Optimal control of sampled-data and stochastic distributed-parameter systems
by Vichit Lorchirachoonkul

A thesis submitted to the Graduate Faculty in. partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University

Abstract:
The subject of this thesis is the development of concepts and techniques of optimal design for linear deterministic as well as stochastic distributed-parameter systems.

The primary purpose of the thesis project is the development of computationally feasible methods for optimal design of distributed-parameter systems under a linear or convex performance measure of the distributed error of the system.

The content of the thesis is summarized as follows: First, a new definition of distributed correlation function is introduced to facilitate the study of stochastic control of distributed-parameter systems. Relationships are derived which paralleled well-known relationships in the statistical analysis of lumped-parameter systems. With this definition of distributed correlation function, an optimal control problem is shown to be a Wiener-Hopf spectrum-factorization problem.

Second, a computationally feasible method is developed for optimal control of distributed-parameter systems. By use of the method, the optimization problem is reduced to one of linear programming for which well-known solution algorithms are available. Performance is optimized with respect to absolute values of distributed errors weighted both in time and in space. Magnitude constraints on both control and state-variable functions are incorporated. Both distributed and non-distributed inputs to the system are treated. Third, the optimal control of distributed-parameter systems is studied from dynamic programming point of view. A functional equation associated with a convex performance measure is derived with properties analoguous to those of the lumped-parameter case. A computationally feasible method is developed for minimizing the spatial-average-integral-square error of a class of distributed-parameter systems. With this method, feedback control of distributed-parameter systems can be achieved. Fourth, the theory of deadbeat response with minimal overshoot compromise is generalized to include systems enfolding time delay in the loop. The optimal trade-off between deadbeat-response time for a ramp input and step-response overshoot is derived for a single-loop sampled-data system with time delay.

In addition, sensitivity of the output response is studied as a function of parameter variations in the optimal controller, and stability limits are derived in terms of a change-in-gain parameter.

OPTIMAL CONTROL OF SAMPLED-DATA AND STOCHASTIC DISTRIBUTED-PARAMETER SYSTEMS

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 PURPOSE OF THE THESIS PROJECT

The purpose of the thesis project is fivefold:

1. To review, evaluate and summarize the published work on optimization of distributed-parameter control systems.
2. To introduce a new definition of distributed correlation functions to facilitate the study of distributed-parameter control systems with stochastic forcing functions.
3. To develop an optimal design theory, based on linear programming techniques, for distributed-parameter systems with distributed outputs.
4. To develop a computational algorithm for optimal control functions of distributed-parameter systems based on dynamic programming techniques.
5. To generalize the theory of deadbeat responses with minimal overshoot compromise to apply to sampled-data systems enfolding time delay.

1.2 VALUE OF THE THESIS RESEARCH

In many situations, the energy of a system is spatially distributed throughout a medium. Examples of such systems are transmission lines, heat exchangers, nuclear reactors, chemical reactors, distributed RC networks, biological systems, et cetera. Numerous applications of distributed-parameter-systems theory are given in the literature (39, 60). Generally, the characteristics of a distributed-parameter system can be described by a set of partial differential equations with appropriate boundary conditions.

Recently, Kolb (39) noted that the development of a general body of theory for the analysis and synthesis of sampled-data, class-II-type,* distributed-parameter systems similar to that developed by Pierre (60) for class-I-type systems* would provide a worthy subject of investigation. Wang and Tung (86) suggest that a control theory for distributed systems may be developed in parallel with the existing theory for lumped-parameter systems. Also Pierre (60) states that a topic worthy of additional research is the development of a general body of theory for effecting the design of optimal distributed-parameter control systems, based on minimization of integral-square or mean-square error, subject to stated constraints.

Therefore, as will be evident in Chapters 3 and 4, the primary contributions of this thesis research consist of the development of a body of theory which satisfies these needs to a large extent. In addition, the theory of deadbeat response with minimal overshoot compromise (68) is generalized to sampled-data systems enfolding time delay. This enables one to apply this theory to cases where, for example, the speed of an on-line digital computer is not sufficiently fast and, thus, results in a pure time delay in the loop.

