The static solution to which an acceleration factor is added (or on which $\mathfrak{A}_{4s}$ acts), is the multipole solution in Equation (7.17). This seed solution is used because it is general enough to cover a wide range of sources in Weyl ρ-z space. If $\mathfrak{A}_{4s}$ acting on this solution produces an accelerating and radiating space-time, then from the linearity of the field equation (Equation (3.2)) for static space-times, when $\mathfrak{A}_{4s}$ acts on more general sources in ρ-z space, the result should be radiation. The calculations in Chapter 7 are simplified by assuming the linear limit: in both Weyl coordinates (t, ϕ, ρ, z) and the radiation coordinates of Newman and Unti (U, r, θ, ϕ) the assumption is that the observer is far away from the accelerating source.

A comparison of Equations (7.23) and (7.12) suggests cause for initial optimism with respect to the hypothesis concerning acceleration. The Weyl tensor components of the new generated solutions exhibit correct asymptotic radial coordinate behavior. This behavior is sometimes termed 'peeling' and is analogous to the electric and magnetic fields of electromagnetism having a predictable r^{-1} dependence when the sources of the field are accelerating. At any rate, there is a certain gratification at seeing the new static space-times produced by $\mathfrak{Q}_{4s}$ demonstrate the same radial dependence as space-times containing gravitational radiation.

However, to say the new space-times are those of accelerating sources, the other coordinate (U, θ, ϕ) dependence of the Weyl tensor components must be examined. When this is done, the agreement between Equations (7.23) and what the dependence should be for radiating sources is impossible to find. The calculated behavior (with respect to the (U, θ, ϕ) coordinates) of the Weyl tensor for the assumed accelerated solutions produced by $\mathfrak{Q}_{4s}$ does not coincide with the predicted behavior suggested by Newman and Unti (1962) or Janis and Newman (1965). This disagreement is seen by examining Appendices F and G. As a matter of fact the Weyl tensor components of the new accelerated solutions demonstrate some very serious angular infinities and singularities as demonstrated in Equations (7.26).

Therefore, the initial optimism for $\mathfrak{A}_{4s}$ producing accelerated solutions may have to be tempered.

DIFFICULTIES AND FUTURE INVESTIGATIONS

i) Throughout this work, there is always the issue of coordinates. One of the elegant aspects of general relativity is the ability to formulate its laws in a coordinate free fashion. Yet, to examine physical aspects of different space-times, a set of coordinates is needed to perform calculations and express physical results. Chapter 2 showed how flat space-time can appear as in infinite line singularity in Weyl canonical coordinates and Chapter 4 showed how the metric of a Schwarzchild particle can appear as a finite rod source in Weyl canonical coordinates. In Chapter 7 issue becomes important and more difficult. The problem boils down to having to abandon Weyl ρ-z coordinates in favor of radiation coordinates $(U, r, \theta, \emptyset)$. The Weyl coordinates are useful in simplifying the algebra involved in stationary, axially symmetric space-time. The radiation coordinates are useful in considering the possible radiation from accelerating sources. The argument for the validity of the relation between these two sets of coordinates (Equations (7.8),(7.9), and (7.15)) is not as compelling as it might be. For one thing, the coordinate transformations assume a flat space-time background and yet the results are used to describe accelerating with no constraint on the magnitude of the acceleration.

ii) The Weyl tensor components demonstrate very awkward behavior in the limit if the acceleration magnitude ('s', or 'A') going to zero.

iii) One direction for future investigation would be to make a more convincing argument for the connection between the Weyl coordinates and the radiation coordinates. The key expressions summarizing this problem appear in Equations (7.20a,b). The assumptions made in arriving at these equations boil down to assuming that an observer watching the accelerating the accelerated source is in some way at a great distance from the source. The coordinate transformations are based on the assumption that when the radiation coordinate r goes to infinity this means the observer is also a great distance away. Such thinking is certainly valid when the sources of radiation are localized and the accelerations are confined to some well defined spatial region. However, the accelerations here are uniform and linear and are by no means confined to a small region of space-time. The sources here follow a world-line which is a hyperbola in space-time. The spatial motion is that of a source which deaccelerates as it comes in from minus infinity (along the symmetry axis), stops for an instant, and then accelerates back out to spatial infinity. Therefore, one might expect a more strict criterion is needed to define what is meant by an asymptotically distant observer with respect to such a linearly accelerated source.

