



A development of the Navier-Stokes equations for a real fluid from a variational principle  
by George F Dickerson

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree  
of Master of Science in Physics at Montana State College

Montana State University

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Abstract:

We are concerned in this study with the formulation of a Lagrangian which will yield the correct field equations for a viscous fluid. The determination of the Lagrangian is an important operation, for it is the basis for future quantum development of the system. However, this transition to quantum theory is beyond the scope of our present work.

By separating our velocity component into two constituent parts and postulating that the resultant flow will be such that the difference in dissipation of energy between these two constituent systems will be a minimum, we find that it is possible to obtain the equations of motion for a real fluid from a variational principle.

A DEVELOPMENT OF THE NAVIER-STOKES EQUATIONS FOR  
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GEORGE F. DICKERSON

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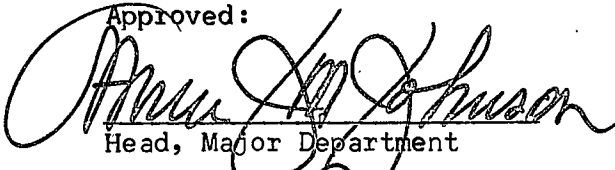
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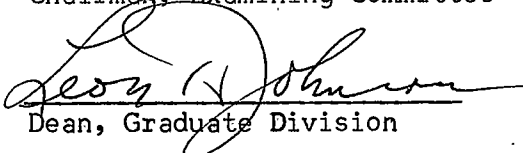
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Approved:

  
Head, Major Department

  
Chairman, Examining Committee

  
Dean, Graduate Division

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ABSTRACT

We are concerned in this study with the formulation of a Lagrangian which will yield the correct field equations for a viscous fluid. The determination of the Lagrangian is an important operation, for it is the basis for future quantum development of the system. However, this transition to quantum theory is beyond the scope of our present work.

By separating our velocity component into two constituent parts and postulating that the resultant flow will be such that the difference in dissipation of energy between these two constituent systems will be a minimum, we find that it is possible to obtain the equations of motion for a real fluid from a variational principle.

## INTRODUCTION

The variational principle supplies one of the most powerful methods of attacking dynamical problems. The formulations developed from the variational principle can be extended beyond classical dynamics and be applied to systems that are not normally considered in dynamics such as the electromagnetic field and field due to elementary particles. In addition these methods of classical dynamics supply the basis for our present-day operations in modern physics.

Experiments performed early in this century showed the need for quantization of electromagnetic radiation. The methods of quantization, however, were first developed for particle mechanics starting essentially from the Lagrangian formulation of classical mechanics. The transition from classical to quantum theory must be stated in terms of the canonical variables describing a system, for it is from the classical Poisson brackets that the axiomatic quantum brackets are constructed. Thus if it is desired to construct a quantum theory of any field, we must first obtain a description of the field in the language of mechanics. The Lagrangian and Hamiltonian formulations form the basis for such a description.

This study is concerned with the application of the variational principle to a viscous fluid in order to obtain a Lagrangian formulation such that this formulation will yield the correct field equations. We are not concerned in this paper with the quantization

of the viscous fluid, but merely in the intermediate formulations that must proceed such a quantization attempt.

To understand the development of the Lagrangian formulation we must examine the variational principle in detail and determine the manner in which the equations of motion are evolved from it.

We first consider the development of Lagrange's equations of motion for a discrete mechanical system and then determine in what manner these equations have been altered as the discrete system approaches the limiting continuous system. By these means are developed the well known Euler-Lagrange equations of motion.

In the consideration of dissipative systems we shall introduce a method which enables us to carry on calculations for a dissipative system as though it were a conservative system. This method is, of course, the well known mirror-image method. However, we find that no such subterfuge is necessary to apply the variational principle to a viscous fluid. It is shown that it is possible to develop the Navier-Stokes equation from a variational principle by dividing our velocity into two elements and postulating that the difference of the rate of dissipation of energy between these two element systems is a minimum for the resultant flow. As classical hydrodynamics is a well established art, we follow well known classic treatments in some introductory developments of this thesis .

### THE VARIATIONAL PRINCIPLE

Hamilton's principle is often regarded as the most fundamental principle for the whole of physics and an important aim is to express it in correct form and to develop from it, by variational methods, the Lagrange equations of motion.

It is necessary when applying Hamilton's principle to a system to distinguish whether the system is conservative or nonconservative. A system is said to be conservative when the external force  $\bar{F}$  is such that the work done around a closed path is zero, i.e.

$$\oint \bar{F} \cdot d\bar{s} = 0.$$

Physically it is clear that a system can not be conservative if friction or other dissipative forces are present, for  $\bar{F} \cdot d\bar{s}$  is then always positive and the integral can not vanish. By Stokes' Theorem, the condition for a conservative system becomes

$$\nabla \times \bar{F} = 0,$$

and since the curl of a gradient always vanishes  $\bar{F}$  will in general be the gradient of some scalar:

$$\bar{F} = -\nabla V.$$

If the external forces acting on a system are conservative the total energy of the system is conserved.

We consider first the application of Hamilton's principle to

conservative systems and then the extension to nonconservative systems.

Lagrange's Equations.

Hamilton's principle for conservative systems is:

The motion of the system from time  $t_1$  to time  $t_2$  is such that the line integral

$$I = \int_{t_1}^{t_2} L dt.$$

where  $L = T - V$ , is an extremum for the path of motion.

In the Hamiltonian method the actual motion of the system between times  $t_1$  and  $t_2$  is compared with a slightly varied motion. Consider the instantaneous configuration of a system described by the generalized coordinates  $q_1 \dots q_n$  so that this configuration corresponds to a point in space where the  $q$ 's form the  $n$  coordinate axes. As time goes on the state of the system changes, and the system point moves tracing out a curve in configuration space. By motion of the system is meant the motion of the system point tracing some path in configuration space. Each point on this path represents the entire system configuration at some time  $t$ . Hence time can be considered as a parameter of the curve.

We observe that the system point will trace a path in configuration space such that the integral  $I$  is an extremum, i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) dt = 0.$$



Applying the variation to the integral gives,

$$\int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right\} dt = 0.$$

Integrating the second term by parts, we obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i dt = \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt$$

and as the variation of  $q_i$  vanishes at  $t_1$  and  $t_2$ , the integral has the form

$$\int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right\} \delta q_i dt = 0.$$

The  $\delta q$ 's are all independent, which means the coefficient of each must be identically zero, i.e.,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, n.$$

These  $n$  equations are Lagrange's equations of motion for a conservative system.

### Canonical Equations.

The Lagrangian formulation can be considered as a description of mechanics in terms of the generalized coordinates and velocities, with time as a parameter. It is often convenient to obtain a formulation

in which the independent variables are the generalized coordinates and the canonically conjugate momenta  $P_i$  defined as:

$$P_i = \frac{\partial L}{\partial \dot{q}_i} .$$

The change in basis from the  $(q, \dot{q}, t)$  set to the  $(q, p, t)$  set can be accomplished by making use of the Legendre transformation.

Consider a function of only two variables  $f(x, y)$ , so that the differential of  $f$  has the form

$$df = u dx + v dy ,$$

where

$$u = \frac{\partial f}{\partial x} , \quad v = \frac{\partial f}{\partial y} .$$

It is wished now to change from  $x, y$  to the independent variables  $u, y$  so that the differential quantities are expressed in terms of the differentials  $du$  and  $dy$ . Let  $g$  be a function of  $u$  and  $y$  defined by the equation

$$g = f - ux .$$

A differential of  $g$  is then given as

$$dg = df - u dx - x du ,$$

or

$$dg = v dx - x du,$$

which is in the form desired. The quantities  $x$  and  $v$  are now functions of the variables  $u$  and  $y$  given by the relations

$$x = - \frac{\partial g}{\partial u}, \quad v = \frac{\partial g}{\partial y}.$$

The transformation from  $(q, \dot{q}, t)$  to  $(q, p, t)$  differs from the type considered above only in that more than one variable is to be transformed.

