The forced vibration of viscus damped, N-beam structures
by Daniel Franklin Prill

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Aerospace and Mechanical Engineering
Montana State University

Abstract:
The equation of motion for a linear beam with viscous damping and dynamic loading is presented. The particular solution for a uniformly distributed load which varies sinusoidally in time is presented for a beam with general boundary conditions. It is recognized that the boundary conditions are functions of the first three spatial derivatives and thus recursion formulas in matrix form are developed.

For a system of beams which are interconnected, knowledge of the boundary conditions allows development of algebraic equations which can be solved for the integration coefficients of the particular solution for each beam of the system. Thus a method, including the appropriate computer program, is presented which results in the exact particular solution for a structural system composed of a large number of beams.

Also, the method is such that any number of the natural frequencies and mode shapes for the undamped structural system can be approximated with a high degree of accuracy.

Example solutions for structural systems are presented for a cantilever beam and for a three beam configuration.

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Date 11-24-70

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DAMPED, N-BEAM STRUCTURES

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

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CHAPTER I

INTRODUCTION

The method of solution to the free-vibration of linear beams is well known [1][1]. When damping and a forcing function are considered, the complexity of the solution increases and, in some cases, an approximation technique has been used to determine a solution [2].

If an undamped, multi-element structure is to be analyzed for free-vibrations, the eigenvalue problem may become complex [3].

This paper presents an exact solution to the problem of a viscous damped beam with a distributed load forcing function. The solution presented is similar to that of Stanek [4]. The solution, as presented, is in matrix form and is applied to an N-beam structure.

Examples are worked for a cantilever beam and for a three-beam structure. Since the computer is used to obtain numerical results, Fortran-4 level programs are presented.

A method of finding natural frequencies and the mode shapes of these frequencies is also presented.

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[1]Numbers in brackets refer to literature consulted.

CHAPTER II

SOLUTION OF THE FORCED VIBRATION OF THE VISCOUS DAMPED BEAM EQUATION

Equation of Motion

The derivation of the linear beam equation for dynamic motion is well known [1]. When viscous damping and a forcing function are included in the derivation, the resulting equation of motion has two terms which are not in the linear equations. These terms are the damping term, $\frac{C}{L}\frac{\partial y}{\partial t}$, and the forcing function, $F_o \sin\omega t$.

Adopting the sign convention of Figure 2.1, the equation of motion is

$$\frac{W}{gL}\frac{\partial^2 y}{\partial t^2} + \frac{C}{L}\frac{\partial y}{\partial t} + E\,I\,\frac{\partial^4 y}{\partial x^4} = F_o \sin\omega t \qquad (2.1)$$

where:
W = Total weight of beam (lb)
g = Acceleration due to gravity (in/sec^2)
L = Length of beam (in)
C = Total damping constant (lb-sec/in)
y = Lateral deflection of beam (in)
x = Axial coordinate of beam (in)
E = Modulus of Elasticity of material of beam (lb/in^2)
I = Moment of Inertia of beam cross-section (in^4)

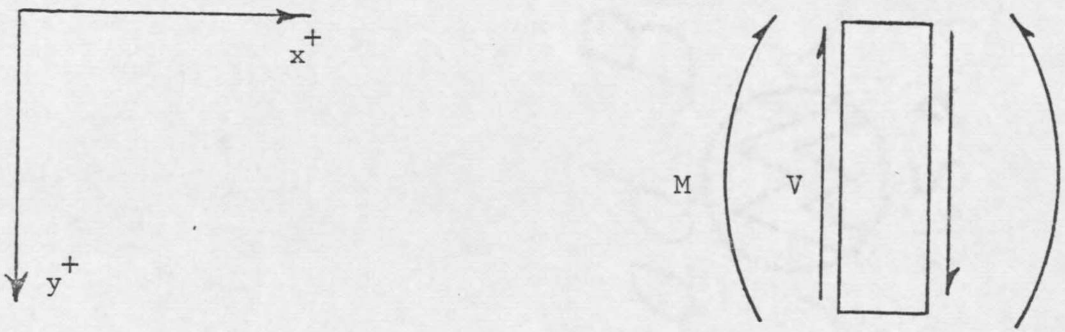

Figure 2.1. Sign Convention

F_o = Maximum load on beam (lb/in)

