



An analysis of a catadioptric system
by Allen L Hess

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree
of Master of Science in Physics
Montana State University
© Copyright by Allen L Hess (1960)

Abstract:

The image aberration produced "by a concentric backsurface reflector is studied in this thesis to determine the design's merit for use in optical instruments. The analysis is only for the case where the object is at infinity. Application of Seidel theory reveals that with a stop at the center of curvature, the design is inherently free of the off-axis primary defects of coma, astigmatism, and distortion. For a concave form primary spherical aberration can be eliminated if the thickness of the refractive element is about $2/3$ the radius of the reflecting surface. Curvature of the image is present as well as a large first order chromatism. A geometric analysis shows further that the monochromatic imaging can be made stigmatic to fifth order if the ratio' of the radius of the refractive surface to that of the reflective surface is $R1/R2 = .35689$ and if the relative index of refraction is 1.44506.

AN ANALYSIS OF A
CATADIOPTRIC SYSTEM

by

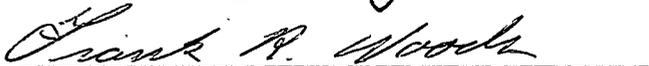
ALLEN L. HESS

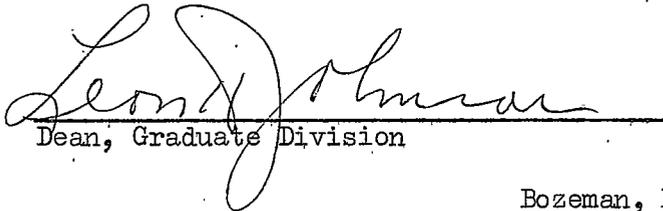
A THESIS

Submitted to the Graduate Faculty
in
partial fulfillment of the requirements
for the degree of
Master of Science in Physics
at
Montana State College

Approved:


Head, Major Department


Chairman, Examining Committee


Dean, Graduate Division

Bozeman, Montana
May, 1960

N378
H463 a
cop. 2

TABLE OF CONTENTS

ABSTRACT 3

I INTRODUCTION 4

II HISTORICAL SURVEY. 5

III ABERRATION THEORY. 15

 The Characteristic Functions of Hamilton. 15

 Schwarzschild's Perturbation Eikonal 18

 Spherical Aberration 22

 Coma 23

 Astigmatism and Curvature. 23

 Distortion 26

 The Primary Aberration Coefficients of an
 Instrument of Revolution 26

IV THE PRIMARY ABERRATIONS OF A CONCENTRIC LENS-MIRROR. 33

 Off-Axis Aberrations 34

 Spherical Aberration 35

 Curvature of Field 38

 Chromatism 40

V GEOMETRIC ANALYSIS 42

VI CONCLUSION 52

LITURATURE CONSULTED 54

ABSTRACT

The image aberration produced by a concentric backsurface reflector is studied in this thesis to determine the design's merit for use in optical instruments. The analysis is only for the case where the object is at infinity. Application of Seidel theory reveals that with a stop at the center of curvature, the design is inherently free of the off-axis primary defects of coma, astigmatism, and distortion. For a concave form primary spherical aberration can be eliminated if the thickness of the refractive element is about $2/3$ the radius of the reflecting surface. Curvature of the image is present as well as a large first order chromatism. A geometric analysis shows further that the monochromatic imaging can be made stigmatic to fifth order if the ratio of the radius of the refractive surface to that of the reflective surface is $R_1/R_2 = .35689$ and if the relative index of refraction is 1.44506.

I INTRODUCTION

Some major advances have been made comparatively recently in the field of image forming optical instrumentation with the use of catadioptric objectives, systems incorporating both refraction and reflection. This paper presents a study of a type of catadioptric system, that of a concentric backsurface reflector. We shall investigate the imaging produced by this type of reflector and determine to what extent aberrations can be reduced or eliminated. The analysis will cover only the case where the object being imaged is sufficiently distant from the objective to be considered at infinity.

A brief survey will be given of past development of reflective, refractive, and catadioptric objectives and their limitations. Next a standard theoretical optical analysis called Seidel theory will be explained and applied to our concentric lens-mirror. A further analysis will then be made by a direct geometric method. Finally the analytical results will be interpreted to evaluate the system's performance compared to other forms of objectives and to determine its possible applications.

II HISTORICAL SURVEY

In earlier development of optical instrumentation image forming objectives were in general composed solely of refractive elements or solely of reflective elements. Terrestrial and small astronomical telescopes usually consist of lenses of suitable combination to obtain the best image formation, however some aberration is always present and this limits the useful size of the instrument. Instruments such as cameras, microscopes, and projectors also are principally systems of refractive elements. Because of the difficulty in obtaining large blanks of optical material, the objectives of larger telescopes are reflective systems, relegating refractive material to the smaller auxiliary equipment.

Generally objectives of either type produce imperfect imaging. It is the object in optical design to reduce to acceptable tolerances all image errors, or aberration. The primary monochromatic aberrations are known as spherical aberration, coma, astigmatism, distortion, and curvature of field, and these enter as the third order in the angles of the light rays within the instrument.

Aside from these defects, refractive systems introduce an additional aberration called chromatism, which is due to the variance of refractive power of optical glass with wave-length. This chromatism can be reduced with proper lens shapes and arrangements, but a residual color error is always present. Reduction of this residual chromatism further requires additional lenses and thus the cost of the instrument increases. Lens combinations known as achromats, where the focal length is identical for two-lengths, or apochromats, corrected for three or more wave-lengths, constitute the objectives of high quality cameras and other instruments.

Reflectors are free from chromatism and the monochromatic aberrations of a spherical mirror are about eight times smaller than for a lens of the same power and aperture. However a difficulty arises with the reversal of the light as the object or image receiver will intercept part of the useful light. If this obstruction could be minimized and the primary aberrations reduced, it would seem preferable to use objectives of reflective elements in optical instruments.

One way of evaluating the aberration of an optical image is to determine the asphericity of the wave front in image space of a ray congruence from a point in object space. Then for good imaging we could adopt the Rayleigh criterium that the asphericity should not exceed one quarter wave-length of the light considered, such a condition is indeed required in astronomy and microscopy. This then introduces a restriction on the allowed apertures, relative apertures, and field angles. The relative aperture can be considered a general figure of merit for an instrument, as a measure of the light gathering power, and the field angle determines the angular extent of view.

Let us examine the wave aberration after reflection of parallel rays incident on a spherical mirror of focal length f . If the rays are parallel to the Z -axis and $Z=0$ is the mirror vertex, a ray at a distance y from the axis intersects the mirror at

$$Z(s) = \frac{y^2}{4f} + \frac{y^4}{64f^3} + \dots \quad (1)$$

Now for a paraboloid of the same curvature at the vertex ($R=2f$) paraxial incident rays have an exactly spherical wave front after reflection, the center at $-f$ from the mirror, and the ray intersection with the mirror is

$$Z^{(p)} = \frac{y^2}{4f} . \quad (2)$$

Thus the advance in wave front from spherical form for zone y of light reflected by a spherical mirror is

$$2 \left(Z^{(s)} - Z^{(p)} \right) = \frac{y^4}{32 f^3} + \dots \quad (3)$$

Applying the Rayleigh condition to (3) gives a maximum allowable y , and the aperture becomes $D=2y$ max. The relative aperture, $\alpha = D/f$, must then satisfy, for $\lambda = 5.5 \times 10^{-5}$ cm. ,

$$\alpha = \frac{.25}{\sqrt[4]{f}} , \quad (f \text{ in cm.}) . \quad (4)$$

For a telescope of aperture 10 cm and with a central obstruction less than 40%, this would limit the relative aperture to $\alpha = .07$ and the field angle to $.6^\circ$.

To improve on the performance of a reflective objective then, some way must be found to reduce the wave asphericity (3).

Of course a paraboloid gives perfect imaging of paraxial rays, but the imaging for oblique incident rays shows considerable coma, and all the classical telescopes suffer from this aberration. Schwarzschild in 1905 was able to improve these telescopes by figuring both mirrors to reduce the off-axis aberrations. He was able to obtain a useful field of 3° with a relative aperture of .33.

Using Schwarzschild's system analysis, Burch (3) later designed an achromatic microscope consisting of one spherical and one aspherical mirror. Numerical apertures greater than .5 are possible with his design, but not

values as high as those possible in conventional microscopes. Further, obstruction by the secondary mirror is high, generally greater than 45%.

Catadioptric systems, or objectives combining reflection and refraction, though not new, were generally neglected until about 1930. Before this some use had been made of the Mangin mirror in projectors and searchlights. This is a thin backsurface mirror where the glass thickness increases radially from the center such that the refraction corrects for the spherical aberration of the mirror.

Around 1930 however Schmidt invented a system where a refractive element is used to correct for the aberrations of a reflector. The corrector is a thin, figured plate at the center of curvature of a spherical mirror. The plate thickness varies with the zone y so as to introduce a retardation of the wave front to compensate for the advance produced by the mirror (eq. 3). Thus for correction of aberration at the Gaussian focus the thickness for zone y must exceed the axial thickness by an amount

$$T(y) - T_0 = \frac{y^4}{32(n-1)f^3} + \dots \quad (5)$$

For correction at focal points chosen near the Gaussian focus other figurings than (5) are obtained. The focal point is slightly arbitrary and can be chosen so as to minimize figuring depths or chromatism. A typical system is shown in figure 1.

Since the plate is thin and the figuring small, only negligible chromatism is introduced. Further, the correction for oblique rays is almost the same as for rays parallel to the system's axis and the residual coma and astigmatism are small.

