ON THE CONNECTEDNESS OF THE RAUZY FRACTAL

by

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DEDICATION

To my parents María Guadalupe Bañuelos Cabral and Roberto Eduardo Soto Laris. I thank you for being my parents.

To the people that brought me hope. Peace to you wherever life may take you.

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GLOSSARY

$\mathbb{N}$ .... The set of natural numbers

$\mathbb{Z}^+$ .... The set of positive numbers

$\mathbb{Z}^-$ .... The set of negative numbers

$\mathbb{R}^+$ .... The set of positive real numbers

$\mathbb{R}^-$ .... The set of negative real numbers

$r_\sigma(A)$ .... The spectral radius of an operator

$B_r(p)$ .... The ball of radius r centered at p

S^1 .... The unit circle

$\mathbb{T}^d$ .... The $d-$dimensional torus

$\#(S)$ .... The cardinality of a set

$[v]_{\mathcal{B}}$ .... The coordinates of the vector v in the basis $\mathcal{B}$

$|w|$ .... length of a word

ε .... The empty word

$\mathcal{A}^*$ .... The set of finite words

$\mathcal{A}^+$ .... The set of non-empty words

w_1w_2 .... The concatenation of two words

$\mathcal{A}^{\mathbb{N}}$ .... Set of forward-infinite-words

GLOSSARY-CONTINUED

$\mathcal{A}^{\mathbb{Z}^-}$ .... Set of backward-infinite-words

$\mathcal{A}^{\mathbb{Z}}$ .... Set of bi-infinite words

$w_1.w_2$ .... Concatenation of two words $w_1 \in \mathcal{A}^{\mathbb{Z}-}$ and $w_2 \in \mathcal{A}^{\mathbb{N}}$

σ .... The shift map

u_J .... Given a word u, u_J is the factor of u given by the restriction $u|_J$ up to a shift of domain.

$w = \ldots v$ ...... v is a right factor of w

$w = u$ .... u is a left factor of w

$w = \ldots v$ ...... The word v is a factor of w

$\mathcal{L}_w$ .... The set of all finite factors of w

$\mathcal{L}_\varphi$ .... The set of all finite factors of words of the form $\varphi^n(i)$ for some $n \in \mathbb{N}$ and $i \in \mathcal{A}$

$|w|_l$ .... The number of letters l in the word w

$[w]$ .... The vector with coordinates $\left(|w|_{l_1}, |w|_{l_2}, \ldots, |w|_{l_d}\right)$

$[\varphi] = (a_{ij})$ ..... The abelianization matrix whose columns are given by $[\varphi(j)]$

X_φ .... The substitutive system for a substitution φ

λ_φ .... The Perron-froebinous eigenvalue for $[\varphi]$

$v_R = (v_\varphi)_R$ .... The right Perron eigenvector for φ

$v_L, (v_\varphi)_L$ .... The left Perron eigenvector for a primitive substitution

GLOSSARY-CONTINUED

$T = \ldots S_{-1}S_0S_1S_2S_3\ldots$ A tiling of the real line

$T = \ldots S_{-1}S_0S_1S_3\ldots$ A tiling of the real line such that $0 \in S_2$

$T = \ldots S_{i_0-2}S_{i_0-1}.S_{i_0}S_{i_0+1}S_{i_0+2}\ldots$ A tiling of the real line such that $0 \in S_{i_0-1} \cap S_{i_0}$

$\mathcal{P}$ .... A patch of a tiling of $\mathbb{R}$

$\mathcal{P}_w + t$ .... A patch of transates of prototiles following the pattern of the word w and translated by a number t

Σ_φ .... the collection of all tilings with prototiles given by segments of length the entries of the left eigenvector of φ

$\mathcal{T}_\varphi$ .... The tiling space associated with φ consisting of all tilings such that each patch is a translate of the patch associated with $\varphi^n(i)$ for some $i \in \mathcal{A}$ and $n \in \mathbb{Z}^+$

$\phi_t = (\phi_\varphi)_t$ ..... The flow translation in $\mathcal{T}_\varphi$ and in Σ_φ

$\mathcal{S}$ .... The set of all segments in $\mathbb{R}^d$

$\mathcal{S}^+$ .... The set of nonempty finite strands

$\mathcal{S}^{\mathbb{N}}$ .... The set of forward infinite strands

$\mathcal{S}^{\mathbb{Z}^-}$ .... The set of backward infinite strands

$\mathcal{F}$ .... The set of bi-infinite strands

$\max(S)$ ...... The maximum of the strand S

$\min(S)$ .... The minimum of the strand S

GLOSSARY-CONTINUED

$\mathcal{S}^*$ .... The set of finite strands

S_{par} .... The parametrization of a strand

V_S .... The set of vertices of a strand S

S_1S_2 .... The concatenation of two strands

$[S]$ .... The word associated with the strand S

$[\cdot]$ .... The map sending a strand to its associated word

$\Pi_{i\in\mathbb{Z}}T_i$ .... A concatenation of strands

$\ldots T_{-2}T_{-1}T_0T_1\ldots$ A concatenation of strands such that $v^\perp \cap (T_2 - \{\max(T_2)\}) \neq \varnothing$

$\ldots T_{-2}T_{-1}T_0T_1.T_2\ldots$ A concatenation of strands such that $\sup T_1 = \min T_2 \in v^\perp$

$(\cdot)$ .... The mapping sending each strand to the state induced by it.

S_J .... The factor of the strand given by the interval J

$\varphi(S)$ .... The image of the strand under the substitution φ

substitution;IUP An irreducible, Pisot and unimodular substitution

RUP .... A reducible, unimodular Pisot substitution

E^s .... The stable space

E^u .... The unstable space

$\|p\|_s$ .... The stable norm

GLOSSARY-CONTINUED

$\|p\|_u$ .... The unstable norm

pr_s .... The projection into the stable space

pr_u .... The projection into the unstable space

$\mathcal{C}^R$ .... The stable cylinder. The set of all points p such that $\|p\|_s < R$

$\mathcal{F}^R$ .... The set of all bi-infinite strands that are allowed and that are contained in the cylinder $\mathcal{C}^R$

$\mathcal{F}_\varphi$ .... The tiling space of the IUP substitution φ

ϕ_t .... The tiling flow

$\mathbb{S}$ .... The set of all states; that is, all segments that intersect E^s

$\mathbb{S}_p$ .... The set of all states with a vertex congruent with p mod $\mathbb{Z}^d$

$\mathbb{S}_0$ .... The set of all integer states.

$\mathbb{S}_R$ .... The states contained in the stable cylinder $\mathcal{C}_R$

$\tilde{h}_\varphi$ .... The first vertex map for φ given by $\tilde{h}_\varphi(S) = \min\left(\hat{S}\right)$

h_φ .... the geometric realization of φ

π .... The quotient map $\pi : \mathbb{R}^d \to \mathbb{T}^d$ given by $\pi(p) = p + \mathbb{Z}^d$

GCC .... The geometric coincidence condition

substitution;IUPC An irreducible Unimodular Pisot Substitution that satisfies GCC

GLOSSARY-CONTINUED

$\{X_i, f_i\}_{i\in\mathbb{N}}$ .... An inverse sequence of spaces with coordinates spaces X_i and bonding maps f_i

$\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$. The inverse limit with inverse sequence $\{X_i, f_i\}_{i\in\mathbb{N}}$

X_∞ .... The inverse limit $\varprojlim \{X_i, f_i\}$

$\varprojlim \{X, f\}$ ..... The inverse limit with only one bonding map $f : X \to X$

R_φ .... The set of d different circles in the plane tangent at the origin with length given by the entries of the vector v_L

f_φ .... The map of the rose, which follows the pattern of the substitution φ

$\tilde{\Omega}_\varphi$ .... The image of the tiling space under the first-vertex map

Ω_φ .... the image of the tiling space under the geometric realization map

$\mathcal{F}^s_\varphi$ .... The set of strands with one vertex in E^s

$\tilde{\Omega}^s_\varphi$ .... The stable part of $\tilde{h}_\varphi(\mathcal{F}_\varphi) \subset \mathbb{R}^d$

Ω^s_φ .... The stable part of $h(\mathcal{F}_\varphi)$

$\sigma_{\mathcal{F}_\varphi}$ .... The shift map

$\mathcal{R}_\varphi$ .... The Rauzy Fractal of the subsitution φ

$\left(\mathcal{F}^s_\varphi\right)^{.w}$ .... The set of strands $S \in \mathcal{F}^s_\varphi$ such that $[S] = \ldots [S]_{-1}.w \ldots$

$\mathcal{R}^{.w}_\varphi$ .... The Rauzy piece corresponding to the word w

GLOSSARY-CONTINUED

$\left(\Omega_{\varphi}^{s}\right)^{\cdot w}$ .... The image, under geometric realization, of the set $\left(\mathcal{F}_{\varphi}^{s}\right)^{\cdot w}$

$P\left(i,j\right)$......... The positions at which the letter j appears in $\varphi\left(i\right)$

$p_{\beta}\left(x\right)$.......... The charracteristic polynomial of $[\beta]$

$V_{\varphi}^{P}\oplus V_{\varphi}^{R}$...... The decomposition of $\mathbb{R}^{d}$ into the Pisot part and the reducible part for a RUP substitution β

$p_{\varphi}^{P}\left(x\right)$ .... The minimal polynomial of λ_{β}. The Pisot part of the characteristic polynomial of $[\beta]$

$p_{\varphi}^{R}\left(x\right)$ .... The reducible part of the characteristic polynomial of $[\beta]$

V_{φ}^{P} .... The Pisot space of the RUP substitution β

V_{φ}^{R}........... The reducible space of β

$[\varphi]_{P}$ .... The restriction of the matrix $[\beta]$ to the Pisot space of β

$[\varphi]_{R}$ .... The restriction of the matrix $[\beta]$ to the reducible space of β

pr_{β}^{P}........... The projection to the Pisot space of β

pr_{β}^{R}........... The projection into the reducible space of β

$p_{\varphi}^{u}=\left(x-\lambda_{\varphi}\right)$.. The unstable part of the characteristic polynomial for $[\beta]$

p_{φ}^{s}............ The stable part of the characteristic polynomial

V_{φ}^{s}............ The stable space associated to the RUP substitution β

V_{φ}^{u} .... The unstable space associated to the RUP substitution β

GLOSSARY-CONTINUED

pr^s_β .... The projection into the stable part of the RUP substitution β

pr^u_β .... The projection into the unstable part of the RUP subsitution β

$[\varphi]_s$ .... The restriction of the matrix $[\beta]_P$ to the stable space of β

$[\varphi]_u$ .... The restriction of the matrix $[\beta]_P$ to the unstable space of β

$\tilde{h}_\beta$ .... The first-vertex map for a RUP substitution

Σ .... The sublatice of $\mathrm{pr}^P_\alpha(\mathbb{Z})$ for geometric realization

$\mathbb{S}^{R_0}_q$ .... The set of states over $q \in V^P_\beta/\Sigma$ in the cylinder $\mathcal{C}^{R_0}$

T .... The state induced by a strand in V^P_β

GCC .... The geometric coincidence condition

substitution;RUPC A reducible unimodular, Pisot substitution satisfying GCC

U_g .... induced isometric operator

() .... A balanced pair

$G^{.w}_n$ .... The graph whose vertex is the set of possible extensions of w of length n, and whose edges are determined by certain proximal pairs

$p(n)$ .... The complexity function: the number of allowed words of lengthe n

$\mathcal{V}_w$ .... The set of words v that cover w minimally

GLOSSARY-CONTINUED

$[\mathcal{B}]$ .... the set $\mathbb{R}^d$

$[\mathcal{B}]_P$ .... The Pisot Space of $\mathbb{R}^d$

$[\mathcal{B}]_s$ .... The stable space in $\mathbb{R}^d$

$[\mathcal{B}]_u$ .... The unstable space in $\mathbb{R}^d$

E_1 .... The indices of the form $\left|\varphi\left(X_{[0,p-1]}\right)\right|$, 0, or $\left|\varphi\left(X_{[-p,-1]}\right)\right|$

P_φ^L .... The set of all pairs $a\left(\right)$, where $|a| = |u| = |v|$, and $u_{[0]} \neq v_{[0]}$

$\text{dis}_\varphi\left(a\left(\right)\right)$ ...... The pair $a'\left(\right)$, where $|a'| = |u'| = |v'| = L$, $u'_{[0]} \neq v'_{[0]}$, and $\varphi\left(a\left(\right)\right) = c'a'\left(\right)\ldots$

ABSTRACT

If ϕ is a reducible unimodular Pisot substitution, the Rauzy fractal associated to ϕ can be studied using the strand space. In this dissertation we are going to provide a characterization of the connectedness of the Rauzy Fractal in terms of infinitely many graphs closely related to the proximal structure of the strands in the strand space. Using this characterization, we show a topological characterization of invertible substitutions on two letters, and show that the Rauzy fractal associated to an Arnoux-Rauzy substitution is connected. We show that if two reducible unimodular Pisot substitutions ϕ and ψ are homemorphic, then there is a subdivision of the Rauzy fractal for ϕ into finitely many pieces, which, after applying suitable linear transformations and a translations to each piece, becomes a set whose union is the Rauzy fractal for ψ. We also found an algorithm to find asymptotic composants.

CHAPTER 1

INTRODUCTION

1.1 Motivation

The study of tiling spaces has had a lot of growth in the last few years. One of the reasons for its popularity is that tiling spaces can be studied using various branches of Mathematics, such as Topology, Dynamical Systems, and Algebraic Number Theory. Some of the standing questions in the area are even related to Physics and the study of Quasicrystals. Tiling spaces are examples where "Topology determines Geometry." Their study still has many questions open for research, including famous ones, like the Pisot conjecture.

Two articles are key to the study of Tiling spaces. One is [Wil70], where Williams classifies the one dimensional attractors of a hyperbolic map in a manifold, and shows that these attractors are precisely the inverse limits on wedges of circles. The second article is [AP98], where Anderson and Putnam show that substitution tiling spaces are inverse limits on a fixed simplicial complex, and that all substitution tiling spaces occur as hyperbolic attractors. In the one dimensional case, using the "Williams Moves" of [Wil70], there is a representation of one dimensional tiling spaces as inverse limits of wedges of circles. Thus, some of the attractors studied by Williams were, in fact, tiling spaces.

A particularly important class of tiling spaces are those arising from a substitution. All we do in this thesis is in the setting of substitution tiling spaces.

In [BD01], Barge and Diamond give a complete invariant for the topology of a tiling space. This result was later improved by Barge and Swanson in [BS07], where they show that if two tiling spaces are homeomorphic, then some powers of the inflation and substitution maps induced by their substitutions are, in fact, conjugate. Furthermore, the study of proximality in [BD07] and [BK11] revealed the structure of the topology of the tiling space.

The results by Barge and Diamond in [BD01], and Williams in [Wil67] and [Wil70] classify the one dimensional orientable attractors in a manifold, as either homogeneous, and thus corresponding to solenoids, or non-homogeneous, in which case they correspond to one-dimensional substitution tiling spaces. Both solenoids and substitution tiling spaces are locally homeomorphic to the product of a Cantor set and an interval.

Rauzy in [Rau82] studied the tribonacci substitution $\varphi : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ and studied what we now call the Rauzy Fractal for that substitution. He gave a presentation of the Rauzy fractal as power series with certain restrictions in the coefficients.

Rauzy Fractals are defined for a class of substitutions called Pisot substitutions, and can be studied as a quotient of a section of the tiling space. The particular section of the tiling space that gets mapped onto the Rauzy Fractal is called the substitutive system. The substitutive system depends on the particular presentation of the substitution. The study of the substitutive system is the main topic of [Fog02].

The Rauzy Fractal has a particularly nice representation in terms of strands. The study of tiling spaces in terms of strands, as well as a definition in terms of strands of the Rauzy Fractal is given by Barge and Kwapisz in [BK06], and in the reducible case by Baker, Barge and Kwapisz in [BBK06]. They also prove, using the language of tilings, the recognizability property of primitive substitutions that was first proved, using combinatorics, by Mossé in [Mos92] and [Mos96].

Rauzy Fractals provide a concrete construction of Markov partitions for certain hyperbolic automorphisms of an n-dimensional torus. The results of this thesis are motivated in part by the standing question of whether a hyperbolic toral automorphism always admits a connected Markov partition.

The question of the connectedness of the Rauzy fractal has been approached from different angles. Many use the prefix-suffix automaton, like in [Can03] and [Sie04], some algebraic number theory like in [Sie04] or the Ito construction like in [BJS11]. Algorithms to compute the connectedness with some conditions are given in [Can03], [Sie04] and [ST$^+$09]. In [Can03] Canterini announced a proof that Arnoux-Rauzy substitutions were connected. Recently, Berthe, Jolivet and Siegel in [BJS11]proved the conjecture using the Ito construction.

Our approach is different in that we will approach these questions using the strand space and proximality. We give a proof that Arnoux-Rauzy substitutions and sturmian substitutions have connected Rauzy fractals with our methods. We also find a characterization of the connectedness of the Rauzy fractal in terms of the topology of the tiling space in the two-letter case. Then we analyze the relationship between the Rauzy fractals of two homeomorphic tiling spaces, where we obtained a shuffling theorem. In the course of the analysis of the topology of the tiling space and proximality, we obtained a method to find asymptotic pairs, and developed some code written in the computer language Sage.

The first four chapters in this work constitute reference material. Chapter 2 provides the necessary background and notation for the remainder of the dissertation. In Chapter 3, we analyze both shift equivalence of substitutions and rewritings, as well as the relationship between their eigenvalues. Chapter 4 discusses the definition of the Rauzy Fractal, as well as the necessary definitions for reducible Pisot substitutions.

In Chapter 5, we give a new characterization of the connectedness of the Rauzy fractal in terms of infinitely many graphs.

We use these graphs in Chapter 6 to give a topological characterization of invertible substitutions on two letters, and to show that Arnoux-Rauzy substitutions have associated connected Rauzy Fractals..

In Chapter 7, we analyze the effect that shift equivalence and rewriting have on the Rauzy fractal. We obtain that if two reducible unimodular Pisot substitutions are homeomorphic, then there are pieces in the Rauzy fractal associated to the substitutions that, after suitable linear transformations and translations, can be assembled to produce the Rauzy fractal of the other substitution.

In Chapter 8, we provide a method to find asymptotic composants, that, unlike the one given in [BD01], does not require solving for a prefix or suffix problem.

Finally, we attach to this dissertation some code in SAGE (see [S$^+$11]) that was used to produce Rauzy fractals for reducible Pisot substitutions, and to manipulate substitutions.

Proposition 2.65, the second part of the proof of Theorem 3.9, Theorem 3.14, and the main results in Chapters 5 through 8, constitute original research.

1.2 Informal Introduction to Tiling Spaces

Informally speaking, a one dimensional tiling space is a topological space whose elements are tilings of the real line. Two tilings are considered close if, inside of a large ball around the origin, one tiling is the same as the other one up to a small translation (see, for example, [AP98]).

Example 1.1 Consider the tiling T_1 of the real line shown in Figure 1.1, with tiles of the form $[k, k+1] \in \mathbb{Z}$. Let $\mathcal{T}_1$ to be the space of all tilings that are translations of T_1.

..

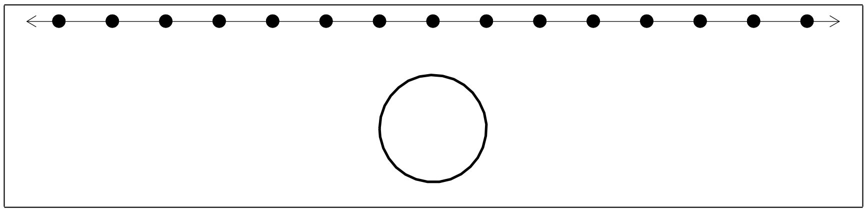


Figure 1.1: The tiling space formed by translates of the tiling above is a circle

After translating the tiling T we obtain different tilings until we translate exactly by one unit, in which case we obtain the tiling T back again. Thus, the tiling space of the translates of this tiling with the tiling metric is a circle.

Example 1.2 Consider the tiling T_2 of the real line consisting of T_1, except that we replace two intervals $[0, 1]$, $[1, 2]$ by a single interval $[0, 2]$. Let $\mathcal{T}_2$ be the tiling space formed by the completion of the space of all translations of T_2.

After translating T_2 by larger and larger integers to the right, we approach the tiling of the first example, as shown in Figure 1.2. The same phenomenon occurs when we translate T_2 to the left. The tiling T_1 of the first example appears in the closure of the orbit of the second tiling when translating it to the right or to the left.

In general, for tiling spaces, the recurrence properties of a tiling T are reflected in the tiling space formed by all translates of T.

Example 1.3 Let T_3 be the tiling with tiles of the form $[2k - 2, 2k]$, for $k \in (-\infty, 0] \cap \mathbb{Z}$ and with tiles of the form $[k, k + 1]$ for $k \in [0, \infty) \cap \mathbb{Z}$. Let $\mathcal{T}_3$ be the closure of the orbit of T_3 with the tiling metric.

If we translate T_3 in one direction we approach more and more the tiling with unit tiles, whereas we approach the tiling with tiles of length two in the other direction. Thus, two different circles appear in the tiling space $\mathcal{T}_3$. This is shown in Figure 1.3

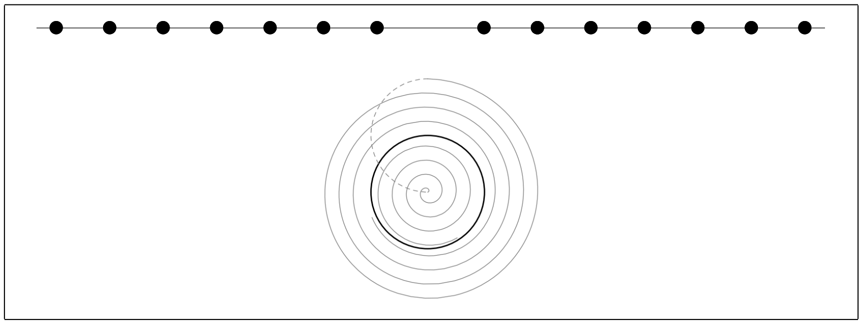


Figure 1.2: The tiling space formed by translates of the tiling above is a line converging onto the circle. The solid lines in the tiling space are in the plane; the dashed line only intersects the plane at its end points.

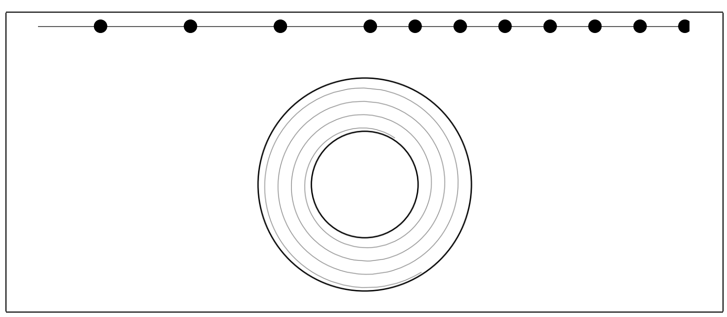


Figure 1.3: The tiling space formed by translates of the tiling above is a circle. The outer circle represents the translates of the tiling formed with intervals of length one, whereas the inner circle represents the translates of the tiling with intervals of length two. The line represents the translates of the tiling on the top of this figure.

1.2.1 Tiling Spaces Arising from Substitutions

A substitution is a mapping that transforms letters into words. For the third example we use a substitution φ on two letters 1 and 2 given by $\varphi(1) = 121$ and $\varphi(2) = 12$. This substitution is commonly known as the *Fibonacci* substitution. The reason for this name is that the length of a word of the form $\varphi^n(1)$ are all Fibonacci numbers (see [Fog02], page 131).

The Fibonacci substitution also has the interesting property (see [Fog02], page 132) that the number of words of length n that are factors of words of the form $\varphi^n(1)$ is $n+1$; that is, the *complexity function* for Fibonacci is $n+1$.

A third interesting property of the Fibonacci substitution is that the extension φ' to the free abelian group on the alphabet $\{1, 2\}$ is an invertible morphism. Substitutions on any number of letters with this property are called *invertible* substitutions.

A fourth interesting property of Fibonacci substitution is the following: Let $\mathcal{L}_\varphi$ be the set containing the empty word and all factors of words of the form $\varphi^n(i)$ for $i \in \mathcal{A}$ and $n \in \mathbb{Z}^+$. Let us call the elements of $\mathcal{L}_\varphi$ *allowed* words for φ, and $\mathcal{L}_\varphi$ the *language* of the φ. The Fibonacci substitution has the property that, for any allowed word of length $n \in \mathbb{Z}^+$, there is exactly one allowed word w of length n such that $w1$ and $w2$ are also allowed. This is referred to as saying that, for each n there is a unique *right-special* factor. Similarly, Fibonacci also has a unique *left-special* factor for each $n \in \mathbb{Z}^+$.

Note that if we start with the word 2.1 and we apply φ over and over we obtain longer and longer words 2.1, $\varphi(2.1) = 12.121$, $\varphi(12.121) = 12112.12112121$,... In each stage we obtain that $W^{i+1} = \varphi(W_i) = P_i W_i S_i$ for some words P_i and S_i. Following this process, we obtain a bi-infinite word $W = \ldots u_{-3}u_{-2}u_{-1}.u_0u_1u_2\ldots = \ldots 12112.12112121\ldots$ where each u_i is either 1 or 2.

Example 1.4 We construct a tiling of the real line whose tiles are arranged like the occurrences of 1 and 2 in the word W in the paragraph above for the Fibonacci substitution: Let $\lambda_1 > 0$ be an irrational number and $\lambda_2 = 1$. Construct the tiling T_φ with translates $\{I'_i\}_{i\in\mathbb{Z}}$ of the intervals $I_1 = [0, \lambda_1]$ and $I_2 = [0, \lambda_2]$ in such a way that:

- $I'_0 = I_1$
- $I'_i \cap I'_{i+1}$ is one point and for $x \in I'_i$ and $y \in I'_{i+1}$, we have that $x \leqslant y$
- I'_i is a translate of I_{w_i}

Let $\mathcal{T}_\varphi$ be the completion of the orbit of T_φ.

The recurrence properties of $\mathcal{T}_\varphi$ are interesting due to the recurrence properties of the word W (see [Fog02]): First, the word $W$is not periodic. Second, any factor of W appears infinitely often to both the left and to the right of 2. Finally, the word W is *uniformly recurrent*: For every factor U_0 that appears in W, there exists a number $k > 0$ such that the gap between any two consecutive occurrences of U_0 is at most k.

Because of the recurrence property of W, we have that, as we translate T_φ to the right, we recurrently get arbitrarily close to T_φ.

In [BD01] a model for this tiling space is presented. The construction starts with a two dimensional torus in $\mathbb{R}^3$, for example, and a dense line in it. Along the dense line the torus is slit, making the separation between the parts smaller and smaller in such a way that the resulting space is bounded. The torus can be recovered by gluing back. A sketch is shown in Figure 1.4.

This slit torus $\mathcal{T}_\varphi$ is locally the product of a Cantor set and an interval, and it is an indecomposable continuum; that is, a compact connected metric set (a continuum) that is not the union of two of its proper subcontinua. Note that $\mathcal{T}_\varphi$ is not a solenoid:

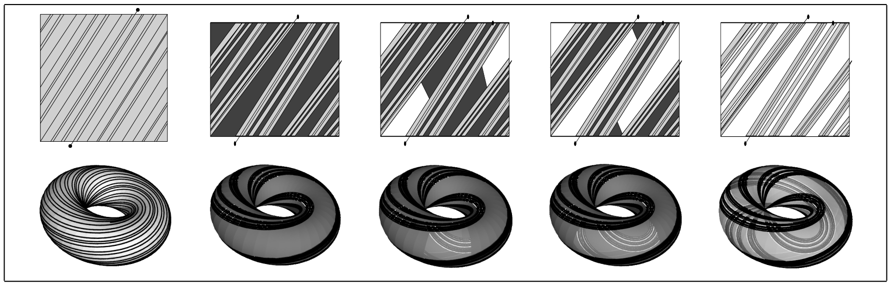


Figure 1.4: Sketches of the construction of the slit torus for the substitution $1 \mapsto 12$, $2 \mapsto 1$. The final drawing shows a sketch of the Tiling space embedded in a torus. We start with the torus and a line wrapping densely in it. We replace the line with a bi-infinite band that is thick in the center, and that is decreasing in thickness in both directions. In the following drawings, we erase the interior of the band until we obtain the tiling space.

The lines along which we sliced provide arc components C_1, C_2, points T, T', and parametrizations $f : \mathbb{R} \to C_1$, $g : \mathbb{R} \to C_2$ such that $f(0) = T$, $g(0) = T'$ and $\lim_{t\to\infty} \|f(t) - g(t)\| = 0$, which is impossible for a solenoid.

