



A method of constrained randomization for $p_1 n_1 \times p_2 n_2 \times \dots \times p_k n_k$ factorials
by Richard Leo Schwaller

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
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Montana State University
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Abstract:

This paper presents a method of constrained randomization for (Formula not captured by OCR) factorial experiments in blocked designs when the requirement (imposed by the nature of the process under investigation) restricts the number of factor levels which may be changed from treatment combination to adjacent treatment combination.

The method of constrained randomization requires the construction of a sequence of treatment combinations in which only one level of one factor is changed for each adjacent pair of treatment combinations and also for the first and last treatment combination, considering them as an adjacent pair. The existence and construction of such sequences is considered in Chapter II. Chapter III shows how these sequences are used to generate random sequences having the same properties. This method of constrained randomization provides a basis for statistical analysis utilizing the randomization model.

In Chapter IV, unbiased estimates for factorial effects are obtained using the method of constrained randomization. The average variance of an estimated effect or interaction is found and analysis of variance tables for both the randomization model and the infinite model are given.

A METHOD OF CONSTRAINED RANDOMIZATION
FOR $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ FACTORIALS

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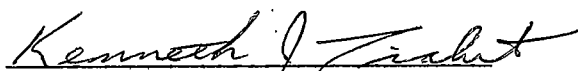
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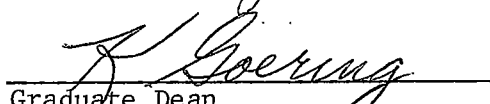
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ABSTRACT

This paper presents a method of constrained randomization for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial experiments in blocked designs when the requirement (imposed by the nature of the process under investigation) restricts the number of factor levels which may be changed from treatment combination to adjacent treatment combination.

The method of constrained randomization requires the construction of a sequence of treatment combinations in which only one level of one factor is changed for each adjacent pair of treatment combinations and also for the first and last treatment combination, considering them as an adjacent pair. The existence and construction of such sequences is considered in Chapter II. Chapter III shows how these sequences are used to generate random sequences having the same properties. This method of constrained randomization provides a basis for statistical analysis utilizing the randomization model.

In Chapter IV, unbiased estimates for factorial effects are obtained using the method of constrained randomization. The average variance of an estimated effect or interaction is found and analysis of variance tables for both the randomization model and the infinite model are given.

CHAPTER I
INTRODUCTION

One of the purposes of the theory of the design of experiments is to insure that the experimenter obtains data relevant to his experiment in the most economical way. If the procedure used by the experimenter results in costly shut down of the equipment or requires that the experiment be terminated before complete data is obtained, the procedure should be replaced by one which avoids such pit-falls and has a valid theoretical basis. This thesis presents the development of a theory and procedure which avoids the above-mentioned difficulties and provides a basis for statistical analysis in experiments involving factorial designs.

Discussion of the Problem

The experimental design considered in this thesis is one in which treatments are applied to experimental units (hereafter denoted by eu's) in a sequential manner, with results of one treatment being observed before application of the following treatment.

This type of experimental design can be applied to the improvement of a manufacturing process which is controlled

by a number of factors. The problem of the statistician is to design and conduct an experiment which will determine that combination of the possible levels of the controlling factors giving optimum output of the product.

Suppose that the controlling factors are such that there are n_1 factors at p_1 levels, n_2 factors at p_2 levels, etc., to n_k factors at p_k levels. A natural method of experimentation is a factorial arrangement of treatments where each factor is set at each level with all possible combinations of the other factor levels. A combination of factor levels is called a treatment combination (hereafter denoted by tc) if all factors under consideration are specified. This is known as a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial.

Terminology which will be used in further describing the problem is now given.

Definition 1-1. A treatment run will be the performance of an operation under the conditions specified by a particular tc. An experiment will consist of a sequence of treatment runs. Usually the number of runs in the sequence will be specified before the experiment begins.

In the type of experiment under consideration, the tc's are run sequentially on the same piece or pieces of equipment. From each treatment run, an observation is taken on the output corresponding to the control settings (or tc.) The process is a continuing one and the tc's are imposed on the process without shutdown, or with only a pause in the operation of the equipment.

One of the problems created by the sequential ordering of the treatment runs is that of incompatibility of the adjacent tc's.

Definition 1-2. Two tc's are adjacent if one follows immediately after the other in the sequential order of running the experiment.

The usual statistical analysis for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial requires that the $\prod_{i=1}^k p_i^{n_i}$ different tc's be

arranged in a random array which is then taken as the sequence of treatment runs in the experiment. This allows for the possibility of many factor changes occurring between adjacent tc's and can be the basis of many potential difficulties. For example, the nature of the equipment may be such that a large number of factor changes between adjacent tc's will

make it difficult to "line out" the apparatus or it may cause the manufacturing process to shut down. In such a situation the number of changes in factor levels between adjacent tc's would need to be restricted and this would, of course, remove the property of full randomization from the design and change the statistical analysis.

Consider a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial experiment with tc i denoted by $(x_{i1}^{(1)}, x_{i2}^{(1)}, \dots, x_{in_1}^{(1)}, x_{i1}^{(2)}, x_{i2}^{(2)}, \dots, x_{in_2}^{(2)}, \dots, x_{i1}^{(k)}, x_{i2}^{(k)}, \dots, x_{in_k}^{(k)})$ where $x_{ij}^{(s)} \in \{0, 1, \dots, p_s - 1\}$ for $s = 1, 2, \dots, k$.

Definition 1-3. Two tc's have order of adjacency Δ if the number of factor levels which are different in the two adjacent tc's is equal to Δ .

Thus, tc's i and i' have order of adjacency

$$\Delta = \sum_{s=1}^k \sum_{j=1}^{n_s} |x_{ij}^{(s)} - x_{i',j}^{(s)}|.$$

Definition 1-4. An operational sequence is any sequence of the tc's which satisfies the compatibility condition imposed on the design by the experimenter and/or the experimental process.

By requiring that the experiment satisfy the condition that $\Delta \leq t$ where $t < \prod_{j=1}^k n_j (p_j - 1)$, one places a compatibility condition on the sequence which prevents full randomization in arranging the sequence of tc's required to conduct the experiment.

The problem thus becomes one of constructing an experimental design which will provide a method of experimentation under the restriction of the compatibility condition and a means of statistically analyzing the data. This thesis investigates the problem for arbitrary values of the p_i 's, n_i 's, and Δ .

Constrained Randomization.

If $\Delta \leq \prod_{j=1}^k n_j(p_j - 1)$, one has full randomization

and the usual statistical analysis can be carried out, see Kempthorne [3] or Ostle [4]. For $\Delta < \sum_{j=1}^k n_j(p_j - 1)$, a

method of constrained randomization which does not destroy the compatibility condition which is placed on adjacent tc's must be used.

A discussion of some aspects of constrained randomization can be found in Kempthorne [3] and Sutter, Zyskind, and Kempthorne [5]. A method of constrained randomization for 2^n factorials was presented in a paper published by Tiahrt and Weeks [6] in August of 1970. Much of the terminology associated with constrained randomization, including that used in this paper, was first introduced in the Tiahrt and Weeks publication. This recently published method of constrained randomization for 2^n factorials requires that the experimenter perform a series of operations at random on a previously obtained operational sequence. These operations preserve the compatibility condition placed on a 2^n factorial but do not preserve the compatibility condition for p^n

factorials in which $p > 2$. Hence, to consider constrained randomization for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial, some other method is needed.

The method of constrained randomization which is presented in this thesis requires that one start with an operational sequence in which adjacent tc's and the first and last tc's in the sequence have order of adjacency $\Delta = 1$. Such a sequence is called a cycle. It is clear that any cycle satisfies the more general condition that $\Delta \leq t$ where $t > 1$ so that any constrained randomization procedure which preserves the compatibility condition for cycles can be applied to factorials in which the compatibility condition requires that $\Delta \leq t$ where $t > 1$.

A systematic method of constructing cycles is presented in Chapter II. A discussion of the comparative advantages of systematic and randomized arrangements may be found in Yates [8].

By performing a series of operations at random on a cycle, other cycles are randomly obtained and provide a random operational sequence which is then used to perform an experiment. This method of constrained randomization is discussed in Chapter III.

The statistical analysis in Chapter IV yields unbiased estimates of factorial effects for both the randomization

model and the infinite model. The usual analysis of variance and unbiased tests for factorial effects are also obtained.

CHAPTER II

EXISTENCE AND CONSTRUCTION OF OPERATIONAL SEQUENCES

This chapter deals with the problem of finding operational sequences which satisfy certain compatibility conditions. The construction of an operational sequence for any factorial of the form $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ will be shown for the compatibility condition $\Delta = 1$ and also for the compatibility condition $\Delta \leq t$ where $t \geq 2$.

In many experimental situations, the statistician will need more than one operational sequence for a given factorial. For example, if the experimenter wishes to conduct a randomization test, the achievable number of significance levels will depend upon the number of operational sequences available.

Of more immediate concern is the problem of having, for a given factorial, several random operational sequences to choose from in order to do the statistical analysis which appears in chapter IV. A systematic method of constructing several such random operational sequences from a given operational sequence, called a cycle, is presented in Chapter III. Most of the theorems in the present chapter pertain to the existence and construction of cycles.

Preliminary Considerations

In finding an operational sequence for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial with a given compatibility condition, all of the $\prod_{i=1}^k p_i^{n_i}$

tc's must be used. Each term in the sequence can be represented as an $\sum_{i=1}^k n_i$ -tuple and thought of as a point in $\sum_{i=1}^k n_i$ -dimensional space. The problem of finding an operational sequence is that of finding a path which connects the vertices of the geometric space. The path must be unbroken, connect all of the vertices, and intersect each vertex only once. To satisfy the compatibility condition, the distance, measured along the lattice lines, between adjacent intersected points must be equal to Δ . If $\Delta \leq \prod_{j=1}^k n_j (p_j - 1)$, there is no constraint on the placement of the tc's and the usual technique of full randomization can be carried out.

Definition 2-1. An operational sequence is said to be a cycle if each tc appears in the sequence and if the first and last tc in the sequence, when placed in adjacent positions, have order of adjacency Δ . Unless otherwise stated, a cycle for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial will contain $\prod_{i=1}^k p_i^{n_i}$ terms. (e.g., each tc will appear once in the operational sequence.)

A factorial is said to be cyclic w.r.t. $\Delta = k$ ($\Delta \leq k$) if there exists a cycle for the factorial which satisfies the compatibility condition $\Delta = k$ ($\Delta \leq k$).

Methods of constructing operational sequences will be based on the results of the next two sections, which investigate the existence or non-existence of a cycle for a given factorial and compatibility condition. Those theorems which establish the existence of a cycle also reveal a method of constructing the cycle and hence an operational sequence for the corresponding factorial.

Cycles for a p^n Factorial

Without loss of generality, the first term in each cycle will be taken as $(0,0,\dots,0)$, the low level of each factor.

It will now be shown that not all p^n factorials are cyclic w.r.t. $\Delta = 1$.

Theorem 2-1. If p is odd, the p^n factorial is not cyclic w.r.t. $\Delta = 1$.

Proof of Theorem 2-1. Let $p = 2k + 1$ where $k \in \mathbb{I}^+$. If a cycle exists for this factorial, it must contain $(2k + 1)^n$ tc's and the last tc, (c_1, c_2, \dots, c_n) , must satisfy the condition $\sum_{i=1}^k |c_i - 0| = 1$ since its order of adjacency with $(0,0,\dots,0)$ must be 1. Moving pointwise through the cycle is equivalent to adding ± 1 to one of the n components in going from one point to the next. Since there are an odd number of points in the cycle, this must be done an even number of times. However, this would mean that the last point,

(c_1, c_2, \dots, c_n) is such that $\sum_{i=1}^n c_i$ is an even number. Since this contradicts the requirement that $\sum_{i=1}^n |c_i - 0| = 1$, the construction of such a cycle is impossible.