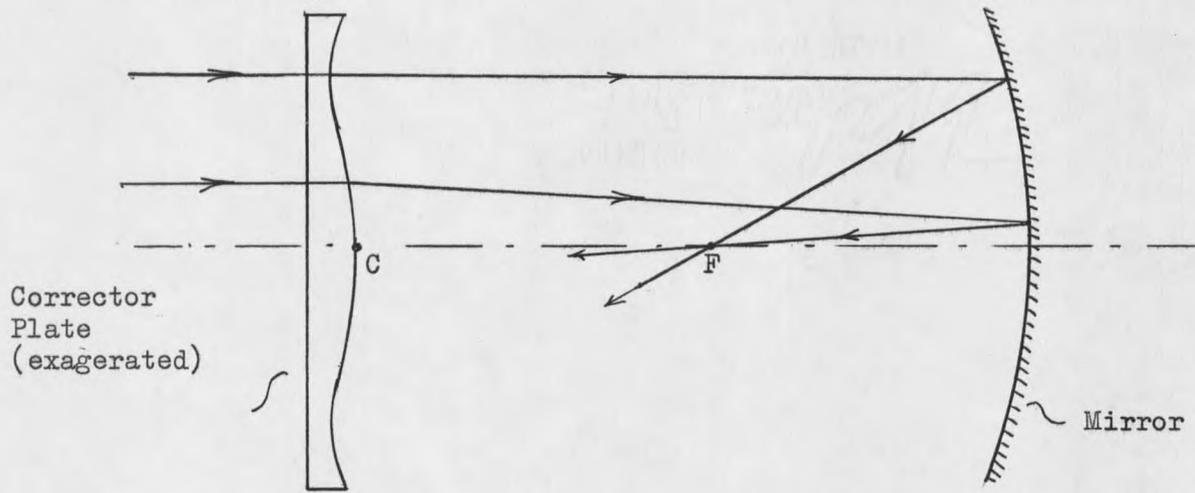


Figure 1.
Schmidt System

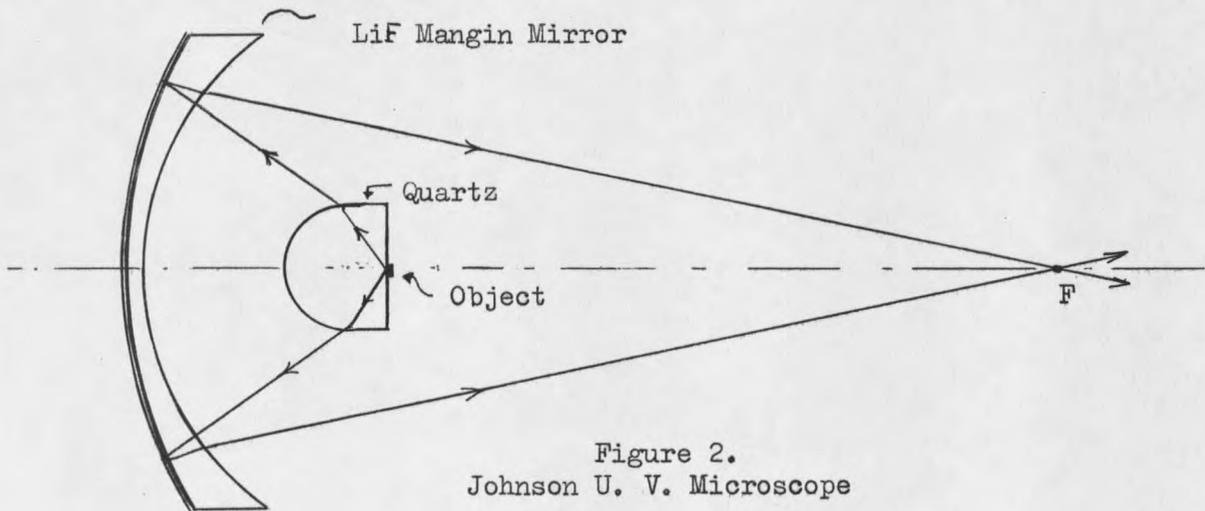


Figure 2.
Johnson U. V. Microscope

Schmidt systems have been developed to a high degree of perfection and are extensively used in astrophotography. Schmidt's first system gave excellent stellar imaging over a field of 16° with a relative aperture of .57, a considerable improvement over previous systems.

The objectives of Schwarzschild, Birch and Schmidt, as we have seen, require elements with aspheric surfaces, surfaces which are difficult to make. It would be desirable for a system to consist entirely of spherical surfaces as these are easy to produce and test. Johnson, (6) 1933 and later, (7), constructed a catadioptric microscope for use in the ultra-violet, using only spherical surfaces, which had a numerical aperture of .84 and was achromatic for two wavelengths. The objective, shown basically in figure 2, consists of a lithium fluoride Mangin mirror in conjunction with other lens elements.

Then about 1945 a new type of catadioptric system was proposed in papers by two separate researchers, Bouwers (2) of Holland, and Maksutov (10) of Russia. The concept is essentially the use of a meniscus lens of low power but large positive spherical aberration to correct the negative aberration of a spherical mirror. Moreover the meniscus, with almost constant thickness, can be made achromatic. The system is shown in figure 3. Bouwers (2) has shown that for this system correction of axial spherical aberration requires the following relationship between parameters,

$$fR_1^2 = \frac{4(n+2)}{n} F^3 \left(1 + \frac{3a-4F}{f} \right), \quad (6)$$

where f is the focal length of the meniscus, R_1 the radius of its first

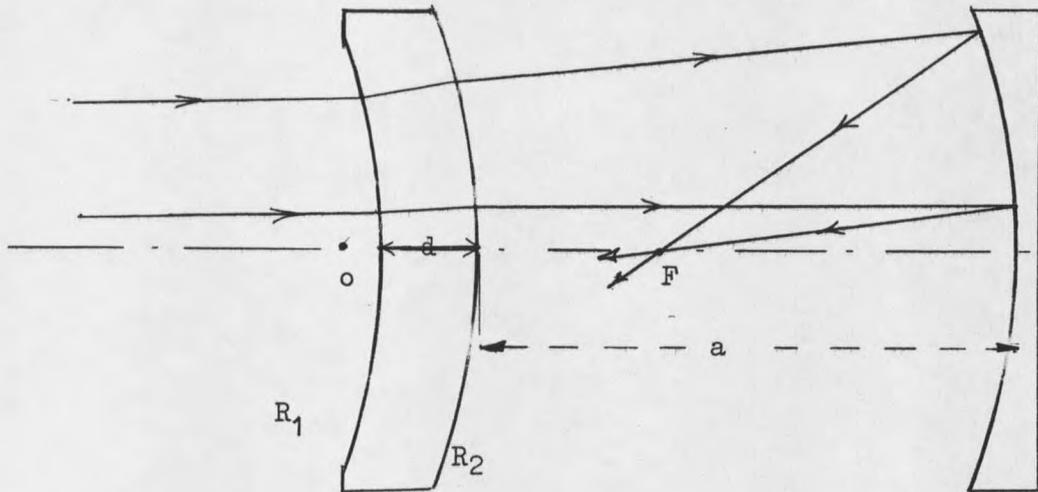


Figure 3.
Meniscus Corrector System

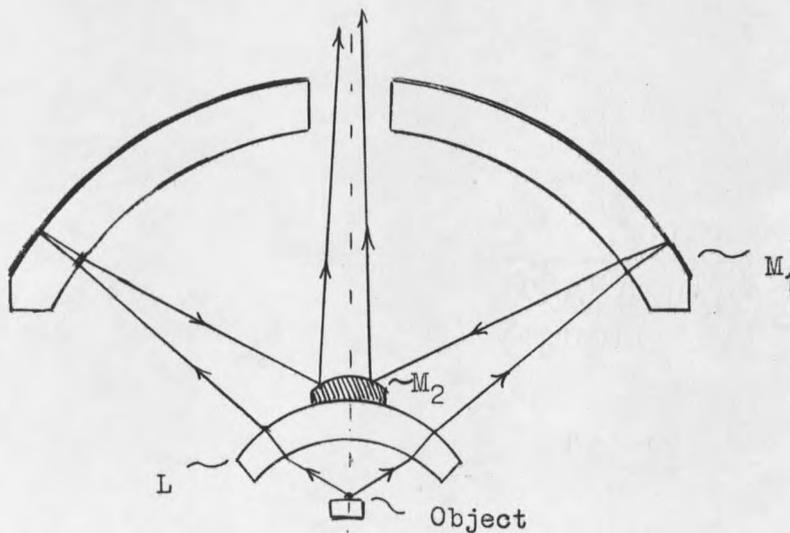


Figure 4.

U.V. Microscope Objective of Grey and Lee

surface, F the focal length of the mirror, and a the mirror-lens separation. For achromatism we need to set equal to zero the derivative of the meniscus focal length with respect to the index of refraction R . If d is the axial thickness, R_1 and R_2 the radii, we then get the condition

$$d = \frac{n^2}{n^2 - 1} (R_1 - R_2) . \quad (7)$$

For a mirror of given radius $2F$, and with conditions (6) and (7) satisfied, there remain three degrees of freedom in design and these three parameters can be chosen to minimize off-axis aberrations and chromatism.

Thus here we have a compact system consisting entirely of spherical surfaces which increases image perfection and allows for larger aperture objectives, though not quite as large as those already attained in Schmidt systems. Both inventors have been able to build satisfactory telescopes and field glasses of 10 cm. aperture with relative apertures up to .7 and field angles to 10° . Bouwers also has adapted the idea to construction of Newtonian-type microscopes where however the numerical aperture is limited to .6 and the obstruction is greater than 30%. An U.V. Microscope of better performance was obtained by Grey and Lee (4), (5), using a meniscus corrector in a Schwarzschild design. This system, with a N.A. of .9 and central obstruction less than 20%, is pictured basically in figure 4. The meniscus L provides correction of aberration for both mirrors M_1 and M_2 . Grey considered the use of a backsurface mirror for M_1 where the refraction would serve to correct for chromatism due to L .

Maksutov and Bouwers also considered meniscus mirror objectives of spherical symmetry, ie. the lens surfaces and mirror would be concentric. This requires departure from the achromatism condition, (equation 7),

but eliminates off-axis aberrations such as coma and astigmatism. This can readily be seen from the symmetry since any ray can be considered parallel to an axis of the system. Thus with a stop at the center of curvature the image pattern for object points will be the same for all field angles. The images will be points if the spherical aberration is eliminated, ie. eq. 6' is satisfied. Then the only remaining primary aberrations would be the image curvature, which is sometimes not objectionable, and chromatism which is minimized by proper choice of free parameters.

A concentric meniscus has its principal planes at the center of curvature, hence it may be considered a Schmidt corrector. That is, it behaves as a plane plate at the center of the mirror introducing a compensating spherical aberration. However it is superior to a regular Schmidt plate in that the correction is the same for oblique rays. This means the image perfection will be the same throughout the field and makes possible very wide angle systems, as in a panorama camera by Bouwers with a 90° field.

Further improvement for objectives has been found, as might have been anticipated, by combining Schmidt plate correctors with concentric menisci. Such "corrected concentric systems" have been incorporated into cameras of very low F numbers, $f/.65$ or better, like the Baker Super-Schmidt camera (14). A comparison of Schmidt, concentric meniscus, and corrected concentric systems for fields up to 60° has been given by Bouwers, (Reference 2. Page 44) and it shows the respective increase in image perfection.

In view of the performance of concentric meniscus systems and the previous use of back surface mirrors to improve imaging, it would appear possible that a concentric back surface mirror of proper design could be made free from spherical aberration. This paper presents a study of such a con-

figuration to determine its possible uses in optical objectives. The third order aberrations will first be examined using standard Seidel theory, and then a geometric method will be used to carry the analysis to 5th order.

III ABERRATION THEORY

Analysis of optical systems is classified according to the power to which ray inclination angles are expanded in terms of coordinates. First order or Gaussian theory predicts accurate imaging for rays of small inclinations and determines the basic system characteristics as the focal lengths, magnifications, etc. Seidel or third order analysis evaluates the departure from Gauss imaging when angle approximations are carried to third order and accounts for the primary aberrations previously mentioned. Instrument design proceeds first from Gaussian and then to Seidel analysis for obtaining optimal performance. For further perfection higher order theory or ray tracing methods are employed.

We shall outline the development of the Seidel theory as presented in Born and Wolf, (reference 1, chapters 4 and 5), and apply the method to study our concentric systems. First however, we need to introduce what are known as Hamilton's characteristic functions.

The Characteristic Functions of Hamilton.

Figure 5 represents a centered optical system. Let $P_0(X_0, Y_0, Z_0)$ be a point in the object medium of refractive index n_0 referred to axis at O_0 . Similarly $P_1(X_1, Y_1, Z_1)$ is a point referred to a parallel set of axis at O_1 in the image medium. A natural ray from P_0 traverses the system and passes through P_1 . We define the point characteristic function V for the points P_0 and P_1 as the optical path length along the ray, or

$$V(x_0, y_0, z_0; x_1, y_1, z_1) = \int_{P_0}^{P_1} n ds. \quad (8)$$

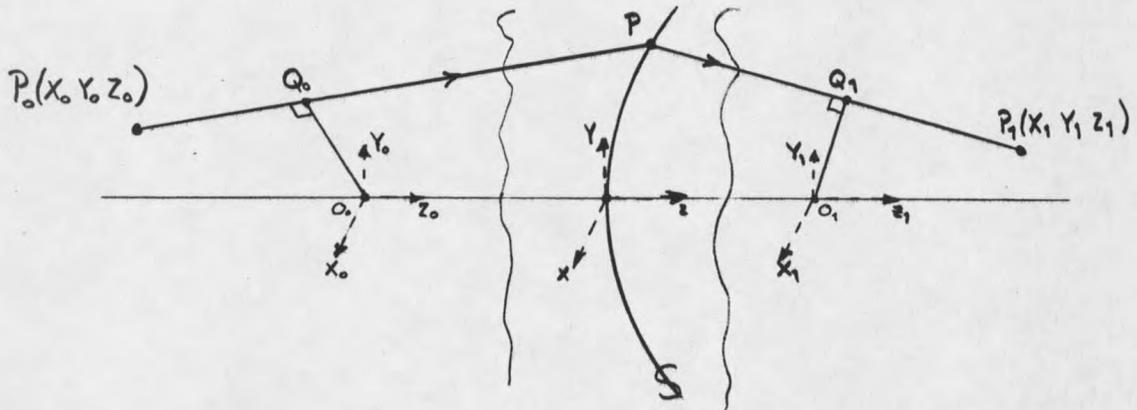


Figure 5.
Illustrating the Characteristic Functions

If \vec{s} is the unit direction vector of the ray, we have

$$\nabla_0 V = -n_0 \vec{s}_0, \quad \nabla_1 V = n_1 \vec{s}_1. \quad (9)$$

Let us call the components of the vector $n_0 \vec{s}_0$ as p_0 , q_0 , and m_0 , and for $n_1 \vec{s}_1$, p_1 , q_1 , m_1 , where in either case

$$n^2 = p^2 + q^2 + m^2. \quad (10)$$

Then clearly

$$p_0 = -\frac{\partial V}{\partial x_0}, \quad p_1 = \frac{\partial V}{\partial x_1}, \quad (11)$$

with similar q_0 , q_1 , m_0 , m_1 relations.