A more accurate description of the tiling space $\mathcal{T}_\varphi$ is obtained once we associate with φ its abelianization matrix $[\varphi] = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, where $[\varphi]_{ij}$ is the number of i's in $\varphi(j)$. Let $\lambda > 1$ be the Perron eigenvalue of $[\varphi]$ and let $\binom{\lambda_1}{\lambda_2}$ be the corresponding eigenvector normalized so that $\lambda_2 = 1$. The values λ_1 and λ_2 have the property that $\lambda[0, \lambda_1] = [0, \lambda\lambda_1] = [0, \lambda_1] \cup ([0, \lambda_2] + \lambda_1) \cup ([0, \lambda_1] + \lambda_1 + \lambda_2)$ and $\lambda[0, \lambda_2] = [0, \lambda\lambda_2] = [0, \lambda_1] \cup ([0, \lambda_2] + \lambda_1)$. By starting with the intervals $[-\lambda_2, 0], [0, \lambda_1]$ and applying the expansion to these intervals and replacing by the new intervals, we obtain the same pattern as what we got using the substitution φ.

We can assign the Markov partition in Fig. 1.5 to the hyperbolic toral automorphism induced by $[\varphi]$. The action of $[\varphi]$ on the Markov partition follows closely that of the substitution, as is shown in Fig. 1.6.

In this particular example, the Markov rectangles are connected. This particular Markov partition can be found using the substitution φ in the following way:

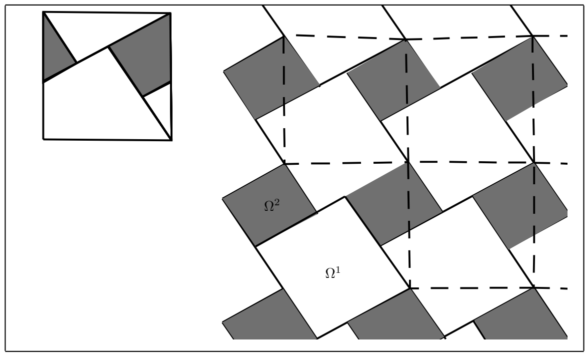


Figure 1.5: Markov partition corresponding to the substitution $\varphi : \mathtt{1} \mapsto \mathtt{1\ 2\ 1}, \mathtt{2} \mapsto \mathtt{1\ 2}$

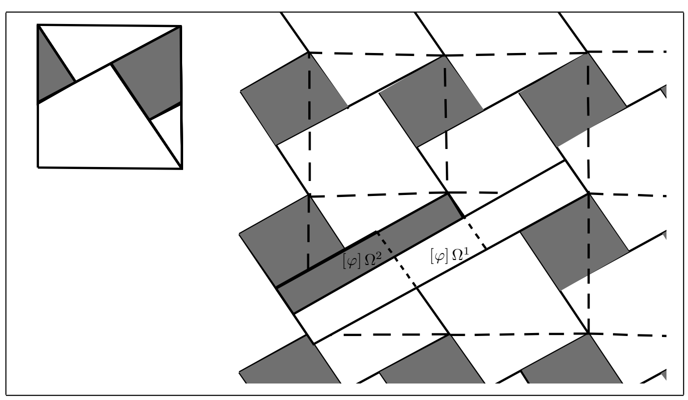


Figure 1.6: The matrix $[\varphi]$ maps the Markov partition rectangles following the pattern of the substitution.

Let E^s and E^u be the stable and the unstable spaces for the matrix $[\varphi]$ with associated projections $\mathrm{pr}^s : \mathbb{R}^2 \to E^s$ and $\mathrm{pr}^u : \mathbb{R}^2 \to E^u$.

As we said before, the substitution φ induces a bi-infinite word

$$W = \ldots u_{-2}u_{-1}.u_0u_1 \ldots = \ldots 112.12112 \ldots$$

Focusing in the right part of the word, we get the word $W' = 12112 \ldots$

Let $e_1 = \binom{1}{0}$, and $e_2 = \binom{0}{1}$ be the canonical basis for $\mathbb{R}^2$. Let $\tilde{S} : [0, \infty] \to \mathbb{R}^2$ be the function such that $\tilde{S}$ restricted to an interval $[k, k+1]$ is an isometry, and $\tilde{S}\,(k+1) - \tilde{S}\,(k) = e_{u_k}$ for $k \in \mathbb{N}$.

Let

$$P_1 = \left\{ \tilde{S}\,(k) : \tilde{S}\,(k+1) - \tilde{S}\,(k) = e_1 \right\},$$

and

$$P_2 = \left\{ \tilde{S}\,(k) : \tilde{S}\,(k+1) - \tilde{S}\,(k) = e_2 \right\}.$$

Let $R_1 = \overline{\mathrm{pr}_s\,(P_1)}$, and let $R_2 = \overline{\mathrm{pr}_s\,(P_2)}$. The set $R_1 \cup R_2$ is called the Rauzy Fractal, and, in this case, is connected. The sets R_1 and R_2 are called the Rauzy pieces corresponding to the letters 1 and 2.

Let f_1 and f_2 be $\mathrm{pr}^u\,(e_1)$ and $\mathrm{pr}^u\,(e_2)$, respectively. The rectangles with base R_1 and height f_1, and base R_2 and height f_2 constitute the Markov rectangles that we were looking for.

There are substitutions on three letters that closely resemble the Fibonacci substitution. One such example is the *Tribonacci* substitution given by $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. The Tribonacci is a substitution with complexity $2n+1$ such that for each n there exists a unique allowed word w such that $w1$, $w2$ and $w3$ are allowed, and for each k there exists a unique allowed word w' of length k such that $1w$, $2w$ and $3w$ are allowed (see [AR91] and [Fog02],pages 6, 232 and 368, for example). Substitutions with those properties are called *Arnoux-Rauzy* substitutions.

The construction of the Markov partition starting from a substitution can be done for certain substitutions on any number of letters (see [Fog02], page 251, for example).

It is an open question whether a hyperbolic toral automorphism always admits a Markov partition with connected rectangles. If the condition of connectedness is dropped, Manning in [Man02] constructed a Markov partition that closely reflects the geometry of the hyperbolic toral automorphism.

Bowen proved that Markov partitions always exist in [Bow70], and showed in [Bow78] that the boundary of Markov partitions on three letters is not smooth. Later, in [Caw89], Cawley proved that, under certain restrictions, for dimension three or higher, the boundary of the Markov rectangles is not smooth.

Since substitutions provide a concrete way to produce Markov partitions that closely reflect the geometry of the hyperbolic toral automorphism, the question of the existence of Markov partitions with connected rectangles is related to the question of whether the Rauzy fractal of a substitution is connected or not.

CHAPTER 2

PRELIMINARIES

2.1 Some Conventions

We denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of natural numbers with zero included, by $\mathbb{Z}$ the set of integers, by $\mathbb{Z}^+ = \{1, 2, \ldots\}$ the set of positive integers, and by $\mathbb{Z}^- = \{-1, -2, \ldots\}$ the set of negative integers. We define $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$ and $\mathbb{R}^- = \{r \in \mathbb{R} : r < 0\}$.

If $f : A \to B$ is a function and $A' \subset A$, we denote the restriction of f to the set A' by $f|_{A'}$.

We denote the spectral radius of a finite dimensional operator A by $r_\sigma(A)$ or simply by r_σ if the context is clear.

The ball $B_r(p)$ in $\mathbb{R}^d$ is the set $\{x : \|x - p\| < r\}$. The unit circle is $S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$, and the d-dimensional torus $\mathbb{T}^d$ is $\prod_{i=1}^{d} S^1 \cong \mathbb{R}^d / \mathbb{Z}^d$.

We denote the difference between two sets A and B as $A \setminus B$

A *singleton* is a one-point set. The cardinality of a set S is denoted by $\#(S)$. If $\mathcal{B}$ is a basis of a vector space V, then the coordinates of a vector v in V are denoted $[v]_{\mathcal{B}}$.

A diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow g_1 & & \downarrow g_2 \\ Y_1 & \xrightarrow[h]{} & Y_2 \end{array}$$

is said to *commute*, if $g_2 \circ f = h \circ g_1$, and a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ & {\scriptstyle h} \searrow & \downarrow g \\ & & Y \end{array}$$

commutes if $g \circ f = h$. In general, a diagram is said to commute if all squares and triangles in it commute.

The following theorem is known as the prime decomposition theorem (See [HK71]).

Proposition 2.1 *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a matrix, and let $q(x) = p_1(x) p_2(x)$ be a factorization in $\mathbb{Z}[x]$ of the characteristic polynomial $q(x)$ of A such that p_1 and p_2 are relatively prime. Let $\lambda_1(x) \in \mathbb{Q}[x]$ and $\lambda_2(x) \in \mathbb{Q}[x]$ be such that $1 = \lambda_1(x) p_1(x) + \lambda_2(x) p_2(x)$. Let $\pi_1 = \lambda_2(A) p_2(A)$ and $\pi_2 = \lambda_1(A) p_1(A)$. Then we have that $\pi_1 : \mathbb{R}^d \to \ker(p_1(A))$, $\pi_2 : \mathbb{R}^d \to \ker(p_2(A))$, $\pi_1 + \pi_2 = I$, and $\mathbb{R}^d = \ker(p_1) \oplus \ker(p_2)$. Furthermore, the characteristic polynomials for $A|_{\ker(p_1(A))}$ and $A|_{\ker(p_2(A))}$ are $p_1(x)$ and $p_2(x)$, respectively.*

2.2 Words and Substitutions

Definition 2.2 *An* interval of integers *I is a set of the form $I' \cap \mathbb{Z}$, where I' is an interval of $\mathbb{R}$. The interval I is called* finite, forward infinite, backward infinite *or* bi-infinite *according to whether I' is of the form $[a, b]$, $[a, \infty)$, $(\infty, b]$ or $(-\infty, \infty)$, respectively. An interval of integers that is not finite is called* infinite

Definition 2.3 *An* alphabet *is an ordered finite set. The elements of an alphabet are called* letters.

Definition 2.4 *Given $k \in \mathbb{N}$ and an alphabet $\mathcal{A}$, a* finite word *w is a function $w : [0, k) \cap \mathbb{Z} \to \mathcal{A}$. The number k is said to be the* length *of w and is denoted by $|w|$. The word of length zero is called the* empty word *and is denoted $\boldsymbol{\varepsilon}$. The set of* *all finite words is denoted $\mathcal{A}^*$, and the set of nonempty words from an alphabet $\mathcal{A}$ is denoted $\mathcal{A}^+$.*

We embed $\mathcal{A}$ in $\mathcal{A}^*$ and $\mathcal{A}^+$ by associating each letter $l \in \mathcal{A}$ with the word $l' : [0,1) \cap \mathbb{Z} = \{0\} \to \mathcal{A}$ given by $l'(0) = l$.

Definition 2.5 *If w_1 and w_2 are words of lengths k_1 and k_2, respectively, the* concatenation *$w_1 w_2$ of w_1 and w_2 is the word $w_1 w_2 : [0, k_1 + k_2) \to \mathcal{A}$ given by* $(w_1 w_2)(i) = \begin{cases} w_1(i) & \text{if } i \in [0, k_1) \\ w_2(i - k_1) & \text{if } i \in [k_1, k_1 + k_2) \end{cases}$

Though we do not show it, concatenation is associative, making $\mathcal{A}^+$ a semigroup. Since the empty word is an identity ε in $\mathcal{A}^*$, we obtain that $\mathcal{A}^*$ is a monoid. In [BK06] Barge and Kwapisz use the language of pointed words, and define words as an equivalence class of pointed words under the shift of domain (See also [GM94], page 132). We simply take a fixed representative by fixing the domain of a pointed finite word to be $[0, k) \cap \mathbb{Z}$.

Definition 2.6 *The set $\mathcal{A}^{\mathbb{N}} = \{w : w : \mathbb{N} \to \mathcal{A}\}$ is called the set of* forward infinite *words; $\mathcal{A}^{\mathbb{Z}^-} = \{w : w : \mathbb{Z}^- \to \mathcal{A}\}$ is the set of* backward infinite *words; and $\mathcal{A}^{\mathbb{Z}} = \{w : w : \mathbb{Z} \to \mathcal{A}\}$ is the set of* bi-infinite *words.*

Concatenation can be defined also for some infinite words; for example, we can concatenate a word $w_1 \in \mathcal{A}^{\mathbb{Z}^-}$ with a word $w_2 \in \mathcal{A}^{\mathbb{N}}$ making the word $w_1 w_2 \in \mathcal{A}^{\mathbb{Z}}$ by defining $(w_1 w_2)(i) = \begin{cases} w_2(i) & \text{if } i \in \mathbb{N} \\ w_1(i) & \text{if } i \in \mathbb{Z}^- \end{cases}$. To emphasize the "location of zero" we use a dot and write $w_1.w_2$ instead of $w_1 w_2$ in this case.

Concatenation is used as a common method to denote words in $\mathcal{A}^+$ and A^*. For example, if $\mathcal{A} = \{1, 2\}$, then 122 denotes the word $w \in \mathcal{A}^+$ such that $w(0) = 1$, $w(1) = 2$ and $w(2) = 2$.

Definition 2.7 *Let Q be $\mathbb{N}$ or $\mathbb{Z}$. We define the* shift map $\sigma : \mathcal{A}^Q \to \mathcal{A}^Q$ *by letting* $(\sigma(w))(i) = w(i+1)$ *for* $w \in \mathcal{A}^Q$

Definition 2.8 *Let w be an infinite word. If $w \in \mathcal{A}^{\mathbb{N}}$, we say that w is* shift-periodic *if there exists an $i \in \mathbb{Z}^+$ such that $w = \sigma^i(w)$. If there is a $k \in \mathbb{Z}^+$ such that $\sigma^k(w)$ is shift-periodic, we call w* eventually periodic.

We make similar definitions of shift-periodicity and eventual shift-periodicity for words in $\mathcal{A}^{\mathbb{Z}}$. Note that $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is invertible.

In the following definition, when $u : I \to \mathcal{A}$ is a word and J is an interval of integers, u_J resembles $u|_{J \cap I}$, up to a shift of domains by the right amount. This shift makes the definitions cumbersome.

Definition 2.9 *Let u be a finite or infinite word defined on an interval of integers I, and let J be an interval of integers. The set D, and the function $u_J : D \to \mathcal{A}$ are given as follows:*

if $I \cap J = \varnothing$, then $D = \varnothing$, and $u_J = \boldsymbol{\varepsilon}$;

if $I \cap J$ is of the form $[b, b+l) \cap \mathbb{Z}$, $b \in \mathbb{Z}, l \in \mathbb{Z}^+$, then $D = [0, l) \cap \mathbb{Z}$, and $(u_J)(i) = u(i+b)$;

if $I \cap J$ is of the form $[b, \infty) \cap \mathbb{Z}$, $b \in \mathbb{Z}$, then $D = \mathbb{N}$, and $(u_J)(i) = u(i+b)$;

if $I \cap J$ is of the form $(-\infty, b) \cap \mathbb{Z}$, $b \in \mathbb{Z}$, then $D = \mathbb{Z}^-$, and $(u_J)(i) = u(i-b)$;

if $I \cap J = \mathbb{Z}$, then $D = \mathbb{Z}$, and $(u_J)(i) = u(i)$

We abuse notation, and write, for example, $[a, b)$ to refer to $[a, b) \cap \mathbb{Z}$ in expressions where the context is clear. We write, for example, $u_{[2,4)}$ instead of $u_{([2,4) \cap \mathbb{Z})}$.

Definition 2.10 *Let u and v be words. If $w = u_J$ for words w, u and an interval of integers J, then w is said to be a* factor *of u.*

Definition 2.11 If *u,v and w are words such that $u = vw$, then we say that v is a* left factor *of u and that w is a* right factor *of u. We write $w = \ldots v$ to denote that v is a right factor of w, and $w = u\ldots$ to denote that u is a left factor of w.*

In general, we use ellipsis to indicate factors. For example, if $w \in \mathcal{A}^{\mathbb{Z}}$ then writing that $w = \ldots 12.32\ldots$. means that $w(-2) = 1$, $w(-1) = 2$, $w(0) = 3$ and $w(1) = 2$. Thus, the word $1232 \in \mathcal{A}^{+}$ is a factor of w.

2.3 Substitutions and Morphisms

Definition 2.12 *A* morphism *is a mapping $\varphi : \mathcal{A} \to \mathcal{B}^{*}$ for some alphabets $\mathcal{A}$ and $\mathcal{B}$.*

Definition 2.13 *A substitution is a morphism $\varphi : \mathcal{A} \to \mathcal{A}^{+}$.*

Note 2.14 In later chapters we assume that all the substitutions in those chapters have extra properties. Extra assumptions are stated at the beginning of the corresponding chapter.

We can extend $\varphi : \mathcal{A} \to \mathcal{B}^{+}$ to a map $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ as $\varphi(\ldots w_{-1}.w_0 w_1 \ldots) = \ldots \varphi(w_{-1}).\varphi(w_0)\varphi(w_1)\ldots$. We abuse notation and denote the extension also by φ.

In a similar way, $\varphi : \mathcal{A} \to \mathcal{B}^{+}$ extends to maps $\varphi : \mathcal{A}^{+} \to \mathcal{B}^{+}$, $\varphi : \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$, $\varphi : \mathcal{A}^{\mathbb{Z}^{-}} \to \mathcal{B}^{\mathbb{Z}^{-}}$ and $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$. For $\varphi : \mathcal{A}^{*} \to \mathcal{B}^{*}$, we also set $\varphi(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}$.

Definition 2.15 *If w is a word, the set $\mathcal{L}_w$ of all finite factors of w is called the* language *of w*

If w is infinite, for any given length n there is a factor of length n. Thus, $\mathcal{L}_w$ is finite only if w is a finite word. Next, we extend the definition of language from a word to a substitution.

Definition 2.16 *If $\varphi : \mathcal{A} \to \mathcal{A}^+$ is a substitution for some alphabet $\mathcal{A}$, then a finite word w is said to be* allowed *provided that there exists a number n and a letter $l \in \mathcal{A}$ such that w is a factor of $\varphi^n(l)$. The set $\mathcal{L}_\varphi$ of all allowed (finite) words for φ is called the* language *of the substitution. We say that an infinite word w is allowed for φ provided that every finite factor of w is allowed.*

For the following proposition see, for example, [BD01]

Proposition 2.17 *Let φ be a substitution. There exists an allowed word $w \in \mathcal{A}^{\mathbb{Z}}$ such that $\varphi^k(w) = w$ for some $k \in \mathbb{Z}^+$.*

Definition 2.18 *If a word w is such that $\varphi(w) = w$ we say that the word w is* fixed*, and if $\varphi^k(w) = w$ for some k we say that word w is $\varphi -$* periodic*.*

For a word w, the terms "shift-periodic" and "φ-periodic" are different. A word w is shift-periodic provided that the word w is preserved when we apply a power of the shift σ; whereas a word w is φ-periodic provided that the word w is preserved when we apply a power of φ to it.

Definition 2.19 *A substitution $\varphi : \mathcal{A} \to \mathcal{A}^+$ is* primitive *if there exists a number $n \in \mathbb{Z}$ such that for any pair of letters $i, j \in \mathcal{A}$ we have that j is a factor of $\varphi^n(i)$.*

Note 2.20 In most chapters we assume that all substitutions are primitive and non shift-periodic

For primitive substitutions we have that either there exists an allowed word for φ that is periodic, or no allowed word for φ is periodic:

Proposition 2.21 *If φ is a primitive substitution, then the following are equivalent:*

a) There exists an allowed bi-infinite non shift-periodic word w.

b) All allowed bi-infinite words for φ are non shift periodic.

Proof See, for example, Propositions 1.2.4 and 5.1.10 in [Fog02]. □

Thus, a primitive substitution is non shift-periodic if at least one (equivalently, each) φ-periodic bi-infinite word is not shift-periodic.

Pansiot, in [Pan86], gives an algorithm that effectively determines if a primitive substitution $\varphi : \mathcal{A} \to \mathcal{A}^+$ is non-periodic. In determining if φ is periodic or not, it is sometimes easier, however, to use the following observation: if there are allowed words $w_1.w_2 \in \mathcal{A}^{\mathbb{Z}}$ and $w_1.w_3 \in \mathcal{A}^{\mathbb{Z}}$, with $w_2 \neq w_3$, then the substitution φ is not shift-periodic. In [BD01] there is a method for finding such words if the substitution is primitive and non-periodic.

The following definition is taken from [Fog02], page 5. (see also [Que10], page 99).

Definition 2.22 *An infinite word u is* uniformly recurrent *if for every factor w of u there exists a number s such that , for every n, we have that w is a factor of $u_n \ldots u_{n+s-1}$.*

That is, a sequence $u = (u_n)$ is uniformly recurrent if every word occurring in u occurs in an infinite number of positions with bounded gaps. Note that if a substitution $\varphi : \mathcal{A} \to \mathcal{A}^+$ is primitive, then

(i) $\lim_{n\to\infty} |\varphi(i)| = \infty$ for all $i \in \mathcal{A}$, and

(ii) there exists $i \in \mathcal{A}$ and $k \in \mathbb{Z}^+$ such that $\varphi^k(i)$ starts with i.

Regarding conditions (i) and (ii) see [Que10], page 126. The following is a nice property of primitive substitutions.

Proposition 2.23 *Let φ be a primitive substitution, and let W be a φ-periodic word. Then W is uniformly recurrent.*

Proof See [Que10], propositions 4.7 and 5.5, and note that primitive substitutions satisfy (i) and (ii) of page 126. □

Definition 2.24 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a substitution. If there exists a number $k \in \mathbb{Z}^+$ and letters $b, e \in \mathcal{A}$ such that, for all $i \in \mathcal{A}$, we have that $\varphi^k(i) = b \ldots e$, then we say that the substitution is* proper*.*

For the proof of the following proposition, see [BD01].

Proposition 2.25 *If φ is a proper substitution, then there exists a unique word $w \in \mathcal{A}^{\mathbb{Z}}$ such that w is $\varphi -$* periodic*, furthermore, such w is fixed under φ.*

The following definitions allow us to derive, from a substitution, a linear mapping.

Definition 2.26 *If $\mathcal{A}$ is an alphabet, $l \in \mathcal{A}$ is a letter, and $w \in \mathcal{A}^*$, we define $|w|_l$ to be the number of times the letter l appears as a factor of w.*

Note that length of w is given by $|w| = \sum_{l \in \mathcal{A}} |w|_l$.

In the following definitions, we use that the alphabet $\mathcal{A}$ is ordered. If the alphabet we are using is comprised of only letter symbols or only number symbols, we generally use the lexicographic ordering.

Definition 2.27 *If $\mathcal{A} = \{l_1, l_2, \ldots, l_d\}$ is an alphabet, and $w \in \mathcal{A}^*$, we define the* abelianization *of w to be the vector $[w] = \left(|w|_{l_1}, |w|_{l_2}, \ldots, |w|_{l_d}\right)$ in $\mathbb{R}^d$.*

Definition 2.28 *If $\varphi : \mathcal{A} = \{l_1, l_2, \ldots, l_d\} \to \mathcal{A}^+$ is a substitution, we define the* abelianization matrix *$[\varphi] = (a_{ij})$ given by $a_{ij} = |\varphi(j)|_{l_i}$, with $i, j \in \{1, 2, \ldots, d\}$.*

Definition 2.29 *A matrix A is said to be* primitive *if all the entries of A^n are positive for some $n \geqslant 1$.*

Definition 2.30 *A matrix A is said to be* unimodular *provided that* $\det(A) = \pm 1$.

Definition 2.31 *A substitution φ is said to be* primitive *or* unimodular *provided that the matrix $[\varphi]$ is primitive or unimodular, respectively.*

Definition 2.32 *The* substitutive system X_φ *for a primitive, non-shift periodic substitution φ, is the space of $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite sequences that are allowed for φ.*

The substitutive system X_φ is taken with the product topology That is, two sequences $x \in X_\varphi$ and $y \in X_\varphi$ are within ε if $x_{[-k,k]} = y_{[-k,k]}$ for $k > \frac{1}{2^\varepsilon}$. With this topology, both σ is a homeomorphisms, and $\varphi : X_\varphi \to X_\varphi$ is continuous. That φ is injective is a result known as *recognizability*, and was first proved by Mosse in [Mos92] and [Mos96].

2.4 Tiling Spaces of Primitive non Shift-periodic Substitutions

This quick review of tiling spaces for primitive substitutions follows [BD01]. In the following, we denote the difference between two sets A and B as $A \setminus B$.

For a primitive substitution φ, we have, by the Perron-Froebinius Theorem, that $[\varphi]$ has an eigenvalue, which we denote by λ_φ, that is greater, in modulus, than its remaining eigenvalues. Since the matrix has integer entries, we can then deduce that $\lambda_\varphi > 1$. The number λ_φ is called the *Perron-Froebinius eigenvalue for* $[\varphi]$. Denote by $v_R = (v_\varphi)_R$ the right Perron eigenvector corresponding to λ_φ. Let $v_L = (v_\varphi)_L = (\lambda_1, \ldots, \lambda_d)$ be the left eigenvector corresponding to λ_φ.

Our tiles will be given by the entries of v_L. Each interval $P_i = [0, \lambda_i]$ is called a *prototile* for φ. A *tile* is a translate of some prototile. A *tiling* T of $\mathbb{R}$ by a set of prototiles $\{P_i\}$ is a collection $T = \{S_i\}_{i \in \mathbb{Z}}$ of tiles S_i for which $\bigcup_{i \in \mathbb{Z}} S_i = \mathbb{R}$, each S_i is a translate of some P_j, and $S_i \cap S_{i+1}$ is a singleton. We denote the tiling T also as $T = \ldots S_{-1} S_0 S_1 S_2 S_3 \ldots$.

To represent the location of zero, we underline the tile S_{i_0} such that $0 \in S_{i_0} \setminus S_{i_0+1}$. For example, when we write $T = \ldots S_{-1}S_0S_1\underline{S_2}S_3\ldots$, we are indicating that $0 \in S_2 \setminus S_3$. In such case, we denote by T_0 the tile that contained zero so that $T = \ldots T_{-1}\underline{T_0}T_1\ldots$. In our example, $T_{-1} = S_1$, $T_0 = S_2$, $T_1 = S_3$, etc. If, further, $\{0\} = S_{i_0-1} \cap S_{i_0}$, then we write $T = \ldots S_{i_0-2}S_{i_0-1}.S_{i_0}S_{i_0+1}S_{i_0+2}\ldots$. If $T = \{S_i\}_{i\in\mathbb{Z}}$ is a tiling of $\mathbb{R}$, then a *patch* $\mathcal{P}$ of T is an ordered subset of the form $\mathcal{P} = \{S_i\}_{i_0}^{i_1}$ for some $i_0 < i_1 \in \mathbb{Z}$

If $\varphi(i) = i_1i_2\ldots i_{|\varphi(i)|}$, then $\lambda_\varphi\lambda_i = \sum_{j=1}^{|\varphi(i)|}\lambda_{i_j}$. Thus $|\lambda_\varphi P_i| = \sum_{j=1}^{|\varphi(i)|}\left|P_{i_j}\right|$, and $\lambda_\varphi P_i$ is tiled by $\{T_j\}_{j=1}^{|\varphi(i)|}$, where $T_j = P_{i_j} + \sum_{k=1}^{j-1}\lambda_{i_k}$. This process is called *inflation and substitution* and extends to a map, which we again denote by φ taking a tiling $T = \{S_i\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ to a new tiling $\varphi(T)$ of $\mathbb{R}$ defined by inflating, substituting, and suitably translating each S_i:

More precisely, for each word $w = l_1,\ldots,l_n \in \mathcal{A}^+$ with $l_i \in \mathcal{A}$, and for each $t \in \mathbb{R}$, we define a *patch associated with the word*, and translated by t as follows

$$\mathcal{P}_w + t = \left\{P_{l_1} + t, P_{l_2} + t + |P_{l_1}|, \ldots, P_{l_n} + t + \sum_{i<n}|P_{l_i}|\right\}$$

Then $\varphi(P_i + t) = \mathcal{P}_{\varphi(i)} + \lambda_\varphi t$ and $\varphi\left(\{P_{k_i} + t_i\}_{i\in\mathbb{Z}}\right) = \bigcup_{i\in\mathbb{Z}}\left\{\mathcal{P}_{\varphi(k_i)} + \lambda_\varphi t_i\right\}$. In general, $\varphi(\mathcal{P}_{w_1} + t) = \mathcal{P}_{\varphi(w_1)} + \lambda_\varphi t$.

There is a natural topology on the collection Σ_φ of all tilings of $\mathbb{R}$ by prototiles: For ε small, two tilings $\{S_i\}_{i\in\mathbb{Z}}$ and $\{S_i'\}_{i\in\mathbb{Z}}$ are within ε if there are translates $t, t' \in \mathbb{R}$ such that $|t|, |t'| < \varepsilon$ and such that $\{S_i\}_{i\in\mathbb{Z}} - t$ and $\{S_i'\}_{i\in\mathbb{Z}} - t'$ are identical inside of the interval $\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right)$. This space Σ_φ is compact and metrizable with this topology and $\varphi : \Sigma_\varphi \to \Sigma_\varphi$ is continuous.(See [AP98] for the details).

The *tiling space* $\mathcal{T}_\varphi$ associated to a substitution φ, is defined as the collection of tilings with the following property: A tiling $T = \{S_i\}_{i\in\mathbb{N}}$ is in $\mathcal{T}_\varphi$ if, whenever

$P = \{S_i\}_{i=i_0}^{i_1}$ is any patch of T, then there are $n \in \mathbb{Z}^+$, $i \in \mathcal{A}$ and $t \in \mathbb{R}$ such that $P \subset \varphi^n (P_i + t)$.

