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Global Dynamics for Steep Nonlinearities in Two Dimensions

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1. Introduction

This paper discusses a novel approach to obtaining mathematically rigorous results on the global dynamics of ordinary differential equations. The motivation is twofold. The first arises from applications, in particular the study of regulatory networks. Because of their centrality in the study of systems biology anything beyond the most cursory comments are beyond the scope of this introduction. Instead we refer the interested reader to [1]. The second arises from ongoing work of the authors to develop a mathematical framework, which we call a Database for Dynamics, that provides a computationally efficient and mathematically rigorous analysis of global dynamics of multiparameter nonlinear systems. Reasonable success has been obtained in the context of nonlinear systems generated by maps [2, 7, 6, 8]. However, extending these methods to ordinary differential equations is proving to be technically challenging since the rigorous evaluation of a map needs to be replaced by the rigorous evaluation of solutions to a differential equation [19, 20]. We return to this topic in the conclusion.

A regulatory network can be represented as an annotated directed graph. The vertices represent a regulatory object, e.g. a protein, and an edge from node m to node n indicates that m directly regulates n. The annotation on this edge indicates whether m activates (up regulates) or represses (down regulates) n. It is natural to think of modeling the dynamics of this system via an ordinary differential equation. In the context of gene regulatory networks, the associated proteins have a natural decay rate, and thus given a network with N genes one is led to a system ordinary differential equation of the form

$$\dot{x}_n = -\gamma_n x_n + f_n(x), \quad n = 1, \ldots, N \tag{1}$$

where $\gamma_n > 0$ and f_n is independent of x_m if there is no edge from node m to node n. In the context of applications at this level of generality there is little that one can hope to say. However, it is typically assumed that the interactions have switch like behavior and thus the nonlinearities used to model interactions between individual nodes are often assumed to have a sigmoidal shape. A common construction is based on Hill functions

$$\frac{x^k}{\theta^k + x^k} \quad \text{or} \quad \frac{\theta^k}{\theta^k + x^k} \tag{2}$$

where the former and latter expressions are used to model activation and repression, respectively. However, it is important to keep in mind that there are other models that are probably more representative of the underlying biochemistry [4, 5]. Even for N of moderate size the analysis of (1) with arbitrary Hill functions is intractable. A standard simplification is to let $k \to \infty$ in which case one obtains nonlinearities that take the form of piecewise constant functions. We refer to this as a *switching system* (for a precise definition see Section 3) and denote it by

$$\dot{x} = -\Gamma x + \Lambda(x), \quad x \in \mathbb{R}^N. \tag{3}$$

We view (3) as a computational model, the purpose of which is to give us insight into the behavior of the biologically motivated model (1) in which the nonlinearities f_n are Lipschitz continuous though a particular analytic form is not known. In particular, we are not concerned with identifying solutions to (3).

We recall [1] that there is a long tradition of associating *state transition diagrams*, which take the form of directed graphs, to regulatory networks. The paths through the state transition diagram are then used to represent the dynamics of the network. As is described in Section 6 we use (3) to define a particular choice of state transition diagram. Because our focus is on dynamics we find it convenient to represent this state transition diagram as a combinatorial multivalued map $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$. In this notation, $\mathcal{V}$ denotes the set of vertices in the state transition diagram and there exists a directed edge $u \to v$ in the state transition diagram if and only if $v \in \mathcal{F}(u)$.

The number of elements in $\mathcal{V}$ can grow rapidly as a function of the size of the regulatory network. In particular, in the approach we take here, if $O(n)$ denotes the number of out edges at node n, then the size of $\mathcal{V}$ is of order

$$\prod_{n=1}^{N} (O(n) + 1).$$

Cataloguing all the paths in a graph of this size is not practical. However, there are efficient (both in time and memory) graph algorithms that allow one to identify essential dynamical structures: the *recurrent dynamics*, i.e. the nontrivial strongly connected components of $\mathcal{F}$; and the *gradient-like dynamics*, i.e. the reachability, defined by paths in $\mathcal{F}$, between the recurrent components (see [2, 7] and references therein and [3] for an application of these techniques in the context of state transition diagrams). We encode this information in the form of a *Morse graph*, $\mathsf{MG}(\mathcal{F})$. This is the minimal directed acyclic graph such that each nontrivial strongly connected component is represented by a distinct node and the edges indicate the reachability information inherited from $\mathcal{F}$ between the nodes.

The switching system (3) has an N dimensional phase space, but an $N + 3E$ dimensional parameter space (E denotes the number of edges). In [10] we describe a set of algorithms that allows us to decompose parameter space into semi-algebraic sets, such that on each set the state transition diagram $\mathcal{F}$ is constant and hence the global dynamics as described by the Morse graph $\mathsf{MG}(\mathcal{F})$ is valid for each parameter value in the set. The algorithms have been implemented [14] and thus for moderate sized N we have the capability of describing via the Morse graphs the global dynamics associated with the state transition diagrams for *all* parameter values.

As is emphasized above we believe that from the biological perspective (1) provides a more realistic model than (3). Therefore, to justify the biological relevance of the combinatorial computations described above that are based on (3), it is important to be able to demonstrate that $\mathsf{MG}(\mathcal{F})$ provides correct and meaningful information about the dynamics of (1). Since a Morse graph is a directed acyclic graph it generates a poset. Our goal is to translate the poset structure associated with $\mathsf{MG}(\mathcal{F})$ into information about the structure of invariant sets for a flow generated by a smooth differential equation. For this we make use of Morse decompositions as defined by Conley [9]. Recall that given a continuous flow $\varphi\colon \mathbb{R} \times X \to X$ defined on a compact metric space X a *Morse decomposition* of X consists of a finite collection of mutually disjoint compact invariant sets called *Morse sets* indexed by a partially ordered set $(\mathsf{P}, \leq)$ with the property that if

$$x \in X \setminus \bigcup_{p \in \mathsf{P}} M(p),$$

where $M(p)$ denotes the Morse set indexed by p, then

$$\alpha(x, \varphi) \subset M(p) \quad \text{and} \quad \omega(x, \varphi) \subset M(q)$$

where $q \leq p$.

The primary goal of this paper is Theorem 8.4, which roughly states that given a regulatory network we can use (3) to construct a state transition diagram for which we can efficiently compute a Morse graph $\mathsf{MG}(\mathcal{F})$ from which we can determine lower bounds for a Morse decomposition for the dynamics defined by a smooth system of the form (1). The following outline indicates the tools and constructions that are used to obtain a proof of Theorem 8.4.

We begin in Section 2 with a brief description of Conley theory. Section 2.1 contains elementary ideas from lattice and poset theory and a statement of Birkhoff's theorem that relates finite distributive lattices and finite posets. Sections 2.2 and 2.3 presents the necessary definitions of Conley theory in the settings of combinatorial and continuous dynamics, respectively. Section 2.4 contains Theorem 2.17, which provides the theoretical framework by which we translate the information from the combinatorial dynamics to the continuous dynamics.

As indicated above Section 3 presents the definition of a switching system (3) and associated notation and definitions. The material in this section is restricted to two-dimensional systems, but can straightforwardly be extended to systems of arbitrary finite dimension. In Section 4 δ-constrained continuous switching systems

$$\dot{x} = -\Gamma x + f^{(\delta)}(x) \tag{4}$$

are defined. The definition begins with a given switching system, which determines Γ and a positive number δ, which indicates the width of a collar around the lines of discontinuity of Λ. The function $f^{(\delta)}$ is obtained by replacing Λ by a continuous function on this collar. Again, this construction is done in the setting of $\mathbb{R}^2$, but can be extended to $\mathbb{R}^n$.

The careful reader will note that we impose extremely weak constraints on the function $f^{(\delta)}$, thus on the level of individual invariant sets the dynamics can vary tremendously. However, what we prove is that the poset structure, which provides a rough description of the nonrecurrent dynamics, obtained from our combinatorial computations guided by Λ remains the same for the continuous dynamics generated by any $f^{(\delta)}$.

Theorem 8.4 states that the $\mathsf{MG}(\mathcal{F})$ derived from the switching system (3) determines a Morse decomposition for any associated δ-constrained continuous switching system (4) for any $0 < \delta < \delta^*$, where δ^* is explicitly determined by Γ and Λ (33). The strategy of the proof is as follows. We use the results of Section 2 and in particular Birkhoff's theorem to pass from the poset structure induced by $\mathsf{MG}(\mathcal{F})$ to a lattice of attractors for the state transition graph $\mathcal{F}$. We use Theorem 2.17 to guide the construction of a lattice of forward invariant sets for the δ-constrained continuous switching system. Birkhoff's theorem is then once more employed to identify the poset structure of the Morse decomposition for the δ-constrained continuous switching system.

The state transition graph $\mathcal{F}$ associated with a switching system (3) is defined in Section 6. This construction is presented in the setting of $\mathbb{R}^2$, but has been extended to $\mathbb{R}^n$ [10].

The major technical work of this paper is to construct a lattice of trapping regions for the δ-constrained continuous switching system (4) that is isomorphic an extension of the lattice of attractors for $\mathcal{F}$. This is done in two steps, both of which are restricted to $\mathbb{R}^2$. The first is to construct elementary regions in $\mathbb{R}^2$ that we call tiles and chips, which are used to construct the trapping regions. Tiles are related to the definition of $f^{(\delta)}$ and defined in Section 4. Chips are defined in Section 5 along with a proof that if the constraints on δ imposed by (33) are satisfied, then the flow of (4) is transverse along the edges of interest of tiles and chips. The second step involves the construction of the desired trapping regions. This is done in Section 7 in slightly more generality than needed as we construct a trapping regions associated with any forward invariant set of $\mathcal{F}$. There are two important remarks that need to be made about the content of Section 7. First, it is a tedious case by case local analysis of the construction and verification of the trapping regions and hence too cumbersome to generalize to higher dimensions. Second, we do not know of counter examples to the construction for higher dimensional systems, thus we believe that with an alternate proof it might be possible to generalize Theorem 8.4 to n-dimensional systems.

The formal statement and proof of Theorem 8.4 along with related results are presented in Section 8.

2. Conley Theory

Our proof that efficient graph theoretic computations can lead to rigorous mathematical results for smooth switching systems is based on new developments in Conley theory as presented in a series of papers [15, 16, 17, 18]. We review the essential ideas of these results along the lines in which we employ these ideas in this paper. We begin with a brief review of posets and lattices, describe Conley theory first in the context of directed graphs (combinatorial dynamics), and then in the setting of continuous flows on compact metric spaces. Finally, we state a theorem using this language that provides the framework by which the state transition graph leads to mathematically rigorous statements about the global dynamics for a smooth switching system.

2.1. Posets and Lattices

We assume the reader is familiar with the concepts of partially ordered sets (posets) and lattices (see [22, 12]), but review some fundamental concepts as a means of establishing notation.

Recall that given two posets $(\mathsf{P}, <_{\mathsf{P}})$ and $(\mathsf{Q}, <_{\mathsf{Q}})$, a function $f\colon \mathsf{P} \to \mathsf{Q}$ is a *poset morphism* if $p <_{\mathsf{P}} p'$ implies that $f(p) <_{\mathsf{Q}} f(p')$ for all $p, p' \in \mathsf{P}$.

We will make use of the following construction.

Definition 2.1. Given a poset $(\mathsf{Q}, <_{\mathsf{Q}})$ define $(\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}})$ to be the poset where $\bar{\mathsf{Q}} = \mathsf{Q} \cup \{\bar{q}\}$ and $<_{\bar{\mathsf{Q}}}$ restricted to Q equals $<_{\mathsf{Q}}$ along with the additional relations $q <_{\bar{\mathsf{Q}}} \bar{q}$ for all $q \in \mathsf{Q}$. We denote the inclusion map, which is a poset monomorphism, by $\bar{\iota}\colon \mathsf{Q} \to \bar{\mathsf{Q}}$.

Given a poset $(\mathsf{P}, <)$, $\mathsf{D} \subset \mathsf{P}$ is a *down set* of P if $p \in \mathsf{P}$, $q \in \mathsf{D}$, and $p < q$ implies that $p \in \mathsf{D}$. The set of down sets of P is denoted by $\mathsf{O}(\mathsf{P})$ and is a lattice under the operations of union and intersection. In fact, O defines a contravariant functor from the category of posets to the category of lattices.

Let $(\mathsf{L}, \vee, \wedge, \mathbf{0}, \mathbf{1})$ be a bounded distributive lattice where $\mathbf{0}$ and $\mathbf{1}$ denote the minimum and maximum elements, i.e.

$$\mathbf{0} \wedge \mathsf{U} = \mathbf{0} \quad \text{and} \quad \mathbf{1} \wedge \mathsf{U} = \mathsf{U}$$

for all $U \in \mathsf{L}$. The lattice algebra induces a partial order on its elements as follows. Given $U, V \in \mathsf{L}$

$$\text{if} \quad U \vee V = V \quad \text{or equivalently} \quad U \wedge V = U \quad \text{then} \quad U \leq V. \tag{5}$$

Recall that a nonzero element $\mathsf{U} \in \mathsf{L}$ is *join irreducible* if $U = A \vee B$ implies that $U = A$ or $U = B$. We denote the set of join irreducible elements of L by $\mathsf{J}^{\vee}(\mathsf{L})$. Observe that in a finite lattice if U is a join irreducible element, then it has a unique immediate predecessor in L with regard to the partial order (5). We denote this unique predecessor by

$$\overleftarrow{U}. \tag{6}$$

Using the partial order (5) $(\mathsf{J}^{\vee}(\mathsf{L}), \leq)$ is a poset and, more generally, $\mathsf{J}^{\vee}$ defines a contravariant functor from the category of bounded distributive lattices to the category of finite posets.

Theorem 2.2 (Birkhoff's theorem)**.** *Let* $(\mathsf{L}, \vee, \wedge)$ *be a finite distributive lattice and let* $(\mathsf{P}, \leq)$ *be a finite poset. Then,*

(i) $\mathsf{O}(\mathsf{J}^{\vee}(\mathsf{L}))$ *is lattice isomorphic to* L.

(ii) $\mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P}))$ *is poset isomorphic to* P.

As is made clear below our interest in Birkhoff's theorem lies in the fact that it guarantees that one can represent the same information either in a poset or a lattice and that there is a well defined transformation between the two.

Definition 2.3. Given two lattices $(\mathsf{L}, \vee_{\mathsf{L}}, \wedge_{\mathsf{L}})$ and $(\mathsf{K}, \vee_{\mathsf{K}}, \wedge_{\mathsf{K}})$ a **0** *join-semilattice morphism* $N\colon \mathsf{L} \to \mathsf{K}$ satisfies

$$N(U \vee_{\mathsf{L}} V) = N(U) \vee_{\mathsf{K}} N(V) \quad \text{and} \quad N(\mathbf{0}) = \mathbf{0}.$$

The following proposition, which follows directly from the definition of a partial order in a lattice (5), shows that join-semilattice morphisms are poset morphisms.

Proposition 2.4. *Let $N\colon \mathsf{L} \to \mathsf{K}$ be a join-semilattice morphism and let $U, V \in \mathsf{L}$. If $U < V$, then $N(U) \leq N(V)$.*

2.2. Combinatorial Conley Theory

As indicated in the introduction, to emphasize the fact that we are interested in dynamical structures we represent a directed graph as a combinatorial multivalued map $\mathcal{F}\colon \mathcal{V} \rightrightarrows \mathcal{V}$, where $\mathcal{V}$ is the finite set of vertices and there is a directed edge $\nu \to \nu'$ if and only if $\nu' \in \mathcal{F}(\nu)$. Note that we allow self edges in our directed graph thus it is possible that $\nu \in \mathcal{F}(\nu)$.

We use the notation $\nu \rightsquigarrow \nu'$ to indicate the existence of a nontrivial path from ν to ν'. Using the multivalued map notation, $\nu \rightsquigarrow \nu'$ is equivalent to the statement that there exists $n \in \mathbb{N}$ such that $\nu' \in \mathcal{F}^n(\nu)$. Backward paths in the graph can be associated with reversal of time. With this in mind define the *transpose* of $\mathcal{F}$, $\mathcal{F}^t\colon \mathcal{V} \to \mathcal{V}$, by

$$\nu' \in \mathcal{F}^t(\nu) \quad \text{if and only if} \quad \nu \in \mathcal{F}(\nu').$$

Elements $\nu, \nu' \in \mathcal{V}$ belong to the same *strongly connected path component* of $\mathcal{F}$ if $\nu \rightsquigarrow \nu'$ and $\nu' \rightsquigarrow \nu$. Since we allow self edges it is possible that a strongly connected path component consists of a single vertex with a self edge. We refer to a strongly connected path component of $\mathcal{F}$ as a *Morse set* of $\mathcal{F}$ and denote it by $\mathcal{M} \subset \mathcal{V}$. The collection of all strongly connected path components of $\mathcal{F}$ is denoted by

$$\mathsf{MD}(\mathcal{F}) := \{\mathcal{M}(p) \subset \mathcal{V} \mid p \in \mathsf{P}(\mathcal{F})\}$$

and forms a *Morse decomposition* of $\mathcal{F}$. We impose a partial order on the indexing set $\mathsf{P}(\mathcal{F})$ of $\mathsf{MD}(\mathcal{F})$ by defining

$$q <_{\mathsf{P}(\mathcal{F})} p \quad \text{if } p \neq q \text{ and there exists a path in } \mathcal{F} \text{ from an element of } \mathcal{M}(p) \text{ to an element of } \mathcal{M}(q).$$

For notational simplicity, we often denote $<_{\mathsf{P}(\mathcal{F})}$ by $<$.

Definition 2.5. The *Morse graph* of $\mathcal{F}$, $\mathsf{MG}(\mathcal{F})$, is the Hasse diagram of the poset $(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})})$. We refer to the elements of $\mathsf{P}(\mathcal{F})$ as the *Morse nodes* of the graph.

As discussed in the introduction given $\mathcal{F}\colon \mathcal{V} \rightrightarrows \mathcal{V}$ identification of Morse sets and the Morse graph is computationally feasible. Thus, these are the objects that we extract from the state transition diagram. However, we know of no direct means of transferring knowledge of the Morse graph to an associated continuous system of the form (3). As indicated above Birkhoff's theorem guarantees that we will not lose information by considering the lattice of down sets $\mathsf{O}(\mathsf{P})$, where $(\mathsf{P}, <)$ is the poset that defines the Morse graph. To identify this lattice in the directed graph $\mathcal{F}\colon \mathcal{V} \rightrightarrows \mathcal{V}$ recall the following concept.

Definition 2.6. A set $\mathcal{N} \subset \mathcal{V}$ is *forward invariant* under $\mathcal{F}$ if $\mathcal{F}(\mathcal{N}) \subset \mathcal{N}$.

Definition 2.7. Let $\mathcal{F}\colon \mathcal{V} \rightrightarrows \mathcal{V}$ be a multivalued map. A set $\mathcal{N} \subset \mathcal{V}$ is *invariant* if $\mathcal{N} \subset \mathcal{F}(\mathcal{N})$ and $\mathcal{N} \subset \mathcal{F}^t(\mathcal{N})$. The set of invariant sets for $\mathcal{F}$ is denoted by $\mathsf{Invset}(\mathcal{F})$.

Proposition 2.8. *$\mathcal{N} \in \mathsf{Invset}(\mathcal{F})$ if and only if for every $\nu \in \mathcal{N}$, there exists $\nu_{-1}, \nu_1 \in \mathcal{N}$ such that $\nu_{-1} \to \nu \to \nu_1$.*

Proof. Observe that $\mathcal{N} \subset \mathcal{F}(\mathcal{N})$ is equivalent to if $\nu \in \mathcal{N}$ then there exists $\nu_{-1} \in \mathcal{N}$ such that $\nu_{-1} \to \nu$. Similarly, $\mathcal{N} \subset \mathcal{F}^t(\mathcal{N})$ is equivalent to if $\nu \in \mathcal{N}$ then there exists $\nu_1 \in \mathcal{N}$ such that $\nu \to \nu_1$. □

Given $\mathcal{U} \subset \mathcal{V}$ we denote the *maximal* (with respect to inclusion) *invariant set under $\mathcal{F}$ in $\mathcal{U}$* by

$$\mathrm{Inv}(\mathcal{U}, \mathcal{F}) \in \mathsf{Invset}(\mathcal{F}).$$

Of central interest is the following special type of forward invariant set.

Definition 2.9. A set $\mathcal{A} \subset \mathcal{V}$ is an *attractor* for $\mathcal{F}$ if $\mathcal{F}(\mathcal{A}) = \mathcal{A}$.

The collection of all attractors in $\mathcal{V}$ under $\mathcal{F}$ is denoted by $\mathsf{Att}(\mathcal{F})$ and as discussed in [17, Section 2] is a bounded distributive lattice where $\mathbf{0} := \emptyset$ and $\mathbf{1} = \max\{\mathcal{A} \mid \mathcal{A} \in \mathsf{Att}(\mathcal{F})\}$. Furthermore, given $\mathcal{A}_0, \mathcal{A}_1 \in \mathsf{Att}(\mathcal{F})$ the lattice operations are defined by

$$\mathcal{A}_0 \vee \mathcal{A}_1 := \mathcal{A}_0 \cup \mathcal{A}_1 \tag{7}$$

and

$$\mathcal{A}_0 \wedge \mathcal{A}_1 := \max\{\mathcal{A} \in \mathsf{Att}(\mathcal{F}) \mid \mathcal{A} \subset \mathcal{A}_0 \cap \mathcal{A}_1\}. \tag{8}$$

As is discussed in detail in [18] Birkhoff's theorem provides a fundamental relationship between Morse sets and attractors. We make use of the following concepts and results from [18].