The angle characteristic T is defined by (12)

$$T = V + n_0 \vec{s}_0 \cdot \vec{r}_0 - n_1 \vec{s}_1 \cdot \vec{r}_1,$$

where the \vec{r} 's are the position vectors of P_0 and P_1 . - Thus T is the optical path length between points Q_0 and Q_1 in figure 5, where Q_0 and Q_1 are the intersections of the normals to the ray from the respective origins. It can be shown that T is a function only of the ray components, and because of relation (10) we have

$$T = T(p_0, q_0; p_1, q_1). \quad (13)$$

The derivatives satisfy

$$\left. \begin{aligned} X_0 - \frac{p_0}{m_0} Z_0 &= \frac{\partial T}{\partial p_0}, & Y_0 - \frac{q_0}{m_0} Z_0 &= \frac{\partial T}{\partial q_0}, \\ X_1 - \frac{p_1}{m_1} Z_1 &= \frac{\partial T}{\partial p_1}, & Y_1 - \frac{q_1}{m_1} Z_1 &= \frac{\partial T}{\partial q_1}. \end{aligned} \right\} \quad (14)$$

If the system consists of the single refracting spherical surface S at the point O a distance a_0 from O_0 and $-a_1$ from O_1 , and if $P(X, Y, Z)$ is the ray intersection, the angle characteristic becomes,

$$\begin{aligned} T &= [Q_0 P] + [P Q_1] \\ &= \{X p_0 + Y q_0 + (Z - a_0) m_0\} - \{X p_1 + Y q_1 - (Z - a_1) m_1\}. \end{aligned} \quad (15)$$

The coordinates X, Y, Z , are eliminated from (15) by use of the law of refraction and the equation of the surface S . If R is the surface radius, to fourth order T will become

$$\begin{aligned}
 T(P_0, q_0, P_1, q_1) = & n_1 a_1 - n_0 a_0 + \frac{R}{2(n_1 - n_0)} [(P_0 - P_1)^2 + (q_0 - q_1)^2] \\
 & + \frac{a_0}{2n_0} (P_0^2 + q_0^2) - \frac{a_1}{2n_1} (P_1^2 + q_1^2) + \frac{a_0}{8n_0^3} (P_0^2 + q_0^2)^2 \\
 & - \frac{a_1}{8n_1^3} (P_1^2 + q_1^2)^2 - \frac{R [(P_0 - P_1)^2 + (q_0 - q_1)^2]^2}{8(n_1 - n_0)^3} \\
 & - \frac{R [(P_0 - P_1)^2 + (q_0 - q_1)^2]}{4(n_1 - n_0)^2} \left[\frac{P_0^2 + q_0^2}{n_0} - \frac{P_1^2 + q_1^2}{n_1} \right].
 \end{aligned} \tag{16}$$

With this function we may proceed to develop a third order theory due to Schwartzschild for the study of aberrations.

Schwartzschild's Perturbation Eikonal

Schwartzschild employed a perturbation method for optical analysis. In it variables are introduced which to Gaussian accuracy have constant values along each ray through a system. The third order perturbations of these variables, or the aberrations, are then found with the help of a perturbation function. Let us illustrate an instrument of revolution as in figure 6, where we have chosen our reference coordinates O_0 and O_1 at the object plane and its Gaussian image respectively. We also introduce an entrance pupil at O_0' a distance D_0 from the object and an exit pupil at O_1' a distance D_1 from the image plane. The positions of these pupils are determined by the system's apertures and the ray aberrations are found with respect to the ray passing through the pupil centers, this ray called the principal ray.

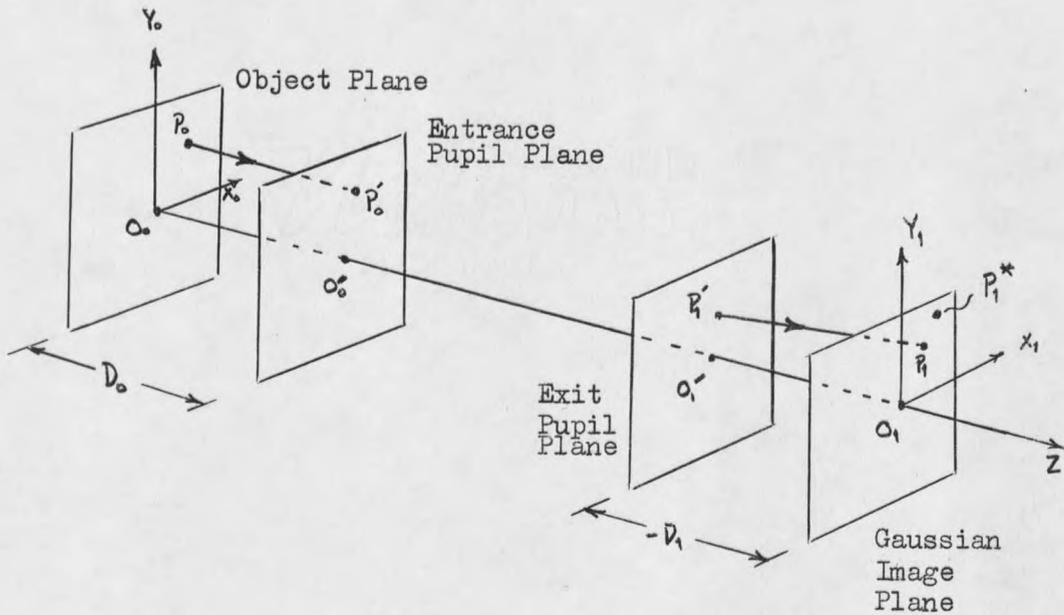


Figure 6.
Object, Image, and Pupil Planes

Thus in figure 6 P_1^* , the Gaussian image of P_0 , is the intersection of the principal ray from P_0 with the image plane. Another ray from P_0 will intersect the entrance pupil at P'_0 , the exit pupil at P'_1 and the image plane at $P_1(X_1, Y_1, Z_1)$. Then the vector $\vec{P_1^* P_1}$ is the ray aberration.

We next introduce a change of variables. Let l_0 and l_1 be the unit lengths in object and image space respectively such that

$$\frac{l_1}{l_0} = M, \quad \text{the lateral magnification.} \quad (17)$$

The new variables will be

$$\left. \begin{aligned} x_0 &= C \frac{X_0}{l_0}, & x_1 &= C \frac{X_1}{l_1}, \\ y_0 &= C \frac{Y_0}{l_0}, & y_1 &= C \frac{Y_1}{l_1}. \end{aligned} \right\} \quad (18)$$

For the pupil planes we choose unit lengths λ_0 and λ_1 , such that

$$\lambda_1 / \lambda_0 = M' , \quad (19)$$

where M' is the magnification between these planes, and use the variables

$$\xi_0 = \frac{X_0'}{\lambda_0} , \quad \xi_1 = \frac{X_1'}{\lambda_1} , \quad \eta_0 = \frac{Y_0'}{\lambda_0} , \quad \eta_1 = \frac{Y_1'}{\lambda_1} . \quad (20)$$

C is then chosen as

$$C = \frac{n_0 \lambda_0 l_0}{D_0} = \frac{n_1 \lambda_1 l_1}{D_1} . \quad (21)$$

The variables (18) and (20) are called the Seidel variables and to Gaussian accuracy

$$x_1 = x_0 , \quad y_1 = y_0 , \quad \xi_1 = \xi_0 , \quad \eta_1 = \eta_0 . \quad (22)$$

From (18), (20) and (21) we get the total transformation, to the approximation required,

$$\left. \begin{aligned} X_0 &= \frac{D_0}{n_0 \lambda_0} x_0 , & X_1 &= \frac{D_1}{n_1 \lambda_1} x_1 , \\ Y_0 &= \frac{D_0}{n_0 \lambda_0} y_0 , & Y_1 &= \frac{D_1}{n_1 \lambda_1} y_1 , \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} P_0 &= \frac{n_0 \lambda_0}{D_0} \xi_0 - \frac{x_0}{\lambda_0} , & P_1 &= \frac{n_1 \lambda_1}{D_1} \xi_1 - \frac{x_1}{\lambda_1} , \\ Q_0 &= \frac{n_0 \lambda_0}{D_0} \eta_0 - \frac{y_0}{\lambda_0} , & Q_1 &= \frac{n_1 \lambda_1}{D_1} \eta_1 - \frac{y_1}{\lambda_1} . \end{aligned} \right\} \quad (24)$$

The perturbation function introduced by Schwarzschild and called by him the Seidel eikonal is defined as

$$\psi = T + \frac{D_0}{2n_0 \lambda_0^2} (x_0^2 + y_0^2) - \frac{D_1}{2n_1 \lambda_1^2} (x_1^2 + y_1^2) + x_0 (\xi_1 - \xi_0) + y_0 (\eta_1 - \eta_0) . \quad (25)$$

Here $T = T(P_0, q_0, P_1, q_1)$ is the angle characteristic referred to origins at O_0 and O_1 . If we transform T to Seidel variables with the help of (23) and (24) and differentiate (25), using equations (14), we get the relations

$$\frac{\partial \Psi}{\partial \xi_1} = (x_1 - x_0) \quad , \quad \frac{\partial \Psi}{\partial \eta_1} = (y_1 - y_0) \quad . \quad (26)$$

Thus the ray aberrations, or deviations from Gauss imaging, can be determined from a knowledge of the eikonal Ψ .

