Calculus in limit spaces
by Kent Franklin Carlson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
The purpose of this paper is to examine the axiomatic characterization of a differential calculus as proposed by Wehrli (8) for the case of linear limit spaces over an arbitrary field in view of the usual notion of a derivative. Whether or not this characterization is satisfactory is partially dependent upon an open question in the theory of limit spaces which I have answered in the form of the following theorem: Let E and F be linear Limit spaces over a field K with limitations Λ and Λ' . respectively, and let Λ0and Λ'0 be the associated principle ideal limitations. Then if Λ x Λ' represents the product limitation of E x F, (Λ x Λ')0 = Λ0 x Λ'0. With this theorem the axiomatic characterization proposed by Wehrli is shown to be quite natural in regard to the usual notion of differentiation on linear limit spaces. It is then shown that the remainders introduced by Binz (1), together with the requirement that the set of derivatives consist of all linear continuous mappings, satisfy these axioms. Finally a generalization of these remainders which I have called the Strong Binz remainders is formulated: Let K be a separated field and suppose that there is some calculus given on K. Define χ= {(E, Λ), (F, Λ'), (G, Λ"),...} to be the class of all linear limit spaces over K, and let α(EΛ ,FΛ') be the set of all linear continuous mappings from EΛ to FΛ' , for all EΛ , FΛ' in χ. Then define the mapping r:EΛ→FΛ' to be a Strong Binz remainder if r(0)=0 for 0 □ E, if r is continuous at 0 □ E, and if for every filter &theta converging to 0 □ E with respect to Λ there exists a filter ψ converging to 0 □ F such that for every N □ ψ there is an M □ θ and a ς such that r(λM) □ ς(λ)N, for all λ in a U-set about 0 □ K. Here ς represents a remainder from the calculus on the base field. Two methods for constructing a calculus containing the Strong Binz remainder are then introduced, which lead to the same result.

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Table of Contents

Abstract

The purpose of this paper is to examine the axiomatic characterization of a differential calculus as proposed by Wehrli (8) for the case of linear limit spaces over an arbitrary field in view of the usual notion of a derivative. Whether or not this characterization is satisfactory is partially dependent upon an open question in the theory of limit spaces which I have answered in the form of the following theorem: Let E and F be linear Limit spaces over a field K with limitations Λ and Λ' respectively, and let Λ_0 and Λ'_0 be the associated principle ideal limitations. Then if $\Lambda \times \Lambda'$ represents the product limitation of $E \times F$, $(\Lambda \times \Lambda')_0 = \Lambda_0 \times \Lambda'_0$. With this theorem the axiomatic characterization proposed by Wehrli is shown to be quite natural in regard to the usual notion of differentiation on linear limit spaces. It is then shown that the remainders introduced by Binz (1), together with the requirement that the set of derivatives consist of all linear continuous mappings, satisfy these axioms. Finally a generalization of these remainders which I have called the Strong Binz remainders is formulated: Let K be a separated field and suppose that there is some calculus given on K. Define $\mathcal{K} = \{(E,\Lambda), (F,\Lambda'), (G,\Lambda''), \ldots\}$ to be the class of all linear limit spaces over K, and let $\mathcal{L}(E_\Lambda, F_{\Lambda'})$ be the set of all linear continuous mappings from E_Λ to $F_{\Lambda'}$, for all E_Λ, $F_{\Lambda'}$ in $\mathcal{K}$. Then define the mapping $r: E_\Lambda \to F_{\Lambda'}$ to be a Strong Binz remainder if $r(0)=0$ for $0 \in E$, if r is continuous at $0 \in E$, and if for every filter ϕ converging to $0 \in E$ with respect to Λ there exists a filter ψ converging to $0 \in F$ such that for every $N \in \psi$ there is an $M \in \phi$ and a σ such that

$$r(\lambda M) \subseteq \sigma(\lambda)N,$$

for all λ in a U-set about $0 \in K$. Here σ represents a remainder from the calculus on the base field. Two methods for constructing a calculus containing the Strong Binz remainder are then introduced, which lead to the same result.

INTRODUCTION

In recent years several attempts have been made to define in general a notion of differentiability for topological spaces. In (3) for example, Dieudonne' has illustrated a derivative structure for Banach Spaces which is very elegant. As an extension of these considerations it seems quite natural to try to generalize the notion of differentiability to mappings between more abstract spaces which do not necessarily possess norms. Limit spaces or so-called pseudo topological spaces were found to be very useful for this development (cf. (1) , (6)). In particular in (1), E. Binz proposed a notion of differentiability for linear limit spaces over the real field.

In (8) Wehrli has offered a collection of sixteen axioms to characterize a differential calculus and he has shown that concepts of differentiation in topological spaces satisfy them. The question of whether or not the remainders introduced by Binz satisfy these axioms is in part equivalent to an open question in the theory of limit spaces. I have answered this question in the affirmative (cf.Theorem 2.2). In this paper I will adopt the axiomatic characterization of a differential calculus proposed by Wehrli and show that it does in fact characterize the usual notion of differentiation very well in the case of linear limit spaces.

In chapter I a few basic ideas of filter theory and limitations are considered, and the sixteen axioms introduced by Wehrli are noted.

In chapter II the above mentioned result is shown in Theorem 2.2, and the axioms proposed by Wehrli are tested in view of the desired general requirements for a differentiation structure.

In chapter III, the remainder introduced by Binz is considered and shown to be an example of a remainder which, together with the requirement that the set of derivatives be made up of all linear continuous functions, satisfies the sixteen axioms which characterize a calculus.

Finally in chapter IV, an extension of the remainder introduced by Binz, which I have called the Strong Binz remainder is proposed, and this new definition is examined in view of the axioms which characterize a calculus.

Chapter I Background

In this chapter some fundamental notions of filter theory and limit spaces are compiled which are needed in the development to follow, (cf. (2), (4)).

Limitations

Definition Let E be a nonempty set and set $\mathcal{P}(E)$ be the power set of E.

A filter ϕ on E is an element of $\mathcal{P}(E)$ with the following properties:

(1) the empty set does not belong to ϕ.

(2) for any two subsets X and Y of E, if $X \in \phi$, and $X \subseteq Y$, then $Y \in \phi$.

(3) for any two elements X and Y of ϕ, $X \cap Y \in \phi$.

A subcollection γ of a filter ϕ on E is called a filter basis of ϕ if

$\phi = \{X \subseteq E \mid \text{for some } Y \in \gamma,\ Y \subseteq X\}$.

It is clear then that a nonempty collection of subsets of E is a filter basis γ for some filter ϕ on E if and only if

(1) the empty set does not belong to γ,

(2) and if $B \in \gamma$, and $B' \in \gamma$, then there exists a $B'' \in \gamma$ such that $B'' \subseteq B \cap B'$.

Now let E be a nonempty set, and consider the following natural order $\leq$ of all filters $\mathcal{J}(E)$ on E. If ϕ and θ are filters in $\mathcal{J}(E)$, then $\phi \leq \theta$ means that for every $M \in \phi$ there exists an $N \in \theta$ such that $N \subseteq M$. We say in this case that θ is <u>finer</u> than ϕ or ϕ is <u>coarser</u> than θ, and θ is called an <u>overfilter</u> of ϕ.

We define the <u>ultrafilter</u> of $\mathcal{J}(E)$ to be the maximal element of $\mathcal{J}(E)$.

<u>Definition</u> An element I of the power set $\mathcal{P}(\mathcal{J}(E))$ of $\mathcal{J}(E)$ is called a <u>Λ Ideal</u> in $\mathcal{J}(E)$ if and only if

(1) whenever $\phi \in I$, all overfilters in $\mathcal{J}(E)$ of ϕ lie in I, and

(2) whenever $\phi, \theta \in I$, $\phi \wedge \theta \in I$, where $\phi \wedge \theta$, called the <u>join</u> of ϕ and θ, consists of all sets in E containing sets from $\{A \cup B \mid A \in \phi, \quad B \in \theta\}$.

We call I a <u>Λ Principle ideal</u> (a <u>Λ-Hauptideal</u> in German) in $\mathcal{J}(E)$ if I consists of all overfilters of a filter $\phi \in \mathcal{J}(E)$. I is then said to be generated by ϕ.

Again let E be a nonempty set.

<u>Definition</u> A mapping $\Lambda : E \rightarrow \mathcal{P}(\mathcal{J}(E))$ is called a <u>limitation</u> or simply a convergence structure on E (<u>Limiterung</u> in German) if and only if

(1) for every $x \in E$, $\Lambda(x)$ is a Λ ideal in $\mathcal{J}(E)$,

(2) and for every $x \in E$, $\dot{x} \in \Lambda(x)$, where $\dot{x}$ is the filter consisting of all subsets of E containing x.

The filters in $\Lambda(x)$ are defined to be <u>convergent</u> to $x \in E$ relative to Λ. Then the set E together with the structure Λ is called a <u>limit</u>

space (Limesraum in German). In this paper (E, Λ) and E_Λ will be used interchangeably to denote this limit space.

We note that every topology is a limitation since the convergent filters to any point clearly form a Λ ideal. Of course the definition of convergence of filters to a point x in a topological space X requires that every neighborhood of the point contain a set of the filter. This is clearly consistent with the requirement that $\dot{x}$ converge to x.

Definition Let Λ be some limitation on set E. If for every $x \in E$, the filters convergent to x relative to Λ form a principle ideal, Λ is called a principle ideal limitation on E. (Hauptideal Limitierung in German)

A principle ideal limitation is a topology on E if and only if for every $x \in E_\Lambda$, the coarsest filter $\phi^\circ(x) \in \Lambda(x)$ satisfies the requirement that to every $U \in \phi^\circ(x)$, there exists a $V \in \phi^\circ(x)$ such that for all $y \in V$, $U \in \phi^\circ(y)$.

