On the structure of locally connected topological spaces
by Spencer Edward Minear

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

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In the last part of the paper we impose the condition that every N-set is an A-set. Under this condition we obtain the result that unicoherence is both a cyclicly extensible and a cyclicly reducible property. We also show that in a compact, connected, locally connected space, that the fixed point property is also cyclicly extensible and reducible.

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Table of Contents

Abstract

In the first part of the paper we prove that every cut point of a connected space is either an open or a closed point. We also investigate other properties of cut points in a connected, locally connected space, and the relations amoung the components of the complements of cut points.

We then start a development of cyclic element theory in connected, locally connected spaces by extending the work of Albert and Youngs. We define an A-set and show that it has the classical property, that if H is connected and A is an A-set, then H ∩ A is connected. We also find necessary and sufficient conditions for a non-empty intersection of A-sets to be an A-set.

In the last part of the paper we impose the condition that every N-set is an A-set. Under this condition we obtain the result that unicoherence is both a cyclicly extensible and a cyclicly reducible property. We also show that in a compact, connected, locally connected space, that the fixed point property is also cyclicly extensible and reducible.

Introduction

The concept of cyclic elements was introduced by G. T. Whyburn in 1926. He said that a cyclic element of a Peano space was either: 1) a point p such that X - {p} is disconnected, (a cut point), 2) a point p with arbitrarily small neighborhoods with singleton boundary, (an end point), or 3) a subcontinuum of the space which is maximal with respect to the property that each pair of points in it lie on a simple closed curve. This idea proved to be useful when it was discovered that many properties of the space depend on the analogous property of all the cyclic elements in the space. In 1929 R. L. Moore suggested [6] an alternate definition for a cyclic element. Using Moore's suggestion in 1930 [3] Kuratowski and Whyburn defined a cyclic element as either: 1) a cut point, or 2) a set consisting of a non-cut point p and all points x such that no point separates between p and x. (A subset Y of X is said to separate between two points a and b if there exist subsets H and K of X such that X - Y = H ∪ K, separated, with a ∈ H and b ∈ K.) This new definition proved to be very important as it simplified the proofs for cyclic element theory in Peano spaces and also permitted the extension of cyclic element theory to more general spaces.

In 1929 C. Kuratowski [2] proved that if each cyclic element of a Peano continuum is unicoherent, then so is the space and if the space is unicoherent, then so is each cyclic element. Such a property is called cyclicly extensible and reducible. In the same paper he suggested that the fixed point property is also a cyclicly extensible and reducible property. This was proved by Borsuk in 1932.

In 1942, Albert and Youngs [1] extended cyclic element theory to connected, locally connected spaces. However they required that each cut point be a closed point. In this paper we will show that cyclic element theory can be extended to connected, locally connected spaces with out this requirement. We will also show that with a suitable restriction on the space that unicoherence and the fixed point property are both cyclicly extensible and reducible.

Chapter I

Let X be a connected topological space and let Y be a subset of X. Then Y is said to be a <u>cutting of X</u> if X - Y is disconnected. In the case when Y is a singleton $\{\alpha\}$, then α is called a <u>cut point</u> of X. If Y is a cutting of X and A and B are subsets of X, then Y is said to separate A and B if there is a separation of $X - Y = H \cup K$, where H and K are separated, such that $A \subset H$ and $B \subset K$. If α is a cut point of X and a and b are distinct elements of X, then α is said to <u>cut between a and b</u> if a and b appear in distinct components of $X - \{\alpha\}$. Note that if α separates a and b, then α also cuts between a and b, but the converse is not true in general, as in the case when there is a quasi-component of $X - \{\alpha\}$ which is not a component. A quasi-component C of $X - \{\alpha\}$ is a set such that for each separation $H \cup K$ of $X - \{\alpha\}$ it is true that $C \subset H$ or $C \subset K$.

Let A and B be subsets of X, then S(A,B) will be used to denote the component of X - B which contains A, if one exists. If $A = \{a\}$ or $B = \{b\}$ we use S(a,b) in place of $S(\{a\},\{b\})$. We shall use ∂A to denote the boundary of the set A, $\overline{A}$ and A° to denote the closure and the interior of A respectively and $\emptyset$ to denote the empty set.

The following theorem makes most of the work to follow possible. It was apparently first observed by my advisor Dr. B. L. McAllister as a corollary to the main theorem of his "A Note On Irreducible Separation" [5]. The proof below is independent of the results in the above paper.

1.1) Theorem: If α is a cut point of a connected space X, then $\{\alpha\}$ is either an open or a closed set.

Proof: Let $X - \{\alpha\} = A \cup B$ separated. Since $\overline{A} \cap B = \emptyset$ it follows that $\overline{A} = A$ or $\overline{A} = A \cup \{\alpha\}$. If $\overline{A} = A$, then $B \cup \{\alpha\} = X - A$ is open. Since $\overline{B} \cap A = \emptyset$ it follows that $B \subset \overline{B} \subset B \cup \{\alpha\}$ and since X is connected, $\overline{B} \neq B \cup \{\alpha\}$, thus $B = \overline{B}$. Hence $X - \{\alpha\} = A \cup B$ is closed which implies that $\{\alpha\}$ is open. If $\overline{A} = A \cup \{\alpha\}$, then $B = X - \overline{A}$ is open and as above $\overline{B} = B \cup \{\alpha\}$. Then $\overline{A} \cap \overline{B} = (A \cup \{\alpha\}) \cap (B \cup \{\alpha\}) = \{\alpha\}$ is a closed subset of X.

Earlier it was pointed out that it is possible for a cut point α to cut between two points yet not separate them. If we require the space to be locally connected as well as connected the above situation may still occur if $\{\alpha\}$ is open; however the following theorem shows that closed cut points no longer have this property.

1.2) Theorem: Let X be a locally connected, connected space. If α is a closed cut point that cuts between a and b, then α separates a and b.

Proof: Since $\{\alpha\}$ is closed, $X - \{\alpha\}$ is open, and thus $S(x,\alpha)$ is an open subset of X for each point x in X. Thus $X - \{\alpha\} = S(a,\alpha) \cup (\cup\{ S(x,\alpha) : x \neq \alpha \text{ and } x \notin S(x,\alpha)\})$ is a disjoint union of open sets, one containing a and the other containing b.

<u>NOTE:</u> For the remainder of this paper the symbol X will denote a

connected, locally connected topological space, unless stated otherwise.

The next two theorems are well-known and will be stated without proof.

1.3) Theorem: If α is a closed cut point of X, then $\{\alpha\} = \partial S$ for each component S of $X - \{\alpha\}$.

1.4) Theorem: If α is an open cut point of X and $x \in \partial S$, where S is a component of $X - \{\alpha\}$, then $x \in \overline{\{\alpha\}}$.

1.5) Corollary: If α is a cut point of X and S is a component of $X - \{\alpha\}$, then $S \cup \{\alpha\}$ is connected.

1.6) Theorem: Let α and β be distinct cut points of X. If S is a component of $X - \{\alpha\}$ distinct from $S(\beta,\alpha)$, then $S \subset S(\alpha,\beta)$.

Proof: If S is distinct from $S(\beta,\alpha)$, then $S \subset X - \{\beta\}$ and by 1.5 we know that $S \cup \{\alpha\}$ is a connected subset of $X - \{\beta\}$ that contains α. Thus $S \subset \{\alpha\} \cup S(\alpha,\beta)$, and hence $S \subset S(\alpha,\beta)$.

1.7) Theorem: If α and β are distinct cut points of X such that $\alpha \in \overline{\{\beta\}}$ and $\{\alpha\} = \partial S(\alpha,\beta)$, then $S(\alpha,\beta) \cap S(\beta,\alpha) = \emptyset$.

Proof: Let $A = S(\alpha,\beta) \cap S(\beta,\alpha)$. Since $S(\alpha,\beta)$ is closed, A is a closed subset of $S(\beta,\alpha)$. Since $\alpha \notin A$, $A \subset S(\alpha,\beta) - \{\alpha\} = S(\alpha,\beta)^\circ$, we have that $A = S(\alpha,\beta)^\circ \cap S(\beta,\alpha)$. Therefore A is also an open subset of $S(\beta,\alpha)$, but since $A \neq S(\beta,\alpha)$ and $S(\beta,\alpha)$ is connected, we have $A = \emptyset$.

1.8) Corollary: If α and β are distinct cut points of X such that $\alpha \in \overline{\{\beta\}}$ and $\{\alpha\} = \partial S(\alpha,\beta)$, then:

i) $S(\beta,\alpha) = \cup\{ S \cup \{\beta\} : S$ a component of $X - \{\beta\}$ and $S \neq S(\alpha,\beta) \}$

ii) $S(\alpha,\beta) = \cup\{ S \cup \{\alpha\} : S$ a component of $X - \{\alpha\}$ and $S \neq S(\beta,\alpha) \}$

Proof: i) Since the right hand side of the equation is connected, contains β and does not contain α it must be contained in the left hand side. If $x \in X$ is such that x is not in the right hand side, then $x \in S(\alpha,\beta)$ and $x \notin S(\beta,\alpha)$ by 1.7. Thus the complement of the right hand side is contained in the complement of the left hand side and the left hand side is contained in the right hand side. The proof of part ii is identical to that of i.