iv) Even when such a good transformation of coordinates is developed, there will remain the problem of trying to show the space-time has a multipole structure as suggested by Janis and Newman (see Appendix G). This would amount to showing a relationship between the expansion constants $A^{(n)}$ and a_N which appear in Appendices F and G.

v) Initially, this work began by considering the physical effect of $\mathfrak{Q}_{4s}$ acting on a stationary, axially symmetric exact solution. The stationary case requires the consideration of an additional metric function ω. As Chapter 6 points out, when $\mathfrak{Q}_{4s}$ acts on stationary space-times, a set of two mirror acceleration factors appear in the new solutions (see Equations (6.25a,b)). The problems concerned with a double set of line singularities are quickly seen. However, stationary, axially symmetric space-times include rotating sources such as the Kerr solution. The action of $\mathfrak{Q}_{4s}$ on such solutions could produce the tantalizing possibility of producing an accelerated and spinning gravitational source. The action of $\mathfrak{Q}_{4s}$ acting on the Kerr solution is included in Chapter 8 and Appendix G. Nonetheless, the algebraic difficulties and problems with interpretations of the coordinates for theses cases should not be underestimated.

v) In all the solution generation schemes, the explicit forms of new solutions should be calculated and examined in different limits. Particular care should be placed on weak field limits, large acceleration factors, small acceleration factors, and the behavior of the solution for points along the symmetry axis. In particular, this work has glossed over the fact of the preferred point in the acceleration factor ($\rho=0, z=-1/(2s)$) slides up and down the z-axis for different values of the parameter 's'. If the parameter 's' is to represent the magnitude of the acceleration imparted to the source, some consideration should to what is going on when this preferred point slides around.

vi) Another interesting test of a space-time is the shape of the trajectories of point test particles or geodesic motions. It may be interesting to see how the action of $\mathfrak{Q}_{4s}$ affects the geodesics of both finite mass particles and photons.

vii) The effect of $\mathfrak{Q}_{4s}$ on different space-times could perhaps be better studied through the use of computer: both as an aid in the tedious algebra as well as visually displaying singularities, red-shift surfaces, and geodesics in the newly generated space-times. Part of the lack of computer assistance in this work is due to the author's lack of expertise in this area.

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* In the above references, the following sources,
Cosgrove (1980)
Misner, Thorne, and Wheeler (1973)
Kramer, Stephani, MacCallum, and Herlt (1980)
contain excellent and very comprehensive bibliographies concerning exact solutions, solution generation techniques, and original papers in the field of general relativity.

APPENDICES

APPENDIX A

THE TETRAD FORMALISM

The tetrad formalism is introduced in a number of places. Treatments which are particularly useful are Chandrasekhar (1979), Misner et.al. (1973, Chapter 14), and Kramer (1980, Chapter 1). Four-dimensional space-time is usually described by coordinates, x^{i}, and a metric tensor, $g_{ij}(x^{k})$. At each point of space-time a basis of four vectors is selected. The contravariant components of these vectors are written

$$e_{(a)}{}^{i} \quad (a, i = 1, 2, 3, 4) \tag{A.1}$$

The index in parenthesis is called a tetrad index and indicates which of the four tetrad vectors is being considered. The index without parenthesis is called the tensor index and indicates the particular component of a tetrad vector with respect to the coordinates x^{i}. In differential geometry, the tetrad vectors are viewed as directional derivatives and are written

$$\underset{\sim}{e}_{(a)} = e_{(a)}{}^{i}(\partial / \partial x^{i}) \tag{A.2}$$

Tensor indices are raised and lowered in the usual manner with the metric tensor g_{ij} and its inverse g^{ij}

$$e_{(a)i} = g_{ij} e_{(a)}{}^{j} \tag{A.3}$$

$$e_{(a)}{}^{i} = g^{ij} e_{(a)j} \tag{A.4}$$

The contravariant components, $e_{(a)}{}^{i}$, can be viewed as a matrix with an inverse, $e^{(a)}{}_{i}$, so that

$$e_{(a)}{}^{i} e^{(b)}{}_{i} = \delta_{(a)}{}^{(b)} \tag{A.5}$$

$$e_{(a)}{}^{i} e^{(a)}{}_{j} = \delta^{i}{}_{j} \tag{A.6}$$

The power in the tetrad formalism comes in selecting the tetrad vector components to obey

$$e_{(a)}{}^{i} e_{(b)i} = \eta_{(a)(b)} \tag{A.7}$$

where $\eta_{(a)(b)}$ is a constant symmetric matrix with signature (+,−,−,−). The definition of $\eta_{(a)(b)}$ means it can be used to raise and lower tetrad indices

$$\eta_{(a)(b)} e^{(b)i} = e_{(a)}{}^{j} e_{(b)j} e^{(b)i} = e_{(a)}{}^{j} \delta_{j}{}^{i}$$
$$= e_{(a)}{}^{i} \tag{A.8}$$