In place of the Lagrangian we deal with a function defined by the equation

$$H(q, p, t) = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t) \quad (1)$$

where  $H$  is known as the Hamiltonian. The differential of  $H$  is given by

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \quad (2)$$

but from Eq. (1) the differential may also be written as

$$dH = \sum_i \dot{q}_i dp_i + \sum_i p_i dq_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \quad (3)$$

The terms in  $d\dot{q}_i$  in Eq. (3) cancel in consequence of the definition of the generalized momentum:

$$\sum_i p_i d\dot{q}_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i = 0$$

and from Lagrange's equations it follows that

$$\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i.$$

Eq. (3) therefore reduces to the form

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt. \quad (4)$$

Comparison with Eq. (2) furnishes the following relations:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned} \quad (5)$$

Eqs. (5) are known as the canonical equation of Hamilton. They constitute a set of  $2n$  equations of motion replacing Lagrange's equations.

#### Transition to a Continuous System.

The Lagrangian and Hamiltonian formulations thus far discussed have been devised for treating systems with a finite number of degrees

of freedom. The problem of fluid motion, however, involves a continuous system. Here the complete motion can only be described by specifying the coordinates of all points. By coordinates are meant those generalized independent coordinates whose number is equal to the number of degrees of freedom of a system.

The previous formulations are readily modified for application to a continuous system. These modifications are best accomplished by approximating a continuous system by one containing a discrete number of particles, and then examining the change in the equations describing the motion as the continuous limit is approached. This procedure is easily applied to an infinitely long elastic rod which can undergo small longitudinal vibrations<sup>1</sup>, (i.e., vibrations parallel to the axis of the rod). As an illustration we consider such a rod.

We consider a continuous elastic rod as an infinite chain of equal mass points spaced a distance  $a$  apart and connected by massless springs having force constants  $K$ . The kinetic energy of the  $i^{\text{th}}$  particle when displaced from its equilibrium position by a

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<sup>1</sup> Herbert Goldstein, Classical Mechanics (Reading: Addison-Wesley Publishing Company, Inc., 1957), p. 347.

distance  $\mathcal{N}_i$  is:

$$T = \frac{1}{2} m \dot{\mathcal{N}}_i^2 .$$

The potential energy of the  $i^{\text{th}}$  particle can be expressed as

$$V = \frac{1}{2} K (\mathcal{N}_{i+1} - \mathcal{N}_i)^2 .$$

Therefore the Lagrangian for the system is

$$L = T - V = \frac{1}{2} \sum_i \left( m \dot{\mathcal{N}}_i^2 - K \{ \mathcal{N}_{i+1} - \mathcal{N}_i \}^2 \right) ,$$

which can be written as

$$L = \frac{1}{2} \sum_i a \left[ \frac{m}{a} \dot{\mathcal{N}}_i^2 - Ka \left( \frac{\mathcal{N}_{i+1} - \mathcal{N}_i}{a} \right)^2 \right] = \sum_i a L_i .$$

The resulting equations of motion for the coordinates  $\mathcal{N}_i$  are

$$\frac{m}{a} \ddot{\mathcal{N}}_i - Ka \left( \frac{\mathcal{N}_{i+1} - \mathcal{N}_i}{a^2} \right) + Ka \left( \frac{\mathcal{N}_i - \mathcal{N}_{i-1}}{a^2} \right) = 0 .$$

It is apparent that  $m/a$  reduces to  $\mu$ , the mass per unit length of the continuous system, and though it is not so apparent it can be shown that  $Ka$  becomes equivalent to Young's modulus,  $Y$ , as  $a$  approaches zero. The integer  $i$  identifying a particular mass point becomes the

continuous position coordinate  $x$  when going from the discrete to the continuous case. The displacement  $\eta_i$  then becomes  $\eta(x)$ . Hence the quantity

$$\frac{\eta_{i+1} - \eta_i}{a} = \frac{\eta(x+a) - \eta(x)}{a}$$

approaches the limit

$$\frac{d\eta}{dx}$$

as  $a$  approaches zero. Further, in the previous expression for the Lagrangian the summation over a discrete number of particles becomes an integral over  $x$  and the Lagrangian can be written as

$$L = \frac{1}{2} \int \left\{ \mu \dot{\eta}^2 - Y \left( \frac{d\eta}{dx} \right)^2 \right\} dx.$$

Consider the last two terms of the equations of motion, i.e.,

$$-Y \left( \frac{\eta_{i+1} - \eta_i}{a^2} \right) + Y \left( \frac{\eta_i - \eta_{i-1}}{a^2} \right).$$

In the limit as  $a$  goes to zero, these two terms become

$$- \frac{Y}{a} \left[ \left( \frac{d\eta}{dX} \right)_x - \left( \frac{d\eta}{dX} \right)_{x-a} \right],$$

which defines a second derivative of  $\eta$ . Therefore the equation of motion for the continuous elastic rod is

$$\mu \frac{d^2\eta}{dt^2} - Y \frac{d^2\eta}{dX^2} = 0.$$

The most important fact to grasp is the role played by the position coordinate  $x$ . It is not a generalized coordinate. Just as each value of  $i$  corresponded to a different one of the generalized coordinates,  $\eta_i$ , so here for each value of  $x$  there is a generalized coordinate  $\eta(x)$ .  $\eta$  depends also on the continuous variable  $t$  and would be more correctly written as  $\eta(x, t)$ . If the continuous system were three-dimensional, the generalized coordinates would have three continuous indices  $x, y, z$  and would be written as  $\eta(x, y, z, t)$ . The Lagrangian for a continuous system will have the form

$$L = \iiint \mathcal{L} dx dy dz,$$

where  $\mathcal{L}$  is known as the Lagrangian density. In the limit as  $a$  approaches zero the Lagrangian, for the longitudinal vibrations of



the continuous rod, becomes

$$L = \int \frac{1}{2} \left\{ \mu \left( \frac{\partial \pi}{\partial t} \right)^2 - Y \left( \frac{\partial \pi}{\partial x} \right)^2 \right\} dx .$$

Hence the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left\{ \mu \left( \frac{\partial \pi}{\partial t} \right)^2 - Y \left( \frac{\partial \pi}{\partial x} \right)^2 \right\} .$$

It is from the Lagrangian density in the continuous case that the equations of motion will be obtained directly rather than from the Lagrangian itself.

Lagrangian Formulation for a Continuous System.

It will be noted from the Lagrangian density for the continuous elastic rod that  $\mathcal{L}$ , besides being a function of  $\pi$ , is also a function of the spatial derivatives of  $\pi$ . Thus  $x$  and  $t$  both play a role as parameters of the Lagrangian density.<sup>2</sup> In any three-dimensional system the Lagrangian density will appear as a function of the form

$$\mathcal{L} = \mathcal{L} \left( \frac{\partial \pi}{\partial x}, \frac{\partial \pi}{\partial y}, \frac{\partial \pi}{\partial z}, \pi, t \right) .$$

---

<sup>2</sup> In classical field theory the field variables are functions of position and time and are regarded as operators on which are imposed commutation rules. The position and time coordinates are not operators, but are known as c-numbers.

The form of the Lagrangian density has been discussed assuming that each point of the system can suffer only one type of displacement, indicated by  $\mathcal{N}$ . In a more complicated problem in three-dimensional space a particle will undergo displacements along all three of the coordinate axes. In such case there will be three types of generalized coordinates, which will be denoted by the integer index  $j$ :  $\mathcal{N}_j(x, y, z, t)$ . The Lagrangian density will now be a function of the form

$$\mathcal{L} = \mathcal{L}(\mathcal{N}_1, \nabla \mathcal{N}_1, \dot{\mathcal{N}}_1, \mathcal{N}_2, \nabla \mathcal{N}_2, \dot{\mathcal{N}}_2, \mathcal{N}_3, \nabla \mathcal{N}_3, \dot{\mathcal{N}}_3, t).$$

This reasoning, of course, could readily be extended to an n-dimensional manifold.