1.9) Corollary: If α and β are distinct cut points of X such that $\alpha \in \overline{\{\beta\}}$ and $\{\alpha\} = \partial S(\alpha,\beta)$ and $x \in X$ is distinct from α, then α cuts between x and β if and only if $x \in S(\alpha,\beta)$.

Proof: Follows immediately from 1.8.

In the classical cyclic element theory in Peano spaces much of the work relied on the fact that the space was arc-connected. As pointed out in the introduction the work was soon extended into spaces that are not arc-connected. In Whyburn's work in separable metric spaces he found a set that possessed the property of tying the space together much as the arc had done in the Peano space. We shall find later that the same type of set exists in a suitably restricted locally connected space and will be used as a tool much the same as it was by Whyburn. The following definition is essentially the same as in the separable metric case and is the key to finding the set mentioned above.

Let a and b be distinct elements of X. Then E(a,b) denotes the set of all the cut points of X which cut between a and b. If a and b are distinct elements of X and α and β are distinct elements of E(a,b), then <u>α is said to proceed β in E(a,b)</u>, $\alpha < \beta$, if α cuts between a and β.

1.10) Theorem: The relation "<" on E(a,b) is a linear order for each pair of distinct points a and b in X.

Proof: Let α and β be distinct elements of E(a,b) and assume that α does not cut between a and β, thus $S(a,\alpha) = S(\beta,\alpha)$. Since $\alpha \in E(a,b)$, α cuts between a and b, so $S(a,\alpha) \neq S(b,\beta)$ and $S(b,\alpha) \cup \{\alpha\}$ is a connected subset of $X - \{\beta\}$ that contains both α and b. Therefore $S(\alpha,\beta) = S(b,\beta)$ and β cuts between α and a.

Let α and β be elements of E(a,b) such that $\alpha < \beta$. Then $S(a,\alpha) \neq S(\beta,\alpha)$ and $S(a,\alpha) \cup \{\alpha\}$ is a connected subset of $X - \{\beta\}$ which contains both a and α. Thus β does not cut between a and α hence $\beta \not< \alpha$.

Let α, β and η be elements of E(a,b) such that $\alpha < \beta$ and $\beta < \eta$. Since $\alpha < \beta$, $S(a,\alpha) \neq S(\beta,\alpha)$. If $\alpha \not< \eta$, we know that $\eta < \alpha$ and thus $S(a,\alpha) = S(\eta,\alpha)$. But then $S(a,\alpha)$ is a connected subset of $X - \{\beta\}$ which contains both a and η, which implies that β does not cut between a and η, contrary to the hypothesis. Therefore $\alpha < \eta$.

Let a and b be distinct elements of X such that $E(a,b) \neq \emptyset$. Then if B is any nonempty subset of $E(a,b)$, B inherits a linear order relation from $E(a,b)$. Thus $(B,<)$ is a directed set and hence $(i_B,<)$, where i_B is the identity mapping of B onto B, is a net. For convenience we shall let $(B,<)$ denote the net $(i_B,<)$.

1.11) Theorem: Let α and β be in $E(a,b)$ with $\alpha < \beta$. Then $S(a,\alpha) \cup \{\alpha\}$ is contained in $S(\alpha,\beta)$.

Proof: Since $\alpha < \beta$ we know that $S(a,\alpha) \neq S(\beta,\alpha)$, thus $S(a,\alpha) \cup \{\alpha\}$ is a connected subset of $X - \{\beta\}$ which contains a. Thus we have that $S(a,\alpha) \cup \{\alpha\} \subset S(a,\beta)$.

1.12) Corollary: Let α and β be in $E(a,b)$ with $\alpha < \beta$. Then:

i) $S(a,\beta) = S(\alpha,\beta)$

ii) $S(b,\alpha) = S(\beta,\alpha)$

iii) $S(b,\beta) \cup \{\beta\} \subset S(b,\alpha)$

iv) $S(a,\alpha) \cap S(b,\beta) = \emptyset$

Proof: i) Follows immediately from 1.11. ii) If $S(b,\alpha) \neq S(\beta,\alpha)$, then $S(a,\alpha) \cup \{\alpha\} \cup S(\beta,\alpha)$ is a connected subset of $X - \{\beta\}$. Thus $S(a,\beta) = S(b,\beta)$, contrary to the hypothesis. iii) By 1.6 we know $S(b,\beta) \subset S(b,\alpha)$ and by ii, $\beta \in S(b,\alpha)$. Thus $S(b,\beta) \cup \{\beta\} \subset S(b,\alpha)$. iv) $S(a,\alpha) \cap S(b,\beta) \subset S(a,\alpha) \cap S(b,\alpha) = \emptyset$.

1.13) Theorem: Let a, b, α, β, and η be distinct elements of X, then:

i) If α and η are in E(a,b) with $\alpha < \eta$ and $\beta \in E(\alpha,\eta)$, then $\beta \in E(a,b)$ and $\alpha < \beta < \eta$.

ii) If α, β and η are in E(a,b) and $\alpha < \beta < \eta$, then $\beta \in E(\alpha,\eta)$.

Proof: i) Assume that $\beta \notin S(b,\alpha) = S(\eta,\alpha)$. Then $S(\eta,\alpha) \cup \{\alpha\}$ is a connected subset of $X - \{\beta\}$ containing α and η which implies that $S(\alpha,\beta) = S(\eta,\beta)$, contrary to the hypothesis. Therefore $\beta \in S(\eta,\alpha)$. Similarily $\beta \in S(\alpha,\eta)$. Thus we have $S(a,\alpha) \cup \{\alpha\}$ and $S(b,\eta) \cup \{\eta\}$ are connected subsets of $X - \{\beta\}$, hence $S(a,\beta) = S(\alpha,\beta)$ and $S(b,\beta) = S(\eta,\beta)$. However since $\beta \in E(\alpha,\eta)$, we have that $S(\alpha,\beta) \neq S(\eta,\beta)$. Thus $S(a,\beta) \neq S(b,\beta)$ and $\beta \in E(a,b)$. ii) By 1.12i, $\alpha < \beta$ implies $S(a,\beta) = S(\alpha,\beta)$ and by 1.12ii, $\beta < \eta$ implies $S(b,\beta) = S(\eta,\beta)$. But since $\beta \in E(a,b)$, $S(a,\beta) \neq S(b,\beta)$ and thus $\beta \in E(\alpha,\eta)$.

1.14) Theorem: Let a and b be distinct elements of X, and let B be a nonempty subset of E(a,b). If $S = \cup\{ S(b,\alpha) : \alpha \in B \}$, then ∂S is contained in the set of limits of (B,<). If B has no first element, then S is open and ∂S is equal to the set of limits of (B,<).

Proof: If B has a first element β, let $x \in \partial S = \partial S(b,\beta)$ and let V be an open connected neighborhood of x. Then $\beta \in V$ and thus x is a limit of (B,<).

Assume that B has no first element and let $x \in S$. We shall show that $x \in S^\circ$ by considering three cases, which are not necessarily distinct. First let $x \in B$ and let $\alpha \in B$ be such that $\alpha < x$. If α is a closed cut point, $S(x,\alpha) = S(b,\alpha)$ is open and thus $x \in S^\circ$. If α is an open cut point, then $X - S(a,\alpha) = \{\alpha\} \cup \{y \in X : S(y,\alpha) \neq S(a,\alpha)\}$ is an open connected neighborhood of x which contains b. Let $\beta \in B$ be such that $\beta < \alpha$. Then $\beta \in S(a,\alpha)$ and thus $X - S(a,\alpha) \subset S(b,\beta)$ and $x \in S^\circ$. Second, if $x \in S(b,\alpha)^\circ$ for some $\alpha \in B$, then $x \in S^\circ$. Finally let $x \in \partial S(b,\alpha)$ for some $\alpha \in B$ and assume that $x \notin B$. Let $\beta \in B$ be such that $\beta < \alpha$. If β is closed, then $S(x,\beta)$ is an open connected neighborhood of x and thus $\alpha \in S(x,\beta)$ and $S(x,\beta) = S(\alpha,\beta)$. Then $S(x,\beta) = S(b,\beta)$, since $S(\alpha,\beta) = S(b,\beta)$, and $x \in S^\circ$. If β is open, then $x \in S(b,\beta)$ since $S(b,\beta)$ is closed and $S(b,\alpha) \subset S(b,\beta)$. Then as in the first case we have that $X - S(a,\beta)$ is an open set contained in S which contains $S(b,\beta)$ and thus $x \in S^\circ$. Therefore S is open.

Again assume that B has no first element. We shall show that ∂S is equal to the set of limits of $(B,<)$. Let $x \in \partial S$ and let $\alpha \in B$. If $\beta \in B$ is such that $\beta < \alpha$ and if $S(x,\alpha) \neq S(a,\alpha)$, then $S(x,\alpha) \subset S(b,\beta)$ and $x \in S$ which is impossible since S is open. Therefore $x \in S(a,\alpha)$ for all $\alpha \in B$. Let V be a connected neighborhood of x. Then $V \cap S(b,\alpha) \neq \emptyset$ for some $\alpha \in B$ and since $x \notin S(b,\alpha)$, $\alpha \in V$. If $\beta \in B$ such that $\beta < \alpha$, then β cuts between x and α since $S(x,\beta) = S(a,\beta) \neq S(b,\beta) = S(\alpha,\beta)$. Thus $\beta \in V$. Therefore x is a limit of $(B,<)$. Finally, let x be a limit of $(B,<)$. Recall that if $y \in S$, then as shown above there is an $\alpha \in B$ such that $y \in S(b,\alpha)^\circ$. But no element

β of B is such that $\beta < \alpha$ and is in $S(b,\alpha)^{\circ}$. Therefore no point of S is a limit of $(B,<)$. Hence $x \in \overline{S} - S = \partial S$, since S is open.

