On the unique factorization of a non-singular matrix
by William M Lowney

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics
Montana State University

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ABSTRACT

This paper describes a sequential method of factoring a square non-singular matrix into two triangular matrices. The purpose of this factorization is to enable the computation of the inverse of the original matrix to be more easily coded for machine computation, for the means of finding the inverse of a triangular matrix is well known. Herein we deal with a matrix whose principal minor determinants do not vanish.

The method of attack is to make use of pivotal condensation of determinants and also the use of extensional identities to form the sequential method of evaluation. The results obtained indicate the possibility of this method being used for machine computation.

INTRODUCTION

The problem of finding the inverse of a matrix is one of utmost importance in practically all branches of applied mathematics. The theoretical solution using the adjoint method is almost worthless in actual computation. Other methods of attack include partitioning, approximation and statistical methods. For a complete outline of methods we refer to Forsythe's article (Paige and Taussky, 6). All such methods, however, require laborious computation and considerable machine time. On the other hand, if the matrix is triangular, the coding for machine computation has been set up and is well known.

This paper treats of the factorization of a non-singular matrix into two triangular matrices. The conditions imposed on the original matrix are relatively weak. We develop a method of sequential computation of the elements of the triangular matrices such that it may be coded for electronic computers.

We shall show that the elements of our triangular matrices are formed by simple operations on the two by two determinants compiled in evaluating the determinant of the matrix by successive pivotal condensations. Then if $A = BC$ we have $A^{-1} = C^{-1}B^{-1}$ and if B and C are triangular our inverse is readily computed.

THE FUNDAMENTAL THEOREM

We take $A=[a_{ij}]$ as our given non-singular matrix with the added condition that none of the first principal minor determinants of any order vanishes. We shall factor A into two triangular matrices, B and C, such that $A=BC$, where B and C are of the form exhibited:

$$(1) \qquad B=\begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & b_{n3} & b_{n4} & \cdots & b_{nn} \end{bmatrix}$$

$$(2) \qquad C=\begin{bmatrix} 1 & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} \\ 0 & 1 & c_{23} & c_{24} & \cdots & c_{2n} \\ 0 & 0 & 1 & c_{34} & \cdots & c_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then

$$BC = \begin{bmatrix} b_{11} & b_{11}\,c_{12} & b_{11}\,c_{13} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot & b_{11}\,c_{1n} \\ b_{21} & b_{21}\,c_{12} + b_{22} & b_{21}\,c_{13} + b_{22}\,c_{23} \cdot \cdot \cdot \cdot & b_{21}\,c_{1n} + b_{22}\,c_{2n} \\ b_{31} & b_{31}\,c_{12} + b_{32} & b_{31}\,c_{13} + b_{32}\,c_{23} + b_{33} \cdot \cdot & b_{31}\,c_{1n} + b_{32}\,c_{2n} + b_{33}\,c_{3n} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{n1} & b_{n1}\,c_{12} + b_{n2} & b_{n1}\,c_{13} + b_{n2}\,c_{23} + b_{n3} \cdot \cdot & \sum_{i=1}^{n-1} b_{ni}\,c_{in} + b_{nn} \end{bmatrix}$$

However, if the two matrices are equal, we know that the corresponding elements must be equal. That is

(3) $$a_{ij} = \sum_{k=1}^{p} b_{ik}\,c_{kj}$$ where p is the minimum of (i,j),

for by our method of construction it is obvious that

$b_{ik} = 0$ for $i<k$, that $c_{kj} = 0$ for $k>j$, and $c_{ii} = 1$.