The quantities $\eta_{(a)(b)}$ and $\eta^{(a)(b)}$ are used to raise and lower tetrad indices in the same manner as the metric tensor g_{ij} and its inverse, g^{ij}, are used to raise and lower tensor indices.

Tensor components are projected on to the tetrad basis to get the tetrad components

$$A_{(a)(b)} = e_{(a)}{}^{i} e_{(b)}{}^{j} A_{ij} \tag{A.9}$$

Tetrad vectors in general have non-zero commutators

$$[\underset{\sim}{e}_{(a)}, \underset{\sim}{e}_{(b)}] = C^{(c)}{}_{(a)(b)} \underset{\sim}{e}_{(c)}. \tag{A.10}$$

The commutation coefficients, $C_{(a)(b)(c)}$ are related to the Ricci rotation coefficients, $\Gamma_{(a)(b)(c)}$

$$\Gamma_{(a)(b)(c)} = e_{(a)}{}^{m} e_{(b)m:n} e_{(c)}{}^{n} \tag{A.11}$$

by the formula

$$2(\Gamma_{(a)(b)(c)}) = C_{(b)(a)(c)} + C_{(c)(a)(b)}$$
$$- C_{(a)(b)(c)} \tag{A.12}$$

APPENDIX B

ORTHONORMAL TETRAD FOR STATIONARY AXIALLY SYMMTERIC SPACE-TIMES

Stationary, axially symmetric space-times are described by the four coordinates $(x^1=t, x^2=\phi, x^3, x^4)$ and by the space-time interval:

$$(ds)^2 = e^{2u}(dt-\omega d\phi)^2 - e^{2(B-u)}d\phi^2 - e^{2(\Omega-u)}[dx^{32} + dx^{42}]. \tag{B.1}$$

An orthornormal tetrad (Misner et.al. 1973, p.354) is a set of four vectors such that the quantity, $\eta_{(a)(b)}$, in Eq.(A.7) has the form:

$$\eta_{(a)(b)} = \begin{matrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \tag{B.2}$$

An orthonormal tetrad with respect to the coordinates and the metric defined by Eq.(B.1) is:

$$e_{(1)}{}^{i} = e^{-u}\delta_1{}^{i} \tag{B.3a}$$

$$e_{(2)}{}^{i} = e^{-B+u}[\omega\delta_1{}^{i}+\delta_2{}^{i}] \tag{B.3b}$$

$$e_{(3)}{}^{i} = e^{-\Omega+u}\delta_3{}^{i} \tag{B.3c}$$

$$e_{(4)}{}^{i} = e^{-\Omega+u}\delta_4{}^{i} \tag{B.3d}$$

When the components of the Einstein tensor, G_{ij}, are projected on to the orthonormal teteraJ of Eqs.(B.3) using Eq.(A.9), the non-zero components of $G_{(a)(b)}$ can be written in the form:

$$-e^{2(\Omega-u)}[G^{(2)}{}_{(2)} - G^{(1)}{}_{(1)}] = -e^{-B}\nabla[e^{2u}\nabla e^{B-2u}] + e^{-2B+4u}\nabla\omega\nabla\omega \tag{B.4}$$

$$2e^{2\Omega}[G^{(2)}{}_{(1)}] = \nabla[e^{-B+4u}\nabla\omega] \tag{B.5}$$

$$-e^{2(\Omega-u)}[G^{(3)}{}_{(3)} + G^{(4)}{}_{(4)}] = e^{-B}\nabla\nabla e^{B} \tag{B.6}$$

$$\begin{aligned} -e^{2(\Omega-u)}[G^{(2)}{}_{(2)} + G^{(1)}{}_{(1)}] = {} & -2^{-1}e^{-2B+4u}\nabla\omega\nabla\omega \\ & +e^{-B}\nabla\nabla e^{B} \\ & +2\nabla u\nabla u-B - 2\nabla\nabla u \\ & +2\nabla\nabla\Omega \end{aligned} \tag{B.7}$$