Chapter 2

Let a and b be distinct points in X, then a and b are said to be <u>conjugate</u>, denoted a 0 b, if there is no point in X which cuts between them. Thus if a 0 b, then a and b are in the same component of X - $\{\alpha\}$ for every α in X distinct from a and b.

2.1) Theorem: Let a and b be distinct elements of X such that $a \in \overline{\{b\}}$. Then a 0 b.

Proof: Assume there is a point α which cuts between a and b. If a is a closed cut point then $S(b,\alpha)$ is an open neighborhood of a which does not contain b. If α is an open cut point, then $S(b,\alpha)$ is a closed set containing b but not a. In either case we have a contradiction to the hypothesis. Therefore a 0 b.

2.2) Theorem: If a 0 x_1 0 ... 0 x_n 0 b and $z \in E(a,b)$, then $z = x_i$ for some i between 1 and n.

Proof: If z is not equal to any x_i, $1 \leq i \leq n$, then $\{a,x_1,\ldots x_n,b\}$ must be contained in a single component of X - $\{z\}$, contrary to the hypothesis.

2.3) Corollary: If a 0 x_1 0 ... 0 x_n 0 b and a 0 y_1 0 ... 0 y_m 0 b and $x_i \neq y_j$ for all i and j, where $1 \leq i \leq n$ and $1 \leq j \leq m$, then a 0 b.

Proof: If a and b are not conjugate there is a cut point α in X which cuts between a and b. By 2.1 $\alpha = x_i$ and $\alpha = y_j$ for some i and j, where i and j are as above. But then $x_i = y_j$ for some i and j contrary to the hypothesis.

Theorem 2.2 and corollary 2.3 along with the following two definitions are due to Radó and Reichelder [7].

A subset A of X is said to be <u>O-coherent</u> (or coherent) if every two points a and b in A are conjugate. If a contains every point which is conjugate to two distinct points in A, then A is said to be <u>O-complete</u> (or complete).

2.4) Theorem: Let A be a coherent subset of X, then $A - \{\alpha\}$ is contained in a single component of $X - \{\alpha\}$ for each point α in X.

Proof: Let a be in $A - \{\alpha\}$ where $\alpha \in X$. If $b \in A - \{\alpha\}$ such that b is distinct from a, then a O b. Thus $S(a,\alpha) = S(b,\alpha)$. Therefore $A - \{\alpha\} \subset S(a,\alpha)$.

Let N be a nondegenerate subset of X which is both complete and coherent, then N is said to be an <u>N-set</u>.

2.5) Theorem: If a and b are distinct elements of X such that a O b, then there is a unique N-set N which contains both a and b.

Proof: Let $N = \{ x \in X : x \text{ O } a \text{ and } x \text{ O } b \}$ and let x and y be distinct elements of N. Then x O a O y and x O b O y. Thus by 2.3 x O y, and N is coherent. Now let $z \in X$ be such that z O x and z O y, where x and y are distinct elements of N. Then a O x O z and a O y O z, thus a O z. Similarly b O z. Hence by the definition of N we have that $z \in N$, and N is complete. Therefore N is an N-set.

Let N' be any other N-set containing a and b, and let $x \in N'$.

Then x O a and x O b since N' is coherent. Thus $x \in N$ and $N' \subset N$. If $x \in N$, then x O a and x O b, but since N' is complete $x \in N'$ and $N \subset N'$. Therefore $N' = N$, and N is unique.

2.6) Corollary: If N and N' are distinct N-sets in X, then $N \cap N'$ is degenerate.

Proof: Follows immediately from 2.5.

2.7) Corollary: If N and N' are distinct N-sets in X and $\{\alpha\} = N \cap N'$, then α is a cut point of X, and if $a \in N - \{\alpha\}$ and $b \in N' - \{\alpha\}$, then α cuts between a and b.

Proof: Since a and α are in N, a O α and since b and α are in N', b O α. But since N is complete and b is not in N we have that a and b are not conjugate. Let β be a cut point which cuts between a and b. By 2.2, $\beta = \alpha$ and thus α cuts between a and b and is a cut point.

We should point out that 2.4, 2.5, 2.6 and 2.7 are essentially the same as in [1] though we did not have to worry about showing that the cut points are closed points. Also note that from 2.1 and 2.5 we have the result that every open cut point is in at least one N-set. The same is not true of closed cut points.

2.8) Theorem: Let N be an N-set in X. If $x \in \overline{N} - N$, then there is a unique open cut point α of X in N such that α cuts between x and each point of $N - \{\alpha\}$, $x\ O\ \alpha$, $x \notin \overline{N - \{\alpha\}}$, and $x \in \overline{\{\alpha\}}$.

Proof: Since x is not in N there is a point n in N and a cut point α of X such that α cuts between x and n. Thus $S(x,\alpha) \neq S(n,\alpha)$. By 2.4, $N - \{\alpha\}$ is contained in $S(n,\alpha)$. If α were a closed cut point $S(x,\alpha)$ would be an open neighborhood of x that is contained in $X - (\{\alpha\} \cup S(n,\alpha))$. However $N \subset (\{\alpha\} \cup S(n,\alpha))$, thus we would have $S(x,\alpha) \cap N = \emptyset$ which is impossible since $x \in \overline{N}$. Therefore α is an open cut point. If $\alpha \notin N$, then $N \subset S(n,\alpha)$ and since $S(n,\alpha)$ is closed, $\overline{N} \subset S(n,\alpha)$. Thus $x \in S(n,\alpha)$ which is impossible since α cuts between x and n. Therefore $\alpha \in N$. Since $S(n,\alpha)$ is a closed set and $N - \{\alpha\} \subset S(n,\alpha)$ we have that $\overline{N - \{\alpha\}} \subset S(n,\alpha)$, thus $x \notin \overline{N - \{\alpha\}}$. However $\overline{N} = \overline{N - \{\alpha\}} \cup \overline{\{\alpha\}}$ and $x \in \overline{N}$, hence $x \in \overline{\{\alpha\}}$ and, by 2.1, $x\ O\ \alpha$. Let β be an open cut point of X which cuts between x and each point of $X - \{\beta\}$. Then since $x\ O\ \alpha\ O\ n$ where n is any point of N and β cuts between x and n we have, by 2.2, that $\alpha = \beta$. Therefore α is unique.

2.9) Corollary: Let N be an N-set in X. If N contains no open cut points, then N is closed.

2.10) Lemma: If $x \in \overline{N} - N$, where N is an N-set, then $S(x,N) = S(x,\alpha)$ where α is the open cut point in N whose existence is guaranteed by theorem 2.8.

Proof: Since $\alpha \in N$, $S(x,N)$ is a connected subset of $X - \{\alpha\}$ containing x. So $S(x,N) \subset S(x,\alpha)$. Let $n \in N - \{\alpha\}$. Then by 2.8 $S(n,\alpha) \neq S(x,\alpha)$ and by 2.4 $S(N - \{\alpha\},\alpha) = S(n,\alpha)$. Thus $S(x,\alpha) \cap N = \emptyset$, which implies $S(x,\alpha)$ is a connected subset of $X - N$ containing x. Therefore $S(x,\alpha) \subset S(x,N)$.

2.11) Theorem: If N is an N-set in X and S is a component of $X - N$, then S is either open or closed.

Proof: Let S be a component of $X - N$ which is not open. Then there is a point x in $\partial S \cap S$. Let V be a connected neighborhood of x. If $V \cap N = \emptyset$, then $V \subset S$ which is impossible since $x \in \partial S$. Therefore $V \cap N \neq \emptyset$. Thus $x \in \overline{N} - N$. Let α be the open cut point in N guaranteed by theorem 2.8. Then by 2.10 $S = S(x,\alpha)$ which implies that S is closed.

2.12) Corollary: Let N be an N-set in X and let S be an open component of $X - N$, then $\partial S \subset N$.

Proof: Since S is relatively closed in $X - N$, S contains $\partial S \cap (X - N)$. But by the prooof of 2.11 if there is one point of the boundary of S in S, then S is closed in X. Thus if S is open in X, then $\partial S \cap (X - N) = \emptyset$ or $\partial S \subset N$.

If S is a closed component of $X - N$, for some N-set N in X we let $b(S)$ denote the open cut point in N whose existence is guaranteed by theorem 2.8. Note that $S = S(S,b(S))$.

Chapter 3

A nonempty subset A of X is called an <u>A-set</u> if it satisfies the following:

i) Every component of $X - A$ is open or closed.

ii) Each open component of $X - A$ has singleton boundary.

iii) For each closed component S of $X - A$ there is a unique open cut point α in A such that S is a component of $X - \{\alpha\}$ and $\partial S \cap \overline{A - \{\alpha\}} = \emptyset$

Let A be an A-set in X and S be a component of $X - A$. Then if S is open and $\partial S = \{\alpha\}$, $b(S)$ denotes α, or if S is closed, then $b(S)$ denotes the unique open cut point α in A whose existence is guaranteed by iii of the definition of an A-set.