Since the system is rotationally symmetric, Ψ depends on four variables only in the three combinations

$$r^2 = x_0^2 + y_0^2 \quad , \quad \rho^2 = \xi_1^2 + \eta_1^2 \quad , \quad \kappa^2 = x_0 \xi_1 + y_0 \eta_1 \quad . \quad (27)$$

Therefore Ψ contains only even powers of the variables. Further, from (26) and the stipulation that the aberration is third order and not linearly dependent on the coordinates, Ψ contains no second order terms and can be written

$$\Psi = \Psi^{(0)} + \Psi^{(4)} + \Psi^{(6)} + \dots \quad . \quad (28)$$

With only combinations (27) possible $\Psi^{(4)}$ must be of the form

$$\begin{aligned} \Psi^{(4)} = & -\frac{1}{4} A r^4 - \frac{1}{4} B \rho^4 - C \kappa^4 \\ & - \frac{1}{2} D r^2 \rho^2 + E r^2 \kappa^2 + F \rho^2 \kappa^2 , \end{aligned} \quad (29)$$

where the notation is in accord with standard practice. Then the aberration (26) become

$$\left. \begin{aligned} \Delta^{(3)}x &= x_1 - x_0 = \frac{\eta_1 \lambda_1}{D_1} (X_1 - X_1^*) \\ &= x_0 (2Ck^2 - Er^2 - F\rho^2) + \xi_1 (B\rho^2 + Dr^2 - 2Fk^2), \\ \Delta^{(3)}y &= y_1 - y_0 = \frac{\eta_1 \lambda_1}{D_1} (Y_1 - Y_1^*) \\ &= y_0 (2Ck^2 - Er^2 - F\rho^2) + \eta_1 (B\rho^2 + Dr^2 - 2Fk^2). \end{aligned} \right\} (30)$$

It is seen that the aberration depends on the five coefficients E, B, C, D, and F, and these evaluate the five primary aberration types heretofore listed. To determine what each means in the way of image error, let us pick a point in the Y_0-Z_0 plane as object and introduce the polar coordinates in the exit pupil

$$\eta_1 = \rho \cos \theta, \quad \xi_1 = \rho \sin \theta. \quad (31)$$

We will determine the image errors $\Delta x, \Delta y$, due to rays intersecting the exit pupil in the zone ρ .

Spherical Aberration, $B \neq 0$.

If all the coefficients but B are zero, (30) gives

$$\Delta^{(3)}x = B\rho^3 \sin \theta, \quad \Delta^{(3)}y = B\rho^3 \cos \theta. \quad (32)$$

Thus the aberration curves are circles centered on the Gaussian image point and whose radii vary as the third power of the zone ρ . This defect is independent of the object position Y_0 and is called primary spherical aberration.

Coma, $F \neq 0$.

F determines the image defect known as coma. When the other coefficients are zero, the image error becomes,

$$\left. \begin{aligned} \Delta^{(3)}x &= -Fy_0 \rho^2 \sin 2\theta \\ \Delta^{(3)}y &= -Fy_0 \rho^2 (2 + \cos 2\theta) \end{aligned} \right\}, \quad (33)$$

or $(\Delta x)^2 + (\Delta y - 2Fy_0 \rho^2)^2 = (Fy_0 \rho^2)^2$. (34)

Rays from a given zone ρ of the exit pupil trace out in the image plane a circle of radius $Fy_0 \rho^2$ whose center is $2Fy_0 \rho^2$ from the Gauss image point. As ρ takes on all values possible, the 60° wedge pattern shown in figure 7. is obtained. The size of this pattern is proportional to the off axis position y_0 of the object:

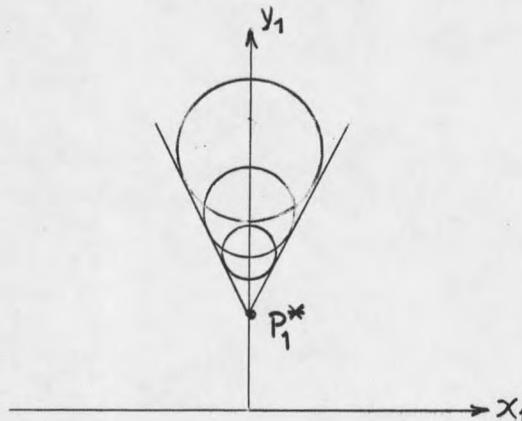


Figure 7. Coma

Astigmatism and Curvature.

If C and D only do not vanish, equations (30) become

$$\Delta^{(3)}x = \xi_1 y_0^2 D \quad , \quad \Delta^{(3)}y = (2C + D) y_0^2 \eta_1 \quad . \quad (35)$$

Generally a pencil of rays from an object point forming a normal congruence in image space will intersect two short focal lines, one lying in the meridional plane (the sagittal focal line), and the other perpendicular to that plane (the tangential focal line). A region of the object plane then gives rise to two focal surfaces, which we will approximate as being spheres. The radii R_s and R_t , of the sagittal and tangential focal surfaces respectively, can be found from the coefficients C and D . Let us find what the image pattern would be in the presence of curvature and compare to (35).

In figure 8 we have stigmatic imaging on the tangential focal sphere and in the Gauss image plane the defect

$$\frac{\Delta^{(3)}Y_1}{Y_1} = \frac{u}{D_1 + u} \quad . \quad (36)$$

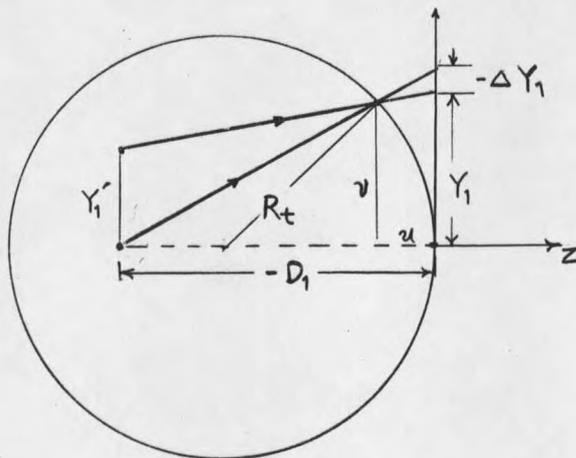


Figure 8. Illustrating Image Curvature

We have the approximations

$$\begin{aligned} v^2 &\approx 2R_t u, & v &\approx Y_1, \\ u &\approx \frac{Y_1^2}{2R_t}, & D_1 &\gg u, \end{aligned} \quad (37)$$

which give

$$\Delta^{(3)} Y_1 = \frac{Y_1^2}{2R_t} \frac{Y_1'}{D_1}. \quad (38)$$

Similarly for the sagittal focusing,

$$\Delta^{(3)} X_1 = \frac{X_1^2}{2R_s} \frac{X_1'}{D_1}. \quad (39)$$

In terms of the Seidel variables these become

$$\Delta^{(3)} y = \frac{y_1^2 \eta_1}{2n_1 R_t}, \quad \Delta^{(3)} x = \frac{y_1^2 \xi_1}{2n_1 R_s}. \quad (40)$$

Comparing to (35) we find the tangential field curvature

$$\frac{1}{R_t} = 2n_1(2C+D), \quad (41)$$

and the sagittal field curvature

$$\frac{1}{R_s} = 2n_1 D. \quad (42)$$

The mean

$$2n_1(C+D) = \frac{1}{R} \quad (43)$$

is called simply the field curvature.

The semidifference

$$\frac{1}{2} \left(\frac{1}{R_t} - \frac{1}{R_s} \right) = 2n_1 C \quad (44)$$

is called the astigmatism. When C vanishes, we see that there is but one focal surface.

Distortion.

When only E is not zero, for a general object point we have the image error

$$\Delta^{(3)} x = -E x_0 (x_0^2 + y_0^2), \quad \Delta^{(3)} y = -E y_0 (x_0^2 + y_0^2). \quad (45)$$

This defect is independent of ρ and θ and hence the imaging is stigmatic. However the off-axis image distance is not proportional to that of the object and the imaging of the object plane is distorted.

The Primary Aberration Coefficients of an Instrument of Revolution.

Now that we know the dependence of aberrations on the eikonal ψ , we must determine this function for a centered system of spherical surfaces. The angle characteristic T for the whole system referred to axial points O_0 and O_1 is obviously the sum of the characteristics for the individual surfaces, referred to the same points. From this fact and the definition of the eikonal, it can be shown that the fourth order part of ψ for a system of α surfaces is

$$\begin{aligned} \psi^{(4)} = & -\frac{1}{4} \sum_{i=1}^{\alpha} A_i r_0^4 - \frac{1}{4} \sum_{i=1}^{\alpha} B_i \rho_{\alpha}^4 - \sum_{i=1}^{\alpha} C_i K_{0\alpha}^4 \\ & - \frac{1}{2} \sum_{i=1}^{\alpha} D_i r_0^2 \rho_{\alpha}^2 + \sum_{i=1}^{\alpha} E_i r_0^2 K_{0\alpha}^2 + \sum_{i=1}^{\alpha} F_i K_{0\alpha}^2 \rho_{\alpha}^2. \end{aligned} \quad (46)$$

Hence each primary aberration coefficient of a centered system is the sum of the corresponding coefficients associated with the individual surfaces of the system.

To find the coefficients for each surface, we note first that, from equation (25),

$$\psi^{(4)}(x_0, y_0; \xi_1, \eta_1) = T^{(4)}(P_0, q_0; P_1, q_1). \quad (47)$$

From equation (16) we have

$$\begin{aligned} T^{(4)}(P_0, q_0; P_1, q_1) &= \frac{a_0}{8n_0^3} (P_0^2 + q_0^2)^2 - \frac{a_1}{8n_1^3} (P_1^2 + q_1^2)^2 \quad (48) \\ &- \frac{R}{4(n_1 - n_0)} [(P_0 - P_1)^2 + (q_0 - q_1)^2] \left[\frac{P_0^2 + q_0^2}{n_0} - \frac{P_1^2 + q_1^2}{n_1} \right] \\ &- \frac{R}{8} \cdot \frac{1}{(n_0 - n_1)^3} \cdot [(P_0 - P_1)^2 + (q_0 - q_1)^2]^2. \end{aligned}$$

Next we choose the axial points $Z = a_0$, $Z = a_1$ as locating the object plane and its Gaussian image and set

$$S = a_0, \quad S' = a_1, \quad t = a_0 + D_0, \quad t' = a_1 + D_1. \quad (49)$$

Then S is the object distance, S' the image distance and the t 's the separations of the pupils from the pole of the refracting surface. (fig.9)

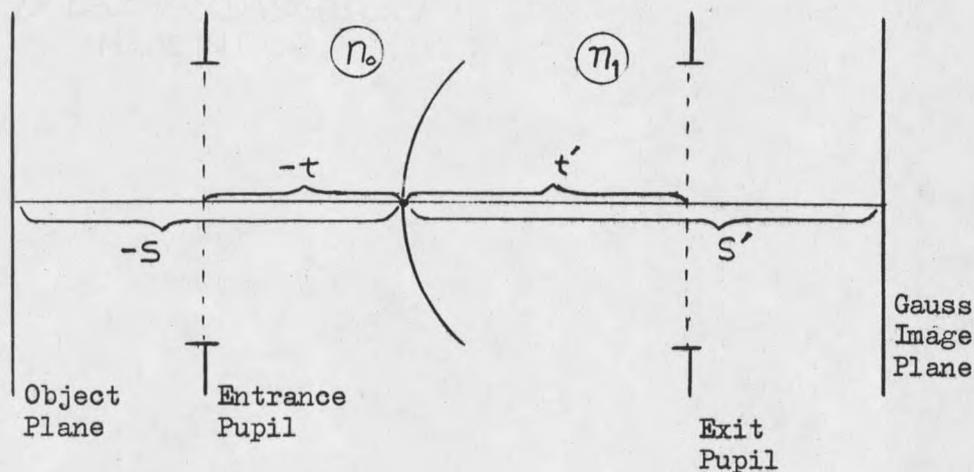


Figure 9.