1.3 SCOPE OF THE THESIS

Chapter 2 enfolds the concept of distributed correlation functions in stochastic linear distributed-parameter systems. A new definition of

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*Pierre and Kolb (67) define class-I-type distributed-parameter systems as those which ostensibly contain no space-distributed inputs or outputs and class-II-type distributed-parameter systems as those containing space-distributed inputs or space-distributed outputs or both.

distributed correlation function is introduced and is shown to be appropriate for distributed-parameter systems with distributed outputs. An optimal stochastic control problem is solved with the aid of this distributed correlation function.

Chapter 3 is devoted to the development of linear programming techniques for use in the optimal design of linear distributed-parameter systems with distributed outputs. The time-and-space summation of absolute values of distributed error, with weighted coefficients, at discrete time-and-space points is minimized subject to linear constraints to obtain desirable control functions.

In Chapter 4, optimization of linear distributed-parameter systems is treated from a dynamic programming point of view. Properties of functionals are derived which are similar to those which apply to lumped-parameter systems. Also, a closed-form expression of the optimal control function is derived for the case of a quadratic performance measure of distributed error.

In Chapter 5, the theory of deadbeat response with minimal overshoot compromise is generalized to apply to both minimal and nonminimal phase plants enfolding time delay.

Finally, the concluding Chapter 6 contains a summary of the essential values of the thesis research and some aspects of the research which merit additional study.

1.4 LITERATURE REVIEW

The literature on distributed-parameter systems is extensively reviewed by Kolb (39) and Pierre (60). In this section, a survey of current literature related to the general scope of this dissertation is made. The work directly related to the material in this dissertation is reviewed appropriately in the introductory sections of the chapters.

1.4.1 Optimization Techniques Applicable to Distributed-Parameter Systems

In 1960, the first major study of optimal control of distributed-parameter systems was made by Butkovskii and Lerner (12). They formulated three general optimal control problems associated with distributed-parameter systems. Later, Butkovskii (16) derived a maximum principle for distributed-parameter systems analogous to Pontryagin's maximum principle (69) for lumped-paramter systems. Solutions which satisfy the necessary conditions associated with this maximum principle are not obtainable in general without use of approximation (67). In reference (15), Butkovskii considered representations of distributed-parameter systems by corresponding spatially discretized models. By use of this approach, sets of partial differential equations characterizing such systems are reduced to sets of ordinary differential equations to which Pontryagin's maximum principle can be applied directly, but the usual difficulties of applying this principle to high-order systems is encountered. Also for this discretized model, Butkovskii (19) showed that a class of the optimal control problems associated with distributed-parameter systems can be solved by the method of moments (1). In reference (20), Butkovskii and Poltavskii solved the time-optimal control problem of the oscillating string with the aid of Krein's L-problem of moments (1) and variational methods.

The classical Krein's L-problem of moments (1) can be stated as follows: given n linearly independent vectors $h_1, h_2, \ldots, h_n$ of a normed linear space E and n real numbers $c^1, c^2, \ldots, c^n$, not all zero, and the positive real number L, find the necessary and sufficient conditions on $c^1, c^2, \ldots, c^n$, and L for there to exist a linear functional f(h) which satisfies

$$f(h_i) = c^i \tag{1.1}$$

and

$$\|f\| = \sup_{\|h\|} \frac{f(h)}{\|h\|} \leq L \tag{1.2}$$

In 1957, Krassovskii (43) reduced the time-optimal control problem in lumped-parameter system theory to Krein's L-problem of moments. He used some properties derived by Krein (1) to obtain the expression for an optimal input which is bounded in amplitude. Two years later, Kulikowskii (45, 46, 47, 48) showed the Krein's results can be directly applied to derive time-optimal control with amplitude, energy, power, and fuel constraints. Also, Kulikowskii (48, 49) used results from functional analysis to treat certain problems associated with nonlinear plants and adaptive systems. The works by Krassovskii and Kulikowskii were generalized by Kranc and Sarachik (42) to apply to the case of a time-optimal n^{th}-order system with r inputs. Recently, Chaudhuri (22) further generalized the works by Kranc and Sarachik (42) to apply to the optimal design of linear distributed-parameter systems. But the difficulties in determining the optimal time of the system (22) are quite apparent. Swiger (80) showed that the problem of minimum-effort control of distributed-parameter systems can be solved by the theory of minimum normed operators (53). For an exact solution, this approach requires solution of an infinite number of linear algebraic equations with an infinite number of unknowns. But if only a finite number of points in the spatial domain is used to represent the entire domain, the problem can be reduced to Krein's L-problem of moments. For a Mayer problem in distributed-parameter systems which contain nondistributed outputs, Katz (38) applied functional analysis and variational methods to formulate a general set of necessary conditions to be satisfied by an optimal system. Andreyer and Butkovskii (2) solved the time-optimal control problem for heating a massive body in a furnace by use of a spatial-harmonic-truncated model and functional analysis.