The lattice of attractors of $\mathcal{F}$ can be recovered from the Morse decomposition of $\mathcal{F}$. To be more precise, given $\mathcal{M}(p) \in \mathsf{MD}(\mathcal{F})$ let

$$\downarrow(\mathcal{M}(p)) = \{\nu \in \mathcal{V} \mid \exists \nu' \in \mathcal{M}(p) \text{ such that } \nu' \rightsquigarrow \nu\}.$$

It is straightforward to check that $\downarrow(\mathcal{M}(p)) \in \mathsf{Att}(\mathcal{F})$. In fact, under the lattice operations $\vee$ and $\wedge$ $\{\downarrow(\mathcal{M}(p)) \mid p \in \mathsf{P}\}$ generates $\mathsf{Att}(\mathcal{F})$ (if there is a unique minimal Morse set then it is necessary to include $\mathbf{0} = \emptyset$).

Similarly, the Morse decomposition of $\mathcal{F}$ can be recovered from the lattice of attractors of $\mathcal{F}$. To see this let $\mathcal{A} \in J^\vee(\mathsf{Att}(\mathcal{F}))$. This implies that there exists a unique immediate predecessor $\overleftarrow{\mathcal{A}}$. Define $m\colon J^\vee(\mathsf{Att}(\mathcal{F})) \to \mathsf{Invset}(\mathcal{F})$ by

$$m(\mathcal{A}) := \mathrm{Inv}(\mathcal{A} \setminus \overleftarrow{\mathcal{A}}, \mathcal{F}).$$

Proposition 2.10. *If $\mathcal{A} \in J^\vee(\mathsf{Att}(\mathcal{F}))$, then*

1. *$m(\mathcal{A}) \neq \emptyset$,*
2. *$\downarrow(m(\mathcal{A})) = \mathcal{A}$,*
3. *$m(\mathcal{A}) \in \mathsf{MD}(\mathcal{F})$,*

4. *If* $m(\mathcal{A}) \in \mathsf{Att}(\mathcal{F})$*, then* $\overleftarrow{\mathcal{A}} = \emptyset$.

5. *If* $\mathcal{A}' \in \mathsf{Att}(\mathcal{F})$ *and* $v \in m(\mathcal{A}) \cap \mathcal{A}'$*, then* $\mathcal{A} \subset \mathcal{A}'$.

Proof. (1) Let $v \in \mathcal{A} \setminus \overleftarrow{\mathcal{A}}$. Since $\mathcal{A}, \overleftarrow{\mathcal{A}} \in \mathsf{Att}(\mathcal{F})$, $\mathcal{F}^t(v) \cap \mathcal{A} \neq \emptyset$ and $\mathcal{F}^t(v) \cap \overleftarrow{\mathcal{A}} = \emptyset$. Let $v_{-1} \in \mathcal{F}^t(v) \cap \mathcal{A}$ and inductively define $v_{-k-1} \in \mathcal{F}^t(v_{-k}) \cap \mathcal{A}$. Since $\mathcal{A}$ has only finitely many elements there exists $n \neq m$ such that $v_{-n} = v_{-m}$. Observe that $v_{-n} \in m(\mathcal{A})$.

(2) By definition $\mathcal{F}(\downarrow(m(\mathcal{A}))) \subset \downarrow m(\mathcal{A})$. Let $v \in \downarrow m(\mathcal{A})$. Again by definition if $v \in \downarrow m(\mathcal{A}) \setminus m(\mathcal{A})$, then $\mathcal{F}^t(v) \cap \downarrow m(\mathcal{A}) \neq \emptyset$. Since $m(\mathcal{A})$ is invariant, if $v \in m(\mathcal{A})$ then $\mathcal{F}^t(v) \cap m(\mathcal{A}) \neq \emptyset$. Therefore, $\downarrow m(\mathcal{A}) \subset \mathcal{F}(\downarrow(m(\mathcal{A})))$.

(3) By definition $\mathcal{F}(\mathcal{A}) = \mathcal{A}$. Therefore, if $\mathcal{M} \in \mathsf{MD}(\mathcal{F})$ then $\mathcal{M} \subset \mathcal{A}$ or $\mathcal{M} \cap \mathcal{A} = \emptyset$. Proposition 2.8 combined with the fact that $\mathcal{A}$ is finite, implies that there exists $\mathcal{M}(p) \in \mathsf{MD}(\mathcal{F})$ such that $\mathsf{M}(p) \subset m(\mathcal{A})$. Assume there exists $q \neq p$ such that $\mathcal{M}(q) \subset m(\mathcal{A})$ and $p \not< q$. Then $\downarrow(\mathcal{M}(q)) \cup \overleftarrow{\mathcal{A}} \in \mathsf{Att}(\mathcal{F})$ and $\overleftarrow{\mathcal{A}} \subsetneq \downarrow(\mathcal{M}(q)) \cup \overleftarrow{\mathcal{A}} \subsetneq \mathcal{A}$, a contradiction. Thus, $\mathcal{M}(p)$ is the unique Morse set in $m(\mathcal{A})$. Finally, observe that $m(\mathcal{A}) \setminus \mathcal{M}(p) \neq \emptyset$ contradictions the fact that $m(\mathcal{A})$ is invariant.

(4) If $m(\mathcal{A}) \in \mathsf{Att}(\mathcal{F})$, then by (2) $\downarrow(m(\mathcal{A})) = \mathcal{A}$ and hence $\overleftarrow{\mathcal{A}} = \emptyset$.

(5) Since $\mathcal{A}' \in \mathsf{Att}(\mathcal{F})$, $\mathcal{A} \cap \mathcal{A}' \in \mathsf{Att}(\mathcal{F})$. By definition

$$\overleftarrow{\mathcal{A}} \cap \mathcal{A}' \subset \mathcal{A} \cap \mathcal{A}' \subset \mathcal{A}.$$

By assumption $v \in (\mathcal{A} \cap \mathcal{A}') \setminus (\overleftarrow{\mathcal{A}} \cap \mathcal{A}')$ and hence

$$\overleftarrow{\mathcal{A}} \cap \mathcal{A}' \subsetneq \mathcal{A} \cap \mathcal{A}' \subset \mathcal{A}.$$

Thus $\mathcal{A} \subset \mathcal{A}'$. □

2.3. Continuous Conley Theory

We assume the reader is familiar with basic concepts from the theory of dynamical systems [21]. Let $\varphi \colon \mathbb{R} \times X \to X$ be a continuous flow defined on a compact metric space. Let $\mathsf{Invset}(X, \varphi)$ denote the collection of invariant sets in X under φ.

Definition 2.11. A compact set $N \subset X$ is an *attracting neighborhood* if $\omega(N, \varphi) \subset \mathrm{int}(N)$ where $\omega(N, \varphi)$ denotes the ω-limit set of N under φ.

The set of all attracting neighborhoods in X under φ is denoted by $\mathsf{ANbhd}(X, \varphi)$ and as shown in [16] is a bounded distributive lattice (but in general is not finite). In this paper we make use of a special class of attracting neighborhoods.

Definition 2.12. Given a flow $\varphi \colon \mathbb{R} \times X \to X$ a compact set $N \subset X$ is an *attracting block* or *trapping region* if $\varphi(t, N) \subset \mathrm{int}(N)$ for all $t > 0$. We denote the set of attracting blocks for φ by $\mathsf{AB}(X, \varphi)$.

We leave it to the reader to show that $\mathsf{AB}(X, \varphi)$ is sublattice of $\mathsf{ANbhd}(X, \varphi)$ and to check the following lemma.

Lemma 2.13. *Assume the flow φ is generated by a differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. Let $N \subset \mathbb{R}^2$ be a regular closed set [23] whose boundary is made up of a finite number of straight line segments. Furthermore, assume that if $x \in \partial N$ and e is a boundary edge containing x, then $f(x) \cdot n_e < 0$ where n_e is denotes the outward normal of e. Then, N is an attracting block.*

Recall that an invariant set $A \in \mathsf{Invset}(X, \varphi)$ is an *attractor* for φ if there exists an attracting neighborhood N such that $A = \omega(N, \varphi)$. The *dual repeller* of an attractor A is defined to be

$$A^* := \{x \in X \mid \omega(x, \varphi) \cap A = \emptyset\}.$$

In what follows we assume that A is a finite sublattice of $\mathsf{ANbhd}(X, \varphi)$. Recall that $\mathsf{J}^\vee(\mathsf{A})$ forms a poset. Define $\mu \colon \mathsf{J}^\vee(\mathsf{A}) \to \mathsf{Invset}(X, \varphi)$ by

$$\mu(N) := \mathrm{Inv}(N, \varphi) \cap \left(\mathrm{Inv}\left(\overleftarrow{N}, \varphi\right)\right)^*$$

where $\mathrm{Inv}(N, \varphi)$ denotes the maximal invariant set in N under φ. As is shown in [18] $\mathsf{J}^\vee(\mathsf{A})$ provides a *labeling* for a *Morse decomposition* $MD(\varphi)$ of X under φ, i.e. a collection of mutually disjoint compact invariant sets $\mu(N)$, $N \in \mathsf{J}^\vee(\mathsf{A})$ with the following property: if

$$x \in X \setminus \bigcup_{N \in \mathsf{J}^\vee(\mathsf{A})} \mu(N),$$

then there exists $N, N' \in \mathsf{J}^\vee(\mathsf{A})$ such that

$$\alpha(x, \varphi) \subset \mu(N) \quad \text{and} \quad \omega(x, \varphi) \subset \mu(N')$$

and furthermore $N' < N$ under the partial order on $\mathsf{J}^\vee(\mathsf{A})$. The Hasse diagram of $\mathsf{J}^\vee(\mathsf{A})$ is the *Morse graph* associated to the Morse decomposition.

2.4. The Translational Theorem

The primary result of this paper makes use of the following ideas and theorems that are explained in detail in [18]. In Section 6 we define the transition graph or combinatorial multivalued map $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$ that is associated with a switching system. As is indicated in Section 2.2 this gives rise to Morse graph $\mathsf{MG}(\mathcal{F})$ and the associated poset $\left(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})}\right)$. In Section 7 we construct a function $N \colon \mathsf{Att}(\mathcal{F}) \to \mathsf{AB}(\varphi)$ and in Section 8 we prove the following proposition making use of δ-constrained continuous switching systems defined in Section 4.

Proposition 2.14. *$N \colon \mathsf{Att}(\mathcal{F}) \to \mathsf{AB}(\varphi)$ is a $\mathbf{0}$ join-semilattice monomorphism where φ is the flow generated by a δ-constrained continuous switching system* (4)*.*

Since, in general, we can only assume that N is a join-semilattice morphism, $N(\mathsf{Att}(\mathcal{F}))$ need not be a sublattice of $\mathsf{AB}(\varphi)$. Therefore, we define $\mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$ to be the minimal sublattice of $\mathsf{AB}(\varphi)$ containing $N(\mathsf{Att}(\mathcal{F}))$ and make note of the following proposition.

Proposition 2.15. *Then, every $K \in \mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$ can be written in the form*

$$K = \bigcap_i N(\mathcal{A}_i)$$

where $\{\mathcal{A}_i\} \subset \mathsf{Att}(\mathcal{F})$.

Proof. Consider $K \in \mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$. Since $\mathsf{Att}(\mathcal{F})$ is a distributive lattice with operations $\cap$ and $\cup$, and $\mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$ is generated by intersection and union of elements of $N(\mathsf{Att}(\mathcal{F}))$, we can assume that

$$K = \bigcap_i \bigcup_j (N(\mathcal{U}_{i,j})) = \bigcap_i N\left(\bigcup_j \mathcal{U}_{i,j}\right) = \bigcap_i N(\mathcal{A}_i)$$

where $\mathcal{A}_i = \bigcup_j \mathcal{U}_{i,j}$. ☐

In Section 8 we prove the following proposition

Proposition 2.16. *If* $\mathsf{A} \in J^{\vee}(\mathsf{Att}(\mathcal{F}))$, *then* $N(\mathsf{A}) \in J^{\vee}(\mathsf{AB}(\varphi))$.

The following diagram is meant to organizes the concepts presented up to this point and indicates the flow of information that leads to the theorem that translates the data of the combinatorial dynamics captured by $\mathcal{F}$ to the continuous dynamics on the global attractor X of an associated δ-constrained continuous switching system (4):

$$\begin{array}{ccccccc}
\left(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})}\right) & \xrightarrow{\text{Birkhoff}} & \mathsf{Att}(\mathcal{F}) & \xrightarrow{N} & \mathsf{L}(N(\mathsf{Att}(\mathcal{F}))) & \hookrightarrow & \mathsf{AB}(\varphi) \\
 & \overset{\iota}{\searrow} & & & \downarrow \text{Birkhoff} & & \\
 & & & & (\mathsf{Q}, <_{\mathsf{Q}}) & & \\
 & & & & \downarrow \bar{\iota} & & \\
 & & & & (\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}}) & \xrightarrow{\mu} & \mathsf{Invset}(\varphi)
\end{array} \tag{9}$$

where $\mathsf{Q} = \mathsf{J}^{\vee}(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$ and $\mu\colon (\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}}) \to \mathsf{Invset}(\varphi)$ (recall Definition 2.1) defines a Morse decomposition of X.

The first arrow follows from Theorem 2.2 and provides an order preserving 1-1 correspondence between the elements of $\mathsf{MD}(\mathcal{F})$ and join irreducible elements of $\mathsf{Att}(\mathcal{F})$. Propositions 2.14 and 2.16 guarantees that N is a poset monomorphism that carries the join irreducible elements of $\mathsf{Att}(\mathcal{F})$ to the join irreducible elements of $\mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$. Another application of Theorem 2.2 produces the poset $(\mathsf{Q}, <_{\mathsf{Q}})$ consisting of the join irreducible elements of $\mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$. The images of μ are determined by N. However, since we are *not* assuming that $N(\mathbf{1}) = X$ we cannot claim that the we have a labeling for a Morse decomposition of all of X. In particular, we cannot capture the dynamics in the region $X \setminus N(\mathbf{1})$. Thus we are forced to allow for the existence of an additional maximal Morse set $\bar{p}$ (which may be the empty set). Observe that the induced map $\bar{\iota} \circ \iota\colon \mathsf{P}(\mathcal{F}) \to \bar{\mathsf{Q}}$ is a poset monomorphism from the labeling of the Morse decomposition of $\mathcal{F}$ to a labeling of a Morse decomposition of the global attractor of φ. Therefore, we obtain the following result.

Theorem 2.17. *Consider a switching system* (3) *in* $\mathbb{R}^2$. *Let* $\mathcal{F}\colon \mathcal{V} \rightrightarrows \mathcal{V}$ *be the associated transition graph. Let* φ *be a flow generated by an associated* δ*-constrained continuous switching system* (4). *Let* X *be the global attractor of* φ. *Then there exists a Morse decomposition* $MD(\varphi)$ *for* X *under* φ *indexed by* $\left(\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}}\right)$, *where* $\mathsf{Q} = \mathsf{J}^{\vee}(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$, *for which there exists a poset monomorphism* $\bar{\iota} \circ \iota\colon \left(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})}\right) \to \left(\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}}\right)$.

3. Two-dimensional Switching Systems

In this section we provide a formal definition of a general two-dimensional switching system and provide elementary results about the associated dynamics. We begin with two sets of non-negative real numbers $\Xi := \{\xi_i \mid i = 0, \dots, I+1\}$ and $\mathsf{H} := \{\eta_j \mid j = 0, \dots, J+1\}$ that we refer to as *threshold values*, with the property that

$$\begin{aligned} 0 = \xi_0 < \xi_1 < \ldots < \xi_I < \xi_{I+1} = \infty \\ 0 = \eta_0 < \eta_1 < \ldots < \eta_J < \eta_{J+1} = \infty. \end{aligned}$$

Let

$$\Pi := \{(\xi_i, \eta_j) \mid i = 0, \dots, I,\ j = 0, \dots, J\} \subset [0, \infty)^2. \tag{10}$$

In addition we assume that we are given partitions of the sets of threshold values

$$\Xi = \Xi^1 \cup \Xi^2 \quad \text{and} \quad \mathsf{H} = \mathsf{H}^1 \cup \mathsf{H}^2.$$

Of primary importance is the following collection of open rectangles, called *cells*, defined in terms of the thresholds as follows:

$$\mathcal{K} := \left\{ \kappa(i,j) := (\xi_i, \xi_{i+1}) \times (\eta_j, \eta_{j+1}) \subset (0,\infty)^2 \mid i = 0, \dots, I,\ j = 0, \dots, J \right\}.$$

Definition 3.1. The *switching system* $\Sigma = \Sigma(\Gamma, \Lambda, \Xi^1, \Xi^2, \mathsf{H}^1, \mathsf{H}^2)$ is defined to be the system of differential equations

$$\dot{x} = -\Gamma x + \Lambda(x), \qquad x \in \bigcup_{\kappa \in \mathcal{K}} \kappa \subset (0,\infty)^2 \tag{11}$$

where

$$\Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \qquad \gamma_i > 0$$

and

$$\Lambda(x) = \begin{bmatrix} \Lambda_1(x) \\ \Lambda_2(x) \end{bmatrix}$$

is constant on each cell of $\mathcal{K}$. Furthermore, Λ satisfies the following constraints for all $i \in \{0, \dots, I\}$, $j \in \{0, \dots, J\}$,

$$\begin{aligned} \xi_i \in \Xi^2 \quad &\Rightarrow \quad \Lambda_1(\kappa(i-1,j)) = \Lambda_1(\kappa(i,j)) \\ \xi_i \in \Xi^1 \quad &\Rightarrow \quad \Lambda_2(\kappa(i-1,j)) = \Lambda_2(\kappa(i,j)) \\ \eta_j \in \mathsf{H}^2 \quad &\Rightarrow \quad \Lambda_1(\kappa(i,j-1)) = \Lambda_1(\kappa(i,j)) \\ \eta_j \in \mathsf{H}^1 \quad &\Rightarrow \quad \Lambda_2(\kappa(i,j-1)) = \Lambda_2(\kappa(i,j)). \end{aligned} \tag{12}$$

For the sake of simplicity we assume that if $\kappa \neq \kappa'$, then

$$\Lambda(\kappa) \neq \Lambda(\kappa').$$

Furthermore, setting

$$\Phi(\kappa) := \Gamma^{-1}\Lambda(\kappa) \tag{13}$$

we assume that for all $\kappa \in \mathcal{K}$

$$\Phi_1(\kappa) \notin \Xi \quad \text{and} \quad \Phi_2(\kappa) \notin \mathsf{H}. \tag{14}$$

A consequence of assumption (14) is that for a fixed switching system Σ

$$\mu = \mu(\Sigma) := \min_{\kappa \in \mathcal{K},\ \xi \in \Xi,\ \eta \in \mathsf{H}} \{|\Phi_1(\kappa) - \xi|, |\Phi_2(\kappa) - \eta|\} > 0. \tag{15}$$

Two other positive constants that are used later in the paper are

$$\rho = \rho(\Sigma) := \max_{\kappa = \kappa(i,j) \in \mathcal{K}} \{|\Phi_1(\kappa) - \xi_i|, |\Phi_1(\kappa) - \xi_{i+1}|, |\Phi_2(\kappa) - \eta_j|, |\Phi_2(\kappa) - \eta_{j+1}|\}, \tag{16}$$

which provides a measurement of the maximal displacement of the attracting fixed point $\Phi(\kappa)$ from the cell κ, and

$$\bar{\gamma} = \bar{\gamma}(\Sigma) := \min\left\{\frac{\gamma_1}{\gamma_2}, \frac{\gamma_2}{\gamma_1}\right\} \leq 1. \tag{17}$$

Given a particular cell $\kappa = \kappa(i,j) \in \mathcal{K}$ there is an associated affine vector field, which we call the κ-*equation*, given by

$$\dot{x} = -\Gamma x + \Lambda(\kappa). \tag{18}$$

We denote the flow generated by the κ-equation by ψ_κ. Observe that $\Phi(\kappa)$, as defined in (13), is an attracting fixed point for the κ-equation.

We label each cell according to the behavior of the associated κ-equation on the cell. In particular,

$$\kappa(i,j) \text{ is of type } \begin{cases} \text{N} & \text{if } \xi_i < \Phi_1(\kappa) < \xi_{i+1} \text{ and } \eta_{j+1} < \Phi_2(\kappa) \\ \text{NE} & \text{if } \xi_{i+1} < \Phi_1(\kappa) \text{ and } \eta_{j+1} < \Phi_2(\kappa) \\ \text{E} & \text{if } \xi_{i+1} < \Phi_1(\kappa) \text{ and } \eta_j < \Phi_2(\kappa) < \eta_{j+1} \\ \text{SE} & \text{if } \xi_{i+1} < \Phi_1(\kappa) \text{ and } \Phi_2(\kappa) < \eta_j \\ \text{S} & \text{if } \xi_i < \Phi_1(\kappa) < \xi_{i+1} \text{ and } \Phi_2(\kappa) < \eta_j \\ \text{SW} & \text{if } \Phi_1(\kappa) < \xi_i \text{ and } \Phi_2(\kappa) < \eta_j \\ \text{W} & \text{if } \Phi_1(\kappa) < \xi_i \text{ and } \eta_j < \Phi_2(\kappa) < \eta_{j+1} \\ \text{NW} & \text{if } \Phi_1(\kappa) < \xi_i \text{ and } \eta_{j+1} < \Phi_2(\kappa) \\ \text{A} & \text{if } \xi_i < \Phi_1(\kappa) < \xi_{i+1} \text{ and } \eta_j < \Phi_2(\kappa) < \eta_{j+1} \end{cases} \tag{19}$$

Assumption (14) implies that every cell κ is of the type indicated above. A cell of type A is called an *attracting cell*; a cell of type W, E, S, or W is called a *focussing cell*; and a cell of type NE, SE, SW, or NW is called a *translating cell.* Furthermore, for bookkeeping purposes we find it convenient to place a star in the cell to indicate its type (see Figure 1).