Illustrating Gaussian Parameters

From Gaussian optics we have the relations between object and image distances known as the Abbe invariants K and L defined by

$$n_0 \left(\frac{1}{R} - \frac{1}{s} \right) = n_1 \left(\frac{1}{R} - \frac{1}{s'} \right) = K, \quad (50)$$

$$n_0 \left(\frac{1}{R} - \frac{1}{t} \right) = n_1 \left(\frac{1}{R} - \frac{1}{t'} \right) = L. \quad (51)$$

Using (49) and (50), (48) becomes

$$T^{(4)} = \frac{1}{8n_0s} \left\{ \frac{n_0 R}{(n_1 - n_0)^2} \left[(p_0 - p_1)^2 + (q_0 - q_1)^2 \right] - \frac{s}{n_0} (p_0^2 + q_0^2) \right\}^2 \quad (52)$$

$$- \frac{1}{8n_1s'} \left\{ \frac{n_1 R}{(n_1 - n_0)^2} \left[(p_0 - p_1)^2 + (q_0 - q_1)^2 \right] - \frac{s'}{n_1} (p_1^2 + q_1^2) \right\}^2.$$

This must be obtained in terms of the Seidel variables. In doing this we may

use the Gaussian approximation for the variables in (52), that is we transform equations (24) to

$$\left. \begin{aligned} p_0 &= \frac{n_0 \lambda_0}{D_0} \xi_1 - \frac{x_0}{\lambda_0} , & p_1 &= \frac{n_1 \lambda_1}{D_1} \xi_1 - \frac{x_0}{\lambda_1} , \\ q_0 &= \frac{n_0 \lambda_0}{D_0} \eta_1 - \frac{y_0}{\lambda_0} , & q_1 &= \frac{n_1 \lambda_1}{D_1} \eta_1 - \frac{y_0}{\lambda_1} . \end{aligned} \right\} (53)$$

We can find the Gaussian magnifications by noting that the imaging by a spherical surface is a projection from the sphere's center or

$$\frac{l_1}{l_0} = \frac{R-s'}{R-s} = \frac{n_0 s'}{n_1 s} , \quad \frac{\lambda_1}{\lambda_0} = \frac{R-t'}{R-t} = \frac{n_0 t'}{n_1 t} , \quad (54)$$

where (50) and (51) were used.

Introducing abbreviations

$$h = \frac{\lambda_0 s}{D_0} = \frac{\lambda_1 s'}{D_1} , \quad H = \frac{t}{\lambda_0 n_0} = \frac{t'}{\lambda_1 n_1} , \quad (55)$$

equations (53) become

$$\left. \begin{aligned} p_0 &= n_0 \left(\frac{h \xi_1}{s} - \frac{H x_0}{t} \right) , & p_1 &= n_1 \left(\frac{h \xi_1}{s'} - \frac{H x_0}{t'} \right) , \\ q_0 &= n_0 \left(\frac{h \eta_1}{s} - \frac{H y_0}{t} \right) , & q_1 &= n_1 \left(\frac{h \eta_1}{s'} - \frac{H y_0}{t'} \right) . \end{aligned} \right\} (56)$$

Substitution of (56) into (52), using also the rotational invariants

$$r^2 = x_0^2 + y_0^2 , \quad \rho^2 = \xi_1^2 + \eta_1^2 , \quad \kappa^2 = x_0 \xi_1 + y_0 \eta_1 , \quad (57)$$

finally gives

$$\begin{aligned}
 T^{(4)} = \psi^{(4)} &= \frac{r^4 H^4}{8} \left\{ (K-L)^2 \left(\frac{S}{n_0 t^2} - \frac{S'}{n_1 t'^2} \right) \right. \\
 &\quad \left. + L^2 \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) - 2L(K-L) \left(\frac{1}{n_0 t} - \frac{1}{n_1 t'} \right) \right\} \\
 &+ \frac{\rho^4 h^4}{8} K^2 \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) + \frac{\kappa^4 H^2 h^2 L^2}{2} \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) \\
 &- \frac{1}{2} \rho^2 \kappa^2 H h^3 K L \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) \tag{58} \\
 &+ \frac{r^2 \rho^2 H^2 h^2}{2} \left\{ K L \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) - K(K-L) \left(\frac{1}{n_0 t} - \frac{1}{n_1 t'} \right) \right\} \\
 &- \frac{r^2 \kappa^2 H^3 h}{2} \left\{ L^2 \left(\frac{1}{n_0 s} - \frac{1}{n_1 s'} \right) - L(K-L) \left(\frac{1}{n_0 t} - \frac{1}{n_1 t'} \right) \right\} .
 \end{aligned}$$

This is the desired expression for one refracting surface. If in a centered system the separation of the i th and $(i+1)$ th surface is d_i , we have the transfer relations

$$S_{i+1} = S'_i - d_i \quad , \quad t_{i+1} = t'_i - d_i \tag{59}$$

Then the progressive Gaussian imaging is found with the help of (59) and the Abbe invariants

$$\left. \begin{aligned} n_{i-1} \left(\frac{1}{R_i} - \frac{1}{S_i} \right) &= n_i \left(\frac{1}{R_i} - \frac{1}{S_i'} \right) = K_i , \\ n_{i-1} \left(\frac{1}{R_i} - \frac{1}{t_i} \right) &= n_i \left(\frac{1}{R_i} - \frac{1}{t_i'} \right) = L_i . \end{aligned} \right\} \quad (60)$$

We have also the relation

$$D_i = t_i' - S_i' = t_{i+1} - S_{i+1} . \quad (61)$$

Choosing the unit length λ_0 in the entrance pupil as unity, from (55) and (61) the following relations are obtained for the determination of H and h for the i th surface:

$$\left. \begin{aligned} H_1 &= t_1 / n_0 , & H_{i+1} &= H_i \cdot \frac{t_{i+1}}{t_i} , \\ h_1 &= \frac{S_1}{t_1 - S_1} , & h_{i+1} &= h_i \cdot \frac{S_{i+1}}{S_i'} . \end{aligned} \right\} \quad (62)$$

Finally from equations (47) and (58) we obtain the expressions for the determination of the primary aberration coefficients;

$$B = \frac{1}{2} \sum_{i=1}^{\alpha} h_i^4 K_i^2 \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right),$$

$$C = \frac{1}{2} \sum_{i=1}^{\alpha} H_i^2 h_i^2 L_i^2 \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right), \quad (63)$$

$$D = \frac{1}{2} \sum_{i=1}^{\alpha} H_i^2 h_i^2 \left\{ K_i L_i \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right) - K_i (K_i - L_i) \left(\frac{1}{n_i t_i'} - \frac{1}{n_{i-1} t_i} \right) \right\},$$

$$E = \frac{1}{2} \sum_{i=1}^{\alpha} H_i^3 h_i L_i \left\{ L_i \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right) - (K_i - L_i) \left(\frac{1}{n_i t_i'} - \frac{1}{n_{i-1} t_i} \right) \right\},$$

$$F = \frac{1}{2} \sum_{i=1}^{\alpha} H_i h_i^3 K_i L_i \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right).$$

IV THE PRIMARY ABERRATIONS OF A CONCENTRIC BACKSURFACE MIRROR

The third order analysis developed in the last sections will now be applied to the optical system consisting of a spherical back surface mirror where refracting and reflecting surfaces are concentric. The treatment will include both the convex and the concave form of this design by following proper sign conventions. By placing a stop at the center of curvature we thereby locate the entrance and exit pupils at that point. The resulting configuration is shown in Figure 10.

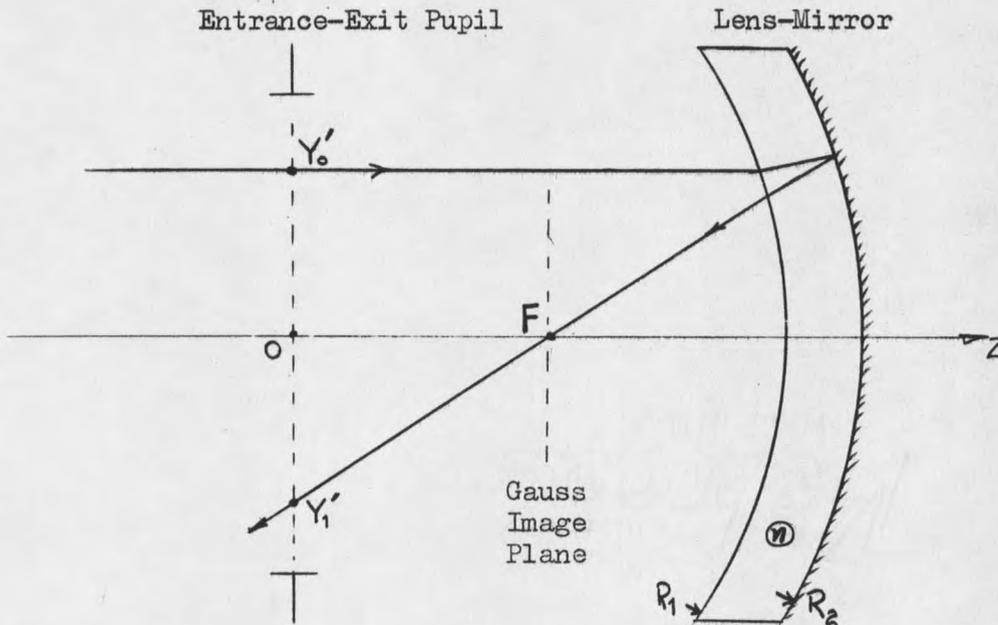


Figure 10.
Concentric Lens-Mirror

It must be pointed out that for a convex system the pupil at the center of curvature would not be a physically realizable stop, but an analytical "virtual stop". With it we would merely define the principal rays as those projectable through the center, and also limit the incident pencil to the rays which could be projected through this "virtual stop". The idea of this approach is to obtain an incident beam which is symmetrical about the principal ray, which for a concentric design would eliminate off-axis aberrations.

Call R_1 the radius of the refracting surface and R_2 the radius of the reflector, both considered positive for a convex system, both negative for the concave type shown. The refractive index is n . The analysis to be used was developed for a system of refractive elements. Reflections are included by considering indices of refraction and distances encountered after reflection as negative.

Off Axis Aberrations

The Abbe invariant L_i for the i th surface of a system is by equation (51)

$$L_i = n_{i-1} \left(\frac{1}{R_i} - \frac{1}{t_i} \right) ,$$

where R_i is the surface radius, t_i its distance from the entrance pupil plane. Thus for the design we are considering, and for any other concentric system with a stop at the center, the L_i for each surface is zero. The aberration coefficients C_i , E_i , and F_i , which all contain the factor L_i , therefore vanish.

We see then, as previously argued, that a concentric system with a

stop at the center of curvature is free of the off-axis aberrations of coma, astigmatism and distortion.

Spherical Aberration

For most applications of spherical reflectors either the object or image is close to the prime focus, so we will restrict our analysis to an object plane at infinity, $S_1 = -\infty$. Let us express the coefficient B as (from equations 63)

$$B = \sum_{i=1}^3 B_i, \quad B_i = \frac{h_i^4 k_i^2}{2} \left(\frac{1}{n_i s_i'} - \frac{1}{n_{i-1} s_i} \right),$$

and proceed to evaluate the h_i, k_i, S_i, S_i', B_i . For the first surface we have $S_1 = -\infty, n_0 = 1, n_1 = n, R = R_1,$

and by eqs. (62) $h_1 = -1$. From (60) we obtain

$$K_1 = n_0 \left(\frac{1}{R_1} - \frac{1}{S_1} \right) = \frac{1}{R_1}, \quad \frac{1}{S_1'} = \frac{1}{R_1} - \frac{K_1}{n_1} = \frac{n-1}{n R_1},$$

and B_1 becomes

$$B_1 = \frac{n-1}{2 n^2 R_1^3} \quad . \quad (64)$$

For the second surface we have ;

$$R = R_2, \quad n_1 = n, \quad n_2 = -n, \quad S_2 = S_1' - d_1,$$

$$S_2 = S_1' - (R_1 - R_2) = \frac{R_1 + R_2 (n-1)}{n-1},$$

$$h_2 = h_1 \frac{S_2}{S_1'} = - \frac{R_1 + R_2 (n-1)}{n R_1},$$

$$K_2 = n \left(\frac{1}{R_2} - \frac{1}{S_2} \right) = \frac{n R_1}{R_2 [R_1 + R_2 (n-1)]} ,$$

$$n_2 \left(\frac{1}{R_2} - \frac{1}{S_2'} \right) = K_2 , \quad \frac{1}{S_2'} = \frac{1}{R_2} - \frac{K_2}{n} ,$$