Let $x \in E_\Lambda$ and consider $\Lambda(x)$. Define the lower bound of $\Lambda(x)$ to be the filter $\phi^\circ(x)$ generated by $\bigcup_\eta N_\eta$, $N_\eta \in \eta(x)$, where the $\eta(x)$ are the filters in $\Lambda(x)$. This filter of course does not always belong to $\Lambda(x)$. If one constructs for every point $x \in E$, the filter $\phi^\circ(x)$, and defines it to belong to $\Lambda(x)$, then a principle ideal limitation is formed on E_Λ which is uniquely determined by Λ. We denote this by Λ_0.

Definition Let (E, Λ) be a limit space. A subset $A \subseteq E$ is said to be Λ open if and only if for every $x \in A$, $A \in \phi$, for every filter $\phi \in \Lambda(x)$.

Fisher has shown the following in (4):

Theorem 1.1. The set of all Λ open sets Q in (E, Λ_o) forms a topology on E, in which the sets of Q are the open sets.

We call this the topology associated with Λ.

Now let A be nonempty subset of set E. All filters of a Λ ideal in $\mathfrak{F}(E)$ which have nonempty intersection with A, generate in $\mathfrak{F}(A)$ a Λ ideal. We note here that nonempty intersection of a filter ϕ with set A means that all sets of ϕ have nonempty intersection with A.

Definition Let E_Λ be a limit space and A a nonempty subset of E. Associate with every $x \in A$ the Λ ideal $\Lambda_A(x)$ in $\mathfrak{F}(A)$ which is generated in A by $\Lambda(x)$ through intersection of filters in $\Lambda(x)$ with A, where of course only filters in $\Lambda(x)$ with nonempty intersection with A are considered. Then $\Lambda_A : A \to \mathcal{P}(\mathfrak{F}(A))$ is a limitation on A and is called the limitation induced on A by Λ.

Suppose that Λ, Λ' are both limitations defined on E.

Definition Λ is said to be finer than Λ' if and only if for every $x \in E$, $\Lambda(x) \subseteq \Lambda'(x)$. We denote this symbolically by $\Lambda' \leq \Lambda$. We also say that Λ' is coarser than Λ.

Now let E and F be two nonempty sets, and let f be a mapping from E to F. If ϕ is a filter in $\mathcal{J}(E)$, it is mapped by f into the set $\{f(M) | M \in \phi\}$ in F, which forms a filter basis on F.

Definition A mapping $f: E_{\Lambda} \rightarrow F_{\Lambda'}$ is continuous at $x \in E$ if and only if for every filter $\phi \in \Lambda(x)$, $f(\phi)$ is a filter basis of a filter in $\Lambda'(f(x))$.

Of course if the mapping f is continuous at all $x \in E$, it is said to be continuous on E.

This definition of continuity is entirely consistent with the definition given in the case of topological spaces. We note the following theorem proven in part by Fischer in (4) :

Theorem 1.2 The definition of continuity for a function in the case of limitations is equivalent to the definition of a continuous function for topologies.

Proof Suppose first of all that we have a continuous function $f: E \rightarrow F$, where E and F are topological spaces. Let $x \xrightarrow{f} y$, where x any point in E, and let ϕ be any filter on E converging to x in the topological sense. Consider then $f(\phi)$. If U is any open neighborhood of $y \in F$ then by definition $f^{-1}(U)$ is an open neighborhood of $x \in E$. Then there exists a $V \in \phi$ such that $V \subseteq f^{-1}(U)$. Therefore $f^{-1}(U) \in \phi$. It follows then that $f(V) \subseteq U$, and $U \in f(\phi)$. We have shown then that for any open neighborhood U of $y \in F$, an image set f(V) of a set V in the filter ϕ converging to x is contained in U. Thus the image of ϕ under f forms a filter convergent to y.

Now suppose again that E and F are topological spaces, and $f: E \to F$ is continuous mapping but now in the sense of limitations. Let N be any open set in F, and let $M = f^{-1}(N)$. Choose an arbitrary point $x \in M$ and let $f(x) = y \in N$. We must show that $f^{-1}(N) \in \Theta$, where Θ is any filter converging to x. By the continuity of f, $f(\Theta)$ generates a filter in F, and therefore, $N \in f(\Theta)$. Since $N \in f(\Theta)$, there exists a set $M' \in \Theta$ such that $f(M') \subseteq N$. It follows then that $M = f^{-1}(N)$ contains $f^{-1}(f(M'))$ and therefore M'. But this means that $M \in \phi$. Therefore M is open in E.

The notion of the limit space can now be extended to product spaces. Let E_{Λ}, $F_{\Lambda'}$ be limit spaces, and on the space E x F construct the coarsest limitation for which the projection mappings $pr_E: E \times F \to E$ and $pr_F: E \times F \to F$ are continuous. This is called the <u>product limitation</u> and is denoted $\Lambda \times \Lambda'$.

The following theorem shown by Binz (1) is quite useful when working with product limitations:

<u>Theorem 1.3</u> Let E_{Λ} and $F_{\Lambda'}$ be two limit spaces. A filter $\phi \in \mathcal{F}(E \times F)$ converges to $(x,y) \in E \times F$ relative to $\Lambda \times \Lambda'$ if and only if $pr_E(\phi) \in \Lambda(x)$

and $pr_F(\phi) \in \Lambda'(y)$.

It can be remarked here that clearly if $\phi \in \Lambda(x)$ and $\theta \in \Lambda'(y)$, then $\phi \times \theta$ generates a filter in $(E \times F, \Lambda \times \Lambda')$ convergent to (x,y). On the other hand if ψ is a filter in $(E \times F, \Lambda \times \Lambda')$ converging to (x,y), then from Theorem 1.3, $pr_X \psi \in \Lambda(x)$ and $pr_Y \psi \in \Lambda'(y)$. This means of course that $pr_X \psi$ generates a filter $\phi \in \Lambda(x)$, and $pr_Y \psi$ generates a filter $\theta \in \Lambda'(y)$, and moreover, ψ is finer than $\phi \times \theta$. To show this let $U \in \psi$. Then $pr_X U = V_1 \in \phi$ and $pr_Y U = V_2 \in \theta$, and it follows that $U \subseteq V_1 \times V_2$. Thus $\psi \geq \phi \times \theta$.

These results on product limitations will be used frequently in the present paper.

<u>Limit Spaces with Algebraic Operations</u>

Fischer in (4) defines a limit group (G, Λ) to be a set G on which is defined a limitation Λ such that

(a) G is a group.

(b) and (G, Λ) is a limit space such that the group operations $(x,y) \to xy$, and $x \to x^{-1}$ are continuous maps from G x G into G, and G into G respectively.

The definitions for limit fields, limit rings, and linear limit spaces are similar. For instance the definition of a limit field is similar to the definition for a limit group in that continuity is required for both operations and both inversion mappings.

Now let E be a linear space over a limit field K.

Definition A limitation Λ on E is said to be admissable for E if and only if the mappings $(\lambda, x) \longrightarrow \lambda x$ and $(x,y) \rightarrow x+y$, from K x E to E and E x E to E respectively, are continuous.

Definition If Λ is an admissable limitation on the linear space E over field K, then (E, Λ) is called a linear limit space (limitierter Vektorraum in German).

Axioms of a Differential Calculus.

In (8) Wehrli proposed the following structure to describe a differential calculus over a field K. Much use of this structure will be made in the present paper.

We say that we have a differential calculus over a field K when the following are given:

I. A class $\mathcal{K}$ of linear spaces $\{E, F, \ldots\}$ over the field K which will be called objects of the differential calculus.

II. For any two objects E,F of $\mathcal{K}$, there are two sets $\mathcal{A}(E,F)$, and $\mathcal{R}(E,F)$ of mappings from E to F. The mappings of $\mathcal{A}(E,F)$ will be called derivatives from E to F, the mappings of $\mathcal{R}(E, F)$ will be called remainders from E to F, and they will satisfy the following axioms:

Axiom 0 The field K is an object of the differential calculus.

Axiom 1 $\mathcal{R}(E,F)$ is a linear space over K for any $E, F \in \mathcal{K}$.

<u>Axiom 2</u> $\mathcal{R}$ (E,F) is a linear space over K. for any E,F $\in \mathcal{K}$.

<u>Axiom 3</u> Let $r \in \mathcal{R}(E,F)$. If $r|$one dimensional subspace L of E is linear, then $r|L = 0$

<u>Axiom 4</u> If $A \in \mathcal{Q}(E,F)$ and $B \in \mathcal{Q}(F,G)$, then $BA \in \mathcal{Q}(E,G)$

<u>Axiom 5</u> $r(A+\rho) \in \mathcal{R}(E,G)$ when $r \in \mathcal{R}(F,G)$, $A \in \mathcal{Q}(E,F)$, and $\rho \in \mathcal{R}(E,F)$.

<u>Axiom 6</u> If $A \in \mathcal{Q}(F,G)$, and $r \in \mathcal{R}(E,F)$, then $Ar \in \mathcal{R}(E,G)$.

<u>Axiom 7</u> Let $r \in \mathcal{R}(E,F)$, and $\lambda \in K$. Then the mapping $f(x) = r(\lambda x)$ is a remainder from E to F, ie. $f \in \mathcal{R}(E,F)$.