This completes the proof but it should be noted for future reference that the proof was based solely on the fact that a cycle cannot contain an odd number of terms if $\Delta = 1$.

The next theorem pertains to factorials of the form p^n where p is even. Its proof, and the proofs of other subsequent theorems, will make use of the following notations.

The symbolism $(a,b)s(a,b+k)$ will denote the sequence $(a,b), (a,b+1), (a,b+2), \dots, (a,b+k)$ and $(a,b+k)s(a,b)$ will denote the sequence $(a,b+k), (a,b+k-1), (a,b+k-2), \dots, (a,b)$.

$(a+k,b)s(a,b)$ and $(a,b)s(a+k,b)$ are defined in a similar manner.

Small Greek letters will be used to denote row vectors. If

$\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_m)$, (α, β) is defined by

$(\alpha, \beta) = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m)$.

Theorem 2-2. If p is even, the p^n factorial is cyclic w.r.t.

$\Delta \leq t$ where $t \geq 1$.

Proof of Theorem 2-2. Let $p = 2k$ where $k \in I^+$. It suffices to show

that the theorem holds for $\Delta = 1$. This will be done by mathematical induction.

(a) If $n = 2$, each tc is denoted by (a,b) where $a,b \in \{0,1,\dots,2k-1\}$.

A cycle for this factorial is the sequence $(0,0)s(2k-1,0),$

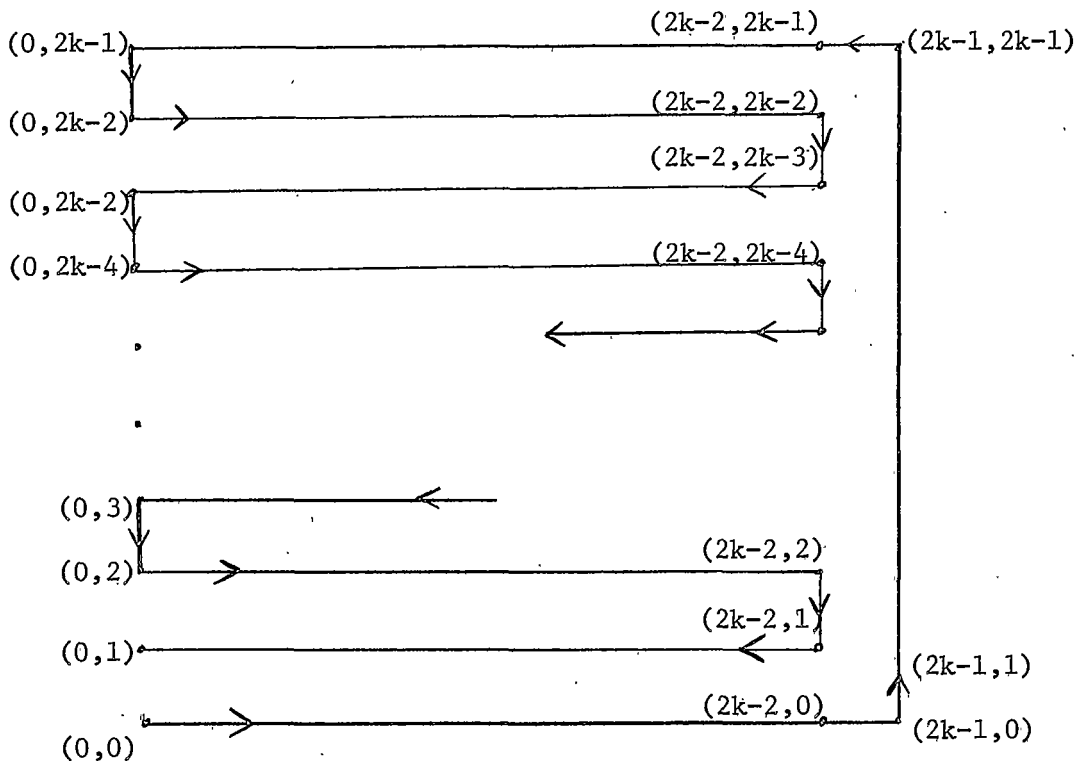
$(2k-1,1)s(2k-1,2k-1), (2k-2,2k-1)s(0,2k-1),$

$(0,2k-2)s(2k-2,2k-2), (2k-2,2k-3)s(0,2k-3),$

$(0,2k-4)s(2k-2,2k-4), \dots, (0,2)s(2k-2,2), (2k-2,1)s(0,1).$

Notice for the last tc to be $(0,1)$, p must be even as it is here.

The geometrical interpretation of this cycle is as shown below.



(b) Suppose the theorem is true for $n = r$. Then each point in the cycle is represented in r -dimensional space by an r -tuple of the form (a_1, a_2, \dots, a_r) where $a_i \in \{0, 1, \dots, 2k - 1\}$. This same point has a representation in $(r + 1)$ -dimensional space as $(a_1, a_2, \dots, a_r, 0)$. Since $\Delta = 1$, a pair of adjacent points in the cycle will have a representation in $(r + 1)$ -dimensional space as $(a_1, a_2, \dots, a_\ell, \dots, a_r, 0)$, $(a_1, a_2, \dots, a_{\ell \pm 1}, \dots, a_r, 0)$. Since there are an even number of points in the cycle, there are $(2k)^r / 2$ disjoint pairs of adjacent points where the first pair consists of the first two points in the cycle and the remaining pairs consist of the remaining points taken two at a time in consecutive order. To construct a cycle in $(r + 1)$ -dimensional space, replace each pair $(a_1, a_2, \dots, a_\ell, \dots, a_r, 0)$, $(a_1, a_2, \dots, a_{\ell \pm 1}, \dots, a_r, 0)$ by the following sequence:
 $(a_1, a_2, \dots, a_\ell, \dots, a_r, 0), (a_1, a_2, \dots, a_\ell, \dots, a_r, 1), \dots,$
 $(a_1, a_2, \dots, a_\ell, \dots, a_r, 2k - 1), (a_1, a_2, \dots, a_{\ell \pm 1}, \dots, a_r, 2k - 1),$
 $(a_1, a_2, \dots, a_{\ell \pm 1}, \dots, a_r, 2k - 2), \dots, (a_1, a_2, \dots, a_{\ell \pm 1}, \dots, a_r, 0)$. Clearly, this construction yields a cycle for the $(2k)^{r+1}$ factorial which satisfies the condition that $\Delta = 1$ and the proof is complete.

For future reference, notice that the last point in the cycle is $(0, 1, 0, 0, \dots, 0)$.

Although it was shown in Theorem 2-1 that a p^n factorial is not cyclic w.r.t. $\Delta = 1$ if p is odd, a cycle for this factorial can be

constructed if the condition that each t_c can appear only once in the operational sequence is removed. With the freedom to repeat points wherever convenient, it is easy to see how any number of cycles could be constructed.

However, due to the time and economic factors involved, the experimenter may wish to have available a cycle in which repetitions are kept to a minimum. This would also be desirable in using the observations to calculate unbiased estimates or in conducting a test of hypothesis.

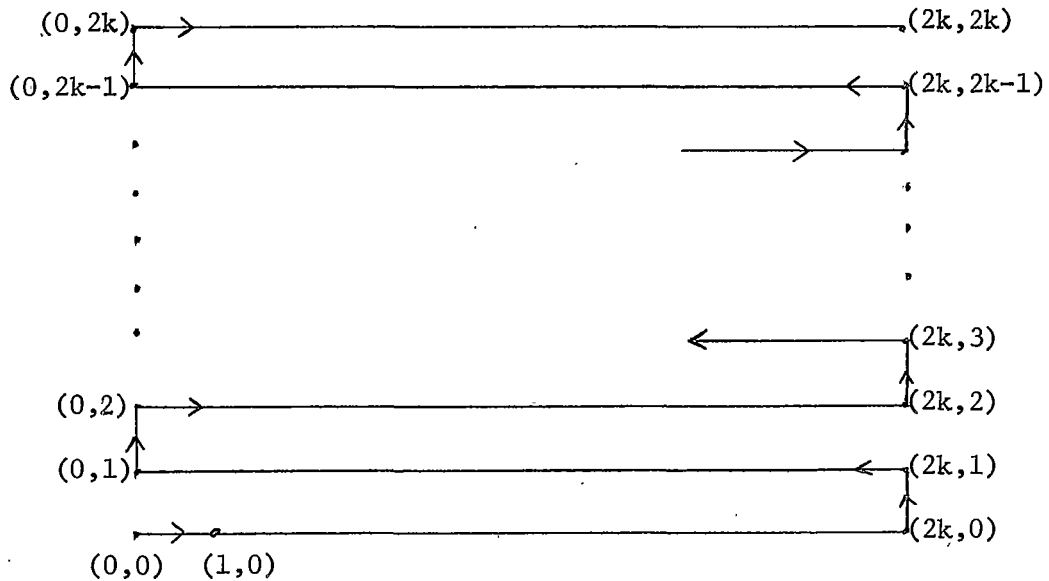
The next theorem shows that such a cycle can be constructed by repeating one t_c once. Information which is crucial to the proof of the theorem is contained in the following lemma.

Lemma 2-1. In a $(2k + 1)^n$ factorial with $\Delta = 1$, a non-cyclic operational sequence can be constructed in such a way that the first point in the sequence is $(0,0,\dots,0)$, the last point is $(2k,2k,\dots,2k)$, and each of the other $(2k + 1)^n - 2$ points appears exactly once.

Proof of Lemma 2-1. The proof is by mathematical induction.

(a) For $n = 2$, the desired operational sequence is the sequence $(0,0)s(2k,0), (2k,1)s(0,1), (0,2)s(2k,2), (2k,3)s(0,3), \dots, (0,2k)s(2k,2k)$. Notice that for the last point to be $(2k,2k)$, there must be an odd number of levels for the second factor as

there are here. The geometrical interpretation of the sequence is as shown.



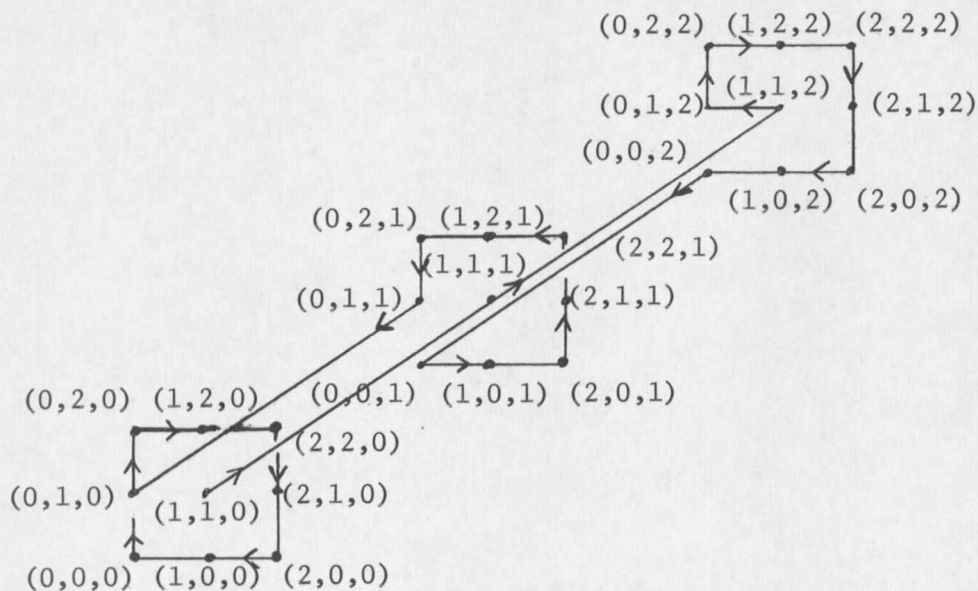
(b) Suppose the lemma is true for $n = r$. Each point in the sequence is represented in r -dimensional space by (a_1, a_2, \dots, a_r) and in $(r + 1)$ -dimensional space by $(a_1, a_2, \dots, a_r, 0)$. The first $(2k + 1)^r - 1$ points can be considered as $[(2k + 1)^r - 1]/2$ disjoint pairs of adjacent points where the pairs are formed in the same manner as those in theorem 2-2. The replacement of each of these pairs by a sequence of points in $(r + 1)$ -dimensional space is identical in notation and method to the procedure used in theorem 2-2. The last point $(2k, 2k, \dots, 2k, 0)$ is replaced by the sequence $(2k, 2k, \dots, 2k, 0)$,

$(2k, 2k, \dots, 2k, 1), (2k, 2k, \dots, 2k, 2), \dots, (2k, 2k, \dots, 2k, 2k)$. Clearly, this construction yields an operational sequence for a $(2k + 1)^{r+1}$ factorial which satisfies the conditions of the lemma. Thus, the lemma holds for $n = r + 1$ and the proof is complete.