We shall prove the following theorem:

Theorem:

(4) $$b_{ik} = \frac{|a_{11}\ a_{22} \cdot \cdot \cdot \cdot a_{k-1,k-1}\ a_{ik}|}{|a_{11}\ a_{22} \cdot \cdot \cdot \cdot a_{k-1,k-1}|} \qquad i \geq k$$

(5) $$c_{kj} = \frac{|a_{11} \cdot \cdot \cdot \cdot a_{k-1,k-1}\, a_{kj}|}{|a_{11}\ a_{22} \cdot \cdot \cdot \cdot a_{kk}|} \qquad j \geq k$$

where we use the notation (Aitken, 1);

$$|a_{11}\ a_{22} \cdot \cdot \cdot a_{jj}\ a_{ik}| = \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{1j} & a_{1k} \\ a_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{2j} & a_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{jj} & a_{jk} \\ a_{i1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{ij} & a_{ik} \end{vmatrix} .$$

To begin our proof, we note that $b_{11} = a_{11}$ and, in fact, that the first column of B equals the first column of A, thus $b_{i1} = a_{i1}$.

Now
$$a_{1j} = b_{11}\, c_{1j}$$
$$c_{1j} = \frac{a_{1j}}{a_{11}} .$$

The solution for b_{i2} is as follows:

$$a_{i2} = b_{i1}\, c_{12} + b_{i2}$$
$$b_{i2} = a_{i2} - b_{i1}\, c_{12} = a_{i2} - \frac{a_{i1}\, a_{12}}{a_{11}}$$
$$b_{i2} = \frac{|a_{11}\ a_{i2}|}{a_{11}} .$$

Solving for c_{2j}

$$a_{2j} = b_{21}\, c_{1j} + b_{22}\, c_{2j}$$

$$b_{22}\, c_{2j} = a_{2j} - a_{21} \frac{a_{1j}}{a_{11}}$$

$$\frac{|a_{11}\ a_{22}|}{a_{11}}\, c_{2j} = \frac{|a_{11}\ a_{2j}|}{a_{11}}$$

$$c_{2j} = \frac{|a_{11}\ a_{2j}|}{|a_{11}\ a_{22}|} .$$

To prove the theorem we shall use the first law of induction, noting that the column of b's is to be computed first. Assume the theorem true for $b_{i,j-1}$ where $i \geqslant j - 1$ and $c_{j-1,k}$ where $j - 1 \leqslant k$. Then we have

$$b_{i,j-1} = \frac{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-2,j-2}\ a_{i,j-1}|}{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-2,j-2}|}$$

$$c_{j-1,k} = \frac{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-2,j-2}\ a_{j-1,k}|}{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-2,j-2}\ a_{j-1,j-1}|} .$$

Now consider

(6) $$Q = \frac{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-1,j-1}\ a_{ij}|}{|a_{11}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{j-1,j-1}|} .$$

By use of our extensional identities (Aitken, 1) we get the above equal to

$$\frac{\begin{vmatrix} |a_{11} \; . \; . \; . a_{j-1,j-1}| & |a_{11} \; . \; . \; . a_{j-2,j-2} \;\; a_{j-1,j}| \\ |a_{11} \; . \; . \; . a_{j-2,j-2} \;\; a_{i,j-1}| & |a_{11} \; . \; . \; . a_{j-2,j-2} \;\; a_{ij}| \end{vmatrix}}{|a_{11} \; . \; . \; . a_{j-1,j-1}| \qquad |a_{11} \; . \; . \; . a_{j-2,j-2}|} \; .$$

Dividing the first row of the numerator by $|a_{11} \; . \; . \; . a_{j-1,j-1}|$ and the second row by $|a_{11} \; . \; . \; . a_{j-2,j-2}|$ we have:

$$\begin{vmatrix} 1 & c_{j-1,j} \\ b_{i,j-1} & \dfrac{|a_{11} \; . \; . \; . a_{j-2,j-2} \;\; a_{ij}|}{|a_{11} \; . \; . \; . a_{j-2,j-2}|} \end{vmatrix}$$

$$= \quad \frac{|a_{11} \; . \; . \; . a_{j-2,j-2} \;\; a_{ij}|}{|a_{11} \; . \; . \; . \; a_{j-2,j-2}|} - b_{i,j-1} \;\; c_{j-1,j} \; .$$