<u>Axiom 8</u> Let E,F, and G be objects in $\mathcal{K}$, and let $G \subseteq E$. If $r \in \mathcal{R}(E,F)$, then $r|G$ is contained in $\mathcal{R}(G,F)$.

<u>Axiom 9</u> Let E,F,G be objects in $\mathcal{K}$, and let E_1 be a one dimensional subspace of E, and F_1 a one dimensional subspace of F. If $r \in \mathcal{R}(E,G)$ and L is a linear isomorphism of F_1 into E_1, then there is a $\rho \in \mathcal{R}(F,G)$ such that $r \cdot L = \rho|F_1$.

<u>Axiom 10</u> Let E,F, and G be objects in $\mathcal{K}$, let $G \subseteq E$, and $r \in \mathcal{R}(G,F)$.

$$\text{Let } \rho(x) = \begin{cases} r(x), & x \in G \\ 0, & x \in E \text{ but } x \notin G \end{cases}$$

Then $\rho \in \mathcal{R}(E,F)$.

<u>Axiom 11</u> Let r_1, $r_2 \in \mathcal{R}(E,F)$, and let $r: E \to F$ such that when $x \in E$, and $r(x) \neq r_1(x)$, then $r(x) = r_2(x)$. Then $r \in \mathcal{R}(E,F)$.

<u>Axiom 12</u> Let E, F, G be objects in $\mathcal{K}$, and let M be a subset of E. Let

$r \in \mathcal{R}(E,F)$ such that $r \mid M = c$, a nonzero constant. Then no matter what $f: M \to G$ is, there exists a $\rho \in \mathcal{R}(E,G)$ with $\rho \mid M = f$.

<u>Axiom 13</u> Let $E, F \in \mathcal{K}$, and let $0 \neq z \in E$, $y \in F$. Then there exists an $r \in \mathcal{R}(E,F)$ such that $r(z) = y$.

<u>Axiom 14</u> Let $E, F, G \in \mathcal{K}$ and let M be a subset of $E \times E$ such that there exists an $r \in \mathcal{R}(E,F)$ with the property that $r(x+y) = c$, a nonzero constant for $(x,y) \in M$. Then there exists a G in $\mathcal{K}$ and $\rho \in \mathcal{R}(E,G)$ such that $\rho(x)$ takes only two values and such that for $(x,y) \in M$ $\rho(x)$ or $\rho(y)$ is <u>not</u> zero.

<u>Axiom 15</u> Let E, F, G, $\bar{G}$ be objects in $\mathcal{K}$ and let M be a subset of $K \times E$, such that there exists an $r \in \mathcal{R}(E,F)$ with $r(\lambda x) = d$, a nonzero constant, for $(\lambda, x) \in M$. Then there is a $\rho \in \mathcal{R}(E, G)$ which assumes only two values, and there exists a $\tilde{\rho} \in \mathcal{R}(K, \bar{G})$, such that for (λ, x) M, $\rho(x)$ and $\tilde{\rho}(\lambda)$ are not zero simultaneously.

Chapter II. Differentiation in Limit Spaces

Preliminary Material Consider first of all the following theorem due to Wehrli (9). Let (X, Λ) and (Y, Λ') be limit spaces.

Theorem 2.1 If $x \in X$, and $y \in Y$, and $f: (X, \Lambda) \rightarrow (Y, \Lambda')$ where $x \xrightarrow{f} y$ is continuous at x, then $f: (X, \Lambda_o) \rightarrow (Y, \Lambda'_o)$ is continuous at x. (Here of course Λ_o, and Λ'_o are the associated principle ideal limitations.)

It is important to remark at this point that if E_Λ and $F_{\Lambda'}$ are limit spaces and f is a continuous function from E_Λ to $F_{\Lambda'}$, then f is of course, by Theorem 2.1, continuous relative to the associated limitations Λ_o and Λ'_o. In view of Theorem 1.2 we see that this means that $f: E_{\Lambda_o} \rightarrow F_{\Lambda'_o}$ is therefore also a continuous mapping in the topological sense.

With the aid of Theorem 2.1 the following important result can be obtained.

Theorem 2.2 Let (X, Λ) and (Y, Λ') be limit spaces, and consider $(X \times Y, \Lambda \times \Lambda')$, where $\Lambda \times \Lambda'$ is the product limitation. Then $(X \times Y, \Lambda_o \times \Lambda'_o) = (X \times Y, (\Lambda \times \Lambda')_o)$.

Proof Let (x, y) be any point in X x Y, and let φ_o be the coarsest filter in $(\Lambda \times \Lambda')_o$ converging to (x, y). Clearly φ_o exists since $(\Lambda \times \Lambda')_o$ is a principle ideal limitation. In addition let $\alpha_o \times \beta_o$ be the coarsest filter in $\Lambda_o \times \Lambda'_o$ converging to (x, y).

Lemma 2.1 $\alpha_o \times \beta_o$ is not finer than ψ_o.

Proof of Lemma 2.1 Let V be any set in $\alpha_o \times \beta_o$. Then V contains a set of the form $V_1 \times V_2$, where $V_1 \in \alpha_o$ and $V_2 \in \beta_o$. From the definition of α_o and β_o we know that $V_1 = \cup V_{1\mu}$ and $V_2 = \cup V_{2\eta}$ where $V_{1\mu}$ and $V_{2\eta}$ are sets from each of the filters μ and η in $\Lambda(x)$ and $\Lambda'(y)$ respectively. Then

$$V_1 \times V_2 = V_{1\mu} \times V_{2\eta} = \bigcup_{\mu,\eta}(V_{1\mu} \times V_{2\eta}).$$

Now what is the nature of ψ_o? The sets of ψ_o contain sets of the form $\cup (W_1 \times W_2)$ where W_1 and W_2 are sets from each of the filters which converge in Λ and Λ' to x and y respectively. The sets then of $\alpha_o \times \beta_o$ are also sets of ψ_o, and $\alpha_o \times \beta_o$ is not finer than ψ_o.

Lemma 2.2 ψ_o is not finer than $\alpha_o \times \beta_o$.

Proof of lemma 2.2 Basically the approach in this proof is to show that there is a limit space (S, Γ), and a mapping f from (S, Γ) into $(X \times Y, \Lambda \times \Lambda')$ which is continuous at a point $u \in S$ with $f(u) = (x,y)$, and which maps a filter converging to u in (S, Γ_o) into the filter $\alpha_o \times \beta_o$ in $(X \times Y, (\Lambda \times \Lambda')_o)$. Then of course ψ_o could not be finer than $\alpha_o \times \beta_o$. To construct this space S first of all form the direct product $S_1 = \prod_\lambda X_\lambda$ for $\lambda \in \mathcal{M}$, where $\mathcal{M}$ is a set which indexes the filters $\phi_\lambda \in \Lambda(x)$, and let $X_\lambda = X$ for all λ. Then similarly form $S_2 = \prod_\eta Y_\eta$ for $\eta \in \mathcal{N}$, the index set of all filters $\theta_\eta \in \Lambda'(y)$, and let $Y_\eta = Y$. Now form the direct product $S = S_1 \times S_2 = \prod_\lambda X_\lambda \times \prod_\eta Y_\eta$. Then $S = X \times X \times \ldots \times Y \times Y \times \ldots$, in the above dimension.

Now define the following limitation Γ on S. Let $u = (x,x,\ldots,y,y,\ldots)$ in S be the point with x in each of the X components, and y in each of the Y components.

I. For filters $\tilde{\phi}_\lambda$ and $\tilde{\theta}_\eta$ on S, define $\tilde{\phi}_\lambda \in \Gamma(u)$ and $\tilde{\theta}_\eta \in \Gamma(u)$ when $\tilde{\phi}_\lambda$ is generated by $(x,x,\ldots,\phi_\lambda,\ldots,y,y,\ldots)$, $\phi_\lambda \in \Lambda(x)$, and when $\tilde{\theta}_\eta$ is generated by $(x,x,\ldots,y,y,\ldots,\theta_\eta,y,\ldots)$, $\theta_\eta \in \Lambda'(y)$, where just one x component is occupied by the points of a filter ϕ_λ converging to x in (X,Λ) and all other components are identical with those of u, or where just one Y component is occupied by the points of a filter θ_η converging to y in (Y,Λ') and all other components are identical with those of u.

II. If $\tilde{\phi}_{\lambda_1},\ldots,\tilde{\phi}_{\lambda_i},\tilde{\theta}_{\eta_1},\ldots,\tilde{\theta}_{\eta_j}$ are a finite set of filters converging to u in the sense of I, define their join to converge to u.

III. Finally define a filter δ to converge to u if δ is finer than a filter convergent to u in the sense of II.

For any other point $z \neq u$ in S define $\dot{z}$ to be the only filter in S converging to z.

The question that must be answered now is whether or not Γ is a limitation on S. Consider then the definition of a limitation. The first condition of this definition requires that $\Gamma(z)$, for all $z \in S$, must be a Λ ideal in $\mathcal{J}(S)$. If $z \neq u$, then $\Gamma(z)$ consists of just $\dot{z}$. Certainly

$\dot{z} \wedge \dot{z} = \dot{z}$, and any filter containing $\dot{z}$ is just $\dot{z}$ because $\dot{z}$ is the ultrafilter, i.e. the finest filter converging to z. Thus $\Gamma(z)$ is a Λ ideal for $z \neq u$. If $z = u$, then the fact that $\Gamma(u)$ is a Λ ideal follows directly from the definition of Γ. The second condition of the definition of a limitation requires that for all $z \in S$, $\dot{z} \in \Gamma(z)$. This is true by the definition of Γ for all $z \neq u$. It is also true for u since $\dot{u}$ is finer that any filter converging to u in the sense of II. Therefore Γ is a limitation of S.