A theorem concerning a cycle with one repeated tc now follows.

Theorem 2-3. If p is odd and $\Delta = 1$, a cycle for the p^n factorial can be constructed by repeating one point once.

Proof of Theorem 2-3. Let $p = 2k + 1$ where $k \in I^+$. To illustrate how the cycle will be constructed, consider the case where $k = 1$ and $n = 3$. The cycle for this factorial, starting at $(1, 1, 0)$ and ending at $(0, 1, 0)$, is as shown below.



In this figure, each tc has a representation as (a_1, a_2, a_3) and the sequences in the 2-dimensional subspaces $a_3 = 0$, $a_3 = 1$, and $a_3 = 2$ can be thought of as cycles in which the $(1, 1, a_3)$ point has been omitted. The repeated point is $(0, 1, 0)$.

To prove the theorem in the general case, first consider a $(2k + 1)^2$ factorial. If the point $(1, 1)$ is omitted, a cycle for this factorial is $(0, 0)s(2k, 0), (2k, 1)s(2k, 2k), (2k - 1, 2k)s(0, 2k), (0, 2k - 1)s(2k - 1, 2k - 1), (2k - 1, 2k - 2)s(2k - 1, 1), (2k - 2, 1)s(2k - 2, 2k - 2), \dots, (2, 2)s(0, 2), (0, 1)$.

Notice that this sequence has been constructed in such a way that any point in the form $(0, a)$ where a is positive and even, is followed by the point $(0, a - 1)$. Hence the last point is $(0, 1)$. If the sequence was reversed, the cycle would omit $(1, 1)$, and start at $(0, 1)$.

In completing the proof, the following notation will be used. If β is an $(n - 2)$ -dimensional constant vector, $(a, b, \beta)c(d, f, \beta)$ will represent a cycle in the 2-dimensional subspace of an n -dimensional space which starts at (a, b, β) , ends at (d, f, β) , and omits the point $(1, 1, \beta)$.

Now let $\alpha_1, \alpha_2, \dots, \alpha_\ell$, where $\ell = (2k + 1)^{n-2}$, be a sequence in $(n - 2)$ -dimensional space which satisfies Lemma 2-1 for a $(2k + 1)^{n-2}$ factorial.

The sequence which satisfies the conditions of Theorem 2-3 is
 $(1,1,\alpha_1), (1,1,\alpha_2), (1,1,\alpha_3), \dots, (1,1,\alpha_\ell), (0,1,\alpha_\ell)c(0,0,\alpha_\ell),$
 $(0,0,\alpha_{\ell-1})c(0,1,\alpha_{\ell-1}), (0,1,\alpha_{\ell-2})c(0,0,\alpha_{\ell-2}), \dots, (0,1,\alpha_1)c(0,0,\alpha_1),$
 $(0,1,\alpha_1).$

Notice that there are an odd number of α 's and that the sequences $(a,b,\alpha_t)c(d,t,\alpha_t)$ have been constructed in such a way that if t is odd, $b = 1$ and $a = d = f = 0$. This justifies having the sequence $(0,1,\alpha_1)c(0,0,\alpha_1)$ in the last 2-dimensional subspace. The repeated point, $(0,1,\alpha_1)$, is compatible with the first point, $(1,1,\alpha_1)$. This completes the proof of Theorem 2-3.

Although a p^n factorial is not cyclic w.r.t. $\Delta = 1$ if p is odd and no point is repeated, this is not the case for other values of Δ . The following corollary deals with such cases.

Corollary 2-1. If p is odd, the p^n factorial is cyclic w.r.t. $\Delta \leq t$ where $t \geq 2$.

Proof of Corollary 2-1. It suffices to show that the corollary holds for $\Delta \leq 2$. If the point $(0,1,\alpha_1)$ is removed as the last point in the sequence constructed in Theorem 2-3, the resultant sequence is clearly cyclic w.r.t. $\Delta \leq 2$.

It should be noted that the resultant sequence is a non-cyclic operational sequence having order of adjacency $\Delta = 1$, $(1,1,0,0,\dots,0)$ as its first term, and $(0,0,\dots,0)$ as its last term. By reversing

the terms in this sequence, an operational sequence is generated which has $(0,0,\dots,0)$ as its first term and $(1,1,0,0,\dots,0)$ as its last term.

Although these last remarks were not germane to the proof of the corollary, they will be needed in the next section in extending these results to a more general case.

Cycles for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ Factorials

Thus far, only factorials of the form p^n have been considered. The problem of constructing cycles for factorials of the form $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ will now be considered. The first theorem will show that not every factorial of this form is cyclic w.r.t. $\Delta = 1$.

Theorem 2-4. If $\prod_{i=1}^k p_i$ is odd, the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial

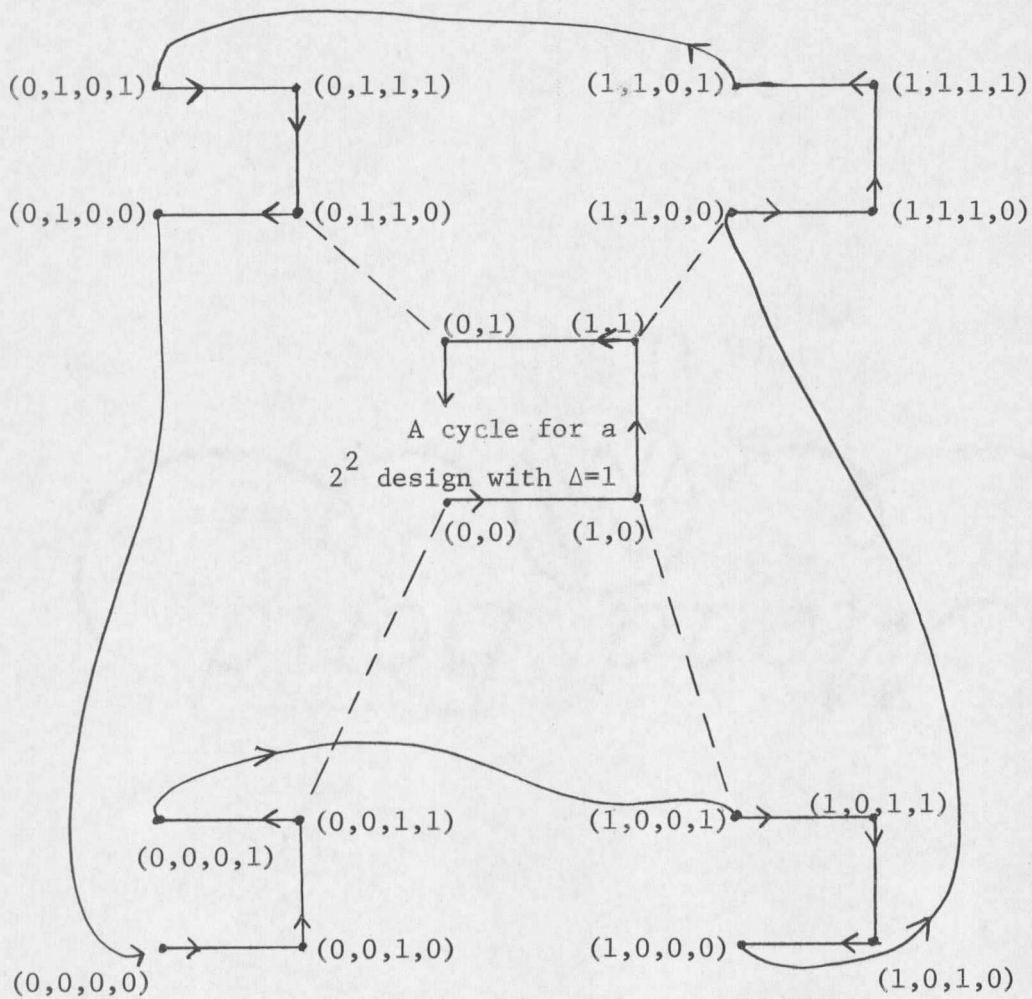
is not cyclic w.r.t. $\Delta = 1$.

Proof of Theorem 2-4. First note that $p_i^{n_i}$ is odd if and only if p_i is odd. Hence $\prod_{i=1}^k p_i^{n_i}$ is odd if and only if $\prod_{i=1}^k p_i$ is odd. Thus,

if a cycle for this factorial exists, it must contain an odd number of terms. However, it was shown in Theorem 2-1, independent of the form of the factorial whose factor levels yield an odd number of

tc's, that a cycle consisting of an odd number of terms cannot be constructed if $\Delta = 1$. This completes the proof of the theorem.

Now consider the following geometrical interpretation of a cycle for a $2^2 \times 2^2$ factorial with $\Delta = 1$. This will illustrate the method to be employed in the proof of the next theorem.



The cycle just illustrated was constructed by replacing each point of a cycle for a 2^2 factorial by a cycle corresponding to another 2^2 factorial and having $\Delta = 1$.

For convenience, the following notation will be used in the proofs of the theorems which follow. If α is a constant vector and $\beta_1, \beta_2, \dots, \beta_n$ is a sequence having order of adjacency $\Delta = 1$,

$(\alpha, \beta_1)C(\alpha, \beta_n)$ will denote the sequence $(\alpha, \beta_1), (\alpha, \beta_2), \dots, (\alpha, \beta_n)$.

The order of adjacency for this sequence is also $\Delta = 1$.

$(\beta_1, \alpha)C(\beta_n, \alpha)$ will denote the sequence $(\beta_1, \alpha), (\beta_2, \alpha), \dots, (\beta_n, \alpha)$.

Theorem 2-5. If p_i is even for $i = 1, 2, \dots, k$, the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial is cyclic w.r.t. $\Delta = 1$.

Proof of Theorem 2-5. The proof is by mathematical induction.

(a) Let $k = 2$. It was shown in Theorem 2-2 that if p_1 and p_2 are even, the factorials $p_1^{n_1}$ and $p_2^{n_2}$ are each cyclic w.r.t. $\Delta = 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_\ell$, where $\ell = p_1^{n_1}$ and $\alpha_\ell = (0, 1, 0, 0, \dots, 0)$ be a cycle for the $p_1^{n_1}$ factorial. Let $\beta_1, \beta_2, \dots, \beta_m$, where $m = p_2^{n_2}$ be a cycle for the $p_2^{n_2}$ factorial. Then a cycle for the $p_1^{n_1} \times p_2^{n_2}$ factorial is given by the sequence $(\alpha_1, \beta_1)C(\alpha_1, \beta_m), (\alpha_2, \beta_m)C(\alpha_2, \beta_1), (\alpha_3, \beta_1)C(\alpha_3, \beta_m), (\alpha_4, \beta_m)C(\alpha_4, \beta_1), \dots, (\alpha_\ell, \beta_m)C(\alpha_\ell, \beta_1)$.

Notice that the sequence was constructed in such a way that if a is even in the subsequence $(\alpha_a, \beta_b)C(\alpha_d, \beta_f)$, $b = m$ and $f = 1$. Hence

the last point must be $(0,1,0,0,\dots,0)$ and is compatible with the first point $(0,0,\dots,0)$. Hence the sequence is a cycle, as claimed.