As indicated in the introduction we use the switching system to construct the state transition diagram. The boundaries of the cells are used in the definition of the set of vertices of the state transition diagram. The precise definition is as follows.

Definition 3.2. Let $\kappa = \kappa(i,j) \in \mathcal{K}$. The set of *faces* of κ is denoted by $\mathcal{V}(\kappa)$ and consists of

$$v_{*,\bar{j}} := \{\xi_*\} \times (\eta_j, \eta_{j+1}), \quad * = i, i+1$$
$$v_{\bar{i},*} := (\xi_i, \xi_{i+1}) \times \{\eta_*\}, \quad * = j, j+1$$

and, additionally,

$$w_\kappa \quad \text{if } \kappa \text{ is of type A.}$$

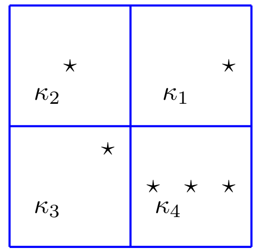


Figure 1: The figure indicates four cells labeled κ_i, $i = 1, 2, 3, 4$. The star in the center of cell κ_2 indicates that it is of type A. The star in the center right of cell κ_1 indicates that it is of type E. The star in the upper right corner of cell κ_3 indicates that it is of type NE. The multiple stars in κ_4 indicate that it is of type W, A, or E.

The complete set of faces is given by

$$\mathcal{V} := \bigcup_{\kappa\in\mathcal{K}} \mathcal{V}(\kappa)$$

where faces arising from distinct cells but representing the same region in $\mathbb{R}^2$ are considered equivalent.

We remark that elements of $\mathcal{V}(\kappa)$ have two interpretations: faces of κ and hence subsets of $[0,\infty)^2$, and vertices as elements of the state transition diagram. Whether we are employing them as topological or combinatorial objects should be clear from the context.

Definition 3.3. Given a cell $\kappa = \kappa(i, j)$ we label the faces $\mathcal{V}(\kappa)$ as follows

$$v_{i,\bar{j}} \text{ is an} \begin{cases} \textit{entrance} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_1(\kappa) > \xi_i \\ \textit{absorbing} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_1(\kappa) < \xi_i \end{cases}$$

$$v_{i+1,\bar{j}} \text{ is an} \begin{cases} \textit{entrance} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_1(\kappa) < \xi_{i+1} \\ \textit{absorbing} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_1(\kappa) > \xi_{i+1} \end{cases}$$

$$v_{\bar{i},j} \text{ is an} \begin{cases} \textit{entrance} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_2(\kappa) > \eta_j \\ \textit{absorbing} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_2(\kappa) < \eta_j \end{cases}$$

$$v_{\bar{i},j+1} \text{ is an} \begin{cases} \textit{entrance} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_2(\kappa) < \eta_{i+1} \\ \textit{absorbing} \text{ face of } \mathcal{V}(\kappa) & \text{if } \Phi_2(\kappa) > \eta_{i+1} \end{cases}$$

$$w_\kappa \text{ is an } \textit{absorbing} \text{ face of } \mathcal{V}(\kappa).$$

Let $\mathcal{V}_e(\kappa)$ and $\mathcal{V}_a(\kappa)$ denote the entrance and absorbing faces of $\mathcal{V}(\kappa)$. Observe that

$$\mathcal{V}_e(\kappa) \cap \mathcal{V}_a(\kappa) = \emptyset.$$

As the following proposition indicates, given a pair of adjacent cells that share a face v there are constraints on the possible cell types. Using the notation of Definition 3.2 we can assume $v = v_{i,\bar{j}} = \{\xi_i\} \times (\eta_j, \eta_{j+1})$ or $v = v_{\bar{i},j} = (\xi_i, \xi_{i+1}) \times \{\eta_j\}$. In an abuse of notation we write

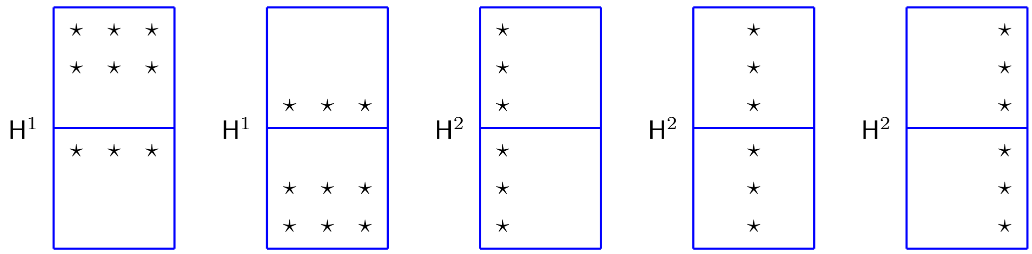
$$v_{i,\bar{j}} \in \Xi^n \quad \text{if} \quad \xi_i \in \Xi^n$$

and

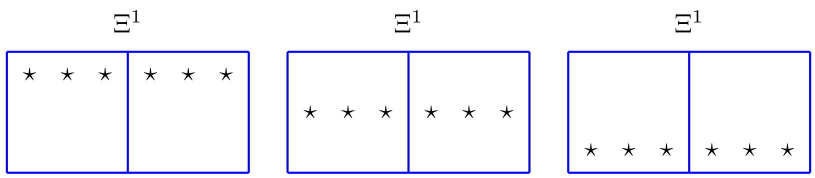
$$v_{\bar{i},j} \in \mathsf{H}^n \quad \text{if} \quad \eta_j \in \mathsf{H}^n$$

for $n = 1, 2$.

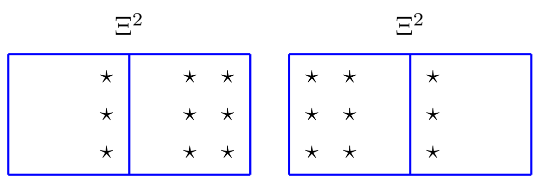
Proposition 3.4. *The following figures show the possible types of two cells that share a face v where it is indicated whether $v \in \mathsf{H}^n$ or $v \in \Xi^n$ for $n = 1, 2$.*



(i) *(ii)* *(iii)* *(iv)* *(v)*



(vi) *(vii)* *(viii)*



(ix) *(x)*

Proof. The arguments for all the cases are essentially the same so we only provide explicit proofs in two cases.

(i) Without loss of generality assume the lower cell is $\kappa(i,j)$. Then the adjacent cell is $\kappa(i,j+1)$. The stars indicate that $\kappa(i,j)$ is of type NW, W or NE. This implies that $\Phi_2(\kappa(i,j)) > \eta_{j+1}$. By assumption $v \in H^1$. Thus $\Phi_2(\kappa(i,j+1) = \Phi_2(\kappa(i,j)) > \eta_{j+1}$, and therefore, $\kappa(i,j+1)$ is of type NW, W, NE, W, A, or E, as indicated.

(vii) Without loss of generality assume the left cell is $\kappa(i,j)$. Then the adjacent cell is $\kappa(i+1,j)$. The stars indicate that $\kappa(i,j)$ is of type W, A or E. This implies that $\eta_j < \Phi_2(\kappa(i,j)) < \eta_{j+1}$. By assumption $v \in \Xi^1$. Thus $\Phi_2(\kappa(i+1,j) = \Phi_2(\kappa(i,j))$, and therefore, $\kappa(i+1,j)$ is of type W, A or E. $\square$

4. δ-Constrained Continuous Switching Systems

As indicated in the introduction the goal of this section is as follows: given a fixed switching system, $\Sigma = \Sigma(\Gamma, \Lambda, \Xi^1, \Xi^2, \mathsf{H}^1, \mathsf{H}^2)$, construct $f^{(\delta)}$ to define an associated δ-constrained continuous switching system (4).

Starting with the observation that

$$\lambda = \lambda(\Sigma) := \frac{1}{2} \min \left\{ \min_{i=1,\dots,I} \{\xi_i - \xi_{i-1}\}, \min_{j=1,\dots,J} \{\eta_j - \eta_{j-1}\} \right\}, \tag{20}$$

is half of the minimal width of any cell in Σ we choose

$$0 < \delta < \lambda.$$

For each $\kappa = \kappa(i,j) = (\xi_i, \xi_{i+1}) \times (\eta_j, \eta_{j+1}) \in \mathcal{K}$, $i = 1, \dots, I$, $j = 1, \dots, J$ define

$$\begin{aligned} G^2(\kappa) = G^2(i,j) &:= [\xi_i + \delta, \xi_{i+1} - \delta] \times [\eta_j + \delta, \eta_{j+1} - \delta] \\ G^2(0,0) &:= (0, \xi_1 - \delta] \times (0, \eta_1 - \delta] \\ G^2(i,0) &:= [\xi_i + \delta, \xi_{i+1} - \delta] \times (0, \eta_1 - \delta] \\ G^2(0,j) &:= (0, \xi_1 - \delta] \times [\eta_j + \delta, \eta_{j+1} - \delta]. \end{aligned} \tag{21}$$

For $i = 1, \dots, I$ and $j = 1, \dots, J$ define

$$\begin{aligned} G^1(v_{i,\bar{j}}) = G^1(i,\bar{j}) &:= [\xi_i - \delta, \xi_i + \delta] \times [\eta_j + \delta, \eta_{j+1} - \delta] \\ G^1(v_{\bar{i},j}) = G^1(\bar{i},j) &:= [\xi_i + \delta, \xi_{i+1} - \delta] \times [\eta_j - \delta, \eta_j + \delta] \\ G^1(v_{i,\bar{0}}) = G^1(i,\bar{0}) &:= [\xi_i - \delta, \xi_i + \delta] \times (0, \eta_1 - \delta] \\ G^1(v_{\bar{0},j}) = G^1(\bar{0},j) &:= (0, \xi_1 - \delta] \times [\eta_j - \delta, \eta_j + \delta] \end{aligned} \tag{22}$$

and let

$$G^0(\pi) := [\xi_i - \delta, \xi_i + \delta] \times [\eta_j - \delta, \eta_j + \delta] \tag{23}$$

where $\pi \in \Pi$ as defined in (10). We refer to these compact sets shown in Figure 2 as *tiles*, and more precisely G^i, $i = 0, 1, 2$ is called an *i-tile*.

We define the continuous nonlinearity $f^{(\delta)}\colon (0,\infty)^2 \to (0,\infty)^2$ in steps. Observe that Λ is constant on any given 2-tile G^2, and hence, $\Lambda(G^2)$ is a unique well defined vector. Thus we define $f^{(\delta)}\colon \cup_{i,j} G^2(i,j) \to (0,\infty)^2$ by

$$f^{(\delta)}(x) := \Lambda(G^2(i,j)) \quad x \in G^2(i,j). \tag{24}$$

To define the action of $f^{(\delta)}$ on the 1-tiles we consider four cases.

- If $\xi_i \in \Xi^1$, then $\Lambda_1(\kappa(i-1,j)) \neq \Lambda_1(\kappa(i,j))$ and $\Lambda_2(\kappa(i-1,j)) = \Lambda_2(\kappa(i,j))$. Thus we define

$$f_2^{(\delta)}(x) := \Lambda_2(\kappa(i,j)) \quad x \in G^1(i,\bar{j}). \tag{25}$$

  and choose $f_1^{(\delta)}$ to be a continuous function on $G^1(i,\bar{j})$ which agrees with $f_1^{(\delta)}$ as defined by (24) on $G^2(i-1,j) \cap G^1(i,\bar{j})$ and $G^1(i,\bar{j}) \cap G^2(i,j)$ with the constraint that for $x \in G^1(i,\bar{j})$

$$\min\{\Lambda_1(\kappa(i-1,j)), \Lambda_1(\kappa(i,j))\} \leq f_1^{(\delta)}(x) \leq \max\{\Lambda_1(\kappa(i-1,j)), \Lambda_1(\kappa(i,j))\}. \tag{26}$$

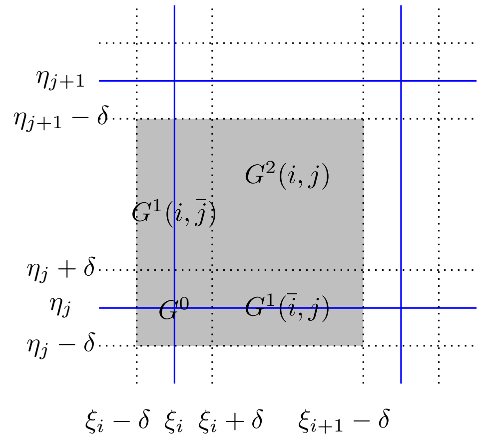


Figure 2: The shaded regions indicate the four tiles $G^2(i,j)$, $G^1(i,\bar{j})$, $G^1(\bar{i},j)$, and $G^0 = G^0(\pi)$ where $\pi = (\xi_i, \eta_j)$.

- If $\xi_i \in \Xi^2$, then $\Lambda_1(\kappa(i-1,j)) = \Lambda_1(\kappa(i,j))$ and $\Lambda_2(\kappa(i-1,j)) \neq \Lambda_2(\kappa(i,j))$. Thus we define

$$f_1^{(\delta)}(x) := \Lambda_1(\kappa(i,j)) \quad x \in G^1(i,\bar{j}). \tag{27}$$

and choose $f_2^{(\delta)}$ to be a continuous function on $G^1(i,\bar{j})$ which agrees with $f_2^{(\delta)}$ as defined by (24) on $G^2(i-1,j) \cap G^1(i,\bar{j})$ and $G^1(i,\bar{j}) \cap G^2(i,j)$ with the constraint that for $x \in G^1(i,\bar{j})$

$$\min\left\{\Lambda_2(\kappa(i-1,j)), \Lambda_2(\kappa(i,j))\right\} \leq f_2^{(\delta)}(x) \leq \max\left\{\Lambda_2(\kappa(i-1,j)), \Lambda_2(\kappa(i,j))\right\}. \tag{28}$$

- If $\eta_j \in \mathsf{H}^1$, then $\Lambda_1(\kappa(i,j-1)) \neq \Lambda_1(\kappa(i,j))$ and $\Lambda_2(\kappa(i,j-1)) = \Lambda_2(\kappa(i,j))$. Thus we define

$$f_2^{(\delta)}(x) := \Lambda_2(\kappa(i,j)) \quad x \in G^1(\bar{i},j). \tag{29}$$

and choose $f_1^{(\delta)}$ to a continuous function on $G^1(\bar{i},j)$ which agrees with $f_1^{(\delta)}$ as defined by (24) on $G^2(i,j-1) \cap G^1(\bar{i},j)$ and $G^1(\bar{i},j) \cap G^2(i,j)$ with the constraint that for $x \in G^1(\bar{i},j)$

$$\min\left\{\Lambda_1(\kappa(i,j-1)), \Lambda_1(\kappa(i,j))\right\} \leq f_1^{(\delta)}(x) \leq \max\left\{\Lambda_1(\kappa(i,j-1)), \Lambda_1(\kappa(i,j))\right\}. \tag{30}$$

- If $\eta_j \in \mathsf{H}^2$, then $\Lambda_1(\kappa(i,j-1) = \Lambda_1(\kappa(i,j))$ and $\Lambda_2(\kappa(i,j-1)) \neq \Lambda_2(\kappa(i,j))$. Thus we define

$$f_1^{(\delta)}(x) := \Lambda_1(\kappa(i,j)) \quad x \in G^1(\bar{i},j). \tag{31}$$

and choose $f_2^{(\delta)}$ to a continuous function on $G^1(\bar{i},j)$ which agrees with $f_2^{(\delta)}$ as defined by (24) on $G^2(i,j-1) \cap G^1(\bar{i},j)$ and $G^1(\bar{i},j) \cap G^2(i,j)$ with the constraint that for $x \in G^1(\bar{i},j)$

$$\min\left\{\Lambda_2(\kappa(i,j-1)), \Lambda_2(\kappa(i,j))\right\} \leq f_2^{(\delta)}(x) \leq \max\left\{\Lambda_2(\kappa(i,j-1)), \Lambda_2(\kappa(i,j))\right\}. \tag{32}$$

At this point we have defined $f^{(\delta)}\colon (0,\infty)^2 \setminus \left(\bigcup_{\pi\in\Pi} \operatorname{int}\left(G^0(\pi)\right)\right) \to (0,\infty)^2$. For each $\pi \in \Pi$ we extend $f^{(\delta)}$ to $G^0(\pi)$ continuously.

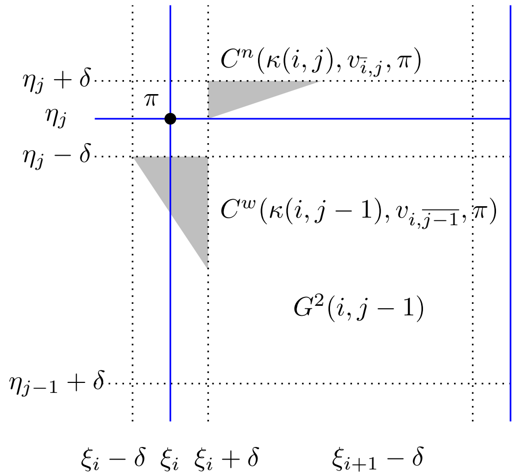


Figure 3: The shaded triangular regions indicate the wide chip $C^w(\kappa(i,j-1), v_{i,\overline{j-1}}, \pi)$ and the narrow chip $C^n(\kappa(i,j), v_{\bar{i},j}, \pi)$. Observe that $C^n(\kappa(i,j), v_{\bar{i},j}, \pi) \subset G^1(v_{\bar{i},j})$ and its edges lie in $G^2(\kappa(i,j))$ and $G^0(\pi)$ where $\pi = (\xi_i, \eta_j)$.

5. Cells, Chips, and Transversality

As described in the introduction we construct a lattice of trapping regions using tiles and chips. Tiles are introduced in the previous section. In this section we define chips, which are closed right triangular regions, and prove results concerning the transversality of the vector field of a δ-constrained switching system on the edges of tiles and the hypotenuse of chips. The transversality results are used in Section 7 to verify the construction of trapping regions.

A *chip* is a closed right triangular subset of a 1-tile $G^1(v)$ and is uniquely identified by the 2-tile $G^2(\kappa)$ and the 0-tile $G^0(\pi)$ that it intersects. With this in mind we denote a chip by

$$C^n(\kappa, v, \pi) \quad \text{or} \quad C^w(\kappa, v, \pi)$$

depending on whether it is a *narrow chip* or *wide chip*, respectively. If $v = v_{i,\bar{j}}$ or $v = v_{\bar{i},j}$, then the lengths of the edges of a narrow chip are δ and $(\eta_{j+1} - \eta_j)/2 - \delta$ or $(\xi_{i+1} - \xi_i)/2 - \delta$, respectively, while the lengths of the edges of a wide chip are 2δ and $(\eta_{j+1} - \eta_j)/2 - \delta$ or $(\xi_{i+1} - \xi_i)/2 - \delta$, respectively. Representative wide and narrow chips are shown in Figure 3; in the figure the vertical (horizontal) edge of the narrow chip has length δ $((\xi_{i+1} - \xi_i)/2 - \delta)$, respectively, and the vertical (horizontal) edge of the wide chip has length $(\eta_{j+1} - \eta_j)/2 - \delta$ (2δ), respectively.

As indicated above, we use the tiles and chips to construct trapping regions for δ-constrained switching systems. For this we need to know that the vector field associated with the δ-constrained switching system are transverse to the appropriate edges of the tiles and chips. We catalogue this information in the following sequence of propositions.

For the remainder of this section $\Sigma = \Sigma(\Gamma, \Lambda, \Xi^1, \Xi^2, \mathsf{H}^1, \mathsf{H}^2)$ is assumed to be a fixed switching system. Choose $\delta > 0$ such that

$$\delta < \delta_*(\lambda, \mu, \rho, \bar{\gamma}) := \min\left\{ \frac{\lambda\mu\bar{\gamma}}{\sqrt{2}(2\lambda + 3\rho)},\ \sqrt{\frac{\lambda\mu\bar{\gamma}}{32}} \right\}, \tag{33}$$

where $\mu = \mu(\Sigma)$, $\lambda = \lambda(\Sigma)$, $\rho = \rho(\Sigma)$, and $\bar{\gamma} = \bar{\gamma}(\Sigma)$ are defined as in (15), (20), (16), and (17), respectively. Note that the above condition implies $\delta < \lambda$, as $\bar{\gamma} \leq 1$ and $\mu \leq \rho$.

Proposition 5.1. *Consider $\kappa \in \mathcal{K}$. If $v \in \mathcal{V}_e(\kappa)$, then the vector field* (11) *is transverse in to $G^2(\kappa)$ along the associated edge.*

Proof. Let $\kappa = \kappa(i,j)$. Assume $v = v(i,\bar{j})$. Since $v \in \mathcal{V}_e(\kappa)$, κ is of type W, NE, A, E, S, or SE and $\Phi_1(\kappa) \geq \xi_i + \mu > \xi_i + \delta$. Thus the vector field (11) is transverse in to $G^2(\kappa)$ along the edge $\{\xi_i + \delta\} \times [\eta_j + \delta, \eta_{j+1} - \delta]$. The arguments for the remain faces are similar. □

Similar arguments lead to the following propositions.