There is a natural flow $\phi_t : \Sigma_\varphi \to \Sigma_\varphi$ defined by $\phi_t \left(\{S_i\}_{i\in\mathbb{Z}}\right) = \{S_i - t\}_{i\in\mathbb{Z}}$. We denote ϕ_t as $(\phi_\varphi)_t$ if we want to indicate the substitution. We write $\phi_t (p)$ or $\phi (t, p)$ indistinctively.

If φ is primitive and non shift-periodic, ϕ_r is minimal on $\mathcal{T}_\varphi$ (see Corollary 3.5 in [AP98]), and thus the closure of the orbit of the flow of any tiling $T \in \mathcal{T}_\varphi$ is T_φ. It follows that $\mathcal{T}_\varphi$ is a continuum. Finally $\varphi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ is a homeomorphism. This latter theorem is not easy, and the proof is closely associated to the notion of *recognizability* for substitutive systems (See [Mos92] and [Mos96]). See [AP98] for details.

2.5 Strands

Definition 2.33 *Let* $B = \{e_1, \ldots, e_d\}$ *be the canonical basis of* $\mathbb{R}^d$*, and let* $I \subset \mathbb{R}$ *be an interval of the form* $[0, b]$ *for some* $b \in \mathbb{Z}^+$*,* $[0, \infty)$*,* $(-\infty, 0]$*, or* $\mathbb{R}$ *. Let* $f : I \to \mathbb{R}^d$ *be such that whenever* $k, k + 1 \in I$*, we have that there is a number* $i \in \{1, \ldots, d\}$ *such that* $f (k + 1) - f (k) = e_i$ *and such that* $f|_{[k,k+1]}$ *is an isometry. A set of the form* $f ([k, k + 1])$*, with* $k, k + 1 \in I$ *is called a* segment*. The set of all segments in* $\mathbb{R}^d$ *is denoted* $\mathcal{S}$ *.The set* $S = \{f ([k, k + 1]) : k, k + 1 \in I\}$ *is called a* strand*, and the elements of a strand are called* edges*. The sets* $\mathcal{S}^+$ *of* nonempty finite *strands,* $\mathcal{S}^{\mathbb{N}}$ *of* forward infinite *strands,* $\mathcal{S}^{\mathbb{Z}^-}$ *of* backward infinite *strands, and* $\mathcal{F}$ *of* bi-infinite *strands correspond to whether the interval* I *taken is* $[0, b]$ *for some* $b \in \mathbb{Z}^+$*,* $[0, \infty)$*,* $(-\infty, 0]$*, or* $\mathbb{R}$*, respectively. The function* f *is called a* parametrization *of the strand.*

Definition 2.34 *If* $f : I \to \mathbb{R}^d$ *is a parametrization of a strand* S*, and* I *is bounded above, we define* $\max (S)$ *to be* $f (\max (I))$*. If* I *is bounded below we define* $\min (S)$ *as* $f (\min (I))$*.*

Definition 2.35 *The empty set* $\varnothing \subset \mathbb{R}^d$ *is called the* empty *strand. The set* $\mathcal{S}^*$ *of* finite *strands is defined as* $\mathcal{S}^* = \{\varnothing\} \cup \mathcal{S}^+$

The parametrization of a strand is unique for strands in $\mathcal{S}^+$, $\mathcal{S}^{\mathbb{N}}$ or $\mathcal{S}^{\mathbb{Z}^-}$. If $\{S_i\}_{i \in I}$ is a strand, we assume that the indexing is such that $\max(S_i) = \min(S_{i+1})$, if $i, i+1 \in I$.

Definition 2.36 *Let* $S = \{S_i\}$ *be a strand in* $\mathcal{S}^+$, $\mathcal{S}^{\mathbb{N}}$ *or* $\mathcal{S}^{\mathbb{Z}^-}$. *We define* $S_{\text{par}} : I \to \mathbb{R}^d$ *to be the unique parametrization for* S *such that* $S_{\text{par}}([k, k+1]) = S_k$ *for* $k, k+1 \in I$.

If a strand $S = \{S_i\}_{i \in \mathbb{Z}}$ is in $\mathcal{F}$, then there are infinitely many parametrizations f of S. Later we choose a particular parametrization for $S \in \mathcal{F}$.

Definition 2.37 *For a finite strand* $S \in \mathcal{S}^+$ *with parametrization* $S_{\text{par}} : [0, b] \to \mathbb{R}^d$, *the number* b *is called the* length *of the strand. The length of an empty strand is defined to be zero.*

Definition 2.38 *Let* $\mathcal{A} = \{l_1, l_2, \dots l_d\}$ *be an alphabet in* d *letters,* $B = \{e_1, e_2, \dots, e_d\}$ *be the canonical basis of* $\mathbb{R}^d$ *and* $S = S_{\text{par}}([0, 1])$ *be a segment with parametrization* $S_{\text{par}} : [0, 1] \to S$. *Let* $e_i = \max(S) - \min(S)$. *We call* i *the* type, l_i *the* letter type, *and* e_i *the* vector type *of the segment* S.

In the case where the alphabet is of the form $\{1, 2, \dots, d\}$, we identify the type with the letter type.

Definition 2.39 *Let* $S_{\text{par}} : I \to \mathbb{R}^d$ *be a parametrization of a strand* S, *where* I *is either* $(-\infty, 0]$, $[0, \infty)$, $(-\infty, \infty)$ *or* $[0, b]$ *for some* $b \in \mathbb{Z}^+$. *We define the set of vertices of* S *as* $V_S = S_{\text{par}}(I \cap \mathbb{Z})$.

Note that the vertices of $S \in \mathcal{F}$ are ordered. The parametrization S_{par} can be recovered from the countable set V_S for $S \in \mathcal{S}^+$, $S \in \mathcal{S}^{\mathbb{Z}^-}$, or $S \in \mathcal{S}^{\mathbb{N}}$. If $S \in \mathcal{F}$ the problem to recover a particular parametrization $f : \mathbb{R} \to \mathbb{R}^d$ for the strand is "the location of zero"; that is, to decide which of the vertices of S should be $f(0)$. We address this problem later.

Since, for a strand S in $\mathcal{S}^+$, $\mathcal{S}^{\mathbb{N}}$ or $\mathcal{S}^{\mathbb{Z}^-}$ there is a unique parametrization S_{par}, we can think of S as a function $g : J \subset \mathbb{Z} \to \mathcal{S}$ with the requirement that $\max(g(i)) = \min(g(i+1))$, where $J = [0, b] \cap \mathbb{Z}$, $J = \mathbb{N}$ or $J = \mathbb{Z}^-$. This justifies the notation $\mathcal{S}^+$, $\mathcal{S}^{\mathbb{N}}$ or $\mathcal{S}^{\mathbb{Z}^-}$.

Definition 2.40 *Let S and T be strands such that* $\max(S) = \min(T)$. *The* concatenation $S_1 S_2$ *is the set $S \cup T$.*

Definition 2.41 *Let $\mathcal{A} = \{l_1, l_2, \dots l_d\}$ be an alphabet in d letters. Let $S \in \mathcal{S}^+$ be a strand with edge set $E_S = \{E_0, E_2, \dots, E_{k-1}\}$, and let $s(i)$ be the type of the edge E_i. We call the word $[S] = l_{s(0)} l_{s(1)} \dots l_{s(k-1)}$ the* word associated *with the strand.*

The previous definition allows us to define a map $[\cdot] : \mathcal{S}^+ \to \mathcal{A}^+$, which we can extend to a map $[\cdot] : \mathcal{S}^* \to \mathcal{A}^*$, by defining $[\varnothing] = \boldsymbol{\varepsilon}$. We also define $[\cdot] : \mathcal{S}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$, and $[\cdot] : \mathcal{S}^{\mathbb{Z}^-} \to \mathcal{A}^{\mathbb{Z}^-}$ similarly. Note that we are using the natural order of the edges of the strand.

2.5.1 Definitions for Bi-infinite Strands

To define $[\cdot] : \mathcal{F} \to \mathcal{A}^{\mathbb{Z}}$, and to choose a parametrization for S_{par} for a strand $S \in \mathcal{F}$, we need to define where "the origin" is. To do that, let $v \in (\mathbb{R}^+)^d$ be a positive vector. Note that, if $S \in \mathcal{F}$ is a strand, then $\left(v^\perp \cap S_{i_0}\right)$ is a singleton for some $S_{i_0} \in S$, where $v^\perp = \{x : x \cdot v = 0\}$.

If a strand $S \in \mathcal{F}$ is a concatenation of other strands $\{T_i\}$, we write concatenation as a product. For example, $S = \Pi_{i\in\mathbb{Z}} T_i = \ldots T_{-2}T_{-1}T_0T_1\ldots$.For a strand $S \in \mathcal{F}$, there is always a unique edge $T \in S$ such that $(T \setminus \{\max(T)\}) \cap v^{\perp} \neq \varnothing$.

If $S \in \mathcal{F}$ is of the form $S = \ldots T_{-2}T_{-1}T_0T_1\ldots$, with $T_i \in \mathcal{S}$ for all $i \in \mathbb{Z}$, and we want to emphasize which of the T_i is such that $(T_i \setminus \{\max(T_i)\}) \cap v^{\perp} \neq \varnothing$, we underline the symbol representing the edge where the intersection lies. Thus, when we write $S = \ldots T_{-2}T_{-1}T_0T_1\underline{T_2}T_3\ldots$, we are indicating that $v^{\perp} \cap (T_2 \setminus \{\max(T_2)\}) \neq \varnothing$. We also write $S = \ldots T_{-2}T_{-1}T_0T_1.T_2\ldots$ to indicate that $\min T_2 \in v^{\perp}$. Note that if $S \in \mathcal{F}$, then there are unique strands $L \in \mathcal{S}^{\mathbb{Z}^-}$, $R \in \mathcal{S}^{\mathbb{N}}$ and a segment $C \in \mathcal{S}$ such that $S = L\underline{C}R$. The mapping that assigns to each strand $S = L\underline{C}R$ the segment C is denoted $(\cdot) : \mathcal{F} \to \$$. We use $\hat{S}$ or (S) interchangeably.

Definition 2.42 *Let $v \in (\mathbb{R}^+)^d$ be a positive vector and $S \in \mathcal{F}$. Let $L \in \mathcal{S}^{\mathbb{Z}^-}$, $R \in \mathcal{S}^{\mathbb{N}}$ be such that $S = L\underline{\hat{S}}R$. We call this factorization of S the* natural decomposition *of the strand. We call $\hat{S}$ the* center *, L the* lower part*, and R the* upper part *of the strand S.*

Definition 2.43 *Let $v \in (\mathbb{R}^+)^d$ be a positive vector. The map $[\cdot] : \mathcal{F} \to \mathcal{A}^{\mathbb{Z}}$ is the map given by $[S] = [L] . \left[\hat{S}\right] [R]$, where $S = L\underline{\hat{S}}R$ is the natural decomposition of the strand.*

Definition 2.44 *Let $v \in (\mathbb{R}^+)^d$ be a positive vector, and $S \in \mathcal{F}$. Let $L\underline{\widehat{S}}R$ be the natural decomposition of S. We define $S_{\text{par}} : \mathbb{R} \to \mathbb{R}^d$ by*

$$S_{\text{par}}(t) = \begin{cases} \left((L)_{\text{par}}\right)(t) & \text{if } t \in (-\infty, 0) \\ \left((SR)_{\text{par}}\right)(t) & \text{if } t \in [0, \infty) \end{cases}.$$

If J is an interval of integers, we define $S_J = \{S_{\text{par}}([k, k+1]) : k, k+1 \in I\}$. Mimicking what we did for words, we can define *factors* of a strand. Note that a finite non-empty strand S is uniquely determined by a pair $(v, [S])$, where $v = \min(S)$. In a similar way, a strand $S \in \mathcal{F}$ is uniquely determined by a pair $(v, [S])$, where $v = \min\left(\hat{S}\right)$.

Definition 2.45 *Given a substitution $\varphi : \mathcal{A} = \{l_1, l_2, \ldots, l_d\} \to \mathcal{A}^+$. We say that a strand $S \subset \mathbb{R}^d$ is* allowed *for φ if the word $[S]$ is allowed for φ.*

Note that if S is a strand, then $[S]$ denotes a word, whereas if w is a word, then $[w]$ denotes a vector.

We can define the *action of the substitution on a segment* in the following way:

Definition 2.46 *Let $\varphi : \mathcal{A} = \{l_1, l_2, \ldots, l_d\} \to \mathcal{A}^+$ be a substitution. Let $S \in \mathcal{S}$ be a segment in $\mathbb{R}^d$ with vector type e_i, and letter type l_i and let $p = \min(S)$. Then we define $\varphi(S)$ to be the unique strand such that $\min(\varphi(S)) = [\varphi](p)$ and such that $[\varphi(S)] = \varphi(l_i)$.*

In the previous definition, note that $\min(\varphi(S)) = [\varphi](p)$, $\max(\varphi(S)) = [\varphi](p + e_i)$, the associated word is $[\varphi(S)] = \varphi(l_i)$, and the vertex set of $\varphi(S)$ is

$$V_{\varphi(S)} = [\varphi](p) + \left\{[\boldsymbol{\varepsilon}], \left[(\varphi(l_i))_{[0,1)}\right], \left[(\varphi(l_i))_{[0,2)}\right], \ldots, \left[(\varphi(l_i))_{[0,|\varphi(l_i)|)}\right] = [\varphi](e_i)\right\}.$$
$$= [\varphi](p) + \left\{\left[(\varphi(l_i))_{[0,k)}\right] : k \in [0, |\varphi(i)|]\right\}$$

We extend the definition from edges to strands by concatenating in the following way.

Definition 2.47 *Let S be a non empty strand, we define $\varphi(S) = \bigcup\{\varphi(S_i) : S_i \in S\}$*

Note that if $S = \prod_{i\in I} S_i = \ldots S_{-1}S_0S_1\ldots$ then the substitution φ *acts on strands* by defining $\varphi(S) = \prod_{i\in I}\varphi(S_i) = \ldots\varphi(S_{-1})\varphi(S_0)(\varphi(S_1))\ldots$. Figure 2.1 exemplifies the definition.

We give a topology to a certain subset of $\mathcal{F}$. For this, we first define a metric in $\mathcal{F}$. Recall that V_S denotes the vertex of a strand S.

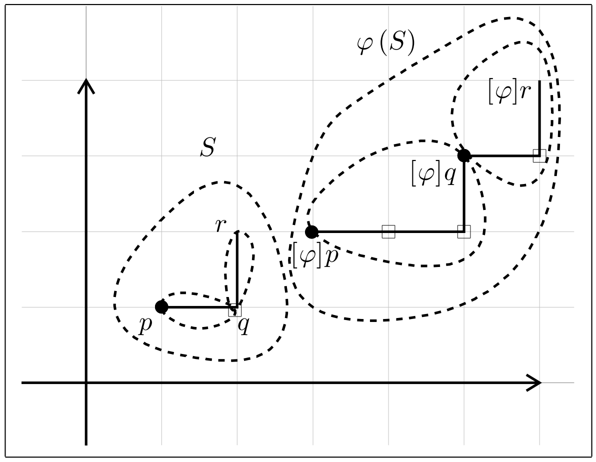


Figure 2.1: The effect of the mapping φ: $1 \mapsto 112$, $2 \mapsto 12$. The pattern of S is the word 12, and the pattern of $\varphi(S)$ is the word $\varphi(12) = 11212$.

Definition 2.48 *Given two bi-infinite strands S_1 and S_2 $\mathbb{R}^d$. We define the distance $d(S_1, S_2) = \min\{1, \rho(S_1, S_2)\}$ where $\rho(S_1, S_2) = \inf_\varepsilon B_\varepsilon$ and B_ε is the set of all numbers ε with the property that there are $v_1, v_2 \in \mathbb{R}^d$ such that $\|v_1\|, \|v_2\| < \varepsilon$ and $(V_{S_1} - v_1) \cap B_{\frac{1}{\varepsilon}}(0) = (V_{S_2} - v_2) \cap B_{\frac{1}{\varepsilon}}(0)$.*

It can be shown that the function just defined is, indeed, a metric.

2.6 Unimodular Pisot Substitutions

Though we have defined the tiling space $\mathcal{T}_\varphi$ for primitive substitutions, we focus most of our attention to special substitutions for which we use the strand spaces to our advantage.

Recall (see [DF91], page 622) that an *algebraic number* is a number in $\mathbb{C}$ that is a solution of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ with $a_n, \ldots a_0 \in \mathbb{Z}$, and an *algebraic integer* is an algebraic number that is a root of a monic polynomial with coefficients from $\mathbb{Z}$.

Definition 2.49 *An algebraic integer p is said to be a* Pisot number *if $p > 1$, and for any other root $q \in \mathbb{C}$ of the minimal polynomial of p, we have that $|q| < 1$.*

Definition 2.50 *A Pisot number is a* Pisot unit *provided that its minimal polynomial is of the form $x^n + a_{n-1}x^{n-1} + \ldots \pm 1$.*

Definition 2.51 *A substitution φ is said to be* Pisot *provided that the Perron eigenvalue λ_φ of $[\varphi]$ is a Pisot number. A substitution φ is said to be* irreducible Pisot *provided that the characteristic polynomial of $[\varphi]$ is the minimal polynomial of λ_φ. A substitution is said to be* reducible Pisot *if the minimal polynomial of λ_φ divides the characteristic polynomial of $[\varphi]$.*

Definition 2.52 *We say that a substitution φ is an* IUP *substitution if φ is an irreducible Pisot substitution, and $[\varphi]$ is a unimodular matrix. We say that φ is an* RUP *substitution if φ is an irreducible Pisot substitution, and $[\varphi]$ is a unimodular matrix.*

Since the Perron Froebinius eigenvalue of a primitive matrix is a simple root of the characteristic polynomial, we have that if φ is a RUP substitution then the characteristic polynomial $q(x)$ of $[\varphi]$ factors into a *reducible part* $r(x)$ and a *Pisot part* $p(x)$. Additionally, $p(x)$ and $r(x)$ do not have any common roots in $\mathbb{C}$.

Note that, in the definition of IUP and RUP substitutions, the requirement that $[\varphi]$ be a unimodular matrix is equivalent to the requirement that the Perron eigenvalue of $[\varphi]$ be a Pisot unit.

It can be shown (for example, in [CS01]) that all IUP substitutions are primitive, and thus, by the Perron-Froebinius Theorem, $[\varphi]$ has positive left and right eigenvectors v_L and v_R, respectively, for λ_φ.

Definition 2.53 *A substitution* $\varphi : \mathcal{A} = \{1, 2, \ldots, d\} \to \mathcal{A}^+$ *is called a* hyperbolic *substitution provided that there are a* stable space E^s *and an* unstable space E^u *such that* $\mathbb{R}^d = E^s \oplus E^u$ *such that* $\lim_{n\to\infty} [\varphi]^n v = 0$ *for all* $v \in E^s$*, and* $\lim_{n\to-\infty} [\varphi]^n w = 0$ *for all* $w \in E^u$*. For a point* $p \in \mathbb{R}^d$*, we define* $\|p\|_s$ *and* $\|p\|_u$ *as* $\|\mathrm{pr}^s(p)\|$ *and* $\|\mathrm{pr}^u(p)\|$*, respectively, where* $\mathrm{pr}^s : \mathbb{R}^d \to E^s$ *and* $\mathrm{pr}^u : \mathbb{R}^d \to E^u$ *are the induced projections.*

For the case of IUP substitutions, defining $E^u =< v_R >$ to be the subspace generated by v_R, and $E^s = (v_L)^\perp$ we note that IUP substitutions are hyperbolic.

2.7 The Strand Space of a IUP Substitution

Definition 2.54 *Let* φ *be a IUP substitution with* $E^u = \langle v_R \rangle$ *and* $E^s = v_L^\perp$*, and induced projections* pr^s *and* pr^u*. We define the* stable-cylinder $\mathcal{C}^R$ *to be* $\mathcal{C}^R = \{p : \|p\|_s < R\}$*.*

For the proof of the following proposition, see [BK06].

Proposition 2.55 *There exists a number* $R > 0$ *such that if* $S \in F_\varphi$*, and* $S_i \subset \mathcal{C}^R$ *for all* $S_i \in S$*, then* $T \subset \mathcal{C}^R$*, for all* $T \in \varphi(S)$*.*

Definition 2.56 *Given* $R > 0$*, we define*

$$\mathcal{F}^R = \left\{S : S \textit{ is a bi-infinite strand, } T \subset \mathcal{C}^R \textit{ for all } T \in S \textit{ and } [S] \textit{ is allowed}\right\}$$

Definition 2.57 *Let* φ *is an IUP substitution, and* R *be a number as in proposition 2.55. We define the* strand space $\mathcal{F}_\varphi$ *of* φ *to be*

$$\mathcal{F}_\varphi = \bigcap_{n\in\mathbb{Z}^+} \left\{\varphi^n\left(\mathcal{F}^R\right)\right\}.$$

In other words, given φ and R as in Proposition 2.55, we say that a strand S is in $\mathcal{F}_\varphi$ provided that S is a bi-infinite allowed strand inside of the cylinder $\mathcal{C}^R$ and such that S has "infinite past" under φ: that is, $S \in \mathcal{F}^R$, and for each $n \in \mathbb{Z}^+$, there is a strand $S_n \in \mathcal{F}^R$ such that $\varphi^n(S_n) = S$.

Definition 2.58 *Let $\mathcal{B} = \{e_1, e_2, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. For $v \in \mathcal{B}$, let S_v be the segment starting at the origin and finishing at v. Let $T = \{T_i\}_{i\in\mathbb{Z}}$ be a strand in $\mathcal{F}_\varphi$, then the* induced tiling *of E^u is the tiling of E^u with prototiles $\mathrm{pr}^u(S_{e_1}), \mathrm{pr}^u(S_{e_2}), \ldots, \mathrm{pr}^u(S_{e_d})$ given by $T = \{\mathrm{pr}^u(S_i)\}_{i\in\mathbb{Z}}$.*

The space of strands $\mathcal{F}_\varphi$ is linked to the space of tilings of the real line $\mathcal{T}_\varphi$ via the previous definition. In [BK06], it is shown, using 2.61, that the mapping that associates strands with tilings of E^u is bijective for strands in the tiling space $\mathcal{F}_\varphi$ for a IUP substitution. Note that $\mathcal{F}_\varphi$ is only defined for IUP substitutions, whereas the tiling space $\mathcal{T}_\varphi$ is defined for any primitive substitution. In Chapter 4, we extend the definition of strand space to RUP substitutions.

We next define the flow on the strand space $\mathcal{F}_\varphi$. In the following definition, v_R is the right Perron eigenvector for $[\varphi]$

Definition 2.59 *Let φ be a IUP substitution, and let v_R be such that $\|v_R\| = 1$ and $< v_R >= E^s$. The* tiling flow *$\phi_t : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ is the function given by $\phi_t(S) = S - tv_R$*

In section 2.4 we mentioned that the tiling flow ϕ for $\mathcal{T}_\varphi$ was continuous and minimal, and that the inflation and substitution map $\varphi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ was a homeomorphism. These results have also been proven in [BK06] for IUP substitutions using completely the language of strands:

Proposition 2.60 *The tiling flow for a IUP substitution is continuous and minimal.*

Proposition 2.61 *The inflation and substitution map φ is a homeomorphism in $\mathcal{F}_\varphi$.*

The main advantage of IUP substitutions is that we have the strands as a tool at our disposal. The next section uses strands in a fundamental way.

2.8 Geometric Realization and the First-Vertex Map

Definition 2.62 *Let φ be a IUP substitution on d letters. A* state *is a segment S such that $S \cap E^s \neq \varnothing$. We denote by $\$$ the set of all states, and by $\$_p$ the set of all states S such that $\min(S) \cong p \bmod \mathbb{Z}^d$. We call $\$_0$ the set of* integer *states. We define $\$_R = \left\{S \in \$: S \subset C^R\right\}$*

Recall that $\widehat{(\cdot)} : \mathcal{F}_\varphi \to \$$ is the function such that $\hat{S}$ is the unique state in S; that is, $\hat{S}$ is the unique edge of S such that $\left(\hat{S} \setminus \left\{\max\left(\hat{S}\right)\right\}\right) \cap E^s \neq \varnothing$.

The following definitions come from [BK06], although the map $\tilde{h}$ is not given a special name there.

Definition 2.63 *If φ is a IUP substitution on d letters, then the* first-vertex map *is the function $\tilde{h}_\varphi : \mathcal{F}_\varphi \to \mathbb{R}^d$ given by $\tilde{h}_\varphi(S) = \min\left(\hat{S}\right)$.*

Definition 2.64 *If φ is a IUP substitution on d letters, then the* geometric realization *is the function $h_\varphi : \mathcal{F}_\varphi \to \mathbb{T}^d$ defined by $h_\varphi(S) = \tilde{h}_\varphi(S) + \mathbb{Z}^d$.*

Note that if $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical mapping $\pi(p) = p + \mathbb{Z}^d$, then $h_\varphi = \pi \circ \tilde{h}_\varphi$.

The next proposition, which explicitly gives a representation for the first-vertex map, is of interest as it might be the link with the treatment given by others in terms of algebraic number theory.

Proposition 2.65 *Let φ be a IUP substitution with projections pr^s and pr^u, and let $S \in \mathcal{F}_\varphi$. For $k \in \mathbb{Z}$, let $S_k = \varphi^k(S)$, $p_k = \min(\widehat{(S_k)})$ and $v_k = p_k - [\varphi]\, p_{k-1}$. Then*

$$p_0 = [\varphi]^n\, p_{-n} + \sum_{k=0}^{n-1} [\varphi]^k\, v_{-k}, \textit{for } n \geqslant 1 \tag{2.1}$$

$$p_0 = [\varphi]^{-n} p_n - \sum_{k=1}^{n} [\varphi]^{-k} v_k, \text{ for } n \geqslant 0 \tag{2.2}$$

$$\mathrm{pr}^s (p_0) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathrm{pr}^s \left([\varphi]^k v_{-k} \right) \tag{2.3}$$

$$\mathrm{pr}^u (p_0) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{pr}^u \left([\varphi]^{-k} v_k \right) \tag{2.4}$$

Proof To show that equation 2.1 holds, we proceed by induction: For $n = 1$, $v_0 = p_0 - [\varphi] p_{-1}$ by hypothesis, and thus $p_0 = [\varphi] p_{-1} + v_0 = [\varphi]^1 p_{-1} + \sum_{k=0}^{1-1} [\varphi]^k v_{-k}$. Assume that $p_0 = [\varphi]^n p_{-n} + \sum_{k=0}^{n-1} [\varphi]^k v_{-k}$. Since $v_{-n} = p_{-n} - [\varphi] p_{(-n)-1}$, we obtain that $p_0 = [\varphi]^n \left([\varphi] p_{(-n)-1} + v_{-n} \right) + \sum_{k=0}^{n-1} [\varphi]^k v_{-k} = [\varphi]^{n+1} p_{-(n+1)} + \sum_{k=0}^{n} [\varphi]^k v_{-k}$. This finishes the proof of equation 2.1.

Notice that, for each n, if (S_n) is associated with the letter i, then we have that $v_{n+1} = [r]$, where r is a prefix of $\varphi(i)$. Thus, the set $\{v_n : n \in \mathbb{Z}\}$ is finite, and, hence, bounded.

Since the set $\{v_{-n} : n \in \mathbb{N}\}$ is bounded (since it is finite), it follows that $\lim_{n\to\infty} \mathrm{pr}^s ([\varphi]^n p_{-n}) = 0$. Applying to both sides of equation 2.1 and taking the limit when $n \to \infty$, we obtain equation 2.3.

The proofs for equations 2.2 and 2.4 are similar to those for 2.1 and 2.3, respectively, and is omitted. □

Corollary 2.66 *Let v_k as in the previous proposition. Then*

$$\tilde{h}_\varphi (S_0) = \left(\lim_{n\to\infty} \sum_{k=0}^{n-1} \mathrm{pr}^s \left([\varphi]^k v_{-k} \right) \right) + \left(\lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{pr}^u \left([\varphi]^{-k} v_k \right) \right)$$

We focus now on stating some properties of the geometric realization map.

Definition 2.67 *A function f from a set A to a set B is said to be* bounded-to-one *provided that $\#\left(f^{-1}(p)\right) < \infty$ for all $p \in B$.*

Definition 2.68 *A function f from a set A to a set B is said to be* uniformly bounded-to-one *if there exist a number k such that $\#\left(f^{-1}(b)\right) < k$ for all $b \in B$.*

The following four propositions are proved in [BK06]

Proposition 2.69 *The geometric realization h of an IUP substitution in d letters is closed, continuous, and uniformly bounded-to-one.*

Proposition 2.70 *The geometric realization of an IUP substitution is onto.*

Let $(\phi_\varphi)_t$ be the tiling flow in $\mathcal{F}_\varphi$, and let $\phi_{[\varphi]}$ be the *Kronecker flow* in $\mathbb{T}^d$ given by

$$\left(\phi_{[\varphi]}\right)_t(p) = p - tv_R + \mathbb{Z}^d$$

Proposition 2.71 *The geometric realization map is a semiconjugacy from the tiling flow $(\phi_\varphi)_t$ to the Kronecker flow $\left(\phi_{[\varphi]}\right)_t$.*

Proposition 2.72 *The geometric realization is a semiconjugacy from the inflation and substitution map φ in $\mathcal{T}_\varphi$ to the hyperbolic toral automorphism $[\varphi]$ in $\mathbb{T}^d$.*

2.9 The Geometric Coincidence Condition

Most of our work is done for substitutions that satisfy the *Geometric Coincidence Condition*. It is not known if all IUP substitutions satisfy this condition. Barge proved in [BD02] that any IUP substitution in two letters satisfies the Geometric Coincidence Condition.