$$\frac{1}{S_2'} = \frac{2R_1 + R_2(n-1)}{R_2 [R_1 + R_2(n-1)]} ,$$

and the coefficient B_i for $i=2$ is

$$B_2 = \frac{h_2^4 K_2^2}{2} \left(\frac{1}{-n S_2'} - \frac{1}{n S_2} \right) = - \frac{h_2^4 K_2^2}{n R_2} ,$$

$$B_2 = \frac{-[R_1 + R_2(n-1)]^2}{n^3 R_1^2 R_2^3} . \quad (65)$$

For the last surface:

$$n_2 = -n , \quad n_3 = -1 , \quad d_2 = -d_1 , \quad R = R_1 ,$$

$$S_3 = S_2' - d_2 = S_2' - (R_2 - R_1) = \frac{R_1 [2R_1 + R_2(n-2)]}{2R_1 + R_2(n-1)} ,$$

$$h_3 = h_2 \frac{S_3}{S_2'} , \quad h_3 = - \frac{2R_1 + R_2(n-2)}{n R_2} ,$$

$$K_3 = n_2 \left(\frac{1}{R_1} - \frac{1}{S_3} \right) , \quad K_3 = \frac{n R_2}{R_1 [2R_1 + R_2(n-2)]} ,$$

$$\frac{1}{S_3'} = K_3 + \frac{1}{R_1} = \frac{2}{R_1} \cdot \frac{R_1 + R_2(n-1)}{2R_1 + R_2(n-2)} ,$$

and B_3 becomes

$$B_3 = \frac{1}{2} h_3^4 K_3^2 \left(\frac{1}{n_3 s_3} - \frac{1}{n_2 s_3} \right) ,$$

$$B_3 = \frac{(n-1)}{n^3 R_2^2 R_1^3} [2R_1 + R_2(n-1)] [R_2(1-2n) - 2R_1] . \quad (66)$$

The S.A. coefficient for the whole system then is

$$B = \sum_{i=1}^3 B_i = \frac{1}{2 n^3 R_1^3 R_2^3} \times \quad (67)$$

$$\left\{ n(n-1)R_2^3 - 2R_1 [R_1 + R_2(n-1)]^2 \right.$$

$$\left. + R_2(n-1)[2R_1 + R_2(n-2)][R_2(1-2n) - 2R_1] \right\} ,$$

which reduces to,

$$B = -\frac{1}{n^3 R_1^3 R_2^3} \cdot \left\{ R_2^3 (n-1)(n^3 - 3n + 1) \right.$$

$$\left. + 4R_2^2 R_1 (n-1)^2 + 4R_2 R_1^2 (n-1) + R_1^3 \right\} . \quad (68)$$

If we call r the ratio of the radii of the refractive and reflective surfaces, (68) becomes

$$B = -\frac{1}{n^3 R_1^3} \cdot \left\{ r^3 + 4r^2(n-1) + 4r(n-1)^2 + (n-1)^3 - n(n-1) \right\} . \quad (69)$$

Then the condition for the absence of primary spherical aberration for a concentric lens-mirror is, for object at infinity,

$$r^3 + 4r^2(n-1) + 4r(n-1)^2 + (n-1)^3 = n(n-1). \quad (70)$$

This can be also written

$$r \left\{ r + 2(n-1) \right\}^2 = (n-1) \left[n - (n-1)^2 \right], \quad r = \frac{R_1}{R_2}. \quad (71)$$

For a practical solution r must be positive; with r less than one the system is concave; r greater than one represents a convex system. By (71) the only possible cases lie between $n=1$ and $n=2.62$ ($n = (n-1)^2$). Further, it can be seen that the condition cannot be satisfied for r values greater than one. Hence a convex form of our system cannot be made free of spherical aberration.

Condition (71) and the variance of B with the thickness of the refractive element will be considered in more detail later. For now let us examine the other aberrations.

Curvature of Field

Assume that spherical aberration has been eliminated. Then the imaging of our system is stigmatic but the focal surface is curved. We find the curvature from the coefficient D , which by eqs. (63) and the fact that the $L_i = 0$ has the form

$$D = -\frac{1}{2} \sum_{i=1}^3 K_i^2 H_i^2 h_i^2 \left(\frac{1}{n_i t_i} - \frac{1}{n_{i-1} t_i} \right). \quad (72)$$

First the parameters H must be found from eqs. (62). They are

$$H_1 = R_1, \quad H_2 = R_2, \quad H_3 = R_1.$$

Using our previous derivations of the h_i , K_i , it is found in each case that

$$K_i^2 H_i^2 h_i^2 = 1 ,$$

thus

$$D = \frac{1}{2} \sum_{i=1}^3 \left(\frac{1}{n_i t_i} - \frac{1}{n_{i-1} t_i} \right) . \quad (73)$$

Since $t_1 = t'_1 = t_3 = t'_3 = R_1$, and $t_2 = t'_2 = R_2$, the coefficient D for our system is

$$D = - \frac{1}{2} \left[\frac{1}{R_1} \left(\frac{1}{n} - 1 \right) + \frac{1}{R_2} \left(\frac{-2}{n} \right) + \frac{1}{R_1} \left(-1 + \frac{1}{n} \right) \right] , \quad (74)$$

$$D = \frac{R_1 + R_2(n-1)}{R_1 R_2 n} .$$

The field curvature then is

$$\frac{1}{R} = 2D = \frac{2[R_1 + R_2(n-1)]}{R_1 R_2 n} , \quad (75)$$

$$\frac{1}{R} = \frac{2}{R_1 n} (r + n - 1) , \quad r = R_1/R_2 .$$

By the symmetry of the system one would expect that the focal surface would be a sphere concentric with the system's surfaces. The radius given by (75) should be the separation between the center and the paraxial focal point. That this is so will be shown later. It will also be found that this radius is equal to the Gaussian focal length of the system, (for a

concave system R_1 , and thus R_2 will be negative).

Chromatism

The refractive element of our system will of course introduce chromatic aberration. Let us investigate how the chromatism varies with the thickness of the element and determine if for some choice of parameters the system can be made achromatic. The first order chromatism is found from the displacement of the Gaussian focus for a change of refractive index.

The distance between the pole of the refractive surface and the Gaussian focal point is given by S'_3 of the previous sections. Call l_1 the separation between the focus and the system's center, then

$$l_1 = R_1 - S'_3 = \frac{R_1 R_2 n}{2 [R_1 + R_2 (n-1)]} \quad (76)$$

Using $r = R_1 / R_2$, this is

$$l_1 = \frac{R_1 n}{2 (r+n-1)} \quad (77)$$

For a variation δn of the refractive index, we will then obtain a displacement of the focal point

$$\delta l_1 = \frac{R_1}{2} \cdot \frac{[r+n-1-n] \delta n}{(r+n-1)^2} = \frac{l_1 (r-1)}{r+n-1} \cdot \frac{\delta n}{n} \quad (78)$$

Hence it is seen that it is not possible to make a concentric lens-mirror achromatic. Keeping chromatic image shift within specific tolerances then provides a restriction on the useful focal length and size of our design.

For a crown-flint objective, designed to eliminate spherical aberration, the chromatic change in the position of the image will be roughly .4% of the focal length. As already noted, Grey (5) has used this arrangement to

introduce a correcting chromatism for other lens elements of a system.

The third order color error, or spherochromatic aberration, can effectively be eliminated for a certain range of refractive index. This can so be determined from a later graph (figure 12).

V GEOMETRIC ANALYSIS

Although our concentric design cannot be made achromatic and the elimination of third order spherical aberration requires a thick refractive element, let us nevertheless determine whether higher order monochromatic aberrations can be eliminated. Because of symmetry, evaluation of spherical aberration for rays parallel to one axis specifies the performance of the whole system when the object is at infinity.

A requirement for exact imaging known as the Abbe sine condition might well be applied in this case. For rays parallel to the axis at a height h_0 from the axis the sine condition takes the form

$$\sin \gamma_1 = h_0 / f_1 , \quad (79)$$

where γ_1 is the inclination of the final ray and f_1 is the focal length. This implies that the locus of intersections of conjugate rays must be a sphere of radius f_1 centered at the focus.

We shall consider only a concave system in our geometric treatment as it is already known that the aberration of the convex type cannot be eliminated. In figure 11 a ray parallel to the axis at a height y is incident on the refractive surface.

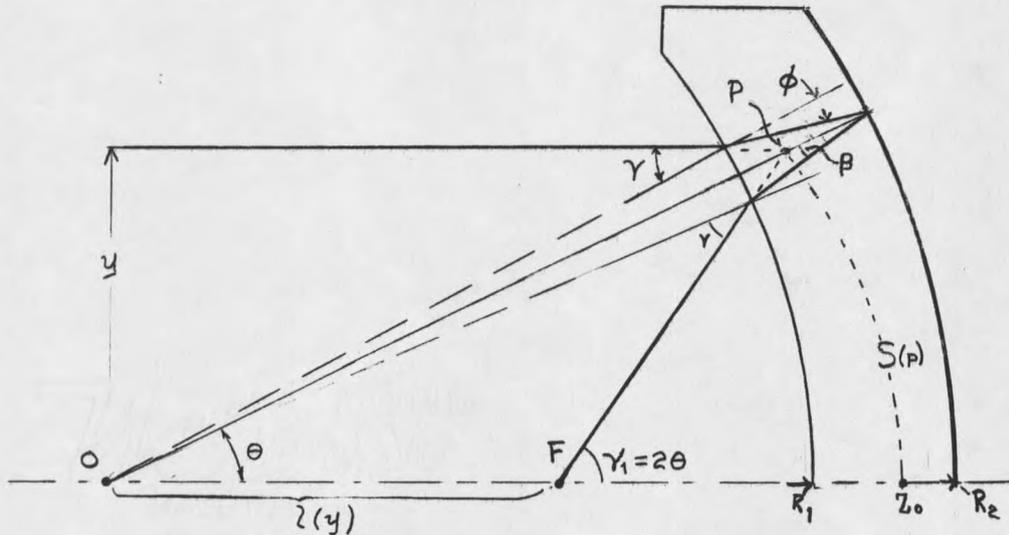


Figure 11.
Geometry of Concentric Lens Mirror

Calling γ the angle of incidence, ϕ the refractive angle, and β the incident angle of reflection, we get the following relations ;

$$\sin \gamma = y/R_1, \quad \sin \phi = \frac{y}{nR_1}, \quad \sin \beta = \frac{y}{nR_2}. \quad (80)$$

Also from the figure we have the inclination of the final ray

$$\gamma_1 = 2\theta, \quad \text{where } \theta = \gamma + \beta - \phi.$$

From (80) we obtain

$$\sin^{-1} \gamma = \frac{y}{R_1} + \frac{1}{6} \left(\frac{y}{R_1} \right)^3 + \frac{3}{40} \left(\frac{y}{R_1} \right)^5 + \frac{5}{112} \left(\frac{y}{R_1} \right)^7 + \dots,$$

$$\sin^{-1} \beta = \frac{y}{R_1} \cdot \frac{1}{n} + \frac{1}{6} \left(\frac{y}{R_1} \right)^3 \frac{1}{n^3} + \dots,$$

$$\sin^{-1} \phi = \frac{y}{R_1} \cdot \frac{1}{n} + \frac{1}{6} \left(\frac{y}{R_1} \right)^3 \frac{1}{n^3} + \dots.$$

Call

$$\alpha_0 = \frac{1}{n}(r+n-1), \quad \alpha_2 = \frac{1}{n^3}(r^3+n^3-1), \quad (81)$$

$$\dots \alpha_k = \frac{1}{n^{k+1}}(n^{k+1} + r^{k+1} - 1),$$

then the expansion for θ is

$$\theta = \left(\frac{y}{R_1}\right)\alpha_0 + \frac{1}{6}\left(\frac{y}{R_1}\right)^3\alpha_2 + \frac{3}{40}\left(\frac{y}{R_1}\right)^5\alpha_4 + \dots \quad (82)$$

