



Sinc domain decomposition methods for elliptic problems
by Nancy Jean Lybeck

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in
Mathematical Sciences
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Abstract:

Sinc basis functions form a desirable basis to use for solving singular problems via domain decomposition. This is because both the Sinc-Galerkin and sinc-collocation methods converge exponentially, even in the presence of boundary singularities. For Poisson's equation with homogeneous Dirichlet boundary conditions posed on a rectangular domain, the Sinc-Galerkin and sinc-collocation methods have been well developed. The sinc methods have also been developed for any domain which can be mapped to a rectangular domain via an invertible or conformal mapping. In order to increase the number and complexity of domains which can be handled via sinc methods, domain decomposition techniques are used.

The Sinc-Galerkin and sinc-collocation domain decomposition methods are first studied for a two-point boundary-value problem. Both of the traditional methods of domain decomposition, overlapping and patching, are developed. This lays the groundwork to readily determine which method is most suited to any given problem. Because the goal is to clearly develop and test the sinc domain decomposition methods, techniques such as subdomain iterations and preconditioning are not employed here. The number of subdomains is limited to two in order to limit the complexity of the presentation. Numerical results are presented for both decomposition methods that exhibit the nearly identical errors achieved whether one uses the sinc-collocation or Sinc-Galerkin method.

Next the patching and overlapping Sinc-Galerkin methods are presented for Poisson's equation presented on a rectangle. For certain parameter choices the sinc-collocation system is identical for these problems, and is thus not presented separately. Again the number of subdomains is limited to two in order to present the material more clearly. Both domain decomposition methods perform well, and this is highlighted in the numerical examples.

Finally, Poisson's equation is studied on an ϵ -shaped domain. In the derivation of the discrete system, it becomes evident that the patching domain decomposition method is the method of choice for this problem. The derivation and numerical examples are presented using three subdomains, although multiple subdomains could certainly be used. Numerical examples illustrate the ability of this method to handle these types of problems.

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ABSTRACT

Sinc basis functions form a desirable basis to use for solving singular problems via domain decomposition. This is because both the Sinc-Galerkin and sinc-collocation methods converge exponentially, even in the presence of boundary singularities. For Poisson's equation with homogeneous Dirichlet boundary conditions posed on a rectangular domain, the Sinc-Galerkin and sinc-collocation methods have been well developed. The sinc methods have also been developed for any domain which can be mapped to a rectangular domain via an invertible or conformal mapping. In order to increase the number and complexity of domains which can be handled via sinc methods, domain decomposition techniques are used.

The Sinc-Galerkin and sinc-collocation domain decomposition methods are first studied for a two-point boundary-value problem. Both of the traditional methods of domain decomposition, overlapping and patching, are developed. This lays the groundwork to readily determine which method is most suited to any given problem. Because the goal is to clearly develop and test the sinc domain decomposition methods, techniques such as subdomain iterations and preconditioning are not employed here. The number of subdomains is limited to two in order to limit the complexity of the presentation. Numerical results are presented for both decomposition methods that exhibit the nearly identical errors achieved whether one uses the sinc-collocation or Sinc-Galerkin method.

Next the patching and overlapping Sinc-Galerkin methods are presented for Poisson's equation presented on a rectangle. For certain parameter choices the sinc-collocation system is identical for these problems, and is thus not presented separately. Again the number of subdomains is limited to two in order to present the material more clearly. Both domain decomposition methods perform well, and this is highlighted in the numerical examples.

Finally, Poisson's equation is studied on an el -shaped domain. In the derivation of the discrete system, it becomes evident that the patching domain decomposition method is the method of choice for this problem. The derivation and numerical examples are presented using three subdomains, although multiple subdomains could certainly be used. Numerical examples illustrate the ability of this method to handle these types of problems.