But this becomes, upon another application of the extensional identity,

$$\frac{\begin{vmatrix} |a_{11} \; . \; . \; . a_{j-2,j-2}| & |a_{11} \; . \; . \; . a_{j-3,j-3} \;\; a_{j-2,j}| \\ |a_{11} \; . \; . \; . a_{j-3,j-3} \; a_{i,j-2}| & |a_{11} \; . \; . \; . a_{j-3,j-3} \; a_{ij}| \end{vmatrix}}{|a_{11} \; . \; . \; . \; . \; a_{j-2,j-2}| \quad |a_{11} \; . \; . \; . \; . \; a_{j-3,j-3}|} - b_{i,j-1} \;\; c_{j-1,j} \; ,$$

and following the same procedure as above this becomes:

$$\begin{vmatrix} 1 & c_{j-2,j} \\ b_{i,j-2} & \dfrac{|a_{11} \cdots a_{j-3,j-3}\ a_{ij}|}{|a_{11} \cdots a_{j-3,j-3}|} \end{vmatrix} - b_{i,j-1}\ c_{j-1,j}$$

$$= \frac{|a_{11} \cdots a_{j-3,j-3}\ a_{ij}|}{|a_{11} \cdots a_{j-3,j-3}|} - \sum_{k=1}^{2} b_{i,j-k}\ c_{j-k,j}$$

$$= \frac{\begin{vmatrix} |a_{11} \cdots a_{j-3,j-3}| & |a_{11} \cdots a_{j-4,j-4}\ a_{j-3,j}| \\ |a_{11} \cdots a_{j-4,j-4}\ a_{i,j-3}| & |a_{11} \cdots a_{j-4,j-4}\ a_{ij}| \end{vmatrix}}{|a_{11} \cdots a_{j-3,j-3}| \qquad |a_{11} \cdots a_{j-4,j-4}|} - \sum_{k=1}^{2} b_{i,j-k}\ c_{j-k,j}$$

$$= \begin{vmatrix} 1 & c_{j-3,j} \\ b_{i,j-3} & \dfrac{|a_{11} \cdots a_{j-4,j-4}\ a_{ij}|}{|a_{11} \cdots a_{j-4,j-4}|} \end{vmatrix} - \sum_{k=1}^{2} b_{i,j-k}\ c_{j-k,j}$$

$$= \frac{|a_{11} \cdots a_{j-4,j-4}\ a_{ij}|}{|a_{11} \cdots a_{j-4,j-4}|} - \sum_{k=1}^{3} b_{i,j-k}\ c_{j-k,j}$$

or in general

$$\frac{|a_{11} \cdots a_{j-1,j-1}\ a_{ij}|}{|a_{11} \cdots a_{j-1,j-1}|}$$

$$= \frac{|a_{11} \cdots a_{j-n,j-n}\ a_{ij}|}{|a_{11} \cdots a_{j-n,j-n}|} - \sum_{k=1}^{n-1} b_{i,j-k}\ c_{j-k,j} \quad .$$

If we let $n = j - 2$ we have

$$Q = \frac{|a_{11}\ a_{22}\ a_{ij}|}{|a_{11}\ a_{22}|} - \sum_{k=1}^{j-3} b_{i,j-k}\ c_{j-k,j}$$

$$= \frac{\begin{vmatrix} |a_{11}\ a_{22}| & |a_{11}\ a_{2j}| \\ |a_{11}\ a_{i2}| & |a_{11}\ a_{ij}| \end{vmatrix}}{|a_{11}\ a_{22}|\ a_{11}} - \sum_{k=1}^{j-3} b_{i,j-k}\ c_{j-k,j}$$

$$= \begin{vmatrix} 1 & c_{2j} \\ b_{i2} & \frac{|a_{11}\ a_{ij}|}{a_{11}} \end{vmatrix} - \sum_{k=1}^{j-3} b_{i,j-k}\ c_{j-k,j}$$

$$= \frac{|a_{11}\ a_{ij}|}{a_{11}} - \sum_{k=1}^{j-2} b_{i,j-k}\ c_{j-k,j}$$

$$= a_{ij} - \frac{a_{i1}\ a_{1j}}{a_{11}} - \sum_{k=1}^{j-2} b_{i,j-k}\ c_{j-k,j}$$