As before the equations of motion must come from Hamilton's principle which now has the form

$$\delta I = \delta \int_{t_1}^{t_2} \int_{\tau} \mathcal{L} d\tau dt,$$

where  $d\tau$  is the volume element  $dx dy dz$ . The nature of the variation is essentially the same as that used previously. There can be no variation of the parameters  $x, y, z$  and  $t$  as the virtual displacements constructing the varied paths are for constant  $x, y, z$  and  $t$ . Just as the variation of  $\mathcal{N}_j$  is taken to be zero at the end points  $t_1$  and  $t_2$ , so the variation of  $\mathcal{N}_j$  on the surface of the volume of integration is also taken to be zero. The variation of  $\mathcal{L}$  can be written as

$$\delta L = \frac{\partial L}{\partial n_y} \delta n_y + \frac{\partial L}{\partial \dot{n}_y} \delta \dot{n}_y + \sum_{k=1}^3 \frac{\partial L}{\partial \left( \frac{\partial n_k}{\partial x_k} \right)} \delta \left( \frac{\partial n_k}{\partial x_k} \right),$$

where  $x, y$  and  $z$  have been replaced by  $x_1, x_2$  and  $x_3$ , and the double indices indicate summation. Therefore Hamilton's principle becomes

$$\int_{t_1}^{t_2} \int_{\tau} \left[ \frac{\partial L}{\partial n_y} \delta n_y + \frac{\partial L}{\partial \dot{n}_y} \delta \dot{n}_y + \sum_{k=1}^3 \frac{\partial L}{\partial \left( \frac{\partial n_k}{\partial x_k} \right)} \delta \left( \frac{\partial n_k}{\partial x_k} \right) \right] d\tau dt = 0.$$

Integrating by parts as before and recalling that  $\delta n_y$  vanishes at the extremities of the integration, Hamilton's principle takes the form

$$\int_{t_1}^{t_2} \int_{\tau} \left[ \frac{\partial L}{\partial n_y} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{n}_y} \right) - \sum_{k=1}^3 \frac{\partial}{\partial x_k} \left( \frac{\partial L}{\partial \left( \frac{\partial n_k}{\partial x_k} \right)} \right) \right] \delta n_y d\tau dt = 0.$$

The integral can vanish identically only if the coefficients of the independent variations  $\delta n_y$  vanish, yielding the equations of motion:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{n}_y} \right) + \sum_{k=1}^3 \frac{\partial}{\partial x_k} \left( \frac{\partial L}{\partial \left( \frac{\partial n_k}{\partial x_k} \right)} \right) - \frac{\partial L}{\partial n_y} = 0.$$

These equations are generally known as Euler's equations and shall henceforth be referred to by that title to avoid confusing them with Lagrange's equations for a discrete case.

The notation may be simplified by introducing a quantity known as a functional derivative. The functional derivative of the Lagrangian  $L$  with respect to  $\eta_f$  is defined as

$$\frac{\delta L}{\delta \eta_f} = \frac{\partial L}{\partial \eta_f} - \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k} \frac{\partial L}{\partial \left( \frac{\partial \eta_f}{\partial x_k} \right)}$$

There is a similar definition for the functional derivative of  $L$  with respect to  $\dot{\eta}_f$ , but  $L$  is not a function of the gradient of  $\eta_f$ , so it is simply

$$\frac{\delta L}{\delta \dot{\eta}_f} = \frac{\partial L}{\partial \dot{\eta}_f}$$

The functional derivative notation enables us to write Euler's equations in a form greatly resembling the ordinary Lagrange equations in appearance, i.e.,

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\eta}_f} - \frac{\delta L}{\delta \eta_f} = 0.$$

Canonical Equations for a Continuous System.

It is possible to obtain a Hamiltonian formulation for continuous systems analogous to that for discrete systems. Consider again the elastic rod for which the transition from a discrete to a continuous case was made. Conjugate to each generalized coordinate  $\eta_i$  is a canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{\eta}_i} = a \frac{\partial L_i}{\partial \dot{\eta}_i},$$

where  $a$  was the distance separating each mass point in the discrete case. Recalling that  $L$  was previously defined as  $\sum_i a L_i$ , the Hamiltonian for the system is

$$H = \sum_i p_i \dot{\eta}_i - L = \sum_i a \frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - L,$$

or

$$H = \sum_i a \left( \frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - L_i \right).$$

In the limit of the continuous rod, as  $a$  approaches zero,  $L_i$  will approach  $\mathcal{L}$  and the summation in the Hamiltonian becomes an integral:

$$H = \int \left( \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \dot{\eta} - \mathcal{L} \right) dx. \tag{6}$$

A new momentum density,  $\pi$ , is defined by

$$\pi = \frac{\partial L}{\partial \dot{\eta}}$$

Eq. (6) is a space integral over a Hamiltonian density,  $\mathcal{H}$ , defined by

$$\mathcal{H} = \pi \dot{\eta} - L.$$

In any three-dimensional system the total Hamiltonian will be

$$H = \int_{\tau} \mathcal{H} d\tau = \int_{\tau} \left( \sum_i \pi_i \dot{\eta}_i - L \right) d\tau.$$

The canonical field equations can be obtained from the Hamiltonian density by following the same procedure used in the discrete case.

An infinitesimal change in H can be written

$$dH = \int_{\tau} \left\{ \sum_i \left[ \frac{\partial \mathcal{H}}{\partial \eta_i} d\eta_i + \frac{\partial \mathcal{H}}{\partial \pi_i} d\pi_i + \sum_{\kappa=1}^3 \frac{\partial \mathcal{H}}{\partial \left( \frac{\partial \eta_i}{\partial X_{\kappa}} \right)} d \left( \frac{\partial \eta_i}{\partial X_{\kappa}} \right) + \frac{\partial \mathcal{H}}{\partial t} dt \right] \right\} d\tau.$$

Integrate the integral

$$\int \frac{\partial \mathcal{H}}{\partial \left( \frac{\partial \eta_i}{\partial X_{\kappa}} \right)} d \left( \frac{\partial \eta_i}{\partial X_{\kappa}} \right) dX_{\kappa}.$$

by parts to get the expression,

$$\left. \frac{\partial \mathcal{H}}{\partial \left( \frac{\partial \pi_i}{\partial X_K} \right)} d\pi_i \right]_{X_K} - \int \frac{\partial}{\partial X_K} \frac{\partial \mathcal{H}}{\partial \left( \frac{\partial \pi_i}{\partial X_K} \right)} d\pi_i dX_K.$$

The volume of integration can be made larger than the system thereby causing the integrated term to vanish when the limits of integration are substituted into it. Then dH can be written

$$dH = \int \left\{ \sum_i \left[ \frac{\partial \mathcal{H}}{\partial \pi_i} d\pi_i + \frac{\partial \mathcal{H}}{\partial \dot{\pi}_i} d\dot{\pi}_i - \sum_{K=1}^3 \frac{\partial}{\partial X_K} \left( \frac{\partial \mathcal{H}}{\partial \left( \frac{\partial \pi_i}{\partial X_K} \right)} \right) + \frac{\partial \mathcal{H}}{\partial t} dt \right] \right\} d\tau.$$

By making use of the functional derivative notation this equation can be written simply as

$$dH = \int \left\{ \sum_i \left( \frac{\delta H}{\delta \pi_i} d\pi_i + \frac{\delta H}{\delta \dot{\pi}_i} d\dot{\pi}_i \right) + \frac{\partial \mathcal{H}}{\partial t} dt \right\} d\tau. \quad (7)$$