CHAPTER 1

INTRODUCTION

The subject of this thesis is the solution of Poisson's equation via domain decomposition coupled with sinc methods. For a general domain $\Omega \in \mathbb{R}^2$, Poisson's equation with homogeneous Dirichlet boundary conditions is given by

$$\begin{aligned} -\nabla^2 u(x, y) &\equiv -\Delta u(x, y) = f(x, y), \quad (x, y) \in \Omega \\ u(x, y) &= 0, \quad (x, y) \in \partial\Omega. \end{aligned} \tag{1.1}$$

Many methods for Poisson's equation have been highly developed. If Ω is a rectangle, for example, using a finite difference method leads to a sparse matrix system which can be efficiently solved with specialized techniques for banded matrices. One such difference method has a quadratic rate of convergence depending on the properties of the fourth-order partials of the solution. See [3] for a development of this method and numerical results. Similarly, one can develop a finite element method (see [12]) that has a quadratic rate of convergence depending on the properties of the second-order partials of the solution. The matrix system has the same structure as that arising from a centered difference method.

Sinc methods for Poisson's equation on a rectangle have been well-studied in [2], [17], [18], [21], [23], and [27]. They are desirable methods to use for a variety of reasons. First, the sinc approximations converge exponentially to the true solution. The tradeoff for this rate of convergence is a full matrix system, as is true of most spectral methods. Since the procedure is a product method, its most convenient application occurs when $\Omega = I \times J$, where I and J are intervals.

Each method for solving partial differential equations excels on a particular class of problems. For example, finite differences yield simple methods which work well for problems with analytic solutions. The finite element methods work well for these problems and can more easily handle complicated geometries and boundary conditions. The sinc methods excel for problems with boundary singularities, as discussed in [27]. The convergence estimates for both the finite difference and finite element methods depend on the smoothness of the partial derivatives of the solution. Thus there is no guarantee that they will perform well on problems with any singularity. This is also true of other spectral methods. Thus the sinc methods are in a class of their own when it comes to dealing with boundary singularities.

If the domain Ω is not a rectangle there are two basic methods for the discretization of (1.1). The first method involves redeveloping the discrete system and the error analysis for each new domain. Finite element methods take this approach. The second approach invokes a change of coordinates to exchange the domain Ω for a domain on which the numerical method has been previously developed. In this case, the Laplacian becomes a more general elliptic operator. This method has received less attention than it warrants due to the fact that the coefficients of the transformed Laplacian may be singular. This has no effect on the sinc implementation or resulting calculations and accuracy. The method developed in this thesis, in combination with both of the above methods, is advocated in [27], and handles any domain whose boundary consists of finitely many analytic arcs.

Extensions to more general domains seem possible only if the domain can be split into two or more pieces, each of which could be mapped to its own rectangle. The solutions in each subdomain must then be matched in some manner. This leads to a need for general domain decomposition methods.

Domain decomposition techniques have been of great interest lately, especially

with the advancement of parallel computing technology. A series of conferences on domain decomposition methods began in Paris in 1987 with the First International Symposium on Domain Decomposition Methods for Partial Differential Equations. The proceedings from each of these conferences is a good source of information on domain decomposition. For example, [9] contains the proceedings from the conference held in Moscow in May, 1990. Applications of these methods include field-scale simulations of fluid flow in porous media and two-dimensional convection-diffusion problems. See [5] and [8] for more details on these applications. By breaking these large-scale problems into multiple subproblems, parallel processors may be used to efficiently solve these problems using iterative techniques.

Having made the decision to decompose the domain Ω , there are two traditional methods of handling the decomposition: patching and overlapping. When the problem at hand does not motivate one method over another it is natural to ask which method is preferable. Such comparisons must include a measure of accuracy balanced with respect to implementation considerations. In certain cases, the two methods can be shown to be related, if not identical. See [6] for more details. When iterative procedures are used to solve the problems on parallel computers, the patching method has a lower overhead cost. On the other hand, the overlapping method is considered to be more robust (see [5]). Due to the potential advantages of each method, this thesis will carry out the discretization using both decompositions for the sinc basis.