$$= a_{ij} - b_{i1}\ c_{1j} - \sum_{k=1}^{j-2} b_{i,j-k}\ c_{j-k,j}$$

$$= a_{ij} - \sum_{k=1}^{j-1} b_{i,j-k}\ c_{j-k,j}\ .$$

Now let $j - k = 1$ and we have

$$a_{ij} - \sum_{l=j-1}^{1} b_{il}\ c_{lj}$$ but this is a finite sum

$$a_{ij} - \sum_{\ell=1}^{j-1} b_{i\ell} \; c_{\ell j}.$$ Finally replace ℓ by k

$$a_{ij} - \sum_{k=1}^{j-1} b_{ik} \; c_{kj}$$ but this is the value of b_{ij},

for from our original matrices $a_{ij} = \sum_{k=1}^{j-1} b_{ik} \; c_{kj} + b_{ij}$

and so we see that

$$Q = \frac{|a_{11} \cdot \cdot \cdot \cdot \cdot a_{j-1,j-1} \quad a_{ij}|}{|a_{11} \cdot \cdot \cdot \cdot \cdot \cdot a_{j-1,j-1}|} = b_{ij} \qquad \text{for } i \geq j.$$

Now we shall examine the C matrix. If we take

$$\frac{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \quad a_{jk}|}{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \quad a_{jj}|}$$

and multiply by b_{jj}, which we now know, we have

$$R = \frac{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \quad a_{jk}|}{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \quad a_{jj}|} \cdot \frac{|a_{11} \cdot \cdot \cdot \cdot a_{jj}|}{|a_{11} \cdot \cdot \cdot a_{j-1,j-1}|}$$

$$= \frac{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \quad a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot \cdot a_{j-1,j-1}|}.$$

Again using our identities, we have

$$\frac{\left|\begin{array}{cc} |a_{11} \cdot \cdot \cdot a_{j-1,j-1}| & |a_{11} \cdot \cdot \cdot a_{j-2,j-2} \quad a_{j-1,k}| \\ |a_{11} \cdot \cdot \cdot a_{j-2,j-2} \quad a_{j,j-1}| & |a_{11} \cdot \cdot \cdot a_{j-2,j-2} \quad a_{jk}| \end{array}\right|}{|a_{11} \cdot \cdot \cdot a_{j-1,j-1}| \qquad |a_{11} \cdot \cdot \cdot a_{j-2,j-2}|}$$

Dividing the first row by $|a_{11} \cdot \cdot \cdot a_{j-1,j-1}|$ and the second row by $|a_{11} \cdot \cdot \cdot a_{j-2,j-2}|$, we have

$$\begin{vmatrix} 1 & c_{j-1,k} \\ b_{j,j-1} & \dfrac{|a_{11} \cdot \cdot \cdot a_{j-2,j-2} \; a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot a_{j-2,j-2}|} \end{vmatrix}$$ which equals

$$\frac{|a_{11} \cdot \cdot \cdot a_{j-2,j-2} \; a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot a_{j-2,j-2}|} - b_{j,j-1} \; c_{j-1,k}$$

$$= \frac{\begin{vmatrix} |a_{11} \cdot \cdot \cdot a_{j-2,j-2}| & |a_{11} \cdot \cdot \cdot a_{j-3,j-3} \; a_{j-2,k}| \\ |a_{11} \cdot \cdot \cdot a_{j-3,j-3} \; a_{j,j-2}| & |a_{11} \cdot \cdot \cdot a_{j-3,j-3} \; a_{jk}| \end{vmatrix}}{|a_{11} \cdot \cdot \cdot a_{j-2,j-2}| \qquad |a_{11} \cdot \cdot \cdot a_{j-3,j-3}|} - b_{j,j-1} \; c_{j-1,k},$$

and following the same procedure we have

$$\begin{vmatrix} 1 & c_{j-2,k} \\ b_{j,j-2} & \dfrac{|a_{11} \cdot \cdot \cdot a_{j-3,j-3} \; a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot a_{j-3,j-3}|} \end{vmatrix} - b_{j,j-1} \; c_{j-1,k}$$