ω = Circular frequency of load (rad/sec)

t = time (sec)

Nondimensional Equation of Motion

The solution of Equation 2.1 may be presented in a more efficient fashion in nondimensional form. The equation of motion is put into nondimensional form by use of the following quantities:

$$z = \frac{x}{L}$$

$$Y = \frac{y}{\dfrac{F_o L^4}{C_o EI}} \qquad (2.2)$$

$$\theta = \omega t$$

where: z = Nondimensional axial coordinate

Y = Nondimensional lateral deflection

θ = Nondimensional time

C_o = Nondimensional constant[2]

---

[2] C_o is a numerical nondimensional constant which is chosen for convenience. The constant may be any number, but is chosen to be eight for the cantilever beam example of this paper and is chosen to be one for the three-beam example. See Chapter IV for reasons for these choices.

The nondimensional equation of motion is

$$\beta^4 \frac{\partial^2 y}{\partial \theta^2} + \alpha \beta^4 \frac{\partial Y}{\partial \theta} + \frac{\partial^4 Y}{\partial z^4} = C_o \sin\theta \tag{2.3}$$

where: $$\beta^4 = \frac{W L^3 \omega^2}{gEI}$$

(2.4)

and: $$\alpha = \frac{Cg}{W\omega}$$

Solution

For this study, only the steady-state solution is of interest. The solution will, therefore, be a particular solution. Thus, assume a solution of the form:

$$Y = Z_o^s \sin\theta + Z_o^c \cos\theta \tag{2.5}$$

where the Z-functions (Z_o^s and Z_o^c) are functions of only the nondimensional axial coordinate, z. The subscripts and superscripts of Equation 2.5 are explained as follows:

1) The superscript on the Z-function in question will be either an "s" or a "c". An "s" will designate that the Z-function is a coefficient of the term sinθ; and a "c" will designate that the Z-function is a coefficient of the term cosθ.

2) The subscripts of the Z-functions will designate the order

of the derivative of the Z-functions with respect to the axial coordinate, z. For example, the nondimensional slope of the beam, $\frac{\partial Y}{\partial z}$, would be written as:

$$Y' = Z_1^s \sin\theta + Z_1^c \cos\theta \tag{2.6}$$

Differentiating Equation 2.5, substituting the results into Equation 2.3 and equating coefficients of sinθ and cosθ yields

$$-\beta^4 Z_o^s - \alpha \beta^4 Z_o^c + Z_4^s = C_o \tag{2.7}$$

$$-\beta^4 Z_o^c + \alpha \beta^4 Z_o^s + Z_4^c = 0 \tag{2.8}$$

Equation 2.5 becomes a solution of Equation 2.3 if the Z-functions satisfy Equations 2.7 and 2.8 simultaneously. Combining Equations 2.7 and 2.8 to eliminate Z_o^s yields the following eighth-order nonhomogeneous differential equation.

$$Z_8^c - 2\beta^4 Z_4^c + \beta^8 Z_o^c (1 + \alpha^2) = -C_o \alpha \beta^4 \tag{2.9}$$

The following symbols are introduced for writing the homogeneous solution of Equation 2.9.

$$\mu = \beta(1 + \alpha^2)^{1/8}$$
$$\phi = \frac{1}{4} \tan^{-1} \alpha \tag{2.10}$$
$$a = \mu \cos\phi$$

$$b = \mu \sin\phi \qquad (2.10 \text{ con't})$$

The complete solution of Equation 2.9 is

$$Z_o^c = \lfloor \cosh az \quad \sinh az \rfloor \; M_o^c \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix}$$

$$+ \lfloor \cosh bz \quad \sinh bz \rfloor \; N_o^c \begin{Bmatrix} \cos az \\ \sin az \end{Bmatrix} + Z_p^c \qquad (2.11)$$

where Z_p^c, the particular solution of Equation 2.9, is

$$Z_p^c = - \frac{C_o \, \alpha \, \beta^4}{\mu^8} \qquad (2.12)$$

The eight integration constants are the elements of the M_o^c and N_o^c matrices where

$$M_o^c = \begin{bmatrix} A & D \\ C & B \end{bmatrix}_o^c$$

and

$$N_o^c = \begin{bmatrix} E & H \\ G & F \end{bmatrix}_o^c \qquad (2.13)$$

The subscripts and superscripts of Equation 2.13 have the same meanings as those of the Z-functions.