As in any product method, there is a clarity of presentation imported by first fully understanding the implementation of the procedure for the one-dimensional problem. For this reason, the patching and overlapping methods are carried out for the sinc methods on an interval. Sinc methods here refer to both the Sinc-Galerkin and sinc-collocation procedures, which are introduced in Chapter 2. These procedures complement one another and provide the link to establish (numerically) the

convergence of the procedure. These two methods are spelled out in Chapter 3 and the examples included show that, with respect to accuracy and implementation, they are numerically equivalent.

Chapter 4 presents both the patching and overlapping methods for Poisson's equation on a rectangle. In this chapter, only two subdomains are used. Because the Sinc-Galerkin and sinc-collocation methods are the same for Poisson's equation with an appropriate choice of weights, only the Sinc-Galerkin method is discussed. Again, both the patching and overlapping methods perform equally well, as seen in the examples.

Chapter 5 addresses the solution of Poisson's equation on an el-shaped domain. This is the final tool needed for solving such equations on more complex domains. The method used for the development of the discrete system mandates that the subdomains must not overlap, and at least three subdomains must be used for this domain. Thus the patching method is developed for use with three subdomains, though multiple subdomains are possible. The numerical results are quite good for this method.

CHAPTER 2

SINC METHODS FOR DIFFERENTIAL EQUATIONS

Introduction

Sinc methods for differential equations have been well-studied since their introduction in [25]. They have been applied to a variety of differential equations such as two-point boundary-value problems, Poisson's equation, the wave equation, the heat equation, the advection-diffusion equation, and Burgers' equation. Both the Sinc-Galerkin and sinc-collocation methods are well-suited for problems with boundary singularities. They also both converge exponentially, even in the presence of such boundary singularities. For an overview of sinc methods for differential equations see [17], [26], and [27].

The second section presents an introduction to sinc interpolation and quadrature methods. These are necessary tools for deriving the Sinc-Galerkin and the sinc-collocation methods for solving differential equations. For problems with constant coefficients, the Sinc-Galerkin method might well be the method of choice. For problems with variable coefficients, the sinc-collocation method is especially convenient because the coefficients are more efficiently handled. In order to leave a clear path for future work; both methods are presented here. The Sinc-Galerkin method is developed in the third section and the sinc-collocation method is derived in the fourth section.

Sinc Interpolation and Quadrature Methods

The following sinc interpolation and quadrature results are presented in detail in [17], [26], and [27]. For $h > 0$ and any integer j , define the sinc translates on the real line by

$$S(j, h)(x) \equiv \operatorname{sinc}\left(\frac{x - jh}{h}\right)$$

where for $z \in \mathbb{C}$

$$\operatorname{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0 \\ 1, & z = 0 \end{cases}.$$

Three examples of these translates are shown in Figure 1.

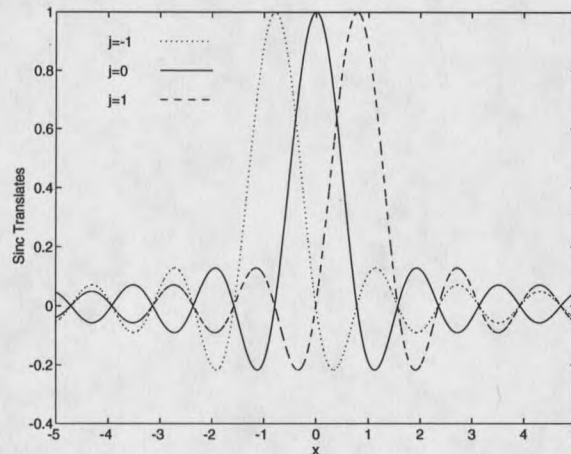


Figure 1: The sinc translates $S(j, h)(x)$ for $h = \pi/4$ shown on $[-5, 5]$

In order to present the interpolation and quadrature results on the real line, the Paley-Wiener class of functions is defined as follows.