The differential of H can also be written as

$$dH = \int \left\{ \sum_i \left( \pi_i d\pi_i + \dot{\pi}_i d\dot{\pi}_i - \frac{\delta L}{\delta \pi_i} d\pi_i - \frac{\delta L}{\delta \dot{\pi}_i} d\dot{\pi}_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt \right\} d\tau.$$

But the first and the last terms in the parentheses cancel because of the definition of  $\pi_i$ . Recalling the expression of Euler's equations

in functional derivative notation,

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\pi}_i} - \frac{\delta L}{\delta \pi_i} = 0,$$

we can write

$$\frac{\delta L}{\delta \dot{\pi}_i} = \frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\pi}_i} = \dot{\pi}_i.$$

Therefore the differential of H has the form

$$dH = \int \left\{ \sum_i (-\pi_i d\pi_i + \dot{\pi}_i d\pi_i) - \frac{\partial L}{\partial t} dt \right\} d\tau. \quad (8)$$

Comparison of Eq. (7) with Eq. (8) results in the equations

$$\frac{\delta H}{\delta \pi_i} = -\dot{\pi}_i, \quad \frac{\delta H}{\delta \dot{\pi}_i} = \pi_i.$$

These equations are Hamilton's canonical equations for a continuous system. In terms of the Hamiltonian density they can be written

$$\frac{\delta \mathcal{H}}{\delta \pi_i} - \sum_{\alpha=1}^3 \frac{\partial}{\partial x_\alpha} \left( \frac{\delta \mathcal{H}}{\delta (\frac{\partial \pi_i}{\partial x_\alpha})} \right) = -\dot{\pi}_i, \quad \frac{\delta \mathcal{H}}{\delta \dot{\pi}_i} = \pi_i.$$



Extension of Hamilton's Principle to a Nonconservative System.

Up to the present we have considered the application of Hamilton's principle only to conservative systems. We shall consider now the extension of Hamilton's principle to a nonconservative, discrete system. The extended principle appears as

$$\delta I = \delta \int_{t_1}^{t_2} (T+W) dt = 0,$$

where  $\delta W$  represents the work done by the forces on the system during the virtual displacement from the actual to the varied path. The variation of  $W$  can be written as

$$\sum_i Q_i \delta q_i,$$

where  $Q_i$  is a generalized force acting on the system. Hence Hamilton's principle can be expressed as

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} \sum_i Q_i \delta q_i dt = 0, \quad (9)$$

or

$$\int_{t_1}^{t_2} \left( \frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + Q_i \delta q_i \right) dt = 0.$$

Integrating the second term by parts, Hamilton's principle takes the form

$$\int_{t_1}^{t_2} \left( \frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} + Q_i \right) \delta q_i dt = 0.$$

Again the integral can vanish only if the separate coefficients vanish so that

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i,$$

which represents the equations of motion for a nonconservative system.

Eq. (9) reduces to the ordinary form of Hamilton's principle if the  $Q_i$ 's are derivable from a generalized potential. The integral of the virtual work then becomes

$$\int_{t_1}^{t_2} \sum_i Q_i \delta q_i dt = - \int_{t_1}^{t_2} \sum_i \delta q_i \left( \frac{\partial V}{\partial q_i} - \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_i} \right) dt,$$

but

$$- \int_{t_1}^{t_2} \sum_i \delta q_i \left( \frac{\partial V}{\partial q_i} - \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_i} \right) dt = - \delta \int_{t_1}^{t_2} V dt.$$

The equation,

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} \sum_i Q_i \delta q_i dt = 0,$$

is then reduced to

$$\delta \int_{t_1}^{t_2} T dt - \delta \int_{t_1}^{t_2} V dt = \delta \int_{t_1}^{t_2} (T - V) dt = 0,$$

which is the ordinary form of Hamilton's principle when the generalized forces are derivable from a potential.

Although Hamilton's principle can be applied to a nonconservative system, this formulation is most useful when a Lagrangian can be set up to supply the complete equations of motion of a system.

We shall not attempt to extend the Hamiltonian formulation to a nonconservative system for its primary use to us is that of an energy conservation theorem for conservative systems. This may be readily shown by making use of the definition

$$H = \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i - L.$$

$T$  is a homogeneous quadratic function of the  $q_i$ 's. Euler's theorem states that if  $f$  is a homogeneous function, of order  $n$ , of a set of variables  $q_i$ , then

$$\sum_i q_i \frac{\partial F}{\partial q_i} = n F.$$

Here  $n$  is 2, so that

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

and

$$H = 2T - (T - V) = T + V,$$

which is the total energy of the system.

#### Advantages of the Lagrangian Formulation.

The variational principle gives a powerful tool in solving any given mechanical problem, for in Hamilton's principle is contained all of the mechanics of conservative systems, (i.e., Hamilton's principle replaces Newton's laws of motion as the basic postulate of mechanics for conservative systems.) Such a formulation has certain advantages; for example, the Lagrangian contains only physical quantities, namely the kinetic and potential energy, which can be defined without reference to a particular set of generalized coordinates. The formulation is therefore automatically invariant with respect to the choice of coordinates for the system.

We cannot speak of an invariant formulation without immediately having the question arise of whether the formulation is invariant

under a Lorentz transformation. The procedure for the derivation the Lagrangian density slips very readily into a Lorentz covariant formulation. In the derivation,  $x_k$  and  $t$  are treated in equal fashion as parameters appearing in  $\mathcal{L}$ . The product  $dx_1 dx_2 dx_3 dt$  is essentially an element of volume in world space and hence is invariant under a Lorentz transformation. Hamilton's principle is Lorentz invariant provided  $\mathcal{L}$  is a world scalar for it has the form

$$\delta I = \delta \int_{t_1}^{t_2} \int_{\tau} \mathcal{L} dt d\tau = 0.$$

Hence Euler's equation is relativistically invariant and in covariant notation is simply

$$\sum_{\mu=1}^4 \frac{\partial}{\partial X_{\mu}} \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \mathcal{N}_i}{\partial X_{\mu}} \right)} - \frac{\partial \mathcal{L}}{\partial \mathcal{N}_i} = 0.$$

In addition to the advantage of the invariant properties afforded by the variational principle, this principle can also be extended to describe systems not normally considered in dynamics, such as the elastic field, the electromagnetic field and the properties of elementary particles. This is to be expected since mathematically, a field is a set of independent functions of a point in space and the generalized coordinates fit this description exactly.

The Lagrangian formulation for a continuous set of generalized coordinates was developed to treat a continuous mechanical system.

But it is readily conceivable that, although historically fields arose as the vibrations of some continuous system, the variational principle can stand independent of a continuous mechanical system and will serve to furnish the equations describing any space-time field.

For this purpose the concept of a field must be extended somewhat. Consider, for example, the transverse displacement from equilibrium of a string under static forces. This is a very simple example of a one-dimensional field. The displacement  $y$  is different for different parts of the string, so that  $y$  can be considered as a function of the distance  $x$  along the string.

Since the Lagrangian density for a field is not associated with a definite mechanical system, it will not necessarily be given as the difference of kinetic and potential energy density. It may be any expression which leads to the desired field equations.

## VARIATIONAL PRINCIPLE APPLIED TO IDEAL FLUIDS

The variational principle has been applied to both perfect and viscous fluids with varying degrees of success. The application to perfect fluids yields the usual equations of motion, but the application to a viscous fluid, as yet, has not been completely satisfactory. Either the equations of motion appear in altered form, or it is necessary to make certain restrictions upon the properties of the viscous fluid.

A few methods of application, of the variational principle to fluids, will be outlined in the following material. Of necessity these outlines will be somewhat sketchy. Initially only methods of application suitable to perfect fluids will be considered. Of three methods considered, the first two are applicable only when certain flow restrictions are placed upon the perfect fluid. The third, however, is suitable for application to the general case of the perfect fluid. This third method is also applicable under the more restricted flow conditions. The methods are of interest because they give us some concept of the first applications of Hamilton's principle to hydrodynamics.