The following definition is from [BK06].

Definition 2.73 *Let φ be an IUP substitution, and $\|\cdot\|_u$ be the unstable norm. We say that two edges S and S' of strands T and T', respectively, are* stably related *if $\|p - p'\|_u = 0$ for some points $p \in S$ and $p' \in S'$.*

In what follows $h_\varphi : \mathcal{F}_\varphi \to \mathbb{T}^d$ represents geometric realization. There are many equivalent formulations of GCC in [BD07]. The following is an equivalent formulation from Proposition 17.1 in [BD07].

Definition 2.74 *An IUP substitution φ satisfies the* geometric coincidence condition *provided that, whenever S and S' are two stable related edges such that $h_\varphi(S) = h_\varphi(S')$, then there is an $n > 0$ and edges $S_i \in \varphi^n(S)$, $S'_j \in \varphi^n(S')$ such that $S_i = S'_j$. We say that φ* satisfies GCC *if φ satisfies the geometric coincidence condition.*

Definition 2.75 *We say that a substitution φ is a* IUPC *substitution if φ is an irreducible unimodular Pisot substitution that satisfies the geometric coincidence condition.*

2.10 Inverse Limits

2.10.1 Basic Properties

We next define inverse limits and some of their properties. In Subsections 2.10.1 and 2.10.2 we follow [Nad92]. We omit the proofs in these two sections

Definition 2.76 *A continuum is a non-empty, compact, connected metric space.*

It is well now (see [Nad92]) that the nested intersection of continua is a continuum.

Definition 2.77 *An* inverse sequence *is a sequence $\{X_i, f_i\}_{i\in\mathbb{N}}$ of spaces X_i called* coordinate spaces, *and continuous functions $f_i : X_{i+1} \to X_i$, called* bonding maps.

Inverses sequences $\{X_i, f_i\}_{i\in\mathbb{N}}$ are written as $\{X_i, f_i\}_{i=0}^{\infty}$ or as

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \dots \xleftarrow{f_{i-1}} X_i \xleftarrow{f_i} X_{i+1} \xleftarrow{f_{i+1}} \dots$$

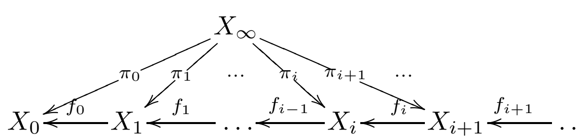
Definition 2.78 *If $\{X_i, f_i\}_{i\in\mathbb{N}}$ is an inverse sequence, then the* inverse limit, *denoted $\varprojlim \{X_i, f_i\}_{i=0}^{\infty}$, is the subspace of the cartesian product space $\prod_{i=0}^{\infty} X_i$ defined by*

$$\varprojlim \{X_i, f_i\}_{i=0}^{\infty} = \left\{ (x_i)_{i=1}^{\infty} \in \prod X_i : f_i(x_{i+1}) = x_i \text{ for all } i \right\}.$$

Another notation for the inverse limit $\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$ is X_∞, and we write $X_\infty = \varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$ as

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \dots \xleftarrow{f_{i-1}} X_i \xleftarrow{f_i} X_{i+1} \xleftarrow{f_{i+1}} \dots X_\infty$$

Let $\pi_i : \prod_{j\in\mathbb{N}} X_j \to X_i$ be the projection into the i-th coordinate. Abusing notation, we also denote by π_i the restriction $\pi_i|_{X_\infty}$. We have the commuting diagram



If X is a non-empty compact space, and $f : X \to X$ is a continuous map, the notation $\varprojlim \{X, f\}$ stands for the inverse limit with constant inverse sequence $\{X_n, f_n\} = \{X, f\}$ for $n \geqslant 0$.

In the case where, for all i, we have that $X_{i+1} \subset X_i$, and the bonding map $f_i : X_{i+1} \to X_i$ is an inclusion, we have a natural homeomorphism between $\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$ and $\bigcap \{X_i : i \in \mathbb{N}\}$. The next proposition states that inverse limits can always be viewed as nested intersections.

Proposition 2.79 *Let $\{X_i, f_i\}_{i\in\mathbb{N}}$ be an inverse sequence. For each $n \in \mathbb{N}$, define $Q_n(X_i, f_i)$ by $Q_n(X_i, f_i) = \left\{ (x_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \leqslant n \right\}$. Then (1)-(3) hold:*

1. $Q_{n+1}(X_i, f_i) \subset Q_n(X_i, f_i)$ *for all* $n \in \mathbb{N}$

2. $Q_n(X_i, f_i)$ *is homeomorphic to* $\prod_{i=n+1}^{\infty} X_i$ *for each* $n \in \mathbb{N}$

3. $\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}} = \bigcap_{n=1}^{\infty} Q_n(X_i, f_i)$

Corollary 2.80 *An inverse limit of continua is a continuum.*

Also, an inverse limit of compact metric spaces is a nonempty compact metric space.

Proposition 2.81 *Let* $X_\infty = \varprojlim \{X_n, f_n\}_{n\in\mathbb{N}}$ *and* $Y_\infty = \varprojlim \{Y_n, g_n\}_{n\in\mathbb{N}}$ *be inverse limits of compact spaces. Suppose that the following diagram commutes:*

$$\begin{array}{ccccccccc} X_0 & \xleftarrow{f_0} & X_1 & \xleftarrow{f_1} \dots & \xleftarrow{f_{i-1}} & X_i & \xleftarrow{f_i} & X_{i+1} & \xleftarrow{f_{i+1}} & \dots \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & & \varphi_i \downarrow & & \varphi_{i+1} \downarrow & & \\ Y_0 & \xleftarrow{g_0} & Y_1 & \xleftarrow{g_1} \dots & \xleftarrow{g_{i-1}} & Y_i & \xleftarrow{g_i} & Y_{i+1} & \xleftarrow{g_{i+1}} & \dots \end{array}$$

Define φ_∞ *by* $\varphi_\infty\left((x_i)_{i=0}^{\infty}\right) = \left(\varphi_i(x_i)\right)_{i=0}^{\infty}$. *Then (a)-(d) hold:*

a) φ_∞ *maps* X_∞ *into* Y_∞*;*

b) *if each* φ_i *is continuous, then* φ_∞ *is continuous;*

c) *if each* φ_i *is one-to-one, then* φ_∞ *is one-to-one*

d) *if each* φ_i *is continuous and onto, then* φ_∞ *is continuous and onto.*

Proposition 2.82 *Let* $\{S_i, f_i\}_{i\in\mathbb{N}}$ *be an inverse sequence where each* S_i *is a nonempty topological space. Assume that* $r : \mathbb{N} \to \mathbb{N}$ *is increasing. Let*

$$f_{r(j),r(j+1)} = \begin{cases} f_{r(j)} \circ f_{r(j)+1} \circ \dots \circ f_{r(j+1)-1} : S_{r(j+1)} \to S_{r(j)} & \text{if } r(j+1) > r(j)+1 \\ f_{r(j)} : S_{r(j+1)} \to S_{r(j)} & \text{if } r(j+1) = r(j)+1 \end{cases}.$$

Then $\varprojlim \left\{S_{r(j)}, f_{r(j),r(j+1)}\right\}_{j\in\mathbb{N}}$ *and* $\varprojlim \{X_n, f_n\}$ *are homeomorphic*

Corollary 2.83 *If $k \in \mathbb{Z}^+$, then* $\varprojlim \{X_n, f_n\}_{n\in\mathbb{N}} \cong \varprojlim \{X_n, f_n\}_{n=k}^{\infty}$

Corollary 2.84 *If $k \in \mathbb{Z}^+$, then* $\varprojlim \{X, f\} \cong \varprojlim \{X, f^k\}$.

Definition 2.85 *An inverse sequence $\{X_i, f_i\}_{i\in\mathbb{N}}$ where each X_i is a continuum is called an* indecomposable *inverse sequence provided that, for each $i \in \mathbb{N}$, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i(A_{i+1}) = X_i$ or $f(B_{i+1}) = X_i$*

Subcontinua of inverse limits can be represented as inverse limits of subcontinua in each factor:

Proposition 2.86 *Let $\{X_i, f_i\}_{i\in\mathbb{N}}$ be an inverse sequence of metric spaces with inverse limit X_∞. For each $i \in \mathbb{N}$, let $\pi_i : X_\infty \to X_i$ be the i-th projection map. Let A be a compact subset of X_∞. Then, $\left\{\pi_i(A), f_i|_{\pi_{i+1}(A)}\right\}_{i=1}^{\infty}$ is an inverse sequence with onto bonding maps and*

$$\varprojlim \{\pi_i(A), f_i|\pi_{i+1}(A)\}_{i\in\mathbb{N}} = A = \left[\prod_{i\in\mathbb{N}} \pi_i(A)\right] \cap X_\infty$$

Definition 2.87 *A continuum X is said to be* indecomposable *provided that, whenever A and B are proper subcontinua of X such that $X = A \cup B$, we have that $A = X$ or $B = X$.*

Inverse limits provide a tool for producing examples of indecomposable continua.

Proposition 2.88 *If $\{X_i, f_i\}_{i\in\mathbb{N}}$ is an indecomposable inverse sequence with inverse limit X_∞, then X_∞ is an indecomposable continuum.*

We use the following property from General Topology known as transgression. Recall that an identification $p : X \to Y$ is a continuous surjection such that, for any

$A \subset Y$, we have that A is open if and only if $p^{-1}(A)$ is open. It is also useful to consider for the following proposition the diagram

$$\begin{array}{ccc} Z & \overset{h}{\leftarrow} & X \\ & \nwarrow^{\eta} & \downarrow p \\ & & Y \end{array}$$

Proposition 2.89 *Let $p : X \to Y$ be an identification, and $h : X \to Z$ be continuous. If, for every $y \in Y$, we have that $h(p^{-1}(y))$ is a singleton (i.e., h is constant on the fibers of p), then the map $\eta : Y \to Z$, where $\eta(y)$ is the unique element in $h(p^{-1}(y))$ is continuous.*

Proof See [Dug66] □

The previous proposition is useful to find continuous maps in quotient spaces. For example, if $\beta : X \to X$, $\alpha : A \to A$ and $\pi : B \to A$ are continuous maps such that the following diagram commutes

$$\begin{array}{ccc} B & \overset{\beta}{\longleftarrow} & B \\ \pi \downarrow & & \pi \downarrow \\ A & \overset{\alpha}{\longleftarrow} & A \end{array}$$

,and such that π is constants in the fibers of β, then there is a continuous map $\eta : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \overset{\beta}{\longleftarrow} & B \\ \pi \downarrow & \nwarrow^{\eta} & \pi \downarrow \\ A & \overset{\alpha}{\longleftarrow} & A \end{array}$$

The importance of this is that, for identifications π satisfying the previous requirements, we obtained induced homeomorphic tiling spaces.

Proposition 2.90 *Suppose that $\beta : X \to X$, $\alpha : A \to A$ and $\pi : B \to A$ are continuous maps, and suppose that*

$$\begin{array}{ccccccccccc} B & \xleftarrow{\beta} & B & \xleftarrow{\beta} \cdots & \xleftarrow{\beta} & B & \xleftarrow{\beta} & B & \xleftarrow{\beta} & \cdots \\ \pi\downarrow & \nwarrow^{\eta} & \pi\downarrow & \nwarrow^{\eta} & \nwarrow^{\eta} & \pi\downarrow & \nwarrow^{\eta} & \pi\downarrow & \nwarrow^{\eta} & \\ A & \xleftarrow{\alpha} & A & \xleftarrow{\alpha} \cdots & \xleftarrow{\alpha} & A & \xleftarrow{\alpha} & A & \xleftarrow{\alpha} & \cdots \end{array}$$

that induces an homeomorphism of the inverse limits $\varprojlim \{\alpha, A\}$ and $\varprojlim \{\beta, B\}$ (See [Nad92], page 26).

2.10.2 The Solenoid

Recall that $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 2.91 *For an integer $p \geqslant 2$, let $f^p : S^1 \to S^1$ be given by $f^p(z) = z^p$. The* p-adic solenoid *is the inverse limit $\varprojlim \{S^1, f^p\}$.*

The p-adic solenoid is an example of an indecomposable continuum by 2.88. Propositions 2.81, 2.89 and 2.90 can be used to show that the p-adic solenoid is homeomorphic to the nested intersection of tori in which each torus is wrapped p times inside of the interior of the previous torus (see [Nad92] for details).

The solenoid is locally the product of a cantor set and an in interval, and it can be shown that it is *homogeneous*, that is, for any two points in the solenoid, there is a homeomorphism of the solenoid onto itself that maps one point to the other (see [Nad92] for details).

2.10.3 Tiling Spaces as Inverse Limits

In [AP98], Anderson and Putnam gave a presentation of tiling spaces of substitution tiling spaces (of any dimension) as inverse limits of simplicial complexes. Their representation has as the space X a simplicial complex built based on the way in which translations of prototiles are adjacent in a particular tiling.

In the article [BD01], Barge and Diamond follow a different approach from that of Anderson and Putnam in [AP98] to study one-dimensional tiling spaces. In their treatment, Barge and Diamond do not use collaring, but rather transform the given substitution into a proper one without altering the tiling space. We follow [BD01], where the proofs of the results of this subsection can be found.

Definition 2.92 *Let $\varphi : \mathcal{A} = \{1, 2, \ldots, d\} \to \mathcal{A}^+$ be a primitive substitution, and let $v_L = (\lambda_1, \ldots, \lambda_d)$ be the left Perron eigenvector for $[\varphi]$. The associated* wedge of d circles*, is the set R_φ consisting of d oriented circles R_i that are disjoint apart from one common point where they are tangent, and such that the perimeter of R_i is λ_i.*

Note that the wedge of circles associated to a primitive substitution can easily be constructed in $\mathbb{R}^3$.

Given a substitution $\varphi : \mathcal{A} \to \mathcal{A}^+$, with $\#(\mathcal{A}) = d > 2$, such that $\varphi(i) = a_{i_1} a_{i_2} \ldots a_{i_{|\varphi(i)|}}$,. the *map of the rose* f_φ is the 'linear' map, with expansion constant λ_φ, that follows the pattern of φ; that is, f_φ is the map that sends the circle R_i to an interval $[0, \lambda_i]$ by cutting it to the common point of the R_i, then expands it by a factor of λ_φ to obtain an interval $I = [0, \lambda_\varphi \lambda_i] = \left[0, \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_{|\alpha(i)|}}\right]$, and then locally isometrically maps the interval I to the circles $R_{i_1}, R_{i_2}, \ldots, R_{i_{|\alpha(i)|}}$ in that order.

A wedge of d circles is also called a rose with d circles, which explains the term "map of the rose". We retain the term "wedge of d circles", though we still call the map f_φ the map of the rose.

The tiling space of a substitution φ in d letters is closely related to the inverse limit $\varprojlim \{R_\varphi, f_\varphi\}$. In the case of proper substitutions we have a homeomorphism. The proofs of the following two propositions are in [BD01]

Proposition 2.93 *Let φ be a primitive proper non-periodic substitution on d letters. Then $\mathcal{T}_\varphi \cong \varprojlim \{R_\varphi, f_\varphi\}$, where R_φ is the wedge of d circles associated to φ.*

Proposition 2.94 *Given a primitive aperiodic substitution φ, there is a primitive aperiodic proper substitution χ whose tiling space $\mathcal{T}_\chi$ is homeomorphic to the tiling space $\mathcal{T}_\varphi$.*

As consequences of the previous propositions, tiling spaces of primitive substitutions are closely related to solenoids in that they are indecomposable by Proposition 2.88 and, locally, the product of a Cantor set and an interval by Proposition 2.86. However, tiling spaces are not homogeneous (see [BD01]). Such inhomogeneities were used by Barge and Diamond in [BD01] to find a complete invariant for tiling spaces.

Definition 2.95 *Let X be a continuum. The* composant *of a point p is the union of all proper compact connected subsets of X that contain p. A* composant *of X is a composant of some point $p \in X$.*

For substitution tiling spaces, the only proper subcontinua are arcs by Proposition 2.86, thus composants and arc components coincide. Further, the composant of a tiling T is the orbit, under the tiling flow, of T.

As a consequence of the first vertex map the composant of a strand S is uniquely determined by the bi-infinite word $[S]$, where we disregard the location of the center.

CHAPTER 3

REWRITINGS, SHIFT EQUIVALENCE AND RIGIDITY

Rewriting and shift equivalence are methods that modify a substitution but leave the tiling space intact up to homeomorphism. In this chapter, we present several methods, taken from [BD01] and [Dur98], to create new substitutions from old. We define shift equivalences and rewritings with respect to starting and stopping rules. The methods to obtain new substitutions from old will be classified as either shift-equivalences or rewritings with starting and stopping rules.

We present a result from [Dur98] that gives the relationship between the eigenvalues of the new substitutions and the old substitutions. We provide a different proof from that of Durand to express for rewritings the eigenvalues induced by the original substitution in terms of the eigenvalues induced by the new substitution.

We also compute explicitly one rewriting of the substitution $\alpha = \varphi^6$, where $\varphi : 1 \to 32, 2 \to 1, 3 \to 2$, into the substitution $\beta : 1 \mapsto 1234356,\ 2 \mapsto 12356,\ 3 \mapsto 136,\ 4 \mapsto 14234356,\ 5 \mapsto 142356,\ 6 \mapsto 14236$. In this example, the characteristic polynomial of the abelianization matrix for α is $x^3 - 5x^2 - 2x - 1$, whereas the characteristic polynomial for β is $x(x-1)^2(x^3-5x^2-2x-1)$. As the result from Durand in [Dur98] shows, this situation is typical.

Finally, we discuss the relevance of this classification in terms of the Barge-Swanson rigidity theorem in [BS07].

3.1 The Idea Behind Rewriting

There are two main methods to create new substitutions. One is via shift equivalence, and the other one is through a method called rewriting. This section discusses rewritings. We also compute a specific example.

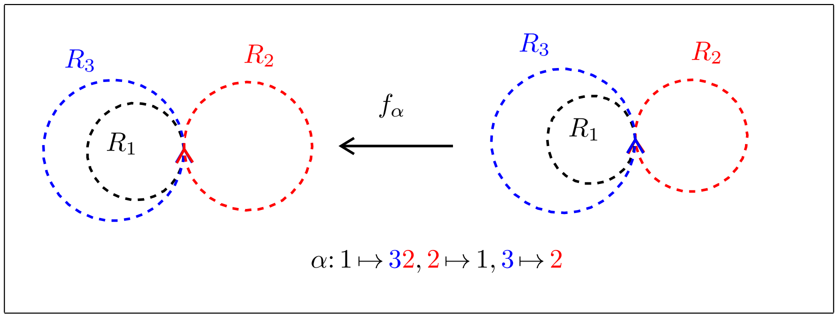


Figure 3.1: The mapping f_α

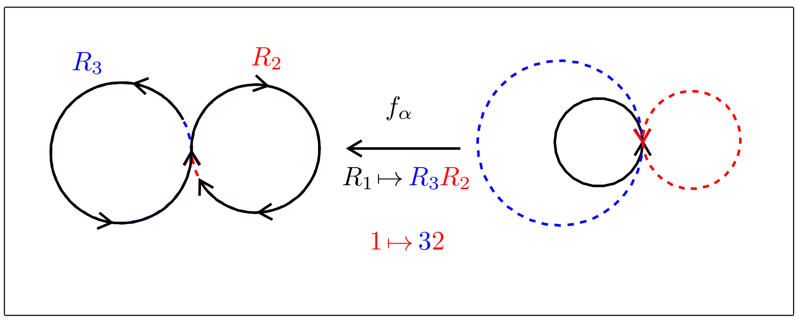


Figure 3.2: The mapping $R_1 \mapsto R_3R_2$

One way to understand rewriting is through the *Williams moves* in [Wil70]. Let $\mathcal{A} = \{1, 2, \dots, d\}$, and consider the substitution $\alpha : \mathcal{A} \to \mathcal{A}^+$. Let R_α be the wedge of d circles and $f_\alpha : R_\alpha \to R_\alpha$ the map of the rose as defined in Section 2.10.3. In the following figures we make some drawings for the substitution $\alpha : \mathcal{A} \to \mathcal{A}^+$, with $\mathcal{A} = \{1, 2, 3\}$, and such that $1 \mapsto 32, 2 \mapsto 1, 3 \mapsto 2$

In general, given a map $\alpha : \mathcal{A} \to \mathcal{A}^+$ on d letters, we let p be a fixed point for the map of the rose $f_\alpha : R_\alpha \to R_\alpha$. and suppose that $p \in R_{i_0}$. Traverse the circle R_{i_0} isometrically starting at the point p and ending in the point p. Since p is fixed, applying f_α to the circle R_{i_0}, maps the circle R_{i_0} into the other circles, but it does so starting and finishing at the point p. Applying the map f_α over and over again, we produce paths each of which can be factored into subpaths starting and finishing at

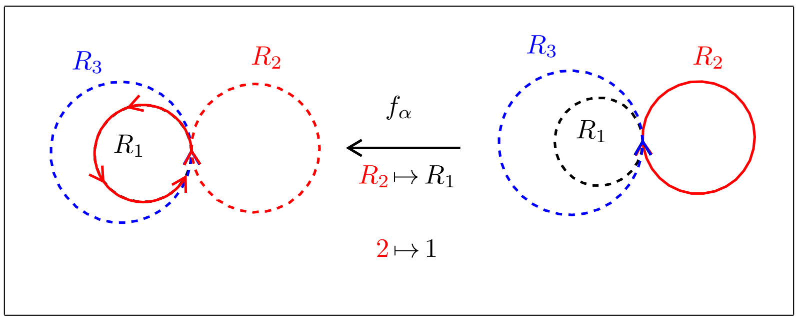


Figure 3.3: The mapping $R_2 \mapsto R_1$

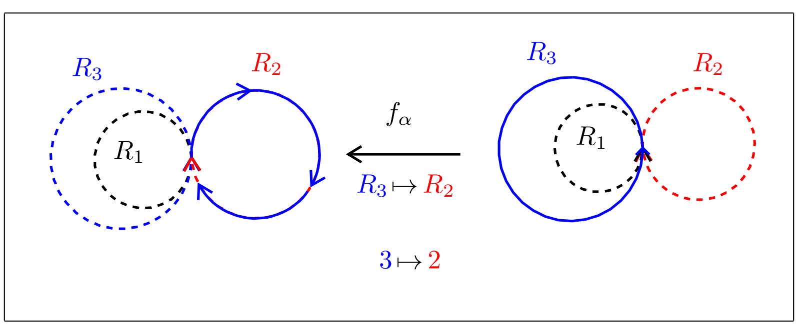


Figure 3.4: The mapping $R_3 \mapsto R_2$

the point p in such a way that the paths only pass through p at the beginning of the path, and on the end. Let P be the set of resulting paths starting and finishing at p, but not visiting p in the middle.

There are finitely many such paths constructed in this way. With each of those paths we can create a map of the rose on a wedge of $\#(P)$ circles. This creates a new substitution. In this chapter we present several ways to create new substitutions from old, and many constructions have an interpretation in terms of the map of the rose.

3.2 Rewriting with Starting and Stopping Rules

In this section, we follow [BD01].

Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a primitive non shift periodic substitution, and suppose $S \subset \mathcal{A}$, $P \subset \mathcal{A}$ are sets of letters such that for all $s \in S, p \in P$, there are $s' \in S, p' \in P$, such that $\alpha(s) = s' \ldots$, and $\alpha(p) = \ldots p'$. Suppose that there are $p_0 \in P, s_0 \in S$ and a bi-infinite word $W_\alpha = \ldots p_0.s_0 \ldots$ such that $\alpha(W_\alpha) = W_\alpha$, and such that $p_0 s_0$ is allowed.

By Proposition 2.23, the word W_α is uniformly recurrent under the shift map. Hence, the word W_α can be factored uniquely as a concatenation of finite words from a collection $\mathcal{W}$, where each $w \in \mathcal{W}$ starts with some $s \in S$, ends with some $p \in P$, and contains no factors of the form ps, where $p \in P$ and $s \in S$.

By Proposition 2.23, the set $\mathcal{W}$ is finite. We can obtain the words of $\mathcal{W}$ in the following way: For any occurrence of a two-letter factor of W_α of the form ps with $p \in P$ and $s \in S$, we put a divider between the letter p and the letter s. The words between any consecutive dividers are the words in $\mathcal{W}$. Figure 3.5 shows how the division is made.

$$x = \ldots | \underbrace{\ldots \overbrace{x_{-7}x_{-6}}^{\in P}}_{\in W} | \underbrace{\overbrace{x_{-5}x_{-4}}^{\in S} x_{-3}x_{-2} \overbrace{x_{-1}x_{0}}^{\in P}}_{\in W} | \underbrace{\overbrace{x_{1}x_{2}x_{3}x_{4}}^{\in S} x_{5} \overbrace{x_{6}x_{7}}^{\in P}}_{\in W} | \underbrace{\overbrace{x_{8}}^{\in S} \ldots}_{\in W} \ldots$$

Figure 3.5: Division of a bi-infinite word according to starting rules and stopping rules.

We enumerate the words $\mathcal{W} = \{w_1, w_2, \ldots w_{\#\mathcal{W}}\}$ thus obtained according to their first appearance in the right hand side of W_α. Let $\mathcal{B} = \{1, 2, \ldots, \#\mathcal{W}\}$, and let $\pi : \mathcal{B} \to \mathcal{A}^+$ be the morphism given by $\pi(i) = w_i$.

We can read $\pi(i)$ as the i-th new word appearing in the factorization of the right hand side of W_α.

For any $i \in \mathcal{B}, w_i \in \mathcal{W}$, we can factor $\alpha(w_i)$ into words $\mathcal{W}$. Let $k(i)$ be the number of factors from $\mathcal{W}$ into which we factor $\alpha(w_i)$.Thus, for any w_i, there are words $w_{i_1}, w_{i_2}, \ldots, w_{i_{k(i)}} \in \mathcal{W}$ such that $\alpha(w_i) = w_{i_1}w_{i_2}\ldots w_{i_{k(i)}}$. We then can form the substitution $\beta : \mathcal{B} \to \mathcal{B}^+$ given by $\beta(i) = i_1 i_2 \ldots i_{k(i)}$ for all $i \in \mathcal{B}$.

The computations done for the following example are used in chapter 6.

Example 3.1 For example, consider the substitution $\alpha : 1 \mapsto 1323221, 2 \mapsto 21132, 3 \mapsto 3221$ (α is the sixth iterate of $1 \mapsto 32, 2 \mapsto 1, 3 \mapsto 2$), and let us note that we can take $S = \{1\}$ as starting rules and $P = \{1\}$ as stopping rules. We consider the fixed point $W_\alpha = \ldots 1.1 \ldots$ under α. Note that we can obtain arbitrary long words of the right hand side of W_α by simply iterating the letter 1 under α.

Next we write $\alpha^3(1)$, cutting whenever we see a 11.

$$...1.\underbrace{1323221322121}_{w_1}\underbrace{132322121}_{w_2}\underbrace{13221}_{w_3}\underbrace{1321323221322121}_{w_4}$$

$$\underbrace{13221}_{w_3}\underbrace{132132322121}_{w_5}\underbrace{1321323221}_{w_6}1...$$

We note that 6 different words w_1, w_2, w_3, w_4, w_5 and w_6 have resulted.