Definition 2.1 *Let h be a positive constant. The Paley-Wiener class of functions $B(h)$ is the family of entire functions f such that on the real line $f \in L^2(\mathbb{R})$ and in the complex plane f is of exponential type π/h , i.e., there exists $K > 0$ so that*

$$|f(z)| \leq K \exp(\pi|z|/h)$$

for all $z \in \mathbb{C}$.

The sinc interpolation and quadrature methods are exact on members of the Paley-Wiener class of functions, as seen in Theorem 2.1.

Theorem 2.1 *If $f \in B(h)$, then for all $z \in \mathbb{C}$,*

$$f(z) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(z) .$$

Furthermore, if $f \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(u) du = h \sum_{k=-\infty}^{\infty} f(kh) .$$

See [22] and [24] for the proof of the first and second parts, respectively, of this theorem.

The Paley-Wiener class of functions is quite restrictive. For practical applications, a larger class of functions on which these methods perform well is desirable. Define the infinite strip D_S by

$$D_S \equiv \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\} . \quad (2.1)$$

D_S is shown in Figure 2.

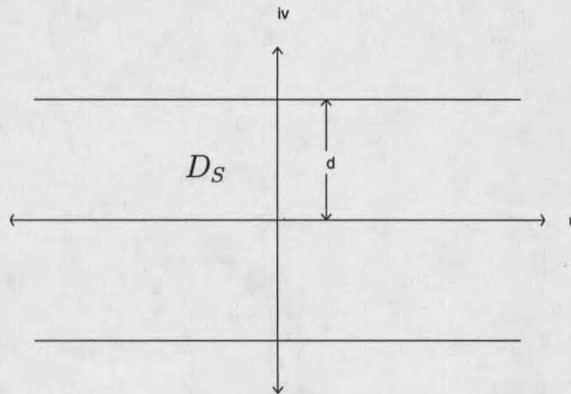


Figure 2: The domain D_S

In order to extend the sinc interpolation and quadrature results, define the class of functions $B^p(D_S)$.

Definition 2.2 Let $B^p(D_S)$ be the set of functions analytic in D_S that satisfy

$$\int_{-d}^d |f(u + iv)| dv = \mathcal{O}(|u|^a), \quad u \rightarrow \pm\infty, \quad 0 \leq a < 1$$

and

$$\begin{aligned} N^p(f, D_S) &\equiv \lim_{v \rightarrow d^-} \left\{ \left(\int_{-\infty}^{\infty} |f(u + iv)|^p du \right)^{1/p} \right. \\ &\quad \left. + \left(\int_{-\infty}^{\infty} |f(u - iv)|^p du \right)^{1/p} \right\} \\ &< \infty. \end{aligned}$$

As seen in Theorem 2.2, proven in [17], the sinc interpolation and quadrature methods perform quite well on this class of functions.

Theorem 2.2 If $f \in B^p(D_S)$, $p = 1$ or 2 then there exists a positive constant K_1 such that

$$\left\| f - \sum_{k=-\infty}^{\infty} f(kh)S(k, h) \right\|_{\infty} \leq K_1 \exp(-\pi d/h).$$

Additionally, if $p = 1$ then there exists a positive constant K_2 such that

$$\left| \int_{-\infty}^{\infty} f(u) du - h \sum_{k=-\infty}^{\infty} f(kh) \right| \leq K_2 \exp(-2\pi d/h).$$

In practice, only finite sums can be calculated. The effect of truncation can be minimized by assuming appropriate growth conditions on f , as summarized in the following theorem proved in [24].