Now define $f: (S, \Gamma) \rightarrow (X \times Y, \Lambda \times \Lambda')$ so that f maps points in the form

(x, x, ..., s, ..., x, ... y, y, ...) into (s, y), where the point is constructed by replacing one of the x's in u by $s \in S$;

points in the form (x, x, ..., y, y, ..., t, ...) into (x, t), where this point is formed by replacing one of the y's in u by $t \in S$, and

points in the form (x, x, ..., s, x, ..., y, y, ..., t. ,,,) into (s, t), where this point is formed by replacing one of the x's and one of the y's in u by s and t respectively. If one defines f to map all other points of S into (x, y) then f is a continuous map from (S, Γ) into $(X \times Y, \Lambda \times \Lambda')$. To show this let $\tilde{\phi}_\lambda$ converge to u according to form I, i.e. $\tilde{\phi}_\lambda$ is generated by $(x, x, \ldots, x, \tilde{\phi}_\lambda, x, \ldots, y, y, \ldots)$, where $\tilde{\phi}_\lambda \in \Lambda(x)$. If $W \in \tilde{\phi}_\lambda$, then W contains a set of the form (x, x, ..., U, x, ... x, y, y, ...), where $U \in \phi_\lambda$. This set is mapped by f into (U, y), so f(W) contains (U, y).

All $U \in \phi_\lambda$ appear in images under f in the form (U, y), i.e. every $U \in \phi_\lambda$ generates a set $W \in \tilde{\phi}_\lambda$, and the image of any such W under f then contains (U, y). It is clear that $\{(U, y) \mid U \in \tilde{\phi}_\lambda\}$ generates a filter $\bar{\gamma}$ in $(X \times Y, \Lambda \times \Lambda')$. At this point it is convenient to use Theorem 1.3 stated in Chapter I. The projection maps of X x Y into X and Y, symbolically $pr_X\ X \times Y \to X$ and $pr_Y\ X \times Y \to Y$, are continuous, and since $pr_X \bar{\gamma}$ gives all $U \in \phi_\lambda \in \Lambda(x)$, which form a basis of ϕ_λ in $\Lambda(x)$, and $pr_Y\ \bar{\gamma}$ gives all sets in (Y, Λ') which contain y, i.e. the ultrafilter $\dot{y} \in \Lambda'(y)$, by Theorem 1.3 it follows that $\bar{\gamma}$ converges to (x,y) in $(X \times Y, \Lambda \times \Lambda')$. A similar argument will suffice if a filter $\mathcal{N}$, convergent in form II, is chosen since only finite joins are involved. A filter $\mathcal{S}$ convergent to u in form III is of course finer than some filter $\mathcal{N}$ convergent to u in form II. But the image of $\mathcal{N}$ under f generates a filter convergent in $(X \times Y, \Lambda \times \Lambda')$ to (x,y). The image of $\mathcal{S}$ under f clearly generates a filter in $(X \times Y, \Lambda \times \Lambda')$ which is finer than $f(\mathcal{N})$ and is therefore convergent to (x,y). f thus is continuous at u.

Introduce now the associated principle ideal limitations Γ_0 and $(\Lambda \times \Lambda')_0$ on S and X x Y respectively. By Theorem 2.1 f is continuous at $u \in S$ relative to these new limitations. The task now is to construct a filter in S which converges to u in Γ_0 and is mapped by f into a filter basis of $\alpha_0 \times \beta_0$. Construct then all filters $\tilde{\phi}_\lambda$ and $\tilde{\theta}_\eta$ in (S, Γ_0). Each of these converges to $u \in S$ by form I. Clearly the sets

$$(x, x, \ldots, \dot{x} \wedge \phi_\lambda, x, \ldots, y, \ldots)$$

$$(x, x, \ldots, y, y, \ldots, \dot{y} \wedge \theta_\eta, y, \ldots)$$

for all ϕ_λ and θ_η in the λ and η components respectively, each generate a filter convergent to u by form II. Denote these filters by $\widetilde{\phi}_{\lambda\lambda}$ and $\widetilde{\theta}_{\eta\eta}$ respectively. Note now that the following sets generate these filters:

$(x,x,\ldots,x \cup U_\lambda, x,\ldots,y,\ldots)$, all $U_\lambda \in \phi_\lambda$ for all $\lambda \in \mathcal{M}$.

$(x,x,\ldots,y,\ldots,y \cup V_\eta, y, y,\ldots)$, all $V_\eta \in \theta_\eta$ for all $\eta \in \mathcal{N}$.

Now construct the filter $\overline{\Psi}$ generated by

$$(\bigwedge_\lambda \widetilde{\phi}_{\lambda\lambda}) \wedge (\bigwedge_\eta \widetilde{\theta}_{\eta\eta}).$$

If f is applied to the set which is formed when one fixes $x \cup U_\lambda$, where $U \in \phi_\lambda$ for some $\lambda \in \mathcal{M}$, in an X component; requires that all other X components be occupied by simply $x \in X$; and allows each Y component to be occupied by a distinct $V_\eta \in \theta_\eta$, for all $\eta \in \mathcal{N}$, then this set is mapped into

$$U_\lambda \times \bigcup_\eta V_\eta .$$

Now if a distinct $x \cup U_\lambda$, $U_\lambda \in \phi_\lambda$ for $\lambda \in \mathcal{M}$, is allowed to occupy each X component of this set, it is mapped by f into $\bigcup_\lambda U_\lambda \times \bigcup_\eta V_\eta$. Of course the original set belonged to $\overline{\Psi}$ and thus when f is applied to this filter all sets in the form

$$\{\bigcup_\lambda U_\lambda \times \bigcup_\eta V_\eta\} \quad \text{for all} \quad U_\lambda \in \phi_\lambda \text{ and } V_\eta \in \theta_\eta,$$

$\lambda \in \mathcal{M}, \eta \in \mathcal{N}$, are obtained. All other sets in $\overline{\Psi}$ map into oversets of these. Clearly $\{\bigcup_\lambda U_\lambda \times \bigcup_\eta V_\eta\}$ generates the filter $\alpha_0 \times \beta_0$ in X x Y, and the projection mappings pr_X and pr_Y of this filter generate α_0 and β_0 respectively. Therefore by the continuity of f, $\alpha_0 \times \beta_0$ converges to (x,y) with respect to $(\Lambda \times \Lambda')_0$, and Ψ_0 can not be finer than $\alpha_0 \times \beta_0$.

Lemmas 2.1 and 2.2 then together imply that $\Psi_0 = \alpha_0 \times \beta_0$, and therefore that $(\Lambda \times \Lambda')_0 = \Lambda_0 \times \Lambda'_0$. This concludes the proof of Theorem 2.2.

The following theorems on limit groups and linear limit spaces as introduced in chapter I. are almost immediate consequences of Theorem 2.2.

<u>Theorem 2.3</u> If (G, Λ) is a limit group, then (G, Λ_0) is a topological group.

<u>Proof</u> First of all, since $x \to x^{-1}$ is continuous relative to Λ, by Theorem 2.1, it is continuous relative to Λ_0. Secondly, if $(x,y) \to xy$ is continuous relative to $\Lambda \times \Lambda$ and Λ, then by Theorem 2.1 it is continuous relative to $(\Lambda \times \Lambda)_0$ and Λ_0. But by Theorem 2.2, $(\Lambda \times \Lambda)_0 = \Lambda_0 \times \Lambda_0$, so $(x,y) \to xy$ is continuous relative to $\Lambda_0 \times \Lambda_0$ and Λ_0. Therefore (G, Λ_0) is a limit group, and of course then a topological group with structure Λ_0 since by Theorem 1.1 Λ_0 is a topology on G.

A similar theorem for linear limit spaces is as follows.

<u>Theorem 2.4</u> If (E, Λ) is a linear limit space over the limit field

$(K, \bar{\Lambda})$, then (E, Λ_o) is a topological space over $(K, \bar{\Lambda}_o)$.

Proof From the definition of a linear limit space the operations of vector addition and scalar multiplication are continuous from E x E to E and E x K to E respectively. Since $x^{-1} = -x$, the fact that $x \rightarrow x^{-1}$ is continuous follows from the continuity of scalar multiplication, and E is a topological group. Therefore (E, Λ_o) is a topological space over $(K, \bar{\Lambda}_o)$ with the topology of the group.

Requirements for a General Differentiation Structure

Suppose we are given a notion of differentiation in the class $\mathcal{K}$ of all linear limit spaces over a limit field K. It is quite natural to ask what conditions must hold, or what requirements really characterize the structure, in view of the axioms proposed for a differential calculus by Wehrli. By a concept of differentiation we mean first of all that between any two linear limit spaces E_Λ and $F_{\Lambda'}$ over K, there exists a set of linear continuous functions $\mathcal{Q}(E_\Lambda, F_{\Lambda'})$ and a set of functions $\mathcal{R}(E_\Lambda, F_{\Lambda'})$ with the property that if $r \in \mathcal{R}$, then $r(0) = 0$, and r is continuous at zero. Then $f: E \rightarrow F$ is said to be differentiable at the point $z \in E$ if and only if there exist functions

$A \in \mathcal{Q}(E_\Lambda, F_{\Lambda'})$ and $r \in \mathcal{R}(E_\Lambda, F_{\Lambda'})$ such that

$$f(z+h) - f(z) = Ah + r(h), \text{ for all } h \in E.$$

A then is said to be the derivative of $f(x)$ at $x = z$, and is usually designated $f'(z)$. $r(h)$ is said to be a remainder. It appears quite natural now to require that the zero mapping from E_{Λ} to $F_{\Lambda'}$ be differentiable for all $z \in E$, and have derivative equal to zero on all such points. Also it seems natural to stipulate that differentiation be a local process. To expand on this notion somewhat, we mean that when a function has a derivative at a certain point this derivative is independent of the nature of the function outside a Λ open neighborhood of the point. It follows then from this and the fact that the zero mapping is differentiable that any function f from E_{Λ} to $F_{\Lambda'}$ which vanishes on a Λ open neighborhood of zero is a remainder. This can be shown by considering the following expression which results from the fact that f is differentiable at zero:

$$f(0 + h) - f(0) = Ah + r(h), h \in E.$$

If f vanishes on the Λ open neighborhood of zero U in E, then we have in U the zero mapping and since the zero mapping has derivative zero, the above equation reduces to $f(h) = r(h)$, $h \in U \subseteq E$. Therefore f is a remainder.