If S is an open component of $X - A$, where A is an A-set, then since $S = (S \cup \{b(S)\}) \cap (X - \{b(S)\}) = \overline{S} \cap (X - \{b(S)\})$, S is a clopen subset of $X - \{b(S)\}$. Therefore S is a component of $X - \{b(S)\}$. If S is a closed component of $X - A$, then by 1.4 each point x of ∂S is in $\overline{\{b(S)\}}$ and by 2.1, x 0 $b(S)$.

We should point out that the notation $b(S)$ has already been used in connection with N-sets. If S is a closed component of $X - N$, where N is an N-set in X, $b(S)$ is an open cut point in N such that S is a component of $X - \{b\}$ and $\partial S \cap \overline{N - \{b\}} = \emptyset$. Since the notation refers to the same type of point in either case no confusion should arise from its use.

If α is a closed cut point of X and $\mathcal{U}$ is a collection of components of X - $\{\alpha\}$, then $A = (\cup \mathcal{U}) \cup \{\alpha\}$ is an A-set since each component of X - A is a component of X - $\{\alpha\}$ and thus is open with singleton boundary. Unfortunately the same is not true in general if α is an open cut point. However, if α is an open cut point we do have the following result.

3.1) Theorem: If α is an open cut point of X and $\mathcal{U}$ is a collection of components of X - $\{\alpha\}$, then $A = \{\alpha\} \cup (\cup \mathcal{U})$ is an A-set in X if and only in $\cup \mathcal{U}$ is closed.

Proof: If $\cup \mathcal{U}$ is closed the only thing to check is that for each component S of X - $\{\alpha\}$ which is not in $\mathcal{U}$, $\partial S \cap (\cup \mathcal{U}) = \emptyset$. But this is obvious since $\cup \mathcal{U}$ and S are disjoint closed sets.

Assume that $A = \{\alpha\} \cup (\cup \mathcal{U})$ is an A-set and let S be a component of X - A. Then S is a component of X - $\{\alpha\}$ and $\partial S \cap \overline{(\cup \mathcal{U})} = \partial S \cap \overline{A - \{\alpha\}} = \emptyset$. Since this is true for each component of X - A and α is an open point, $\overline{\cup \mathcal{U}} = \cup \mathcal{U}$.

3.2) Theorem: Let A be an A-set in X and let x be in X - A. Then $b = b(S(x,A))$ cuts between x and each point of A - $\{b\}$.

Proof: Follows immediately from the fact that $S(x,A) = S(x,b)$ and $S(x,A) \cap A = \emptyset$.

3.3) Theorem: Let N be an N-set in X and let A be an A-set in X. If $N \cap A$ is nondegenerate, then $N \subset A$. If N is not contained in A and $N \cap A \neq \emptyset$, then $N \cap A = \{\alpha\}$, where α is a cut point of X.

Proof: Let $x \in N - A$. Then $b = b(S(x,A))$ cuts x from each point of $A - \{b\}$. Since $N - \{b\} \subset S(x,b) = S(x,A)$ it follows that $N \cap (A - \{b\}) = \emptyset$ and therefore $N \cap A \subset \{b\}$.

A subset C of X is called a <u>true cyclic element of X</u> if it is both an N-set and an A-set. A point $x \in X$ is called an <u>end point of X</u> if x is a non-cut point which is conjugate to no other point of X.

3.4) Theorem: Every end point a of X has a basis of neighborhoods consisting of all sets of the form $S(a,\alpha)$, where α is a cut point of X.

Proof: If α is a cut point, then $a \in S(a,\alpha)^\circ$ since either $\{\alpha\} = \partial S(a,\alpha)$ or each point of $\partial S(a,\alpha)$ is in $\overline{\{\alpha\}}$ and thus is conjugate to α. Therefore each set of the form $S(a,\alpha)$ is a neighborhood of a. By 2.1 we know that $\{a\}$ is a closed set, thus any neighborhood of a is nondegenerate. Let V be a connected neighborhood of a and let $y \neq a$ be an element of V. Since y is not conjugate to a there is a cut point α in V such that α cuts between a and y. Suppose that $S(a,\alpha)$ is not contained in V, and let $z \in S(a,\alpha) - V$. Then $E(a,z)$ is non-empty and has no first element. By 1.14, ∂S is the set of limits of $(E(a,z),<)$ and $S = \cup\{ S(z,\alpha) : \alpha \in Ea,z) \}$ is open. Suppose there is a point $x \in \partial S$ such that $x \neq a$. Then there is a cut point α which cuts between x and a and $x \in S(x,\alpha)^\circ$. Thus $S(x,\alpha)^\circ \cap E(a,z) \neq \emptyset$, which implies that α cuts between a and some point of $E(a,z)$, hence $\alpha \in E(a,z)$. Let $\beta \in E(a,z)$ such that

$\beta < \alpha$, then $S(a,\beta) = S(z,\beta)$. But this implies $x \in S$, which is impossible since S is open. Therefore $\partial S = \{a\}$ and S is a clopen proper subset of $X - \{a\}$. Thus a is a cut point, contrary to the hypothesis. Therefore $S(a,\alpha) \subset V$ and we have the results.

Let A be an A-set in X and let B be a subset of A. Let B' denote the union of all the components S of X - A such that $b(S) \in B$, and let $B^* = B' \cup B$. If B is a subset of X and B is not contained in A, then $B' = (B \cap A)'$ and $B^* = (B \cap A)^*$ as long as there is no possiblity of confusion as to which A-set A is being used.

3.5) Lemma: Let A be an A-set in X. Then if $B \subset A$ is closed in A, $\overline{B} \subset B^*$.

Proof: Since $B = \overline{B} \cap A$, we need only consider the points in $\overline{B} - A$. Let $x \in \overline{B} - A$, then $S(x,A)$ is closed since $x \in \partial S \cap S$. Let $b = b(S(x,A))$. If $b \notin B$, then $B \subset A - \{b\}$ and thus $\overline{B} \subset \overline{A - \{b\}}$. But by the definition of an A-set, $x \notin \overline{A - \{b\}}$ thus $x \notin \overline{B}$, contrary to the hypothesis. Therefore $b \in B$ and $x \in S(x,A) \subset B^*$.

3.6) Lemma: Let A be an A-set in X. Then if $B \subset A$ is closed in A, $\overline{B'} \subset B^*$.

Proof: Let $x \in \overline{B'} - B'$ and let V be a connected neighborhood of x. Since $x \in \overline{B'}$, $V \cap B' \neq \emptyset$. Thus there is a component S of X - A in B' such that $V \cap S \neq \emptyset$. Let $b = b(S)$. Since $V \neq V \cap S \neq \emptyset$ and V is connected, $b \in V$. However $S \subset B'$ implies that $b(S) \in B$.

Therefore $V \cap B \neq \emptyset$ and thus $x \in \overline{B} \subset B^*$ by 3.4.

3.7) Theorem: Let A be an A-set in X. Then if $B \subset A$ is closed in A, B^* is closed in X.

Proof: $\overline{B^*} = \overline{B} \cup \overline{B'} \subset B^* \cup B^* = B^*$.

3.8) Theorem: Let A be an A-set in X and let B be a subset of A. Then $X - B^* = (X - B)^*$.

Proof: Follows immediately from the definition of B^*.

3.9) Corollary: Let A be an A-set in X and let B and C be disjoint subsets of A, then B^* and C^* are disjoint subsets of X.

3.10) Corollary: Let A be an A-set in X and let $B \subset A$ be open in A, then B^* is open in X.

Proof: $(X - B)^* = ((X - B) \cap A)^*$ is closed in X, thus $B^* = X - (X - B)^*$ is open in X.

3.11) Corollary: Let A be an A-set in X, then the mapping f from X onto A defined by $f(x) = x$, if $x \in A$, and $f(x) = b(S(x,A))$, if $x \notin A$, is a retraction of X onto A.

3.12) Lemma: Let A be an A-set in X and let B be a subset of A. Then $\overline{B^*} \subset \overline{B}^*$.

Proof: Since $B \subset \overline{B}$, it follows that $B^* \subset \overline{B}^*$. Since $\overline{B}^*$ is closed in X we have $\overline{B^*} \subset \overline{B}^*$.

3.13) Theorem: Let A be an A-set in X and let B and C be subsets of A which are separated in A. Then B* and C* are separated in X.

Proof: B and C separated in A implies that $\overline{B} \cap C = \emptyset$, thus $\overline{B}^* \cap C^* = \emptyset$. Since $\overline{B^*} \subset \overline{B}^*$, we have that $\overline{B^*} \cap C^* = \emptyset$. Similarly $B^* \cap \overline{C^*} = \emptyset$.

3.14) Theorem: Let A be an A-set in X and let H be a connected subset of X. Then $H \cap A$ is connected.

Proof: If $H \subset A$ or $H \cap A = \emptyset$ the result is immediate. So assume that $\emptyset \neq H \cap A \neq H$. Let $x \in H - A$ and let $S = S(x,A)$. Since H is connected and $\emptyset \neq H \cap S \neq H$, $b = b(S) \in H$ and thus $x \in S(x,A) \subset (H \cap A)^*$. Therefore $H \subset (H \cap A)^*$. If $H \cap A = B \cup C$, where B and C are non-empty and separated in A, then $H \subset B^* \cup C^*$, where B* and C* are non-empty and separated in X. Thus $H = (B^* \cap H) \cup (C^* \cap H)$ separated and $B \subset B^* \cap H$ and $C \subset C^* \cap H$. Thus neither $B^* \cap H$ nor $C^* \cap H$ is empty. Thus H is not connected, contrary to the hypothesis. Therefore $H \cap A$ is connected.