$$\begin{aligned} -e^{2(\Omega-u)}[G^{(3)}{}_{(3)} - G^{(4)}{}_{(4)}] = {} & 2^{-1}e^{-2B+4u}[(\omega_3)^2-(\omega_4)^2)] \\ & +B_{44}+(B_4)^2-B_{33}-(B_3)^2 \\ & +2[B_3\Omega_3-B_4\Omega_4 \\ & -(u_3)^2+(u_4)^2] \end{aligned} \tag{B.8}$$

$$\begin{aligned} -e^{(\Omega-u)}[G^{(3)}{}_{(4)}] = {} & -B_{34}-B_3B_4-2u_3u_4+2^{-1}e^{-2B+4u}\omega_3\omega_4 \\ & +\Omega_3B_4+\Omega_4B_3 \end{aligned} \tag{B.9}$$

For convenience, a symbol, ∇, is used as a two-dimensional gradient operator:

$$\nabla M\nabla N = M_3N_3 + M_4N_4 \tag{B.10a}$$

$$\nabla\nabla M = M_{33} + M_{44} \tag{B.10b}$$

APPENDIX C

NULL TETRADS AND THE NEWMAN-PENROSE FORMALISM

A null tetrad is a set of four vectors such that the quantity, $\eta_{(a)(b)}$, in Eq.(A.7) has the form:

$$\eta_{(a)(b)} = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix} \qquad (C.1)$$

The Ricci rotation coefficients are defined as in Eq.(A.11) and for a null tetrad are given the names (Newman and Penrose 1962):

$$\Gamma_{(3)(1)(1)} = k \qquad \Gamma_{(3)(1)(4)} = \rho$$

$$\Gamma_{(3)(1)(3)} = \sigma \qquad \Gamma_{(2)(4)(3)} = \mu$$

$$\Gamma_{(2)(4)(4)} = \lambda \qquad \Gamma_{(3)(1)(2)} = \tau$$

$$\Gamma_{(2)(4)(2)} = v \qquad \Gamma_{(2)(4)(1)} = \pi$$

$$(1/2)(\Gamma_{(2)(1)(1)}+\Gamma_{(3)(4)(1)}) = \varepsilon$$

$$(1/2)(\Gamma_{(2)(1)(2)}+\Gamma_{(3)(4)(2)}) = \gamma$$

$$(1/2)(\Gamma_{(2)(1)(4)}+\Gamma_{(3)(4)(4)}) = \alpha$$

$$(1/2)(\Gamma_{(2)(1)(3)}+\Gamma_{(3)(4)(3)}) = \beta \qquad (C.2)$$

The five independent components of the vacuum space Weyl tensor are (Newman and Penrose 1962):

$$\Psi_0 = -R_{(1)(3)(1)(3)} \tag{C.3a}$$

$$\Psi_1 = -R_{(1)(2)(1)(3)} \tag{C.3b}$$

$$\Psi_2 = -R_{(1)(3)(4)(2)} \tag{C.3c}$$

$$\Psi_3 = -R_{(1)(2)(4)(2)} \tag{C.3d}$$

$$\Psi_4 = -R_{(2)(4)(2)(4)} \tag{C.3e}$$

where $R_{(a)(b)(c)(d)}$ are the tetrad components of the Riemann curvature tensor.

APPENDIX D

NULL TETRAD FOR STATIONARY AXIALLY SYMMETRIC SPACE-TIMES

A null tetrad with respect to the coordinates and the metric defined by Eq.(B.1) is:

$$(2)^{1/2}\ e_{(1)}{}^{i} = (e^{-u} + \omega e^{-B+u})\delta_1{}^{i} + (e^{-B+u})\delta_2{}^{i} \qquad \text{(D.1a)}$$

$$(2)^{1/2}\ e_{(2)}{}^{i} = (e^{-u} - \omega e^{-B+u})\delta_1{}^{i} + (-e^{-B+u})\delta_2{}^{i} \qquad \text{(D.1b)}$$

$$(2)^{1/2}\ e_{(3)}{}^{i} = e^{-\Omega+u}(\delta_3{}^{i} + i\delta_4{}^{i}) = \underset{\sim}{m} \qquad \text{(D.1c)}$$

$$(2)^{1/2}\ e_{(4)}{}^{i} = e^{-\Omega+u}(\delta_3{}^{i} - i\delta_4{}^{i}) \qquad \text{(D.1d)}$$