Proposition 5.2. *Consider $G^1(v_{i,\bar{j}})$.*

(i) If $v_{\bar{i},j} \in \mathcal{V}_e(\kappa(i,j))$, then the vector field (11) *is transverse in to $G^1(v_{i,\bar{j}})$ along the edge $[\xi_i, \xi_i + \delta] \times \{\eta_j + \delta\}$.*

(ii) If $v_{\overline{i-1},j} \in \mathcal{V}_e(\kappa(i-1,j))$, then the vector field (11) *is transverse in to $G^1(v_{i,\bar{j}})$ along the edge $[\xi_i - \delta, \xi_i] \times \{\eta_j + \delta\}$.*

(iii) If $v_{\bar{i},j+1} \in \mathcal{V}_e(\kappa(i,j))$, then the vector field (11) *is transverse in to $G^1(v_{i,\bar{j}})$ along the edge $[\xi_i, \xi_i + \delta] \times \{\eta_{j+1} - \delta\}$.*

(iv) If $v_{\overline{i-1},j+1} \in \mathcal{V}_e(\kappa(i-1,j))$, then the vector field (11) *is transverse in to $G^1(v_{i,\bar{j}})$ along the edge $[\xi_i - \delta, \xi_i] \times \{\eta_{j+1} - \delta\}$.*

Proposition 5.3. *Consider $G^1(v_{\bar{i},j})$.*

(i) If $v_{i,\bar{j}} \in \mathcal{V}_e(\kappa(i,j))$, then the vector field (11) *is transverse in to $G^1(v_{\bar{i},j})$ along the edge $\{\xi_i + \delta\} \times [\eta_j, \eta_j + \delta]$.*

(ii) If $v_{i,\overline{j-1}} \in \mathcal{V}_e(\kappa(i,j-1))$, then the vector field (11) *is transverse in to $G^1(v_{\bar{i},j})$ along the edge $\{\xi_i + \delta\} \times [\eta_j - \delta, \eta_j]$.*

(iii) If $v_{i+1,\bar{j}} \in \mathcal{V}_e(\kappa(i,j))$, then the vector field (11) *is transverse in to $G^1(v_{\bar{i},j})$ along the edge $\{\xi_{i+1} - \delta\} \times [\eta_j, \eta_j + \delta]$.*

(iv) If $v_{i+1,\overline{j-1}} \in \mathcal{V}_e(\kappa(i,j-1))$, then the vector field (11) *is transverse in to $G^1(v_{\bar{i},j})$ along the edge $\{\xi_{i+1} - \delta\} \times [\eta_j - \delta, \eta_j]$.*

Proposition 5.4. *Consider $G^0(i,j)$.*

(i) If $\kappa(i,j)$ is of type NW, W, *or* SW, *then the vector field* (11) *is transverse in to $G^0(i,j)$ along the edge $\{\xi_i + \delta\} \times [\eta_j, \eta_j + \delta]$.*

(ii) If $\kappa(i,j)$ is of type SW, S, *or* SE, *then the vector field* (11) *is transverse in to $G^0(i,j)$ along the edge $[\xi_i, \xi_i + \delta] \times \{\eta_j + \delta\}$.*

(iii) If $\kappa(i-1,j)$ is of type SW, S, *or* SE, *then the vector field* (11) *is transverse in to $G^0(i,j)$ along the edge $[\xi_i - \delta, \xi_i] \times \{\eta_j + \delta\}$.*

(iv) If $\kappa(i-1,j)$ is of type NE, E, *or* SE, *then the vector field* (11) *is transverse in to $G^0(i,j)$ along the edge $\{\xi_i - \delta\} \times [\eta_j, \eta_j + \delta]$.*

(v) *If* $\kappa(i-1,j-1)$ *is of type* NE*,* E*, or* SE*, then the vector field* (11) *is transverse in to* $G^0(i,j)$ *along the edge* $\{\xi_i-\delta\}\times[\eta_j-\delta,\eta_j]$*.*

(vi) *If* $\kappa(i-1,j-1)$ *is of type* NW*,* W*, or* NE*, then the vector field* (11) *is transverse in to* $G^0(i,j)$ *along the edge* $[\xi_i-\delta,\xi_i]\times\{\eta_j-\delta\}$*.*

(vii) *If* $\kappa(i,j-1)$ *is of type* NW*,* W*, or* NE*, then the vector field* (11) *is transverse in to* $G^0(i,j)$ *along the edge* $[\xi_i,\xi_i+\delta]\times\{\eta_j-\delta\}$*.*

(viii) *If* $\kappa(i,j-1)$ *is of type* NW*,* W*, or* SW*, then the vector field* (11) *is transverse in to* $G^0(i,j)$ *along the edge* $\{\xi_i+\delta\}\times[\eta_j-\delta,\eta_j]$*.*

We now give a proposition which shows the transversality on the hypotenuse of a chip. Consider a δ-constrained continuous switching system (4) associated with a switching system $\Sigma = \Sigma(\Gamma,\Lambda,\Xi^1,\Xi^2,\mathsf{H}^1,\mathsf{H}^2)$, and suppose a narrow chip $C^n(\kappa,v,\pi)$ or a wide chip $C^w(\kappa,v,\pi)$ is introduced for Σ. Recall that $\delta>0$ is chosen to satisfy the condition (33).

Without loss of generality, one can assume, by applying rotation if necessary, that a chip C appears associated with $\kappa=\kappa(i,j),\ v=v_{\bar{i},j},\ \pi=(\xi_i,\eta_j)$. Let H be the hypotenuse of the chip C. Under these circumstance, the transversality results can be formulated as follows:

Proposition 5.5. *(i) If* $C=C^n(\kappa,v,\pi)$ *for* $\kappa=\kappa(i,j),\ v=v_{\bar{i},j},\ \pi=(\xi_i,\eta_j)$*,* $\kappa=\kappa(i,j)$ *is of type* A*,* W*,* NE*,* E*, and* $\kappa'=\kappa(i,j-1)$ *is of type* NW*, then the vector field* (4) *is transverse in to* C *along the hypotenuse* H*.*

(ii) If $C=C^w(\kappa,v,\pi)$ *for* $\kappa=\kappa(i,j),\ v=v_{\bar{i},j},\ \pi=(\xi_i,\eta_j)$*,* $\kappa=\kappa(i,j)$ *is of type* A*,* W*,* NE*, or* E*, and* $\kappa'=\kappa(i,j-1)$ *is of type* W *or* NE*, then the vector field* (4) *is transverse in to* C *along the hypotenuse* H*.*

To prove the above proposition, note that the vector field (4) can be rewritten as

$$\dot{x}=-\Gamma x+f^{(\delta)}(x)=-\Gamma(x-\Phi^{(\delta)}(x)).$$

From the above definition of $f^{(\delta)}(x)$, we see that $\Phi^{(\delta)}(x)=\Phi(\kappa)$ for any $x\in G^2(\kappa)$ with $\kappa=\kappa(i,j)$ or $\kappa'=\kappa(i,j-1)$. We first show that the transversality of the vector field (4) on H is reduced to the transversality of an affine vector field. Let $p(t)=(1-t)H_0+tH_1$ $(0\le t\le 1)$ where H_0 and H_1 are the end points of H, and let $\theta(t)$ be the angle (measured counter-clockwise from the positive direction of the x_1-axis) of the vector $V(x)=-\Gamma x+f^{(\delta)}(x)$ evaluated at $p(t)$. In other words,

$$V(p(t))=R(t)\cdot\begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}\quad (R(t)>0).$$

Observe that the types of $\kappa(i,j)$ and $\kappa(i,j-1)$ being considered and the definition of $f^{(\delta)}(x)$ on $G^1(v)$ imply that the angle $\theta(t)$ must satisfy $0<\theta(t)<\pi$ for any $t\in[0,1]$.

Let $\nu=r\cdot\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$ be a normal vector to H pointing upward, hence $\pi/2<\varphi<\pi$. The vector field (4) is transverse in to the chip C along H if and only if the inner product $V(p(t))\cdot\nu$ at any point of $p(t)\in H$ $(t\in[0,1])$ is positive. Let $\theta_{\min}$ be the minimum of $\theta(t)$, and let $t_{\min}\in[0,1]$ be such that $\theta(t_{\min})=\theta_{\min}$.

Lemma 5.6. *Let* $V_{\min}=V(p(t_{\min}))$*. If* $V_{\min}\cdot\nu>0$*, then* $V(p(t))\cdot\nu>0$ *for any* $t\in[0,1]$*.*

Proof. Observe that

$$V(p(t)) \cdot \nu = R(t) \cdot \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \cdot r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = R(t) r \cos(\theta(t) - \varphi),$$

where $-\pi < \theta_{\min} - \varphi \le \theta(t) - \varphi < \pi/2$. If $0 \le \theta(t) - \varphi < \pi/2$, then we immediately obtain the positivity of $V(p(t)) \cdot \nu$. So we consider the case $-\pi < \theta_{\min} - \varphi \le \theta(t) - \varphi < 0$. In this range, the cosine function is monotone increasing, and hence $\cos(\theta_{\min} - \varphi) \le \cos(\theta(t) - \varphi)$. Therefore, if $V_{\min} \cdot \nu > 0$, then we have $0 < \cos(\theta_{\min} - \varphi) \le \cos(\theta(t) - \varphi)$, and therefore $V(p(t)) \cdot \nu > 0$. □

Now define $\Phi^* = -\Gamma^{-1} V_{\min}$ and consider the affine vector field $V^*(x) = -\Gamma(x - \Phi^*)$. Observe that, from the definition of $f^{(\delta)}(x)$ on $G^1(v)$, $\Phi^* = \begin{pmatrix} \Phi_1^* \\ \Phi_2^* \end{pmatrix}$ satisfies

$$|\Phi_1^* - \xi| < \rho, \quad |\Phi_2^* - \eta| < \rho$$

for any $\xi \in \Xi,\ \eta \in \mathsf{H}$. Similarly, it also satisfies

$$\mu < |\Phi_2^* - \eta|$$

for $\eta = \eta_j$.

In order to prove $V_{\min} \cdot \nu > 0$, we first consider the narrow chip $C = C^n(\kappa, v, \pi)$.

i

In this case, we can use coordinate system with origin at (ξ_i, η_j) to evaluate

$$p(t) = (1-t)\begin{pmatrix} \delta \\ 0 \end{pmatrix} + t \begin{pmatrix} a/2 \\ \delta \end{pmatrix} = \begin{pmatrix} \delta + (a/2 - \delta)t \\ t\delta \end{pmatrix}$$

and one can choose $\nu = \begin{pmatrix} -\delta \\ a/2 - \delta \end{pmatrix}$, where $a = \xi_{i+1} - \xi_i (> 2\lambda)$. Therefore the inner product $V^*(p(t)) \cdot \nu$ defines a function $T(t, \delta)$ which is affine in $t \in [0,1]$ and quadratic in δ. More explicitly, we can obtain $T(t,\delta) = K(t)\delta^2 + L(t)\delta + M(t)$, where

$$K(t) = (1-t)\gamma_1 + t\gamma_2 \tag{34}$$
$$L(t) = (\gamma_1 - \gamma_2)(a/2)t - \gamma_1(\Phi_1^* - \xi_i) - \gamma_2(\Phi_2^* - \eta_j) \tag{35}$$
$$M(t) = (a/2)\gamma_2(\Phi_2^* - \eta_j) \tag{36}$$

In fact, M does not depend on t, $M(t) = M$, and is strictly positive. Also $K(t)$ is strictly positive because $\gamma_1, \gamma_2 > 0$ and $t \in [0,1]$. So the quadratic function $T(t,\delta) = K(t)\delta^2 + L(t)\delta + M$ is always positive for $\delta > 0$ if the discriminant $D = D(t) = L(t)^2 - 4K(t)M$ is negative, or if $D \ge 0$ and $L > 0$. So the only case where $T(t,\delta)$ can take a negative value is the case $D \ge 0$ and $L < 0$. In this case, we can easily see that $T(t,\delta) > 0$ for any $\delta \in (0, \delta_-(t))$, where $\delta_-(t)$ is a smaller root of the quadratic function $T(t,\delta) = 0$, namely

$$\delta_-(t) = \frac{-L(t) - \sqrt{D(t)}}{2K(t)}.$$

In order to obtain an estimate which does not depend on t, we observe that $\delta_-(t) > -M/L(t)$ in case $L(t) < 0$. This easily follows by comparing the graph of $T(t,\delta)$ and its tangent line at $\delta = 0$. Substituting the above and using the estimates (15), (16), (17), (20), we obtain

$$\begin{aligned}
-\frac{M}{L(t)} &= \frac{(a/2)\gamma_2(\Phi_2^* - \eta_j)}{-(\gamma_1-\gamma_2)(a/2)t + \gamma_1(\Phi_1^* - \xi_i) + \gamma_2(\Phi_2^* - \eta_j)} \\
&\geq \frac{(a/2)\gamma_2(\Phi_2^* - \eta_j)}{|\gamma_1-\gamma_2|(a/2) + \gamma_1(\Phi_1^* - \xi_i) + \gamma_2(\Phi_2^* - \eta_j)} \\
&= \frac{|\Phi_2^* - \eta_j|}{|(\gamma_1/\gamma_2) - 1| + (2/a)\{(\gamma_1/\gamma_2)|\Phi_1^* - \xi_i| + |\Phi_2^* - \eta_j|\}} \\
&\geq \frac{\mu}{|(1/\bar{\gamma}) - 1| + (1/\lambda)\{(1/\bar{\gamma})\rho + \rho\}} \\
&= \frac{\lambda\mu\bar{\gamma}}{(1-\bar{\gamma})\lambda + (1+\bar{\gamma})\rho} \\
&\geq \frac{\lambda\mu\bar{\gamma}}{\lambda + 2\rho}
\end{aligned}$$

Therefore, in the case of the narrow chip, if we choose δ so that $0 < \delta < \lambda\mu\bar{\gamma}/(\lambda+2\rho)$, then we can conclude that the vector field $V^*(x)$, and hence $V^{(\delta)}(x)$ as well, is transverse in to the chip along its hypotenuse.

For the wide chip, we can argue in the same manner, but the estimates become more complicated. Using

$$p(t) = (1-t)\begin{pmatrix}\delta \\ -\delta\end{pmatrix} + t\begin{pmatrix}a/2 \\ \delta\end{pmatrix} = \begin{pmatrix}\delta + (a/2-\delta)t \\ (2t-1)\delta\end{pmatrix}$$

and $\nu = \begin{pmatrix}-2\delta \\ a/2-\delta\end{pmatrix}$, one can similarly define

$$T(t,\delta) := V^*(p(t)) \cdot \nu = K(t)\delta^2 + L(t)\delta + M$$

where

$$\begin{aligned}
K(t) &= 2\gamma_1 - \gamma_2 - 2(\gamma_1-\gamma_2)t, && (37)\\
L(t) &= (\gamma_1-\gamma_2)at + (a/2)\gamma_2 - 2\gamma_1(\Phi_1^* - \xi_i) - \gamma_2(\Phi_2^* - \eta_j), && (38)\\
M(t) &= (a/2)\gamma_2(\Phi_2^* - \eta_j). && (39)
\end{aligned}$$

As in the case of a narrow chip, M does not depend on t and is positive. However, $K(t)$ can change its sign. In the case $K(t) > 0$, the same argument works, and we obtain that $-M/L(t)$ gives a bound for δ in case $L(t) < 0$, otherwise $T(t,\delta) > 0$ for any $\delta > 0$.

Observe that $K(t)$ vanishes at some $t_0 \in [0,1]$ only when $\gamma_2 \geq 2\gamma_1$. If $K(t_0) = 0$, $T(t_0,\delta) = 0$ when $\delta = -M/L(t_0)$ and

$$L(t_0) = (\gamma_1-\gamma_2)at_0 + (a/2)\gamma_2 - 2\gamma_1(\Phi_1^* - \xi_i) - \gamma_2(\Phi_2^* - \eta_j) = (\gamma_1-\gamma_2)a - 2\gamma_1(\Phi_1^* - \xi_i) - \gamma_2(\Phi_2^* - \eta_j) < 0,$$

so the upper bound of δ for $T(t_0,\delta) > 0$ is $\delta < -M/L(t_0)$ which is the same as before, or more precisely,

$$
\begin{aligned}
-\frac{M}{L(t_0)} &= \frac{(a/2)\gamma_2(\Phi_2^* - \eta_j)}{(\gamma_2-\gamma_1)a + 2\gamma_1(\Phi_1^* - \xi_i) + \gamma_2(\Phi_2^* - \eta_j)} \\
&= \frac{|\Phi_2^* - \eta_j|}{2|1-(\gamma_1/\gamma_2)| + (2/a)\{2(\gamma_1/\gamma_2)|\Phi_1^* - \xi_i| + |\Phi_2^* - \eta_j|\}} \\
&> \frac{\lambda\mu\bar{\gamma}}{2(1-\bar{\gamma})\lambda + (2+\bar{\gamma})\rho} \\
&\geq \frac{\lambda\mu\bar{\gamma}}{2\lambda + 3\rho}.
\end{aligned}
$$

In the case $K(t) < 0$, which occurs when $\gamma_2 > 2\gamma_1$ and $0 < t < t_0$, we have $T(t,\delta) > 0$ if $0 < \delta < \delta_+(t) = (-L(t) - \sqrt{D(t)})/2K(t)$, where $\delta_+(t)$ is a larger root of the quadratic equation $T(t,\delta) = 0$. Using the inequality

$$
\sqrt{x+h} \geq \sqrt{x} + \frac{1}{2}\frac{h}{\sqrt{x+h}} \quad (x > 0, h > 0)
$$

we have, independent of whether $L(t) > 0$ or not,

$$
\delta_+(t) \geq \frac{L(t) + |L(t)| + \frac{1}{2}\frac{4(-K(t))M}{\sqrt{L(t)^2 + 4(-K(t))M}}}{2(-K(t))} \geq \frac{M}{\sqrt{L(t)^2 + 4(-K(t))M}}.
$$

Since $L(t)^2 + 4(-K(t))M < \max\{2L(t)^2, 2 \times 4(-K(t))M\}$, we finally obtain the estimate

$$
\delta_+(t) \geq \min\left\{\frac{M}{\sqrt{2}|L(t)|},\ \frac{1}{2\sqrt{2}}\sqrt{\frac{M}{-K(t)}}\right\}
$$

The first term in the min can be treated exactly the same way, up to the constant $1/\sqrt{2}$. The second term can be treated as follows:

$$
\frac{M}{-K(t)} > \frac{(a/2)\gamma_2(\Phi_2^* - \eta_j)}{|2\gamma_1 - \gamma_2| + 2|\gamma_1 - \gamma_2|} = \frac{(a/2)(\Phi_2^* - \eta_j)}{|2(\gamma_1/\gamma_2) - 1| + 2|(\gamma_1/\gamma_2) - 1|} > \frac{\lambda\mu\bar{\gamma}}{(2-\bar{\gamma}) + 2(1-\bar{\gamma})} > \frac{\lambda\mu\bar{\gamma}}{4}
$$

Putting all the above estimates together, for both the narrow chip case and the wide chip case, we define

$$
\delta_* = \delta_*(\lambda, \mu, \rho, \bar{\gamma}) := \min\left\{\frac{\lambda\mu\bar{\gamma}}{\sqrt{2}(2\lambda + 3\rho)},\ \sqrt{\frac{\lambda\mu\bar{\gamma}}{32}}\right\}. \tag{40}
$$

The above argument proves that, for any δ with $0 < \delta < \delta_*$, the vector field $V^*(x)$, and hence $V^{(\delta)}(x)$ as well, is transverse in to the (both narrow and wide) chip along its hypotenuse. This completes the proof of Proposition 5.5.

6. State Transition Diagram

Definition 6.1. Given a cell κ the directed *κ-graph*, $\mathcal{F}_\kappa \colon \mathcal{V}_e(\kappa) \rightrightarrows \mathcal{V}_a(\kappa)$, is defined by

$$
v \in \mathcal{F}(u) \quad \text{if and only if} \quad v \in \mathcal{V}_a(\kappa) \text{ and } u \in \mathcal{V}_e(\kappa).
$$

In addition, if $w_\kappa \in \mathcal{V}(\kappa)$, then $\mathcal{F}_\kappa(w_\kappa) = w_\kappa$.

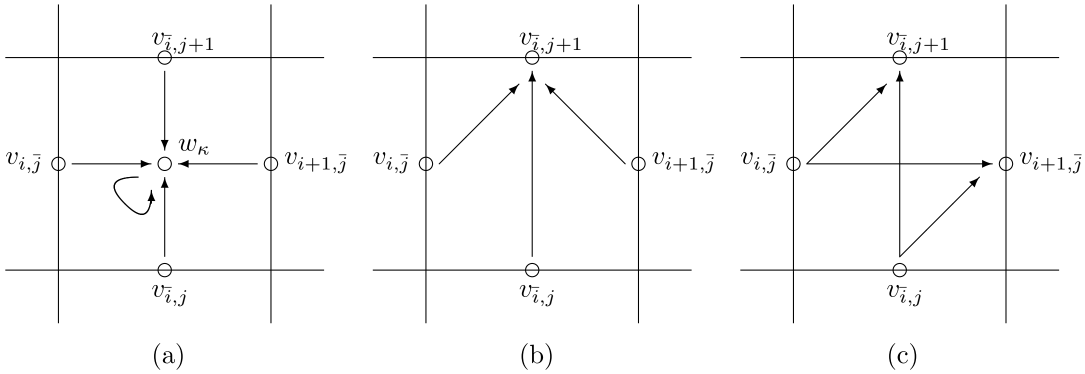


Figure 4: The directed graphs or multivalued maps $\mathcal{F}_\kappa \colon \mathcal{V}_e(\kappa) \rightrightarrows \mathcal{V}_a(\kappa)$ for $\kappa = \kappa(i,j)$. (a) κ is attracting and hence of type A. (b) κ is focusing of type W. (c) κ is translating cell of type NE.

To give some geometric perspective to these definitions the reader is referred to Figure 4 where the directed graphs $\mathcal{F}_\kappa$ are shown for various types of cells κ. The following proposition, whose proof is left to the reader, relates the definitions of entrance and absorbing faces to the dynamics of the associate κ-equation.