Of systems enfolding time delay, Newton, et al. (57), demonstrated the use of the Wiener-Hopf spectrum-factorization technique (64, 57) to effect minimum-integral-square or mean-square-error design. In order to effect the factorization, Pade approximations (84) were used to enable the application of the Wiener-Hopf technique in the usual manner. Pierre (62, 66) generalized the approach of Newton, et al. (57), to apply to linear distributed-parameter systems which contain no space-distributed inputs or outputs. By applying the Wiener-Hopf spectrum-factorization technique, Pierre derived necessary and sufficient conditions for the design of linear controllers to obtain a minimum-mean-square error. Kolb (39) and Kolb and Pierre (40) further generalized the Wiener-Hopf technique for the design of linear distributed-parameter systems with spatially distributed inputs or outputs or both. To effect the factorization, Pierre (62, 66), Kolb (39) and Kolb and Pierre (40) used rational fraction approximations (21, 41, 61) for the nonrational terms. Moreover, Kolb (39) discussed general classes of performance measures applicable in distributed-parameter systems. These performance measures paralleled those for lumped-parameter systems considered by Schultz and Rideout (75). Also, Kolb (39) clarified Senin's work (77) on the statistical design of distributed-parameter systems. In essense, the concepts of correlation function analysis were extended to include distributed-parameter systems.

Goodson and Khatri (30) used the calculus of variations to obtain a necessary condition for optimality of a distributed system described by an integral equation. The output of the system is not spatially distributed. To obtain an explicit form of the optimal feedback control system, truncation of an infinite series is required. For another approach (30), the transcendental Laplace-transform transfer function of the distributed-parameter plant is approximated by a few terms in the infinite product approximations used by Oldenburger and Goodson (58); then Pontryagin's maximum principle is applied to obtain the optimal feedback control law for the approximate model.

Using dynamic programming techniques, Brogan (11) reduced the optimal control problems for a class of distributed-parameter systems to the solution of a pair of two-point boundary-value partial differential equations. The specific form of the boundary conditions depends upon the particular problem being considered and must be obtained by considering transversality conditions. Numerical algorithms for solving the two-point boundary-value problems are developed. Also Brogan (11) showed that requirements for controllability (37) of lumped-parameter systems guarantee only ε-controllability when applied to distributed-parameter systems. A system is said to be ε-controllable if the state of the system can be forced into an epsilon neighborhood of the desired state within a finite amount of time, but it is not necessary for the system to be maintained in this neighborhood. Miranker (56) also derived necessary and sufficient conditions for ε-controllability of a certain class of distributed-parameter systems. To clarify the meaning of controllability of lumped-parameter systems Lemay (51, 52) defined the term reachability and maintainability. In essence, a system is said to have the reachable property if it can be driven from its initial state to the desired final state in finite time, and a system is said to have the maintainability property if controlling actions are capable of maintaining the system at this desired state throughout the remainder of the process. For distributed-parameter systems, Kolb (39) introduced the terms ε-reachability and ε-maintainability as follows: A system is said to be ε-reachable for a set of states if admissible control signals exist which will take the system from a given initial state to within an ε-neighborhood of any given desired state from the set of desired states in a finite time T(ε); and a system is said to be ε-maintainable for a set of states if it is ε-reachable for the set of states and if, in addition, admissible control signals exist which will maintain any of the states in the set within a specified ε-neighborhood for all T greater than T(ε). Also, Wang and Tung (86) investigated controllability and observability (37) of distributed-parameter systems.

McCausland (54, 55) developed three approximate methods, namely, a subdivision or finite difference method, a harmonic or Fourier series method, and a parabolic method, for synthesizing bang-bang controllers to obtain minimum-time performance for a linear diffusion process in which control inputs, applied at the spatial boundary, are subject to absolute-value constraints. In addition, McCausland (54, 55) investigated conditions for controllability of linear distributed-parameter systems.