For this null tetrad, the non-zero spin coefficients (Eqn(D.2)-Eqn(D.4) have the form:

$$k = (X-Y)^{-1}\underset{\sim}{m}(X) \qquad \text{(D.2a)}$$

$$v = (X-Y)^{-1}\underset{\sim}{\overset{*}{m}}(Y) \qquad \text{(D.2b)}$$

$$\tau = -2^{-1}\underset{\sim}{m}(B) \qquad \text{(D.2c)}$$

$$\pi = -\overset{*}{\tau} \qquad \text{(D.2d)}$$

$$\alpha = 2^{-2}(\overset{*}{k}+v) - 2^{(-3/2)}e^{-2(\Omega-u)}[(e^{\Omega-u})_3 - i(e^{\Omega-u})_4] \qquad \text{(D.2e)}$$

$$\beta = 2^{-2}(k+\overset{*}{v}) + 2^{(-3/2)}e^{-2(\Omega-u)}[(e^{\Omega-u})_3 + i(e^{\Omega-u})_4] \qquad \text{(D.2f)}$$

where the auxiliary functions:

$$X = \omega + e^{B-2u} \qquad \text{(D.3)}$$

$$Y = \omega - e^{B-2u} \qquad \text{(D.4)}$$

have been used.

The components of the Weyl tensor are:

$$\Psi_0 = -\underset{\sim}{m}(k) + 2k\tau + k(3\beta+\overset{*}{\alpha}) \qquad \text{(D.5a)}$$

$$\Psi_1 = 0 \qquad \text{(D.5b)}$$

$$\Psi_2 = \tau\overset{*}{\tau} + kv \qquad \text{(D.5c)}$$

$$\Psi_3 = 0 \qquad \text{(D.5d)}$$

$$\Psi_4 = \underset{\sim}{\overset{*}{m}}(v) - 2v\overset{*}{\tau} + v(3\alpha+\overset{*}{\beta}) \qquad \text{(D.5e)}$$

In the case of static, axially symmetric space-times

$$\omega = 0 \tag{D.6}$$

the spin coefficients reduce to:

$$k = 2^{-1}\underset{\sim}{m}(X) \tag{D.7a}$$

$$\nu = -\overset{*}{k} \tag{D.7b}$$

$$\tau = -2^{-1}\underset{\sim}{m}(B) \tag{D.7c}$$

$$\pi = -\overset{*}{\tau} \tag{D.7d}$$

$$\alpha = -2^{-1}\underset{\sim}{\overset{*}{m}}(\Omega-u) \tag{D.7e}$$

$$\beta = -\overset{*}{\alpha} \tag{D.7f}$$

where:

$$X = e^{B-2u} \tag{D.8}$$

The Weyl tensor components reduce to:

$$\Psi_0 = -\underset{\sim}{m}(k) + 2k(\tau-\overset{*}{\alpha}) \tag{D.9a}$$

$$\Psi_2 = \tau\overset{*}{\tau} - k\overset{*}{k} \tag{D.9b}$$

$$\Psi_4 = -(\Psi_0)^{*} \tag{D.9c}$$

$$\Psi_1 = \Psi_3 = 0 \tag{D.9d}$$

APPENDIX E

PROLATE SPHEROIDAL COORDINATES

Prolate spheroidal coordinates (x,y) are related to the cylindrical coordinates (ρ,z) by the relations

$$\rho = k(x^2-1)^{1/2}(1-y^2)^{1/2} \tag{E.1}$$

$$z = kxy+c \tag{E.2}$$

Geometrically,

$$x = (2k)^{-1}(r_- + r_+) \tag{E.3}$$

$$y = (2k)^{-1}(r_- - r_+). \tag{E.4}$$

where r_+ and r_- are the distances from an arbitrary point P in the (ρ,z) plane to two preferred points along the z-axis. The preferred points are centered on the point z = c and separated by a distance 2k. Figure 8 shows the constant coordinate curves for x and y. From the relations above (Equations (E.3) and (E.4)), the range of the x and y

$$1 < x < \infty \quad , \quad -1 < y < +1 . \tag{E.5}$$

The square of the distance between two points in the (ρ,z) plane with an infintesimal separation is

$$(d\rho)^2 + (dz)^2 =$$

$$k^2(x^2-y^2)[(dx)^2/(x^2-1) + (dy)^2/(1-y^2)]. \tag{E.6}$$

The relations between differential operators are

$$\partial_\rho = (x^2-1)^{1/2}(1-y^2)^{1/2}/(k(x^2-y^2))[x\partial_x - y\partial_y], \tag{E.7}$$

$$\partial_z = 1/(k(x^2-y^2))[(x^2-1)y\partial_x + (1-y^2)x\partial_y]. \tag{E.8}$$

These allow equations in Weyl canonical (ρ,z) coordinates to be expressed in prolate spheroidal coordinates.