For an incident zone y , the final ray will intersect the axis at a point $F(y)$. If we call l the distance from the center O to $F(y)$,

$$l = \frac{R_1 \sin \gamma}{\sin 2\theta} = \frac{y}{\sin 2\theta} \quad (83)$$

Expanding $\sin 2\theta$ gives

$$\sin 2\theta = \frac{y}{R_1}C_0 + \left(\frac{y}{R_1}\right)^3C_2 + \left(\frac{y}{R_1}\right)^5C_4 + \dots, \quad (84)$$

where

$$C_0 = 2\alpha_0, \quad (85)$$

$$C_2 = \frac{1}{3}(\alpha_2 - 4\alpha_0^3),$$

$$C_4 = \frac{1}{60}(9\alpha_4 - 40\alpha_0^2\alpha_2 + 16\alpha_0^5),$$

$$C_6 = \left(\frac{5}{56}\alpha_6 - \frac{\alpha_2^2\alpha_0}{9} - \frac{3}{10}\alpha_4\alpha_0^2 + \frac{2\alpha_0^4\alpha_2}{9} - \frac{8\alpha_0^7}{315}\right),$$

From this we then have

$$l(y) = \frac{R_1}{2\alpha_0} \left\{ 1 + \left(\frac{y}{R_1}\right)^2 \frac{C_2}{C_0} + \left(\frac{y}{R_1}\right)^4 \frac{C_4}{C_0} + \left(\frac{y}{R_1}\right)^6 \frac{C_6}{C_0} + \dots \right\}^{-1}, \quad (86)$$

$$l(y) = \frac{R_1}{2\alpha_0} \left\{ 1 - \left(\frac{y}{R_1}\right)^2 \frac{C_2}{C_0} - \left(\frac{y}{R_1}\right)^4 \left(\frac{C_4}{C_0} - \frac{C_2^2}{C_0^2}\right) - \dots \right\}. \quad (87)$$

Next let us determine the locus (surface S in fig.11) of points where the initial and final rays intersect. For a zone y the coordinate Z of intersection will be

$$Z(y) = y \operatorname{ctn} \theta = y \frac{(1 + \cos 2\theta)}{\sin 2\theta}, \quad (89)$$

$$Z(y) = l (1 + \cos 2\theta). \quad (90)$$

The point Z_0 where the surface S cuts the axis is called the principal point. From (90) and (87), we obtain, as y and θ go to zero,

$$Z_0 = R_1 / \alpha_0. \quad (91)$$

In Gaussian optics the focal length, f_1 , is defined as the first order distance of the principal point to the final ray intersection F. Thus by eqs. (91), and (87) the Gaussian focal length is, as previously asserted,

$$f_1 = -Z_0 + l_1 = -\frac{R_1}{2\alpha_0} = \frac{-R_1 n}{2(r+n-1)}. \quad (92)$$

Calling f the focal distance from Z_0 for an incident zone y we have, using (91) and (87)

$$f(y) = f_1 \left[1 + \left(\frac{y}{R_1}\right)^2 \frac{C_2}{C_0} + \left(\frac{y}{R_1}\right)^4 \left(\frac{C_4}{C_0} - \frac{C_2^2}{C_0^2}\right) + \dots \right]. \quad (93)$$

For the system to be free of spherical aberration to all orders, f must be independent of the zone y of incidence. We must then require,

$$C_2 = C_4 = C_6 = \dots C_K = 0, \quad (94)$$

or by eqs. (85)

$$\alpha_2 = 4\alpha_0^3, \quad \alpha_4 = 16\alpha_0^5, \quad \dots, \quad \alpha_K = 2^K \alpha_0^{K+1}. \quad (95)$$

If (94) were satisfied to all orders, we would have, from eqs. (87) and (92),

$l = -f_1$, and the surface S of conjugate ray intersections would be defined by (eq. 90),

$$z = -f_1(1 + \cos 2\theta). \quad (96)$$

Using (83) this would give

$$(z + f_1)^2 + y^2 = f_1^2, \quad (97)$$

or a sphere of radius $|f_1|$ centered at the Gaussian focus. Thus we see that condition (94) implies the sine condition. The equations (94) or (95) are then requirements on the system parameters for obtaining perfectly stigmatic imaging.

By eqs. (81) and (95), the condition for absence of spherical aberration of all orders becomes

$$(n^{K+1} + r^{K+1} - 1) = 2^K (n + r - 1)^{K+1}, \quad K = 2, 4, 6, \dots \quad (98)$$

But since the design is characterized by only two parameters, r and n , condition (94) cannot be satisfied for more than two values of K . This then will select r and n so as to eliminate third and fifth order aberration.

For $K = 2$, (98) requires

$$n^3 + r^3 - 1 = 4(n+r-1)^3. \quad (99)$$

This is the same requirement given by the Seidel method for vanishing of third order S.A.,

$$r^3 + 4(n-1)r^2 + 4(n-1)^2r + (n-1)^3 - n(n-1) = 0. \quad (100)$$

It was noted earlier that r must be positive and that this can only be so for n between one and 2.62. For a given value of n the solution of (100) for r can be found numerically. In fig.12 are plotted the values of r which satisfy eq. (96) for corresponding values of the refractive index ranging from one to two. Since r is the ratio R_1/R_2 between the radii of the refractive and reflective surfaces, we see from the graph that elimination of third order spherical aberration requires a thick glass element.

If eq. (98) is to be satisfied also for $K = 4$, we then have two equations for the selection of r and n . They are (99) and

$$\alpha_4 = \frac{1}{n^5}(n^5 + r^5 - 1) = \frac{16}{n^5}(n+r-1)^5. \quad (101)$$

Again the solution is obtained numerically, and for stigmatic imaging up to fifth order for a concentric backsurface mirror we must have

$$R_1/R_2 = .35689, n = 1.4451, \alpha_4 = .55496.$$

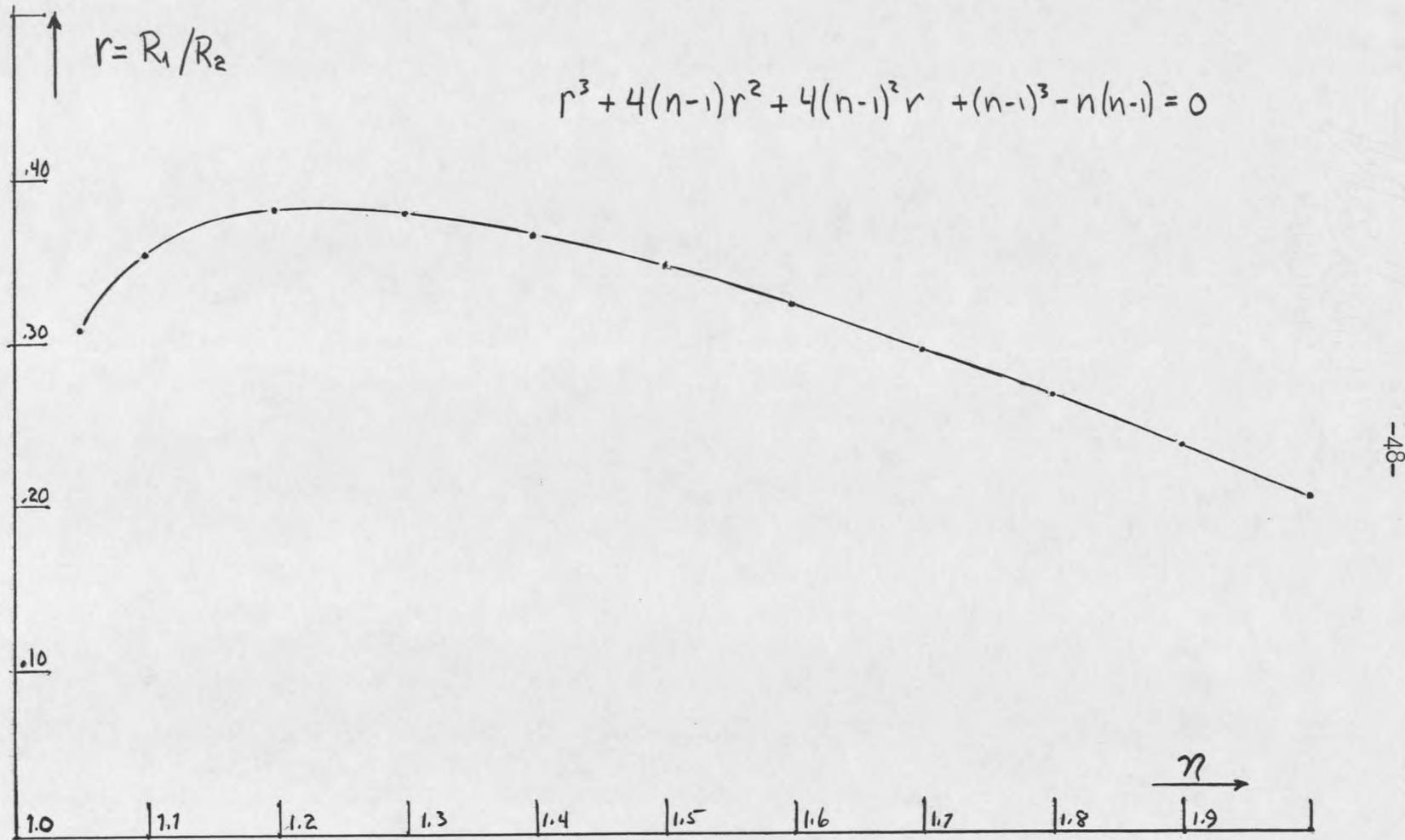


Figure 12.
Relation of r to n for Elimination of Primary Spherical Aberration

At this point there remains to be determined the residual spherical aberration for the system studied when the third and fifth order defect has been eliminated. From eq. (93) we would have in this case

$$f = f_1 \left[1 + \frac{C_6}{2\alpha_0} \left(\frac{y}{R_1} \right)^6 + \dots \right], \quad (102)$$

and C_6 will be

$$C_6 = \frac{5}{56} (\alpha_0 - 64\alpha_0^7).$$

With $r = .3569$, $n = 1.445$, $\alpha_0 = .5550$, $\alpha_6 = \frac{1}{n^2} (n^2 + r^2 - 1)$,

we would have $\alpha_6 = .9228$ $\frac{C_6}{2\alpha_0} = .0077$.

Then if y/R_1 is restricted by a relative aperture of

$$2y_{\max}/f_1 = 1.0, \quad f = f_1 \left(1 + .0077 \cdot \left(\frac{1}{2} \right)^6 \right),$$

the maximum aberration would be roughly .015% of the focal length.

Having thus investigated the aberrations of our system up to fifth order, we may proceed to evaluate its merit for use in optical objectives. First, however, it would be well to express one of the analytical results in graphical form for ease of interpretation. From Section IV we have the expression for the coefficient of primary spherical aberration.

$$B = -\frac{1}{n^3 R_2^3} \left\{ 1 + \frac{4(n-1)}{r} + \frac{4(n-1)^2}{r^2} + \frac{(n-1)(n^2-3n+1)}{r^3} \right\}.$$

Let us select the refractive index as 1.600 and choose R_2 as unity. Then the variation of B with the thickness of the element will be that shown in figures 13 and 14. Figure 13 is for a concave design and figure 14 is for the convex form. For the concave type the result is revealing in that the thickness must be large before there is any improvement in imaging over a conventional spherical reflector.