(b) Assume the theorem holds for $k = t$. Then the

$p_1^{n_1} \times p_2^{n_2} \times \dots \times p_t^{n_t}$ factorial is cyclic w.r.t. $\Delta = 1$. Now consider

the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_t^{n_t} \times p_{t+1}^{n_{t+1}}$ factorial. Replacing the cycle for the $p_1^{n_1}$ factorial in part (a) by the cycle for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_t^{n_t}$ factorial, and the cycle for the $p_2^{n_2}$ factorial in part (a) by the cycle for the $p_{t+1}^{n_{t+1}}$ factorial which ends at $(0,1,0,0,\dots,0)$, the construction described in part (a) yields a cycle for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_t^{n_t} \times p_{t+1}^{n_{t+1}}$ factorial. That the last point is $(0,1,0,0,\dots,0)$ again follows from the fact that there are an even number of terms in the cycle for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_t^{n_t}$ factorial.

This completes the proof of Theorem 2-5.

It was previously shown that for $\Delta = 1$, the p^n factorial is not cyclic if p is odd and the $p_1^{n_1} \times p_2^{n_2}$ factorial is not cyclic if $p_1 p_2$ is odd. The next theorem treats the case where p_2 is odd but $p_1 p_2$ is even in the $p_1^{n_1} \times p_2^{n_2}$ factorial.

Theorem 2-6. If p_1 is even and p_2 is odd, the $p_1^{n_1} \times p_2^{n_2}$ factorial is cyclic w.r.t. $\Delta = 1$.

Proof of Theorem 2-6. It was shown in Theorem 2-3 and Corollary 2-1 that a non-cyclic operational sequence having order of adjacency $\Delta = 1$ exists for a $p_2^{n_2}$ factorial. Let this operational sequence be denoted by $\beta_1, \beta_2, \dots, \beta_m$, where $m = p_2^{n_2}$ and $\beta_m = (1, 1, 0, 0, \dots, 0)$. Let a cycle for the $p_1^{n_1}$ factorial again be denoted by the α 's in Theorem 2-5. Then a cycle for the $p_1^{n_1} \times p_2^{n_2}$ factorial is $(\alpha_1, \beta_1)C(\alpha_1, \beta_m), (\alpha_2, \beta_m)C(\alpha_2, \beta_1), (\alpha_3, \beta_1)C(\alpha_3, \beta_m), \dots, (\alpha_\ell, \beta_m)C(\alpha_\ell, \beta_1)$.

Here again, as in Theorem 2-5, that the last term in the sequence is $(0, 1, 0, 0, \dots, 0)$ follows from the fact that $p_1^{n_1}$ is even. Since the last term is compatible with the first term $(0, 0, \dots, 0)$, the sequence is a cycle, as claimed, and the proof is complete.

It was established in Theorem 2-4 that the $p_1^{n_1} \times p_2^{n_2}$ factorial is not cyclic w.r.t. $\Delta = 1$. However, as in the case for the p^n factorial where p is odd, it would be desirable to have a cycle for this factorial which repeats a minimum number of points. The next theorem deals with this problem.

Theorem 2-7. If p_1 and p_2 are both odd and $\Delta = 1$, a cycle for the $p_1^{n_1} \times p_2^{n_2}$ factorial can be constructed by repeating one point once.

To illustrate the method which will be used in the proof of this theorem, consider a factorial of the form $3^2 \times 3^2$. Starting with the sequence $(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), (0, 1),$

(1,1), each point is replaced by a sequence of points in 4-dimensional space as follows: if $a + b$ is even, (a,b) is replaced by $(a,b,0,0)$, $(a,b,1,0)$, $(a,b,2,0)$, $(a,b,2,1)$, $(a,b,2,2)$, $(a,b,1,2)$, $(a,b,0,2)$, $(a,b,0,1)$; if $a + b$ is odd, (a,b) is replaced by $(a,b,0,1)$, $(a,b,0,2)$, $(a,b,1,2)$, $(a,b,2,2)$, $(a,b,2,1)$, $(a,b,2,0)$, $(a,b,1,0)$, $(a,b,0,0)$. Complete the sequence with the points $(1,1,1,1)$, $(0,1,1,1)$, $(0,2,1,1)$, $(1,2,1,1)$, $(2,2,1,1)$, $(2,1,1,1)$, $(2,0,1,1)$, $(1,0,1,1)$, $(0,0,1,1)$, $(0,0,0,1)$. Notice that the repeated point is the last point $(0,0,0,1)$.

Proof of Theorem 2-7. Let $\beta_1, \beta_2, \dots, \beta_m$ be the operational sequence of β 's described in Theorem 2-6. The order of adjacency for this sequence is $\Delta = 1$ and the second from the last term is

$\beta_{m-1} = (0,1,0,0, \dots, 0)$. Let $\alpha_1, \alpha_2, \dots, \alpha_\ell$, where $\ell = p_1^{n_1}$ be a similar operational sequence for the $p_1^{n_1}$ factorial. A cycle for the $p_1^{n_1} \times p_2^{n_2}$ factorial is $(\alpha_1, \beta_1)C(\alpha_1, \beta_{m-1}), (\alpha_2, \beta_{m-1})C(\alpha_2, \beta_1), (\alpha_3, \beta_1)C(\alpha_3, \beta_{m-1}), \dots, (\alpha_\ell, \beta_1)C(\alpha_\ell, \beta_{m-1}), (\alpha_\ell, \beta_m)C(\alpha_1, \beta_m), (\alpha_1, \beta_{m-1})$.

The repeated point is the last point

$(\alpha_1, \beta_{m-1}) = (0,0, \dots, 0; 0,1,0,0, \dots, 0)$ where the semi-colon separates the components of α from the components of β .

This completes the proof but it should be observed that the second from the last term is $(0,0, \dots, 0; 1,1,0,0, \dots, 0)$. This will be needed in the next theorem, which is a generalization of Theorem 2-7.

Theorem 2-8. If p_i is odd for $i = 1, 2, \dots, k$ and $\Delta = 1$, a cycle for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial can be constructed by one repetition of the point $(0, 0, \dots, 0; 0, 1, 0, 0, \dots, 0)$, where the semi-colon separates the last n_k components from the preceding ones. The point preceding the last (or repeated) point can be taken as $(0, 0, \dots, 0; 1, 1, 0, 0, \dots, 0)$.

Proof of Theorem 2-8. The proof is by mathematical induction.

(a) Theorem 2-7 establishes the case for $k = 2$.

(b) Suppose the Theorem holds for $k = r$. Then there exists a cycle for the $p_2^{n_2} \times p_3^{n_3} \times \dots \times p_r^{n_r} \times p_{r+1}^{n_{r+1}}$ factorial which repeats one point. Denote this cycle by $\beta_1, \beta_2, \dots, \beta_t, \beta_{t+1}$ where $t = \prod_{i=2}^{r+1} p_i^{n_i}$, $\beta_{t+1} = (0, 0, \dots, 0; 0, 1, 0, 0, \dots, 0)$ and

$\beta_t = (0, 0, \dots, 0; 1, 1, 0, 0, \dots, 0)$. Let $\alpha_1, \alpha_2, \dots, \alpha_\ell$ be the operational sequence used in Theorem 2-7 for the $p_1^{n_1}$ factorial. Replacing the points β_{m-1} and β_m in Theorem 2-7 by β_{t-1} and β_t respectively, the construction of a cycle for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_r^{n_r} \times p_{r+1}^{n_{r+1}}$ factorial is identical to the construction shown in Theorem 2-7.

The repeated point is β_{t+1} . This completes the proof.

To complete the discussion of the problem of constructing cycles for factorials of the form $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$, we need to consider the case where $\prod_{i=1}^k p_i$ is even and some of the p_i 's are odd.

Since the statistical design is essentially unchanged by permuting the $p_i^{n_i}$'s, we can assume that those p_i 's which are even occur first in the notation $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$.

Theorem 2-9. If p_i is even for $i = 1, 2, \dots, r$ and p_j is odd for $j = r + 1, r + 2, \dots, k$, the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial is cyclic w.r.t. $\Delta = 1$:

Proof of Theorem 2-9. Let $\alpha_1, \alpha_2, \dots, \alpha_q$ where $q = \prod_{i=1}^r p_i^{n_i}$ be a

cycle for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_r^{n_r}$ factorial. That such a cycle exists for $\Delta = 1$ was established in Theorem 2-5. Let $\beta_1, \beta_2, \dots, \beta_\ell$ where $\ell = \prod_{i=r+1}^k p_i^{n_i}$ be the sequence constructed in Theorem 2-8

omitting the last term, so that there is no repeated point. Replacing the α 's and β 's in Theorem 2-6 with the α 's and β 's just described, the construction of the desired cycle is identical to the construction shown in Theorem 2-6.

This completes the problem of constructing cycles for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorials with $\Delta = 1$. The following corollary treats the case of constructing cycles for this factorial if no points are to be repeated and the restriction that $\Delta = 1$ is relaxed.

Corollary 2-2. Any $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial is cyclic w.r.t. $\Delta \leq t$ where $t \geq 2$.

Proof of Corollary 2-2. It suffices to show that the corollary holds

for $\Delta = 2$. If $\prod_{i=1}^k p_i$ is even, Theorem 2-9 establishes the desired

result. If $\prod_{i=1}^k p_i$ is odd, the result follows from considering the

sequence constructed in Theorem 2-8 with the last term (the repeated point) omitted.

This completes the proof of the corollary.

In order to construct a cycle for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial with $\Delta = 1$ and $\prod_{i=1}^k p_i$ odd, it was necessary to repeat at

least one point. If the operational sequence is non-cyclic, no repetitions are required in order to construct the sequence.

Theorem 2-10. An operational sequence having no repeated terms exists for any $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial having compatibility condition $\Delta = 1$.

Proof of Theorem 2-10. If $\prod_{i=1}^k p_i$ is even, Theorem 2-9 establishes

the desired result. If $\prod_{i=1}^k p_i$ is odd, the operational sequence

constructed in Theorem 2-8 with the last (or repeated) term omitted is an operational sequence having $\Delta = 1$ and no repeated points.

CHAPTER III

CONSTRAINED RANDOMIZATION FOR A $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ FACTORIAL IN A RANDOMIZED NON-CONSECUTIVE BLOCK DESIGN

This chapter presents methods of constrained randomization for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorials in randomized complete block designs where each block is independent of the others.

An example illustrating the randomization procedure is presented and some results are obtained which will be basic to the statistical analysis developed in subsequent chapters.

Cycles and Constrained Randomization

As previously defined, an operational sequence satisfying the compatibility condition and in which the first and last tc's in the sequence are compatible is called a cycle. It is necessary to have a cycle for the factorial under consideration in order to generate other operational sequences in the randomization procedure described in this section. If k is large, the construction of a cycle for the factorial $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ would be a formidable task if some systematic method of construction was not available. The theorems in Chapter 2 of this thesis insure the existence and describe a method of constructing a cycle for the factorial under consideration. It should be noted however, that any cycle for the factorial could be used and might even be preferred if it appears to contain a less systematic arrangement of the tc's.

Constrained randomization for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial in non-consecutive replicates is performed by the following procedure.

- (1) A cycle for the factorial under consideration is constructed.

This can be done directly by using the methods of construction developed in Chapter 2 of this thesis.

- (2) For each replication, randomly assign n_1 factor names to the first n_1 components x_1, x_2, \dots, x_{n_1} of each tc in the cycle,

randomly assign n_2 factor names to the next n_2 components

$x_{n_1+1}, x_{n_1+2}, \dots, x_{n_1+n_2}$ of each tc in the cycle, etc. Since this

is done once in each replication, each tc in a given replicate

has the same assignment of factor names.

- (3) For each replication, pick an integer at random from the set $\{1, 2, \dots, P\}$ where P is the number of tc's in the cycle obtained in step (1).