Theorem 2.3 Assume $f \in B^p(D_S)$ for $p = 1$ or 2 and that there are positive constants α , β , and C so that

$$|f(u)| \leq C \begin{cases} \exp(-\alpha|u|), & u \in (-\infty, 0) \\ \exp(-\beta u), & u \in [0, \infty) \end{cases}$$

Make the selections

$$N = \left\lceil \left\lceil \frac{\alpha}{\beta} M + 1 \right\rceil \right\rceil$$

and

$$h = \left(\frac{\pi d}{\alpha M} \right)^{1/2},$$

where $[\cdot]$ denotes the greatest integer function. Then there exists $K_3 > 0$, independent of M , so that

$$\left\| f - \sum_{k=-M}^N f(kh)S(k, h) \right\|_{\infty} \leq K_3 \exp(-(\pi d \alpha M)^{1/2}).$$

Additionally, if $p = 1$ and

$$h = \left(\frac{2\pi d}{\alpha M} \right)^{1/2},$$

then there exists $K_4 > 0$ independent of M so that

$$\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^N f(kh) \right| \leq K_4 \exp(-2\pi d \alpha M)^{1/2}.$$

To solve problems on the finite interval (a, b) , use the conformal map

$$\phi(z) = \ln \left(\frac{z-a}{b-z} \right). \quad (2.2)$$

This map carries the eye-shaped region

$$D_E \equiv \left\{ z = x + iy : \left| \arg \left(\frac{z-a}{b-z} \right) \right| < d \leq \frac{\pi}{2} \right\} \quad (2.3)$$

onto the infinite strip D_S in (2.1). An example of D_E is shown in Figure 3.

To describe the sinc quadrature and interpolation rules, the function space $B(D_E)$ is defined as follows.

Definition 2.3 Let D_E be the domain described in (2.3) in the $z = x + iy$ plane with boundary points $a \neq b$ on the real line. Let $w = \phi(z)$ be the conformal map of D_E onto the infinite strip D_S given in (2.2). Denote by $z = \tau(w)$ the inverse of the mapping ϕ and let

$$\Gamma \equiv \{z \in \mathbb{C} : z = \tau(u), u \in \mathbb{R}\} = \tau(\mathbb{R}) = (a, b).$$

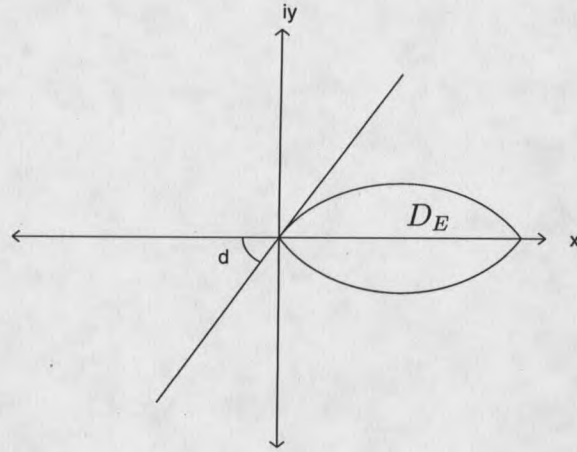


Figure 3: The domain D_E for $d = \pi/3$, $a = 0$, and $b = 1$

Then $B(D_E)$ is defined as the class of functions analytic in D_E which satisfy for some constant a with $0 \leq a < 1$,

$$\int_{\tau(u+L)} |F(z)dz| = \mathcal{O}(|u|^a), \quad u \rightarrow \pm\infty$$

where $L = \{iv : |v| < d\}$ and for γ a simple closed contour in D_E

$$N(F, D_E) \equiv \lim_{\gamma \rightarrow \partial D_E} \int_{\gamma} |F(z)dz| < \infty.$$

Further, for $h > 0$ and $k = 0, \pm 1, \pm 2, \dots$, denote the nodes

$$x_k \equiv \tau(kh) = \phi^{-1}(kh) = \frac{a + be^{kh}}{e^{kh} + 1}. \quad (2.4)$$

The sinc quadrature rule on (a, b) is contained in the following result.