The form of the Z_o^s function is the same as that for the Z_o^c function. There are eight additional integration constants for the Z_o^s function. Since the Z-functions contain sixteen integration constants, the eight integration constants of one Z-function must be dependent upon the integration constants of the other Z-function. To find this relationship the first four derivatives of the Z-functions must be developed. These derivatives are

$$Z_i^j = \lfloor \cosh az \quad \sinh az \rfloor \, M_i^j \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix} + \lfloor \cosh bz \quad \sinh bz \rfloor \, N_i^j \begin{Bmatrix} \cos az \\ \sin az \end{Bmatrix} \qquad \begin{matrix} i = 1,2,3,\ldots n \\ j = s \text{ or } c \end{matrix} \tag{2.14}$$

The difference between successive derivatives will be the elements of the M_i^j and N_i^j matrices. When derivatives of the Z-functions are taken, recursion relationships for the M_i^j and N_i^j matrices become evident. These relations are

$$\begin{aligned} M_i^j &= a\,J\,M_{i-1}^j + b\,M_{i-1}^j\,K \\ N_i^j &= b\,J\,N_{i-1}^j + a\,N_{i-1}^j\,K \end{aligned} \qquad \begin{matrix} i = 1,2,3,\ldots n \\ j = s \text{ or } c \end{matrix} \tag{2.15}$$

where the J and K matrices are defined as

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.16}$$

Expansion of Equations 2.15 enables one to write the M_i^j and N_i^j matrices which occur in the first four derivatives of the Z-function as

$$M_i^j = \mu\,(\cos\phi\; J\, M_o^j + \sin\phi\; M_o^j\, K) \tag{2.17}$$

$$M_2^j = \mu^2 \; (\cos 2\phi \; M_o^j + \sin 2\phi \; J \; M_o^j \; K)$$

$$M_3^j = \mu^3 \; (\cos 3\phi \; J \; M_o^j + \sin 3\phi \; M_o^j \; K)$$

$$M_4^j = \mu^4 \; (\cos 4\phi \; M_o^j + \sin 4\phi \; J \; M_o^j \; K) \qquad (2.17 \text{ con't})$$

$$j = s \text{ or } c$$

and:

$$N_1^j = \mu(\sin\phi \; J \; N_o^j + \cos\phi \; N_o^j \; K)$$

$$N_2^j = \mu^2 \; (-\cos 2\phi \; N_o^j + \sin 2\phi \; J \; N_o^j \; K)$$

$$N_3^j = -\mu^3 \; (\sin 3\phi \; J \; N_o^j + \cos 3\phi \; N_o^j \; K)$$

$$N_4^j = \mu^4 \; (\cos 4\phi \; N_o^j - \sin 4\phi \; N_o^j \; K) \qquad j = s \text{ or } c \qquad (2.18)$$

Substitution of Equations 2.17 and 2.18 into Equation 2.17 gives the Z_o^s function, in terms of the integration constants of the Z_o^c function, as

$$Z_o^s = - \lfloor \cosh az \quad \sinh az \rfloor \; J \; M_o^c \; K \begin{Bmatrix} \cos bz \\ \sin bz \end{Bmatrix}$$
$$+ \lfloor \cosh bz \quad \sinh bz \rfloor \; J \; N_o^c K \begin{Bmatrix} \cos az \\ \sin az \end{Bmatrix} - \frac{C_o \beta^4}{\mu^8} \qquad (2.19)$$

The relationships between the integration constants of the Z-functions are:

$$M_o^s = - J \; M_o^c \; K \quad , \quad N_o^s = J \; N_o^c \; K \qquad (2.20)$$