Sakawa (74) derived necessary and sufficient conditions for optimality of one-dimensional linear stationary distributed-parameter systems which are controlled by boundary control functions. The performance measure used by Sakawa (74) is the spatial integral of the squared error at the terminal time, augmented by an energy measure of the control variables. When the control variables are not required to satisfy additional constraints, the optimal control is in the form of Fredholm integral equations of the second kind. But if the control functions are magnitude-limited, the optimal control is in the form of nonlinear integral equations. Egorov (25, 26) derived necessary and sufficient conditions, in the form of integral inequalities, for optimal, linear distributed-parameter systems. These same conditions are merely necessary conditions if the system is nonlinear.

Recently, several computational algorithms (3, 4, 11, 24 80) were developed to solve optimal control problems associated with distributed-parameter systems. Axelband (3, 4) applied function-space methods to effect the optimal control of distributed-parameter systems. In reference (4), Axelband suggested a convex programming algorithm (33) to solve the problem of minimizing the norm of the difference between desired system response and actual system response of linear distributed-parameter systems with bounded inputs. Denn, et al. (24), developed a computational algorithm based on the method of steppest ascent (33, 64) to solve an optimal control problem in conjunction with the necessary conditions for optimality which are derived by variational methods.

Erzberger and Kim (27) considered the problem of designing an optimum controller for boundary control of distributed-parameter systems based on a quadratic error criterion. By applying dynamic programming, Erzberger and Kim derived a Hamilton-Jacobi equation for distributed-parameter systems, under a quadratic error criterion, the general solution of which is not obtainable by use of existing methods.

Recently Yeh and Tou (88) used Butkovskii's maximum principle (16, 18) to derive an optimal control function for a class of distributed-parameter systems in terms of integral equations. Uzgiris and D'Souza (83) used the properties developed by Krein (1) to obtain a time-optimal control function for a plant described by a linear partial differential equation of parabolic type with nonlinear boundary conditions. The model used in reference (83) is the spatially discretized one. However, Uzgiris and D'Souza did not justify application of the Krein's L-problem (1) to a class of nonlinear functional equations.

1.4.2 Modeling

In recent years, much attention has been devoted to modeling distributed-parameter systems. Pierre (60) and Pierre and Higgins (65) used sampled-data systems (35, 50, 72, 82) to represent distributed-parameter systems. Distributed-parameter elements can be isolated from the lumped-parameter elements by suitably selected sample-and-hold circuits. Selection of the sampling period is discussed in the literature (36, 65). Recently, Gravey and D'Souza (32) modified one of Pierre's sampled-data models (60, 65) by using only one sample-and-hold circuit. Gaither, et al., (28) proposed two techniques, namely, quantization and integral transformation, for modeling distributed-parameter systems. If the spatial variables are quantized, i.e., broken into "lumps" while the time variable is continuous, partial differential equations are reduced to sets of ordinary differential equations. All the lumped-parameter techniques of modeling can then be applied. In the integral transformation method, only a finite number of spatial harmonics is considered in the model. Gaither, et al., found close agreement between the results in the frequency domain for both methods.

In his Ph.D. dissertation (39), Kolb used four different models for linear distributed-parameter systems. For distributed-parameter systems with boundary inputs and a distributed output, two models -- a series-form representation and a parallel-form representation -- are developed on the basis that only values of a distributed state variable at specific spatial points are of interest. In the series-form representation, the value of distributed state variable at a given point in the spatial domain is expressed in terms of the value of the distributed state variable at an adjoining point in the spatial domain. But in the parallel-form representation, the value of distributed state variable at each point of interest in the spatial domain is written in terms of the boundary inputs. The third model, the spatial-harmonic representation, is one in which a distributed-parameter system is represented by a harmonic series in the spatial variable. Generally, because of the presence of an infinite number of spatial harmonics, an infinite number of blocks is required to represent a distributed-parameter system exactly with this model. The fourth model developed by Kolb for linear distributed-parameter systems with distributed inputs and outputs involves implicit transfer functions between distributed input variables and outputs.