In fact, to find all words in $\mathcal{W}$, it suffices to find one word w in $\mathcal{W}$, then factor the word $\alpha(w)$ obtaining more words of $\mathcal{W}$ and repeat the process until no new words are obtained. The word $\alpha(w_i)$, $i \in \{1,2,3,4,5,6\}$ factors as follows:

$$\begin{array}{rcl} w_1 & \mapsto & w_1w_2w_3w_4w_3w_5w_6 \\ w_2 & \mapsto & w_1w_2w_3w_5w_6 \\ w_3 & \mapsto & w_1w_3w_6 \\ w_4 & \mapsto & w_1w_4w_2w_3w_4w_3w_5w_6 \\ w_5 & \mapsto & w_1w_4w_2w_3w_5w_6 \\ w_6 & \mapsto & w_1w_4w_2w_3w_6 \end{array}$$

From the factorization above, we set $\mathcal{B} = \{1,2,3,4,5,6\}$, and $\pi : \mathcal{B} \to \mathcal{A}^+$ given by $\pi(i) = w_i$, for $i \in \mathcal{B}$. The resulting substitution $\beta : \mathcal{B} \to \mathcal{B}$ is

$$1 \mapsto 1234356, 2 \mapsto 12356, 3 \mapsto 136, 4 \mapsto 14234356, 5 \mapsto 142356, 6 \mapsto 14236.$$

Example 3.2 Consider again the substitution $\alpha : 1 \mapsto 1323221, 2 \mapsto 21132, 3 \mapsto 3221$, but this time set as starting rules $S = \{1,2,3\}$ and as stopping rules $P = \{1,2\}$. And take the fixed word obtained by iterating 1.1 under α, and cutting it whenever we see a word of the form $11, 12, 13, 21, 22, 23$:

$$...1.\underbrace{1}_{w_1}\underbrace{32}_{w_2}\underbrace{32}_{w_2}\underbrace{2}_{w_3}\underbrace{1}_{w_1}\underbrace{32}_{w_2}\underbrace{2}_{w_3}\underbrace{1}_{w_1}...$$

We note that only three words appear: $w_1 = 1$, $w_2 = 32$ and $w_3 = 2$; and we get:

$\alpha(1) = 132\ 32\ 2\ 1$, $\alpha(32) = 32\ 2\ 1\ 2\ 1\ 1\ 32$ and $\alpha(2) = 2\ 1\ 1\ 32$. Hence, we get $\mathcal{B} = \{1,2,3\}$ with $\pi : \mathcal{B} \to \mathcal{A}^+$, given by $\pi(1) = 1$, $\pi(2) = 32$ and $\pi(3) = 2$. Thus, the induced substitution $\beta : \mathcal{B} \to \mathcal{B}$ is given by $\beta(1) = 12231$, $\beta(2) = 2313112$ and $\beta(3) = 3112$. This is the same as the one we originally had by making the change of letters $\mathcal{B} \to \mathcal{A}$: $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3$.

In [BD01] it is shown that there are only finitely many possible rewritings using starting rules and stopping rules. This mimics the result of Durand for return words in [Dur98].

3.3 Rewriting Using Return Words

The following discussion follows [Dur98].

A right-infinite word $W = x_0 x_1 \ldots$ is said to be *primitive* provided that any prefix of W appears infinitely often in W, and is said to be *minimal* provided that for any factor u of W there exists $L > 0$ such that any factor of W of length greater than L contains u. Given a prefix u, we can determine a way to "cut" W into certain words: Let u be a prefix of W, and let $C_u = \left\{i \in \mathbb{N} : \ u = x_i x_{i+1} \ldots x_{i+|u|-1}\right\}$. We order the elements of C_u in increasing order so that $C_u = \{i_0, i_1, \ldots, \}$ is an increasing sequence of numbers. A word of the form $x_{i_n} x_{(i_n+1)} \ldots w_{(i_{n+1}-1)}$ is called a *return word*.

For a minimal primitive word W, denote the collection of all return words as $\mathcal{W}_u$, and note that $\#\mathcal{W}_u$ is finite so we can enumerate the words according to their appearance in W; that is, if the first occurrence of a return word x appears before the first appearance of the return word y, then we assign an index to x smaller than the one we assign to y. Thus, we order the return words as $\mathcal{W} = \{w_1, w_2, \ldots w_{\#\mathcal{W}}\}$. Now, let $\pi : \{1, \ldots, \#\mathcal{W}\} \to \mathcal{W}$ given by $\pi(i) = w_i$, and let $D_u(W)$ to be the unique sequence on $\{1, \ldots, \#\mathcal{W}\}$ with the property that $\pi(D_u(W)) = W$. The sequence $D_u(W)$ is called the *derived* sequence. In [Dur98], it is shown that if W is primitive and minimal, then so is $D_u(W)$.

It is shown also in [Dur98], though we do not use it, that a right infinite word W is a fixed point of some substitution φ if and only if $\{D_u(W) : \ u \text{ is a prefix of } W\}$ is a finite set.

Given a substitution φ, a right-infinite word W fixed by φ and a prefix u of W, we find the *derived substitution* $\varphi_{W,u}$ as follows: For each $i \in \mathcal{A}$ there are $k(i)$ unique return words $w_{i_0}, w_{i_1}, \ldots, w_{i_{k(i)}}$ such that $\varphi(w_i) = w_{i_1} \ldots w_{i_{k(i)}}$. We define $\varphi_{W,u}(i) = i_1, \ldots, i_{k(i)}$. If φ is primitive and non shift-periodic, then so is $\varphi_{W,u}$ (see [Dur98]). We sometimes denote $\varphi_{W,u}$ as φ_u if it is clear which fixed word W we are using

In what follows we use the following property:

Proposition 3.3 *Let W be a right-infinite minimal word, and let v be a non-empty prefix of $D_u(W)$, and let $w = \pi(v)u$. Then w is a prefix of W, and $D_v(D_u(W)) = D_w(W)$*

Proof See [Dur98] □

3.4 Rewriting with Starting and Stopping Sets of Words

This section closely follows section 3.2, with the exception that words take the place of letters. Recall that if W is a right infinite word $W = x_0x_1\ldots$, then $W_{[a,b)} = x_ax_{a+1}\ldots x_{b-1}$

Suppose we are given a substitution $\alpha : \mathcal{A} \to \mathcal{A}^+$, and sets $P = \{p_1, \ldots, p_k\}$ and $S = \{s_1, \ldots, s_k\}$ of nonempty allowed words. We say that P is a set of *stopping rules* if for each $p \in P$ there is a word $p' \in P$ such that $\alpha(p) = \ldots p'$. We say that S is a set of *starting rules* if for each $s \in S$ there is a word $s' \in S$ such that $\alpha(s) = s' \ldots$.

Let S and P be starting and stopping rules, respectively. Let W be a bi-infinite word W allowed for α, fixed by α, and of the form $W = \ldots p_0.s_0\ldots$, with $p_0 \in P$ and $s_0 \in S$. Let $C = \{i \geqslant 0 : \ \sigma^i(W) {=} \ldots p.s \ldots$ for some $p \in P$ and $s \in S\}$, that is, C is the set of all indices in the right part of W at which a word of the form ps occurs for some $p \in P$ and $s \in S$. Figure 3.5 shows how the subdivision is done also in this case.

Just as we did for return words, we can arrange the indices in C in increasing order $0 = i_0 < i_1 < i_2 < \ldots$. The set $\mathcal{W} = \left\{u : \ u = W_{[i_k,i_{k+1})}\right\} \subset \mathcal{L}_\alpha$ of factors of W is, by uniform recurrence, finite, and we can enumerate it as $\mathcal{W} = \{w_1, \ldots, w_{\#\mathcal{W}}\}$. Order the set $\mathcal{W}$ in such a way that w_1 is the word $W_{[0,i_1)}$, and w_{k+1} is the word $W_{[i_k,i_{k+1})}$ with the smallest index i_k such that $W_{[i_k,i_{k+1})} \notin \left\{W_{[i_l,i_{l+1}]} : \ l < k\right\}$; that is, the order of the words in $\mathcal{W}$ is such that the first occurrence of w_k in $W_{[0,\infty)}$ appears to the left of the first occurrence of w_{k+1} in $W_{[0,\infty)}$. Notice that if a word is of the form $psup's' \in \mathcal{L}_\alpha$, with $u \in \mathcal{L}_\alpha$ $p, p' \in P$ and $s, s' \in S$, then there exists a unique set of words $\{w_{i_1}, \ldots, w_{i_k}\} \subset \mathcal{W}$ such that $sup' = w_{i_1} w_{i_2} \ldots w_{i_k}$.

Note that, by primitivity, the set $\mathcal{W}$ does not depend on the word W. In fact, any bi-infinite word allowed for α would lead to the same set $\mathcal{W}$. It is, however, customary to order the words in $\mathcal{W}$ according to their appearance in W.

Let $\mathcal{B} = \{1, \ldots, \#\mathcal{W}\}$. We define the morphism $\pi : \mathcal{B} \to \mathcal{A}^+$ as follows: $\pi(i) = w_i$, for $i \in \mathcal{B}$. We define the *rewriting* of α by starting rules S and stopping rules P to be the unique substitution $\beta : \mathcal{B} \to \mathcal{B}^+$ satisfying that $\alpha \circ \pi = \pi \circ \alpha$; that is, we define $\beta(i) = i_1 i_2 \ldots i_k$ provided that $\alpha(w_i) = w_{i_1} w_{i_2} \ldots w_{i_k}$, and we have the following commuting diagram

$$\begin{array}{ccc} \mathcal{B} & \stackrel{\beta}{\to} & \mathcal{B}^+ \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{A} & \stackrel{\alpha}{\to} & \mathcal{A}^+ \end{array}$$

There is a relationship between rewriting with respect to starting and stopping rules and the substitution on return words. Suppose that $\alpha : \mathcal{A} \to \mathcal{A}^+$ is a primitive substitution and that W is a right infinite word that is fixed under alpha. Let u be a prefix of W, and let W' be a bi-infinite word, allowed by α, of the form $\ldots x.W$ for some $x \in \mathcal{A}$. Cutting the fixed right-infinite word W at the occurrences of u is the same as cutting W' at the occurrence of words of the form au, where $a \in \mathcal{A}$. Thus, we obtain the following well known result:

Proposition 3.4 *Let $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ be a primitive substitution. The substitution α_u on return words associated with the prefix u of a fixed right-infinite word W is the same as the rewriting β obtained by taking as stopping rules the set $P = \mathcal{A}$, and as set of starting rules the set $S = \{u\}$.*

There is a diagram for rewritings with starting rules and stopping rules that closely follows the formulas II.2 and II.3 that Durand obtained in [Dur98] for return words. The arguments closely follow Durand's, and we provide just a sketch of the argument of this well-known result.

Proposition 3.5 *Let $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ be a substitution, and let $\beta : \mathcal{B} \to \mathcal{B}^{+}$ be a rewriting of α with starting rules and stopping rules. Let $\pi : \mathcal{B} \to \mathcal{A}^{+}$ be the induced mapping such that $\alpha \circ \pi = \pi \circ \beta$. Then there is an R such that for each m sufficiently large, there is a substitution $\eta_m : \mathcal{A} \to \mathcal{B}^{+}$ such that in the non-commutative diagram*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta^m} & \mathcal{B}^{+} \\ \pi \downarrow & {}^{\eta_m} \nearrow & \downarrow \pi \\ \mathcal{A} & \xrightarrow[\alpha^m]{} & \mathcal{A}^{+} \end{array}$$

we have that the matrices $[\eta_m]\,[\pi] - [\beta]^m$ and $[\pi]\,[\eta_m] - [\alpha^m]$ have entries bounded by R.

Sketch of proof Let α be a primitive substitution, and let S and P be sets of starting rules and stopping rules, respectively. Let $W = \ldots p_0.s_0 \ldots$ be an allowed bi-infinite word for α with $p_0 \in P$ and $s_0 \in S$. Since α is primitive, by Proposition

2.23, we have that the word W is uniformly recurrent, and thus words of the form ps, with $p \in P$ and $s \in S$, appear infinitely often with bounded gaps.

Thus, there is a gap $R_0 > 0$ such that for any sufficiently large m, and for any letter i, the word $\alpha^m(i)$ can be factored as $\alpha^m(i) = \underbrace{\ldots p_1}_{p} \underbrace{s_1 \ldots p_n}_{w_{i,m}} \underbrace{s_n \ldots}_{q}$, where $|p|, |q| < R_0$ and $p_1 s_1$ and $p_n s_n$ are the first appearance and the last appearance, respectively, of words of the form ps with $p \in P$ and $s \in S$.

We can define the mapping $\eta_m : \mathcal{A} \to \mathcal{B}^+$ by $\eta_m(i) = \pi^{-1}(w_{i,m})$, where π^{-1} denote cutting according to the starting and stopping rules the word $w_{i,m}$, and obtaining the subindeces. That is, η_m is the unique substitution such that $\pi(\eta_m(i)) = w_{i,m}$.

Consider, for any given m, the non-commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta^m} & \mathcal{B}^+ \\ \pi \downarrow & {}^{\eta_m} \nearrow & \downarrow \pi \\ \mathcal{A} & \xrightarrow[\alpha^m]{} & \mathcal{A}^+ \end{array}$$

In this diagram, the entries of the matrix $[\eta_m][\pi] - [\beta]^m$ and of $[\pi][\eta_m] - [\alpha^m]$ are uniformly bounded by some number R not depending on m, since $|p|, |q| < R_0$. $\square$

3.5 Reduction of Alphabet and Other Identifications

It is useful in this section to think of the representations of inverse limits in terms of the map of the rose, in the light of Propositions 2.89 and 2.90.

3.5.1 Identifying Letters

Suppose that $\beta : \mathcal{B} \to \mathcal{B}^+$ is a substitution, and we have letters $i, j \in \mathcal{B}$ such that $\beta(i) = \beta(j)$, and let $\mathcal{A} = \mathcal{B} \setminus \{j\}$. Define the morphism $\pi : \mathcal{B} \to \mathcal{A}$ given by $\pi(k) = \begin{cases} k & \text{if } k \neq j \\ i & \text{if } k = j \end{cases}$, and define the morphism $\eta : \mathcal{A} \to \mathcal{B}^+$ given by $\eta(k) = \beta(k)$. Let $\alpha : \mathcal{A} \to \mathcal{A}^+$ be the substitution $\pi \circ \eta$.

The passage to this new substitution α corresponds, with regards of the mapping of the rose on the wedge of circles to identifying two circles if they map identically. Compare this construction with the identifications mentioned in Proposition 2.90. The substitutions α, β, π and η satisfy this commutative diagram:

$$\begin{array}{ccc} \mathcal{B} & \overset{\beta}{\to} & \mathcal{B}^{+} \\ \pi\downarrow & {}^{\eta}\nearrow & \downarrow\pi \\ \mathcal{A} & \underset{\alpha}{\to} & \mathcal{A} \end{array}$$

3.5.2 Solving for Prefix and Suffix Problems

Recall that a proper substitution is a substitution $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ such that there exists $n \in \mathbb{Z}^{+}$ and $b, e \in \mathcal{A}$ such that $\alpha^{n}(i) = b \ldots e$ for all $i \in \mathcal{A}$.

Definition 3.6 *We say that a substitution $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ has a* prefix problem *if there are letters $i, j \in \mathcal{A}$ and a word w such that $\varphi(i) = \varphi(j)\,w$. We say that α has a* suffix problem *if there are letters $i, j \in \mathcal{A}$ and a word W such that $\varphi(i) = w\varphi(j)$.*

In [BD01], Barge and Diamond describe a procedure that transforms a proper substitution α with a prefix problem into a proper substitution with no prefix problem. See [BD01] for details and proofs. We describe their procedure, which applies to any substitution with a prefix or a suffix problem, proper or not, though in the case that the substitution is not proper, this procedure will not be helpful to fix a prefix problem.

Let $\mathcal{A}$ be an alphabet, and let $i, j \in \mathcal{A}$ such that $i \neq j$. Let $\tau_{i}^{j} : \mathcal{A} \to \mathcal{A}^{+}$ be given by $\tau_{i}^{j}(i) = ij$ and $\tau_{i}^{j}(k) = k$, for $k \neq i$ which can be read as "suffixing j at i".

In [BD01], Barge and Diamond observe that any substitution with a suffix problem of the form $\alpha(i) = w\alpha(j)$ can be written as $\alpha = \eta \circ \tau_{i}^{j}$, where $\eta(i) = w$ and $\eta(k) = \alpha(k)$ for $k \neq i$, and then show that if the substitution α is proper, primitive and has a suffix problem, then the substitution $\beta_1 = \tau_i^j \circ \eta$ is proper and primitive. If, further, β is a proper substitution, then repeating this process several times, and applying reduction of alphabet when needed, leads to a substitution β_k that does not have a suffix problem.

For every substitution α with a suffix problem, proper or not, we have the commuting diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta_1} & \mathcal{B}^+ \\ \tau_i^j \downarrow & {}^{\eta}\nearrow & \downarrow \tau_i^j \\ \mathcal{A} & \underset{\alpha}{\rightarrow} & \mathcal{A} \end{array}$$

In a similar way, letting $\sigma_i^j(i) = ji$ and $\sigma_i^j(k) = k$ for $k \neq i$, which can be read as "prefixing j at i", we have that a substitution α having a prefix problem $\alpha(i) = \alpha(j)W$ can be written as $\alpha = \eta \circ \sigma_i^j$, where $\eta(k) = \alpha(k)$ for $k \neq i$, and $\eta(i) = W$. Let $\beta_1 = \sigma_i^j \circ \eta$. This procedure, using can be used to solve a prefix problem just as τ_i^j was used to fix a suffix problem. Regardless of the substitution α being proper, the substitution β_1 satisfies the following commutative diagram.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta_1} & \mathcal{B}^+ \\ \sigma_i^j \downarrow & {}^{\eta}\nearrow & \downarrow \sigma_i^j \\ \mathcal{A} & \underset{\alpha}{\rightarrow} & \mathcal{A} \end{array}$$

3.6 Splitting with Respect to Interior Points

Let $\beta : \mathcal{B} \to \mathcal{B}^+$ be a substitution such that $\beta(i_0) = w_1 i_0 w_2$, with $w_1, w_2 \in \mathcal{B}^+$. Let i_1 and i_2 be two symbols not in $\mathcal{B}$. Let $\mathcal{A} = (\mathcal{B} \setminus \{i_0\}) \cup \{i_1, i_2\}$. We define the map $\pi : \mathcal{B} \to \mathcal{A}^+$ as

$$\pi(i) = \begin{cases} i_1 i_2 & \text{if } i = i_0 \\ i & \text{if } i \neq i_0 \end{cases}.$$

Finally, define $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ given by

$$\alpha(i) = \begin{cases} \pi(w_1)\, i_1 & \text{if } i = i_1 \\ i_2 \pi(w) & \text{if } i = i_2 \\ \pi(\beta(i)) & \text{if } i \notin \{i_1, i_2\} \end{cases} .$$

We obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta} & \mathcal{B}^{+} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow[\alpha]{} & \mathcal{A} \end{array}$$

The previous procedure is also called *splitting the letter* i according to $\beta(i) = w_1 i_0 w_2$, *splitting with respect to an interior point*, or simply *splitting*. The following is noted in [BD01].

Proposition 3.7 *If a substitution* $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ *is obtained from* $\beta : \mathcal{B} \to \mathcal{B}^{+}$ *via splitting, then* β *is a rewriting of* α *with starting and stopping rules.*

Sketch of Proof Suppose that $\beta(i_0) = w_1 i_0 w_2$, and that $\alpha : \mathcal{A} \to \mathcal{A}^{+}$ is the substitution resulting from splitting the letter i_0. Rewrite α according to the stopping rules $P = \mathcal{A} \setminus \{i_2\}$ and the starting rules $S = \mathcal{A} \setminus \{i_1\}$. Note that every two letter word in $\mathcal{L}_\alpha$, except for the word $i_1 i_2$, is of the form ps for some word letter $p \in P$ and $s \in S$. Thus, any word in the substitutive system X_α factors in terms of the words $(\mathcal{B} \setminus \{i_0\}) \cup \{i_1 i_2\}$. By renaming the word $i_1 i_2$ as i_0, we obtain the original substitution β. $\square$

3.7 Shift Equivalence of Substitutions

In the previous sections we discussed several methods to create new substitutions and in all of them we had a diagram of the form

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta} & \mathcal{B}^+ \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{A}^+ & \xrightarrow[\alpha]{} & \mathcal{A}^+ \end{array}$$

where, $\alpha, \beta$are substitutions, and π is a morphism. In some cases, like reduction of alphabet and solving a prefix problem, there exists a morphism $\eta : \mathcal{A} \to \mathcal{B}^+$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta} & \mathcal{B}^+ \\ \pi \downarrow & {}^{\eta} \nearrow & \downarrow \pi \\ \mathcal{A}^+ & \xrightarrow[\alpha]{} & \mathcal{A}^+ \end{array}$$

When α and β are related in a diagram like the one above, we say that the substitutions α and β are *shift-equivalent with lag one*.

In the case of rewriting with respect to starting and stopping rules, and, as special cases, rewriting via return words and splitting, we do not obtain a shift equivalence of substitutions, but rather, by Proposition 3.5, for each m sufficiently large, the following non-commuting diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\beta^m} & \mathcal{B}^+ \\ \pi \downarrow & {}^{\eta_m} \nearrow & \downarrow \pi \\ \mathcal{A}^+ & \xrightarrow[\alpha^m]{} & \mathcal{A}^+ \end{array}$$

satisfies that the entries of the matrices $[\eta_m]\,[\pi] - [\beta]^m$ and of $[\pi]\,[\eta_m] - [\alpha^m]$ have a bound independent of m.

3.8 Eigenvalues for Rewritings and Shift Equivalencies

In this section we show that the eigenvalues of the abelianization matrix of a rewriting or a shift equivalence are related to those of the original substitution.

We start with a lemma whose proof can be found in [ME05], page 100. We present the proof adding some detail.

Lemma 3.8 *Suppose that $\alpha \in \mathbb{C}$ is an algebraic integer in the unit circle $S^1 \subset \mathbb{C}$, and that all other conjugates of α are also in the unit circle. Then α is a root of unity.*

Proof Let $G = \{\alpha = \alpha_1, \ldots, \alpha_d\}$ be the Galois conjugates for α, and let $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_d)$ be the minimal polynomial for $\alpha = \alpha_1$ and such that $|\alpha_i| = 1$ for all $i \in \{1, \ldots, d\}$.

For each $n \in \mathbb{Z}^+$, since $|\alpha_i| = 1$, the polynomial

$$\begin{aligned} P_n &= (x - \alpha_1^n)(x - \alpha_2^n) \cdots (x - \alpha_d^n) \\ &= x^d + b_{n,d-1}x^{d-1} + b_{n,d-2}x^{n-2} + \cdots + b_{n,0} \end{aligned}$$

is a polynomial such that $|b_{n,i}| \leqslant \binom{d}{i} \leqslant 2^d$. The polynomial has integer entries, since any symmetric polynomial in $\alpha_1, \ldots, \alpha_d$ can be written uniquely as a polynomial in the elementary symmetric functions in $\alpha_1, \ldots, \alpha_d$ (see [DF91], page 537). Since there are only finitely many possible polynomials with bounded integer coefficients, we must have that $P_n = P_m$ for some n and m, and, thus, there is a permutation $\sigma \in S_d$ such that, for all i, we have that $\alpha_i^m = \alpha_{\sigma(i)}^n$. Hence $P_{n(d!)} = P_{m(d!)}$, and thus $\alpha_i^{m(d!)} = \alpha_i^{m(d!)}$, which implies that α_i is a root of unity. □

The first part of the proof of the following theorem appears, for the special case of matrices associated to substitutions on return words, in [Dur98]. This first part of the proof is still valid with trivial modifications for rewritings.

For proving the converse we provide a more elaborate argument and do not follow Durand's work, since the author could not follow Durand's arguments.

Theorem 3.9 *Let $B : \mathbb{R}^m \to \mathbb{R}^m$, $A : \mathbb{R}^n \to \mathbb{R}^n$, and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be matrices with integer entries such that the diagram*

$$\begin{array}{ccc} \mathbb{R}^m & \stackrel{B}{\to} & \mathbb{R}^m \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R}^n & \underset{A}{\to} & \mathbb{R}^n \end{array} \tag{3.1}$$

commutes. Suppose that there exists $K > 0$ such that for all $m \in \mathbb{Z}^+$, there is a matrix η_m such that in the diagram

$$\begin{array}{cccl} \mathbb{R}^m & \stackrel{B^m}{\longrightarrow} & \mathbb{R}^m & D_m = B^m - \eta_m \pi \\ \pi \downarrow & {}^{\eta_m} \nearrow & \downarrow \pi & \\ \mathbb{R}^n & \underset{A^m}{\longrightarrow} & \mathbb{R}^n & C_m = A^m - \pi \eta_m \end{array} \tag{3.2}$$

we have that each of the entries $(D_m)_{ij}$ of the matrix $D_m = B^m - \eta_m \pi$ satisfies that $\left|(D_m)_{ij}\right| < K$, and each of the entries $(C_m)_{ij}$ of the matrix $C_m = A^m - \pi\eta_m$ satisfy that $\left|(C_m)_{ij}\right| < K$. Let λ be a complex number such that $\lambda \neq 0$ and λ is not a root of unity. Then λ is an eigenvector for B if and only if λ is an eigenvalue of A.

Proof With the notation as in the theorem, let λ be an eigenvalue for B that is not zero nor a root of unity. We need to show that λ is an eigenvalue for A.

We reproduce the argument of Durand in [Dur98] with the necessary trivial adaptations to show that λ is an eigenvalue of A.

Suppose that (λ, v) is an eigenpair for B such that $\lambda \neq 0$ and λ is not a root of unity. We show next that λ is an eigenvalue for A, with eigenvector πv.

Since there are finitely many possibilities for the matrices D_m there must be numbers $l \in \mathbb{Z}^+$, $m \in \mathbb{Z}^+$ such that $D_m = D_l$. Now, since the diagram 3.1 commutes, we must have that $A\pi v = \pi B v = \lambda \pi v$. To show that λ is an eigenvalue for A it suffices, thus, to show that $\pi v \neq 0$.

Suppose, on the contrary that $\pi v = 0$. Then, since $D_l = D_m$ we must have that $(B^m - \eta_m \pi)\, v = \left(B^l - \eta_l \pi\right) v$, and, hence, we must have that $\lambda^m v = \lambda^l v$. Since $v \neq 0$, we obtain that $\lambda^m = \lambda^l$, which implies that $\lambda = 0$ or λ is a root of unity, which contradicts the hypothesis. Thus, we conclude that λ is an eigenvalue of A.

For the converse, we do not follow Durand, but provide a different proof. We start with some considerations.

Since diagram 3.1 commutes, then so does diagram 3.3 below,

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{B^m - \lambda^m} & \mathbb{R}^m \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R}^n & \xrightarrow[A^m - \lambda^m]{} & \mathbb{R}^n \end{array} \tag{3.3}$$

Claim: Let (λ_1, v_1) be an eigenpair for A, and suppose that v_1 is in the range of π. Then, for all $m \in \mathbb{N}$, λ_1^m is an eigenvalue for B^m.

Proof of Claim Suppose, on the contrary, that $B^m - \lambda_1^m$ is invertible. Since, by hypothesis, there is a vector w such that $\pi(w) = v_1$, and thus the vector $w' = (B^m - \lambda_1^m)(w)$ is not zero. On the other hand, we also have that $(A^m - \lambda_1^m)\, \pi(w) = (A^m - \lambda_1^m)\, v_1 = 0$, which is a contradiction. Therefore, $B^m - \lambda_1^m$ is singular, and λ_1^m is an eigenvalue for B^m. □

Let (λ, v) be an eigenpair for A with λ

Case 1: $|\lambda| > 1$. Then we have that $\lim_{m\to\infty} \frac{\pi \eta_m v}{\lambda^m} = \lim_{m\to\infty} \frac{\lambda^m v - C_m v}{\lambda^m} = v$, since the entries of C_m are bounded. Since the range of π is closed and $\frac{\pi \eta_m v}{\lambda^m} \to v$, we have that v is in the range of π, and, by the claim, we have that λ is an eigenvector for B.

Case 2: Suppose that A has an eigenvalue λ such that $|\lambda| < 1$. Since A has integer entries, and the minimal polynomial for λ divides the minimal polynomial for

A, we must have that there is a conjugate λ' of λ such that $|\lambda'| > 1$. Let (λ', v') be a corresponding eigenpair for λ'. By Case 1, we have that λ' is an eigenvalue of A. Hence all other roots of the minimal polynomial of λ' must also be eigenvalues for B. Hence λ is an eigenvalue for B.

Case 3: Suppose that all the conjugates of λ are in the unit circle. Since λ is an algebraic integer in the unit circle all of whose conjugates are also in the unit circle, we must have by Lemma 3.8 that λ is a root of unity, contradicting the hypothesis.

We conclude that λ must be an eigenvalue for B. □

In the light of 3.5, we obtain the following corollary.

Corollary 3.10 *Let β be obtained from α using starting and stopping rules, return words or splitting a letter . Let $\lambda \neq 0$ and λ not a root of unity. Then λ is an eigenvalue of $[\alpha]$ if and only if λ is an eigenvalue for $[\beta]$.*

Corollary 3.11 *Let β be obtained from α via shift equivalence of substitutions. Let $\lambda \neq 0$ and λ not a root of unity. Then the λ is an eigenvalue of $[\alpha]$ if and only if λ is an eigenvalues for $[\beta]$.*

In what follows we use the notion of shift equivalence for matrices. The definition is very similar to that of shift equivalence for substitutions.

Definition 3.12 *Let A and B be non-negative integral matrices. A* shift equivalence *(over $\mathbb{Z}$) from A to B with lag one is a pair (R, S) of rectangular integral matrices such that the following diagram commutes:*

$$\begin{array}{ccc} \cdot & \overset{A}{\rightarrow} & \cdot \\ \downarrow R & \nearrow S & \downarrow R \\ \cdot & \overset{B}{\rightarrow} & \cdot \end{array}$$

Any matrix A can be put in the Jordan canonical form $PAP^{-1} = J = J_{m_1} \oplus J_{m_2} \oplus \ldots \oplus J_{m_k}$, where J_{m_1} are Jordan blocks, each associated to an eigenvalue of A. The Jordan canonical form away from zero is the Jordan canonical form excluding the Jordan blocks corresponding to zero eigenvalues.