Now assume the following separation axiom proposed by Fischer (4) for our spaces E_{Λ}, $F_{\Lambda'}$, etc.: If $x, y \in E \in \mathcal{Y}$, and $x \neq y$, then $y \notin \Lambda(x)$. With this we can prove the following lemma:

<u>Lemma 2.3</u> if $x, y \in E$, such that $x \neq y$, then Cy, where Cy is the complement of y in E, is a Λ open neighborhood of x.

<u>Proof of lemma 2.3</u> Let $z \in Cy$, and show that $Cy \in \phi \in \Lambda(z)$, where ϕ is arbitrary in $\Lambda(z)$. Suppose then that there exists a $\phi \in \Lambda(z)$ such that $Cy \notin \phi$. Then $U \in \phi$ implies that $y \in U$. But then $\dot{y}$ is finer than ϕ and $\dot{y} \in \Lambda(z)$. This of course contradicts the separation axiom since $y \neq z$. Therefore $Cy \in \phi \in \Lambda(z)$, and since z was arbitrary in Cy and ϕ was arbitrary in $\Lambda(z)$, Cy is a Λ open neighborhood of x.

Thus for any two points x and y in E, where $x \neq y$, a Λ open neighborhood of x can be found which doesn't contain y.

<u>Axiom 13</u> proposed by Wehrli for a differential calculus can be easily shown by first applying Lemma 2.3 and finding a Λ open neighborhood U of zero which does not contain z.

Now define

$$r(x) = \begin{cases} 0, & x \in U \\ y, & x = z \\ \text{arbitrary}, & x \notin U \text{ and } x \neq z \end{cases}$$

Certainly $r \in \mathcal{R}(E_\Lambda, F_{\Lambda'})$ since it vanishes on a Λ open neighborhood of zero.

In <u>Axiom 12</u> for a differential calculus it is given that $E_\Lambda, F_{\Lambda'}$, and $G_{\Lambda''}$ are objects in $\mathcal{K}$ and M is a subset of E such that for some r in $\mathcal{R}(E_\Lambda, F_{\Lambda'})$, $r|M = c$, a nonzero constant. Then for any $f: M \to G$, we must show that there exists a $\rho \in \mathcal{R}(E, G)$ with the property that $\rho|M = f$.

Drawing upon Lemma 2.3 a Λ open neighborhood U of zero in (E, Λ'_o), the associated topological space, is obtained which doesn't contain $c \in F$. Since r is continuous at $0 \in F$ by Theorem 1.2, the inverse image of U under r is a Λ open neighborhood W of zero in (E, Λ_o), and clearly $W \cap M$ is empty. Now define

$$\rho(x) = \begin{cases} 0, & x \in W \\ f(x), & x \notin W \end{cases}$$

Certainly $\rho \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ since it vanishes on W, and $\rho | M = f$.

Axiom 14 proposed by Wehrli can also be shown: $E_\Lambda, F_{\Lambda'}, G_{\Lambda''}$ are to be objects in $\mathcal{K}$ and M is a subset of E x E such that there exists an $r \in \mathcal{R}(E_\Lambda, F_{\Lambda'})$ with the property that $r(x + y) = c$, a nonzero constant, for $(x,y) \in M$. If this is the case Axiom 14 requires the existence of a $G_{\Lambda''}$ in $\mathcal{K}$ and a $\rho \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ such that $\rho(x)$ or $\rho(y)$ is not zero.

Let then δ be the map from E x E into E where $(x,y) \xrightarrow{\delta} x + y$. This of course is continuous. Now construct on (E, Λ), (F, Λ'), and $(E \times E, \Lambda \times \Lambda)$ the associated principle ideal limitations Λ_o, Λ'_o, and $(\Lambda \times \Lambda)_o$. By Theorem 2.1, r and δ are still continuous at zero, and by Theorem 1.1, (E, Λ_o), (F, Λ'_o), and $(E \times E, (\Lambda \times \Lambda)_o)$ are topological spaces. Theorem 2.2 implies that $(\Lambda \times \Lambda)_o = \Lambda_o \times \Lambda_o$ and we see that $(E \times E, (\Lambda \times \Lambda)_o)$ is just a topological space with the product topology. By lemma 2.3 we can find a Λ open neighborhood W of zero in F such that $c \notin W$. Let U be the inverse image of W under r. Since r is continuous, by Theorem 1.2 U is a Λ open neighborhood of zero in E. Then if the image of M under δ is $N \subseteq E$, clearly $U \cap N$ is empty. Because δ is continuous at (0,0) with respect to the topologies, the inverse image of U under r is a Λ open neighborhood U' of (0,0) in E x E. Certainly $U' \cap M$ is empty. Since we are dealing with the product topology on E x E, and U' is open in this topology, U' contains a set in the form $V_1 \times V_2$, where V_1 and V_2 are Λ open neighborhoods of zero in (E, Λ_0). Then $V_1 \cap V_2 = V$ is a Λ open neighborhood of zero in (E, Λ_0). Now define $\rho(x)$ to be such that

$$\rho(x) = \begin{cases} 0 \in G, & x \in V \\ c \neq 0 \in G, & x \notin V \end{cases}$$

Clearly $\rho(x) \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ since it vanishes on a Λ open neighborhood of zero, and if $\rho(x)$ and $\rho(y)$ both are zero, then $(x,y) \notin M$.

Axiom 15 can be verified in a similar fashion. Duplicate the previous argument to the point at which a Λ open neighborhood of (0,0), containing $V_1 \times V_2$ in K x E is found, where V_1 and V_2 are Λ open neighborhoods of zero in K and E respectively.

Now define

$$\rho(x) = \begin{cases} 0 \in G, & x \in V_2 \\ c \neq 0 \in G, & x \notin V_2 \end{cases}$$

and
$$\bar{\rho}(\lambda) = \begin{cases} 0 \in G, & \lambda \in V_1 \\ d \neq 0 \in G, & \lambda \notin V_1 \end{cases}$$

Clearly $\rho(x) \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ and $\bar{\rho}(\lambda) \in \mathcal{R}(K, G_{\Lambda''})$, and if $\rho(x) = \bar{\rho}(\lambda) = 0$, $(\lambda, x) \notin M$.

Thus Axioms 12, 13, 14, 15 follow as a consequence of the continunity of remainders at zero, and of the fact that a function vanishing on a Λ open neighborhood of zero is a remainder.

Under the given conditions Axiom 11 can at least be verified for the subset of our set of remainders made up of those which vanish on a Λ open neighborhood of zero. Denote this set by $\mathcal{R}_0(E_\Lambda, F_{\Lambda'})$. If r_1, r_2 are in $\mathcal{R}_0(E, F)$, where $r_1 | U_1 = 0$ and $r_2 | U_2 = 0$, U_1, U_2 Λ open neighborhoods of zero in E, then clearly $r_1 | U_1 \cap U_2 = 0$ and $r_2 | U_1 \cap U_2 = 0$. Of course $U_1 \cap U_2$ is a Λ open neighborhood of zero in E. Therefore regardless of how r is defined, as long as $r(x) = r_1(x)$ or $r(x) = r_1(x)$, certainly $r | U_1 \cap U_2 = 0$, and r is a remainder.

Axiom 10 can be quickly verified for remainders in our set $\mathcal{R}_0(E_\Lambda, F_{\Lambda'})$ since the limitation Λ'' on G is induced by the limitation on E, i.e. the topology Λ''_o on G is of course then just the relative topology with respect to (E, Λ_o). Since $r \in \mathcal{R}_o(G_{\Lambda''}, E_{\Lambda'})$, r vanishes on a Λ open neighborhood U containing zero in G. But $U = U' \cap G$ for some Λ open neighborhood of zero U' in E. Then if

$$\rho(x) = \begin{cases} r(x), & x \in G \\ 0, & x \in E, \text{ but not in } G, \end{cases}$$

clearly $\rho \in \mathcal{R}_o(E_\Lambda, F_{\Lambda'})$ since ρ vanishes on U'.