3.15) Corollary: If A is an A-set in X, then A is connected.

3.16) Theorem: Let $\mathcal{U}$ be a collection of A-sets in X such that $A = \cap\, \mathcal{U} \neq \emptyset$. If S is a component of X - A, then S is either open or closed and for each closed component S of X - A there is a unique open cut point β in A such that S is a component of $X - \{\beta\}$ and $S \cap \overline{A - \{\beta\}} \neq \emptyset$.

Proof: Let S be a component of X - A such that there is a point x in $\partial S \cap S$, (i.e. assume that S is not open) thus $x \in \overline{A} - A$. Since $x \notin A$, there is an A-set A_1 in $\mathcal{U}$ such that $x \notin A_1$. Let $S_1 = S(x,A_1)$ and let $b = b(S_1)$, since $\{b\}$ is open, $\{b\}^*$ is an open neighborhood of x, thus $\{b\}^* \cap A \neq \emptyset$. Since $A \subset A_1$, $\{b\}^* \cap A \subset \{b\}^* \cap A_1 = \{b\}$ thus $\{b\}^* \cap A = \{b\}$. Since A_1 is an A-set in X, $S_1 = S(x,A_1) = S(x,b)$ and since $b \in A$ it follows that $S \subset S(x,b)$. Also since $A \subset A_1$ it follows that $S = S(x,A) \supset S(x,A_1) = S(x,b)$. Therefore $S = S(x,b)$, which implies that S is closed. Also $S \cap \overline{A - \{\alpha\}} \subset S \cap \overline{A_1 - \{\alpha\}} = \emptyset$.

3.17) Corollary: Let $\mathcal{U}$ be a collection of A-sets in X such that $A = \cap\mathcal{U} \neq \emptyset$, and let S be a closed component of X - A. Then there is an A_1 in $\mathcal{U}$ such that S is also a component of $X - A_1$.

3.18) Theorem: Let $\mathcal{U}$ be a collection of A-sets in X such that $A = \cap\mathcal{U} \neq \emptyset$ and let $x \in X - A$. Then $S(x,A) = \cup_i S(x,A_i)$, where $A_i \in \mathcal{U}$ such that $x \notin A_i$.

Proof: Let $S = S(x,A)$. Since $A \subset A_i$ for all $A_i \in \mathcal{U}$ we have $S(x,A_i) \subset S(x,A)$ for all $A_i \in \mathcal{U}$. Thus $\cup_i S(x,A_i) \subset S$. If S is closed then $S = S(x,A_i)$ for some $A_i \in \mathcal{U}$ by 3.17 and thus $S \subset \cup_i S(x,A_i)$. So in the following we shall assume that S is open. Let $y \in \partial S$ and consider the case where there is a point $z \in S$ such that $y \; O \; z$. Since $z \in S$, there is an $A_1 \in \mathcal{U}$ such that $z \notin A_1$. Then, by 3.2, $b = b(S(z,A_1))$ cuts between z and each point of $A_1 - \{b\}$. Thus $b = y$, since $y \in A_1$

and $y \, O \, z$. But $y \in A$, so $b \in A$. Since $b \in A$ we have that $S(z,A) \subset S(z,b) = S(z,A_1)$. But $z \in S$ implies that $S(z,A) = S(x,A_1)$. Hence $S(x,A) \subset S(z,A_1)$, which implies that $S(z,A_1) = S(x,A_1)$. Therefore $S(x,A) \subset S(x,A_1)$ and $S \subset \cup_i S(x,A_i)$. Next, still letting $y \in \partial S$, consider the case where y is not conjugate to any point in S. Let A_i be any element of $\mathcal{U}$ such that $x \notin A_i$. If $b_i = b(S(x,A_i)) = y$, then as in the case when S is closed $S(x,A_i) = S$ and $S \subset \cup_i S(x,A_i)$. So assume that for all $A_i \in \mathcal{U}$, such that $x \notin A_i$, $b_i \neq y$, where b_i is as above. Since b_i cuts between x and each point of $A_i - \{b_i\}$ and since $S \cup \{y\}$ is a connected set containing both y and x it follows that $b_i \in S$. But $b_i \in S$ implies that there is an $A_j \in \mathcal{U}$ such that $b_i \notin A_j$. Let $b = b(S(b_i,A_j))$. Then b cuts between y and b_i, so by 1.13, b cuts between y and x. Thus $b < b_i$ in $E(y,x)$ and $S(b_i,b) = S(x,b) = S(x,A_j)$ so $b = b_j = b(S(x,A_j))$. Therefore $B = \{ b_i : b_i = b(S(x,A_i)), A_i \in \mathcal{U}, \text{ and } x \notin A_i \}$ is a subset of $E(y,x)$ with no first element. By 1.14, $\cup_i S(x,A_i) = \cup_i S(x,b_i)$, $b_i \in B$, is an open subset of X and $\partial(\cup_i S(x,A_i))$ is equal to the set of limits of $(B,<)$. Let $z \in \partial(\cup_i S(x,A_i))$. We shall show that $z \notin S$. Hence suppose $z \in S$. Then we know that $E(y,z) \neq \emptyset$. Let $\alpha \in E(y,z)$. Since $S \cup \{y\}$ is connected and contains both y and z. $\alpha \in S$. Therefore α is not conjugate to y. But if β cuts between y and α, then $\beta \in E(y,z)$ with $\beta < \alpha$. Hence $E(y,z)$ has no first element. By 1.14, we know that $\cup\{ S(z,\alpha) : \alpha \in E(y,z) \}$ is an open neighborhood of z. Therefore

$(\cup\{ S(z,\alpha) : \alpha \in E(y,z) \}) \cap (\cup_i S(x,A_i)) \neq \emptyset$. Thus there is an $\alpha \in S(y,z)$ and an $A_1 \in \mathcal{U}$ such that $S(z,\alpha) \cap S(x,A_1) \neq \emptyset$. If $\alpha \notin A_1$, then $b = b(S(\alpha,A_1))$ cuts between y and α. Thus $S(z,\alpha) \subset S(\alpha,b) = S(\alpha,A_1)$. But $S(z,\alpha) \cap S(x,A_1) \neq \emptyset$ and $S(z,\alpha) \subset S(\alpha,A_i)$ implies that $S(z,A_1) = S(x,A_1)$. Therefore $z \in S(x,A_1)$ which is impossible. If $\alpha \in A_1$, then $S(x,A_1) \subset S(z,\alpha)$. Since $\alpha \in S$ there is an A_2 in $\mathcal{U}$ such that $\alpha \notin A_2$. Then $S(z,\alpha) \subset S(z,A_2)$ since $b = b(S(z,A_2))$ cuts between y and α, and $S(z,A_2) = S(z,b)$. Thus $S(x,A_1) \subset S(z,A_2)$. So $S(x,A_2) = S(z,A_2)$, and $z \in \cup_i S(x,A_i)$. But this is impossible. Therefore $z \notin S$.

3.19) Corollary: If $\mathcal{U}$ is a finite family of A-sets in X, and if $A = \cap\, \mathcal{U}$ is non-empty, then A is an A-set in X.

Proof: Let A and B be A-sets in X such that $C = A \cap B$ in non-empty. By the definition of an A-set in X we need only prove that if S is a component of X - A and T is a component of X - B such that $S \cap T \neq \emptyset$, then either $S \subset T$ or $T \subset S$. For then each component of the complement of X - $(A \cap B)$ will have the necessary properties to make $A \cap B$ and A-set.

Let $x \in S \cap T$ and assume that S is not contained in T. Then $b(T) \in S$. If $b(S) \in A \cap B$, then $b(S) \notin T$ so T is a connected subset of $X - \{b(S)\}$ containing x and thus $T \subset S(x,b(S)) = S$. Assume that $b(S) \notin A \cap B$ and let $y \in A \cap B$. Then b(S) cuts between b(T) and y and b(T) cuts between x and y, thus b(S) and b(T) are in E(x,y) with $b(T) < b(S)$. By 1.11, $S(x,b(T)) \subset S(b(T),b(S))$. By 1.12i,

$S(b(T),b(S)) = S(x,b(S))$. Therefore $T = S(x,b(T)) \subset S(x,b(S)) = S$.

In 3.22 and 3.23 we shall find a necessary and sufficient condition to extend the above result to the infinte case.

3.20) Theorem: Let X be compact. If $\mathcal{U}$ is a family of A-sets in X with the property that each finite subfamily of sets has non-empty intersection then $\cap\{ A_i : A_i \in \mathcal{U} \} \neq \emptyset$.