### Perfect Fluid, Irrotational and Incompressible.

Perhaps the best known application of the variational principle to a hydrodynamical problem is the consideration of an inviscid

fluid which is both irrotational and incompressible.<sup>3</sup> The velocity of the fluid is derivable from the velocity potential

$$\Phi = \sum_i \varphi_i \dot{q}_i,$$

where the  $\varphi_i$  are functions of the generalized coordinates,  $q_i$ .

Hence the velocity  $\beta$  is of the form,

$$\beta = - \sum_i \frac{\partial \varphi_i}{\partial r} \dot{q}_i,$$

where  $r(q_1, \dots, q_n)$  is, as usual, a position vector. From the above equation it is readily apparent that

$$\frac{\partial \beta}{\partial \dot{q}_i} = - \frac{\partial \varphi_i}{\partial r}.$$

The velocity of the fluid can also be given as

$$\beta = \frac{dr}{dt},$$

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<sup>3</sup> L. M. Milne-Thomson, Theoretical Hydrodynamics (New York: The Macmillan Company, 1955), p. 509-13.



Now the time rate of change of the virtual displacement is

$$\frac{d}{dt}(\delta r),$$

which can be written

$$\delta\left(\frac{dr}{dt}\right),$$

and this gives the relationship,

$$\delta B = \frac{d}{dt}(\delta r),$$

of which we must make use later.

Presently we will also make use of the expression for the kinetic energy of the fluid:

$$T = \frac{1}{2} \int_V \rho B^2 d\tau,$$

which when differentiated partially with respect to  $\dot{q}_i$  gives

$$\frac{\partial T}{\partial \dot{q}_i} = \rho \int_V B \frac{\partial B}{\partial \dot{q}_i} d\tau = -\rho \int_V B \frac{\partial \varphi_i}{\partial r} d\tau.$$

Let  $F$  be the total force, including forces derivable from the pressure, acting on each fluid particle. Then the equation

of motion per unit mass is

$$F = \frac{dB}{dt}$$

and the virtual work of the forces on the liquid is

$$\delta W = \int_{\tau} \rho F \delta r d\tau = - \sum_i \int_{\tau} \rho F \frac{\partial \varphi_i}{\partial r} \delta q_i d\tau = \sum_i Q_i \delta q_i$$

where

$$Q_i = - \int_{\tau} \rho F \frac{\partial \varphi_i}{\partial r} d\tau.$$

Now consider the relationship,

$$\int_{\tau} \rho B \delta r d\tau = - \sum_i \int_{\tau} \rho B \frac{\partial \varphi_i}{\partial r} \delta q_i d\tau = \sum_i \frac{\partial T}{\partial q_i} \delta q_i.$$

Operating on this expression with  $\frac{d}{dt}$  gives

$$\int_{\tau} \rho \frac{dB}{dt} \delta r d\tau + \int_{\tau} \rho B \frac{d}{dt} (\delta r) d\tau = \sum_i \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) \delta \dot{q}_i + \sum_i \frac{\partial T}{\partial q_i} \delta \dot{q}_i.$$

But

$$\int_{\tau} \rho B \frac{d}{dt} (\delta r) d\tau = \delta \int_{\tau} \frac{1}{2} \rho B^2 d\tau = \delta T = \sum_i \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_i \frac{\partial T}{\partial q_i} \delta q_i.$$

Combining the preceding two equations gives

$$\sum_i \left\{ Q_i - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial T}{\partial q_i} \right\} \delta q_i = 0.$$

Since the  $\delta q_i$  are independent, all but one can be put equal to zero, and Lagrange's equations appear, i.e.,

$$Q_i = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i}.$$

No restrictions have been made on the generalized force  $Q_i$ .

If it is derivable from a generalized potential, then

$$Q_i = - \frac{\partial V}{\partial q_i} + \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_i} \right),$$

and Lagrange's equations will have the form:

$$\frac{d}{dt} \left( \frac{\partial (T-V)}{\partial \dot{q}_i} \right) - \frac{\partial (T-V)}{\partial q_i} = 0,$$

which will yield the equations of motion for a perfect fluid which is both irrotational and incompressible.

#### Perfect Fluid, Incompressible.

The equations of motion for an incompressible, perfect fluid can be deduced from Hamilton's principle without assuming that the velocity of the fluid is derivable from a velocity potential<sup>4</sup>

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<sup>4</sup> A. G. Webster, The Dynamics of Particles and of Rigid Elastic, and Fluid Bodies (Leipzig: B. G. Teubner, 1904), p. 499-501.

(i.e., without assuming that the fluid is irrotational).

The kinetic energy of an incompressible, perfect fluid contained in a fixed volume is

$$T = \frac{1}{2} \int_{\tau} \rho \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} d\tau.$$

The potential energy, neglecting external forces, is that due to the pressure. The work done by the pressure on the fluid is

$$\delta W' = - \int_A \{ P(\delta x + \delta y + \delta z) \cdot n \} dA - \int_{\tau} \gamma (\delta x + \delta y + \delta z) d\tau,$$

where  $\gamma$  is a volume-force. Making use of the divergence theorem we can express the surface integral as

$$- \int_{\tau} \left\{ \frac{\partial}{\partial x} (P \delta x) + \frac{\partial}{\partial y} (P \delta y) + \frac{\partial}{\partial z} (P \delta z) \right\} d\tau$$

or

$$\delta W' = \int_{\tau} \left[ P \left\{ \frac{\partial}{\partial x} \delta x + \frac{\partial}{\partial y} \delta y + \frac{\partial}{\partial z} \delta z \right\} + \left\{ \left( \frac{\partial P}{\partial x} + \gamma \right) \delta x + \left( \frac{\partial P}{\partial y} + \gamma \right) \delta y + \left( \frac{\partial P}{\partial z} + \gamma \right) \delta z \right\} \right] d\tau.$$

But since we are dealing with an incompressible fluid the coefficients of  $\delta x, \delta y, \delta z$  must vanish. Hence the potential energy is

$$\delta W' = - \int_{\tau} P \left\{ \frac{\partial}{\partial x} \delta x + \frac{\partial}{\partial y} \delta y + \frac{\partial}{\partial z} \delta z \right\} d\tau,$$

Hamilton's principle is then

$$\int_{t_1}^{t_2} (\delta T - \delta W') dt = 0$$

or

$$\int_{t_1}^{t_2} \delta \int_{\tau} \frac{\rho}{2} \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} d\tau dt - \int_{t_1}^{t_2} \delta W' dt.$$

Performing the variations gives

$$\int_{t_1}^{t_2} \int_{\tau} \left[ \rho \left\{ \frac{dx}{dt} \delta \left( \frac{dx}{dt} \right) + \frac{dy}{dt} \delta \left( \frac{dy}{dt} \right) + \frac{dz}{dt} \delta \left( \frac{dz}{dt} \right) \right\} \right.$$

$$\left. + P \left( \frac{\partial}{\partial x} \delta x + \frac{\partial}{\partial y} \delta y + \frac{\partial}{\partial z} \delta z \right) \right] d\tau dt = 0. \tag{10}$$

Integrating the first three terms by parts with respect to  $t$  gives the expression,

$$\int_{\tau} \left[ \rho \left\{ \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right\} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho \left\{ \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right\} dt d\tau.$$

The integrated terms vanish as usual at the limits of integration.

Now integrating the last three terms of Eq. (10) by parts with respect to the space variables gives

$$\begin{aligned}
 & - \int_A P \{ \delta x \cos nx + \delta y (\cos ny) + \delta z (\cos nz) \} dA \\
 & - \int_{\tau} \left( \frac{\partial P}{\partial x} \delta x + \frac{\partial P}{\partial y} \delta y + \frac{\partial P}{\partial z} \delta z \right) d\tau.
 \end{aligned}$$

We assume that  $\delta x, \delta y, \delta z$  vanish for the particles of the fluid at the bounding surface, hence the surface integral vanishes.