Proposition 6.2. *Let $v \in \mathcal{V}_e(\kappa)$. For every $x \in v$, there exists a unique $t = t(x) \geq 0$ and a $v' \in \mathcal{V}_a(\kappa)$ such that*

$$\psi_\kappa(t,x) \in \mathrm{cl}(v') \quad \textit{or} \quad \lim_{t\to\infty} \psi_\kappa(t,v) = \Phi(\kappa).$$

In the latter case, κ is an attracting cell and $\mathcal{F}_\kappa(v) = w_\kappa$.

Definition 6.3. Given a switching system Σ the associated state transition diagram $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$ is the directed graph on the set

$$\mathcal{V} = \bigcup_{\kappa\in\mathcal{K}} \mathcal{V}(\kappa)$$

of vertices which was introduced just after Definition 3.2. Note that we consider two vertices $v \in \mathcal{V}(\kappa)$ and $w \in \mathcal{V}(\kappa')$ are the same if $\kappa \cap \kappa'$ intersect in a 1-dimensional segment that correspond to both v and w.

We label the possible configurations of $\mathcal{F}_\kappa$ and $\mathcal{F}_{\kappa'}$ under the assumption that κ and κ' share a common face.

Definition 6.4. Consider the state transition diagram $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$ for a switching system. Let $v \in \mathcal{V}$. If $v = w_{i,j} \in \mathcal{V}(\kappa(i,j))$, then v is called a *minimal vertex*. Assume $v \in \mathcal{V}(\kappa) \cap \mathcal{V}(\kappa')$. If there exists $u \in \mathcal{V}(\kappa)$ and $w \in \mathcal{V}(\kappa')$ such that $u \to v \to w$, then v is a *transparent vertex*. If there exists $u \in \mathcal{V}(\kappa)$ and $w \in \mathcal{V}(\kappa')$ such that $u \to v$ and $w \to v$, then v is a *black vertex*. If there exists $u \in \mathcal{V}(\kappa)$ and $w \in \mathcal{V}(\kappa')$ such that $v \to u$ and $v \to w$, then v is a *white vertex*. The sets of minimal, transparent, black, and white vertices are denoted by $\mathcal{M}$, $\mathcal{T}$, $\mathcal{B}$, and $\mathcal{W}$, respectively.

Proposition 6.5. *Consider a state transition diagram $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$ for a switching system. The minimal, transparent, black, and white vertices partition $\mathcal{V}$.*

Proof. We need to demonstrate that $\mathcal{M} \cup \mathcal{T} \cup \mathcal{B} \cup \mathcal{W} = \mathcal{V}$ and that $\mathcal{M}$, $\mathcal{T}$, $\mathcal{B}$, and $\mathcal{W}$ are mutually disjoint. By definition $\mathcal{M} \cap (\mathcal{T} \cup \mathcal{B} \cup \mathcal{W}) = \emptyset$. Thus we can restrict our attention to vertices that are associated with faces of cells.

Observe that from Figure 4 if $v \in \mathcal{V}(\kappa)$, then there is either an edge to v or an edge from v in $\mathcal{F}_\kappa$, but not both. The first statement follows from the existence of the edges. The second statement follows from the fact that it is not possible to have both types of edges within one cell. □

For the remainder of the paper we make the following assumption

$\not\mathbf{B}$ The state transition diagram $\mathcal{F} \colon \mathcal{V} \rightrightarrows \mathcal{V}$ does not contain a black vertex.

We hasten to add that this is not an unreasonable assumption. To understand why, consider the following result that is easily checked.

Lemma 6.6. *If $v(i, \bar{j})$ is a black vertex, then*

$$\Lambda_1(\kappa(i-1, j)) > \gamma_1 \xi_i > \Lambda_1(\kappa(i, j)).$$

Observe that starting with a regulatory network this implies that x_1 has a self-edge that corresponds to repression. For many biological applications this type of self-regulation is better modeled by two nodes x_1 and y where x_1 activates y and y represses x_1.

From a mathematical perspective, Lemma 6.6 can be used to show that given a system of the form (1) it is possible to approximate the nonlinear functions f_1 using piecewise constant functions in such a way that the state transition graph for resulting switching system does not contain black walls. This approach is discussed in [13] and in future work [11].

We conclude this section with three results concerning the structure of forward invariant sets under $\mathcal{F}$.

Proposition 6.7. *Consider $\mathcal{A} \in \mathsf{Att}(\mathcal{V}, \mathcal{F})$ under assumption $\not\mathbf{B}$. If $v \in \mathcal{A}$, then $v \in \mathcal{M} \cup \mathcal{T}$. Furthermore, if $v \in \mathcal{T}$, then there exist distinct $u, w \in \mathcal{A}$ such that $u \to v \to w$.*

Proof. By Proposition 6.5 and assumption $\not\mathbf{B}$, it is sufficient to show that $v \notin \mathcal{W}$. If $v \in \mathcal{W}$, then $\mathcal{F}^t(v) = \emptyset$, hence $v \notin \mathcal{A}$, a contradiction.

Since $v \in \mathcal{A}$, by definition there exists $u \in \mathcal{F}^t(v) \cap \mathcal{A}$. If $v \in \mathcal{T}$, then v does not have a self-edge hence $u \neq v$. By definition, if $v \in \mathcal{T}$, then there exists $w \in \mathcal{V}$ such that $v \to w$. This implies that $w \in \mathcal{F}(\mathcal{A})$. By definition of a forward invariant set $\mathcal{F}(\mathcal{A}) \subset \mathcal{A}$ and hence $w \in \mathcal{A}$. Since $v \to w$, $w \in \mathcal{A}$. □

Proposition 6.8. *Let $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$. If $\mathcal{V}(\kappa) \cap \mathcal{A} \cap \mathcal{T} \neq \emptyset$, then $\mathcal{V}_e(\kappa) \cap \mathcal{A} \neq \emptyset$ and $\mathcal{V}_a(\kappa) \subset \mathcal{A}$.*

Proof. Let $v \in \mathcal{V}(\kappa) \cap \mathcal{A} \cap \mathcal{T}$. By Proposition 6.7 there exist distinct $u, w \in \mathcal{A}$ such that $u \to v \to w$. If $v \in \mathcal{V}_e(\kappa)$, then $\mathcal{V}_a(\kappa) = \mathcal{F}_\kappa(v)$ and hence $\mathcal{V}_a(\kappa) \subset \mathcal{A}$. If $v \in \mathcal{V}_a(\kappa)$, then $u \in \mathcal{A} \cap \mathcal{V}_e(\kappa)$ and $\mathcal{F}_\kappa(u) = \mathcal{V}_a(\kappa)$. Therefore, $\mathcal{V}_a(\kappa) \subset \mathcal{A}$. □

Proposition 6.9. *Consider $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$ under assumption $\not\mathbf{B}$. If $v \in \mathcal{A} \cap \mathcal{V}(\kappa)$, then*

$$\mathcal{V}_a(\kappa) \subset \mathcal{A}.$$

Furthermore, if $\mathcal{A} \cap \mathcal{V}(\kappa) \neq \{w_\kappa\}$, then $\mathcal{V}_e(\kappa) \cap \mathcal{A} \neq \emptyset$.

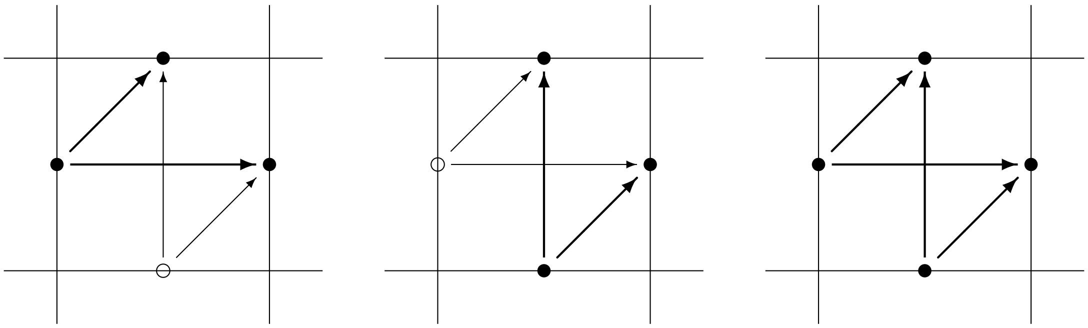


Figure 5: Possible intersections of $\mathcal{A}$ with a translation cell. Vertices of $\mathcal{A}$ are indicated as solid circles.

Proof. First, assume $v \in \mathcal{V}_e(\kappa)$. Then, $\mathcal{V}_a(\kappa) \subset \mathcal{F}_\kappa(v)$, and hence, by Proposition 6.8 $\mathcal{V}_a(\kappa) \subset \mathcal{A}$.

Now assume $v \in \mathcal{V}_a(\kappa)$. To indicate the line of reasoning we provide the proofs of two of the three cases. Assume κ is an attracting cell. This implies that $\mathcal{V}_a(\kappa) = \{w_\kappa\} = \{v\} \subset \mathcal{V}_a(\kappa)$. Assume κ is a translating cell. By Proposition 6.7 there exists $u, w \in \mathcal{N}\mathbb{A}$ such that $u \to v \to w$. Since $v \in \mathcal{V}_a(\kappa)$, by $\nexists$ $u \in \mathcal{V}(\kappa)$. Furthermore, $u \to v$ implies that $u \in \mathcal{V}_e(\kappa)$. Thus by Proposition 6.8 $\mathcal{F}_\kappa(u) = \mathcal{V}_a(\kappa) \subset \mathcal{A}$. □

7. Constructing Trapping Regions for δ-Continuous Switching Systems

Our goal is to define a function $N\colon \mathsf{Att}(\mathcal{F}) \to \mathsf{AB}(\varphi)$ where φ is a flow generated by

$$\dot{x} = -\Gamma x + f^{(\delta)}(x). \tag{41}$$

As is made clear below, given $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, the region $N(\mathcal{A})$ is constructed locally using the above defined tiles and chips. Furthermore, it is shown that at all points of the boundary of $N(\mathcal{A})$ the vector field given by (41) $N(\mathcal{A})$ points strictly into the interior of $N(\mathcal{A})$, which implies that $N(\mathcal{A})$ is a trapping region.

Definition 7.1. For each (i,j), $i = 1, \dots, I$, $j = 1, \dots, J$ the associated *elementary domain* is defined to be

$$E(i,j) := (\xi_{i-1} + \delta, \xi_{i+1} - \delta) \times (\eta_{j-1} + \delta, \eta_{j+1} - \delta)\,.$$

The set of associated vertices is denoted by $\mathcal{E}(i,j) \subset \mathcal{V}$ and is defined to be the union of vertices for which the associated face v satisfies $v \cap E(i,j) \neq \emptyset$ or a face w_κ where $\kappa \cap E(i,j) \neq \emptyset$. We use $\bar{\mathcal{E}}(i,j)$ to denote the union of vertices in $\mathcal{E}(i,j)$ and the vertices associated with the faces of the cells in $[\xi_{i-1}, \xi_{i+1}] \times [\eta_{j-1}, \eta_{j+1}]$.

We leave it to the reader to check that

$$(\delta, \infty)^2 = \bigcup_{\substack{i=1,\dots,I \\ j=1,\dots,J}} E(i,j) \tag{42}$$

Fix an elementary domain $E(i,j)$. In applying the following rules, unless otherwise specified, it is assumed that the cells κ, the faces v, and $\pi = (\xi_i, \eta_j) \in \Pi$ all intersect $E(i,j)$ nontrivially.

For all of the following rules $\mathcal{A}$ denotes an element of $\mathsf{Att}(\mathcal{F})$.

Rule 0 If $w_\kappa \in \mathcal{A} \cap \mathcal{M} \cap \mathcal{E}(i,j)$, then

$$G^2(\kappa) \subset N(\mathcal{A}).$$

Rule 1 Let $u \in \mathcal{A} \cap \mathcal{T} \cap \mathcal{E}(i,j)$ and let $r, s \in \bar{\mathcal{E}}(i,j)$ such that $r \in \mathcal{A}$ and $r \to u \to s$. Consider the grid elements κ_r and κ_s such that $r, u \in \mathcal{V}(\kappa_r)$ and $u, s \in \mathcal{V}(\kappa_s)$. Then

$$G^2(\kappa_r) \cup G^2(\kappa_s) \cup G^1(u) \subset N(\mathcal{A}).$$

Rule 2 Assume

$$G^2(\kappa_\alpha) \cup G^2(\kappa_\beta) \cup G^2(\kappa_\gamma) \cup G^1(v_{\alpha\beta}) \cup G^1(v_{\beta\gamma}) \subset N(\mathcal{A}),$$

where $v_{\alpha\beta} = \kappa_\alpha \cap \kappa_\beta$ and $v_{\beta\gamma} = \kappa_\beta \cap \kappa_\gamma$. If $v_{\alpha\beta} \in \mathcal{V}_a(\kappa_\beta)$ or $v_{\beta\gamma} \in \mathcal{V}_a(\kappa_\beta)$, then

$$G^0(\pi) \subset N(\mathcal{A}).$$

Rule 3 Assume $G^2(\kappa) \cup G^1(v_\alpha) \cup G^0(\pi) \subset N(\mathcal{A})$ where $v_\alpha \in \mathcal{V}(\kappa) \cap \mathcal{E}(i,j)$. If $\mathcal{V}(\kappa) \cap \mathcal{E}(i,j) \subset \mathcal{V}_e(\kappa)$, then

$$C^n(\kappa, v_\beta, \pi) \subset N(\mathcal{A})$$

where $v_\beta \in \mathcal{V}(\kappa) \cap \mathcal{E}(i,j)$.

Rule 4 Assume $G^0(\pi) \subset N(\mathcal{A})$ and $G^2(\kappa_\alpha) \not\subset N(\mathcal{A})$. Let $\{v_{\alpha\beta}, v_{\alpha\gamma}\} = \mathcal{V}(\kappa_\alpha) \cap \mathcal{E}(i,j)$. If $v_{\alpha\beta} \notin \mathcal{V}_a(\kappa)$, then

$$C^w(\kappa_\gamma, v_{\alpha\gamma}, \pi) \subset N(\mathcal{A})$$

where $v_{\alpha\gamma} = \kappa_\alpha \cap \kappa_\gamma$.

Rule 5 Assume $G^2(\kappa_\alpha) \cup G^2(\kappa_\beta) \cup G^0(\pi) \subset N(\mathcal{A})$ and $v_{\alpha\beta} = \kappa_\alpha \cap \kappa_\beta$. If $\mathcal{V}(\kappa_\alpha) \cap \mathcal{E}(i,j) \subset \mathcal{V}_e(\kappa_\alpha)$ and $\mathcal{V}(\kappa_\beta) \cap \mathcal{E}(i,j) \subset \mathcal{V}_e(\kappa_\beta)$, then

$$G^1(v_{\alpha\beta}) \subset N(\mathcal{A}).$$

Definition 7.2. For each $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$ define $N(\mathcal{A}) \subset (0,\infty)^2$ to be the union of the minimal collection of tiles and chips that satisfies **Rules 0 - 5** over all elementary domains $E(i,j)$, $i = 1, \dots I$, $j = 1, \dots, J$.

For future reference we highlight the following remarks.

Lemma 7.3. *If $\mathcal{A} = \emptyset$, then $N(\mathcal{A}) = \emptyset$.*

Lemma 7.4. *If* **Rule 5** *is applied, then $v_{\alpha\beta} \in \mathcal{W}$.*

Proposition 7.5. *If $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, then $N(\mathcal{A}) \in \mathsf{AB}(\varphi)$.*

The goal for the remainder of this section is to prove Proposition 7.5. We will do this by explicitly proving that for each $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, $N(\mathcal{A})$ is a trapping region. This in turn is done by considering all possible forms of intersection of $N(\mathcal{A})$ with all possible elementary domains $E(i,j)$ and checking for transversality as we proceed. For this we make use of the following objects.

Definition 7.6. Let e denote a boundary edge of a 2-tile $G^2(\kappa)$, a 1-tile $G^1(v)$, or a 0-tile $G^0(i,j)$. We say that e is *interior* to $E(i,j)$ if

$$e \cap E(i,j) \neq \emptyset$$

and *exterior* to $E(i,j)$ if e is not interior to $E(i,j)$, but

$$e \subset \mathrm{cl}(E(i,j)).$$

A boundary edge e of a chip is *interior* to $E(i,j)$ if $e \cap E(i,j) \neq \emptyset$.

As the following lemma indicates, for a set R to be a trapping region, it is sufficient to show that for every $E(i,j)$ along any boundary edge of $R \subset [0,\infty)^2$ that is interior to $E(i,j)$ the vector field of (41) is transverse in with respect to R.

Lemma 7.7. *If $R \subset [0,\infty)^2$ is a union of tiles and chips and for every elementary domain $E(i,j)$, $i = 1,\ldots,I$, $j = 1,\ldots,J$, the interior edges of R with respect to $E(i,j)$ are transverse in, then R is a trapping region.*

Proof. By (42) the collection of elementary domains covers the phase space $(0,\infty)^2$. By definition $\delta < \lambda$ which is half the minimal cell width (see 20) and thus every point $x \in (0,\infty)^2$ lies in the interior of some $E(i,j)$. Thus if the interior edges of R with respect to $E(i,j)$ are transverse in for all elementary domains, then every boundary point of R is transverse in. Therefore R is a trapping region. □

To simplify the application of the rules we use the symmetry of the elementary domain. By Proposition 6.7, if $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, then $\mathcal{A} \subset \mathcal{M} \cup \mathcal{T}$. Assume that there exists $v \in \mathcal{A} \cap \mathcal{T}$. This implies that v represents the intersection of two cells κ_1 and κ_2. Since $v \in \mathcal{T}$, there exists $u, w \in \mathcal{V}$ such that $u \to v \to w$. Without loss of generality we assume that $u \in \mathcal{V}(\kappa_2)$ and $w \in \mathcal{V}(\kappa_1)$. Symmetry, i.e. a reflection or rotation, allows us without loss of generality to assume that $\kappa_2 = \kappa(i-1,j)$ and $\kappa_1 = \kappa(i,j)$ and thus we will refer to v as a transparent face moving east. We will study the implications of **Rules 0 - 5** on the elementary domain $E(i,j)$ shown in Figure 6 where the neighboring cells of interest are $\kappa_3 = \kappa(i-1,j-1)$ and $\kappa_4 = \kappa(i,j-1)$.

We begin by making simple observations concerning **Rules 0 - 5**. If $\mathcal{A} \cap \mathcal{T} \cap \mathcal{E}(i,j) \neq \emptyset$, then **Rule 1** implies that

$$G^2(\kappa_1) \cup G^2(\kappa_2) \cup G^1(v) \subset N(\mathcal{A}) \tag{43}$$

as indicated in Figure 6(b).

Rule 2 determines if G^0 tiles belong to $N(\mathcal{A})$. Therefore, a necessary condition for $G^0(i,j) \subset N(\mathcal{A})$ is that at least three of the four G^2 and at least two of the four G^1 tiles in the elementary domain $E(i,j)$ belong to $N(\mathcal{A})$. Observe that **Rule 4** is only applicable if given an elementary domain E exactly three of the four associated G^2 tiles belong to $N(\mathcal{A})$. Furthermore, if **Rule 3** implies that $C^n(\kappa, v_\beta, (i,j)) \subset N(\mathcal{A})$ and **Rule 4** implies that $C^w(\kappa, v_\beta, (i,j)) \subset N(\mathcal{A})$, then we can ignore **Rule 3** since

$$C^n(\kappa, v_\beta, (i,j)) \subset C^w(\kappa, v_\beta, (i,j)) \subset N(\mathcal{A}).$$

Observe that if **Rule 5** applies then, **Rule 3** applies to κ_1 and κ_4. However

$$C^n(\kappa_n, v_1, \pi) \subset G^1(v_1), \quad n = 1,4$$

and hence we can ignore **Rule 3**.

Proposition 7.8. *Let $\{v_{\alpha\beta}\} = \mathcal{V}(\kappa_\alpha) \cap \mathcal{V}(\kappa_\beta)$ where $\kappa_n \cap E(i,j) \neq \emptyset$, $n = \alpha, \beta$. If $G^2(\kappa_\alpha) \not\subset N(\mathcal{A})$, then $G^1(v_{\alpha\beta}) \not\subset N(\mathcal{A})$.*

Proof. Only **Rule 1** and **Rule 5** require that $G^1(v_{\alpha\beta}) \subset N(\mathcal{A})$. Both these rules are based on $G^2(\kappa_\alpha) \subset N(\mathcal{A})$. Thus the minimality of $N(\mathcal{A})$ implies that $G^1(v_{\alpha\beta}) \not\subset N(\mathcal{A})$. □

Proposition 7.9. *If $\mathcal{A} = \{\kappa_k \mid k = 1, \ldots, K\} \subset \mathcal{M}$, then $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$,*

$$N(\mathcal{A}) = \bigcup_{k=1,\ldots,K} G^2(\kappa_k),$$

and $N(\mathcal{A})$ is a trapping region.

Proof. By **Rule 0**

$$\bigcup_{k=1,\ldots,K} G^2(\kappa_k) \subset N(\mathcal{A}).$$

We now show that this is the minimal collection of tiles and chips that satisfy **Rules 0 - 5**. Given that $\mathcal{A} \cap \mathcal{T} = \emptyset$, the only way to require the existence of a G^1 tile is through **Rule 5**. However, **Rule 5** requires the existence of $G^0(i,j)$. The requirement for the existence of $G^0(i,j)$ follows from **Rule 2**, which in turn requires the existence of a G^1 tile. Thus,

$$N(\mathcal{A}) = \bigcup_{k=1,\ldots,K} G^2(\kappa_k). \tag{44}$$

For each κ_k, $k = 1, \ldots, K$, each face is identified with a vertex that belongs to $\mathcal{V}_e(\kappa_k)$. Thus, by Proposition 5.1, $G^2(\kappa_k)$ is a trapping region and hence $N(\mathcal{A})$ is a trapping region. □

For the remainder of the argument we assume that $\mathcal{A} \not\subset \mathcal{M}$ or equivalently that $\mathcal{A} \cap \mathcal{T} \neq \emptyset$. Furthermore, taking advantage of the above described symmetry we always assume that $v = v_{i,\bar{j}} \in \mathcal{A} \cap \mathcal{T}$ is transparent east.