Let K be the integer so chosen. From the sequence

$\alpha_1, \alpha_2, \dots, \alpha_P$ obtained in step (2), construct the sequence

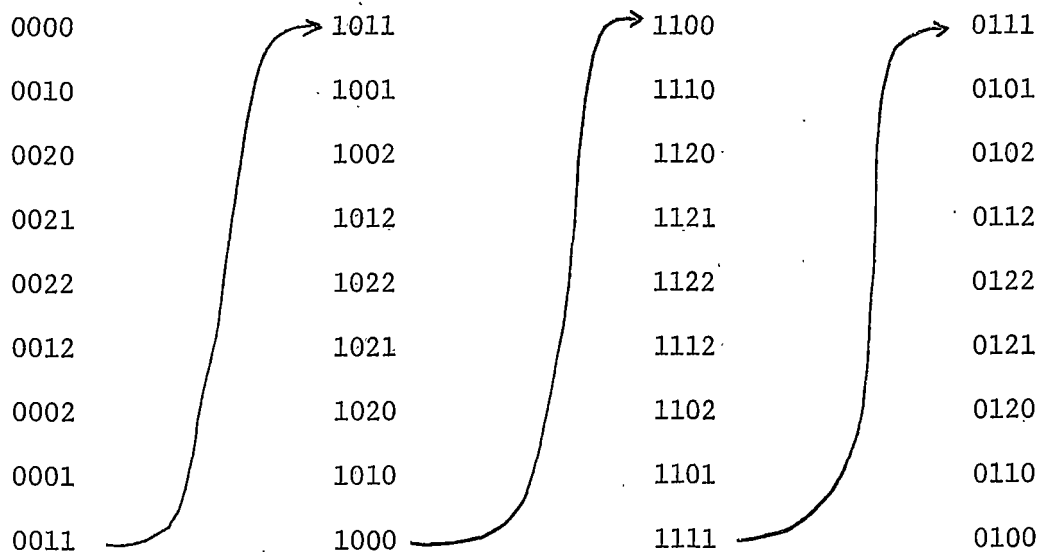
$\alpha_K, \alpha_{K+1}, \dots, \alpha_P, \alpha_1, \alpha_2, \dots, \alpha_{K-1}$.

The resulting cycle is the random sequence of tc's which was desired.

Example

Example 3-1. As an example of this method, consider the following randomization for the replication of a $2^2 \times 3^2$ factorial with $\Delta = 1$.

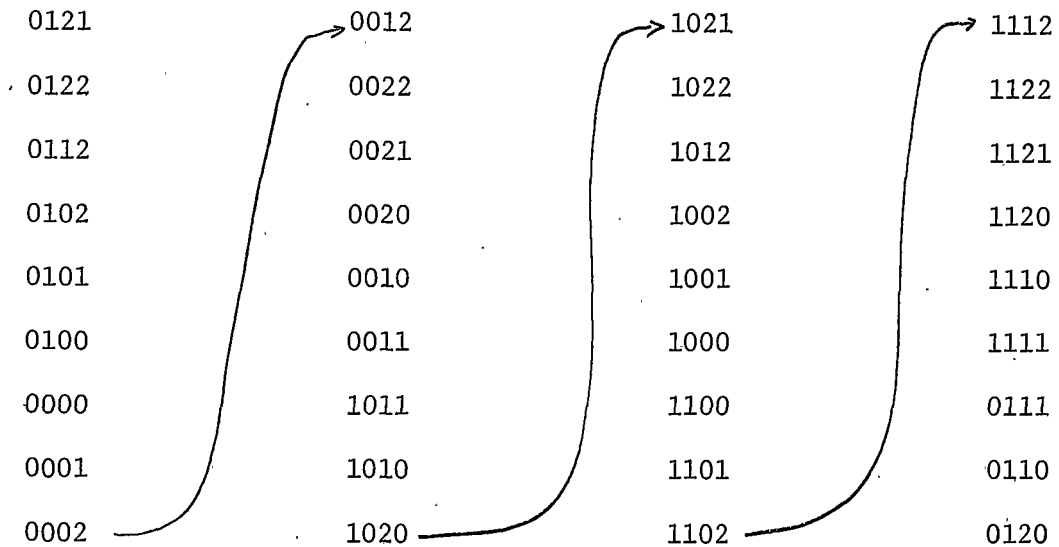
- (1) Since $2 \cdot 3 = 6$ is even, a cycle containing 36 tc's, none of which are repeated, is needed. Such a cycle can be constructed using the method discussed in the proof of Theorem 2-6. This requires a cycle for a 2^2 factorial in the form shown in Theorem 2-2 and an operational sequence for a 3^2 factorial in the form discussed in Theorem 2-3 and Corollary 2-1. The cycle thus constructed for the 2^2 factorial is $(0,0), (1,0), (1,1), (0,1)$. The operational sequence obtained for the 3^2 factorial is $(0,0), (1,0), (2,0), (2,1), (2,2), (1,2), (0,2), (0,1), (1,1)$. Using the method shown in Theorem 2-6, a cycle for the $2^2 \times 3^2$ factorial is



- (2) Randomly assign the factor names A,B to the first two components x_1, x_2 and randomly assign the factor names C,D to the next two components x_3, x_4 . Suppose this random assignment yields $A = x_1, B = x_2, C = x_4, D = x_3$. This replaces each tc (x_1, x_2, x_3, x_4) by the tc (x_1, x_2, x_4, x_3) . The resultant sequence of tc's is

0000	→ 1011	→ 1100	→ 0111
0001	1010	1101	0110
0002	1020	1102	0120
0012	2021	1112	0121
0022	1022	1122	0122
0021	1012	1121	0112
0020	1002	1120	0102
0010	1001	1110	0101
0011	1000	1111	0100

- (3) Randomly choose one of the elements of the set $\{1, 2, \dots, 36\}$. Suppose the number chosen is 31. Then a new sequence of tc's is formed by writing the terms of that cyclic permutation of the sequence in step (2) which moves the 31st term into the first position and the 32nd term into the second position of the new sequence. This new sequence is



Properties Induced by the Constrained Randomization Procedure

Note that the last sequence in the example is again a cycle with $\Delta = 1$. The theorem which follows will show that the randomization procedure will always preserve this property.

Theorem 3-1. The sequence of tc's resulting from constrained randomization of a cycle having order of adjacency $\Delta = k$ ($\leq k$) is a cycle having the same order of adjacency.

Proof of Theorem 3-1. Let the cycle constructed in step (1) be denoted by $\alpha_1, \alpha_2, \dots, \alpha_p$. Step (2) is a rearrangement of the components of each tc and therefore leaves invariant the order of adjacency of any adjacent pair of tc's, α_i , and α_{i^+} , say, where $i^+ = i \text{ mod}(P) + 1$. This is a consequence of the fact that every

tc in the sequence receives identically the same assignment of factor names. Thus, x_{ir} and x_{i+r} , the r^{th} components of α_i and α_{i+} , become the r^{th} components $x_{ir'}$ and $x_{i+r'}$ of α_i and α_{i+} after step (2).

Hence $|x_{ir} - x_{i+r}| = |x_{ir'} - x_{i+r'}|$ and so

$$\sum_r |x_{ir} - x_{i+r}| = \sum_{r'} |x_{ir'} - x_{i+r'}|. \quad \text{This holds for every}$$

pair of adjacent tc's in the sequence and thus the sequence obtained from step (2) is a cycle having the same order of adjacency as the cycle in step (1).

Step (3), as a cyclic permutation of the sequence obtained from step (2), preserves the relative position of the tc's in the cycle. Hence a pair of adjacent tc's are again adjacent after step (3) and so their order of adjacency is unchanged. Thus, the sequence of tc's resulting from the randomization procedure of a cycle having order of adjacency $\Delta = k$ ($\leq k$) is a cycle having the same order of adjacency. This completes the proof.

A result which is basic to the development of the theory to be presented in the next chapter follows.

Theorem 3-2. For a factorial of the form $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$, let $R = \prod_{i=1}^k n_i!$. Then, over all distinct cycles obtained by

the constrained randomization procedure, each tc appears R

times in each position of the sequence if there are no repeated points. If $\prod_{i=1}^k p_i^{n_i}$ is odd and $\Delta = 1$, the repeated tc will occur

2R times in each position of the sequence.

Proof of Theorem 3-2. Let β and γ be two tc's in the cycle constructed in step (1) of the randomization process. Let β' and γ' be the tc's which result from applying step (2) to β and γ . Since step (2) assigns the same factor names to every tc in the cycle, $\beta' = \gamma'$ if and only if $\beta = \gamma$. Thus, if β and γ are distinct points, β' and γ' are also distinct and the number of distinct points is unchanged after applying step (2) of the randomization procedure.

Assuming each tc appears only once in the sequence, then over all cycles generated by applying step (3) to one of the R cycles obtained from step (2), each tc will appear once in each position. Hence in applying step (3) to all of the R cycles resulting from step (2), each tc will appear R times in each position of the cycle.

In the case where the cycle contains a repeated tc, over all cycles generated in applying step (3) to one of the R cycles obtained from step (2) the repeated tc will appear twice in each position. Applying this to all of the R cycles resulting from step (2), the repeated tc will appear 2R times in each position of the cycle.

This completes the proof of Theorem 3-2.

CHAPTER IV
 MODELS FOR $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ FACTORIALS
 IN A NON-CONSECUTIVE BLOCK DESIGN

In this chapter the randomization model corresponding to the constrained randomization method is developed for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorials in randomized complete block designs, where each block is independent of the others.

The randomization model for the case where $\prod_{i=1}^k p_i$ is odd and $\Delta = 1$ is considered separately from the other cases where the cycles employed do not contain any repeated points.

Unbiased estimates of factorial effects are obtained in both models. For the model in which no repeated points occur, variance estimates are also obtained and an analysis of variance table is given, both for infinite model and randomization model assumptions. The model which an experimenter may use will be determined by the process under investigation and the underlying population.

The Randomization Model for Cycles with no Repeated Points

Consider a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial where $\prod_{i=1}^k p_i$ is

even or $\Delta > 1$, or both. The population of inference under the randomization model is the set of eu's (process runs) actually used, or a larger population from which eu's are randomly chosen.

Conceptually, each tc appears in one eu in each replication since there are no repeated points for this factorial under the method of constrained randomization.

Let y_{ijk} denote the population response (conceptual yield) to tc k on eu j in replication i. Letting $N = \prod_{i=1}^k p_i^{n_i}$, there are

N tc's and $k = 1, 2, \dots, N$. Since there are N eu's in each replication $j = 1, 2, \dots, N$ and, supposing r replications, $i = 1, 2, \dots, r$.

Consider the identity

$$y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.}) + (\bar{y}_{ij.} - \bar{y}_{i..}).$$

Assuming the treatment effects are additive, the identity may be written as

$$y_{ijk} = \mu + b_i + t_k + (\bar{y}_{ij.} - \bar{y}_{i..}) \text{ where}$$

$$\mu = \bar{y}_{...}, \quad b_i = \bar{y}_{i..} - \bar{y}_{...}, \quad t_k = y_{ijk} - \bar{y}_{ij.}.$$

$$\text{Note that } \sum_i b_i = \sum_k t_k = 0.$$

Now, in fact, one does not observe the yield of tc k on eu j but merely the yield of tc k on a randomly chosen eu in each replication. To relate the conceptual population of responses to the observed responses, consider the random variable δ_{ij}^k which is defined as follows:

$$\delta_{ij}^k = \begin{cases} 1 & \text{if tc k is on eu j of rep i} \\ 0 & \text{otherwise} \end{cases}$$

Since t_k is on only one eu in each replication, it follows that

$$\sum_j \delta_{ij}^k = \sum_k \delta_{ij}^k = 1. \text{ Denoting the observed response of } t_k$$

in the replicate i by y_{ik} , one may write

$$\begin{aligned} y_{ik} &= \mu + b_i + t_k + \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..}) \\ &= \mu + b_i + t_k + e_{ik} \end{aligned}$$

where $e_{ik} = \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..})$. Note that $\sum_k e_{ik} = 0$. The

properties of the random variables δ_{ij}^k and y_{ik} will be investigated in the following lemmas and theorems.