Lemma 3.13 *Let A and B be integral matrices. If A is shift equivalent to B, then their Jordan canonical forms away from zero are the same.*

Proof See [LM95], page 236. □

The following Proposition contains a weaker form of Theorem 3.9, but the proof is considerably easier using Lemma 3.13.

Proposition 3.14 *With the same hypothesis of Theorem 3.9, we have that there are infinitely many numbers $m > n \in \mathbb{Z}^+$ such that diagram 3.4 below*

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{B^m - B^n} & \mathbb{R}^m \\ \pi \downarrow & {}^{\eta_m - \eta_n} \nearrow & \downarrow \pi \\ \mathbb{R}^n & \xrightarrow[A^m - A^n]{} & \mathbb{R}^n \end{array} \tag{3.4}$$

commutes. Let E_A and E_B be the set of eigenvalues of A and B, respectively, that are not zero nor roots of unity. Then

$$\{\lambda^m - \lambda^n : \lambda \notin E_A\} = \{\mu^m - \mu^n : \mu \notin E_B\}$$

Proof Since there are only finitely many possibilities for $D_m = B^m - \eta_m \pi$, then there is a matrix D, and there are infinitely many m_i such that $D = D_{m_0} = D_{m_1} = D_{m_2} = \ldots$. Now, since, for this sequence there are only finitely many possible values for $C_{m_i} = A^{m_i} - \pi \eta_{m_i}$, we conclude that there is a matrix C, and there are infinitely many m_{i_k} such that $C = C_{m_{i_0}} = C_{m_{i_1}} = \ldots$. Thus, there is a fixed number n and infinitely many numbers m such that both $\pi \eta_m = B^m - C$ and $\pi \eta_n = B^n - C$, from which we obtain that $B^m - B^n = \pi(\eta_m - \eta_n)$. In a similar way we obtain that $A^m - A^n = (\eta_m - \eta_n)\pi$. We obtain the shift equivalence with lag one shown in diagram 3.4.

If $PAP^{-1} = J$, then the matrix $P(A^m - A^n)P^{-1} = J^m - J^n$ is an upper triangular matrix whose eigenvalues are in the diagonal and are of the form $\{\lambda^m - \lambda^n | \ \lambda \text{ is an eigenvalue for } A\}$, and, thus, an eigenvalue for $A^m - A^n$ is zero exactly when $\lambda = 0$ or λ is a root of unity. Thus the eigenvalues for $A^m - A^n$ corresponding to the Jordan canonical form away from zero for $A^m - A^n$ are of the form

$$\{\lambda^m - \lambda^n : \ \lambda \text{ is an eigenvalue for } A, \lambda \neq 0, \text{ and } \lambda \text{ is not a root of unity.}\}.$$

In a similar way, we obtain that the eigenvalues for $B^m - B^n$ are

$$\{\mu^m - \mu^n : \ \mu \text{ is an eigenvalue for } B, \mu \neq 0, \text{ and } \mu \text{ is not a root of unity}\}.$$

By Lemma 3.13, we obtain the desired equality of sets. □

3.9 The Barge-Swanson Rigidity Theorem

Our interest in rewriting and shift equivalence stems from the Rigidity Theorem in [BS07], which implies that whenever two substitutions α and β induce two homeomorphic tiling spaces, then there is a sequence of rewritings, unrewritings and shift equivalences of substitutions that start in α and finish in β.

The procedure to construct from a substitution φ the derived substitution φ^* that is mentioned in the statement of the Rigidity Theorem can be found in [BD01]. For our purposes it is enough to know that φ^* is obtained from φ via a sequence of steps, each step being either a rewriting with starting and stopping rules, or a shift equivalence of substitutions.

Theorem 3.15 *(Barge-Swanson Rigidity Theorem) Let* $\alpha : \mathcal{A} \to \mathcal{A}^+$ *and* $\beta : \mathcal{B} \to \mathcal{B}^+$ *be primitive, non shift-periodic substitutions such that the tiling spaces* $\mathcal{T}_\alpha$ *and* $\mathcal{T}_\beta$ *are homeomorphic. Let* α^* *and* β^* *be the derived substitutions of* α *and* β*, respectively. Then, there are numbers* n_1 *and* n_2*, a substitution* $\gamma : \mathcal{A} \to \mathcal{A}^+$*, and a prefix* u *of a common fixed word for some power of* α *and* η *such that the return substitutions* $(\alpha^*)_u^{n_1}$*,* $(\gamma)_u^{n_2}$ *satisfy that* $(\alpha^*)_u^{n_1} = (\gamma)_u^{n_2}$*, and* γ *is shift equivalent to some power of* β^**.*

Corollary 3.16 *If* α *and* β *are primitive, non shift-periodic substitutions such that* $\mathcal{T}_\alpha \cong \mathcal{T}_\beta$*, then there is a sequence of substitutions* $\alpha = \varphi_1, \varphi_2, \ldots, \varphi_k = \beta$ *such that ether (1)* φ_{i+1} *is obtained from* φ_i *via splitting or stopping and starting rules, or (2)* φ_i *is obtained from* φ_i *via shift equivalence or stopping and starting rules.*

CHAPTER 4

THE RAUZY FRACTAL AND RUP SUBSTITUTIONS

In the following definitions, we follow [BK06]. Let φ be an IUP substitution with corresponding projections $\mathrm{pr}_{\varphi}^{s}$ and $\mathrm{pr}_{\varphi}^{u}$ into the stable space E^s and the unstable space E^u, respectively. Recall that $h_{\varphi} : \mathcal{F}_{\varphi} \to \mathbb{T}^d$ denotes the geometric realization map, and that $\tilde{h}_{\varphi} : \mathcal{F}_{\varphi} \to \mathbb{R}^d$ denotes the first-vertex map. Also recall that if w is the the bi-infinite word $\ldots w_{-1}.w_0 w_1 \ldots$, we denote by $w_{[a,b)}$ the word $w_a w_{a+1} \ldots w_{b-1}$

Definition 4.1 *We let* $\tilde{\Omega}_{\varphi} = \tilde{h}_{\varphi}\left(\mathcal{F}_{\varphi}\right)$, *and* $\Omega_{\varphi} = h_{\varphi}\left(\mathcal{F}_{\varphi}\right) = \mathbb{T}^d$

The following diagram shows the relationship between the geometric, realization , the first-vertex map, and the tiling flow.

Definition 4.2 *Define* $\mathcal{F}_{\varphi}^{s} = \left\{S \in \mathcal{F}_{\varphi} :\ \min\left(\hat{S}\right) \in E^s\right\}$, $\tilde{\Omega}_{\varphi}^{s} = \tilde{h}_{\varphi}\left(\mathcal{F}_{\varphi}^{s}\right)$, $\Omega_{\varphi}^{s} = h\left(\mathcal{F}_{\varphi}^{s}\right)$.

The space $\left(\mathcal{F}_{\varphi}^{s}\right)$ is another presentation of *the substitutive system* X_{φ} as described in section 2.3, but in terms of strands. The shift, for strands in $\mathcal{F}_{\varphi}^{s}$ is given as follows:

Definition 4.3 *The* shift *map* $\sigma_{\mathcal{F}_{\varphi}}$ *is the function* $\sigma : \mathcal{F}_{\varphi}^{s} \to \mathcal{F}_{\varphi}^{s}$ *given by* $\sigma(S) = S - \mathrm{pr}_{\varphi}^{u}\left(\max\left(\hat{S}\right)\right)$

Let $S = \ldots S_{-1}.\hat{S} S_1 S_2 \ldots \in \mathcal{F}_{\varphi}^{s}$, and let $v = \mathrm{pr}_{\varphi}^{u}(\max(S))$. Note that

$$\sigma(S) = \ldots (S_{-1} - v)\left(\hat{S} - v\right).(\widehat{S_1 - v})(S_2 - v)\ldots$$

and $\left[\sigma_{\mathcal{F}_{\varphi}}(S)\right] = \sigma_{X_{\varphi}}([S])$, where $\sigma_{\mathcal{F}_{\varphi}}$ is the shift defined for strands in $\mathcal{F}_{\varphi}^{s}$, and $\sigma_{X_{\varphi}}$ is the shift defined on bi-infinite words of section 2.2.

Definition 4.4 *The* Rauzy fractal $\mathcal{R}_{\varphi}$ *is the set* $\mathcal{R}_{\varphi} = \tilde{\Omega}_{\varphi}^{s} = \tilde{h}_{\varphi}\left(\mathcal{F}_{\varphi}^{s}\right)$

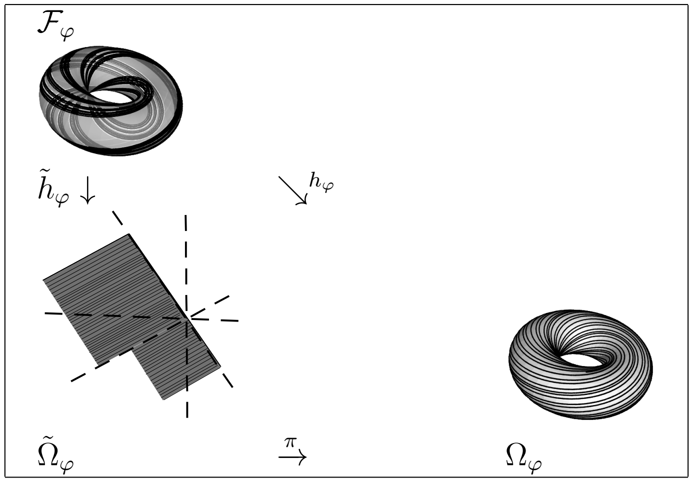


Figure 4.1: The tiling space $\mathcal{F}_\varphi$ and its image under both the first-vertex map and the geometric realization map. The tiling flow lines are depicted in the images. The images are corresponding to the substitution φ given by $\varphi(1) = 121$, $\varphi(2) = 12$.

Definition 4.5 *Let* $w \in \mathcal{L}_\varphi$. $\left(\mathcal{F}^s_\varphi\right)^{.w} = \left\{S \in \mathcal{F}^s_\varphi : [S]_{[0,|w|)} = w\right\}$, $\mathcal{R}^{.w}_\varphi = \tilde{h}_\varphi\left(\left(\mathcal{F}^s_\varphi\right)^{.w}\right)$, *and* $\left(\Omega^s_\varphi\right)^{.w} = h\left(\left(\mathcal{F}^s_\varphi\right)^{.w}\right)$.

Note that the restriction $h_\varphi|_{\mathcal{F}^s_\varphi}$ of the geometric realization to the stable space is a continuous map whose image is in $\pi\left(C^R \cap E^s\right) \subset \mathbb{T}^d$. Also note that the stable space $E^s \subset \mathbb{R}^d$ is totally irrational since φ is a IUP substitution, and, consequently, we have that $\pi(E^s)$ is immersed in the torus $\mathbb{R}^d/\mathbb{Z}^d$. Since $C^R \cap E^s$ is compact, we obtain that $C^R \cap E^s$ and $\pi\left(C^R \cap E^s\right)$ are homeomorphic. Thus, we obtain the following

Proposition 4.6 *The restricted first-vertex map* $\tilde{h}_\varphi|_{\mathcal{F}^s_\varphi}$ *is continuous.*

4.1 Properties of the Rauzy Fractal

In this section we use strands to show some well-known properties of the Rauzy Fractal.

There is not a generally agreed upon definition of *fractal* in the literature (see [Fal03] page xxv). Some definitions require that the Hausdorff dimension of the object be greater than the topological dimension, but in our case that is not the case. It is true, though, that for three letters, the boundary of the Rauzy Fractal is not smooth by a result of Rufus Bowen in [Bow78] that states that Markov partitions on three letters are not smooth. Bowen's result was later generalized by Elise Cawley in [Caw89] to more letters under some conditions. Cawley, in [Caw89], also announced that in a forecoming paper she shows that , under some conditions on the eigenvalues, the Hausdorff dimension of the boundary of a Markov Partition is not an integer. The author could not find a reference to this latter article.

Proposition 4.9 shows that the Rauzy pieces are solutions to what Mauldin and Williams call *geometric graph directed construction in* $\mathbb{R}^d$ in [MW88], and what Mauldin and Urbanski later generalized to *graph directed Markov systems* in [MU03]. By a *fractal*, in this work, we mean a fixed point of a geometric graph directed construction, also known as *graph directed iteration function systems*, which, in turn, generalize the *iterated function systems* of [Fal03], page 123.

In the following proof, the set $P(i,j)$ consists of the positions at which the letter j appears in $\varphi(i)$.

Definition 4.7 *Let* $\varphi : \mathcal{A} \to \mathcal{A}^+$ *be a IUP substitution, and let* $i, j \in \mathcal{A}$. *Let* $\varphi(i) = a_{i1}a_{i2}\ldots a_{i|\varphi(i)|}$. *We define* $P(i,j)$ *to be the set* $\{k : a_{ik} = j\}$.

Lemma 4.8 *Suppose* φ *is a IUP substitution,* $S \in \mathcal{F}_\varphi$ *is a strand, and* S *is a state of type* i *with initial vertex* p *such that* $\varphi(S)$ *is a state of type* j *with initial vertex in* E^s, *then there is* $k \in P(i,j)$ *and* $v = \left[\varphi(i)_{[0,k-1)}\right]$ *such that* $p \in \mathcal{R}^{\cdot i} - \mathrm{pr}^u\left([\varphi]^{-1} v\right)$.

Proof Let φ, S, i and p be as in the hypotheses. Let v_R be the right Perron eigenvector for $[\varphi]$ such that $< v_R >= E^u$ and such that $\|v_R\| = 1$. Since S is a state, and $\mathcal{F}_\varphi$ is invariant under translations parallel to E^u, it follows that there is a unique $x > 0$ such that $S + xv_R$ is a state of type i with initial vertex in E^s. Hence $p \in \mathcal{R}^{\cdot i} - xv_R$. Now, $\varphi(S)$ is a finite strand with a vertex $q \in E^s$ and such that $\varphi(S)$ has the pattern of $\varphi(i)$. Thus, there is a k such that $[\varphi] p + \left[\varphi(i)_{[0,k-1)}\right] = q$. Let $v = \left[\varphi(i)_{[0,k-1)}\right]$ Applying $\mathrm{pr}^u [\varphi]^{-1}$ to both sides, we get $-xv_R + \mathrm{pr}^u \left([\varphi]^{-1} v\right) = 0$. Hence $p \in \mathcal{R}^{\cdot i} - \mathrm{pr}^u \left([\varphi]^{-1} v\right)$. □

The following proposition is well known (see, for example, theorem 2 of[BS+05] and section 18 in [BK06]).

Proposition 4.9 *Let φ be an IUP substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$, with stable projection* pr^s*. Then*

$$\mathcal{R}^{\cdot j} = \bigcup_{i \in \mathcal{A}} \bigcup_{k \in P(i,j)} [\varphi]\left(\mathcal{R}^{\cdot i}\right) + \mathrm{pr}^s \left(\left[\varphi(i)_{[0,k-1)}\right]\right)$$

Proof Let $q \in \mathcal{R}^{\cdot j}$, and let S' a state of type j with initial vertex q and belonging to some strand in $\mathcal{F}_\varphi$. Since φ is bijective in $\mathcal{F}_\varphi$, then there is a state S of some type i with initial vertex p such that $S' = (\varphi(S))$. By Lemma 4.8, there is a $k \in P(i, j)$ such that $p \in \mathcal{R}^{\cdot i} - \mathrm{pr}^u \left([\varphi]^{-1} v\right)$ with $v = \left[\varphi(i)_{[0,k-1)}\right]$.

Since pr^u is a polynomial in $[\varphi]$ by 2.1, we have that pr^u commutes with $[\varphi]$. Hence pr^u also commutes with $[\varphi]^{-1}$. Then $[\varphi] p + v = q$, and hence

$$q \in [\varphi]\left(\mathcal{R}^{\cdot i} - \mathrm{pr}^u \left([\varphi]^{-1} v\right)\right) + v = [\varphi]\mathcal{R}^{\cdot i} - \mathrm{pr}^u(v) + v = [\varphi]\mathcal{R}^{\cdot i} + \mathrm{pr}^s(v).$$

Conversely, suppose that $i \in \mathcal{A}$, $k \in P(i, j)$, and let $v = \left[\varphi(i)_{[0,k-1)}\right]$. Suppose that $q \in [\varphi]\mathcal{R}^{\cdot i} + \mathrm{pr}^s(v) = [\varphi]\mathcal{R}^{\cdot i} - \mathrm{pr}^u(v) + v = [\varphi]\left(\mathcal{R}^{\cdot i} - \mathrm{pr}^u\left([\varphi]^{-1} v\right)\right) + v$. We have to show that $q \in \mathcal{R}^{\cdot j}$.

Since $[\varphi]^{-1}(q - v) + \mathrm{pr}^u\left([\varphi]^{-1} v\right) \in \mathcal{R}^{\cdot i}$, there is a state S, belonging to some strand in $\mathcal{F}_\varphi$, of type i with a vertex $p \in \mathcal{R}^{\cdot i}$ such that $[\varphi]^{-1}(q - v) + \mathrm{pr}^u\left([\varphi]^{-1} v\right) = p$.

Thus $q = [\varphi]\left(p - \mathrm{pr}^u\left([\varphi]^{-1} v\right)\right) + v$, which implies that $\varphi(S)$ is of type j, and hence $q \in \mathcal{R}^{\cdot j}$. □

In Chapter 7 we give a more general version of the previous theorem.

In [MW88], Mauldin and Williams show that graph directed constructions have a unique solution. Although we know that the Rauzy Fractal exists a priori, Proposition 4.9 shows that the Rauzy fractal is the fixed point of a graph directed construction, and justifies the use of the term fractal.

The following properties of the Rauzy Fractal can be found in [SW02] and also in [BK06].

Proposition 4.10 *The Rauzy fractal is equal to the closure of its interior.*

The following proposition states that there is a sequence of vectors $(v_k)_{k\in\mathbb{Z}}$ such that the union $\bigcup\{\mathcal{R} + v_k : k \in \mathbb{Z}\} = E^s$, and such that the measure of $(\mathcal{R} + v_k) \cap (\mathcal{R} + v_{k'})$ is zero for $k \neq k'$.

Proposition 4.11 *Assuming GCC, The Rauzy fractal can tile periodically the stable space.*

Proposition 4.10 implies that the Hausdorff dimension and the topological dimension of the Rauzy fractal are the same.

4.2 Reducible Unimodular Pisot (RUP) Substitutions

In this section we present the necessary changes in definitions to define the Rauzy fractal for reducible Unimodular Pisot substitutions. We follow [BBK06].

Let φ be a RUPC substitution on d letters with Perron eigenvalue λ_φ. The characteristic polynomial $p_\varphi(x)$ of $[\varphi]$ decomposes into irreducible (over $\mathbb{Q}$) factors $p_\varphi(x) = p_\varphi^P(x)\, p_\varphi^R(x)$, where $p_\varphi^P(x)$ is the minimal polynomial for λ_φ. By Proposition

2.1, there is an $[\varphi]$-invariant decomposition of $\mathbb{R}^d$ as $V_\varphi^P \oplus V_\varphi^R$ induced by the polynomials p_φ^P and p_φ^R. The polynomial $p_\varphi^P(x)$ is called the *Pisot part* of the characteristic polynomial, and the polynomial $p_\varphi^R(x)$ is called the *reducible part* of the characteristic polynomial of $[\varphi]$. Similarly, the space V_φ^P is called the *Pisot space* of φ, and V_φ^R is called the *reducible space* of φ

We denote the restriction of the matrix $[\varphi]$ to V_φ^P and to V_φ^R as $[\varphi]_P$ and $[\varphi]_R$ respectively. The decomposition $V_\varphi^P \oplus V_\varphi^R$ has associated projections $\mathrm{pr}_\varphi^P; \mathbb{R}^d \to V_\varphi^P$ and $\mathrm{pr}_\varphi^R : \mathbb{R}^d \to V_\varphi^R$ such that $[\varphi]|_{V_\varphi^P}$ and $[\varphi]|_{V_\varphi^R}$ have characteristic polynomials $p_\varphi^P(x)$ and $p_\varphi^R(x)$, respectively. The matrix $[\varphi]_P$ has characteristic polynomial $p_\varphi^P(x) \in \mathbb{Z}[x]$.

The polynomial p_φ^P factors (over $\mathbb{Q}(\lambda_\varphi)$) as $p_\varphi^P = (x - \lambda_\varphi)\, p_\varphi^s(x)$. The polynomial $p_\varphi^u = (x - \lambda_\varphi)$ is called the *unstable part* of the polynomial p_φ^P, whereas the polynomial p_φ^s is called the stable part of the polynomial p_φ^P. Applying again Proposition 2.1, we obtain that there is a decomposition $V_\varphi^P = V_\varphi^s \oplus V_\varphi^u$ into the *stable space* V_φ^s and the *unstable space* V_φ^u such that $\ker\left(p_\varphi^u([\varphi]_P)\right) = V_\varphi^u$ and $\ker\left(p_\varphi^s([\varphi]_P)\right) = V_\varphi^s$, and with induced projections $\mathrm{pr}_\varphi^s : V_\varphi^P \to V_\varphi^s$ and $\mathrm{pr}_\varphi^u : V_\varphi \to V_\varphi^u$. Again, we denote the restrictions of $[\varphi]_P$ to V_φ^s and to V_φ^u as $[\varphi]_s$ and $[\varphi]_u$, respectively. The characteristic polynomials of $[\varphi]_u$ and $[\varphi]_s$ are $p_\varphi^u = (x - \lambda_\varphi)$ and $p_\varphi^s(x)$, respectively.

The projection pr_φ^P is of the form $\lambda(A)\, p_\varphi^R(A)$ for $\lambda(x) \in \mathbb{Q}[x]$, and thus projects $\mathbb{Q}^d$ into $\mathbb{Q}^d \cap V_\varphi^P$.

4.2.1 Labeled Strands

A strand for a RUPC substitution $\varphi : \{1, \ldots, d\} = \mathcal{A} \to \mathcal{A}^+$ is a labeled sequence $\{S_i\}_{i \in \mathbb{Z}}$ of segments such that $S_i = \mathrm{pr}_\varphi^P(T_i)$, for some segment $T_i \subset \mathbb{Z}^d$. We define the max and the min of a segment T_i in the Pisot space V_φ^P of φ as the $\mathrm{pr}_\varphi^P(\max(T_i))$ and $\mathrm{pr}_\varphi^P(\min(T_i))$, respectively. The label of each edge S_i is the same label of the corresponding segment T_i. We proceed to define strands just as in the irreducible case.

We can then define $\mathcal{S}^+$ and $\mathcal{S}^{\mathbb{Z}}$ also. We denote by $\mathcal{F}$ the collection of all bi-infinite strands γ such that each edge S_k of $\gamma = \{S_k\}_{k\in\mathbb{Z}}$ is contained in V_φ^P, and the edges S_k are indexed in such a way that $\max(S_k) = \min(S_{k+1})$.

One problem of considering a strand just as a sequence of projections into $V_\varphi^P$of segments without labels is that it is possible to have two elements e_i and e_j of the canonical base for $\mathbb{R}^d$ such that $\mathrm{pr}_\varphi^P(e_i) = \mathrm{pr}_\varphi^P(e_j)$ even if $i \neq j$.

The substitution φ also acts in the set of segments $\mathcal{S}$ by defining it as $\varphi\left(\mathrm{pr}_\varphi^P(T_i)\right) = \left\{\mathrm{pr}_\varphi^P(T') : T' \in \varphi(T_i)\right\}$, and then extending the definition to strands.

The definition of the stable cylinder $\mathcal{C}^R \subset V_\varphi^P$ is done in the same way as for strands in the irreducible case: $\mathcal{C}^R = \left\{p \in V_\varphi^P : \ \left\|\mathrm{pr}_\varphi^s(p)\right\| < R\right\}$.

The word $[S]$ associated to a finite or infinite strand is given by the sequence of labels associated to each edge of the strand.

We can define

$$\mathcal{F}_\varphi^{R_0} := \left\{S \in \mathcal{F} : \ S_k \subset \mathcal{C}^{R_0} \text{ for all } S_k \in S\right\}$$

Define the *strand space* as

$$\mathcal{F}_\varphi = \left\{S : \ S \in \bigcap_{n\in\mathbb{N}} \varphi^n\left(\mathcal{F}^{R_0}\right), \text{ and } [S] \text{ is allowed}\right\}$$

Definition 4.12 *The* Rauzy fractal *for a reducible Pisot substitution is the set*

$$\mathcal{R}_\varphi = \left\{S : \ S \in \mathcal{F}_\varphi, \textit{ and } S \textit{ has a vertex in } V_\varphi^s\right\}$$

For reducible substitutions φ there is also *geometric realization* h_φ, but the torus into which geometric realization projects is not necessarily $V_\varphi^P / \mathrm{pr}_\varphi^P\left(\mathbb{Z}^d\right)$, but V_φ^P/Σ, for a sub-lattice Σ of $\mathrm{pr}_\varphi^P\left(\mathbb{Z}^d\right)$. The definition of the sublattice Σ is given in [BBK06].

There is also a *first-vertex map* $\tilde{h}_\varphi : \mathcal{F}_\varphi \to V_\varphi^P$ just as in the reducible case. Since, $V_\varphi^s \cap \mathrm{pr}_\varphi^P\left(\mathbb{Z}^d\right) = \{0\}$ (see [BBK06]), we have that V_φ^s is immersed in V_φ^s/Σ. Thus, as in the irreducible case, $\tilde{h}|_{\mathcal{F}_\varphi^s}$ is continuous, and we can consider the Rauzy fractal as either in the torus V_φ^s/Σ or in V_φ^s.

We define $\mathcal{F}_\varphi^s$, $\left(\mathcal{F}_\varphi^s\right)^{\cdot w}$, $\tilde{\Omega}_\varphi$, Ω_φ, $\pi : V_\varphi^P \to V_\varphi^P/\Sigma$, $\left(\tilde{\Omega}_\varphi^s\right)^{\cdot w}$, $\mathcal{R}_\varphi$ and $\mathcal{R}_\varphi^{\cdot w}$ just as in the irreducible case.

In Remark 4.3 in [BBK06], a definition of the Geometric coincidence condition for the reducible case is very similar to that of the Geometric coincidence condition for the irreducible case.

Let Σ be the invariant sublattice of $\mathrm{pr}_\varphi^P\left(\mathbb{Z}^d\right)$, and R_0 be such that if a strand $S \in \mathcal{C}^{R_0}$ then $\varphi^n(S) \in \mathcal{C}^{R_0}$ for all $n \in \mathbb{Z}^+$.

For $q \in V_\varphi^P$, we define the set of *states over* q.

$$\mathbb{S}_q^{R_0} = \left\{ I |\ \min(I) = q \pmod{\Sigma},\ (I \setminus \{\max(I)\}) \cap V_\varphi^s \neq \varnothing \text{ and } I \subset \mathcal{C}^{R_0} \right\}$$

And every strand T in $\mathcal{F}_\varphi$ determines a state, denoted by T, just as in the irreducible case.

Definition 4.13 *A RUP substitution is said to satisfy the* geometric coincidence condition *provided that for any pair of states $S_1, S_2 \in \mathbb{S}_0^{R_0}$ there is a number n such that the finite strands $\varphi^n(S_1)$ and $\varphi^n(S_2)$ share an edge. We say that a RUP substitution* satisfies GCC *provided that φ satisfies the geometric coincidence condition.*

Definition 4.14 *A* RUPC *substitution is a reducible unimodular substitution that satisfies the geometric coincidence condition.*

CHAPTER 5

PROXIMALITY AND THE GRAPH RELATIONS

5.1 The Barge-Kellendonk Theorem

We state two important theorems appearing in [BK11], along with some definitions. We state those theorems for the special case that we are interested in.

For our purposes, the measure space we consider is the tiling space of a RUPC substitution φ, and the group acting on it is the flow $T_p(S) = S - tv_R$, where v_R is the right Perron eigenvalue of $[\varphi]$. The tiling space with the flow is uniquely ergodic (See Theorem 3.1 in [Sol97]), so there is a unique Borel probability measure for which the tiling flow is measure preserving.

Definition 5.1 *Let (X, G, μ) be a measure dynamical system. For any $g \in G$, the function $U_g : L^2(X, \mu) \to L^2(X, m)$ given by $U_g(f)(x) = f(g \cdot x)$ is the* induced isometric operator *associated to g.*

The general definition for an eigenvalue and an eigenfunction requires the notion of a group character. In our case, the induced operator is $U_t(f)(S) = f(S - tv_R)$, and the group acting on the tiling space is $\mathbb{R}$. Since $\mathbb{R}$ is the group acting in $L^2(X, \mu)$ in our setting, we have the following definition:

Definition 5.2 *For a tiling space with the tiling space flow, an L^2-function $f \neq 0$ is said to be an* eigenfunction *provided there exists a real number λ for which we have that $U_t(f) = e^{2\pi i \lambda t} f$ for all $t \in \mathbb{R}$.*

Definition 5.3 *The measure dynamical system (X, G, μ) has pure point dynamical spectrum if the L^2-eigenfunctions span $L^2(X, \mu)$*

Let φ be a RUPC substitution and let S and S' be two strands in $\mathcal{F}_{\varphi}$. Let v_R be the right Perron eigenvector in the Pisot Space, and let d be the metric in the strand space. We say that S ad S' are *forward proximal* if $\liminf_{t\to\infty} d\left(S - tv_R, S' - tv_R\right) = 0$. If $\liminf_{t\to-\infty} d\left(S - tv_R, S' - tv_R\right) = 0$, then S and S' are called *backward proximal*. If $\inf_{t\in\mathbb{R}} d\left(S - tv_R, S' - tv_R\right) = 0$, we say that S and S' are *proximal*. If two strands are proximal, we say that they are a *proximal pair*. In the next section, in Proposition 5.9, we quote a theorem from [BD07] that states that all those notions are equivalent for RUPC substitutions.