CHAPTER 2

CORRELATION FUNCTIONS AND OPTIMAL CONTROL OF DISTRIBUTED-PARAMETER SYSTEMS

2.1 INTRODUCTION

Methods of statistical analysis and synthesis are of importance for the design of controllers for systems which contain random parameters, measurement noises, or random forcing functions. A special form of correlation function for distributed-parameter systems was introduced by Itskovich (34) in 1963. Itskovich used a discrete form of correlation function to correlate output responses at two particular points in the space. Using this concept, he determined the optimum location of pickups in a distributed field. In 1964, Senin (76) extended Itskovich's ideas by defining a continuous form of correlation function. With his definition of correlation function, Senin (76,77) derived relationships between the correlation functions of the input and the output signals in a system characterized by the heat equation with several types of boundary conditions. Recently, Kolb (38) clarified and expanded the theory developed by Senin in reference (76).

In this chapter, a new definition of correlation function is introduced for use with distributed-paramter systems that are time and spatial in variant and that are excited by random signals which satisfy the ergodic hypothesis. In Section 2.2, relationships are derived which parallel well-known relationships in the statistical analysis of lumped-parameter systems. The optimal control problem of Section 2.3 is shown to be a Wiener-Hopf spectrum-factorization problem, the study of which is facilitated by the use of the new definition of distributed correlation function.

2.2 PROPERTIES OF THE DISTRIBUTED CORRELATION FUNCTION

In order to develop a method of statistical analysis in distributed-parameter systems in parallel to an existing method in lumped-parameter systems, the autocorrelation function of a distributed signal $y_1(x,t)$ whose domain of definition is $[0,L] \times (-\infty,\infty)$ is defined as

$$R_{y_1y_1}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_1(\xi,t)y_1(x+\xi,t+\tau)\, d\xi dt \tag{2.1}$$

and the cross-correlation function of two distributed signals $y_1(x,t)$ and $y_2(x,t)$ is defined as

$$R_{y_1y_2}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_1(\xi,t)y_2(x+\xi,t+\tau)\, d\xi dt \tag{2.2}$$

The definitions of the distributed correlation functions in Equations 2.1 and 2.2 are somewhat different from those used by Senin (76, 77). Senin (76) defined the autocorrelation function of a distributed signal as

$$R_{y_1y_1}(x,z,\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_1(x,t)y_1(z,t+\tau)\, dt \tag{2.3}$$

The advantage to the definition of the distributed autocorrelation function in Equation 2.1 is that when $x = 0$ and $\tau = 0$, the distributed autocorrelation function reduces to the time-and-space mean-square value of $y_1(x,t)$. This is not the case for the distributed autocorrelation function of Equation 2.3. When $x = z = x_1$ and $\tau = 0$, the distributed correlation function of Equation 2.3 yields only the time mean-square value of $y_1(x,t)$ at the point $x = x_1$. Therefore, the definition of the correlation function in Equation 2.3 is more appropriate if the output response of a system is not distributed; if the output is distributed, the definition of the correlation function in Equation 2.1 is more appropriate.

In conjunction with the definition of distributed cross-correlation function in Equation 2.2, the cross-correlation function associated with $y_1(x,t)$ and $y_2(x,t)$ can be written as

$$R_{y_2y_1}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_2(\xi,t)y_1(x+\xi,t+\tau)\, d\xi dt \tag{2.4}$$

By change of dummy variables of integration, it can be shown that

$$R_{y_1y_2}(x,\tau) = R_{y_2y_1}(-x,-\tau) \tag{2.5}$$

A correlation function between a distributed signal and a lumped signal is of use in statistical analysis when a boundary input of a distributed-parameter system is stochastic in nature. Therefore, the cross-correlation function associated with a distributed signal $y_1(x,t)$ and a lumped signal $y_3(t)$ is defined here as

$$R^M_{y_1y_3}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_1(x,t)y_3(t+\tau)\, dt \tag{2.6}$$

The superscript M of R in Equation 2.6 denotes the mixture of the distributed and lumped signals in the correlation function. It can be shown that

$$R^M_{y_1y_3}(x,\tau) = R^M_{y_3y_1}(x,-\tau) \tag{2.7}$$

where

$$R^M_{y_3y_1}(x,\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_3(t)y_1(x,t+\tau)\, dt \tag{2.8}$$

To use correlation functions in the analysis and synthesis of linear distributed-parameter systems, relationships between correlation functions of distributed output signals and input signals of a system must be established. For simplicity, consider a distributed-parameter stable linear system, having a single stochastic input u(t) and a single distributed output y(x,t), which is characterized by

$$y(x,t) = \int_{-\infty}^{\infty} g(x,\sigma)u(t-\sigma)\, d\sigma \tag{2.9}$$

where g(x,t) is the Green's function or impulse response of the system.