Lemma 4-1. Under constrained randomization for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial in a randomized complete block design, the random variable δ_{ij}^k has the following properties:

- (1) $E[\delta_{ij}^k] = E[(\delta_{ij}^k)^2] = \frac{1}{N}$
- (2) $E[\delta_{ij}^k \delta_{ij'}^k] = 0 \quad j \neq j'$
- (3) $E[\delta_{ij}^k \delta_{i'j'}^{k'}] = \frac{1}{N^2} \quad i \neq i'$
- (4) $E[\delta_{ij}^k \delta_{ij}^{k'}] = 0 \quad k \neq k'$
- (5) $0 \leq E[\delta_{ij}^k \delta_{ij'}^{k'}] \leq \frac{1}{N} \quad k \neq k', j \neq j'$
- (6) Average $E[\delta_{ij}^k \delta_{ij'}^{k'}] = \frac{1}{N(N-1)} \quad k \neq k', j \neq j'$

Proof of Lemma 4-1. All probabilities are determined by the conditions of the constrained randomization procedure with $\Delta \leq k$. Each tc appears only once in each replication and each eu contains only one tc , all subject to the constraint that no more than k factors be allowed to change from tc to adjacent tc .

Proof of (1). By Theorem 3-2, each tc k appears on each eu j with equal frequency and therefore $P[tc$ k is on eu $j] = P[\delta_{ij}^k = 1] = \frac{1}{N}$. Using the definition of expectation and the fact that $\delta_{ij}^k = (\delta_{ij}^k)^2$, one has

$$E[\delta_{ij}^k] = E[(\delta_{ij}^k)^2] = P(\delta_{ij}^k = 1) = \frac{1}{N}$$

Proof of (2). If $j \neq j'$, tc k cannot be on eu j' if it is on eu j .

Hence, for $j \neq j'$,

$$\begin{aligned} E[\delta_{ij}^k \delta_{ij'}^k] &= P(\delta_{ij'}^k = 1, \delta_{ij}^k = 1) = P(\delta_{ij'}^k = 1 | \delta_{ij}^k = 1) \cdot P(\delta_{ij}^k = 1) \\ &= 0 \cdot \frac{1}{N} = 0 \end{aligned}$$

Proof of (3). If $i \neq i'$, randomization in replication i is independent of randomization in replication i' . Thus, if $i \neq i'$,

$$\begin{aligned} E[\delta_{ij}^k \delta_{i'j'}^{k'}] &= P(\delta_{ij}^k = 1, \delta_{i'j'}^{k'} = 1) \\ &= P(\delta_{ij}^k = 1)P(\delta_{i'j'}^{k'} = 1) = \frac{1}{N} \cdot \frac{1}{N} = \frac{1}{N^2} \end{aligned}$$

Proof of (4). In each replication i , $tc\ k$ and $tc\ k'$ are not both on $eu\ j$ if $k \neq k'$. Hence for $k \neq k'$,

$$\begin{aligned} E[\delta_{ij}^k \delta_{ij}^{k'}] &= P(\delta_{ij}^k = 1, \delta_{ij}^{k'} = 1) = P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1) \cdot P(\delta_{ij}^k = 1) \\ &= 0 \cdot \frac{1}{N} = 0 \end{aligned}$$

Proof of (5). For $k \neq k'$, $j \neq j'$,

$$0 \leq P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1) \leq 1$$

and $P(\delta_{ij}^k = 1) = \frac{1}{N}$. Hence

$$0 \leq E[\delta_{ij}^k \delta_{ij}^{k'}] \leq \frac{1}{N}$$

If $eu\ j$ is adjacent to $eu\ j'$ and $tc\ k$ is not compatible with $tc\ k'$, $P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1) = 0$ and $E[\delta_{ij}^k \delta_{ij}^{k'}]$ actually assumes the value 0. Appropriate choices for j, j', k , and k' in a given design will yield $P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1) = 1$ and, consequently,

$E[\delta_{ij}^{k'} \delta_{ij}^k] = \frac{1}{N}$. As a consequence of this dependence of

$E[\delta_{ij}^{k'} \delta_{ij}^k]$ on the layout of the block, the usual factorial effects estimates are not minimum variance estimates and an exact expression for the variance is not found.

Proof of (6). Assign weight w to $P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1)$ where $k \neq k'$

and $j \neq j'$ and $\sum_{\substack{j \\ j \neq j'}} \sum_{\substack{k \\ k \neq k'}} w = 1$. It follows that

$w = \frac{1}{(N-1)^2}$. For j and k given,

$$\begin{aligned} \text{Average } P(\delta_{ij}^{k'} = 1 | \delta_{ij}^k = 1) &= \sum_{\substack{j' \\ j' \neq j}} \sum_{\substack{k' \\ k' \neq k}} P(\delta_{ij'}^{k'} = 1 | \delta_{ij}^k = 1) / (N-1)^2 \\ &= \sum_{\substack{j' \\ j' \neq j}} (1) / (N-1)^2 = \frac{N-1}{(N-1)^2} = \frac{1}{N-1}. \text{ Hence} \end{aligned}$$

$$\text{Average } E[\delta_{ij}^k \delta_{ij'}^{k'}] = \text{Average } P(\delta_{ij'}^{k'} = 1 | \delta_{ij}^k = 1) \cdot P(\delta_{ij}^k = 1) = \frac{1}{N-1} \cdot \frac{1}{N}.$$

This completes the proof of Lemma 4-1.

The following lemma relates the properties of the random variable δ_{ij}^k to the randomization model,

Lemma 4-2. Let $S^2 = \sum_j (\bar{y}_{ij.} - \bar{y}_{i..})^2 / N$ and assume homo-

of S^2 over blocks. Then

$$(1) \quad E[e_{ik}] = 0$$

$$(2) \quad E[e_{ik}^2] = S^2$$

$$(3) \quad E[e_{ij} e_{i'k'}] = 0 \quad i \neq i'$$

$$(4) \quad \text{Average } E[e_{ik} e_{ik'}] = -S^2 / (N-1).$$

Proof of Lemma 4-2.

Proof of (1). Using $\sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) = 0$, and Lemma 4-1, (1),

$$E[e_{ik}] = E\left[\sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..})\right] = \frac{1}{N} \sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) = 0$$

Proof of (2). Using Lemma 4-1, (1) and (2),

$$\begin{aligned} E[e_{ik}^2] &= E\left[\sum_j (\delta_{ij}^k)^2 (\bar{y}_{ij.} - \bar{y}_{i..})^2 + \right. \\ &\quad \left. \sum_j \sum_{\substack{j' \\ j \neq j'}} \delta_{ij}^k \delta_{ij'}^k (\bar{y}_{ij.} - \bar{y}_{i..}) (\bar{y}_{ij'.} - \bar{y}_{i..})\right] \\ &= \frac{1}{N} \sum_j (\bar{y}_{ij.} - \bar{y}_{i..})^2 + 0 = s^2. \end{aligned}$$

Proof of (3). Using Lemma 4-1, (3) and the fact that

$$\sum_{j'} (\bar{y}_{i'j'.} - \bar{y}_{i'..}) = 0, \text{ for } i \neq i',$$

$$\begin{aligned} E[e_{ik} e_{i'k'}] &= E\left[\sum_j \sum_{j'} \delta_{ij}^k \delta_{i'j'}^{k'} (\bar{y}_{ij.} - \bar{y}_{i..}) (\bar{y}_{i'j'.} - \bar{y}_{i'..})\right] \\ &= \frac{1}{N^2} \sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) \sum_{j'} (\bar{y}_{i'j'.} - \bar{y}_{i'..}) = 0 \end{aligned}$$

Proof of (4). From Lemma 4-1, (4) and (6) and using the fact

$$\text{that } \sum_j \sum_{\substack{j' \\ j \neq j'}} (\bar{y}_{ij'.} - \bar{y}_{i..}) = -(\bar{y}_{ij.} - \bar{y}_{i..}), \text{ for } k \neq k'$$

$$\begin{aligned} \text{Average } E[e_{ik} e_{ik'}] &= \sum_j E[\delta_{ij}^k \delta_{ij}^{k'}] (\bar{y}_{ij.} - \bar{y}_{i..})^2 + \\ &\quad \sum_j \sum_{\substack{j' \\ j \neq j'}} \text{Average } E[\delta_{ij}^k \delta_{ij'}^{k'}] (\bar{y}_{ij.} - \bar{y}_{i..}) (\bar{y}_{ij'.} - \bar{y}_{i..}) \end{aligned}$$

$$= 0 + \frac{1}{N(N-1)} \left[- \sum_j (\bar{y}_{ij} - \bar{y}_{i.}) \right]^2 = -S^2/(N-1)$$

This completes the proof of Lemma 4-2.

The distributional properties of the random variable y_{ik} are given in the following theorem.

Theorem 4-1. If $i \neq i'$, $k \neq k'$, then

$$(1) \quad E(y_{ik}) = \mu + b_i + t_k$$

$$(2) \quad \text{Var}(y_{ik}) = S^2$$

$$(3) \quad \text{Cov}(y_{ik}, y_{i'k'}) = \text{Cov}(y_{ik}, y_{i'k'}) = 0$$

$$(4) \quad \text{Average Cov}(y_{ik}, y_{i'k'}) = -S^2/(N-1)$$

Proof of Theorem 4-1.

Proof of (1). Using Lemma 4-2, (1),

$$E[y_{ik}] = E[\mu + b_i + t_k + e_{ik}] = \mu + b_i + t_k$$

Proof of (2). Using Lemma 4-2, (2),

$$\text{Var}(y_{ik}) = E[(e_{ik})^2] = S^2$$

Proof of (3). Using Lemma 4-2, (3),

$$\text{Cov}(y_{ik}, y_{i'k'}) = E(e_{ik} e_{i'k'}) = 0 \text{ and}$$

$$\text{Cov}(y_{ik}, y_{i'k'}) = E(e_{ik} e_{i'k'}) = 0.$$

Proof of (4). Using Lemma 4-2, (4),

$$\text{Average Cov}(y_{ik}, y_{ik'}) = \text{Average } E(e_{ik} e_{ik'}) = -S^2/(N-1).$$

This completes the proof of the theorem.

Corollary 4-1. Some distributional properties of $\bar{y}_{i.}$

and $\bar{y}_{.k}$ are

$$(1) \quad E[\bar{y}_{.k}] = \mu + t_k$$

$$(2) \quad E[\bar{y}_{i.}] = \mu + b_i$$

$$(3) \quad \text{Var}(\bar{y}_{.k}) = S^2/r$$

$$(4) \quad \text{Average Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) = -S^2/r(N-1) \quad k \neq k'$$

Proof of Corollary 4-1.

Proof of (1). Since $\sum_i b_i = 0$,

$$\begin{aligned} E[\bar{y}_{.k}] &= E\left[\frac{1}{r} \sum_i y_{ik}\right] = \frac{1}{r} \sum_i (\mu + b_i + t_k) \\ &= \mu + t_k \end{aligned}$$

Proof of (2). Since $\sum_k t_k = 0$,

$$\begin{aligned} E[\bar{y}_{i.}] &= E\left[\frac{1}{N} \sum_k y_{ik}\right] = \frac{1}{N} \sum_k (\mu + b_i + t_k) \\ &= \mu + b_i \end{aligned}$$

Proof of (3). Consider

$$\begin{aligned}\text{Var}(\bar{y}_{.k}) &= \frac{1}{r^2} \left[\sum_i \text{Var}(y_{ik}) + \sum_{\substack{i \ i' \\ i \neq i'}} \sum \text{Cov}(y_{ik}, y_{i'k}) \right] \\ &= \frac{1}{r^2} \left[\sum_i s^2 + 0 \right] = s^2/r\end{aligned}$$

Proof of (4). For $i \neq i'$, $k \neq k'$,

$$\begin{aligned}\text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) &= \frac{1}{r^2} \left[\sum_i \text{Average Cov}(y_{ik}, y_{i'k'}) \right. \\ &\quad \left. \sum_i \sum_{i'} \text{Average Cov}(y_{ik}, y_{i'k'}) \right] \\ &= \frac{1}{r^2} \cdot r \left(\frac{-s^2}{N-1} \right) + 0 = -s^2/r(N-1).\end{aligned}$$

This completes the proof of the corollary.