Let Σ be the sublattice of $\mathrm{pr}_{\varphi}^{P}\left(\mathbb{Z}^d\right)$ given in [BBK06]. Recall that the geometric realization $h_{\varphi} : \mathcal{F}_{\varphi} \to V_{\varphi}^{P}/\Sigma$ is given by mapping each strand to any of its vertices (mod Σ).

Theorem 5.4 *(Barge-Kellendonk) Let $(\mathcal{F}_{\varphi}, \mathbb{R}, \mu)$ be a RUP tiling space with the tiling flow, and let h_{φ} be the geometric realization into V_{φ}^{P}/Σ. Let* $\mathrm{cr}_{\varphi} = \min\left\{h^{-1} : \ z \in V_{\varphi}^{P}/\Sigma\right\}$*. Then the following are equivalent:*

1. *The continuous eigenfunctions generate $L^2(X, \mu)$.*

2. $\min\left\{h^{-1} : \ z \in V_{\varphi}^{P}/\Sigma\right\} = 1$ *(That is, φ satisfies GCC, see)*

3. *Proximality is a closed relation*

Though the Barge-Kellendonk theorem is far more general, we stated it here for the particular case of a tiling flow arising from a RUP substitution, for which the flow is minimal, and for the measure given by unique ergodicity of the tiling flow.

Let $\sim_s$ in $\mathcal{F}_{\varphi} \times \mathcal{F}_{\varphi}$ be the relation such that $x \sim_s y$ provided that $h_{\varphi}(x) = h_{\varphi}(y)$. Also in [BK11], we have the following theorem.

Theorem 5.5 *The proximal relation $\mathcal{P}$ coincides with the relation $\sim_s$ if and only if $\mathcal{P} \subset X \times X$ is closed (in the product topology)*

From [BK11] and [BBK06], it follows that any Reducible Pisot substitution satisfying the Geometric Coincidence condition, must have pure point spectrum, and thus proximality coincides with the relation $\sim_s$. For our purposes the main consequence of [BK11] and [BBK06] that is used throughout is the following.

Proposition 5.6 *Let φ be a RUPC substitution. Let S_1, S_2 two strands in $\mathcal{F}_\varphi$, then their geometrical realization is the same, if and only if S_1 is proximal to S_2. That is, there is a vertex v_1 in S_1 and a vertex v_2 in S_2 such that $v_1 - v_2 \in \Sigma$ if and only if S_1 is proximal to S_2.*

Let φ be a RUPC substitution, and let Σ the invariant lattice in the Pisot space V_φ^P. Since the stable space $V_\varphi^s \subset \mathbb{R}^d$ is immersed in the torus, and the Rauzy Fractal is a compact set in V_φ^s, we obtain that the Rauzy fractal can either be thought of as inside the torus, or as a subset of $V_\varphi^s \subset \mathbb{R}^d$.

Proposition 5.7 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a RUPC substitution, and let $S_1, S_2 \in \mathcal{F}_\varphi^s$. Then S_1 and S_2 have a common vertex in V_φ^s if and only if S_1 and S_2 are proximal.*

Proposition 5.7 is the key to most results in the dissertation.

Corollary 5.8 *Suppose that φ is an RUPC substitution. Then $T, T' \in \mathcal{F}_\varphi^s$ are proximal if and only if $\widetilde{h_\varphi}(T) = \widetilde{h_\varphi}(T')$.*

Proof Suppose that T and T' are proximal. Then, by Proposition 5.7, $h_\varphi(T) = h_\varphi(T')$, and so there is a vector $v \in \Sigma$ such that $\widetilde{h_\varphi}(T) = \widetilde{h_\varphi}(T') + v$, so $\widetilde{h_\varphi}(T) - \widetilde{h_\varphi}(T') = v \in \Sigma$. Since $V_\varphi^s \cap \Sigma = \{0\}$, we obtain that $v = 0$, and so $\widetilde{h_\varphi}(T) = \widetilde{h_\varphi}(T')$.

For the converse, take T and T' such that $\widetilde{h_\varphi}(T) = \widetilde{h_\varphi}(T')$, then $h_\varphi(T) = h_\varphi(T')$, and so, by Proposition 5.7, T and T' are proximal. $\square$

5.2 Proximality and Essential Balanced Pairs

Though the following theorems, as stated in [BD07], have more restrictions in the hypothesis, because of [BK11] and [BK06], the conclusion follows for any Reducible Pisot substitution. This is the case for all results coming from [BD07] that we quote in this section.

The following proposition states that for reducible unimodular Pisot substitutions satisfying GCC, the notions of proximal, backward-proximal and forward-proximal are all equivalent.

Proposition 5.9 *Suppose that φ is a RUP substitution, and let $S, S' \in \mathcal{F}_\varphi$. If S and S' are either forward or backward proximal, then they have the same geometric realization. If φ satisfies GCC, and S and S' have the same geometric realization, then S and S' are proximal in both directions.*

Let φ be a substitution. A *balanced pair* $\binom{u}{v}$ is a pair of words u, v such that $u, v \in \mathcal{L}_\varphi$ and such that $[u] = [v]$. We can apply the substitution φ to a balanced pair as $\varphi\binom{u}{v} = \binom{\varphi(u)}{\varphi(v)}$. We can also concatenate balanced pairs in the following way $\binom{u_1}{v_1}\binom{u_2}{v_2} = \binom{u_1u_2}{v_1v_2}$. We say that a balanced pair $\binom{u}{v}$ is *reducible* if there are words $u_1, u_2, v_1, v_2 \in \mathcal{L}_\varphi$ such that $\binom{u}{v} = \binom{u_1}{v_1}\binom{u_2}{v_2}$, and both $\binom{u_1}{v_1}$ and $\binom{u_2}{v_2}$ are balanced; otherwise we say that the balanced pair is *irreducible*.

Definition 5.10 *Let φ be a substitution. An irreducible balanced pair $\binom{u}{v}$ is said to be* essential *provided that, for each n there is a balanced pair $\binom{u^{-n}}{v^{-n}}$ such that $\binom{u}{v}$ is a factor of $\varphi^n\binom{u^{-n}}{v^{-n}}$.*

The proof of Lemma 4.10 in [BD07] that shows that there are finitely many irreducible balanced pairs, equally applies to the case of RUPC substitutions. We obtain, then, the following proposition.

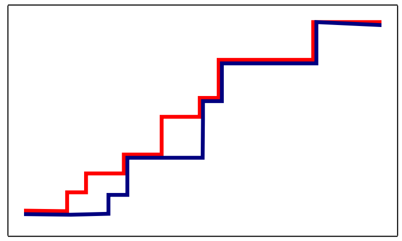


Figure 5.1: A proximal pair forming two bubbles

Proposition 5.11 *Let φ be a RUPC substitution, then there are finitely many essential balanced pairs.*

Suppose that $\varphi : \mathcal{A} \to \mathcal{A}^+$ is a RUPC substitution and let $S = \ldots S_{-1}S_0S_1\ldots$ and $T = \ldots T_{-1}T_0T_1$ be two strands where S_i and T_i are labeled segments in the Pisot space V_φ^P of φ. If S and T are proximal, then S and T have the same geometric realization, by 5.6, and, since, by 5.9, S and T are also backward proximal in the strand metric, we must have that there is a bi-infinite sequence $\{n_k\}_{k\in\mathbb{Z}}$ and a sequence $\{l_k\}_{k\in\mathbb{Z}}$, with $\lim_{k\to\infty} l_k = \infty$ and $\lim_{l\to-\infty} l_k = \infty$ such that we have that $S_{n_k} = T_{n_k}$, $S_{n_k+1} = T_{n_k+1}$,...,$S_{n_k+l_k} = T_{n_k+l_k}$. Thus, the word pair associated with the pair of strands (S,T) must be of the form $\ldots c_{-1}\binom{b_{-1}}{b'_{-1}}c_0\binom{b_0}{b'_0}c_1\binom{b_1}{b'_2}c_2\ldots$, where the c_i represent possibly empty coincidence pairs of the form $\binom{u}{u}$, where $u \in \mathcal{A}^*$, and the non-empty words b_i and b'_i do not agree in the last nor the first letter. Each pair $\binom{b_i}{b'_i}$ can be factored in terms of irreducible balanced pairs $\binom{p_{i,j}}{p'_{i,j}}$. Each word pair of the form $\binom{p_{i,j}}{p'_{i,j}}$ is said to be *obtained from a bubble in a proximal pair*, and the pair of finite strands that correspond to each $\binom{p_{i,j}}{p'_{i,j}}$ is called a *bubble*. A depiction of a bubble appears in Figure 5.1.

For the substitution φ given by $1 \mapsto 12112121$, $2 \mapsto 12112$, we have the proximal pair

$$\ldots\varphi^2(12112121121)\,\varphi(12112121121)\,12112121121\binom{12}{21}121\varphi(121)\,\varphi^2(121)\ldots$$

This proximal pair contains exactly one bubble and is fixed under φ and is such that the origin is located right after the bubble. In general, proximal pairs might not be fixed under the substitution and could contain infinitely many bubbles. This substitution is a power of the Fibonacci substitution.

Proposition 5.12 *Suppose that φ is a RUPC substitution. Let $\binom{u}{v}$ be an irreducible balanced pair for φ. Then $\binom{u}{v}$ is an essential balanced pair for φ if and only if $\binom{u}{v}$ is obtained from a bubble in a proximal pair, or is a coincidence pair $\binom{i}{i}$.*

Definition 5.13 *Let $\mathcal{A}_{\text{EBP}} = \left\{\binom{u}{v} \mid \binom{u}{v} \text{ is an essential balanced pair}\right\}$. And let $\varphi_{\text{EBP}} : \mathcal{A}_{\text{EBP}} \to \mathcal{A}^*_{\text{EBP}}$ the morphism that assigns to each essential pair $\binom{u}{v}$ the factorization of $\binom{\varphi(u)}{\varphi(v)}$ into essential balanced pairs.*

In [BD07] there are examples where φ_{EBP} fails to be primitive. It is customary, though, to call φ_{EBP} a substitution, but we avoid it to emphasize that φ_{EBP} might not be primitive.

One consequence of Proposition 5.12 is proximal pairs are exactly those that are constructed from φ_{EBP} just in the same way as allowed bi-infinite words were constructed for φ.

5.3 The Graph Relations

Throughout this section we assume that all substitutions satisfy GCC. We use Proposition 5.12 throughout.

Let φ be a RUPC substitution with Perron eigenvector v_R. Given a word $w \in \mathcal{L}_\varphi$, denote by $G_n^{;w}$ the graph whose vertex set is the set of possible extensions of w in $\mathcal{L}_\varphi$ of length n; that is,

$$V = \{v \in \mathcal{L}_\varphi : \ |v| = n \text{ and } wv \in \mathcal{L}_\varphi\}, \tag{5.1}$$

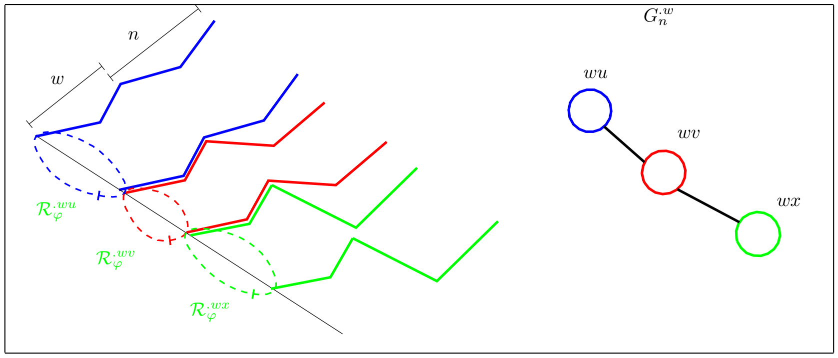


Figure 5.2: Relationship between the connectivity graph and the Rauzy pieces.

and such that there is an edge between v_1 and v_2 provided there are strands $T_{v_1} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_1}$, $T_{v_2} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_2}$ such that T_{v_1} and T_{v_2} are forward proximal, that is, if $\liminf_{t\to\infty} d\left(T_{v_1} - tv_R, T_{v_2} - tv_R\right) = 0$ using the strand distance.

Remark 5.14 Note that there is a path $v_1, v_2, \ldots, v_k$ in $G_n^{.w_2}$ connecting the vertices v_1 and v_k if and only if there is a list of strand pairs

$$E_{(1,2)} = (T_{(1,2)}, T'_{(1,2)}), E_{(2,3)} = (T_{(2,3)}, T'_{(2,3)}), \ldots, E_{(k-1,k)} = (T_{(k-1,k)}, T'_{(k-1,k)}),$$

such that

1. $T_{(i,i+1)} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_i}$, $T'_{(i,i+1)} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_{i+1}}$, and the two are proximal, for $i = 1, 2 \ldots, k-1$; and

2. $T'_{(i,i+1)}, T_{(i+1,i+2)} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_{i+1}}$ for $i = 1, 2, \ldots, k-2$.

Note that in the previous remark, the pair $E_{(i,i+1)} = \left(T_{(i,i+1)}, T'_{(i,i+1)}\right)$ represent an edge in the graph connecting vertices v_i and v_{i+1}.

Definition 5.15 *Let φ be a substitution, $w \in \mathcal{L}_\varphi$ and $m > n$, the* forgetful map *from the vertices of $G_m^{.w}$ to the vertices of $G_n^{.w}$, is the map that sends any vertex $wv_1v_2 \ldots v_n \ldots v_m \in G_m^{.w}$ to the vertex $wv_1v_2 \ldots v_n \in G_n^{.w}$.*

Proposition 5.16 *Let f be the forgetful map from the vertices of $G_m^{\cdot w}$ to the vertices of $G_n^{\cdot w}$. Then there is an edge between the vertices α and β in $G_n^{\cdot w}$, if and only if there exist vertices $\alpha' \in f^{-1}(\alpha)$ and $\beta' \in f^{-1}(\beta)$ such that α' and β' are joined by an edge.*

Proof Suppose that there is an edge between the vertices $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ and $\beta = \beta_1\beta_2 \ldots \beta_n$ in $G_n^{\cdot w}$. Then there are forward proximal strands $T_\alpha \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\alpha}$ and $T_\beta \in \left(\mathcal{F}_\varphi^w\right)^{\cdot w\beta}$. Let α' and β' be right extensions of α and β such that $T_\alpha \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\alpha'}$ and $T_\beta \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\beta'}$. Since α' and β' are extensions of α and β, respectively, we have that $\alpha' \in f^{-1}(\alpha)$ and that $\beta' \in f^{-1}(\beta)$. The strands T_α and T_β provide the edge between α' and β'.

Conversely, let $\alpha, \beta \in G_n^{\cdot w}$ and assume that α' and β' are vertices in $G_m^{\cdot w}$ such that $\alpha' \in f(\alpha)$ and $\beta' \in f(\beta)$. Suppose that there is an edge between α' and β'. Then α' and β' are extensions of α and β, respectively, and there is a forward proximal pair of strands $T_{\alpha'} \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\alpha'}$ and $T_{\beta'} \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\beta'}$. As α' and β' are extensions of α and β, respectively, we have that $T_{\alpha'} \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\alpha'} \subset \left(\mathcal{F}_\varphi^s\right)^{\cdot w\alpha}$ and $T_{\beta'} \in \left(\mathcal{F}_\varphi^s\right)^{\cdot w\beta'} \subset \left(\mathcal{F}_\varphi^s\right)^{\cdot w\beta}$. The pair of strands $T_{\alpha'}$ and $T_{\beta'}$ provide, thus the required edge between α and β. □

Corollary 5.17 *Let $G_m^{\cdot w}$ be a connected graph, then $G_n^{\cdot w}$ is connected for $n < m$.*

5.4 Some Results on Connectedness

We recall that a closed covering $\{C_\alpha : \alpha \in \mathcal{A}\}$ of a space X is *neighborhood finite* provided that, for every point $p \in X$, there exists an open neighborhood U of p such that there are at most finitely many indices α for which $U \cap C_\alpha \neq \emptyset$.

Proposition 5.18 *A topological space Y is connected if and only if every neighborhood-finite closed covering $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of Y has the following property: For each* *pair of sets* $U, U' \in \mathcal{U}$ *there are* $n > 0$ *and sets* $U = U_1, U_2, \ldots, U_n = U' \in \mathcal{U}$ *such that* $U_i \cap U_{i+1} \neq \emptyset, i = 1, \ldots, n-1$.

Proof See exercise 8b in [Dug66], page 117. See also IV 1.6 in [New61] where the requirement is that the closed covering be finite instead of neighborhood finite. □

Definition 5.19 *An* ε-chain *is a finite succession of points,* $a_1, a_2, \ldots, a_q$ *such that* $d(a_i, a_{i+1}) \leqslant \varepsilon$ *for* $i = 1, 2, \ldots, q-1$. *A space is* $\varepsilon -$ connected *if every pair of points in it can be joined by an* ε*-chain of points in the set.*

Proposition 5.20 *A necessary and sufficient condition for a compact space* S *to be connected is that it be* ε*-connected for every positive* ε.

Proof This is proposition IV 5.1 in [New61]. □

Proposition 5.21 *Let* $p : X \to Y$ *be a quotient map, and assume that* $p^{-1}(y)$ *is connected for each* $y \in Y$. *Then an open (or closed)* $F \subset Y$ *is connected if an only if* $p^{-1}(F)$ *is connected.*

Proof This is proposition VI, 3.4 in [Dug66]. □

Proposition 5.22 *Suppose that* φ *is a unimodular Pisot substitution. Then there is an* $M \in \mathbb{N}$ *such that the geometric realization* h_φ *is at most* M*-to-1.*

Proof This is Proposition 6.1 in [BK06]. □

We recall that a *path* of vertices in a graph G, is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that there is an edge between v_i and v_{i+1} for $1 \leqslant i \leqslant n-1$. A graph G is *connected* if for any two vertices of G there is a path between them.

Theorem 5.23 *Let* $\varphi : \mathcal{A} \to \mathcal{A}^+$ *be a RUPC substitution and let* $w \in \mathcal{L}_\varphi$. *Then* $\mathcal{R}^{\cdot w}$ *is connected if and only if for all* $n \in \mathbb{Z}^+$ *the graph* $G_n^{\cdot w}$ *is connected.*

Proof ($\Leftarrow$) It suffices to show by Proposition 5.20 that for any $p \neq q \in \mathcal{R}^{\cdot w}$, and for any $\varepsilon > 0$, there are $n \in \mathbb{Z}^+$, and a sequence of points $p = p_0, p_1, \ldots, p_n = q$ in $\mathcal{R}^{\cdot w}$ such that $d(p_i, p_{i+1}) < \varepsilon$ for $i \in \{0, \ldots, n-1\}$.

Let $p, q \in \mathcal{R}^{\cdot w}$ and let n large enough so that, for any word α with $|\alpha| \geqslant n$, any two strands in $\left(\mathcal{F}^s_\varphi\right)^{\cdot \alpha}$ are within ε of each other.

Let $P, Q \in \left(\mathcal{F}^s_\varphi\right)^{\cdot w}$ be such that $\widetilde{h_\varphi}(P) = p$ and $\widetilde{h_\varphi}(Q) = q$. Let v_p and v_q be the extensions of w such that $P \in \left(\mathcal{F}^s_\varphi\right)^{\cdot wv_p}$ and $Q \in \left(\mathcal{F}^s_\varphi\right)^{\cdot wv_q}$, respectively, with $|wv_p| = |wv_q| = n$.

As the graph $G^{\cdot w}_{|v_p|}$ is connected by hypothesis, there are $k \in \mathbb{N}$, and a path $v_0 = v_p, v_1, \ldots, v_k = v_q$ in $G^{\cdot w}_n$.

We obtain in this way a sequence of pairs of proximal strands

$$\left(S_{(0,1)}, S'_{(0,1)}\right), \left(S_{(1,2)}, S'_{(1,2)}\right), \ldots, \left(S_{(k-1,k)}, S'_{(k-1,k)}\right)$$

with the properties stated in Remark 5.14. Let $p_i = \widetilde{h_\varphi}\left(S_{(i,i+1)}\right)$, for $i = 0, 1, \ldots, k-1$. As $S_{(i,i+1)}$ and $S'_{(i,i+1)}$ are proximal, Corollary 5.8 gives us that $p_i = \widetilde{h_\varphi}\left(S_{(i,i+1)}\right) = \widetilde{h_\varphi}\left(S'_{(i,i+1)}\right)$. Since $S'_{(i,i+1)}, S_{(i+1,i+2)} \in \left(\mathcal{F}^s_\varphi\right)^{\cdot wv_{i+1}}$, we obtain that $d\left(p_i, p_{i+1}\right) < \varepsilon$ by our choice of n.

($\Rightarrow$) Let $n \in \mathbb{N}$ and $v_0, v'_0 \in G^{\cdot w}_n$ such that $wv_0, wv'_0 \in \mathcal{L}_\varphi$ and $|v_0| = |v'_0| = n$.

For any allowed extension v of w with $|v| = n$, the set $\left(\mathcal{F}^s_\varphi\right)^{\cdot wv}$ is a cylinder set, and is, thus, open and closed in $\left(\mathcal{F}^s_\varphi\right)^{\cdot w}$. Also, the set $\mathcal{C} = \left\{\left(\mathcal{F}^s_\varphi\right)^{\cdot wv} : wv \in \mathcal{L}_\varphi, |v| = n\right\}$ is a finite closed covering of $\left(\mathcal{F}^s_\varphi\right)^{\cdot w}$. Since $\widetilde{h_\varphi}|_{\left(\mathcal{F}^s_\varphi\right)^{\cdot w}}$ is continuous and $\left(\mathcal{F}^s_\varphi\right)^{\cdot w}$ is compact, we obtain that $\left\{\widetilde{h_\varphi}(C) : C \in \mathcal{C}\right\}$ is a finite closed covering of $\mathcal{R}^{\cdot w}$.

Let $p \in \widetilde{h_\varphi}\left(\left(\mathcal{F}^s_\varphi\right)^{\cdot wv_0}\right) \subset \mathcal{R}^{\cdot w}$ and $q \in \widetilde{h_\varphi}\left(\left(\mathcal{F}^s_\varphi\right)^{\cdot wv'_0}\right) \subset \mathcal{R}^{\cdot w}$. By hypothesis, $\mathcal{R}^{\cdot w}$ is connected, thus, by Proposition 5.18, there is a sequence of word extensions $v_1, v_2, \ldots, v_k$ such that $p \in \widetilde{h_\varphi}\left(\left(\mathcal{F}^s_\varphi\right)^{\cdot wv_1}\right)$, $q \in \widetilde{h_\varphi}\left(\left(\mathcal{F}^s_\varphi\right)^{\cdot wv_k}\right)$ and $\widetilde{h_\varphi}\left(\left(\mathcal{F}^s_\varphi\right)^{\cdot wv_i}\right) \cap$

$\widetilde{h_\varphi}\left(\left(\mathcal{F}_\varphi^s\right)^{.wv_{i+1}}\right) \neq \varnothing$ for $i \in \{1, 2, \ldots, k-1\}$. Note also that $p \in \widetilde{h_\varphi}\left(\left(\mathcal{F}_\varphi^s\right)^{.wv_0}\right) \cap \widetilde{h_\varphi}\left(\left(\mathcal{F}_\varphi^s\right)^{.wv_1}\right)$ and $q \in \widetilde{h_\varphi}\left(\left(\mathcal{F}_\varphi^s\right)^{.wv_k}\right) \cap \widetilde{h_\varphi}\left(\left(\mathcal{F}_\varphi^s\right)^{.wv_0'}\right)$.

Hence, there is a finite set $\mathcal{E}$ of strand pairs

$$\mathcal{E} = \left\{\left(S_{(0,1)}, S'_{(0,1)}\right), \left(S_{(1,2)}, S'_{(1,2)}\right), \ldots, \left(S_{(k-1,k)}, S'_{(k-1,k)}\right), \left(S_{(k,0')}, S'_{(i,0')}\right)\right\}$$

such that $S_{(i,j)} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_i}$, $S'_{(i,j)} \in \left(\mathcal{F}_\varphi^s\right)^{.wv_j}$ and $\widetilde{h_\varphi}\left(S_{(i,j)}\right) = \widetilde{h_\varphi}\left(S'_{(i,j)}\right)$ for $\left(S_{(i,j)}, S'_{(i,j)}\right) \in \mathcal{E}$. By Corollary 5.8, any strand pair $\left(S_{(i,j)}, S'_{(i,j)}\right) \in \mathcal{E}$ is formed by two forward proximal strands.

By the characterization given in remark 5.14, we obtain that the graph $G_n^{.w}$ is connected. □

From the Theorem 5.23, we obtain easily that if for all w and for all n the graph $G_n^{.w}$ is connected, then all the Rauzy pieces of $\mathcal{R}^{.w}$ are connected. We prove below a somewhat stronger result.

Definition 5.24 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a RUPC substitution, $w \in \mathcal{L}_\varphi$ and $k \in \mathbb{Z}^+$. Let $G_{k+1}^{.w}/\sim$ be the quotient of the graph $G_{k+1}^{.w}$ obtained from identifying any two vertices of the form $wsp \in \mathcal{L}_\varphi$ and $wsq \in \mathcal{L}_\varphi$, with $|s| = n$ and $p, q \in \mathcal{A}$ to a single point, and collapsing any edge between wsp and wsq.*

Proposition 5.25 *For any word w and $k \geqslant 1$, $\left(G_{k+1}^{.w}/\sim\right) \cong G_k^{.w}$, the fibers of $\sim$ are isomorphic to graphs of the form $G_1^{.ws}$ for some word s.*

Proof Let w be a fixed word. We simply note that if there is a proximal pair $\ldots ws_1a\ldots$ and $\ldots ws_2b\ldots$ determining an edge in $G_{k+1}^{.w}$, then there is a proximal pair $\ldots ws_1\ldots$ and $\ldots ws_2\ldots$ determining an edge in $G_k^{.w}$. This shows that the quotient is $G_k^{.w}$.

Consider all the vertices of the form wsp for fixed s, and p is a letter such that wsp is an allowed word. There is a correspondence with the vertices of the graph $G._1^{.ws}$, and the existence of a proximal pair of the form $\ldots wsp_1\ldots$ and $wsp_2\ldots$ is equivalent to the existence of an edge between wsp_1 and wsp_2 in $G_1^{;ws}$, and w in $G_{k+1}^{;w}$. □

Theorem 5.26 *If $\varphi : \mathcal{A} \to \mathcal{A}^+$ is a RUPC substitution, then the following are equivalent:*

1. *For all $w \in \mathcal{L}_\varphi$, $G_1^{;w}$ is connected.*

2. *For all $n \in \mathbb{Z}^+$ and $w \in \mathcal{L}_\varphi$, we have that $G_n^{;w}$ is connected.*

Proof (2) $\Rightarrow$ (1) is trivial.

Next, we show (1) $\Rightarrow$ (2). Let $w \in \mathcal{L}_\varphi$. We prove, by induction, that $G_n^{;w}$ is connected for $n \in \mathbb{Z}^+$.

By hypothesis $G_1^{;w}$ is connected. Now, suppose that $G_n^{;w}$ is connected. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the vertices of $G_n^{;w}$, $G_n^{;w}$ that is, $w\alpha_i \in \mathcal{L}_\varphi$ and $|\alpha_i| = n$ for $1 \leqslant i \leqslant k$. By induction hypothesis $G_n^{;w}$ is connected, and by (1), all the graphs $G_1^{;w\alpha_i}$ are connected for $1 \leqslant i \leqslant k$.

By Proposition 5.25, the graph $G_n^{;w}$ is the quotient of the graph $G_{n+1}^{;w}$ obtained by collapsing in the graphs $G_1^{;w\alpha_i}$ the vertices $\{w\alpha_i k : w\alpha_i k \in \mathcal{L}_\varphi, |k| = 1\}$ into the vertex $\alpha_i \in G_n^{;w}$ for $1 \leqslant i \leqslant k$. Since the quotient $G_1^{;w}$ is connected by induction hypothesis, and the fibers $G_1^{;w\alpha_i}$ are connected by (1) for $1 \leqslant i \leqslant k$, we obtain that the graph $G_{n+1}^{;w}$ is connected by Proposition 5.21. □

Proposition 5.27 *If, for all words w, the graph $G_1^{;w}$ is connected, then all the Rauzy pieces are connected.*

Proof Let w be fixed. We proceed by induction. By hypothesis, $G_1^{;w}$ is connected. Now, suppose that $G_k^{;w}$ is connected. By the Proposition 5.25, $G_k^{;w}$ is the quotient of

$G^{\cdot w}_{k+1}$ with fibers of the form $G^{\cdot ws}_1$. Since the fibers $G^{\cdot ws}_1$ are connected by hypothesis, then $G^{\cdot w}_{k+1}$ is connected since both the quotient and the fibers are connected.