Theorem 2.4 Let $F \in B(D_E)$ with $h > 0$, let ϕ be the one-to-one conformal map given in (2.2), and let $x_k = \phi^{-1}(kh)$ be as given in (2.4). Assume that there are positive constants α , β , and C , so that

$$|F(x)| \leq C \begin{cases} (x-a)^{\alpha-1}, & x \in (a, (a+b)/2) \\ (b-x)^{\beta-1}, & x \in [(a+b)/2, b) \end{cases}.$$

Make the selections

$$N = \left\lceil \left\lfloor \frac{\alpha}{\beta} M + 1 \right\rfloor \right\rceil$$

and

$$h = \left(\frac{2\pi d}{\alpha M} \right)^{1/2}$$

Then there exists a constant $K_5 > 0$, independent of M , so that

$$\left| \int_a^b F(x) dx - h \sum_{k=-M}^N \frac{F(x_k)}{\phi'(x_k)} \right| \leq K_5 \exp \left(-(2\pi d \alpha M)^{1/2} \right).$$

Two-Point Boundary-Value Problem

The linear two-point boundary-value problem with homogeneous Dirichlet boundary conditions on the finite interval (a, b) is given by

$$\begin{aligned} \mathcal{L}u(x) &\equiv -u''(x) + p(x)u'(x) + q(x)u(x) \\ &= f(x), \quad a < x < b \\ u(a) &= u(b) = 0. \end{aligned} \tag{2.5}$$

Sinc methods for problems of this type are discussed in detail in [16], [17], [25], and [26].

The basis functions used in solving (2.5) are defined by

$$S_j(x) = S(j, h) \circ \phi(x), \tag{2.6}$$

where ϕ is given in (2.2). A graph of these basis functions can be seen in Figure 4.

The approximate solution is given by

$$u_m(x) = \sum_{k=-M}^N u_k S_k(x), \quad m = M + N + 1, \tag{2.7}$$

where $h > 0$ is fixed.

The Sinc-Galerkin Method

The Sinc-Galerkin method for (2.5) is clearly developed in [16], [17], [25], and [26]. The Galerkin method requires orthogonalizing the residual $\mathcal{L}u_m - f$ against each

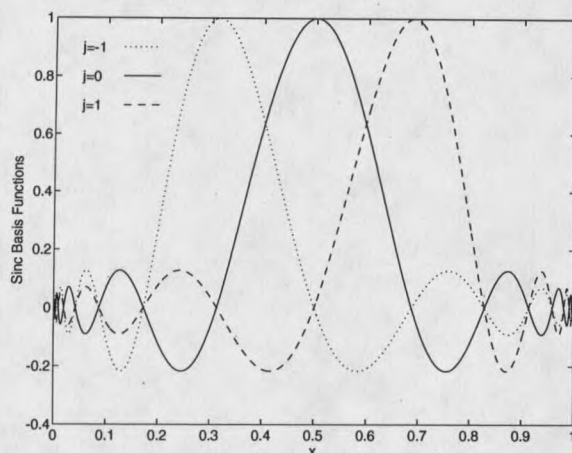


Figure 4: The sinc basis functions $S_j(x)$ for $h = \pi/4$ shown on $(0, 1)$

basis function using a weighted L^2 inner product. To simplify notation, orthogonalize $\mathcal{L}u - f$ against each basis function yielding

$$(\mathcal{L}u - f, S_j) = 0, \quad -M \leq j \leq N,$$

where

$$(f, g) = \int_a^b f(x)g(x)\omega(x)dx$$

and

$$\omega(x) = (\phi'(x))^{-r}, \quad r \geq 0.$$

Thus for $-M \leq j \leq N$,

$$\int_a^b (-u''(x) + p(x)u'(x) + q(x)u(x) - f(x)) S_j(x)\omega(x)dx = 0.$$

Integrating by parts to remove all derivatives from u yields

$$\begin{aligned} \int_a^b f(x)S_j(x)\omega(x)dx &= - \int_a^b u(x) (S_j\omega)''(x)dx \\ &- \int_a^b u(x) (pS_j\omega)'(x)dx + \int_a^b u(x)q(x)S_j(x)\omega(x)dx + BT \end{aligned} \tag{2.8}$$

where

$$BT = (upS_j\omega)(x) \Big|_a^b - (u'S_j\omega)(x) \Big|_a^b + (u(S_j\omega)')(x) \Big|_a^b.$$