$$= \frac{|a_{11} \cdot \cdot \cdot a_{j-3,j-3} \; a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot a_{j-3,j-3}|} - \sum_{l=1}^{2} b_{j,j-l} \; c_{j-l,k}$$

or in general, $\dfrac{|a_{11} \cdot \cdot \cdot a_{j-1,j-1} \; a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot a_{j-1,j-1}|}$

$$= \frac{|a_{11} \cdot \cdot \cdot a_{j-n,j-n}\ a_{jk}|}{|a_{11} \cdot \cdot \cdot \cdot \cdot a_{j-n,j-n}|} - \sum_{\ell=1}^{n-1} b_{j,j-\ell}\ c_{j-\ell,k}\ .$$

If we let $n = j - 2$, we have

$$\frac{|a_{11}\ a_{22}\ a_{jk}|}{|a_{11}\ a_{22}|} - \sum_{\ell=1}^{j-3} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= \frac{\begin{vmatrix} |a_{11} & a_{21}| & |a_{11} & a_{2k}| \\ |a_{11} & a_{j2}| & |a_{11} & a_{jk}| \end{vmatrix}}{|a_{11}\ \ a_{22}|\ \ a_{11}} - \sum_{\ell=1}^{j-3} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= \begin{vmatrix} 1 & c_{2k} \\ b_{j2} & \frac{|a_{11}\ \ a_{jk}|}{a_{11}} \end{vmatrix} - \sum_{\ell=1}^{j-3} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= \frac{|a_{11}\ \ a_{jk}|}{a_{11}} - \sum_{\ell=1}^{j-2} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= a_{jk} - \frac{a_{j1}\ a_{1k}}{a_{11}} - \sum_{\ell=1}^{j-2} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= a_{jk} - b_{j1}\ c_{1k} - \sum_{\ell=1}^{j-2} b_{j,j-\ell}\ c_{j-\ell,k}$$

$$= a_{jk} - \sum_{\ell=1}^{j-1} b_{j,j-\ell}\ c_{j-\ell,k}$$

which by a similar method as applied to the previous case reduces to

$$a_{jk} - \sum_{i=1}^{j-1} b_{ji} \quad c_{ik}$$

but from the matrix product

$$a_{jk} - \sum_{i=1}^{j-1} b_{ji} \quad c_{ik} = b_{jj} \quad c_{jk} \qquad\qquad k > j$$

$$b_{jj} \quad c_{jk} = \frac{|a_{11} \; . \; . \; . a_{jj}|}{|a_{11} \; . \; . \; . a_{j-1,j-1}|} \cdot \frac{|a_{11} \; . \; . \; . a_{j-1,j-1} \; a_{jk}|}{|a_{11} \; . \; . \; . \; . \; . \; a_{jj}|}$$

or $$R = \frac{|a_{11} \; . \; . \; . a_{j-1,j-1} \quad a_{jk}|}{|a_{11} \; . \; . \; . \; . \; . \; a_{jj}|} \quad c$$ as desired.

Hence, if the theorem is true for the (j-1)th column of B and the (j-1)th row of C, it will be true for the jth column of B and the jth row of C. But we have seen it is true for the first and second columns of B and the first and second rows of C, and our induction is complete.

SEQUENTIAL COMPUTATION

We have now proved it is possible to factor a non-singular matrix whose first principal minor determinants do not vanish into two triangular matrices. Let us now consider the determinant A from the aspect of pivotal condensation (Aitken, 1).

$$(7) \qquad |A| = \begin{vmatrix} a_{11}^{0} & a_{12}^{0} & \cdots & a_{1n}^{0} \\ a_{21}^{0} & a_{22}^{0} & \cdots & a_{2n}^{0} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}^{0} & a_{n2}^{0} & \cdots & a_{nn}^{0} \end{vmatrix} \qquad \text{where } a_{ij}^{0} = a_{ij}.$$