The autocorrelation function associated with the distributed signal y(x,t) is given by the definition in Equation 2.1. After appropriate substitution of the right-hand member of Equation 2.9 into Equation 2.1, followed by an interchange in the order of integration,* it reads

$$R_{yy}(x,\tau) = \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt u(t-\sigma_1)u(t+\tau-\sigma_2)$$

$$= \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2)R_{uu}(\tau+\sigma_1-\sigma_2) \tag{2.10}$$

where $R_{uu}(\tau)$ is the autocorrelation function associated with the lumped signal u(t).

Operationally, Equation 2.10 can be simplified by use of transform theory. Following a similar definition used for lumped-parameter systems,

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*In this chapter, the integrand is assumed to be integrable so that the interchange of the order of integration is valid.

the distributed spectral density $S_{yy}(x,s)$ is defined as the two-sided Laplace transform with respect to τ of $R_{yy}(x,\tau)$. $R_{yy}(x,\tau)$ is, of course, assumed to be Laplace transformable. Thus,

$$S_{yy}(x,s) = \int_{-\infty}^{\infty} d\tau R_{yy}(x,\tau)e^{-s\tau} \tag{2.11}$$

When the right-hand member of Equation 2.10 is substituted for $R_{yy}(x,\tau)$ in Equation 2.11, the result is

$$S_{yy}(x,s) = \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2) \int_{-\infty}^{\infty} d\tau R_{uu}(\tau+\sigma_1-\sigma_2)e^{-s\tau}$$

$$= \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1)e^{-s\sigma_1} \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2)e^{s\sigma_2}$$

$$\int_{-\infty}^{\infty} d\sigma_3 R_{uu}(\sigma_3)e^{-s\sigma_3}$$

$$= \frac{1}{L}\int_0^L d\xi G(\xi,s)G(x+\xi,-s)S_{uu}(s) \tag{2.12}$$

Another important property of the distributed correlation function merits consideration: the relationship of the autocorrelation function of a distributed error $e(x,t)$ in terms of correlations of an actual output $y(x,t)$ and a desired output $y^d(x,t)$. The distributed error $e(x,t)$ is defined as the difference between the desired response and the actual output. Mathematically,

$$e(x,t) = y^d(x,t) - y(x,t) \tag{2.13}$$

By definition,

$$R_{ee}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_0^L d\xi e(\xi,t)e(x+\xi,t+\tau)$$

$$= \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi [y^d(\xi,t) - y(\xi,t)]\, [y^d(x+\xi,t+\tau) - y(x+\xi,t+\tau)]$$

$$= R_{y^dy^d}(x,\tau) - R_{yy^d}(x,\tau) - R_{y^dy}(x,\tau) + R_{yy}(x,\tau) \tag{2.14}$$

The corresponding distributed spectral density $S_{ee}(x,s)$ is

$$S_{ee}(x,s) = S_{y^dy^d}(x,s) - S_{yy^d}(x,s) - S_{y^dy}(x,s) + S_{yy}(x,s) \tag{2.15}$$

The last two relationships to be derived are the relationship between $S_{yy^d}(x,s)$ and $S_{uy^d}(x,s)$ and that between $S_{y^dy}(x,s)$ and $S_{uy^d}(x,s)$. By definition in Equation 2.1,

$$R_{yy^d}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi y(\xi,t) y^d(x+\xi,t+\tau)$$

$$= \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi y^d(x+\xi,t+\tau) \int_{-\infty}^{\infty} d\sigma u(t-\sigma) g(\xi,\sigma)$$

$$= \frac{1}{L} \int_{0}^{L} d\xi \int_{-\infty}^{\infty} d\sigma g(\xi,\sigma) R^M_{uy^d}(x+\xi,\tau+\sigma) \tag{2.16}$$