Collecting the terms in  $\delta x, \delta y, \delta z$  gives

$$\begin{aligned}
 & \int_{t_1}^{t_2} \left[ \int_{\tau} \left\{ \left( \rho \frac{d^2 x}{dt^2} + \frac{\partial P}{\partial x} \right) \delta x + \left( \rho \frac{d^2 y}{dt^2} + \frac{\partial P}{\partial y} \right) \delta y \right. \right. \\
 & \left. \left. + \left( \rho \frac{d^2 z}{dt^2} + \frac{\partial P}{\partial z} \right) \delta z \right\} d\tau \right] dt = 0.
 \end{aligned}$$

By the usual reasoning the coefficients of  $\delta x, \delta y, \delta z$  must vanish giving the equations of motion:

$$\rho \frac{d^2 x}{dt^2} + \frac{\partial P}{\partial x} = 0,$$

$$\rho \frac{d^2 y}{dt^2} + \frac{\partial P}{\partial y} = 0,$$

$$\rho \frac{d^2 z}{dt^2} + \frac{\partial P}{\partial z} = 0,$$

which are exactly the equations of motion of a perfect fluid.

Perfect Fluid, General Case.

Hamilton's principle will now be applied to a perfect fluid which is compressible and has both irrotational and solenoidal components. Consider the Lagrangian density<sup>5</sup>

$$\mathcal{L} = \rho \left[ \frac{\partial \varphi}{\partial t} + a \frac{\partial \theta}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) \right] + f(\rho)$$

where

$$u = \frac{\partial \varphi}{\partial x} + a \frac{\partial \theta}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y} + a \frac{\partial \theta}{\partial y}, \quad w = \frac{\partial \varphi}{\partial z} + a \frac{\partial \theta}{\partial z}.$$

The terms  $u, v, w$  are the  $x, y, z$  components of the velocity respectively.

Varying the quantities  $\varphi, \theta, a, \rho$  in such a manner that the variations of  $\varphi$  and  $\theta$  vanish on the boundary of the region of integration, Euler's equations give

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \quad (11)$$

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<sup>5</sup> H. Bateman, Partial Differential Equations (New York: Dover Publications, 1944), pp. 164-5.

$$\frac{d\bar{a}}{dt} = 0, \quad \frac{d\theta}{dt} = 0,$$

$$\frac{\partial\varphi}{\partial t} + \bar{a}\frac{\partial\theta}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + f'(\rho) = 0.$$

If  $P = \rho f'(\rho) - f(\rho)$ , where P is interpreted as the pressure, it is readily seen that

$$P = -\rho \left[ \frac{\partial\varphi}{\partial t} + \bar{a}\frac{\partial\theta}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) \right] - f(\rho).$$

Partial differentiation with respect to X gives

$$\begin{aligned} \frac{\partial P}{\partial X} = & -\rho \left[ \frac{\partial^2\varphi}{\partial t\partial X} + \bar{a}\frac{\partial^2\theta}{\partial t\partial X} + \frac{\partial\bar{a}}{\partial X}\frac{\partial\theta}{\partial t} + u\frac{\partial}{\partial X}\left(\frac{\partial\varphi}{\partial X} + \bar{a}\frac{\partial\theta}{\partial X}\right) \right. \\ & + v\frac{\partial}{\partial X}\left(\frac{\partial\varphi}{\partial Y} + \bar{a}\frac{\partial\theta}{\partial Y}\right) + w\frac{\partial}{\partial X}\left(\frac{\partial\varphi}{\partial Z} + \bar{a}\frac{\partial\theta}{\partial Z}\right) \\ & \left. - \frac{\partial\rho}{\partial X}\left[\frac{\partial\varphi}{\partial t} + \bar{a}\frac{\partial\theta}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + f'(\rho)\right], \right. \end{aligned}$$

which can be written

$$\frac{\partial P}{\partial X} = -\rho \left[ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} + w\frac{\partial u}{\partial Z} + \frac{\partial\bar{a}}{\partial X}\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial X}\frac{\partial\bar{a}}{\partial t} \right],$$



or

$$\frac{\partial P}{\partial x} = -\rho \frac{du}{dt}, \quad (12)$$

since  $\frac{d\theta}{dt}$  and  $\frac{d\alpha}{dt}$  equal zero.

Similarly we get

$$\frac{\partial P}{\partial y} = -\rho \frac{dv}{dt}, \quad \frac{\partial P}{\partial z} = -\rho \frac{dw}{dt}, \quad (13)$$

by partial differentiation with respect to y and z respectively.

Eqs. (12) and (13) are the dynamical equations of motion of a perfect fluid. Notice that Eq. (11) is nothing more than the equation of continuity.

## THE VARIATIONAL PRINCIPLE APPLIED TO DISSIPATIVE SYSTEMS

It is sometimes necessary when applying the variational method to a dissipative system to develop some formalism by which we are able to carry on calculations for a dissipative system as though it were a conservative system. Frequently, however, it is possible to attack a dissipative system directly and, by choosing the components of the Lagrangian judiciously and applying subsidiary boundary conditions, to construct a Lagrangian which will yield the desired field equations. As an example of this direct attack let us consider an attempted application of the variational principle that has been made to a viscous fluid.

### Viscous Fluid, Incompressible with Steady Motion.

One of the first major efforts to apply the variational principle to a viscous fluid was made by C. B. Millikan.<sup>6</sup> The problem that Millikan set up consists of determining if a Lagrangian density exists for an incompressible, viscous fluid with steady motion such that Euler's equations will yield the Navier-Stokes equations. However this Lagrangian density is an energy dissipation function rather than a difference between energies. We shall examine the application of the variational principle to this problem and the conclusions to which Millikan was led.

Assume that Hamilton's principle will have the following form:

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<sup>6</sup> C. B. Millikan, *Phil. Mag.* 7, 641 (1929).

$$\delta \int_{\tau} \left\{ \frac{1}{2} \phi + R - \lambda (\nabla \cdot \beta) \right\} d\tau,$$

where  $\lambda$  is the Lagrangian undetermined multiplier and  $\beta$  is the velocity of the fluid.  $\phi$  is called the dissipation function<sup>7</sup> and for an incompressible fluid is given by

$$\phi = \mu \left\{ 2u_1^2 + 2v_2^2 + 2w_3^2 + (u_2 + v_3)^2 + (u_3 + w_1)^2 + (v_1 + u_2)^2 \right\},$$

where  $\mu$  is the viscosity coefficient, and the subscripts on the velocity components denote differentiation with respect to the space variables  $x, y, z$ . The quantity  $R$  is the density of the time rate at which kinetic energy is carried out of the region being considered.