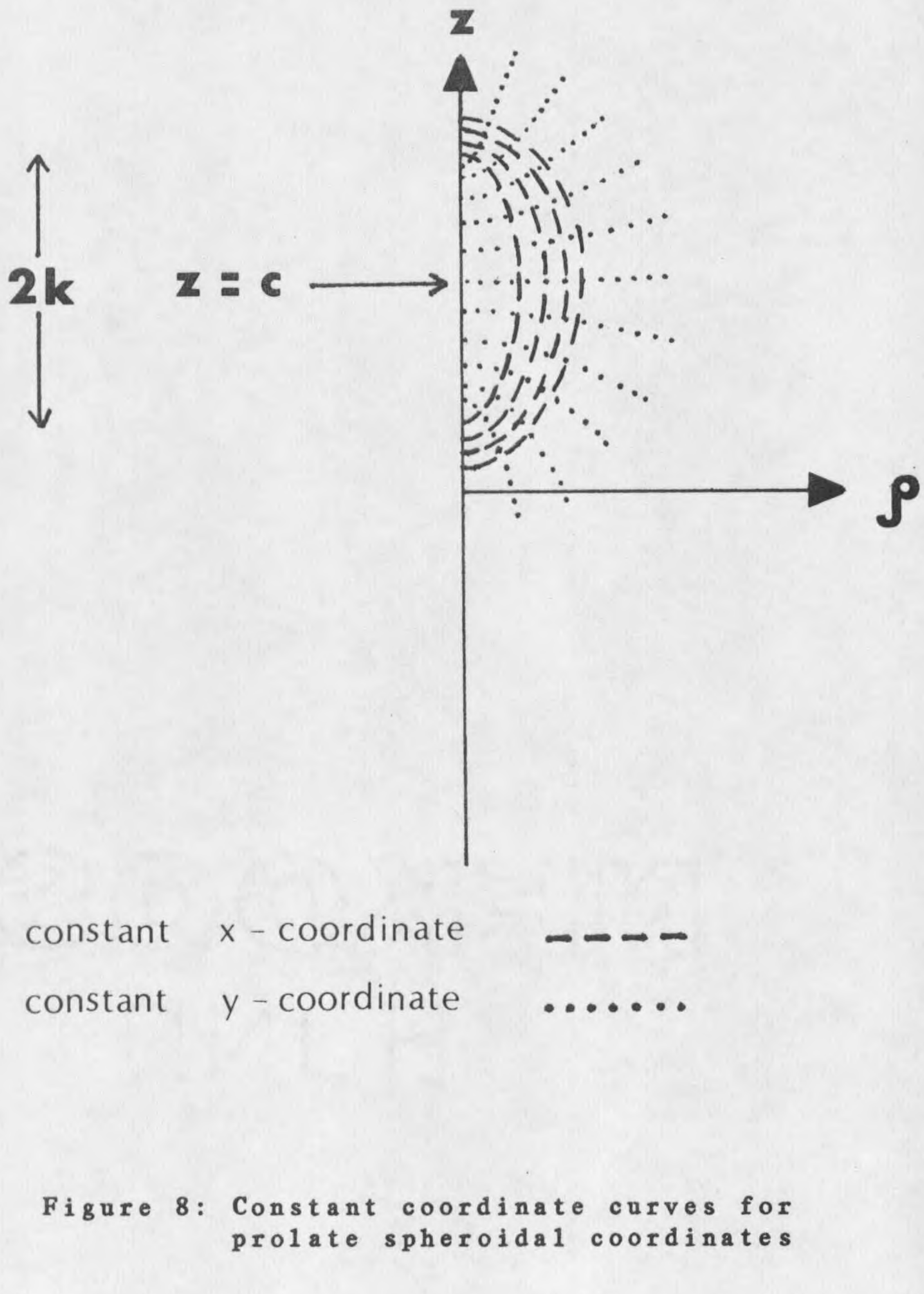

Figure 8: Constant coordinate curves for prolate spheroidal coordinates

APPENDIX F

EXPANSION COEFFICIENTS IN EQUATIONS (7.21a)-(7.21c)