And by use of Equation 2.5, it can be shown that

$$R_{y^dy}(x,\tau) = \frac{1}{L} \int_{0}^{L} d\xi \int_{-\infty}^{\infty} d\sigma g(x+\xi,\sigma) R^M_{y^du}(\xi,\tau-\sigma) \tag{2.17}$$

The cross-spectral density $S_{yy^d}(x,s)$ is obtained by Laplace transformation of both members of Equation 2.16, with the result that

$$S_{yy^d}(x,s) = \frac{1}{L} \int_{0}^{L} d\xi G(\xi,s) S^M_{uy^d}(x+\xi,s) \tag{2.18}$$

and similarly,

$$S_{y^dy}(x,s) = \frac{1}{L} \int_{0}^{L} d\xi G(x+\xi,-s) S^M_{y^du}(\xi,s) \tag{2.19}$$

The relationship between $S^M_{uy_d}(x,s)$ and $S^M_{y_d u}(x,s)$ is readily established on the basis of the corresponding correlation function relationship in Equation 2.7; thus,

$$S^M_{y_d u}(x,s) = S^M_{uy_d}(x,-s) \qquad (2.20)$$

With the result of Equation 2.20, Equation 2.19 becomes

$$S_{y_d y}(x,s) = \frac{1}{L} \int_0^L d\xi G(x+\xi,-s) S^M_{uy_d}(\xi,-s) \qquad (2.21)$$

2.3 OPTIMAL STOCHASTIC CONTROL

For lumped-parameter systems, a basic statistical design criterion is the mean-square-error performance measure. By analogy, statistical design of distributed-parameter systems with distributed outputs may be effected with respect to time-and-space mean-square error; namely,

$$P = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_0^L \{[e(x,t)]^2\} \, dxdt \qquad (2.22)$$

An augmented version of the mean-square-error criterion is

$$P_A = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_0^L \{[e(x,t)]^2 + \lambda u^2(t)\} \, dxdt \qquad (2.23)$$

where λ is a "cost factor" associated with the input $u(t)$.

Use of Parseval's theorem gives the frequency-domain version of Equation 2.23:

$$P_A = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [S_{ee}(0,s) + \lambda S_{uu}(s)] \, ds \qquad (2.24)$$

where $S_{ee}(x,s)$ is the distributed spectral-density function of $e(x,t)$, and $S_{uu}(s)$ is the lumped spectral-density function of $u(t)$.

Figure 2.1 depicts a general linear stable distributed-parameter system with a stochastic boundary control signal. It is assumed that the external input $v(t)$ is a stochastic signal whose spectral density $S_{vv}(s)$ is known. Also, the cross-correlation function $R^M_{vy^d}(x,\tau)$ is assumed to be known, where $R^M_{vy^d}(x,\tau)$ is the cross-correlation function of the external input $v(t)$ and the desired response $y^d(x,\tau)$. $R^M_{vy^d}(x,\tau)$ can be determined experimentally, if necessary, by application of well-established techiques (5,79).

The problem now is to determine the transfer function $G_c(s)$ which yields the minimum of the augmented performance measure P_A in Equation 2.23.

Before proceeding with the minimization, the relationship between $R^M_{uy^d}(x,\tau)$ and $R^M_{vy^d}(x,\tau)$ must be derived. By the convolution integral, the characteristic equation of the lumped-parameter compensator in Figure 2.1 is

$$u(t) = \int_{-\infty}^{\infty} g_c(\sigma)v(t-\sigma)\, d\sigma \qquad (2.25)$$

Based on Equation 2.6, the cross-correlation function $R^M_{uy^d}(x,\tau)$ can be written as

$$\begin{aligned} R^M_{uy^d}(x,\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, u(t) y^d(x,t+\tau) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, y^d(x,t+\tau) \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma) v(t-\sigma) \\ &= \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma) \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, y^d(x,t+\tau) v(t-\sigma)\right] \\ &= \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma) R^M_{vy^d}(x,\tau+\sigma) \end{aligned} \qquad (2.26)$$