A theorem regarding unbiased estimates of the treatment effects is now stated.

Theorem 4-2. An unbiased estimate for the effect of any

treatment k is $\bar{y}_{.k} - \bar{y}_{..}$.

Proof of Theorem 4-2. Consider

$$\begin{aligned}E[\bar{y}_{.k} - \bar{y}_{..}] &= E[\bar{y}_{.k}] - E[\bar{y}_{..}] \\ &= \mu + t_k - \frac{1}{N} \sum_k E[\bar{y}_{.k}] = \mu + t_k - \frac{1}{N} \left(\sum_k \mu + \sum_k t_k \right) \\ &= \mu + t_k - \mu = t_k.\end{aligned}$$

This completes the proof of Theorem 4-2.

Now consider the problem of estimating factorial effects for the $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial design. A factorial effect estimate is given by a linear contrast of the t_k .

Denoting a factorial effect by $X_1 X_2 \dots X_n$,

$$X_1 X_2 \dots X_n = \sum_k \pi_k t_k$$

where $\sum_k \pi_k = 0$.

Unbiased estimates and average variances for factorial effects are given in the following theorem.

Theorem 4-3. An unbiased estimate for $X_1 X_2 \dots X_n$ is given by $\sum_k \pi_k \bar{y}_{.k}$. The average variance for

$X_1 X_2 \dots X_n$ is

$$\frac{1}{r} S^2 \left[\sum_k \pi_k^2 - \frac{1}{N-1} \sum_{\substack{k, k' \\ k \neq k'}} \pi_k \pi_{k'} \right].$$

Proof of Theorem 4-3. With $X_1 X_2 \dots X_n = \sum_k \pi_k t_k$,

$$E \left[\sum_k \pi_k \bar{y}_{.k} \right] = \sum_k \pi_k E[\bar{y}_{.k}] - \sum_k \pi_k E[\bar{y}_{..}]$$

$$= \sum_k \pi_k E[\bar{y}_{.k} - \bar{y}_{..}] = \sum_k \pi_k t_k.$$

$$\text{Average Var} \left(\sum_k \pi_k \bar{y}_{.k} \right) =$$

$$\text{Average} \sum_k \pi_k^2 \text{Var}(\bar{y}_{.k}) + \text{Average} \sum_{\substack{k, k' \\ k \neq k'}} \pi_k \pi_{k'} \text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'})$$

$$= \sum_k \pi_k^2 \frac{S^2}{r} + \sum_{\substack{k, k' \\ k \neq k'}} \pi_k \pi_{k'} \left(\frac{-S^2}{r(N-1)} \right)$$

$$= \frac{S^2}{r} \left[\sum_k \pi_k^2 - \frac{1}{N-1} \sum_{\substack{k, k' \\ k \neq k'}} \pi_k \pi_{k'} \right].$$

This completes the proof of Theorem 4-3.

The usual analysis of variance for a blocked experimental design can now be employed to obtain an unbiased estimate of S^2 and conduct unbiased tests for factor effects and interactions.

Consider the analysis of variance table

Source	df	Sum of Squares	E(Sum of Squares)
Total	$rN-1$	$\sum_i \sum_k (y_{ik} - \bar{y}_{..})^2$	$N \sum_i b_i^2$ $+ r \sum_k t_k^2 + NrS^2$
Blocks	$r-1$	$N \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2$	$N \sum_i b_i^2$
Treatments	$N-1$	$r \sum_k (\bar{y}_{.k} - \bar{y}_{..})^2$	$r \sum_k t_k^2 + NS^2$
Error	$(r-1)(N-1)$	$\sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$	$(r-1)NS^2$

The expected sums of squares in the analysis of variance table are found using Lemma 4-2, Theorem 4-1, and Corollary 4-1. For the total sum of squares,

$$\begin{aligned}
 E\left[\sum_i \sum_k (y_{ik} - \bar{y}_{..})^2 \right] &= E\left[\sum_i \sum_k (b_i + t_k + e_{ik})^2 \right] \\
 &= E\left[N \sum_i b_i^2 + r \sum_k t_k^2 + \sum_i \sum_k e_{ik}^2 + 2 \sum_i \sum_k b_i t_k \right. \\
 &\quad \left. + 2 \sum_i \sum_k b_i e_{ik} + \sum_i \sum_k t_k e_{ik} \right]
 \end{aligned}$$

$$= N \sum_i b_i^2 + r \sum_k t_k^2 + NrS^2$$

For the block sum of squares,

$$E[N \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2] = N \cdot E[\sum_i b_i^2] = N \sum_i b_i^2$$

For the treatment sum of squares,

$$\begin{aligned} E[r \sum_k (\bar{y}_{.k} - \bar{y}_{..})^2] &= r \cdot E[\sum_k (t_k + \frac{1}{r} \sum_i e_{ik})^2] \\ &= r \left[\sum_k t_k^2 + 2 \sum_k t_k \frac{1}{r} E[\sum_i e_{ik}] + \frac{1}{r^2} \sum_k E[(\sum_i e_{ik})^2] \right] \\ &= r \sum_k t_k^2 + \frac{N}{r} E[\sum_i e_{ik}^2 + \sum_i \sum_{i'} e_{ik} e_{i'k}] \\ &= r \sum_k t_k^2 + NS^2 \end{aligned}$$

The error sum of squares is found, as usual, by subtraction.

From the error sum of squares, it is seen that an unbiased estimate of S^2 is

$$\frac{1}{r(N-1)} \sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2.$$

If all treatment effects, t_k , are zero, the design gives an unbiased test of treatment effects in the analysis of variance.

The Infinite Model for Cycles with no Repeated Points

The results of this section again apply to those factorials for which $\prod_{i=1}^k p_i$ is even or $\Delta > 1$, or both.

If an experimenter can assume that the errors are normally and independently distributed rather than being correlated as in the randomization model, then the results in the following pages may be used.

For this analysis one assumes the model is

$$y_{ik} = \mu + b_i + t_k + e_{ik}$$

where $i = 1, 2, \dots, r$, $k = 1, 2, \dots, N$, y_{ik} is the observed response to t_k in block i , μ is the overall mean, b_i is the effect of block i , t_k is the effect of t_k , and e_{ik} is the failure of observed response y_{ik} to be explained by μ , b_i , and t_k . The errors e_{ik} are assumed to be independently and normally distributed with mean 0 and variance σ^2 .

The following theorem gives estimates for factorial effects in this model.

Theorem 4-6. An unbiased estimate of any main effect or interaction $X_1 X_2 \dots X_n$ where $X_1 X_2 \dots X_n = \sum_k \pi_k t_k$ is

$$\text{given by } \sum_k \pi_k \bar{y}_{.k}.$$

Proof of Theorem 4-6.

$$\text{Since } \sum_k \pi_k = 0,$$

$$\begin{aligned} E\left[\sum_k \pi_k \bar{y}_{.k}\right] &= \sum_k \pi_k E\left[\mu + t_k + \bar{b}_{.} + \frac{1}{r} \sum_i e_{ik}\right] \\ &= \sum_k \pi_k (\mu + t_k + \bar{b}_{.}) = (\mu + \bar{b}_{.}) \sum_k \pi_k + \sum_k \pi_k t_k = \sum_k \pi_k t_k. \end{aligned}$$

This completes the proof of the theorem.

The next theorem considers the variance of a factorial effect.

Theorem 4-7. The variance of an estimated main effect or interaction $X_1 X_2 \dots X_n$ is given by

$$\text{Var}(X_1 X_2 \dots X_n) = \frac{\sigma^2}{r} \sum_k \pi_k^2,$$

Proof of Theorem 4-7. First consider

$$\text{Var}(\bar{y}_{.k}) = E[\bar{y}_{.k} - E(\bar{y}_{.k})]^2 = E\left[\left(\frac{1}{r} \sum_i e_{ik}\right)^2\right]$$

$$\begin{aligned}
&= \frac{1}{r^2} \sum_i E[e_{ik}^2] + \frac{1}{r^2} \sum_i \sum_{\substack{i' \\ i' \neq i}} E[e_{ik} e_{i'k}] \\
&= \frac{1}{r^2} \sum_i \sigma^2 = \sigma^2/r.
\end{aligned}$$

Also consider

$$\begin{aligned}
\text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) &= E[(\bar{y}_{.k} - E(\bar{y}_{.k}))(\bar{y}_{.k'} - E(\bar{y}_{.k'}))] \\
&= E\left[\frac{1}{r} \left(\sum_i e_{ik} \right) \left(\sum_{i'} e_{i'k'} \right)\right] = \frac{1}{r^2} \sum_i \sum_{\substack{i' \\ i' \neq i}} E(e_{ik} e_{i'k'}) = 0
\end{aligned}$$

Using the two expressions just obtained, one has

$$\begin{aligned}
\text{Var}\left(\sum_k \pi_k \bar{y}_{.k}\right) &= \sum_k \pi_k^2 \text{Var}(\bar{y}_{.k}) + \sum_k \sum_{k'} \pi_k \pi_{k'} \text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) \\
&= \frac{\sigma^2}{r} \sum_k \pi_k^2.
\end{aligned}$$

This completes the proof of Theorem 4-7.

The usual analysis of variance for a replicated experimental design can be employed to obtain an estimate of σ^2 . The analysis of variance table for this design is as follows:

Source	df	Sums of Squares	E[Sums of Squares]
Total	$rN-1$	$\sum_i \sum_k (y_{ik} - \bar{y}_{..})^2$	
Reps	$r-1$	$\sum_i \sum_k (\bar{y}_{i.} - \bar{y}_{..})^2$	$(r-1)\sigma^2 + N \sum_i b_i^2$
Trtts	$N-1$	$\sum_i \sum_k (\bar{y}_{.k} - \bar{y}_{..})^2$	$(N-1)\sigma^2 + r \sum_k t_k^2$
Error	$(r-1)(N-1)$	$\sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$	$\sigma^2(r-1)(N-1)$

From the error sums of squares it is seen that an unbiased estimate of σ^2 is $\frac{1}{(r-1)(N-1)} \sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$.

The Randomization Model for Cycles with One Repeated Point

When $\sum_{i=1}^k p_i$ is odd and $\Delta = 1$, the cycle which is obtained

by the randomization process contains one repeated point.

The repeated point is not randomly chosen from all the t_i 's available but is determined by the methods for construction of cycles as developed in Chapter 2. Moreover, after the

repeated point has been determined, the relative positions of the two eu's to which the repeated tc is applied remain unchanged after step (2) in the randomization process.

This lack of randomness in the selection of tc's and eu's deprives the corresponding model of many of the properties obtained in the previous sections where none of the tc's were repeated. However, it will be shown in this section that unbiased estimates for factorial effects can be obtained using a randomization model for the case where one of the tc's is repeated.

Since after step (2) in the randomization process the repeated tc is known, the expectations which yield the unbiased estimates are conditional expectations, the conditional event being that the repeated tc is the one observed.

Using the notations and definitions presented in the first section for the randomization model with no repeated points, one may write $y_{ijk} = \mu + b_i + t_k + (\bar{y}_{ij.} - \bar{y}_{i..})$ where $i = 1, 2, \dots, r$, $j = 1, 2, \dots, N + 1$, $k = 1, 2, \dots, N$.

As before, $\sum_i b_i = \sum_k t_k = 0$.