The statement of <u>Axiom 9</u> depends on whether or not one assumes that all linear subspaces of the spaces in $\mathcal{X}$ are isomorphic. To see that this is in general not the case consider the following example:

Let $\mathbb{R}$ be the real numbers and define a filter ϕ to converge to a point when it is generated by a countable sequence of numbers that converge to that point in the natural topology of $\mathbb{R}$. The collection of all filters convergent to points in $\mathbb{R}$ in this sense establishes a limitation Λ on $\mathbb{R}$. Clearly the natural topology $\mathcal{J}$ on $\mathbb{R}$ also establishes a limitation on $\mathbb{R}$. Consider then the identity map id: $(\mathbb{R}, \Lambda) \to (\mathbb{R}, \mathcal{J})$. If this is an isomorphism of course continuity must hold in both directions. Certainly id: $(\mathbb{R}, \Lambda) \to (\mathbb{R}, \mathcal{J})$ is continuous, but id: $(\mathbb{R}, \mathcal{J}) \to (\mathbb{R}, \Lambda)$ is not since the neighborhood filter convergent to zero in $\mathbb{R}$ with respect to $\mathcal{J}$ under the identity mapping, doesn't converge to zero in $\mathbb{R}$ with respect to Λ.

If we assume for our system that all one dimensional subspaces are isomorphic we can show <u>Axiom 9</u> for remainders $\mathcal{R}_o(E_\Lambda, F_{\Lambda'})$ as follows:

We have that E_1 and F_1 are one dimensional subspaces of E and F respectively, that L is a linear isomorphism of F_1 into E_1, and that r is in $\mathcal{R}_0(E_{\Lambda'}, G_{\Lambda''})$. Then $r \mid U = 0$, where U is a Λ open neighborhood of zero in E. But $U \cap E_1$ is a Λ open neighborhood of zero in E_1, and by continuity, the inverse image under L of $U \cap E_1$ is a Λ open neighborhood N_1 of zero in F_1. Then there is a Λ open set N in F such that $N \cap F_1 = N_1$. Now define

$$\rho(x) = \begin{cases} r \cdot L(x), & x \in F_1 \\ 0, & x \notin F_1 \end{cases}$$

Clearly $\rho(x) \in \mathcal{R}_0(F, G)$ since $\rho | N = 0$, where N is a Λ open neighborhood of zero in F.

If we do not assume that linear one dimensional subspaces are isomorphic, Wehrli has suggested in (8) that the axiom be altered to

Axiom 9' Let E, G be objects in $\mathcal{K}$, and E_1 be a one-dimensional subspace of E. Let L, where $L: K \to E_1$, be an algebraic linear isomorphic mapping from K to E_1 and $r \in \mathcal{R}(E, G)$. Then $r \cdot L$ is a remainder from K to G.

The verification of axiom 9' is similar to that for axiom 9. The continuity in one direction for $L: K \to E_1$ is of course a consequence of scalar multiplication.

Axiom 8 is true also for our remainders $\mathcal{R}_0(E_\Lambda, F_{\Lambda'})$. If r is in

$\mathcal{R}_o(E_\Lambda, F_\Lambda)$, then $r|U = 0$, U a Λ open neighborhood of zero. But then $U \cap G$ is a Λ open neighborhood of zero in G and clearly $r|U \cap G = 0$. Therefore $r \in \mathcal{R}(G_{\Lambda''}, F_{\Lambda'})$.

Now consider Axiom 2. This too can be verified for our remainders $\mathcal{R}_o(E_\Lambda, F_{\Lambda'})$. We take of course $(r_1 + r_2)(x) = r_1(x) + r_2(x)$, $r_1, r_2 \in \mathcal{R}_o(E_\Lambda, F_{\Lambda'})$. If $r_1|U_1 = 0$ and $r_2|U_2 = 0$ where U_1 and U_2 are Λ open neighborhoods of zero, then clearly $r_1 + r_2 | U_1 \cap U_2 = 0$, and since $U_1 \cap U_2$ is a Λ open neighborhood of 0, $r_1 + r_2 \in \mathcal{R}_o(E_\Lambda, F_{\Lambda'})$. If $\lambda \in K$, then consider λr, $r \in \mathcal{R}_o(E_\Lambda, F_{\Lambda'})$. If $r|U = 0$, U a Λ open neighborhood of zero, clearly $\lambda r|U = 0$ and $\lambda r \in \mathcal{R}_o(E_\Lambda, F_{\Lambda'})$ for $\lambda \in K$. The remaining properties of a linear space follow easily.

Axiom 7 can also be shown for $\mathcal{R}_o(E_\Lambda, F_{\Lambda'})$ since scalar multiplication is continuous. Let $r|U = 0$, where U is a Λ open neighborhood of zero. Then also $\lambda^{-1}U$, $\lambda^{-1} \in K$, is a Λ open neighborhood of zero by the continuity of scalar multiplication. It follows that $r\lambda|\lambda^{-1}U = 0$ and $r\lambda$ is in $\mathcal{R}(E_\Lambda, F_{\Lambda'})$.

We now see that axioms 2, 7, 8, 9, 10, 11 are all true for our remainders $\mathcal{R}_o(E_\Lambda, F_{\Lambda'})$. It seems reasonable then that we should require that these axioms hold for remainders in general. From this point on we make this requirement.

We note that Axiom 0 follows since we took $\mathcal{K}$ as the set of all linear spaces over K.

Axiom 1 is true since $\mathcal{A}(E_{\Lambda}, F_{\Lambda'})$ was taken as the set of all linear continuous functions from E_{Λ} to $F_{\Lambda'}$.

Axiom 3 deals with uniqueness of differentiation and some expansion is in order. When uniqueness of differentiation is required, this means that for any differentiable function f(x) from E_{Λ} to $F_{\Lambda'}$ a unique derivative and remainder pair A and r exist such that

$$f(x + h) - f(x) = Ah + r(h), \; h \in E.$$

This implies that if $\mathcal{A}(E_{\Lambda}, F_{\Lambda'})$ is the set of all linear continuous functions from E_{Λ} to $F_{\Lambda'}$, $\mathcal{A}(E_{\Lambda}, F_{\Lambda'}) \cap \mathcal{R}(E_{\Lambda}, F_{\Lambda'})$ is zero for all $E_{\Lambda}, F_{\Lambda'} \in \mathcal{K}$. To verify this suppose that there is an $A \in \mathcal{A}(E_{\Lambda}, F_{\Lambda'})$ such that $A \in \mathcal{A} \cap \mathcal{R}$ and $A \neq 0$. Since $\mathcal{R}(E_{\Lambda}, F_{\Lambda'})$ is a linear space, then $-A \in \mathcal{R}(E_{\Lambda}, F_{\Lambda'})$. It follows then that $A(h) + (-A)(h) = A(h) - A(h) = 0$. But then the zero mapping has derivatives 0 and A and thus differentiation is not unique. With this notion then consider Axiom 3 under the requirement that differentiation be unique. Let $r \in \mathcal{R}(E_{\Lambda}, F_{\Lambda'})$ be such that r restricted to a one dimensional subspace $L \subseteq E$ is linear. Of course $r \in \mathcal{R}(E_{\Lambda}, F_{\Lambda'})$ implies that $r \mid L \in \mathcal{R}(L_{\Lambda_L}, F_{\Lambda'})$ by Axiom 8, where Λ_L is the limitation induced on L by Λ. Then r is a linear continuous mapping and $r \in \mathcal{A}(L_{\Lambda_L}, F_{\Lambda'})$. But $\mathcal{A}(L_{\Lambda_L}, F_{\Lambda'}) \cap \mathcal{R}(L_{\Lambda_L}, F_{\Lambda'})$ contains only the zero mapping. Therefore $r \mid L = 0$.

It seems natural for us to require the chain rule to hold for our derivatives. Some expansion is in order to determine what implications are involved in the chain rule.

Let E_Λ, $F_{\Lambda'}$, $G_{\Lambda''} \in \mathcal{Y}$, and let $f: E \to F$, be a differentiable function at a point $x \in E$. Then by our definition there exist $A_1 \in \mathcal{Q}(E_\Lambda, F_{\Lambda'})$ and r_1 in $\mathcal{R}(E_\Lambda, F_{\Lambda'})$ such that

$$f(x+h) - f(x) = A_1 h + r_1(h), \quad h \in E.$$

Let also $g: F \to G$ be a differentiable function at the point $y = f(x)$ in F. Then there exist $A_2 \in \mathcal{Q}(F_{\Lambda'}, G_{\Lambda''})$ and $r_2 \in \mathcal{R}(F_{\Lambda'}, G_{\Lambda''})$ such that

$$g(y+k) - g(y) = A_2 k + r_2(k), \quad k \in F.$$

Setting $y + k = f(x+h)$ and $y = f(x)$ it follows that

$$g(f(x+h)) - g(f(x)) = A_2 k + r_2(k).$$

Since $k = f(x+h) - f(x)$,

$$\begin{aligned} A_2 k + r_2(k) &= A_2(f(x+h) - f(x)) + r_2(f(x+h) - f(x)) \\ &= A_2(A_1 h + r_1(h)) + r_2(A_1 h + r_1(H)) \\ &= A_2(A_1 h) + A_2(r_1(h)) + r_2(A_1 h + r_1(h)) \end{aligned}$$

The chain rule of course states that $A_2 A_1$ is the derivative of $g(f(x))$, so $A_2(r_1(h)) + r_2(A_1 h + r_1(h))$ should be a remainder. Notehere that if we take the function g to be such that $g(y+k) = g(y) + r_2(k)$, then g is still differentiable at y and $g(y+k) - g(y) = A_2 k + r_2(k)$ for $k \in F$, reduces to $g(y)+r_2(k)-g(y)=(A_2k+r_2(k))$ and therefore $A_2(k) = 0$, $k \in F$. Then $A_2(r_1(h))+r_2(A_1h+r_1(h))$ reduces to $r_2(Ah+r(h))$ which in the case must be a remainder. Since $\mathcal{R}(E_\Lambda, F_\Lambda')$ is a linear space, the requirement that $r_2(Ah+r(h))$ be a remainder under certain circumstances together with the requirement that $A_2(r_1(h))+r_2(A_1h+R_1(h))$ be a remainder, gives motivation for requiring that Axioms 4, 5, and 6 for a differential calculus hold and together represent the implications of the chain rule. We assume then that the chain rule holds, and therefore that Axioms 4, 5 and 6 are valid.