Proposition 7.10. *Under the assumption that $v = v_{i,\bar{j}} \in \mathcal{A} \cap \mathcal{T}$ the cells that intersect $E(i,j)$ must be of the types indicated in Figure 6(a) and as indicated in Figure 6(b) it must be the case that*

$$G^2(\kappa_1) \cup G^2(\kappa_2) \cup G^1(v) \subset N(\mathcal{A}).$$

Proof. The assumption that $v = v_{i,\bar{j}}$ is transparent east implies that $\Phi_1(\kappa_1) > \xi_i$ and $\Phi_1(\kappa_2) > \xi_i$. Thus, κ_1 is of type N, A, S, NE, E, or SE and κ_2 is of type NE, E, or SE.

Rule 1 implies that $G^2(\kappa_1) \cup G^2(\kappa_2) \cup G^1(v) \subset N(\mathcal{A})$. □

Proof of Proposition 7.5. 1. *Assume $v_1 \in \mathcal{A}$ and v_1 is transparent north.* By **Rule 1**, $G^2(\kappa_4) \cup G^1(v_1) \subset N(\mathcal{A})$. The possible cell types are indicated in Figure 7(a).

(a) *Assume $v_2 \notin \mathcal{A}$.* By Proposition 6.9 κ_2 cannot be of type SE and thus by Figure 7(a) must be of type E or NE. By Proposition 3.4(i) if $\eta_j \in \mathsf{H}^1$, then κ_2 being of type E or NE implies that κ_3 is of type NW, N or NE. Similarly, by Proposition 3.4(v) if $\eta_j \in \mathsf{H}^2$, then κ_3 is of type NE, E, or SE. The possible cell types are indicated in Figure 7(c). Since $v \notin \mathcal{V}_e(\kappa_2)$, **Rule 3** does not apply to κ_2.

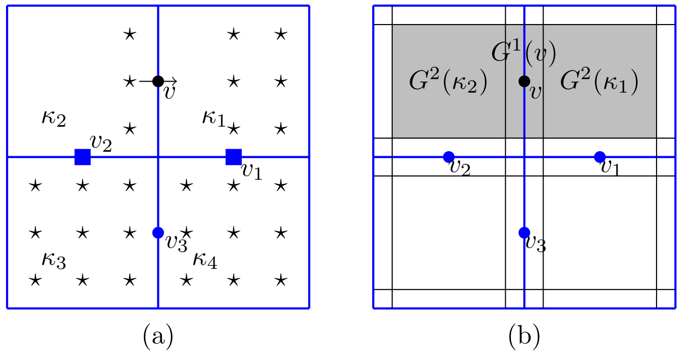


Figure 6: Black filled dot indicates the vertex that belongs to $\mathcal{A}$. Blue filled dots indicate vertices that may or may not belong to $\mathcal{A}$. (a) The vertex v is a transparent face moving east. The associated four cells are κ_i, $i = 1, \ldots, 4$. The possible cell types of κ_1 and κ_2 are indicated by the stars, e.g. center is A, upper right corner is NE, middle right is E. Without further assumptions there are no restrictions on the cell types of κ_3 and κ_4. (b) Shaded region indicates tiles that belong to $N(\mathcal{A})$ under the assumption that $\mathcal{A} \cap \mathcal{T} \neq \emptyset$.

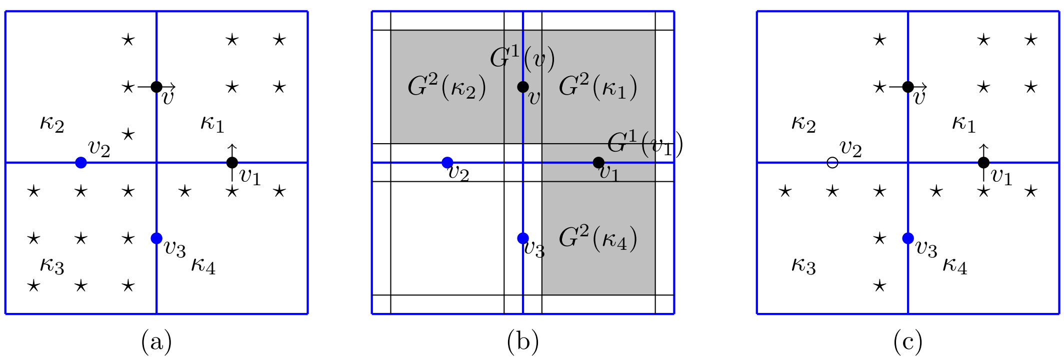


Figure 7: Unfilled dot implies that vertex does not belong to $\mathcal{A}$. (a) Possible cells types in the setting of Case 1. (b) Minimal set of tiles in $N(\mathcal{A})$ in Case 1. (c) Possible cell types in the setting of Case 1(a).

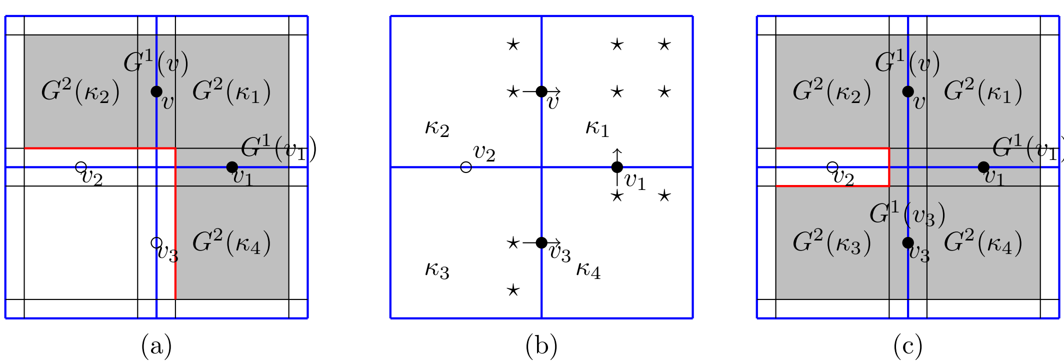
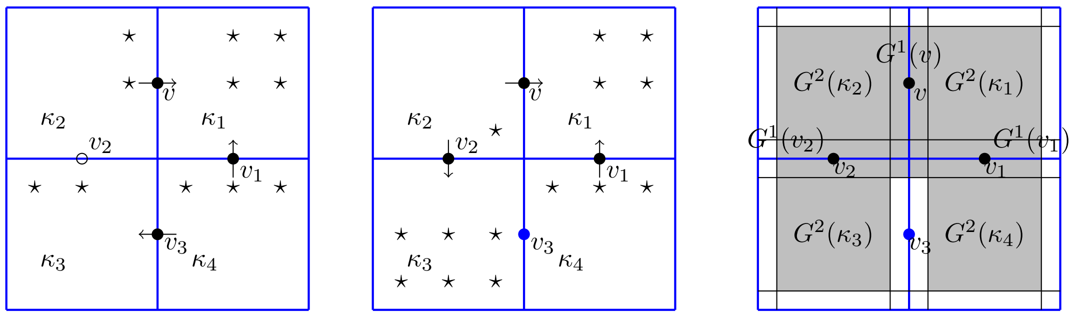


Figure 8: (a) Cells in $N(\mathcal{A})$ associated with $E(i,j)$ under the assumptions of Case 1(a)(i). (b) Possible cell types in the setting of Case 1(a)(ii). (c) Tiles for $N(\mathcal{A})$ associated with $E(i,j)$ in Case 1(a)(ii).



(a) (b) (c)

Figure 9: (a) Possible cell types in the setting of Case 1(a)(iii). Observe that the possible cell types of κ_2 and κ_3 cannot occur simultaneously. Thus this case cannot occur. (b) Possible cell types in the setting of Case 1(b). (c) Minimal set of tiles for Case 1(b).

i. *Assume* $v_3 \notin \mathcal{A}$. Because of the cell type of κ_1, **Rule 2** does not apply. Because $G^0(\pi) \not\subset N(\mathcal{A})$, **Rule 3**, **Rule 4** and **Rule 5** do not apply. We claim that we are in the setting of Figure 8(a). In particular, we need to argue that $G^2(\kappa_3) \not\subset N(\mathcal{A})$. Observe that only **Rule 0** and **Rule 1** require the introduction of a 2-tile. By Figure 7(c) κ_3 is not of type A and hence **Rule 0** does not apply. Assume **Rule 1** forces the introduction of $G^2(\kappa_3)$. Then there exists $u \in \mathcal{A} \cap \mathcal{T}$ such that $u \in \mathcal{V}(\kappa_3)$. Again, by Figure 7(b)$v_2$ or v_3 belong to $\mathcal{V}_a(\kappa_3)$ and hence v_2 or v_3 belong to $\mathcal{A}$, a contradiction. Thus, we are in the setting of Figure 8(a). Proposition 5.1 guarantees the desired transversality on the edges of $G^2(\kappa_2)$ and $G^2(\kappa_4)$. Furthermore, Proposition 5.2(i - ii) and Proposition 5.3(i - ii) guarantees the desired transversality on the edges of $G^0(\pi)$.

ii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent east.* The assumption that v_3 is transparent east implies that κ_4 is of type N or NE and κ_3 is of type NE, E, or SE. By Proposition 6.9 if κ_3 is of type NE or κ_2 is of type SE, $v_2 \in \mathcal{A}$, a contradiction. Thus, the possible cell types are as shown in Figure 8(b).
By **Rule 1** $G^2(\kappa_3) \cup G^1(v_3) \subset N(\mathcal{A})$. Applying **Rule 2** to $G^2(\kappa_3) \cup G^2(\kappa_4) \cup G^2(\kappa_1) \cup G^1(v_3) \cup G^1(v_1)$ implies that $G^0(\pi) \subset N(\mathcal{A})$. This implies that $v_3 \notin \mathcal{V}_e(\kappa_3)$ and hence, **Rule 3** does not apply to κ_3. **Rule 4** and **Rule 5** do not apply and hence we are in the setting of Figure 8(c). Proposition 5.1 guarantees the desired transversality on the edges of $G^2(\kappa_2)$ and $G^2(\kappa_3)$. Proposition 5.4(iv-v) guarantees the desired transversality on the edges of $G^0(\kappa_2)$.

iii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent west.* Then κ_3 is of type NW or N. See Figure 9(a). Since $v_3 \in \mathcal{A}$ Proposition 6.9 implies that $v_2 \in \mathcal{A}$ a contradiction. Thus this case cannot occur.

(b) *Assume* $v_2 \in \mathcal{A}$ *and* v_2 *is transparent south.* This implies that κ_2 is of type SE and κ_3 is of type E or SE. See Figure 9(b). By **Rule 1**, $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. By **Rule 2**, $G^0(\pi) \subset N(\mathcal{A})$. See Figure 9(c).

i. *Assume* $v_3 \notin \mathcal{A}$. Since $v_2 \in \mathcal{A}$, if κ_3 is of type E or SE, then $v_3 \in \mathcal{V}_a(\kappa_3)$, a contradiction. Thus, κ_3 is of type W, A, SW or S, and hence, by Proposition 3.4(x)

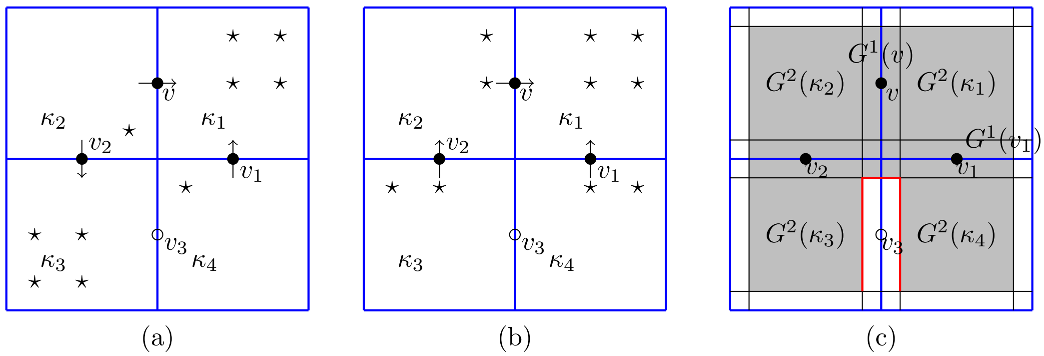


Figure 10: (a) Possible cell types in the setting of Case 1(b). (b) Possible cell types in the setting of Case 1(c)(i). (c) Tiles in $N(\mathcal{A})$ that are related to $E(i,j)$ in Case 1(c)(i). The vector field is transverse in along the interior edges indicated in red.

κ_4 is not of type N or NE. Therefore, the set of possible types is as in Figure 10(a) from which we note that $v_3 \in \mathcal{V}_a(\kappa_4)$. Since $v_1 \in \mathcal{A}$, by Proposition 6.9 $v_3 \in \mathcal{A}$, a contradiction. Therefore, this case cannot occur.

ii. *Assume* $v_3 \in \mathcal{A}$. By **Rule 1**, $G^1(v_3) \subset N(\mathcal{A})$. Combining this with the information from Figure 9(c) we observe that there are no interior boundary edges to check.

(c) *Assume* $v_2 \in \mathcal{A}$ *and* v_2 *is transparent north.*

i. *Assume* $v_3 \notin \mathcal{A}$. By Proposition 6.9 the assumption that $v_3 \notin \mathcal{A}$ implies that κ_3 is of type NW or N and κ_4 is of type N or NE (see Figure 10(b)). By **Rule 1** $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. By **Rule 2** $G^0(\pi) \subset N(\mathcal{A})$. **Rules 3-5** do not apply, thus we are in the setting of Figure 10(c).

Thus, by Proposition 5.1 the vector field is transverse in along the boundary edges of $G^2(\kappa_3)$ and $G^2(\kappa_4)$. By Proposition 5.4(vi-vii) the vector field is transverse in along the boundary edges of $G^0(\pi)$.

ii. *Assume* $v_3 \in \mathcal{A}$. By **Rule 1**, $G^1(v_3) \subset N(\mathcal{A})$. Observe that there are no interior boundary edges to check.

2. *Assume* $v_1 \in \mathcal{A}$ *and* v_1 *is transparent south.* Recall that Figure 6(a) provides an upper bound of the types of cells. Since v_1 is transparent down, $v_1 \in \mathcal{V}_a(\kappa_1)$ and $v_1 \in \mathcal{V}_e(\kappa_4)$. Therefore, κ_1 is of type S or SE and κ_4 is of type W, A, E, SW, S or SE. The possible types of κ_4 preclude the possibility of κ_3 being of type NW or N. Thus we are in the setting of Figure 11(a).

By **Rule 1**, $G^2(\kappa_4) \cup G^1(v_1) \subset N(\mathcal{A})$. By **Rule 2**, $G^0(\pi) \subset N(\mathcal{A})$. See Figure 11(b).

(a) *Assume* $v_2 \notin \mathcal{A}$. Figure 11(a) provides an upper bound on the cell types. By Proposition 6.9 κ_2 is of type E or NE. This in turn, by Proposition 3.4, implies that κ_3 must be of type E, NE or SE (see Figure 11(c)).

Therefore, $v \in \mathcal{V}_a(\kappa_2)$ and hence **Rule 3** does not apply to κ_2. Thus we are still in the setting of Figure 11(b).

i. *Assume* $v_3 \notin \mathcal{A}$. By Proposition 6.9 there cannot be an edge $v_1 \to v_3$. Thus, κ_4 is of type A, S, E, or SE (see Figure 12(a)). We claim that $G^2(\kappa_3) \not\subset N(\mathcal{A})$. Observe

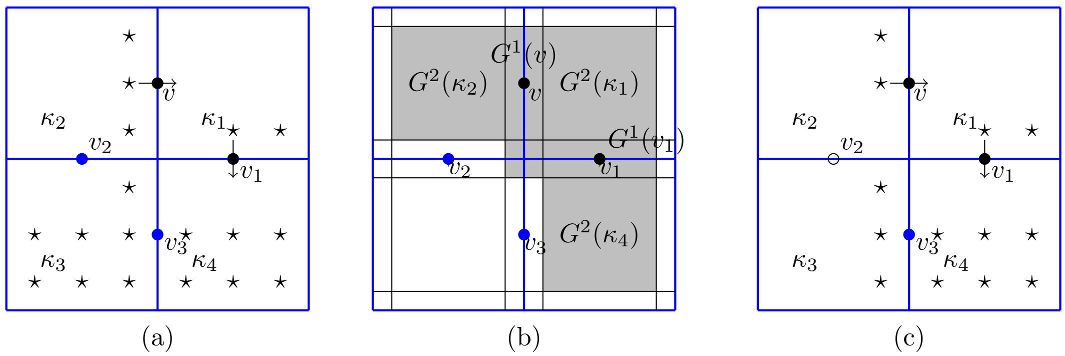


Figure 11: (a) Possible cell types for Case 2. (b) Minimal set of grid elements in $N(\mathcal{A})$ related to $E(i,j)$ in Case 2(a). (c) Possible cell types for Case 2(a).

that only **Rule 0** and **Rule 1** require the introduction of a 2-tile. κ_3 is not of type A and hence **Rule 0** does not apply. Assume **Rule 1** forces the introduction of $G^2(\kappa_3)$. Then there exists $u \in \mathcal{A} \cap \mathcal{T}$ such that $u \in \mathcal{V}(\kappa_3)$. Since κ_3 is of type E, NE or SE, $v_3 \in \mathcal{V}_a(\kappa_3)$ and hence $v_3 \in \mathcal{A}$, a contradiction. By Proposition 7.8, $G^1(v_2)$ and $G^1(v_3)$ are not in $N(\mathcal{A})$. Furthermore, **Rule 5** does not apply. Thus we remain in the setting of Figure 11(b).

A. *Assume κ_3 is of type* NE. Then **Rule 4** does not apply to κ_3. By **Rule 3** applied to κ_4, $C^n(\kappa_4, v_3, \pi) \subset N(\mathcal{A})$. See Figure 12(b). We need to check that the interior edges of $\partial N(\mathcal{A})$ are transverse in. Observe that $v_2 \in \mathcal{V}_e(\kappa_2)$ and $v_3 \in \mathcal{V}_e(\kappa_4)$, hence by Proposition 5.1 we have the desire transversality along the edge of $G^2(\kappa_2)$ and $G^2(\kappa_4)$. By Proposition 5.1(iv-vi) we have the desired transversality for $G^0(\pi)$. By Proposition 5.5 we have the desired transversality for $C^n(\kappa_4, v_3, \pi)$.

B. *Assume κ_3 is of type* E *or* SE. **Rule 4** applied to κ_3 implies that $C^w(\kappa_4, v_3, \pi) \subset N(\mathcal{A})$. See Figure 12(c). The desired transversality follows from the argument used in the previous case.

ii. *Assume $v_3 \in \mathcal{A}$ and v_3 is transparent east.* If we take this configuration and rotate it counterclockwise by 90° then we are in case 1(c)(i) for which we have already shown the desired transversality.

iii. *Assume $v_3 \in \mathcal{A}$ and v_3 is transparent west.* Figure 11(c) provides an upper bound on the cell types. However, v_3 is transparent west implies that κ_3 cannot be of type NE, E, or SE. Thus this case cannot occur.

(b) *Assume $v_2 \in \mathcal{A}$ and v_2 is transparent south.* Figure 11(a) provides an upper bound on the cell types. Since v_2 is transparent south, κ_2 is of type SE and κ_3 is of type W, A, E, SW, S, or SE as indicated in Figure 13(a).

By **Rule 1**, $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. By **Rule 2**, $G^0(\pi) \subset N(\mathcal{A})$. Thus $N(\mathcal{A})$ must contain at least the tiles indicated in Figure 13(b).

i. *Assume $v_3 \notin \mathcal{A}$.* This implies that κ_3 cannot be of type E or SE and κ_4 cannot be of type W or SW (see Figure 13(c)). Therefore $v_3 \in \mathcal{W}$. Thus **Rule 5** applies and

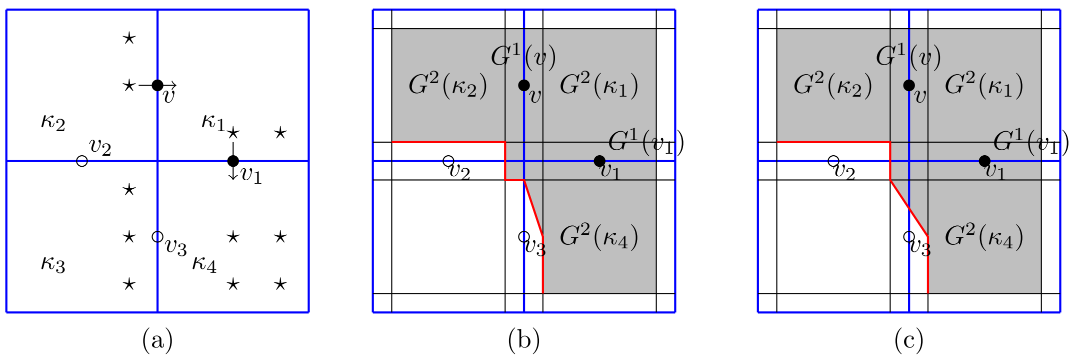


Figure 12: (a) Possible cell types for Case 2(a)(i). (b) Set of grid elements and chips in $N(\mathcal{A})$ associated with $E(i,j)$ in Case 2(a)(i)(A). (c) Set of grid elements and chips in $N(\mathcal{A})$ associated with $E(i,j)$ in Case 2(a)(i)(B).