The random variable δ_{ij}^k is again defined by

$$\delta_{ij}^k = \begin{cases} 1 & \text{if tc } k \text{ is on eu } j \text{ of rep } i \\ 0 & \text{otherwise} \end{cases}$$

Denoting the repeated tc by k^* ,

$$\sum_{k=1}^N \delta_{ij}^k = 1, \quad \sum_{j=1}^{N+1} \delta_{ij}^k = \begin{cases} 2 & \text{if } k = k^* \\ 1 & \text{if } k \neq k^* \end{cases}$$

With y_{ik} the observed response of tc k in replication i , one

has

$$y_{ik} = \sum_j \delta_{ij}^k y_{ijk} = \begin{cases} 2\mu + 2b_i + 2t_k + \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..}) & \text{if } k = k^* \\ \mu + b_i + t_k + \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..}) & \text{if } k \neq k^* \end{cases}$$

$$= \begin{cases} 2\mu + 2b_i + 2t_k + e_{ik} & \text{if } k = k^* \\ \mu + b_i + t_k + e_{ik} & \text{if } k \neq k^* \end{cases}$$

$$\text{where } e_{ik} = \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..})$$

$$\text{Note that } \sum_k e_{ik} = 0.$$

Distributional properties of δ_{ij}^k and y_{ik} will now be investigated.

Lemma 4-3. Under the constrained randomization procedure for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial with $\sum_{i=1}^k p_i$ odd

and $\Delta = 1$,

$$(1) \quad E[\delta_{ij}^k | k^* \text{ is the repeated point (rp)}] = \begin{cases} \frac{2}{N+1} & \text{if } k = k^* \\ \frac{1}{N+1} & \text{if } k \neq k^* \end{cases}$$

$$(2) \quad E[e_{ik} | k^* \text{ is the rp}] = 0$$

Proof of Lemma 4-3. All probabilities are conditional probabilities determined by the constrained randomization procedure after step (2), in which the repeated tc, k^* , is observed.

Proof of (1). By Theorem 3-2, each non-repeated tc appears on each eu j with equal frequency and therefore

$$P[\text{tc } k \text{ is on eu } j | k \neq k^*] = \frac{1}{N+1}. \text{ That the repeated tc appears}$$

twice as frequently as each non-repeated tc was also established

$$\text{in Theorem 3-2. Hence } P[\text{tc } k \text{ is on eu } j | k = k^*] = \frac{2}{N+1}.$$

$$E[\delta_{ij}^k | k^* \text{ is the rp}] = 1 \cdot P[\delta_{ij}^k = 1 | k^* \text{ is the rp}]$$

$$= \begin{cases} \frac{2}{N+1} & \text{if } k = k^* \\ \frac{1}{N+1} & \text{if } k \neq k^* \end{cases}$$

Proof of (2). Consider

$$\begin{aligned}
 & E[e_{ik} | k^* \text{ is the rp}] \\
 &= E\left[\sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..}) | k^* \text{ is the rp} \right] \\
 &= \sum_j E[\delta_{ij}^k | k^* \text{ is the rp}] (\bar{y}_{ij.} - \bar{y}_{i..}) \\
 &= \begin{cases} \frac{2}{N+1} \sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) & \text{if } k = k^* \\ \frac{1}{N+1} \sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) & \text{if } k \neq k^* \end{cases} \\
 &= 0 \text{ for all } k.
 \end{aligned}$$

This completes the proof of the lemma.

The following theorem presents estimates for tc's.

Theorem 4-8. Unbiased estimates for tc's under the conditional event that k^* is the repeated tc are given by

- (1) $E\left[\frac{N+1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} | k^* \text{ is the rp}\right] = t_{k^*}$
- (2) $E\left[\bar{y}_{.k} + \frac{1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} | k^* \text{ is the rp}\right] = t_k \text{ if } k \neq k^*$

Proof of Theorem 4-8. First consider $\bar{y}_{.k}$.

$$\bar{y}_{.k} = \frac{1}{N+1} \sum_j y_{ik} = \begin{cases} 2\mu + 2t_k + \frac{1}{N+1} \sum_i e_{ik} & \text{if } k = k^* \\ \mu + t_k + \frac{1}{N+1} \sum_i e_{ik} & \text{if } k \neq k^* \end{cases}$$

Since, by Lemma 4-3 $E(e_{ik} | k^* \text{ is the rp}) = 0$,

$$E[\bar{y}_{.k} | k^* \text{ is the rp}] = \begin{cases} 2\mu + 2t_k & \text{if } k = k^* \\ \mu + t_k & \text{if } k \neq k^* . \end{cases}$$

Now consider $\bar{y}_{..}$.

$$\begin{aligned} \bar{y}_{..} &= \frac{1}{N} \sum_k \bar{y}_{.k} = \frac{1}{N} [(N+1)\mu + \sum_k t_k + t_{k^*} + \frac{1}{N+1} \sum_k \sum_i e_{ik}] \\ &= \frac{N+1}{N} \mu + \frac{1}{N} t_{k^*} . \end{aligned}$$

Proof of (1).

$$\begin{aligned} &E\left[\frac{N+1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} \mid k^* \text{ is the rp}\right] \\ &= \frac{N+1}{2N} E[\bar{y}_{.k^*} \mid k^* \text{ is the rp}] - \frac{N+1}{N} \mu - \frac{1}{N} t_{k^*} \\ &= \frac{N+1}{2N} (2\mu + 2t_{k^*}) - \frac{N+1}{N} \mu - \frac{1}{N} t_{k^*} = \left(\frac{N+1}{N} - \frac{1}{N}\right) t_{k^*} = t_{k^*} \end{aligned}$$

Proof of (2). If $k \neq k^*$,

$$\begin{aligned} &E\left[\bar{y}_{.k} + \frac{1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} \mid k^* \text{ is the rp}\right] \\ &= E[\bar{y}_{.k} \mid k^* \text{ is the rp}] + \frac{1}{2N} E[\bar{y}_{.k^*} \mid k^* \text{ is the rp}] \\ &\quad - \frac{N+1}{N} \mu - \frac{1}{N} t_{k^*} \\ &= (\mu + t_k) + \frac{1}{2N} (2\mu + t_{k^*}) - \frac{N+1}{N} \mu - \frac{1}{N} t_{k^*} = t_k . \end{aligned}$$

This completes the proof of Theorem 4-8.

The following theorem gives estimates for factorial effects.

Theorem 4-9. Given that k^* is the repeated tc, an unbiased estimate of any main effect or interaction

$$X_1 X_2 \dots X_n \text{ where } X_1 X_2 \dots X_n = \sum_k \pi_k t_k \text{ is}$$

$$\sum_k \pi_k \bar{y}_{.k} - \frac{1}{2} \pi_{k^*} \bar{y}_{.k^*}$$

Proof of Theorem 4-9. Denote the estimates for t_k found in

Theorem 4-8 by \hat{t}_k . Then

$$\begin{aligned} E\left[\sum_k \pi_k \hat{t}_k \mid k^* \text{ is the rp} \right] &= \sum_k \pi_k E[\hat{t}_k \mid k^* \text{ is the rp}] \\ &= \sum_k \pi_k t_k = X_1 X_2 \dots X_n. \end{aligned}$$

To complete the proof it must only be shown that the con-

trast of the t_k 's is $\sum_k \pi_k \bar{y}_{.k} - \frac{1}{2} \pi_{k^*} \bar{y}_{.k^*}$.

Thus, consider

$$\begin{aligned} \sum_k \pi_k \hat{t}_k &= \pi_{k^*} \left(\frac{N+1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} \right) + \sum_{\substack{k \\ k \neq k^*}} \pi_k \left(\bar{y}_{.k} + \frac{1}{2N} \bar{y}_{.k^*} - \bar{y}_{..} \right) \\ &= \frac{1}{2N} (\pi_{k^*} + \sum_k \pi_k) \bar{y}_{.k^*} + (\pi_{k^*} + \sum_{\substack{k \\ k \neq k^*}} \pi_k) \bar{y}_{..} \\ &+ \frac{1}{2} \pi_{k^*} \bar{y}_{.k^*} + \sum_{\substack{k \\ k \neq k^*}} \pi_k \bar{y}_{.k} \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_k \pi_k \right) \left(\frac{1}{2N} \bar{y}_{.k*} + \bar{y}_{..} \right) - \frac{1}{2} \pi_{k*} \bar{y}_{.k*} + \sum_k \pi_k \bar{y}_{.k} \\
&= \sum_k \pi_k \bar{y}_{.k} - \frac{1}{2} \pi_{k*} \bar{y}_{.k*}
\end{aligned}$$

Due to the randomization constraints imposed on the design, variance estimates for this case have not been obtained.

CHAPTER V

SUMMARY

The statistical analysis which is carried out in conducting a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial experiment in blocked designs requires that the tc's be randomly arranged for each treatment run. When the nature of the process under investigation restricts the number of factor levels which can be changed from tc to tc, the usual technique of full randomization cannot be carried out.

This thesis presents methods of constrained randomization for $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial experiments in blocked designs when the requirement on adjacent tc's is that the number, Δ , of factor levels which can be changed is less than or equal to t , where $t < \prod_{i=1}^k n_i (p_i - 1)$.

The method of constrained randomization requires the construction of an operational sequence in which $\Delta = 1$ for the first and last tc's in the sequence as well as for all adjacent tc's. The existence and construction of such operational sequences, called cycles, present interesting graph theory problems whose formulations and solutions are found in this paper. Various geometrical examples which illustrate methods for constructing cycles are presented.

It is shown that if $\prod_{i=1}^k p_i$ is odd, the construction

of a cycle requires one of the tc's in the operational sequence to be repeated. The statistical analysis which results in this case is not as complete as in the case where tc's are not repeated since the underlying probability distributions are conditioned by the relative positions of the repeated tc in the sequence.

The constrained randomization procedure using cycles also applies to the case where $\Delta \leq t$ and $t > 1$. By permuting some of the tc's in the cycle, other operational sequences in which $1 \leq \Delta \leq t$ can be constructed in which Δ can achieve any one of its allowable values for some pair of adjacent tc's.

Unbiased estimates of factorial effects are obtained using the methods of constrained randomization which are contained in this thesis. The average variance of an estimated effect or interaction is found and analysis of variance tables for both the randomization model and the infinite model are given for factorials in which $\prod_{i=1}^k p_i$ is even and/or $\Delta > 1$.

Preliminary investigations indicate that the constrained randomization procedures contained in this paper can be used to design confounding schemes for simple p^n factorials (for instance, a 3^3 factorial) under the restriction that $\Delta \leq t$ for some values of t .

The possibility of extending these results to confounding schemes for a $p_1^{n_1} \times p_2^{n_2} \times \dots \times p_k^{n_k}$ factorial which is restricted by the condition $\Delta \leq t$ is open to further study.

BIBLIOGRAPHY

1. Fisher, R. A. The Design of Experiments. Oliver and Boyd, Edinburgh. 1947.
2. Kempthorne, O. Analysis of Variance Procedures. Aeronautical Research Laboratory: Wright-Patterson Air Force Base, Ohio. 1961.
3. Kempthorne, O. The Design and Analysis of Experiments. John Wiley and Sons, New York. 1961.
4. Ostel, B. Statistics in Research. Iowa State University Press, Ames, Iowa. 1963.
5. Sutter, Zyskind, and Kempthorne, O. Some Aspects of Constrained Randomization. Technical Report, ARL 63-18; Wright-Patterson Air Force Base, Ohio. 1963.
6. Tiahrt, K. J. and Weeks, D. L. A Method of Constrained Randomization for 2^n Factorials. Technometrics, 12, 1970.
7. Wilk, M. B. The Randomization Analysis of a Generalized Randomized Block Design. Biometrika, 42, 1955.
8. Yates, F. The Comparative Advantages of Systematic and Randomized Arrangements in the Design of Agricultural and Biological Experiments. Biometrika, 30, 1939.