To evaluate this expression of Hamilton's principle, it is necessary to find the value of  $R$ . To do this we must make use of the expression,

$$\alpha = \int_{\tau} R d\tau,$$

where  $\alpha$  is the total rate at which kinetic energy is carried out of the region of integration. From the preceding definition then  $\alpha$  can be represented, where  $S$  is the bounding surface of the region,  $ds$  is the surface element, and  $\bar{n}$  is an outwardly-drawn unit

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<sup>7</sup> Horace Lamb, Hydrodynamics (Dover Publications, New York, 1924) 6th edition. p. 580.

vector at  $ds$ , by

$$\alpha = \frac{\rho}{2} \int_S \beta^2 \bar{n} \cdot \bar{\beta} ds = \frac{\rho}{2} \int_S \bar{n} \cdot (\beta^2 \bar{\beta}) ds.$$

Applying the divergence theorem,  $\alpha$  becomes

$$\alpha = \frac{\rho}{2} \int_T \nabla \cdot (\beta^2 \bar{\beta}) d\tau = \frac{\rho}{2} \int_T (\beta^2 \nabla \cdot \bar{\beta} + \bar{\beta} \cdot \nabla \beta^2) d\tau,$$

but because of the continuity equation

$$\alpha = \frac{\rho}{2} \int_T (\bar{\beta} \cdot \nabla \beta^2) d\tau$$

or

$$R = \frac{\rho}{2} (\bar{\beta} \cdot \nabla \beta^2).$$

We now have expressions, in terms of the velocity components, for all of the variables entering into the statement of Hamilton's principle. Hence the Lagrangian density, in terms of the velocity components, becomes

$$\begin{aligned} \mathcal{L} = & \mu \left\{ u_1^2 + v_2^2 + w_3^2 + \frac{1}{2} (w_2 + v_3)^2 + \frac{1}{2} (u_3 + w_1)^2 \right. \\ & \left. + \frac{1}{2} (v_1 + u_2)^2 \right\} + \rho \left\{ u^2 u_1 + u v v_1 + u w w_1 \right. \end{aligned}$$

$$+ u v u_2 + v v v_2 + v w u_2 + u w u_3 + v w v_3 + w w u_3 \} - \lambda \{ u_1 + v_2 + w_3 \} .$$

Substituting the expression for the Lagrangian density into Euler's equation

$$\sum_{k=1}^3 \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \pi_i}{\partial x_k} \right)} - \frac{\partial \mathcal{L}}{\partial \pi_i} = 0 ,$$

gives, when  $\pi_i$  equals  $u$  ,

$$u \{ 2u_{11} + u_{22} + v_{12} + u_{33} + w_{13} \} + \rho \{ u v_2 + u w_3 - v v_1 - w w_1 \} - \lambda_1 = 0 .$$

Two analogous equations are given when  $\pi_i$  equals  $v$  and  $w$ .

The above expression determined from Euler's equation when  $\pi_i$  equals  $u$ , can be written

$$u \left\{ \nabla^2 u + \frac{\partial}{\partial x} (\nabla \cdot \vec{B}) \right\} + \rho \left\{ u (\nabla \cdot \vec{B}) - \frac{1}{2} \frac{\partial B^2}{\partial x} \right\} - \frac{\partial \lambda}{\partial x} = 0 ,$$

but introducing the continuity equation reduces it to

$$\mu \nabla^2 u - \frac{\rho}{2} \frac{\partial \beta^2}{\partial x} - \frac{\partial \lambda}{\partial x} = 0.$$

Combining this equation with the two analogous equations for v and w deduced from Euler's equation, we can write the vector equation:

$$\frac{\rho}{2} \nabla \beta^2 + \nabla \lambda = \mu \nabla \cdot \nabla \bar{\beta}.$$

If now P is written in place of  $\lambda$ , the above equation is just the Navier-Stokes equation, except that  $(\bar{\beta} \cdot \nabla) \bar{\beta}$  is replaced by  $1/2 \nabla \beta^2$ . This condition is satisfied only if the fluid is irrotational, for

$$(\bar{\beta} \cdot \nabla) \bar{\beta} = \frac{1}{2} \nabla \beta^2 - \bar{\beta} \times (\nabla \times \bar{\beta}).$$

Therefore it appears that the variational principle does not give all of the desired terms.

We are now led to the consideration of the possibility of obtaining these missing terms from any Lagrangian density involving only the velocity components and their first-order space derivatives. The problem is then to find a function

$$L = L(u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_z, v_z, w_z),$$

such that if it is introduced into Euler's equation, it will supply the missing terms. This is too long a procedure to be presented here, but Millikan followed this procedure to completion and it was found that the conclusion must be adopted, that if  $L$  is restricted

to be a function of the velocity components and their first-order space derivatives only, then it is impossible to find any  $\mathcal{L}$  which will give the general equations for steady flow of an incompressible fluid through the application of the variational principle.

The Mirror-Image System.

Returning again to the concept of introducing some formalism which will enable us to carry on calculations for dissipative systems as though they were conservative systems, we introduce a mirror-image system.<sup>8</sup> This system is introduced by considering simultaneously with a system having friction, a mirror-image system with negative friction. The mirror-image system gains as much energy as the dissipative system loses and thus the total energy is conserved.

For an example consider the one-dimensional oscillator with friction. The equation of motion is

$$m\ddot{x} + R\dot{x} + Kx = 0.$$

We take as the Lagrangian the expression

$$L = m(\dot{x}\dot{x}^*) - \frac{1}{2}R(x^*\dot{x} - x\dot{x}^*) - Kxx^*,$$

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<sup>8</sup> P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill Book Company, Inc., New York, 1953), Part I, p. 298

where the coordinate  $X^*$  represents the mirror-image oscillator with negative friction. Making use of Lagrange's equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

gives, when  $q_i$  equals  $X^*$ ,

$$m\ddot{X} + R\dot{X} + KX = 0,$$

for the initial system and when  $q_i$  equals  $X$

$$m\ddot{X}^* - R\dot{X}^* + KX^* = 0$$

for the mirror-image system. The equation in  $X$  is just the desired equation of motion for the dissipative system. Note the equation in  $X^*$ , for the mirror-image system contains a negative frictional term, as was assumed at the beginning.

To obtain the Hamiltonian, we must first determine the two canonical momenta:

$$P = \frac{\partial L}{\partial \dot{X}} = m\dot{X} - \frac{1}{2}RX^*,$$

and

$$P^* = \frac{\partial L}{\partial \dot{X}^*} = m\dot{X}^* + \frac{1}{2}RX^*.$$



Using these canonical momenta, the Hamiltonian can be written

$$H = P\dot{X} + P^*\dot{X}^* - L = m\dot{X}\dot{X}^* + KXX^*.$$

The Hamiltonian will remain constant for the oscillator with friction as may be seen by taking the time derivative of H, i.e.,

$$\frac{dH}{dt} = m\ddot{X}\dot{X}^* + m\dot{X}\ddot{X}^* + K\dot{X}X^* + KX\dot{X}^*$$

or

$$\frac{dH}{dt} = (-R\dot{X} - KX)\dot{X}^* + (R\dot{X}^* - KX^*)\dot{X} + K\dot{X}X^* + KX\dot{X}^* = 0.$$

$X^*$  increases in amplitude as rapidly as X decreases.

#### Diffusion Equation.

As a further example of the method of application of the variational principle when a mirror-image system is introduced let us consider the diffusion equation,

$$\nabla^2\psi = a^2 \frac{\partial\psi}{\partial t},$$

where  $\psi$  is the density of the diffusing fluid and  $a^2$  is the diffusion constant. Assume a Lagrangian density of the form:<sup>9</sup>

$$\mathcal{L} = -(\nabla\psi \cdot \nabla\psi^*) - \frac{1}{2}a^2\left(\psi^* \frac{\partial\psi}{\partial t} - \psi \frac{\partial\psi^*}{\partial t}\right),$$

---

<sup>9</sup> Morse and Feshbach, Op. cit., p. 313.

where  $\psi^*$  is the density of the fluid in the mirror-image system where the fluid is undiffusing.

Substituting the Lagrangian density into Euler's equation gives the expressions,

$$\nabla^2 \psi = a^2 \frac{\partial \psi}{\partial t}$$

and

$$\nabla^2 \psi^* = -a^2 \frac{\partial \psi^*}{\partial t}.$$

The first equation, for  $\psi$ , is the usual diffusion equation. The second equation, in terms of  $\psi^*$ , is for the mirror-image system.

Perhaps the introduction of the mirror-image system does sacrifice a certain amount of reality, however, the Lagrangian developed by this method does yield the equations of motion when subjected to the proper variational techniques.