Proof: Let $\overline{\mathcal{U}} = \{ \overline{A}_i : A_i \in \mathcal{U} \}$. Then $\overline{\mathcal{U}}$ also has the property that each finite subcollection has non-empty intersection. Since X is compact, $\cap\overline{\mathcal{U}} \neq \emptyset$. Let $b \in \cap\overline{\mathcal{U}}$, then $b \in \overline{A}_i$ for all $A_i \in \mathcal{U}$. If $b \in A_i$ for all $A_i \in \mathcal{U}$ we are through, so assume that $b \notin A_1$ for some $A_1 \in \mathcal{U}$. Let $b_1 = b(S(b,A_1))$. If b_1 is a closed cut point then $S(b,A_1)$ is a neighborhood of b and $S(b,A_1) \cap A_1 = \emptyset$ which is impossible. Thus b_1 is an open cut point, and hence $\{b_1\}^*$ is a neighborhood of b. Therefore $\{b_1\}^* \cap A_i \neq \emptyset$ for all $A_i \in \mathcal{U}$. If $b_1 \notin \cap\mathcal{U}$, then there is an $A_2 \in \mathcal{U}$ such that $b_1 \notin A_2$. But since $\{b_1\}^* \cap A_2 \neq \emptyset$ and $b_1 \notin A_2$ we have that $\{b_1\}' \cap A_2 \neq \emptyset$. Thus by the definition of $\{b_1\}'$ there is a closed component S of $X - A_1$ such that $A_2 \cap S \neq \emptyset$. Recall that S is also a component of $X - \{b_1\}$. But since A_2 is connected and $b_1 \notin A_2$, we have that $A_2 \subset S$. Therefore $A_1 \cap A_2 \neq \emptyset$, contrary to the hypothesis. Thus $b \in A_i$ for all $A_i \in \mathcal{U}$ and $\cap\mathcal{U} \neq \emptyset$.

3.21) Theorem: Every N-set in X is an intersection of A-sets in X.

Proof: Let S be a component of $X - N$. We shall show that there is a collection of A-sets, $A(S)$, such that S is a component of $X - A(S)$ and N is contained in each member of $A(S)$.

If S is closed, then $b = b(S)$ is an open cut point and by 3.1, $A(S) = S(N - \{b\},b) \cup \{b\}$ is an A-set containing N and S is a closed component of $X - A(S)$.

If S is open we consider the following two cases. First assume that S is an open component of $X - N$ such that there is a point s in S which is conjugate to some point $x \in \partial S$. If $y \in N - \{x\}$, then s is not conjugate to y. So there is a cut point α which cuts between s and y. But since $s\ O\ x\ O\ y$ we have $x = \alpha$. Since $x \in \partial S$, $\{x\}$ is not open so x is a closed cut point and hence $S(N - \{x\},x)$ is an open set containing $N - \{x\}$ not meeting S, so that no point of $N - \{x\}$ is in ∂S. But $\partial S \subset N$, so that $\{x\} = \partial S$. Consequently $A(S) = S(N - \{x\},x) \cup \{x\}$ is an A-set in X containing N and S is an open component of $X - A(S)$.

Second let S be an open component of $X - N$ such that no point of S is conjugate to any point $x \in \partial S$. Let $s \in S$ and $x \in \partial S$. Then $E(x,s) \neq \emptyset$. If $\alpha \in E(x,s)$, then $\alpha \in S$ since $\{\alpha\} \cup S$ is connected and contains both x and α. But $\alpha \in S$ implies that there is a cut point β which cuts between x and α. Then $\beta \in E(x,s)$ and $\beta < \alpha$. Therefore $E(x,s)$ has no first element. For each $\alpha \in E(x,s)$, $S(x,\alpha) \cup \{\alpha\}$ is an A-set containing N. By 1.14, $T = \cup\{ S(s,\alpha) : \alpha \in E(x,s) \}$ is an open set, and since $S(s,\alpha) \subset S$ for each $\alpha \in E(x,s)$, $T \subset S$. Let $z \in \partial T$

and assume that $z \in S$. Then as above $E(x,z)$ has no first element and thus $R = \bigcup\{ S(z,\alpha) : \alpha \in E(x,z) \}$ is an open neighborhood of z. Thus $R \cap T \neq \emptyset$. So there is an $\alpha \in E(x,s)$ and a $\beta \in E(x,z)$ such that $S(s,\alpha) \cap S(z,\beta) \neq \emptyset$. If $\beta \notin S(s,\alpha)$, then $S(s,\alpha) \subset S(z,\beta)$. Then β cuts between x and s so $\beta \in E(x,s)$ with $\beta < \alpha$. But then $S(z,\beta) = S(s,\beta)$ and $s \in T$ which is impossible since T is open. If $\beta \in S(s,\alpha)$, then α cuts between x and β. Thus $\alpha \in E(x,z)$ with $\alpha < \beta$. Therefore $S(s,\alpha) = S(\beta,\alpha) = S(z,\alpha)$ and again we have that $z \in T$ which is impossible. Thus if $z \in \partial T$, then $z \notin S$. Therefore T is a non-empty clopen subset of S. But since S is connected $T = S$. So $A(S) = \{ S(x,\alpha) \cup \{\alpha\} : \alpha \in E(x,s) \}$ is a collection of A-sets, each of which contains N and, by 3.18, S is a component of $X - (\cap A(S))$.

Then $N \subset \cap\{ A : A \in A(S),\ S \text{ a component of } X - N \}$ and since each component of $X - N$ is contained in $X - (\cap A(S))$, we have that $X - N \subset X - \cap\{ A : A \in A(S),\ S \text{ a component of } X - N \}$. Therefore $N = \cap\{ A : A \in A(S),\ S \text{ a component of } X - N \}$.

3.22) Theorem: A necessary and sufficient condition that every N-set in X is a true cyclic element in X is that every non-empty intersection of A-sets in X is an A-set in X.

Proof: First assume that every N-set in X is a true cyclic element, and let $\mathcal{U}$ be a family of A-sets such that $A = \cap\,\mathcal{U}$ is non-empty. In view of theorem 3.16, it is sufficient to show that each open component of $X - A$ has singleton boundary. Let S be an open component of X - A and let x and y be distinct elements of ∂S. If $x \, O \, y$, then there is a unique cyclic element C containing both x and y which, by 3.3, is contained in A. Since x and y are in ∂S, $S(S,C)$ is not closed. Hence $S(S,C)$ is an open component of X - C with at least two points in its boundary. But it is impossible since C is an A-set. If x is not conjugate to y, let α be a cut point which cuts between x and y. Since $S \cup \{x,y\}$ is connected and contains x and y, $\alpha \in S$. Similarly since x and y are in A_i for all $A_i \in \mathcal{U}$ and each A_i is connected we have $\alpha \in A_i$ for all $A_i \in \mathcal{U}$ and thus $\alpha \in A$. But $\alpha \in A$ and $\alpha \in S$ implies that $\alpha \in A \cap S$ which, since $S \subset X - A$, is impossible. Therefore ∂S is a singleton (note that $\partial S \neq \emptyset$ is guaranteed since X is connected.)

If every non-empty intersection of A-sets is an A-set, then by 3.21, we have that every N-set in X is an A-set in X and hence every N-set is a true cyclic element.

3.23) Theorem: A necessary and sufficient condition that every non-empty intersection of A-sets in X is an A-set in X is that for each two points a and b of X, if B is any subset of $E(a,b)$, such that B has no first element, then $(B,<)$ has a unique limit which is either a closed cut point or an end point.

Proof: First assume that every intersection of A-sets in X is an A-set in X. Then for each $\alpha \in B$ we know that $S(a,\alpha) \cup \{\alpha\}$ is an A-set in X containing a. Thus $A = \cap\{ S(a,\alpha) \cup \{\alpha\} : \alpha \in B \}$ is an A-set in X. By 3.18, $S = S(b,A) = \bigcup\{ S(b,\alpha) : \alpha \in B \}$. By 1.14, S is open and ∂S is equal to the set of limits of $(B,<)$. Since S is an open component of X - A, then $\partial S = \{b(S)\}$ and thus $(B,<)$ has a unique limit which is an end point if it is equal to A and it is a closed cut point if it is not all of A.

Let $\mathcal{U}$ be a family of A-sets and assume that $A = \bigcap \mathcal{U}$ is non-empty. As in the proof of 3.22, in order to show that A is an A-set in X we need only show that if S is an open component of X - A such that no point of S is conjugate to any point of ∂S, then ∂S is a singleton. Let $s \in S$, and let $x \in \partial S$. Then x is not conjugate to s and by 3.18, $S = \bigcup\{ S(s,A_i) : A_i \in \mathcal{U} \}$. But for each $A_i \in \mathcal{U}$, $S(s,A_i) = S(s,b_i)$ where $b_i = b(S(s,A_i))$. If there is an A_i such that $b_i = x$ we have the result, so assume that $b_i \neq x$ for each $A_i \in \mathcal{U}$. Thus $b_i \in E(x,s)$ and $b_i \in S$ for each i. Since $b_i \in S$ there is an A-set $A_j \in \mathcal{U}$ such that $b_i \notin A_j$. Then $b(S(b_i,A_j))$ cuts between x and b_i, thus $b(S(b_i,A_j)) = b_j$ with $b_j < b_i$ in $E(x,s)$. Therefore $B = \{ b_i : b_i = b(S(s,A_i)) \}$ is a subset of $E(x,s)$ with no first element. But $\partial(\bigcup_i S(s,b_i))$ is the set of limits of $(B,<)$ and therefore ∂S is the singleton set consisting of the unique limit of $(B,<)$ which is necessarily a closed cut point or an end point.

Chapter 4

A property is said to be <u>cyclicly extensible</u> if whenever each cyclic element has the property then the entire space has the property. A property is said to be <u>cyclicly reducible</u> if whenever the space has the property then each cyclic element has the property.