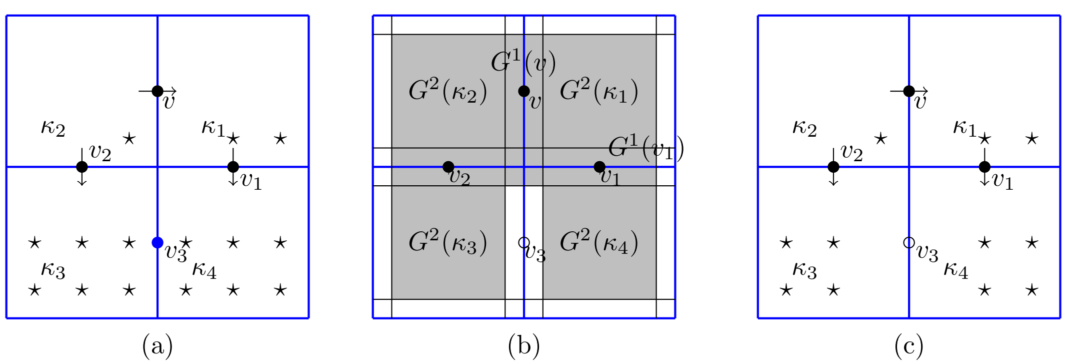


Figure 13: (a) Possible cell types for Case 2(b). (b) Necessary tiles in $N(\mathcal{A})$ for Case 2(b)(i). (c) Possible cell types for Case 2(c).

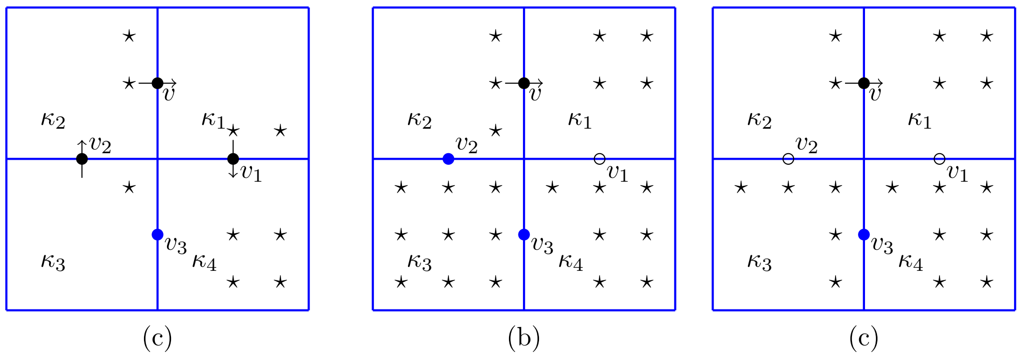


Figure 14: (a) Possible cell types for Case 2(c). (b) Possible cell types for Case 3. (c) Possible cell types for Case 3(a).

$G^1(v_3) \subset N(\mathcal{A})$. Observe that there are no internal edges against which to check internal tangencies.

ii. *Assume* $v_3 \in \mathcal{A}$. By **Rule 1**, $G^1(v_3) \subset N(\mathcal{A})$. Observe that there are no interior boundary edges to check.

(c) *Assume* $v_2 \in \mathcal{A}$ *and* v_2 *is transparent north.* Figure 11(a) provides an upper bound on the cell types. Since v_2 is transparent north, κ_3 must be of type NE and κ_2 cannot be of type SE. The adjacency of κ_3 and κ_4 implies that κ_4 cannot be of type W or SW. See Figure 14(a).

By **Rule 1**, $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. By **Rule 2**, $G^0(\pi) \subset N(\mathcal{A})$. The necessary tiles in $N(\mathcal{A})$ are indicated in Figure 13(b).

i. *Assume* $v_3 \notin \mathcal{A}$. Because $v_2 \in \mathcal{A}$ and κ_3 is of type NE, by Proposition 6.9 $v_3 \in \mathcal{A}$, a contradiction. Thus this case cannot occur.

ii. *Assume* $v_3 \in \mathcal{A}$. By **Rule 1**, $G^1(v_3) \subset N(\mathcal{A})$. Observe that there are no interior boundary edges to check.

3. *Assume* $v_1 \notin \mathcal{A}$. Figure 7(a) provides an upper bound on the cell types. Since $v_1 \notin \mathcal{A}$ there is no edge $v \to v_1$. Therefore the cell κ_1 is of type A, NE, N or E and $v_1 \in \mathcal{V}_e(\kappa_1)$. Adjacency of κ_1 and κ_4 implies that κ_4 is not of type W or SW. See Figure 14(b).

(a) *Assume* $v_2 \notin \mathcal{A}$. Figure 14(b) provides an upper bound on the cell types. By Proposition 6.9 we conclude that κ_2 is of type NE or E. By Proposition 3.4(i) or (iv) the adjacency of κ_2 and κ_3 implies that κ_3 must be of type NW, N, NE, E or SE. See Figure 14(c). The necessary set of tiles in $N(\mathcal{A})$ is given by Figure 6(b).

i. *Assume* $v_3 \notin \mathcal{A}$.

A. *Assume* $G^2(\kappa_3) \not\subset N(\mathcal{A})$ *and* $G^2(\kappa_4) \not\subset N(\mathcal{A})$. By Proposition 7.8, $G^1(v_n) \not\subset N(\mathcal{A})$, $n = 1, 2, 3$. **Rule 2** is not applicable thus, $G^0(\pi) \not\subset N(\mathcal{A})$. Thus the set of tiles is not changed from that of Figure 6(b). By Proposition 5.1 the lower face of G^2_2 and by Proposition 5.2 the lower left face of G^1_v is transverse in.

B. *Assume* $G^2(\kappa_3) \subset N(\mathcal{A})$. By Figure 14(c) κ_3 is not of type A. Thus there exists $u \in \mathcal{V}(\kappa_3) \cap \mathcal{A} \cap \mathcal{T}$. By Proposition 6.8 there exists $u \in \mathcal{V}_e(\kappa_3) \cap \mathcal{A}$. By

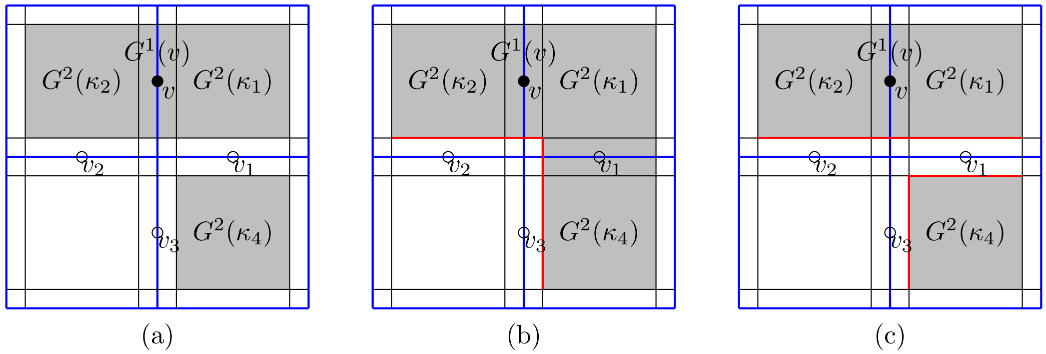


Figure 15: (a) Necessary grid elements for Case 3(a)(i)(C). (b) Tiles in Case 3(a)(i)(C)(I). (c) Tiles in Case 3(a)(i)(C)(II).

assumption $u \in \mathcal{V}(\kappa_3) \setminus \{v_2, v_3\}$. We leave it to the reader to check that given the possible types of κ_3 by Proposition 6.9, either v_2 or v_3 is in $\mathcal{A}$, a contradiction. Thus, it is not possible for $G^2(\kappa_3) \subset N(\mathcal{A})$.

C. *Assume* $G^2(\kappa_4) \subset N(\mathcal{A})$. By Case 3(a)(i)(B), $G^2(\kappa_3) \not\subset N(\mathcal{A})$. Therefore, by Proposition 7.8, $G^1(v_3) \not\subset N(\mathcal{A})$. We are in the setting of Figure 15(a).

(I) *Assume* $G^1(v_1) \subset N(\mathcal{A})$. **Rule 2** does not apply to κ_1, hence $G^0(\pi) \not\subset N(\mathcal{A})$. We are in the setting of Figure 15(b). The desired transversality follows from Propositions 5.1, 5.2, 5.3, and 5.4.

(II) *Assume* $G^1(v_1) \not\subset N(\mathcal{A})$. Assume κ_4 is not of type A. Thus there exists $u \in \mathcal{V}(\kappa_4) \cap \mathcal{A} \cap \mathcal{T}$. By Proposition 6.8 there exists $u \in \mathcal{V}_e(\kappa_4) \cap \mathcal{A}$. By assumption $u \in \mathcal{V}(\kappa_4) \setminus \{v_1, v_3\}$

We leave it to the reader to check that if κ_4 is not of type A, then by Proposition 6.9, either v_1 or v_3 is in $\mathcal{A}$, a contradiction. Thus, if $G^2(\kappa_4) \subset N(\mathcal{A})$, then κ_4 is of type A and we are in the setting of Figure 15(c) and the desired transversality follows from Propositions 5.1 and 5.2.

ii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent east.* Figure 14(c) provides an upper bound on the possible cell types. Observe that κ_3 cannot be of type NW or N and κ_4 cannot be of type NW. Furthermore, by Proposition 6.9 κ_3 cannot be of type NE and since the edge $v_3 \to v_1$ cannot exist κ_4 cannot be of type N or NE. See Figure 16(a). By **Rule 1** $G^2(\kappa_3) \cup G^2(\kappa_4) \cup G^1(v_3) \subset N(\mathcal{A})$. See Figure 16(b). The existence of $G^0(\pi)$ is determined by **Rule 2**, hence a necessary condition for $G^0(\pi) \subset N(\mathcal{A})$ is the existence of $G^1(v_1) \subset N(\mathcal{A})$ or $G^1(v_2) \subset N(\mathcal{A})$.

A. *Assume* $G^1(v_1) \not\subset N(\mathcal{A})$ *and* $G^1(v_2) \not\subset N(\mathcal{A})$. The desired transversality follows from Propositions 5.1 and 5.2.

B. *Assume* $G^1(v_2) \subset N(\mathcal{A})$. Observe that $v \in \mathcal{V}_a(\kappa_2)$. Thus **Rule 2** applied to κ_2 implies that $G^0(\pi) \subset N(\mathcal{A})$. **Rule 5** applies and hence $G^1(v_1) \subset N(\mathcal{A})$. There are no interior boundary edges to check.

C. *Assume* $G^1(v_2) \not\subset N(\mathcal{A})$ *and* $G^1(v_1) \subset N(\mathcal{A})$. From Figure 16(a) we deduce that **Rule 2** does not apply to κ_1 or κ_4. Thus $G^0(\pi) \not\subset N(\mathcal{A})$. This puts us into the setting shown in Figure 16(c) and the desired transversality follows from

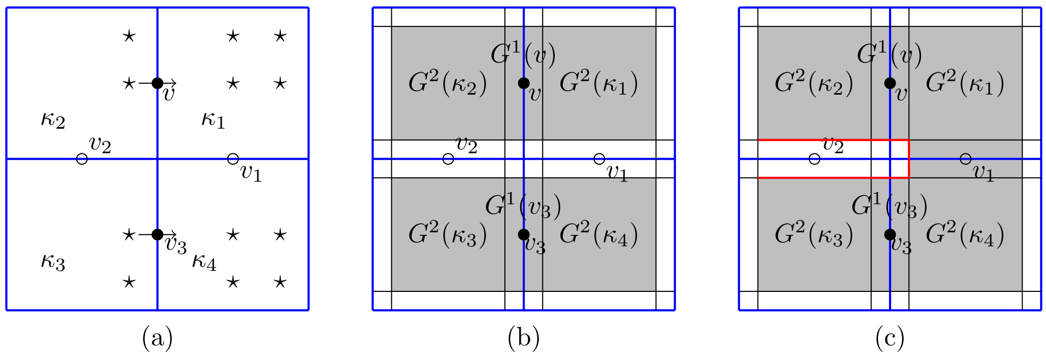


Figure 16: (a) Possible cell types in Case 3(a)(ii). (b) Possible cell types in Case 3(a)(iii). (c) Possible cell types in Case 3(b).

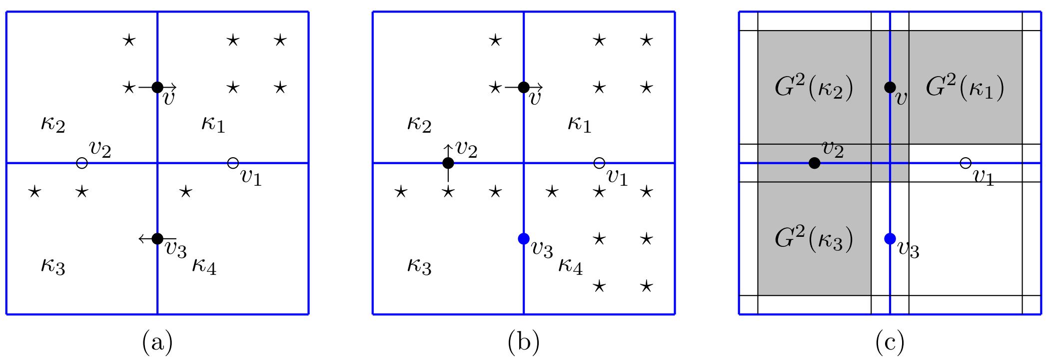


Figure 17: (a) Possible cell types in Case 3(a)(iii). (b) Possible cell types in Case 3(b). (c) Necessary grid elements in $N(\mathcal{A})$ in Case 3(b).

Propositions 5.1, 5.2, and 5.3.

iii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent west.* Figure 14(c) provides an upper bound on the possible cell types. Observe that κ_3 cannot be of type NE, E, or SE and κ_4 must be of type NW. Thus, we are in the setting of Figure 17(a). By Proposition 3.4 the cell types of κ_2 and κ_3 are not compatible, a contradiction. Thus this case cannot occur.

(b) *Assume* $v_2 \in \mathcal{A}$ *and* v_2 *is transparent north.* Figure 14(b) provides an upper bound on the possible cell types. The assumption that $v_2 \in \mathcal{A}$ forces κ_2 to be of type NE or E, and κ_3 to be of type NE, N or NE. By **Rule 1**, $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. By **Rule 2** applied to κ_2, $G^0(\pi) \subset N(\mathcal{A})$. Thus we are in the setting of Figure 17(c).

i. *Assume* $v_3 \notin \mathcal{A}$. This implies that κ_3 is of type A, W, S, or SW and hence $v_2, v_3 \in \mathcal{V}_e(\kappa_3)$. Since $G^2(\kappa_3) \cup G^1(v_2) \cup G^0(i,j) \subset N(\mathcal{A})$, **Rule 3** applies and hence $C^n(\kappa_3, v_3, \pi) \subset N(\mathcal{A})$. Similarly, $C^n(\kappa_1, v_1, \pi) \subset N(\mathcal{A})$. By Proposition 3.4 κ_4 is of type NW. This implies that $v_1, v_3 \in \mathcal{V}_a(\kappa_4)$ Thus **Rule 4** does not apply to κ_4. Figure 17(c) indicates $N(\mathcal{A}) \cap E(i,j)$. That the vector field is transverse in at

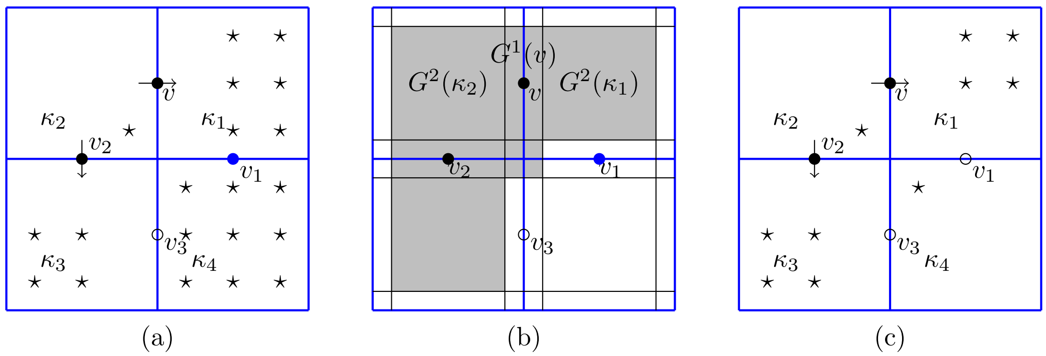


Figure 18: (a) The possible cell types for Case 3(c). (b) Necessary tiles for $N_{\mathcal{A}}$ in Case 3(c). (c) The possible cell types for Case 3(c)(iii).

all the interior boundary edges (beginning with the top right and moving down to the bottom left) follows from Propositions 5.1, 5.5, 5.4, 5.4, 5.5, and 5.1.

ii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent east.* Observe that rotating clockwise by 90° results in Case 2(b)(i) for which the desired transversality has been demonstrated.

iii. *Assume* $v_3 \in \mathcal{A}$ *and* v_3 *is transparent west.* Observe that rotating clockwise by 90° results in Case 2(c)(i), which has already been shown not to occur.

(c) *Assume* $v_2 \in \mathcal{A}$ *and* v_2 *is transparent south.* We begin by rotating clockwise by 90° and then performing a reflection in the east-west direction. Figure 6(a) provides an upper bound on the possible cell types. Since v_2 is transparent south, κ_2 is of type SE and κ_3 cannot be of type NW, N, or NE. The assumption that $v_3 \notin \mathcal{A}$ implies that κ_3 cannot be of type E or SE This results in cell types as shown in Figure 18(a). **Rule 1** implies that $G^2(\kappa_3) \cup G^1(v_2) \subset N(\mathcal{A})$. **Rule 2** implies that $G^0(\pi) \subset N(\mathcal{A})$. Thus the necessary set of tiles in $N(\mathcal{A})$ is shown in Figure 18(b).

i. *Assume* $v_1 \in \mathcal{A}$ *and* v_1 *is transparent north.* This is the same as Case 1(b)(i), hence the desired transversality has been demonstrated.

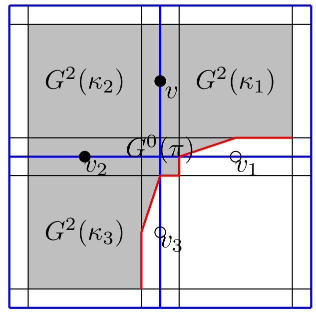
ii. *Assume* $v_1 \in \mathcal{A}$ *and* v_1 *is transparent south.* This is the same as Case 2(b)(i), hence the desired transversality has been demonstrated.

iii. *Assume* $v_1 \notin \mathcal{A}$. Figure 18(a) provides an upper bound on the possible cell types. Since $v_1 \notin \mathcal{A}$, κ_1 cannot be of type S or SE. Since κ_2 is of type SE, Proposition 3.4(ii) and (ix) implies that $\eta_j \in \mathsf{H}^1$ and $\xi_i \in \Xi^2$, respectively. With these restrictions Proposition 3.4(i) implies that κ_4 is of type NW, N, or NE, while Proposition 3.4(x) implies that κ_4 is of type NW, W, or SW. Thus κ_4 is of type NW as is indicated in Figure 18(c).

□

8. Completion of the Proof

The focus of this section is on the proofs of Proposition 2.14 and 2.16. We begin with a preliminary result concerning the existence of global attractors.



(a)

Figure 19: Tiles and chips for Case 3(c)(iii)

Proposition 8.1. *Consider*

$$\dot{x} = -\Gamma x + f(x), \quad x \in (0,\infty)^n \tag{45}$$

where $f\colon (0,\infty)^n \to (0,\infty)^n$ *is Lipschitz continuous and satisfies the conditions that there exist positive constants* $a_i^{\pm}$ *such that*

$$0 < a_i^- < f_i(x) < a_i^+, \quad i = 1,\dots,n$$

for all $x \in (0,\infty)^n$*. Furthermore, assume that* Γ *is a diagonal matrix with diagonal elements* $\gamma_i > 0$*. Then, there exists a global compact attractor* $X \subset (0,\infty)^n$ *for the flow generated by* (45)*.*

We leave the proof of Proposition 8.1 to the reader noting that for positive a^- sufficiently small and a^+ sufficiently large, the vector field (45) is transverse in on the boundary of $[a^-, a^+]^n$.

We turn our attention to the proof of Proposition 2.14.

Lemma 8.2. *For all* $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$*,* $N(\mathcal{A}_0 \cup \mathcal{A}_1) = N(\mathcal{A}_0) \cup N(\mathcal{A}_1)$*.*

Proof. We begin with the observation that given $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, the construction of $N(\mathcal{A})$ can be done in two steps. First apply **Rule 0** and **Rule 1** and then determine if **Rules 2 - 5** apply.

Let $\mathcal{A}_0, \mathcal{A}_1 \in \mathsf{Att}(\mathcal{F})$ and let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$. We need to show that $N(\mathcal{A}) = N(\mathcal{A}_0) \cup N(\mathcal{A}_1)$. Since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, the set of tiles obtained by the application of **Rule 0** and **Rule 1** to $\mathcal{A}$ equals the union of the set of tiles obtained by the application of **Rule 0** and **Rule 1** to $\mathcal{A}_0$ and $\mathcal{A}_1$. Since **Rules 2 - 5** only add tiles based on tiles that are already required by **Rule 0** and **Rule 1**, $N(\mathcal{A}_0) \cup N(\mathcal{A}_1) \subset N(\mathcal{A})$. Thus we need to prove that $N(\mathcal{A}) \subset N(\mathcal{A}_0) \cup N(\mathcal{A}_1)$. We prove a few representative cases leaving the rest to the reader.