We note at this point that an alternative argument for the validity of Axiom 7 can now be given. It is simply a consequence of the fact that scalar multiplication is a linear continuous mapping, and of Axiom 5.

Thus the requirements specified for derivatives and remainders agree essentially with the sixteen axioms, and indeed the axiomatic characterization of a differential calculus seems quite satisfactory in the case of linear limit spaces.

An important concept to be considered in connection with remainders is that of the U-set. Consider the following definition:

Definition A subset U of an object E in a calculus $\mathcal{K}$ is called a U-set with respect to object G_Λ'', where G_Λ'' has nonzero dimension, if whenever there exist $r \in \mathcal{R}(E_\Lambda, G_\Lambda'')$ and $f: E_\Lambda \to G_\Lambda''$, such that

$$f \mid U = r \mid U,$$

then $f \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$.

$G_{\Lambda''}$ of course is any object in $\mathcal{K}$, and the definition in no way depends on a special object.

The question now arises -- what are the U-sets for the structure just considered?

Theorem 2.3 A set $U \subseteq E$ is a U-set if and only if there exists an $r \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ such that $r \mid CU = c$, a nonzero constant.

Proof First of all let $r \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ such that $r \mid CU = c$, a nonzero constant, and let f be a mapping from E_Λ to $G_{\Lambda''}$ such that $f \mid U = \rho_1 \mid U$ where $\rho_1 \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$. Axiom 12 implies that there exists a ρ_2 in $\mathcal{R}(E_\Lambda, G_{\Lambda''})$ such that $\rho_2 \mid CU = f$. Then Axiom 11 implies that $f \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ since f either agrees with ρ_1 or ρ_2.

If again $\mathcal{R}_o(E_\Lambda, G_{\Lambda''})$ is the set of remainders which vanish on a Λ open neighborhood of zero, it is clear that:

Theorem 2.4 U is a U-set if and only if it contains a Λ open neighborhood of zero.

Chapter- III The Binz Remainder

In a recent paper (1), Binz introduced the following definition for a remainder r defined on the class of all limit spaces $\mathcal{K} = \{ (E, \Lambda),$ $(F, \Lambda'), \ldots \}$ over the real number field $\mathbb{R}$. This is a good example of a remainder as described in chapter II, and it will now be shown that this definition satisfies the sixteen axioms used to characterize a differential calculus.

Note that if ϕ is a filter in $\Lambda(0)$, ϕ_u in the following definition, where U is a Λ open set containing zero in E, will mean the filter generated by the intersection of the filter sets in $\phi \in \Lambda(0)$ with U. $\Lambda_u(0)$ will represent the Λ -ideal generated by the intersection of U with the sets of the filters in $\Lambda(0)$.

<u>Defniition (Binz)</u> Let $U \subseteq E_\Lambda$ be a Λ open set containing zero. The mapping $r: U \to F_{\Lambda'}$, which is continuous at $0 \in U$ is called a remainder when for every $\phi_u \in \Lambda_u(0)$ there exists a $\psi \in \Lambda(0)$ satisfying the following condition:

(B) For every $N \in \psi$ there is an $M \in \phi_u$ and a σ such that $r(\lambda M) \subseteq \sigma(\lambda) N$ for all λ in the domain of definition $[-\varepsilon, \varepsilon]$ of σ.

σ here represents a remainder, in the usual sense of analysis, over the calculus of real numbers, ie. where $\lim_{\lambda \to 0} \left| \frac{\sigma(\lambda)}{\lambda} \right| = 0$, and $\sigma(0) = 0$.

From (B) $r(0) = 0$, and the zero mapping is a remainder.

Early in (1), Binz proves the following theorem:

Theorem 3.1 Let $U \subseteq E$ be a Λ open set containing zero. A mapping $r: E_\Lambda \to F_{\Lambda'}$ is a remainder if and only if $r|U$ is a remainder.

With this, the above definition can be altered by simply replacing U by E, and this will be the form for the definition considered in the present paper.

The problem now is to show that the Binz remainders satisfy the axioms associated with a differential calculus. Let, as before, $\mathcal{L}(E_\Lambda, F_{\Lambda'})$, $E_\Lambda, F_\Lambda \in \mathcal{K}$, consist of the set of all linear continuous mappings from E_Λ to $F_{\Lambda'}$, and let $\mathcal{R}(E_\Lambda, F_{\Lambda'})$ be made up of the functions which are remainders in the sense of Binz.

First of all we note that Axioms 12, 13, 14, and 15 follow as consequences of Theorem 2.2 since the zero mapping is a remainder and any function vanishing on a Λ open neighborhood of zero is a remainder, (cf. proof of Theorem 3.2). In his paper Binz verifies Axioms 2,3,4,5, and 6. Clearly Axioms 0 and 1 hold by definition of $\mathcal{K}$ and $\mathcal{L}(E_\Lambda, F_{\Lambda'})$ respectively. Axioms 7,8,9.10, and 11 then remain to be verified for these remainders.

Axiom 7 of course follows since real number multiplication is linear and continuous, and the chain rule holds.

Axiom 8 can be verified in the following way:

The continuity requirement at zero is certainly clear. It must now be shown that for any $\phi \in \Lambda(0)$, there exists an $\psi \in \Lambda'(0)$ such that for any $N \in \psi$ there is an $M \in \phi$ and a σ such that

$$r(\lambda M) \subseteq \sigma(\lambda)N, \text{ for } \lambda \in [-\varepsilon, \varepsilon] .$$

The filters in $\Lambda(0)$ are of course those induced by $\Lambda'(0)$. Let ϕ be any filter in $\Lambda(0)$. The sets of ϕ are generated by the intersection of the sets of a filter $\phi' \in \Lambda'(0)$ with E. But $r \in \mathcal{R}(F_{\Lambda'}, G_{\Lambda''})$. Therefore there exists an $\psi \in \Lambda''(0)$ corresponding to ϕ' so that for any $N \in \psi$ there is a set $W \in \phi'$ and a σ such that

$$r(\lambda W) \subseteq \sigma(\lambda)N, \quad \lambda \in [-\varepsilon, \varepsilon] .$$

But $W \cap E \in \phi$. Let then $M = W \cap E$ and clearly

$$r(\lambda M) \subseteq \sigma(\lambda)N, \quad \lambda \in [-\varepsilon, \varepsilon] .$$

Thus $r|E \in \mathcal{R}(E_{\Lambda}, G_{\Lambda''})$.

From Koethe's book on linear topologies (7) we know that all one dimensional subspaces over the real numbers are isomorphic. Consider then Axiom 9. Define

$$\rho(x) = \begin{cases} (r \cdot L)(x), & x \in F_1 \subseteq F \\ & , \\ 0, & \text{otherwise} . \end{cases}$$

and show that $\rho \in \mathcal{R}(F_{\Lambda'}, G_{\Lambda''})$. It is noted first of all that continuity at zero is clear for $\rho(x)$. Now suppose that ϕ is any filter in $\Lambda'(0)$ such that ϕ has nonempty intersection with F_1. Then ϕ generates a filter ϕ_1 in F_1 which converges to zero in F_1. Since L is a linear isomorphism $L(\phi)$ is a filter θ_1 converging to zero in E_1. This means of course that there is a filter $\theta \in \Lambda(0)$ which generates θ_1. Since $r \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$ there exists an $\psi \in \Lambda''(0)$ corresponding to $\theta \in \Lambda(0)$ such that for any $N \in \psi$ there exists an $M \in \theta$ and a σ so that

$$r(\lambda M) \subseteq \sigma(\lambda)N, \ \lambda \in [-\varepsilon, \varepsilon].$$

Clearly then

$$r\,[\lambda(M \cap E_1)] \subseteq \sigma(\lambda)N,$$

Let $W_1 \in \theta_1$ be the preimage of $M \cap E_1$ under L. Then

$$r\,[\lambda(L(W_1))] \subseteq \sigma(\lambda)N, \text{ and}$$

since L is linear

$$r\,[L(\lambda W_1)] \subseteq \sigma(\lambda)N.$$

Also because $W_1 \in \phi_1$, there exists a W such that $W \cap F_1$ is contained in W_1. Then

$$\rho(\lambda W) \subseteq \sigma(\lambda)N, \text{ for } \lambda \in [-\varepsilon, \varepsilon],$$

follows directly from the definition of ρ since $\rho = (r \cdot L)(x)$ for $x \in F_1 \subseteq F$, and zero otherwise. Therefore $\rho \in \mathcal{R}(F_{\Lambda'}, G_{\Lambda''})$ with the property that $r \cdot L = \rho$ on F_1.

Now if ϕ has empty intersection with F, condition (B) follows trivially, i.e. if ψ is any filter in $\Lambda''(0)$, choose for any $N \in \psi$, an arbitrary set $M \in \phi$ and an arbitrary σ. Then by definition of ρ

$$\rho(\lambda M) = 0,$$

and therefore satisfies the requirement that

$$\rho(\lambda M) \subseteq \sigma(\lambda)N.$$

To verify Axiom 10, first of all consider the question of continuity at zero. Let Λ, Λ', and Λ'' again be the limitations on E, F, and G respectively, and let $\phi \in \Lambda(0)$. If ϕ has empty intersection with G, clearly $\rho(\phi)$ generates the ultrafilter in $\Lambda'(0)$. If ϕ has nonempty intersection with G and therefore generates a filter ϕ_1 in G it is clear from the definition of ρ that $\rho(\phi) \equiv r(\phi_1)$, and $r(\phi_1)$, since $r \in \mathcal{R}(G_{\Lambda''}, F_{\Lambda'})$, generates a filter in $\Lambda'(0)$. Thus ρ is continuous at zero.