A space X is said to be <u>unicoherent</u> if whenever X is expressed as a union of two closed connected sets A and B, then $A \cap B$ is connected. The cyclic extensiblity and reducablity of unicoherence for Peano spaces was first proved by Kuratowski. We shall prove that unicoherence is a cyclicly reducible property in a general connected, locally connected space. Then we shall restrict the space in order to utilize some of the results of chapter three and obtain the result, in 4.3, that in these restricted cases unicoherence is cyclicly extensible.

4.1) Theorem: Unicoherence is a cyclicly reducible property.

Proof: Let C be a cyclic element in X and let $C = A \cap B$, where A and B are connected and closed in C. Then A^* and B^* are closed connected sets in X and since $X = A^* \cup B^*$ we have that $A^* \cap B^*$ is connected. Then, by 3.14, $C \cap (A^* \cap B^*)$ is connected. However $C \cap A^* \cap B^* = C \cap (A \cup A') \cap (B \cup B') = (C \cap A \cap B) \cup (C \cap A' \cap B') = C \cap A \cap B = A \cap B$. Therefore C is unicoherent.

<u>NOTE</u>: For the remaining part of chapter 4 we will assume that X has the property that each N-set in X is a cyclic element.

Note that when a space has the property that every N-set is a cyclic element then any point of the space is either a cut point, an end point, or a member of a true cyclic element.

If a and b are two distinct points of X and C is a cyclic element then $C \cap (\{a,b\} \cup E(a,b)) = D$ has at most two points. If $a \notin C$ and $b \notin C$ and $S(a,C) = S(b,C)$, then $D = \emptyset$. If $a \notin C$ and $b \notin C$ and $S(a,C) \neq S(b,C)$, then $D = \{ b(S(a,C)), b(S(b,C)) \}$, which could be a singleton. If $a \in C$ and $b \notin C$, then $D = \{ a, b(S(b,C)) \}$ and if both a and b are in C, then $D = \{a,b\}$.

If a and b are distinct elements of X let <u>$G(a,b) = \{a,b\} \cup E(a,b) \cup \{ C : C$ is a cyclic element in X such that $C \cap (\{a,b\} \cup E(a,b))$ is exactly two points$\}$</u>.

4.2) Theorem: If a and b are distinct elements of X then $G = G(a,b)$ is an A-set in X.

Proof: If a O b, then a and b are in a true cyclic element and the result is immediate. If $E(a,b) = \{\alpha\}$, then $G = C_1 \cup C_2$, where a and α are in C_1 and α and b are in C_2 and $C_1 \cap C_2 = \{\alpha\}$. Then any component of X - G is a component of $X - C_1$ distinct from $S(b,C_1)$ or a component of $X - C_2$ distinct from $S(a,C_2)$; thus G is an A-set in X.

For the remaining part of the proof we shall assume that E(a,b) is nondegenerate. If α and β are distinct elements of E(a,b), then $\beta \in S(a,\alpha)$ or $\beta \in S(b,\alpha)$; thus no component of $X - \{\alpha\}$ distinct from both $S(a,\alpha)$ and $S(b,\alpha)$ meets $E(a,b) \cup \{a,b\}$. If C is any true cyclic element of X in G and $\alpha \in E(a,b)$, then $(C - \{\alpha\}) \cap (E(a,b) \cup \{a,b\}) \neq \emptyset$ and thus $C - \{\alpha\}$ is contained in $S(a,\alpha)$ or in $S(b,\alpha)$. Hence $G - \{\alpha\} \subset S(a,\alpha) \cup S(b,\alpha)$. Therefore if S is a component of $X - \{\alpha\}$ distinct from $S(a,\alpha)$ and from $S(b,\alpha)$, then $S \cap G = \emptyset$ and S is a component of X - G. Let C be a true cyclic element contained in G. If $a \in C$ let $\{\alpha\} = C \cap E(a,b)$; then $E(a,b) - \{\alpha\}$ is contained in $S(b,C) = S(b,\alpha)$ and thus $G - C \subset S(b,C)$. Similarly if $b \in C$, then $G - C \subset S(a,C)$. If $C \cap E(a,b) = \{\alpha,\beta\}$ with $\alpha < \beta$, then $E(a,b) - \{\alpha,\beta\} \subset S(a,\alpha) \cup S(b,\beta)$ and as above $G - C \subset S(a,C) \cup S(b,C)$. Thus if S is a component of X - C distinct from $S(a,C)$ and $S(b,C)$, $S \cap G = \emptyset$ and S is a component of X - G. If a is a cut point, then $G \subset S(b,a) \cup \{a\}$. Thus if S is a component of $X - \{a\}$ distinct from $S(b,a)$, then S is a component of X - G. A similar result occurs if b is a cut point.

At this point we have that for each cut point α in $E(a,b) \cup \{a,b\}$ every component of $X - \{\alpha\}$, distinct from both $S(a,\alpha)$ and $S(b,\alpha)$, is a component of X - G. Also for each cyclic element C of X in G every component S of X - C, distinct from both $S(a,C)$ and $S(b,C)$, is a component of X - G. In the remaining part of the proof we shall show that every point of X - G is in one of the above types of components. We will then have that G is an A-set of X since each component of X - G will have the necessary properties.

Let $x \in X - G$. If there is a cut point α in $E(a,b) \cup \{a,b\}$ such that $S(x,\alpha)$ is distinct from both $S(a,\alpha)$ and $S(b,\alpha)$, then $S(x,G)$ is as required. If there is a true cyclic element C of X in G such that $S(x,C)$ is distinct from $S(a,C)$ and $S(b,C)$, we have the same result. So assume that for each $\alpha \in E(a,b)$ we have $x \in S(a,\alpha)$ or $x \in S(b,\alpha)$. Let $H = \{ \alpha \in E(a,b) : x \in S(b,\alpha) \}$ and $K = \{ \alpha \in E(a,b) : x \in S(a,\alpha) \}$. Then $E(a,b) = H \cup K$, and if $\alpha \in H$ and $\beta \in K$, then $\alpha < \beta$; for if $\beta < \alpha$, then $S(a,\beta) \cap S(b,\alpha) = \emptyset$ but $x \in S(a,\beta) \cap S(b,\alpha)$. If H has a largest element α and K has a first element β, then $\alpha \; O \; \beta$. So in this case let C be the cyclic element containing α and β which is contained in G. Since $x \notin S(a,\alpha) = S(a,C)$ and $x \notin S(b,\beta) = S(b,C)$, $S(x,C)$ is distinct from both $S(a,C)$ and $S(b,C)$ and we have the result. If K has no first element let y be the unique limit of $(K,<)$. If $\alpha \in K$, then $\{ \beta \in K : \beta \leq \alpha \} \subset S(a,\alpha)$. If $a \notin S(y,\alpha)$, then either $S(y,\alpha)$ or $X - S(a,\alpha)$ is a neighborhood of y which cannot contain any $\beta \in K$ such that $\beta < \alpha$. But this contradicts the fact that y is a limit of $(K,<)$. Therefore $a \in S(y,\alpha)$ for all $\alpha \in K$. For each $\alpha \in K$, $S(y,\alpha) \cup \{\alpha\}$ is an A-set of X which contains both a and y. Thus $A = \bigcap\{ S(y,\alpha) \cup \{\alpha\} : \alpha \in K \}$ is an A-set of X containing a and y. By 3.18, $S(b,A) = \bigcup\{ S(b,\alpha) : \alpha \in K \}$, which by 1.14, is open and $y = b(S(b,A))$. If $y = a$, then a is a cut point. But $x \in S(b,a) = \bigcup\{ S(b,\alpha) : \alpha \in K \}$, thus $x \in S(b,\alpha)$ for some $\alpha \in K$, which is impossible. If $a \neq y$, then $y \in E(a,b)$ and $y \notin K$ since K has no first element. But then $y \in H$ which implies that $x \in S(b,y) = \bigcup\{ S(b,\alpha) : \alpha \in K \}$, which leads to the same contradiction. A similar argument will lead to the same contradiction if H has no largest element. Thus in each case, $S(x,G)$ must be a component of the complement of a cut point in $E(a,b) \cup \{a,b\}$ or of a true cyclic element of X in G, and we have that G is an A-set of X.

4.3) Theorem: Unicoherence is a cyclicly extensible property.

Proof: Assume that each cyclic element in X is unicoherent and let $X = A \cup B$, where A and B are closed, connected subsets of X. Then for each true cyclic element C of X, $C = (C \cap A) \cup (C \cap B)$ where $C \cap A$ and $C \cap B$ are both closed, connected subsets of C. Thus $(C \cap A) \cap (C \cap B) = C \cap A \cap B$ is connected. Let a and b be distinct elements in $A \cap B$. If a O b, then a and b are elements of a true cyclic element C of X and $C \cap A \cap B$ is a connected subset of $A \cap B$ containing a and b. If α is a cut point of X which cuts between a and b, then α is necessarily in A and in B, so that $E(a,b) \subset A \cap B$. Assume that $E(a,b) \neq \emptyset$ and let $G = G(a,b)$ and let $G_1 = G \cap A \cap B =$ $\{a,b\} \cup E(a,b) \cup (\cup\{ C \cap A \cap B : C \text{ is a true cyclic element in } G \})$.