The expansion coeficcients in the summations of Equations (7.21a-c) are (to order n=2),

$$A_0^{(0)} = -3m_0[1+(AU\csc\theta)^2]^2[8A^7U^5]^{-1} \quad (F.1)$$

$$A_0^{(1)} = -3m_1[1+(AU\csc\theta)^2]^2 \times [2(AU)^2-\sin^2\theta][8A^8U^7]^{-1} \quad (F.2)$$

$$A_0^{(2)} = -3m_2\sin^2\theta[1+(AU\csc\theta)^2]^2 \times [3\sin^2\theta-4(AU)^2][16A^9U^9]^{-1} \quad (F.3)$$

$$A_2^{(0)} = m_0[2A^3U^3]^{-1} \quad (F.4)$$

$$A_2^{(1)} = m_1[2(AU)^2-\sin^2\theta][2A^4U^5]^{-1} \quad (F.5)$$

$$A_2^{(2)} = -3m_2\sin^2\theta[-5\sin^2\theta+4(AU)^2] \times [4A^5U^7]^{-1} \quad (F.6)$$

$$A_4^{(0)} = 3m_0\sin^4\theta[2A^3U^5]^{-1} \quad (F.7)$$

$$A_4^{(1)} = 3m_1\sin^4\theta[2(AU)^2+5\sin^2\theta] \times [2A^4U^7]^{-1} \quad (F.8)$$

$$A_4^{(2)} = 3m_2\sin^6\theta[35\sin^2\theta-20(AU)^2] \times [8A^5U^9]^{-1} \quad (F.9)$$

The symbols m_0, m_1, and m_2 are the multipole moments in the summation of Equation (7.17).

APPENDIX G

ASYMPTOTIC STRUCTURE OF GRAVITATIONAL FIELDS IN TERMS OF LINEARIZED MULTIPOLES

In an attempt to define a multipole structure for the sources of a gravitational field, Janis and Newman (1965) show the Weyl tensor components, Ψ_a, have the general form:

$$\Psi_0 = [3a_2\sin^2\theta]r^{-5} + \text{terms of order } r^{-6} \quad (G.1)$$

$$\Psi_1 = [a_1\sin\theta-3(2)^{1/2}a_2^{(1)}\sin\theta\cos\theta]r^{-4} + \text{terms of order } r^{-5} \quad (G.2)$$

$$\Psi_2 = [a_0-(2)^{1/2}a_1^{(1)}\cos\theta+a_2^{(2)}(3\cos^2\theta 1)] + \text{terms of order } r^{-4} \quad (G.3)$$

$$\Psi_3 = [(2)^{1/2}a_2^{(3)}\sin\theta\cos\theta]r^{-2} + \text{terms of order } r^{-3} \quad (G.4)$$

$$\Psi_4 = [2^{-1}a_2^{(4)}\sin^2\theta]r^{-1} + \text{terms of order } r^{-2} \quad (G.5)$$

These expressions are written in terms of the $(U,r,,\theta,\emptyset)$ coordinates used in Chapter 7 and with respect to the tetrad defined in Equations (7.11). The symbol a_N is defined in their work to represent a 2^N-pole whose moment is proportional to a_N. The symbol a_N is, in general, a function of the coordinate U. When a the symbol a_N appears with a superscript (n), this represents the n-th partial derivative of a_N with respect to the coordinate U. The real part of the function a_n is identified as an 'electric' type of pole corresponding to masses at rest. The imaginary part of the function a_N is identified as a 'magnetic' type of pole corresponding to time-independent mass 'currents'. In their analysis the following physical assumptions are used:

First,

$$a_0^{(1)} = 0$$

physically states the conservation of 'electric' and 'magnetic' type monopoles. The condition:

$$a_1^{(2)} = 0$$

physically states there is the absence of gravitational dipole radiation. Both these conditions came about in the detailed mathematical analysis of the gravity field: in particular by considering the initial data on the surface U=constant.