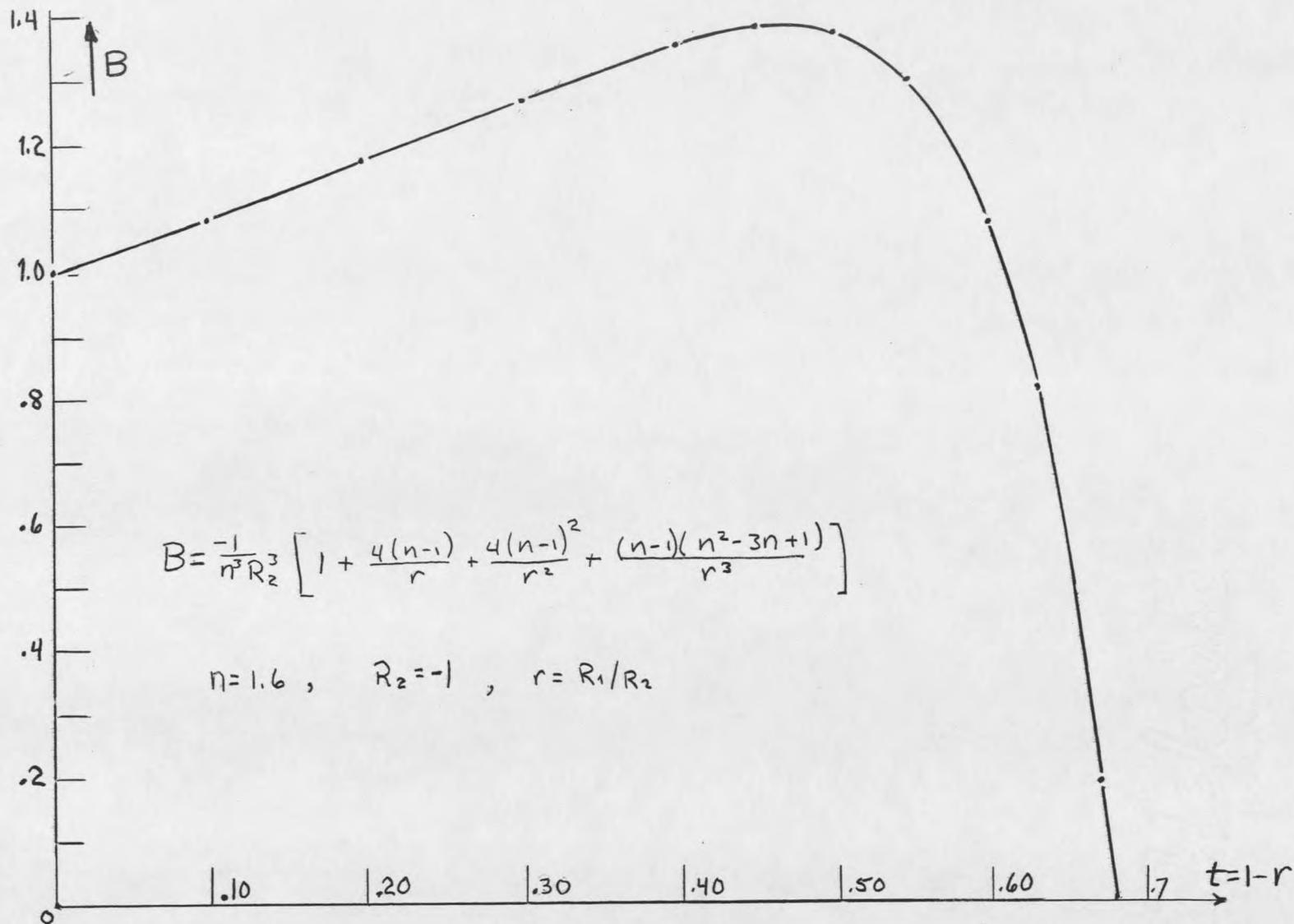


Figure 13.
Aberration Coefficient B for a Concave Concentric Lens-Mirror

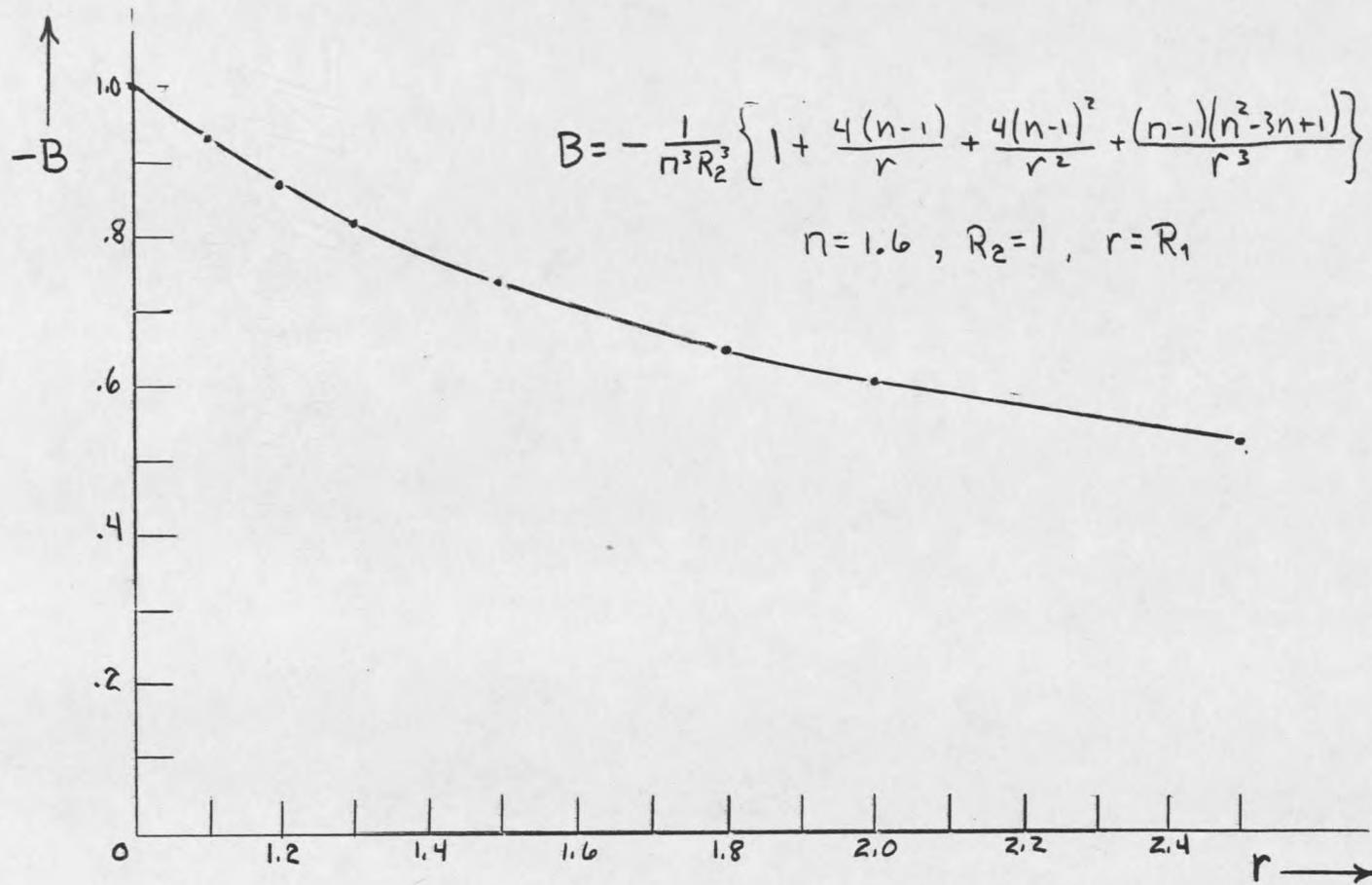


Figure 14.
Aberration Coefficient B for a Convex Concentric Lens-Mirror

-52-
VI CONCLUSION

The analysis has shown, in the first place, that a concentric backsurface mirror with a stop at the center of curvature would be free of the off-axis aberrations known as coma, astigmatism, and distortion. Further, for the concave form, it is possible to eliminate third order spherical aberration in a range of refractive index of the glass element from $n = 1.0$ to $n = 2.62$. The primary aberration called image curvature, however, is inherent for this design. Fifth order spherical aberration can also be corrected if the relative refractive index has a value of 1.445, and in this case the higher order monochromatic defects would be negligible for up to generally large relative apertures. First order chromatism would be the principal defect of imaging for this system.

Before attempting application of this objective, it must be noted from figure 10 that for primary spherical aberration to vanish, the thickness of the refractive element must be better than half the radius of the reflecting surface. Hence, systems of appreciable size would be ruled out because of bulk. Also, optical perfection of the concentric element would require considerable skill.

However, our design might be adaptable as a microscope objective if the chromatism isn't objectionable or can be reduced by other means. The index of 1.445 could even be realized if the microscope was an immersion type: Here the n would be considered the ratio of refractive indices of the glass and the immersion fluid. Some difficulty though might be encountered with the auxiliary optics in this objective.

Considering graphs 13 and 14, other uses for our backsurface reflector may be in the introduction of a specified spherical aberration to

compensate for aberration of other parts of an optical system, as in a Cassegrain telescope or Burch-type microscope.

We may conclude that a concentric backsurface mirror, especially the concave form, exhibits some interesting properties of which some could be incorporated in optical systems. However, because of the required element dimensions for corrected imaging, the use of this design as an objective would offer no improvement over the meniscus systems of Bouwers and Maksutov. This is considering not only image quality, but also the convenience of the construction.

The analysis we have made has been for the case where either the object or image plane is at infinity. Elimination of image defects for finite conjugate points would doubtless require a different relationship between parameters. It would appear that for conjugate points near the center of curvature the glass thickness needed would be smaller. This sort of focal arrangement would be like that used in a Schwarzschild-design microscope. For an investigation of this type of system, as well as the use of the concentric lens-mirror in conjunction with other optical elements, the Seidel theory could well be used.

In conclusion I wish to express my gratitude to Professor Frank Woods for proposing this problem and for his continued interest and valuable advice. I am also thankful for the assistance of the Physics Department of Montana State College.

LITERATURE CONSULTED

1. M. Born and E. Wolf, Principles of Optics (Pergamon Press, London, 1950).
2. A. Bouwers, Achievements in Optics (Elsevier Publishing Co., Amsterdam, 1950).
3. C.R. Burch, "Reflecting Microscopes", Nature 152, 749 (1943).
4. D.S. Grey and P.H. Lee, "A New Series of Microscope Objectives, I, II", J. Opt. Soc. Am. 39, 719 (1949).
5. D.S. Grey, "A New Series of Microscope Objectives, III", J. Opt. Soc. Am. 40, 283 (1950).
6. B.K. Johnson, J. Sci. Instr. 11, 384 (1933).
7. B.K. Johnson, "A Compound Reflecting Microscope of High Aperture for Use in the Ultraviolet", J. Sci. Instr. 26, 148 (1949).
8. R.T. Jones, "Coma of Modified Gregorian and Cassegrainian Mirror Systems", J. Opt Soc. Am. 44, 630 (1954).
9. E.H. Linfoot, Recent Advances in Optics (Oxford University Press, London, 1955).
10. D.D. Maksutov, "New Catadioptric Meniscus Systems", J. Opt Soc. Am. 34, 270 (1944).
11. K.P. Norris and W.E. Seeds, "Reflecting Microscopes with Spherical Mirrors", J. Opt. Soc. Am. 41, 111 (1951).
12. K.P. Norris, "A Reflecting Microscope for Infra-Red Absorption Measurements", J. Sci. Inst. 31, 284 (1954).
13. J. L. Synge, Geometrical Optics (Cambridge University Press, London, 1937).

14. F.L. Whipple, Sky and Telescope 8, 90 (1949).
15. C.G. Wynne, "An Extension of the Lens-Mirror System of Maksutov", Nature 158, 583 (1946).