By pivotal condensation we have

$$(8) \qquad |A| = \frac{1}{(a_{11}^{0})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11}^{0} & a_{22}^{0} \end{vmatrix} & \begin{vmatrix} a_{11}^{0} & a_{23}^{0} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}^{0} & a_{2n}^{0} \end{vmatrix} \\ \begin{vmatrix} a_{11}^{0} & a_{22}^{0} \end{vmatrix} & \begin{vmatrix} a_{11}^{0} & a_{33}^{0} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}^{0} & a_{3n}^{0} \end{vmatrix} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \begin{vmatrix} a_{11}^{0} & a_{n2}^{0} \end{vmatrix} & \begin{vmatrix} a_{11}^{0} & a_{n3}^{0} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}^{0} & a_{nn}^{0} \end{vmatrix} \end{vmatrix}$$

Let

$$(9) \qquad a_{ij}^{1} = \begin{vmatrix} a_{11}^{0} & a_{i+1,j+1}^{0} \end{vmatrix}$$

then (8) takes form

$$|A| = \frac{1}{(a_{11}^{0})^{n-2}} \begin{vmatrix} a_{11}^{1} & a_{12}^{1} & \cdots & a_{1,n-1}^{1} \\ a_{21}^{1} & a_{22}^{1} & \cdots & a_{2,n-1}^{1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,1}^{1} & a_{n-1,2}^{1} & \cdots & a_{n-1,n-1}^{1} \end{vmatrix}.$$

Hence, we see that

$$(10) \qquad |A| = \frac{1}{(a_{11}^{k-1})^{n-k-1}} \begin{vmatrix} a_{11}^{k} & a_{12}^{k} & \cdots & a_{1,n-k}^{k} \\ a_{21}^{k} & a_{22}^{k} & \cdots & a_{2,n-k}^{k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-k,1}^{k} & a_{n-k,2}^{k} & \cdots & a_{n-k,n-k}^{k} \end{vmatrix}$$

where

$$(11) \qquad a_{ij}^{k} = \frac{|a_{11}^{k-1} \quad a_{i+1,j+1}^{k-1}|}{a_{11}^{k-2}} \, . \qquad k \geq 1, \quad a_{11}^{-1} = 1$$

In this notation we shall see that B takes the form

$$B = \begin{vmatrix} a_{11}^{0} & 0 & 0 & \cdots & 0 \\ a_{21}^{0} & \frac{a_{11}^{1}}{a_{11}^{0}} & 0 & \cdots & 0 \\ a_{31}^{0} & \frac{a_{21}^{1}}{a_{11}^{0}} & \frac{a_{11}^{2}}{a_{11}^{1}} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1}^{0} & \frac{a_{n-1,1}^{1}}{a_{11}^{0}} & \frac{a_{n-2,1}^{2}}{a_{11}^{1}} & \cdots & \frac{a_{11}^{n-1}}{a_{11}^{n-2}} \end{vmatrix}$$

and that C takes the form

$$C = \begin{vmatrix} 1 & \frac{a_{12}^{0}}{a_{11}^{0}} & \frac{a_{13}^{0}}{a_{11}^{0}} & \cdots & \frac{a_{1n}^{0}}{a_{11}^{0}} \\ 0 & 1 & \frac{a_{12}^{1}}{a_{11}^{1}} & \cdots & \frac{a_{1,n-1}^{1}}{a_{11}^{1}} \\ 0 & 0 & 1 & \cdots & \frac{a_{1,n-2}^{2}}{a_{11}^{2}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

Hence, we see that

(12) $$b_{ij} = \frac{a_{i-j+1,1}^{j-1}}{a_{11}^{j-2}}$$

(13) $$c_{ij} = \frac{a_{1,j-i+1}^{i-1}}{a_{11}^{i-1}} .$$

To prove the latter true we refer to (11)

$$a_{ij}^{k} = \frac{|a_{11}^{k-1} \quad a_{i+1,j+1}^{k-1}|}{a_{11}^{k-2}}$$

and following the extensional identities of Aitken we have

$$|a_{11}^{k-1} \quad a_{i+1,j+1}^{k-1}| = \Big|\, |a_{11} \cdot \cdot \cdot a_{kk}| \;\; |a_{11} \cdot \cdot \cdot a_{k-1,k-1} \;\; a_{i+k,j+k}| \,\Big|$$