The main point about the above formulJ is that it should be possible to write the expressions derived for the Ψ's of an accelerated object (Equations (7.23)) in the form of the expressions appearing in Equations (G.1)-(G.5). This would make it possible to examine the multipole structure of the radiation from the accelerated solutions generated by $\tilde{Q}_{4s}$.

APPENDIX H

TRANSFORMATION OF THE KERR SOLUTION

In the parametrization of Lewis,(Equation (1.9)) the metric functions for the Kerr solution (1963) are of the form,

$$e^{2u} = ((px)^2 + (qy)^2 - 1)((px+1)^2 + (qy)^2)^{-1}, \qquad \text{(H.1a)}$$

$$\omega = 2mq(1-y^2)(px+1)((px)^2 + (qy)^2 - 1)^{-1}, \qquad \text{(H.1b)}$$

$$e^{2\Omega} = [(px)^2 + (qy)^2 - 1][p^{-2}(x^2-y^2)]^{-1}, \qquad \text{(H.1c)}$$

with x-y being prolate spheroidal coordinates related to the Weyl coordinates by,

$$\sigma = mp, \qquad \text{(H.2a)}$$

$$\rho = \sigma((x^2-1)(1-y^2))^{1/2}, \quad z = \sigma xy, \qquad \text{(H.2b)}$$

$$q^2 = 1 - p^2. \qquad \text{(H.2c)}$$

When the transformation Q_{4s} of Cosgrove (1979) acts on the Kerr solution metric functions, the results are,

$$\begin{aligned} Q_{4s}(e^{2u}) = {} & (1-4s^2\sigma^2) \times \\ & [[(pX+\tilde{S})(p\tilde{x}-1)+q^2Y\tilde{y}](\tilde{T}_1)^2 \\ & -q^2[(pX+\tilde{S})\tilde{y}-(p\tilde{x}-1)Y](\tilde{T}_2)^2] \times \\ & [(pX)^2+(qY)^2-(\tilde{S})^2]^{-1}[(pX+\tilde{S})^2+(qY)^2]^{-1}, \end{aligned} \qquad \text{(H.3a)}$$

$$\begin{aligned} Q_{4s}(\omega) = {} & qs^{-1}(1-4s^2\sigma^2)^{-1} \times \\ & [-[(pX+\tilde{S})(p\tilde{x}-1)+q^2Y\tilde{y}][(p\tilde{x}+1)Y-(pX+\tilde{S})\tilde{y}](\tilde{T}_1)^2 \\ & +[(pX+\tilde{S})\tilde{y}-(p\tilde{x}-1)Y][q^2\tilde{y}Y+(pX+\tilde{S})(p\tilde{x}+1)](\tilde{T}_2)^2] \times \\ & [[(pX+\tilde{S})(p\tilde{x}-1)+q^2Y\tilde{y}]^2(\tilde{T}_1)^2 \\ & -q^2[(pX+\tilde{S})\tilde{y}-(p\tilde{x}-1)Y]^2(\tilde{T}_2)^2]^{-1}, \end{aligned} \qquad \text{(H.3b)}$$

$$\begin{aligned} Q_{4s}(e^{2(\Omega-u)}) = {} & (1-4s^2\sigma^2)[(pX+\tilde{S})^2+(qY)^2] \times \\ & p^{-2}(\tilde{S})^{-1}[X^2-Y^2]^{-1}, \end{aligned} \qquad \text{(H.3c)}$$

where,

$$X = \tilde{x}+2s\sigma\tilde{y}, \qquad Y = \tilde{y}+2s\sigma\tilde{x}, \tag{H.4a,b}$$

$$(\tilde{S})^2 = 1 + 4s\sigma\tilde{x}\tilde{y} + 4s^2\sigma^2(\tilde{x}^2+\tilde{y}^2-1), \tag{H.4c}$$

$$2(\tilde{T}_1)^2 = 1 + 2s\sigma\tilde{x}\tilde{y} + \tilde{S}, \tag{H.4d}$$

$$2(\tilde{T}_2)^2 = -1 - 2\sigma\tilde{x}\tilde{y} + \tilde{S}, \tag{H.4e}$$

with $\tilde{x}$ and $\tilde{y}$ being a set of prolate spheroidal coordinates centered on the two points,

$$\rho = 0, \; z = +\sigma(1-2s\sigma)^{-1}, \tag{H.5a}$$

$$\rho = 0, \; z = -\sigma(1+2s\sigma)^{-1}. \tag{H.5b}$$

DATE	ISSUED TO