Assume that $G_1 = H_1 \cup K_1$ separated. If C is a true cyclic element in G, then $C_1 = C \cap A \cap B$ is connected and hence $C_1 \subset H_1$ or $C_1 \subset K_1$. Let $R = (\{a,b\} \cup E(a,b)) \cap H_1$, $S = (\{a,b\} \cup E(a,b)) \cap K_1$, $H = R \cup (\cup\{ C : C_1 \subset H_1 \})$ and $K = S \cup (\cup\{ C : C_1 \subset K_1 \})$. By the definition of H and K we have immediately that $G = H \cup K$, where both H and K are non-empty since $H_1 \subset H$ and $K_1 \subset K$.

If $x \in S$, then there is a connected neighborhood W of X that does not meet H_1. Thus $W \cap R \neq \emptyset$. Let C be a true cyclic element of X with $C_1 \subset H_1$; then $x \notin C$. If $W \cap C \neq \emptyset$, then $W \cap C \cap (E(a,b)) \neq \emptyset$ which implies that $W \cap R \neq \emptyset$, which is impossible. Thus $x \notin \overline{H}$. If $x \in C$, where C is a true cyclic element of X in K, let $\{\alpha,\beta\} = C \cap (\{a,b\} \cup E(a,b))$. If $x = \alpha$ or $x = \beta$, then $x \in S$, so by the above, $x \notin \overline{H}$. Hence we may suppose that $\alpha \neq x \neq \beta$. If $\alpha = a$, then either $S(a,\beta)$ or $X - S(b,\beta)$ is a neighborhood of x which does not meet $G - C$ and thus does not meet H. Similarly if $\beta = b$, there is a neighborhood of x which does not meet H. If $\alpha, \beta \in E(a,b)$ with $\alpha < \beta$, then let $U = S(b,\alpha)$ if α is a closed cut point of X or $U = X - S(a,\alpha)$ if α is an open cut point. Similarly define V to be $S(a,\beta)$ or $X - S(b,\beta)$ depending on whether β is a closed or open cut point respectively. Then $U \cap V$ is a neighborhood of x and $U \cap V \cap G \subset C$. Therefore again $x \notin \overline{H}$. Thus, in any case, $\overline{H} \cap K = \emptyset$. Similarly $H \cap \overline{K} = \emptyset$, and thus $G = H \cup K$ separated, which is impossible. Therefore G_1 is connected.

Hence given $a \in A \cap B$, we have that $A \cap B = \cup\{ G_1(a,b) : b \in A \cap B \}$ and thus $A \cap B$ is connected.

Another of the classical cyclicly extensible and reducible properties is the fixed point property as was first proved by Borsuk. The question of reducibility is easily answered in view of the fact that the fixed point property is invariant under retraction in general spaces and that each cyclic element in X is a retract of X as proved in 3.11.

Let f be a continuous map from X into X and let A be an A-set of X. A is said to have <u>property-A</u> <u>relative</u> <u>to</u> <u>f</u> if for each X in A such that $x = b(S)$ for some component S of X - A, $f(x) \notin S$. Generally we shall just use the term property-A unless there are several mappings in question. The idea for the definition of property-A is supplied by a similar unnamed property used by Whyburn, see page 241 of [8].

4.4) Theorem: If X is compact and if f is a continuous map from X into X, then there is a cyclic element in X with property-A.

Proof: Let $\mathcal{U}$ be the family of all A-sets in X with property-A. Since $X \in \mathcal{U}$ we know that $\mathcal{U}$ is not empty. Let F be a nested family in $\mathcal{U}$, then, by 3.20, $A = \cap F$ is non-empty and thus, by 3.23, A is an A-set of X. If S is a closed component of X - A and $a = b(S)$, then, by 3.17, S is a component of $X - A_1$ for some $A_1 \in F$ and thus $f(a) \notin S$. If S is an open component of $X - A_1$ such that a O s for some $s \in S$, then as in 3.18 (first paragraph of the proof), S is again a component of $X - A_1$ for some $A_1 \in F$ and again $f(a) \notin S$. Now assume that S is open and a is not conjugate to any point $s \in S$. Then by 3.18, $S = \cup_i S(s,A_i)$ where s is a fixed element of S and each A_i is an element of F. If $B = \{ \alpha_i : \alpha_i = b(S(s,A_i)), A_i \in F \}$, then $(B,<)$ is a subnet of $(E(a,s),<)$ with no first element and a is the only limit of $(B,<)$. If $f(a) \in S$, then again as in the proof of 3.18,

$f(a) \in S(s,A_1)^\circ$ for some $A_1 \in F$. Thus there is a neighborhood W of a such that $f[W] \subset S(s,A_1)^\circ$. Since W is a neighborhood of a there is an $\alpha_i \in B \cap W$ with $\alpha_i < \alpha_1$, thus $f(\alpha_i) \in S(s,A_1)$. However $\alpha_i < \alpha_1$ implies that $S(s,A_1) \subset S(s,A_i)$ and thus $f(\alpha_i) \in S(s,A_1)$ which is impossible, since A_i has property-A. Therefore $f(a) \notin S$ and $A \in \mathcal{U}$. Thus by Zorn's Lemma the family $\mathcal{U}$ of A-sets of X has a minimal element C.

If C is a singleton, then C is a cut point or an end point, and we have the result. So let a and b be two distinct elements in C and assume that there is a cut point α which cuts between a and b. Since C is connected α must be in C. If $f(\alpha) = \alpha$, then $\{\alpha\}$ is an A-set of X which is properly contained in C and which has property-A. But this is impossible since C is minimal. Hence we may let $S = S(f(\alpha),\alpha)$. Let $D = C \cap (S \cup \{\alpha\})$. Then D is an A-set and any component S_1 of $X - D$ is either a component of $X - C$ or of $X - \{\alpha\}$ distinct from S and thus $f(b(S_1)) \notin S_1$. Therefore D is an A-set with property-A which is properly contained in C. But this is impossible. Thus no point α cuts between a and b. But then C is a 0-coherent A-set of X, hence it is a cyclic element.

4.5) Corollary: If the cyclic element C is a singleton, f has a fixed point.

Proof: Follows immediately from the fact that if α is a cut point, then $\alpha = b(S)$ for each component S of $X - \{\alpha\}$.

Our terminology has been so arranged that the proofs of the next four theorems are exactly the same as in the classical theory and thus need not be included here. See [8] page 242.

4.6) Theorem: If f is a continuous map from X into X with no fixed point and A is an A-set of X with property-A relative to f, and g is the retraction of X onto A then $g \circ f|_A$ is a continuous map from A into A with no fixed points.

4.7) Theorem: Let X be compact, and such that every N-set of X is a cyclic element. Then the fixed point property is cyclicly extensible.

4.8) Theorem: Let X be compact. If f is a continuous map from X into X and C is a cyclic element of X such that f[C] is nondegenerate and C has property-A relative to f and no end point or cut point is fixed under f, then $C \cap f[C]$ is nondegenerate.

4.9) Theorem: Let X be compact. If h is a homeomorphism of X onto X, then there is a cyclic element C in X such that h[C] = C.

For the following we remove all conditions from the space X. We do not even need to require that the space is connected or locally connected.

4.10) Theorem: If x and y are distinct points of X such that {x} is closed, $x \in \overline{\{y\}}$ and f(y) = x, where f is a continuous map from

X into X, then $f(x) = x$.

Proof: $f^{-1}(x)$ is a closed set containing y and thus $x \in \overline{\{y\}} \subset f^{-1}(x)$.

4.11) Theorem: If a and y are distinct points of X such that $\{x\}$ is open, $y \in \overline{\{x\}}$ and $f(y) = x$, where f is a continuous map from X into X, then $f(x) = x$.

Proof: $f^{-1}(x)$ is a neighborhood of y and thus $x \in f^{-1}(x)$.

4.12) Theorem: Let x and y be distinct points of X such that $\{x\}$ is closed and let f be a continuous map from X into X. If $f(y) \notin \overline{\{x\}}$, then $f(x) \notin \overline{\{x\}}$.

Proof: $f^{-1}[X - \overline{\{x\}}]$ is a neighborhood of y and thus $x \in f^{-1}[X - \{x\}]$.

Bibliography

[1] G. E. Albert and J. W. T. Youngs, The Structure of Locally Connected Topological Spaces, Trans. Amer. Maty. Soc., 51 (1942), 637 - 654.

[2] C. Kuratowski, Quelques applications d'éléments cycliques de M. Whyburn, Fund. Math., 14 (1929), 138 - 144.

[3] _____ and G. T. Whyburn, Sur les éléments cycliques et leurs applications, Fund. Math., 16 (1930), 305 - 331.

[4] B. L. McAllister, Cyclic Elements in Topology, A History, Amer. Math. Monthly, 73 (1966), 337 - 350.

[5] _____, A Note On Irreducible Separation, Fund. Math., 63 (1968), 143 - 144.

[6] R. L. Moore, Concerning Upper Semi-continuous Collections, Monatsh. fur Math. und Phys., 36 (1929), 81 - 88.

[7] T. Rado and P. Reichelder, Cyclic Transitivity, Duke Math. J., 6 (1940), 474 - 485.

[8] G. T. Whyburn, Analytic Topology, Amer. Math. Soc. Colloq. Pub., 28 (1942).

[9] J. W. T. Youngs, Arc Spaces, Duke Math. J., 7 (1940), 68 - 84.