Assume that a wide chip C^w is contained in $N(\mathcal{A})$. Without loss of generality we can assume that this implies that there is an elementary region $E(i,j)$ such that $N(\mathcal{A}) \cap E(i,j)$ gives rise to Case 2(a)(i)(B) in the proof of Proposition 7.5. In particular, $v, v_1 \in \mathcal{A}$ and $v_2, v_3 \notin \mathcal{A}$. Without loss of generality assume that $v \in \mathcal{A}_0$. Observe that $v_1 \in \mathcal{F}(v)$ and hence $v_1 \in \mathcal{A}_0$. Furthermore, $v_2, v_3 \notin \mathcal{A}_0$ which implies by **Rule 4** that $C^w \subset N(\mathcal{A}_0)$.

Assume that a narrow chip C^n is contained in $N(\mathcal{A})$. Without loss of generality we can assume that this implies that there is an elementary region $E(i,j)$ such that $N(\mathcal{A}) \cap E(i,j)$ gives rise to Case 2(a)(i)(A) or Case 3(c)(iii). In particular, $v, v_1 \in \mathcal{A}$ and $v_2, v_3 \notin \mathcal{A}$. Without loss of generality assume that $v \in \mathcal{A}_0$. Observe that $v_1 \in \mathcal{F}(v)$ and hence $v_1 \in \mathcal{A}_0$. Thus the arguments used in Case 2(a)(i)(A) apply to $\mathcal{A}_0$ and hence $C^n \subset N(\mathcal{A}_0)$. A similar argument applies to Case 3(c)(iii).

Assume that a 0-tile $G^0(\pi)$ is contained in $N(\mathcal{A})$. If $\pi = (\xi_i, \eta_j)$, then **Rule 2** is applicable to $E(i,j)$. Reviewing the proof of Proposition 7.5 we see that this happens in Cases 1(a)(ii), 1(b), 1(c)(i), 2(a), 2(b), 3(b), and 3(c). In Case 1(a)(ii) $v, v_1, v_3 \in \mathcal{T} \cap \mathcal{A}$ and $N(\mathcal{A})$ restricted to $E(i,j)$ is a shown in Figure 8(c). If $v, v_1, v_3 \in \mathcal{T} \cap \mathcal{A}_0$, then $G^0(\pi) \subset N(\mathcal{A}_0)$. So assume $v_3 \in \mathcal{A}_0$. Observe that $v_1 \in \mathcal{F}(v_3)$ and hence $v_1 \in \mathcal{A}_0$. Thus, we can assume $v \in \mathcal{A}_1 \setminus \mathcal{A}_0$. Applying the appropriate symmetric action, this implies that restricted to $E(i,j)$ $\mathcal{A}_0$ is in the setting of Case 2(a)(i)(B). This implies that $N(\mathcal{A}_0)$ restricted to $E(i,j)$ consists of

$$G^2(\kappa_1) \cup G^2(\kappa_3) \cup G^2(\kappa_4) \cup G^1(v_1) \cup G^1(v_3) \cup G^0(\pi) \cup C^w(\kappa_1, v, \pi).$$

Since **Rule 1** applies to $\mathcal{A}_1$, $N(\mathcal{A}_0)$ restricted to $E(i,j)$ contains $G^2(\kappa_2) \cup G^1(v)$. Thus restricted to $E(i,j)$, $N(\mathcal{A}) = N(\mathcal{A}_0) \cup N(\mathcal{A}_1)$. The remaining cases follow from similar arguments.

Assume that a 1-tile $G^1(v_\alpha)$ is contained in $N(\mathcal{A})$. This implies that **Rule 1** or **Rule 5** applies. If $G^1(v_\alpha)$ is introduced because of **Rule 1**, then $v_\alpha \in \mathcal{T} \cap \mathcal{A} \cap \mathcal{E}(i,j)$. Hence, without loss of generality we can assume $v_\alpha \in \mathcal{A}_0$ in which case **Rule 1** implies that $G^1(v_\alpha)$ is contained in $N(\mathcal{A}_0)$. If $G^1(v_\alpha)$ is introduced by **Rule 5**, then up to symmetry we are in the setting of Case 2(b)(i) or Case 3(a)(ii)(B). For Case 2(b)(i), without loss of generality assume that $v \in \mathcal{A}_0$. By Proposition 6.9, $v_2 \in \mathcal{A}_0$. Furthermore, $v_1 \in \mathcal{F}(v)$, hence $v_1 \in \mathcal{A}_0$. Therefore restricted to $E(i,j)$, $\mathcal{A} = \mathcal{A}_0$ and hence $N(\mathcal{A}) = N(\mathcal{A}_0) \cup N(\mathcal{A}_1)$.

Case 3(a)(ii)(B) is slightly more subtle. The assumption is that with respect to the elementary region $E(i,j)$, $G^1(v_2) \subset N(\mathcal{A})$, but $v_2 \notin \mathcal{A}$. Since $v_2 \notin \mathcal{A}$, the inclusion of $G^1(v_2)$ implies that **Rule 5** was applied in the elementary region $E(i-1,j)$ to obtain $G^1(v_2)$. This in turn implies that we have Case 2(b)(i) or Case 3(a)(ii)(B) in elementary region $E(i-1,j)$. If it is Case 2(b)(i), then the argument above implies, without loss of generality, that $\mathcal{A}_0$ and $\mathcal{A}$ agree on $E(i-1,j)$. Thus we obtain the desired result. If it is Case 3(a)(ii)(B), then we must repeat the argument in the elementary region $E(i-2,j)$. Since there are only finitely many elementary regions, at some point we are in Case 2(b)(i), from which we can inductively conclude that $N(\mathcal{A}_0 \cup \mathcal{A}_1) = N(\mathcal{A})$ on $E(i,j)$.

Finally, assume that a 2-tile $G^2(\kappa_\alpha)$ is contained in $N(\mathcal{A})$. This implies that **Rule 0** and/or **Rule 1** applies. In both cases the fact that $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ implies that **Rule 0** and/or **Rule 1** also applies to $N(\mathcal{A}_i)$, $i = 0, 1$. ☐

Lemma 8.3. *N is 1-1.*

Proof. Let $\mathcal{A}_0, \mathcal{A}_1 \in \mathsf{Att}(\mathcal{F})$ such that $N(\mathcal{A}_0) = N(\mathcal{A}_1)$. Assume $G^2(\kappa) \subset N(\mathcal{A}_0)$.

If κ is an attracting cell, then $w_\kappa \in \mathcal{A} \cap \mathcal{M}$ and $G^2(\kappa)$ is introduced using **Rule 0**. Thus, if $N(\mathcal{A}_0) = N(\mathcal{A}_1)$, then $\mathcal{A}_0 \cap \mathcal{M} = \mathcal{A}_1 \cap \mathcal{M}$.

If κ is a focusing or translating cell, then $G^2(\kappa)$ is introduced using **Rule 1** and by Proposition 6.9, $\mathcal{V}_a(\kappa) \subset \mathcal{A}$. Thus, $N(\mathcal{A}_0) = N(\mathcal{A}_1)$ implies

$$\bigcup_{G^2(\kappa) \subset N(\mathcal{A}_i)} \mathcal{V}_a(\kappa) \subset \mathcal{A}_j, \quad i, j = 0, 1.$$

However, if $v \in \mathcal{A}_i \cap \mathcal{T}$, then $v \in \mathcal{V}_a(\kappa)$ for some κ such that $G^2(\kappa) \subset N(\mathcal{A}_i)$. Thus, $N(\mathcal{A}_0) = N(\mathcal{A}_1)$ implies $\mathcal{A}_0 \cap \mathcal{T} = \mathcal{A}_1 \cap \mathcal{T}$.

Finally, recall that given $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, the application of **Rules 2 - 5** is determined by the results of the applications of **Rules 0** and **1**. Therefore, $N(\mathcal{A}_0) = N(\mathcal{A}_1)$ implies $\mathcal{A}_0 = \mathcal{A}_1$. ☐

Proof of Proposition 2.14. By Lemma 7.3 $N(\emptyset) = \emptyset$, i.e. $N(\mathbf{0}) = \mathbf{0}$. Lemma 8.2 shows that N is a join-semilattice morphism and Lemma 8.3 show that it is a monomorphism. □

Proof of Proposition 2.16. Let $\mathcal{A} \in \mathsf{J}^\vee(\mathsf{Att}(\mathcal{F}))$ and assume $N(\mathcal{A}) = K_0 \cup K_1$ where $K_0, K_1 \in \mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$. We need to show that $K_0 = N(\mathcal{A})$ or $K_1 = N(\mathcal{A})$. Obviously $K_0, K_1 \subset N(\mathcal{A})$. Without loss of generality we assume that $K_1 \subsetneq N(\mathcal{A})$. Since $K_0, K_1 \in \mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$, there exist $\{\mathcal{K}_{0,i}\}, \{\mathcal{K}_{1,j}\} \subset \mathsf{Att}(\mathcal{F})$ such that

$$K_0 = \bigcap_i N(\mathcal{K}_{0,i}) \quad \text{and} \quad K_1 = \bigcap_j N(\mathcal{K}_{1,j}).$$

Observe that given a cell κ and $\mathcal{B} \in \mathsf{Att}(\mathcal{F})$ either

$$\mathrm{int}(G^2(\kappa)) \subset N(\mathcal{B}) \quad \text{or} \quad \mathrm{int}(G^2(\kappa)) \cap N(\mathcal{B}) = \emptyset.$$

Let $v \in m(\mathcal{A})$. If $v = w_\kappa \in \mathcal{M}$, then by Proposition 2.10.3 $m(\mathcal{A}) = \{w_\kappa\} \in \mathsf{Att}(\mathcal{F})$, and hence by Proposition 2.10.4, $\mathcal{A} = \{w_\kappa\}$. By **Rule 0**, $N(\mathcal{A}) = G^2(\kappa)$. Since we assume that $K_1 \subsetneq N(\mathcal{A})$, there exists $N(\mathcal{K}_{1,j})$ such that $\mathrm{int}(G^2(\kappa)) \cap N(\mathcal{K}_{1,j}) = \emptyset$. This in turn implies that $K_1 = \emptyset$. Hence, $K_0 = N(\mathcal{A})$.

Thus we only need to consider $v \in m(\mathcal{A}) \cap \mathcal{T}$. Let $v \in m(\mathcal{A})$. Since **Rule 1** applies, $G^2(\kappa_r) \subset N(\mathcal{A})$. Furthermore, since $v \in \mathcal{V}_a(\kappa_r)$, if $G^2(\kappa_r) \subset N(\mathcal{B})$, then $v \in \mathcal{B}$. Therefore, if there exists $\mathcal{K}_{1,j}$ such that $v \notin \mathcal{K}_{1,j}$, then $\mathrm{int}(G^2(\kappa_r)) \cap N(\mathcal{K}_{1,j}) = \emptyset$. This in turn implies that $\mathrm{int}(G^2(\kappa_r)) \cap K_1 = \emptyset$. Thus, without loss of generality we can assume that $v \in \mathcal{K}_{0,i}$ for all i. By Proposition 2.10.3, $m(\mathcal{A}) \subset \mathcal{K}_{0,i}$ and thus by Proposition 2.10.2, $\downarrow(m(\mathcal{A})) = \mathcal{A} \subset \mathcal{K}_{0,i}$. Thus

$$N(\mathcal{A}) \subset \bigcap_i N(\mathcal{K}_{0,i}) = K_0 \subset N(\mathcal{A})$$

and hence $K_0 = N(\mathcal{A})$. □

For the sake of clarity we restate Theorem 2.17 using the language developed in this paper.

Theorem 8.4. *Let $\Sigma = \Sigma(\Gamma, \Lambda, \Xi^1, \Xi^2, \mathsf{H}^1, \mathsf{H}^2)$ be a switching system (see Definition 3.1). Let $\mathcal{F}$ be the associated state transition diagram (see Definition 6.3). Let $\left(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})}\right)$ be the poset indexing the Morse decomposition $\mathsf{MD}(\mathcal{F})$ of $\mathcal{F}$. Choose $0 < \delta < \delta^*$, where δ^* satisfies (40), and let*

$$\dot{x} = -\Gamma x + f^{(\delta)}(x), \quad x \in (0, \infty)^2 \tag{46}$$

be an associated δ-constrained continuous switching system as defined in Section 4. Let φ be the flow associated with (46) and let $X \subset (0,\infty)^2$ be the associated global attractor.

Then, there exists a Morse decomposition for X under φ labeled by

$$\mu\colon \bar{\mathsf{Q}} \to \mathsf{Invset}(\varphi)$$

where $\mathsf{Q} = \mathsf{J}^\vee(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$ and for which there exists a poset monomorphism

$$\bar{\iota} \circ \iota\colon \left(\mathsf{P}(\mathcal{F}), <_{\mathsf{P}(\mathcal{F})}\right) \to (\bar{\mathsf{Q}}, <_{\bar{\mathsf{Q}}}).$$

We conclude this section by noting that the assumption that $f^{(\delta)}$ be constant on the tiles is not necessary. Applying the classical continuation theorem for Morse decompositions [9] to Theorem 8.4 gives rise to the following result.

Theorem 8.5. *Let*

$$\dot{x} = -\Gamma x + f^{(\delta)}(x), \quad x \in (0,\infty)^2 \tag{47}$$

be a δ-constrained continuous switching system with an associated Morse graph MG *for the global attractor of* (47)*. If*

$$\sup_{x\in(0,\infty)^2} \|f(x) - f^{(\delta)}(x)\| < \epsilon$$

for sufficiently small $\epsilon > 0$, then MG *is a Morse graph for the global attractor of*

$$\dot{x} = -\Gamma x + f(x), \quad x \in (0,\infty)^2.$$

9. Conclusion

We conclude with a few comments concerning the results of this paper and potential future directions.

We begin by providing additional insight into the conclusion of Theorem 8.4. Observe that $\mathsf{L}(N(\mathsf{Att}(\mathcal{F})))$ is a bounded lattice where

$$\mathbf{1} = \bigcup_{\mathcal{A}\in\mathsf{Att}(\mathcal{F})} N(\mathcal{A}) \in \mathsf{AB}(\varphi) \subsetneq (0,\infty)^2.$$

It is possible that $\mathrm{Inv}\left([a^-,a^+]^2 \setminus \mathbf{1}, \varphi\right) \neq \emptyset$ where $a^\pm$ is as in the comment following Proposition 8.1. The element $\bar{q} \in \bar{\mathsf{Q}}$ is used to label this Morse set.

To see that this possibility can be realized consider the switching system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \begin{cases} l_{1,1} & \text{if } x_1 < \theta_{1,1} \\ u_{1,1} & \text{if } x_1 > \theta_{1,1} \end{cases} \\ \dot{x}_2 &= -x_2 + \begin{cases} l_{2,1} & \text{if } x_1 < \theta_{2,1} \\ u_{2,1} & \text{if } x_1 > \theta_{2,1} \end{cases} \end{aligned} \tag{48}$$

where $l_{1,1} < \theta_{1,1} < u_{1,1} < \theta_{2,1}$. The phase portrait is shown in Figure 20. Observe that κ_1 and κ_2 are attracting cells. Thus $\mathsf{MD}(\mathcal{F}) = \{\{w_1\},\{w_2\}\}$. However, if one considers a particularly simple δ-constrained continuous switching system obtained by a linear interpolation between the lower and upper values, then it is clear that for the resulting flow there will be an unstable fixed point near the $\theta_{1,1}$ line that does not lie in the tiles $G^2(\kappa_1) \cup G^2(\kappa_2)$.

A similar remark applies to the fact that the poset monomorphism from $\mathsf{P}(\mathcal{F})$ to $\mathsf{J}^\vee(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$ induced by N need not be surjective. Consider cells in an elementary domain $E(i,j)$ as shown in Figure 21(a). Note that for the associated transition graph $\mathcal{F}$, $\{w_1\},\{w_3\} \in \mathsf{Att}(\mathcal{F})$ and by **Rule 0**, $N(\{w_1\}) = G^2(\kappa_1)$ and $N(\{w_3\}) = G^2(\kappa_3)$. Assume that there exist distinct attractors $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $v \in \mathcal{A}_0$ and $v_1 \in \mathcal{A}_1$. Observe that $N(\mathcal{A}_i) \cap E(i,j)$ is given by Case 3c(iii). Thus $N(\mathcal{A}_0) \cap N(\mathcal{A}_1) = G^2(\kappa_3) \cup G^2(\kappa_1) \cup G^0(\pi) \notin N(\mathsf{Att}(\mathcal{F}))$.

Turning now to the dynamics generated by a vector field for an associated δ-constrained continuous switching system as shown in Figure 21(b), we see that

$$G^0(\pi) = N(\mathcal{A}_0) \cap N(\mathcal{A}_1) \setminus (N(\{w_1\}) \cup N(\{w_1\}))$$

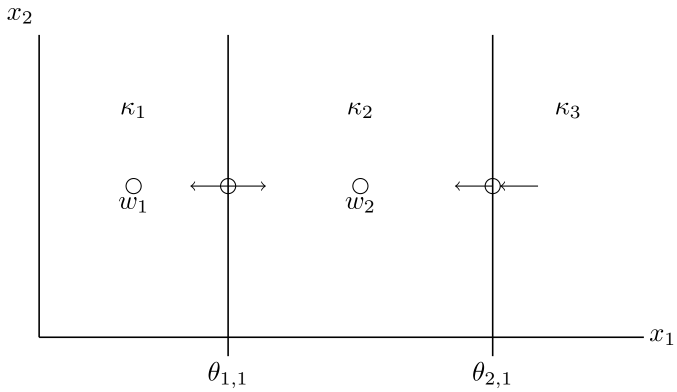


Figure 20: Phase portrait for (48). κ_1 and κ_2 are attracting cells.

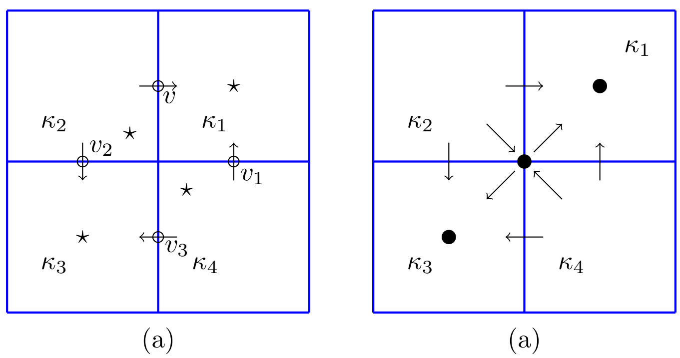


Figure 21: (a) Cells in an elementary domain $E(i, j)$. κ_1 and κ_3 are attracting cells. κ_2 and κ_4 are translating cells. (b) Idealized associated vector field for a δ-constrained continuous switching system.

is an isolating neighborhood for the Morse set consisting of an unstable fixed point. Furthermore, observe that $N(\mathcal{A}_0) \cap N(\mathcal{A}_1) \in \mathsf{J}^\vee(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$ where the immediate predecessor is $N(\{w_1\}) \cup N(\{w_1\})$. Thus, given the construction of $\mathcal{F}$ we cannot hope to improve on the conclusion that N induces an order preserving monomorphism from $\iota \colon \mathsf{P}(\mathcal{F}) \to \mathsf{J}^\vee(\mathsf{L}(N(\mathsf{Att}(\mathcal{F}))))$

With regard to δ-constrained continuous switching systems there are three points worth emphasizing. First, within the δ collars we impose minimal assumptions on the form of $f^{(\delta)}$. Thus we do not need to assume that $f^{(\delta)}$ is based on a particular nonlinearity, e.g. a Hill function, and therefore the dynamics we are recovering is valid for an extremely wide set of potential models. Second, the larger δ is the less steep $f^{(\delta)}$ needs to be, and hence, the less switch-like the system needs to be. Third, we give explicit bounds on δ^* in terms of the parameters of Λ and Γ. A possible consequence of this is that the computational tools developed for switching systems can be used to guide the study of the local and global dynamics of arbitrary systems of the form (1) for fixed families of nonlinearities. To be more precise, assume that the f_n are given in terms of Hill functions and we are interested in particular dynamical structures. Letting the Hill coefficients $k \to \infty$ produces a switching system, the global dynamics of which can be analyzed over all of parameter space using $\mathcal{F}$ [10]. We then can identify parameter values at which the desired nonlinear dynamics occurs and determine the maximal size of perturbation δ^*. Given δ^* one can choose k sufficiently large so that the Hill function approximates a δ-constrained continuous switching system for $\delta < \delta^*$. More standard numerical methods can then be used to identify the desired Morse set for (1) with this large Hill coefficient k. Finally, numerical continuation techniques can be employed to determine if the dynamics continues to lower biologically motivated values of k.

The focus of this paper is on translating information obtained from piecewise constant models in the form of switching systems that are motivated by regulatory networks, to information about the dynamics generated by Lipschitz continuous differential equations. However, as is mentioned in the introduction part of the motivation for this paper is our interest in the mathematically rigorous analysis of global dynamics for multiparameter systems over large regions of parameter space and the question of whether for these purposes it is computationally efficient to use the techniques described here. With this in mind the results presented in Section 7 are much more general than those required to study regulatory networks with two nodes. Observe that Theorem 8.5 suggests that given a system of ordinary differential equations of the form (1) one could try to compute the associated dynamics by approximating f via a linear term Γ and a piecewise constant function Λ, and then, computing the associated Morse graph for the associated switching system. We plan to explore the effectiveness of such a procedure. However, there are at least two related issues that need to be addressed. First, we need explicit results for bounds on ϵ, the acceptable size of perturbation in Theorem 8.5. Second, we need to understand how to determine the threshold values used to define the domains of the piecewise constant functions.

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