The corresponding spectral density $S^M_{uy^d}(x,s)$ is

$$S^M_{uy^d}(x,s) = G_c(s)S^M_{vy^d}(x,s) \quad (2.27)$$

From statistical analysis methods of lumped-parameter theory, the relationship between $S_{uu}(s)$ and $S_{vv}(s)$ can be written as

$$S_{uu}(s) = G_c(s)G_c(-s)S_{vv}(s) \quad (2.28)$$

With the results in Equations 2.12, 2.15, 2.18, 2.21, 2.27, and 2.28, the expression for $S_{ee}(0,s)$ can be written as

$$S_{ee}(0,s) = S_{y^dy^d}(0,s) - \frac{1}{L}\int_0^L d\xi G(\xi,s)S^M_{vy^d}(\xi,s)G_c(s) + - \frac{1}{L}\int_0^L d\xi G(\xi,-s)S^M_{vy^d}(\xi,-s)G_c(-s) + + \frac{1}{L}\int_0^L d\xi G(\xi,s)G(\xi,-s)S_{vv}(s)G_c(s)G_c(-s) \quad (2.29)$$

With the results in Equations 2.28 and 2.29, Equation 2.24 can be arranged as

$$P_A = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [A_4(s) + A_3(s)G_c(-s) + A_2(s)G_c(s) + A_1(s)G_c(s)G_c(-s)]\, ds \quad (2.30)$$

where

$$A_1(s) = [\lambda + \frac{1}{L}\int_0^L d\xi G(\xi,s)G(\xi,-s)]S_{vv}(s) \quad (2.31)$$

$$A_2(s) = -\frac{1}{L}\int_0^L d\xi G(\xi,s)S^M_{vy^d}(\xi,s) \quad (2.32)$$

$$A_3(s) = -\frac{1}{L}\int_0^L d\xi G(\xi,-s) S^M_{vy^d}(\xi,-s) \qquad (2.33)$$

$$A_4(s) = S_{y^d y^d}(0,s) \qquad (2.34)$$

It is shown by Pierre (64), with calculus of variations and the Wiener-Hopf spectrum-factorization technique, that if $g_c(t)$ is to be zero for t less than zero and not both of $A_2(s)$ and $A_3(s)$ are zero, the optimal expression $G_c^*(s)$ of $G_c(s)$ is given by

$$G_c^*(s) = \frac{-\left\{\frac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\right\}_+}{[A_1(s) + A_1(-s)]^+} \qquad (2.35)$$

where, in this case,

$$A_2(-s) + A_3(s) = -\frac{2}{L}\int_0^L d\xi G(\xi,-s) S^M_{vy^d}(\xi,-s) \qquad (2.36)$$

and

$$A_1(s) + A_1(-s) = 2\left[\lambda + \frac{1}{L}\int_0^L d\xi G(\xi,s) G(\xi,-s)\right] S_{vv}(s) \qquad (2.37)$$

The interpretation of the + and - superscripts and subscripts is as follows:

$$A(s) = A(s)^+ A(s)^- \qquad (2.38)$$

and

$$A(s) = A(s)_+ + A(s)_- \qquad (2.39)$$

where the inverse Laplace transforms $L^{-1}[A(s)^+]$ and $L^{-1}[A(s)_+]$ equal zero for t less than zero, and where $L^{-1}[A(s)^-]$ and $L^{-1}[A(s)_-]$ equal zero for t greater than zero. In addition, for $G_c^*(s)$ to be physically realizable, $1/A(s)^+$ must be analytic in the right half-plane of the s domain.

When all A_i's are rational fraction functions, this process of spectrum factorization is straightforward (57, 64). When nonrational fraction terms appear in the A_i's, rational fraction approximations may be used to replace the nonrational terms (21, 41, 58, 61 and 84), and then the process of spectrum factorization can be effected to obtain an approximate solution. Examples of such approximate solutions can be found in the literature (38, 62).

2.4 CONCLUSION

A correlation-function concept for use with distributed-parameter systems is introduced to parallel the existing concept which applies to lumped-parameter systems. Relationships between correlation functions of distributed output signals and input signals of a system are established. With the use of the new definition of distributed correlation functions, an optimal control problem associated with distributed-parameter systems with random signals is shown to be solvable by the Wiener-Hopf spectrum-factorization technique. Though the spatial domain of the system considered in this chapter is of one dimension, the general approach is applicable to systems of higher dimensions, in which case the autocorrelation function is defined as

$$R_{yy}(X,\tau) = \frac{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{V} y(\xi,t)\, y(X+\xi,t+\tau)\, d\xi dt}{\int_{V} d\xi}$$