$$= |a_{11} \cdot \cdot \cdot a_{k-1,k-1}| \quad |a_{11} \cdot \cdot \cdot a_{kk} \quad a_{i+k,j+k}|$$

but $\quad a_{11}^{k-2} = |a_{11} \cdot \cdot \cdot a_{k-1,k-1}| \quad$ so

(14) $$a_{ij}^{k} = |a_{11} \cdot \cdot \cdot a_{kk} \quad a_{i+k,j+k}|$$

$$a_{i1}^{k} = |a_{11} \cdot \cdot \cdot a_{kk} \;\; a_{i+k,k+1}| \quad .$$

Now from (4)

$$b_{i+k,k+1} = \frac{|a_{11} \cdot \cdot \cdot a_{kk} \; a_{i+k,k+1}|}{|a_{11} \cdot \cdot \cdot a_{kk}|}$$

$$= \frac{a_{i1}^{k}}{a_{11}^{k-1}}$$ or the same form as (12).

From (14) we see that

$$a_{1j}^{k} = |a_{11} \cdot \cdot \cdot a_{kk} \; a_{1+k,j+k}|$$

but from (5)

$$c_{1+k,j+k} = \frac{|a_{11} \cdot \cdot \cdot a_{kk} \; a_{k+1,j+k}|}{|a_{11} \cdot \cdot \cdot \cdot a_{k+1,k+1}|}$$

$$= \frac{a_{1j}^{k}}{a_{11}^{k}} \quad \text{or (13) as desired.}$$

Hence, we see that the elements of B and C may be computed sequentially as a product of 2x2 determinants by computing the jth column of B and then the jth row of C. We see that the kth column of B is found by dividing the first column of the (k-1)th pivotal condensation of A by the first principal minor of the (k-2)th pivotal condensation. Likewise, the kth row of C is found by dividing the first row of the (k-1)th pivotal condensation by its first principal minor.

AN EXAMPLE

Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

Now we have, by pivotal condensation

$$|A| = \frac{1}{2^3} \begin{vmatrix} 1 & -1 & 4 & 1 \\ 2 & 2 & 2 & 4 \\ -9 & -1 & 4 & -1 \\ -1 & 1 & 2 & 1 \end{vmatrix} =$$

$$\frac{1}{2^3} \cdot \frac{1}{1^2} \begin{vmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -9 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -9 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -9 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \end{vmatrix} = \frac{1}{1^2} \begin{vmatrix} 2 & -3 & 1 \\ -5 & 20 & 4 \\ 0 & 3 & 1 \end{vmatrix} =$$

$$\frac{1}{1^2} \cdot \frac{1}{2} \begin{vmatrix} \begin{vmatrix} 2 & -3 \\ -5 & 20 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -5 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 25 & 13 \\ 6 & 2 \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{25^0} |-28| = -14$$

$$BC = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 3 & -\frac{9}{2} & -5 & \frac{25}{2} & 0 \\ 1 & -\frac{1}{2} & 0 & 3 & -\frac{14}{25} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 4 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{13}{25} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$$

CONCLUSIONS

In this paper, the results derived show a simple sequential method of factorization of a non-singular matrix whose first principal minors do not vanish. A method for computing the inverses of the resulting triangular matrices has been coded for machine computation (Frazer, Duncan and Collar, 1938).

For computational purposes, we note that we are evaluating two by two determinants to solve for the jth column of B, and using these results to solve for the jth row of C, wherein again we have only to evaluate two by two determinants. The significant advantage of this method, then, is the simple sequential method of attack.

We also note that by weakening the conditions of our original matrix further work can be done. For example, changing the non-vanishing condition of the first principal minors would lead us, in effect, to results obtained from interchanging rows by elementary operations.

The possibility of computing the inverses of the triangular matrices in a similar sequential method could lead to another line of research. This method may have an advantage in enabling the computation of A inverse directly through proper codification.

The next condition to be eliminated could be the nonsingularity of our original matrix. This method probably could be applied to singular matrices. In this case it would seem that the B matrix would have $n-r$ null column vectors, where r denotes the rank of the matrix.

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