### VARIATIONAL PRINCIPLE APPLIED TO A VISCOUS FLUID

In the preceding sections we reviewed the attempts to obtain the Navier-Stokes equations from a variational principle as well as illustrating the well known mirror-image method for analysis of dissipative systems. We will show that it is possible to apply the variational principle to a viscous fluid without recourse to a mirror-image system. Before doing this we wish to make a statement concerning the image system method for handling dissipative systems. Various system-image Lagrangians were employed of the form

$$\frac{1}{2} \rho \left( \frac{\partial u_i^k}{\partial t} + u_j^l \frac{\partial u_i^k}{\partial x_j} \right) u_i^l - \frac{1}{2} \rho \left( \frac{\partial u_i^l}{\partial t} + u_j^k \frac{\partial u_i^l}{\partial x_j} \right) u_i^k$$

*+ dissipation terms*

without success. For awhile it was puzzling as to why the method did not net the expected results as it has been used so extensively for analysis of dissipative systems. However, when we consider the systems with which the method has had success, systems for harmonic oscillators, scalar meson, and those described by Schrodinger and diffusion equations, the answer manifests itself. All solutions of the aforementioned equations satisfy the superposition theorem while solutions to the Navier-Stokes hydrodynamical equations do not

because of the substantive derivative

$$\frac{d}{dt} + u_j \frac{\partial}{\partial x_j}$$

appearing in the equations of motion. This nonlinearity is the cause of much trouble in hydrodynamics. In the following paragraphs we develop a method which circumvents this difficulty.

We divide our velocity component into two parts

$$V_j = u_j + u_j'$$

and postulate that the resultant flow of our systems will be such that the total dissipation of energy between the two component systems shall be a minimum. That is, if  $R$  is the time rate of the dissipation of energy of the unprimed system and  $R'$  that of the primed system then

$$\delta I = \int (R - R') d^3x dt.$$

We note that by dividing flow velocity in this manner we have essentially introduced a Clebsch transformation which is written usually as

$$u_j = \frac{\partial \varphi}{\partial x_j} + a \frac{\partial \theta}{\partial x_j}.$$

Here  $\varphi$  and  $\theta$  are scalar potential functions, and  $a$ , if it is taken as a function of the coordinates, will generate the rotational effects of the flow. The components of our velocity could

be considered in the same manner. However, in this thesis the suggested division will suffice as we are concerned with the derivation of the equations of motion of a real fluid from a variational principle. We leave a more detailed analysis of the interpretation of the components for future work.

The time rate of dissipation of the energy per unit volume due to viscosity and pressure is well known<sup>10</sup> and is given by

$$\begin{aligned}
 & -P \sum \frac{\partial u_j}{\partial x_j} + \mu \left\{ -\frac{2}{3} \left( \sum \frac{\partial u_j}{\partial x_j} \right)^2 + 2 \sum \left( \frac{\partial u_j}{\partial x_j} \right)^2 \right\} \\
 & + \mu \left\{ \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right)^2 + \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right)^2 \right\} \\
 & = -P \frac{\partial u_j}{\partial x_j} + \mu \Phi.
 \end{aligned}$$

We have used the summation sign in the above to avoid confusion, however, whenever double indices appear summation is always implied.

We now write our postulate in the form

$$\delta I = \int \left[ \rho \left\{ \frac{\partial u_i}{\partial t} + (u_j + u_j') \frac{\partial u_i}{\partial x_j} \right\} u_i' - \rho \left\{ \frac{\partial u_i}{\partial t} \right. \right.$$

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<sup>10</sup> Lamb, loc. cit., p. 508.

$$+(u_j + u'_j) \frac{\partial u_i}{\partial x_j} \} u_i - \rho \left\{ \frac{\partial u_j}{\partial x_j} - \frac{\partial u'_j}{\partial x_j} \right\} + \mu (\Phi - \Phi') d^3x dt.$$

The sum of the first and second terms of the integral is just the product of the density and the total time derivative of  $U_i$  dotted into  $U'_i$ . This term represents the time rate of doing work of the unprimed system on the primed system as we travel with the fluid.

In a like manner, the sum of the next two terms is the time rate of doing work of the primed system on the unprimed system and is taken with a negative sign in keeping with our hypothesis.  $\Phi'$  is formed by replacing  $u_j$  by  $u'_j$  in  $\Phi$  and the variation of the integral is with respect to  $\rho$ ,  $U$  and  $U'$ .

Varying  $U'_i$  we obtain

$$2\rho \frac{\partial u_i}{\partial t} + 2\rho(u_j + u'_j) \frac{\partial u_i}{\partial x_j} + u_i \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial \rho(u_j + u'_j)}{\partial x_j} \right\} + \rho u'_j \frac{\partial u_j}{\partial x_i}$$

$$= \rho u_j \frac{\partial u_j}{\partial x_i} = - \frac{\partial \rho}{\partial x_i} - 2 \frac{\partial}{\partial x_i} \left[ \mu \left( 2 \frac{\partial u'_i}{\partial x_i} - \frac{2}{3} \frac{\partial u_j}{\partial x_j} \right) \right]$$

$$- 2 \frac{\partial}{\partial x_2} \left[ \mu \left( \frac{\partial u'_1}{\partial x_2} + \frac{\partial u'_2}{\partial x_1} \right) \right] - 2 \frac{\partial}{\partial x_3} \left[ \mu \left( \frac{\partial u'_1}{\partial x_3} + \frac{\partial u'_3}{\partial x_1} \right) \right].$$

Which becomes, on employing the continuity equation,

$$2\rho \frac{\partial u_i}{\partial t} + 2\rho(u_j + u'_j) \frac{\partial u_i}{\partial x_j} + \rho u'_j \frac{\partial u_j}{\partial x_i} - \rho u_j \frac{\partial u'_j}{\partial x_i}$$

$$= -\frac{\partial P}{\partial x_i} - 2 \frac{\partial}{\partial x_j} \tau'_{ij}$$

where  $\tau'_{ij}$  is the usual hydrodynamical stress tensor.

The variation in  $U_1$  gives

$$2\rho \frac{\partial u'_i}{\partial t} + 2\rho(u_j + u'_j) \frac{\partial u'_i}{\partial x_j} - \rho u'_j \frac{\partial u_j}{\partial x_i}$$

$$+ \rho u_j \frac{\partial u'_j}{\partial x_i} = -\frac{\partial P}{\partial x_i} - 2 \frac{\partial}{\partial x_j} \tau'_{ij}$$

Adding these two derived equations, we obtain

$$\rho \frac{\partial}{\partial t} (u_i + u'_i) + \rho(u_j + u'_j) \frac{\partial}{\partial x_j} (u_i + u'_i) = -\frac{\partial P}{\partial x_i}$$

$$- \frac{\partial}{\partial x_j} (\tau_{ij} + \tau'_{ij})$$

or in general

$$\rho \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j} T_{ij}$$

which is the Navier-Stokes equation for a viscous fluid. Thus we have obtained the equation of motion for a viscous fluid from a variational principle which was constructed on the hypothesis that the difference of the dissipation of energy between the two elements of the components of the flow must be a minimum.



SUMMARY

In this study we have attempted to demonstrate the basic formalism developed for the application of the variational principle to dynamical systems and to exemplify the application of the variational principle to both conservative and nonconservative systems. This procedure supplies the basis for our ultimate purpose; which is to obtain the Navier-Stokes equations for a viscous fluid from a variational principle.

When the variational problem for a viscous fluid was first introduced, it was thought probable that we would have to apply the image method to it in order to obtain the desired results. However, we have demonstrated that it is possible to obtain the desired equations of motion by means of another rather interesting artifice. Namely, a separation of the velocity component into two parts which is in effect equivalent to a Clebsch transformation.

By means of the separation of a velocity into two elements, we are able to consider two different element systems. From this we find that if we consider the difference in the total dissipation of energy between the two element systems to be a minimum, the general equations of motion for a real fluid may be obtained from a variational principle.

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