Let ϕ_1 be a filter in $\Lambda''(0)$. Then ϕ_1, is generated by some filter $\phi \in \Lambda(0)$. Since $r \in \mathcal{R}(G_{\Lambda''}, F_{\Lambda'})$, there exists an $\psi \in \Lambda''(0)$ such that for any $N \in \psi$ there exists an $M_1 \in \phi$ and a σ such that

$$r(\lambda M_1) \subseteq \sigma(\lambda)N, \lambda \in [-\varepsilon, \varepsilon].$$

Let $M \cap G \subseteq M_1$, $M \in \phi$. Note here that if $\lambda y \in \lambda M$ and $\lambda y \in G$, then $\lambda y \in \lambda M_1$, $\lambda \in [-\varepsilon, \varepsilon]$. To verify this let $\lambda y \in \lambda M$. Then $y \in M$ and if $\lambda y \in G$, $y \in G$ since G is a linear space. Therefore $y \in M_1$ and $\lambda y \in \lambda M_1$. Consider now $\rho(\lambda M)$. Clearly $\rho(\lambda M) = r(\lambda M) \subseteq \sigma(\lambda)N$, $\lambda \in [-\varepsilon, \varepsilon]$, and condition (B) is satisfied. If $\phi \in \Lambda(0)$ with empty intersection with G, condition (B) is trivial and $\rho \in \mathcal{R}(E_{\Lambda}, F_{\Lambda'})$.

Before considering Axiom 11 it is useful to introduce the following notation used by Binz in (1). Let $\phi^o_{\mathbb{R}}$ be the neighborhood filter of zero in $\mathbb{R}$; where we think of $\phi^o_{\mathbb{R}}$ as being generated by the system of closed symmetric intervals $\{[-\varepsilon,\varepsilon] \mid \varepsilon>0\}$. Now let $E_{\Lambda} \in \mathcal{X}$ and define $\phi^o_{\mathbb{R}}\phi$, where ϕ is any filter in Λ, to be the filter generated by $\{R \cdot M \mid R \in \phi^o_{\mathbb{R}}, M \in \phi\}$. Note that if $\phi \in \Lambda$, then $\phi \wedge \phi^o_{\mathbb{R}}\phi$ is the finest filter coarser than both ϕ and $\phi^o_{\mathbb{R}}\phi$, and it is generated by $\{M \cup [-\varepsilon,\varepsilon]M \mid \varepsilon>0, M \in \phi\}$. Suppose now that $\phi \in \Lambda(0)$. In (4) Fischer has proven that $\phi^o_{\mathbb{R}}\Lambda(0) \leq \Lambda(0)$, so $\phi^o_{\mathbb{R}}\phi \in \Lambda(0)$. Therefore since Λ is a Λ ideal, $\phi \wedge \phi^o_{\mathbb{R}}\phi \in \Lambda(0)$.

Now consider <u>Axiom 11</u>. The proof here involves essentially the procedure used by Binz in (1) when he verified Axiom 2. For r_1, r_2 in $\mathcal{R}(E_{\Lambda}, F_{\Lambda'})$, we have that for any filter $\phi \in \Lambda(0)$, there exist filters ψ_1 and ψ_2 in $\Lambda(0)$ such that for any $N_1 \in \psi_1$, $N_2 \in \psi_2$ there exists $M_1, M_2 \in \phi$ and σ_1, σ_2 such that $r_1(\lambda M_1) \leq \sigma_1(\lambda) N_1$, $\lambda \in [-\varepsilon_1,\varepsilon_1]$, and $r_2(\lambda M_2) \leq \sigma_2(\lambda) N_2$, $\lambda \in [-\varepsilon_2,\varepsilon_2]$. Consider the filter $\psi = \psi_1 \wedge \psi_2 \in \Lambda(0)$. From the previous discussion it follows that the filter $\psi \wedge \phi^o_{\mathbb{R}}\psi \in \Lambda(0)$ is coarser than ψ. Let K be any set in $\psi \wedge \phi^o_{\mathbb{R}}\psi$. It contains a set of the form $K' \cup [-\varepsilon',\varepsilon']K'$, for $K' \in \psi$, and $0 < \varepsilon' \leq 1$.

Now for any filter ϕ construct a corresponding filter in the above way. Then any set K in $\psi \wedge \phi^o_{\mathbb{R}}\psi$ contains a set in the form $K' \cup [-\varepsilon',\varepsilon']K'$, $K' \in \psi$, i.e. $K' \supseteq N_1 \cup N_2$, $N_1 \in \psi_1$, $N_2 \in \psi_2$. Select then, corresponding to N_1 and N_2 with respect to r_1 and r_2, the sets M_1 and M_2 in ϕ, and form $M = M_1 \cap M_2$. Clearly $M_1 \cap M_2 \in \phi$. Then it follows that

$$r_1(\lambda M) \subseteq \sigma_1(\lambda)K, \ \lambda \in [-\varepsilon_1, \varepsilon_1]$$

$$\text{and} \quad r_2(\lambda M) \subseteq \sigma_2(\lambda)K, \ \lambda \in [-\varepsilon_2, \varepsilon_2].$$

Choosing $[-\varepsilon, \varepsilon] = \text{Min}\{[-\varepsilon_1, \varepsilon_1], [-\varepsilon_2, \varepsilon_2]\}$, the above are satisfied for $\lambda \in [-\varepsilon, \varepsilon]$. Now for nonzero $\mu, \mu' \in \mathbb{R}$ such that $|\mu/\mu'| < \varepsilon'$ we have that

$$(\mu/\mu')(K' \cup [-\varepsilon', \varepsilon']K') \subseteq (K' \cup [-\varepsilon', \varepsilon']K'),$$

and therefore $\mu K \subseteq \mu' K$.

Now for $\lambda \in [-\varepsilon, \varepsilon]$, define

$$\sigma(\lambda) = \begin{cases} \sigma_1(\lambda)/\varepsilon', & \text{when } |\sigma_1(\lambda)| \geq |\sigma_2(\lambda)| \\ \sigma_2(\lambda)/\varepsilon', & \text{when } |\sigma_2(\lambda)| \geq |\sigma_1(\lambda)| \end{cases}$$

$$\text{and } |\varepsilon'\sigma(\lambda)| \geq \text{Max}(|\sigma_2(\lambda)|, |\sigma_2(\lambda)|).$$

Clearly $\lim_{\lambda \to 0} |\sigma(\lambda)/\lambda| = 0$, $\sigma_1(\lambda)K \subseteq \sigma(\lambda)K$, and $\sigma_2(\lambda)K \subseteq \sigma(\lambda)K$. Therefore $r(\lambda M) \subseteq \sigma(\lambda)K, \lambda \in [-\varepsilon, \varepsilon]$.

It remains to show that r is continuous at zero, i.e. that for $\phi \in \Lambda(0)$. $r(\phi) \in \Lambda'(0)$. Clearly $r(\phi) \leq \psi_1 \cup \psi_2$ if ψ_1 and ψ_2 are the images of ϕ under r_1 and r_2 respectively. Since $F_{\Lambda'}$ is a limit space we know that $\psi_1 \wedge \psi_2 \in \Lambda'(0)$. Certainly then r generates a filter and $r(\phi) \geq \psi_1 \cup \psi_2$. Therefore $r(\phi) \in \Lambda'(0)$.

Thus the Binz remainders satisfy the axioms which characterize a differential calculus.

Consider now the following theorem:

Theorem 3.2 A set $U \subseteq E_\Lambda$, $E_\Lambda \in \mathcal{K}$, is a U-set with respect to the Binz remainder if and only if it contains a Λ open neighborhood of zero.

Proof First of all, suppose that U is a U-set with respect to the Binz remainders. Since these remainders satisfy the axioms for a differential calculus, by Theorem 2.3 there exists an $r \in \mathcal{R}(E_\Lambda, G_{\Lambda''})$, $G_{\Lambda''} \in \mathcal{K}$, such that $r | CU = c$, a nonzero constant. Select then a Λ open set V in G such that $c \notin V$. It follows easily from the fact that r is continuous at zero that $r^{-1}(V)$ is an Λ open neighborhood of zero in E_Λ and that $r^{-1}(V) \subseteq U$.

Conversely, it must be shown that any set $W \subseteq E$ containing a Λ open neighborhood of zero is a U-set. Let then $U \subseteq E$ be a Λ open neighborhood of zero where $U \subseteq W$. Define the mapping r from E_Λ to $F_{\Lambda'}$ so that

$$r \mid W = 0$$

and $r \mid CW = c$, where c is a nonzero constant. We now would like to show that $r \in \mathcal{R}(E_\Lambda, F_{\Lambda'})$. Continuity at zero is clear, i.e. let ϕ be any filter in $\Lambda(0)$. Since $U \in \phi$ and $r(U) = 0$, $r(\phi)$ generates the ultrafilter $\mathring{0} \in \Lambda'(0)$. Now let N be any set in $\mathring{0}$. It must be shown that there exists an $M \in \phi$ and a remainder σ such that $r(\lambda M) \subseteq \sigma(\lambda)N$, for $\lambda \in [-\epsilon, \epsilon]$.

Let Θ be the filter generated by the Λ open sets containing $0 \in E$.