Thus all the graphs $G^{\cdot w}_k$ are connected for $k \in \mathbb{N}$, and hence the Rauzy piece $\mathcal{R}^{\cdot w}$ is connected. Since w is arbitrary, this completes the proof. $\square$

CHAPTER 6

INVERTIBLE AND ARNOUX-RAUZY SUBSTITUTIONS

6.1 Two-letter Substitutions

Our interest in the reducible case stems from the question whether connectedness of the Rauzy Fractal is a topological invariant. This section is devoted to the two-letter case.

If v is a vector in $\mathbb{R}^d$, we write $v > 0$ to indicate that all the entries of v are positive. Recall that if φ is a substitution, then $\mathcal{L}_\varphi$ is the set of all allowed words for φ.

Definition 6.1 *The* complexity *function of a substitution φ is the function $p(n) = \#\{w|\ w \in \mathcal{L}_\varphi$ and $|w| = n\}$.*

Definition 6.2 *A* sturmian *substitution is a substitution φ with complexity $n+1$.*

Sturmian substitutions have only two words of length one, and, hence sturmian substitutions have a two-letter alphabet. Sturmian substitutions are also primitive and non shift-periodic, since they have unbounded complexity (See [Fog02], page 3). Any sturmian substitution is unimodular by part (2) of Proposition 6.3 below. By the Perron Froebinius theorem, one eigenvalue must be bigger than one and real, thus the other eigenvalue must be a non-zero eigenvalue of modulus less than one. Thus, Sturmian substitutions are Pisot substitutions, and we can define the Rauzy fractal for them. Recall that an invertible substitution is a substitution whose induced map on the free group generated by its alphabet is invertible. For the definition of coding of a rotation, see [Fog02], page 151.

Proposition 6.3 *Let φ be a substitution on two letters. The following statements are equivalent:*

1. *The substitution φ is Sturmian.*

2. *The substitution φ is invertible.*

3. *The Rauzy Fractal R_φ is connected.*

4. *The substitution φ is generated by the substitutions*

$$\alpha : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}, \beta : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 1 \end{cases}, \ \textit{and}, \gamma : \begin{cases} 1 \mapsto 21 \\ 2 \mapsto 1 \end{cases}$$

5. *For any prefix u of a φ-periodic word W_φ, we have that*

$$\# \{ w : \ w \textit{ is a return word for } W_\varphi \textit{ for the prefix } u \} = 2.$$

6. *A right-infinite word allowed for φ is the coding of a rotation in a circle divided into two intervals for some initial point and some angle.*

Proof (1) $\Leftrightarrow$ (3) is proved in [MH40] and [CH73] ; (2) $\Leftrightarrow$ (4) is proved in [WW94], and (1) $\Leftrightarrow$ (4) is proved in [Séé98] (see also [Fog02], page 300). The equivalence (1) $\Leftrightarrow$ (5) is proved in [Vui01]. See [Fog02], pages 152 and 182, for a proof of (6) $\Leftrightarrow$ (1). Also, (1) $\Leftrightarrow$ (2) is proved in [MZ93]. □

Corollary 6.4 *Let φ be invertible on two letters, and let W be a right-infinite fixed point for φ. If u is a prefix of w, then the substitution $\varphi_{W,u}$ is invertible.*

Proof Let φ be w and v as above. Let $\mathcal{W} = \{w_1, w_2, \ldots, w_d\}$ be the set of return words, and let $\theta : \{1, 2, \ldots, d\} \to \mathcal{W}$ given by $\theta(i) = w_i$. Let W be a right-infinite sequence fixed under φ. By (5) of Proposition 6.3, we have that W has exactly two return words for any finite prefix u of W. Now, take a finite prefix v of $D_u(W)$. Then, by Proposition 3.3, we have that $D_v(D_u(W)) = D_{\theta(v)u}(W)$, which by (5) of Proposition 6.3, only uses two letters; i.e. there are exactly two return words for any v. An application of (2) of Proposition 6.3 implies that φ_u is invertible. □

We proved Corollary 6.4 using Durand's result of Proposition 3.3. It would be desirable to have a proof of Corollary 6.4 relying entirely in the characterization of an invertible matrices in terms of a coding of an irrational rotation in a circle partitioned into two intervals; as we would then gain an understanding of the Poincare map resulting from restricting the irrational rotation to the set of all points whose itineraries follow the word u. In Chapter 7, we discuss how the Rauzy Fractal is changed after we rewrite a substitution, a process that is closely related to the process of finding the substitution on return words.

Proposition 6.5 *Let $\lambda > 0$. Then there are only finitely many matrices A with positive entries for which λ is the Perron eigenvalue for A.*

Proof By the spectral theorem, and taking as norm $\|v\|_\infty = \max\{|v_i|\}$, which induces the norm $\|A\|_\infty = \max_i \sum_j |a_{ij}|$, we have that $\lambda = \lim_{n\to\infty} \sqrt[n]{\|A^n\|_\infty}$. Hence, there exists $n_0 \in \mathbb{Z}^+$ such that $\left|\lambda - \sqrt[n]{\|A^{n_0}\|_\infty}\right| < 1$, and thus $\|A^{n_0}\|_\infty < (1+\lambda)^{n_0}$, from which we obtain that $\max_{i,j}\left|(A^{n_0})_{ij}\right| < \max_i \sum_j \left|(A^{n_0})_{ij}\right| < (1+\lambda)^{n_0}$, which implies that there are finitely many possible options for the entries of A^{n_0}. Since each element $(A^{n_0})_{ij} > 0$ is the sum of n_0 products of positive elements of A, we obtain that $|a_{ij}| \leqslant \max_{i,j}(A^{n_0})_{ij} < (1+\lambda)^{n_0}$. Thus, there are finitely many such matrices A having λ as the Perron eigenvalue. □

Recall that a substitution φ is said to have a prefix problem provided that there are letters $i \neq j$ such that $\varphi(i) = \varphi(j)\ldots$, and φ is said to have a suffix problem provided that there are letters $i \neq j$ such that $\varphi(i) = \ldots\varphi(j)$.

Lemma 6.6 *Let φ be a primitive, non shift-periodic substitution on two letters. If φ does not have a prefix problem, then φ^2 is of the form $\varphi^2(1) = x1x_1$, $\varphi^2(2) = x2x_2$ for some $x, x_1, x_2 \in \mathcal{A}^*$. If φ does not have a suffix problem, then φ^2 is of the form $\varphi^2(1) = y_1 1y$, $\varphi^2(2) = y_2 2y$ for some $y, y_1, y_2 \in \mathcal{A}^*$*

Proof Suppose φ does not have a prefix problem. Since φ has no prefix problem there are $a \neq b \in \mathcal{A}$ and $x' \in \mathcal{A}^*$ such that $\varphi(1) = x'a\ldots$ and $\varphi(2) = x'b\ldots$. That is, the pair (a, b) is the first disagreement. Then either $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$. In either case $(\varphi(a), \varphi(b)) = (x'1\ldots, x'2\ldots)$. Thus, $\varphi^2(1) = \varphi(x')\varphi(a)\ldots = \varphi(x')x'1\ldots$ and $\varphi^2(2) = \varphi(y)x'2\ldots$. Setting $x = \varphi(x')x'$ we obtain the form required.

The proof for when φ does not have a suffix problem is similar and will be omitted.

□

For the following proposition, note that the symbolic representation of a strand determines it uniquely up to translation parallel to the right Perron eigenvector v_R

Proposition 6.7 *Suppose that ψ is a IUP substitution in two letters with exactly one proximal pair up to translation. Suppose that there exists exactly one bubble in the proximal pair for ψ. Then the substitution $\varphi = \psi^2$ has a prefix or a suffix problem.*

Proof Suppose, on the contrary, that φ has neither a prefix nor a suffix problem. Note that φ is also a substitution with exactly one proximal pair, and that there is exactly one bubble B in the proximal pair for φ.

Thus, φ is of the form listed in Lemma 6.6.

The bubble B must be of the form $B = \binom{1u2}{2v1}$ with $u, v \in \mathcal{A}^*$, for, if B were of the form $\binom{1\ldots1}{2\ldots2}$, then, because the alphabet has only two letters, B would be a reducible pair.

Because the substitution does not have a prefix problem, then, by Lemma 6.6 there are words $x, x_1, x_2 \in \mathcal{A}^*$ such that $\varphi(1) = x1x_1$, $\varphi(2) = x2x_2$. Also, since φ does not have a suffix problem, there are words $y, y_1, y_2 \in \mathcal{A}^*$ such that$\varphi(1) = y_1 1y$, $\varphi(2) = y_2 2y$

Also, since there is exactly one bubble, which occurs in exactly one proximal pair, we must have that $\varphi(B) = c_1 B c_2$, where c_1, c_2 are coincidences. Since φ is primitive, either c_1 or c_2 is non-empty.

Since $\varphi(B) = \binom{\varphi(1)\varphi(u)\varphi(2)}{\varphi(2)\varphi(v)\varphi(2)} = x\binom{1x_1\varphi(u)y_1 2}{2x_2\varphi(v)y_2 1}y$, and $\varphi(B) = c_1 B c_2 = c_1\binom{1u2}{2v1}c_2$, we must have that $x = c_1$, $y = c_2$, $x_1\varphi(u)\,y_1 = u$ and $x_2\varphi(v)\,y_1 = v$. This is only possible if $x_1 = x_2 = y_2 = u = v = \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is the empty word. Thus, $B = \binom{12}{21}$ and $\varphi(B) = \binom{\varphi(1)\varphi(2)}{\varphi(2)\varphi(1)} = x\binom{12}{21}y$.

Thus, $\varphi(12) = x12y$ and $\varphi(21) = x21y$. Since φ does not have a prefix nor a suffix problem, and $\varphi(12) = x12y$, we obtain that $\varphi(1) = x1$, and $\varphi(2) = 2y$; and from $\varphi(21) = x21y$, we must have that $\varphi(2) = x2$, and $\varphi(1) = 1y$. Thus $x1 = 1y$ and $2y = x2$. From $x1 = 1y$, we obtain that $x = 1\ldots$ and $y = \ldots1$; from $2y = x2$ we obtain that $y = \ldots2$ and $x = 2\ldots$, which is a contradiction.

Hence, φ must have either a suffix or a prefix problem. $\square$

The procedure in [BD01] to solve for a prefix or suffix problem described in Section 3.5.2 is simplified for an alphabet of two letters. The reason is that, for a substitution φ (primitive and non shift-periodic), we must have that $\varphi(1) \neq \varphi(2)$. We simplify the notation as follows.

Let $i, j \in \mathcal{A} = \{1, 2\}$ and assume that $i \neq j$. Let $\sigma_i : \mathcal{A} \to \mathcal{A}^+$ be given by $\sigma_i(i) = ji$ and $\sigma_i(j) = j$, and let $\tau_i(i) = ij$ and $\tau_i(j) = j$. Note that σ is invertible with inverse σ_i^{-1} given by $\sigma_i^{-1}(i) = j^{-1}i$ and $\sigma_i^{-1}(j) = j$. Similarly, τ_i is invertible.

Proposition 6.8 *Let $\varphi : \mathcal{A} = \{1, 2\} \to \mathcal{A}^+$ be a substitution on two letters and of the form $\varphi = \varphi' \circ p$, where p is one of $\sigma_1, \sigma_2, \tau_1, \tau_2$, and $\varphi' : \mathcal{A} \to \mathcal{A}^+$. Let $\varphi_1 = p \circ \varphi'$. Then φ_1 is a two letter substitution, and $[\varphi_1]$ has the same eigenvalues.*

Proof The diagram

$$\begin{array}{ccc} & \xrightarrow{\varphi} & \\ p\downarrow & \nearrow_{\varphi'} & \downarrow p \\ & \xrightarrow[\varphi_1]{} & \end{array},$$

induces a diagram of matrices

$$\begin{array}{ccc} & \xrightarrow{[\varphi]} & \\ [p]\downarrow & \nearrow_{[\varphi']} & \downarrow [p] \\ & \xrightarrow[{[\varphi_1]}]{} & \end{array}.$$

If v is an eigenvector for an eigenvalue λ, we obtain $[\sigma_1][\varphi] v_R = \lambda [p] v$, and $[p] v \neq 0$ since $[p]$ is invertible. Thus, the eigenvalues and their corresponding eigenvectors for $[\varphi_1]$ are the same as those for $[\varphi]$. Since the application of p does not alter the number of letters, we have that φ_1 is a substitution on two letters. $\square$

Definition 6.9 *Let φ be a substitution. We say that φ is* proper on the left *if there are $n \in \mathbb{Z}^+$ and $b \in \mathcal{A}$ such that $\varphi^n(i) = b\ldots$ for all $i \in \mathcal{A}$. We say that φ is* proper on the right *if there are $n \in \mathbb{Z}^+$ and $e \in A$ such that $\varphi^n(i) = \ldots e$ for all $i \in \mathcal{A}$. We say that φ* preserves the first letter *if $\varphi(i) = i\ldots$ for all $i \in \mathcal{A}$. We say that φ* preserves the last letter *if $\varphi(i) = \ldots i$ for all $i \in \mathcal{A}$.*

Lemma 6.10 *If φ is a substitution on two letters, then $|\varphi^2(i)| \geqslant 2$ for all i, and either (1) φ^2 is proper from the left, and there is a letter b such that $\varphi^2(i) = b\ldots$ for all i, or (2) φ^2 preserves the first letter. Also, we have that either (3) φ^2 is proper from the right, and there is a letter e such that $\varphi^2(i) = \ldots e$ for all i, or (4) φ^2 preserves the last letter.*

Proof Let $\mathcal{A} = \{1, 2\}$, and let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a substitution.

To show that $|\varphi^2(i)| \geqslant 2$ for all $i \in \mathcal{A}$, suppose, on the contrary, that there is a letter $i_0 \in \mathcal{A}$ such that $|\varphi^2(i_0)| = 1$, which implies that $|\varphi(i_0)| = 1$. If $\varphi(i_0) = i_0$, then the substitution φ is not primitive, since $|\varphi^n(i_0)| = 1$ for all $n \in \mathbb{N}$, and we reach a contradiction. If $\varphi(i_0) = j_0$ where $j_0 \neq i_0$ then we have that $|\varphi(j_0)| = 1$, and thus $|\varphi(i)| = 1$ for all i, which implies that φ is not primitive, and we reach a contradiction. Thus $|\varphi^2(i)| \geqslant 2$ for all $i \in \mathcal{A}$.

We show next that φ satisfies either (1) or (2).

Let $f : \mathcal{A} \to \mathcal{A}$ be the function such that $f(i)$ is the first letter of $\varphi(i)$. If f is injective, then φ^2 has the form (1). If f is injective, then f is either the identity or a transposition, in either case, we have that f^2 is the identity, in which case φ^2 preserves the first letter, and, thus, φ^2 has form (2)

The proof that φ^2 satisfies either form (3) or form (4) is similar, and is omitted.

□

The following lemma is well known.

Lemma 6.11 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a substitution. If φ^n is invertible, then φ is invertible. If there is an invertible substitution p such that $p \circ \varphi$ is invertible, then φ is invertible.*

Proof If φ^n is invertible, then the extension of φ^n to the free group generated by $\mathcal{A}$ has an inverse q. Denote the extension of φ to the free group as φ again, and denote the identity homomorphism of the free group as I. That is, there is an automorphism q such that $q \circ \varphi^n = \varphi^n \circ q = I$ where I is the identity map. If $n = 1$, then φ is invertible as required. If $n > 1$, let $l = q \circ \varphi^{n-1}$, which is a left inverse for φ, and let $r = \varphi^{n-1} \circ q$ which is a right inverse for φ. Now, $(l \circ \varphi) \circ r = I \circ r = r$ and $l \circ (\varphi \circ r) = l \circ I = l$, from which we obtain that $l = r$, and, thus, φ is invertible.

If $p \circ \varphi$ is invertible, then there is an automorphism q such that $q \circ (p \circ \varphi) = (p \circ \varphi) \circ q = I$, from which we have that $(q \circ p) \circ \varphi = p \circ (\varphi \circ q) = I$ thus $q \circ p$ is a left inverse for φ, and $\varphi \circ q$ is a right inverse for p. Since $\varphi \circ q$ is a right inverse of p and p is invertible, we obtain that $\varphi \circ q$ is a left inverse for p, thus $(\varphi \circ q) \circ p = \varphi \circ (q \circ p) = I$, which implies that $q \circ p$ is a right inverse for φ. This implies that $q \circ p$ is an inverse of φ and, thus, φ is invertible. $\square$

The following proposition, whose proof is an adaptation of the proof of Theorem 3.16 in [BD01], equally applies to proper or non proper substitutions.

Proposition 6.12 *Let $\mathcal{A} = \{1, 2\}$, and let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a sequence of substitutions $\varphi_i : \mathcal{A} \to \mathcal{A}^+$ such that φ_0 is primitive, and for each $i \in \mathbb{N}$, there are $\varphi'_i : \mathcal{A} \to \mathcal{A}^+$ and $p_i \in \{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ such that $\varphi_i = \varphi'_i \circ p_i$, and $\varphi_{i+1} = p_i \circ \varphi'_i$. Then there are numbers $k \in \mathbb{N}$ and $l \in \mathbb{Z}^+$ such that $\varphi_k = \varphi_{k+l}$, the substitution φ_0 is invertible, and the Rauzy Fractal for φ_0 is connected.*

Proof Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a sequence of substitutions satisfying the hypothesis of the proposition.

For each $i \in \mathbb{N}$, by hypothesis, we obtain a diagram

$$\begin{array}{ccc} & \xrightarrow{\varphi_i} & \\ p_i \downarrow & \nearrow_{\varphi_i'} & \downarrow p_i \\ & \xrightarrow[\varphi_{i+1}]{} & \end{array} ,$$

where $p_0 \in \{\sigma_1, \sigma_2, \tau_1, \tau_2\}$. Since $[\varphi_{i+1}]$ has the same Perron eigenvalue by Proposition 6.8 and since, by Proposition 6.5, there are only finitely many substitutions with positive entries having a given $\lambda > 0$ as its Perron eigenvalue, we obtain that there exist $n \in \mathbb{Z}^+$ and $l \in \mathbb{Z}^+$ such that $\varphi_{n+l} = \varphi_n$.

That is, we have that $(p_{n+l-1} \circ \ldots \circ p_n) \circ \varphi_n = \varphi_n \circ (p_{n+l-1} \circ \ldots p_n)$, and, thus, $(p_{n+l-1} \circ \ldots \circ p_n)^2 \circ (\varphi_n)^2 = (\varphi_n)^2 \circ (p_{n+l-1} \circ \ldots p_n)^2$. Let $p = (p_{n-1} \circ \ldots \circ p_1)^2$ and $\varphi = (\varphi_n)^2$. Then $p \circ \varphi = \varphi \circ p$, and p is an invertible substitution. By Lemma 6.10, both p and φ have the forms stated there.

Claim: There exists a fixed point W for φ, and a suffix u of W such that the induced return word substitutions φ_u and p_u satisfy $(\varphi_u)^n = (p_u)^m$ for some $m, n \in \mathbb{Z}^+$.

Proof of claim By Lemma 6.10, we have that, for all $i \in \mathcal{A}$, $|\varphi(i)| > 1$.

Case 1: Suppose that φ is left-proper, and let b be such that, for all $i \in \mathcal{A}$, $\varphi(i) = b \ldots$. Let W be the right infinite word of the form $W = b \ldots$ that is fixed by φ. Then $W = b \ldots = \varphi(b \ldots) = \varphi^n(b \ldots) = \varphi^n(b) \ldots$, and $\varphi^n(b)$ is a prefix of W such that $\lim_{n \to \infty} |\varphi^n(b)| = \infty$. From $p \circ \varphi = \varphi \circ p$, we obtain that $p(W) = p(b \ldots) = p \circ \varphi^n(b \ldots) = p \circ (\varphi^n(b)) \ldots = \varphi^n(r(b)) \ldots = \varphi^{n-1}(\varphi(r(b))) \ldots = \varphi^{n-1}(b \ldots) \ldots = \varphi^{n-1}(b) \ldots$ Thus, W is fixed by p and, by Proposition 21 in [Dur98], there is a prefix u of W such that $(\varphi_u)^n = (p_u)^m$

Case 2: Suppose that p is left-proper. The proof is just as that of Case 1, interchanging the roles of p and φ.

Case 3; Suppose that both p and φ preserve the first letter. Let $W = 1\ldots = \varphi^n(1)\ldots$ be a fixed word for φ. Then $p(W) = p \circ \varphi^n(1)\ldots = \varphi^n(p(1))\ldots = \varphi^n(1)\ldots$ and, thus, p also fixes the same word, which, by Proposition 21 in [Dur98] implies that there is a prefix u of W such that $(\varphi_u)^n = (p_u)^m$. This finishes the proof of the claim. □

Since $(\varphi_u)^n = (p_u)^m$, then $\varphi^n(w) = p^m(w)$ for any w that is a return word for the prefix u of the common fixed word W. Since the strand spaces for $\varphi_u^n$and p_u^m are the same as those for φ_u and p_u respectively, and any strand in $\mathcal{F}_\varphi$ and $\mathcal{F}_p$ can be decomposed uniquely into return words, we conclude that the strand space for φ is the same as that for p. Since p is invertible, we obtain that the Rauzy Fractal is connected. Since the Rauzy Fractals are the same, we obtain that the Rauzy fractal for φ is connected, which, in turn, implies that φ is invertible.

Next, we use Lemma 6.11 repeatedly. Since φ is invertible, then we have that $(\varphi_n)^2$ is invertible, which implies that φ_n is invertible. Now, $\varphi_0^n = (\varphi_0' \circ \ldots \circ \varphi_n') \circ (p_{n-1} \circ \ldots \circ p_0)$, and $\varphi_n^n = (p_{n-1} \circ \ldots \circ p_0) \circ (\varphi_0' \circ \ldots \circ \varphi_n')$. As φ_n is invertible, we get that φ_n^n is invertible, which implies that $(p_{n-1} \circ \ldots \circ p_0) \circ (\varphi_0' \circ \ldots \circ \varphi_n')$ is invertible, from which, since $(p_{n-1} \circ \ldots \circ p_0)$ is invertible, we obtain that $(\varphi_0' \circ \ldots \circ \varphi_n')$ is invertible. Thus $\varphi_0^n = (\varphi_0' \circ \ldots \circ \varphi_n') \circ (p_{n-1} \circ \ldots \circ p_0)$ is invertible. As φ_0^n is invertible, we obtain that φ_0 is invertible.

Finally, we obtain by Proposition 6.3 that the Rauzy Fractal is connected. □

Notice that, a priori, we do not know if a substitution is invertible when its return word substitution is.

Proposition 6.13 *Let φ be a IUP substitution on two letters. If $\mathcal{R}_\varphi$ is connected, then there exists exactly one proximal pair containing exactly one bubble.*

Proof By [BD02], φ satisfies GCC, and thus two strands share the same geometric realization if and only if the strands are proximal.

Suppose that R_φ is connected. Since $\mathcal{R}_\varphi$ is unidimensional and connected, there exists exactly one point $p \in \mathcal{R}_\varphi^{\cdot 1} \cap \mathcal{R}_\varphi^{\cdot 2}$. Any bubble in any proximal pair, must contain edges associated to 1 and 2, and, thus, the projection of any vertex of the beginning of a bubble must project into p.

Let B_1 and B_2 be two bubbles. By translating, we can assume that B_1 and B_2 have p as their initial point and having as final points q_1 and q_2. Since R_φ is connected, $R_\varphi^{1\cdot} \cap R_\varphi^{2\cdot}$ is a singleton $\{p'\}$. Thus, by translating the final points must project into p'. Since they have the same beginning, this implies that the Perron eigenvector is in $\mathbb{Z}^2$, which is impossible for a Pisot substitution on two letters. Thus, the final point of the bubbles is also the same.

Suppose that $B_1 \neq B_2$. Take four strands S_1, S_2, S_3 and S_4, where S_1 and S_2 form the first bubble, and S_3 and S_4 form the second bubble, and suppose that the bubbles start at the point p. Since they all have a vertex in common, any two of S_1, S_2, S_3 and S_4 are proximal. Since $B_1 \neq B_2$ and the alphabet has only two letters, we have that there is strand $S' \in \{S_{1,}, S_2\}$ and a strand $S'' \in \{S_3, S_4\}$ so that S' and S'' form a bubble with a beginning vertex, or a final vertex, p' different from those for B_1 and B_2. The point p' must project to p, but that would imply that the Perron eigenvector is in $\mathbb{Z}^2$, which is impossible for a Pisot Substitution. Hence $B_1 = B_2$.

Since, by Proposition 5.12, proximal pairs can always be found by iterating on the bubbles, and there is only one bubble, we obtain that there is exactly one pair of proximal strands up to translation. $\square$

Theorem 6.14 *If φ is a IUP substitution on two letters, then the Rauzy fractal R_φ is connected if and only if φ has exactly one bubble, and, thus, exactly one proximal pair with one bubble in it.*

Proof It is immediate from Propositions 6.7, 6.12, and 6.13. □

We recall, from Chapter 2, that a composant in a tiling space is the same as an arc component, it is the orbit of a tiling under the translation flow, and is uniquely represented by a bi-infinite word. Proximal pairs up to translation flow determine a pair of *proximal composants*.

Theorem 6.15 *Connectedness of the Rauzy Fractal is a topological invariant among IUP substitutions on two letters.*

Proof Let α and β be IUP substitutions on two letters with homeomorphic tiling spaces, and suppose that $\mathcal{R}_\alpha$ is connected. We need to show that $\mathcal{R}_\beta$ is connected.

Since all IUP substitutions on two letters satisfy the coincidence condition (See [BD02]), proximal composants are sent to proximal composants under any homeomorphism. Hence β has exactly one pair of proximal composants.

Note that β_{EBP} is non shift-periodic, for, otherwise, β would be periodic.

Claim: There exists exactly one bubble in the proximal pair of β.

Proof of Claim If $P = (S, S')$ is the proximal pair for β, and (S, S') contained several bubbles of the same symbolic representation, then the composant of S would contain more than one fixed point, which is impossible, since that would imply that $[S]$ is a proper suffix of itself, and hence $[S]$ is periodic, which is a contradiction.

If $P = (S, S')$ contained more than one bubble of different symbolic representation, then using the substitution β_{EBP}, we would obtain two essential irreducible balanced pairs B_1 and B_2 corresponding to the two different bubbles such that $(\beta_{\text{EBP}})^k(B_1)$ has both B_1 and B_2 as a factor, and similarly for B_2. Just as is the case with regular substitutions, we can produce, applying β_{EBP}, uncountably many allowed symbolic pairs each corresponding to proximal pairs. This implies that there are uncountably many proximal pairs for β. $\square$

Now, if β has exactly one bubble in its proximal pair, we obtain that $\mathcal{R}_\beta$ is connected as desired. $\square$

If S_1 and S_2 are members of a proximal pair with symbolic form

$$\begin{matrix} \dots c_{-1}u_{-1}c_0u_0c_1u_1\dots \\ \dots c_{-1}v_{-1}c_0v_0c_1v_1\dots \end{matrix},$$

we represent them symbolically as

$$\dots c_{-1}\binom{u_{-1}}{u_{-1}}c_0\binom{u_0}{v_0}c_1\binom{u_1}{v_1}c_2\dots$$

to emphasize the bubbles $\binom{u_n}{v_n}$. At least one bubble of the $\binom{u_n}{v_n}$ is not empty.

Definition 6.16 *A* k-right-extension *of a word w is a word of the form ws for some word s with $|s| = k$.*

Definition 6.17 *If φ is a substitution, a* right special factor *$w \in \mathcal{L}_\varphi$ is a word that admits more than one 1-right-extension.*

We include the following proposition, due to the similarities with the case for Arnoux-Rauzy substitutions that we analyze in Section 6.2 below.

Proposition 6.18 *Let φ be an IUP substitution on two letters. Suppose that for all n there exists exactly one left special factor $w_n \in \mathcal{A}^+$ such that $|w_n| = n$. Suppose that there is exactly one pair of proximal strands (S_1, S_2), and that they form exactly* *one bubble. Then $\mathcal{R}_\varphi$ is connected, and all the Rauzy pieces $\mathcal{R}_\varphi^{;w}$ are connected for any $w \in \mathcal{L}_\varphi$.*

Proof By hypothesis, there is exactly one special factor. Since the strands S_1 and S_2 are proximal with exactly one bubble of the form $\binom{1u_1}{2u_2}$, then pair of strands is of the form $\ldots w_n \binom{1u_1}{2u_2} \ldots$ for some $u_1, u_2 \in \mathcal{A}^+$, where w_n is the unique special factor of length n.

By Theorem 4.19, it suffices to show that, for all $w \in \mathcal{A}^+$, $G_1^{;w}$ is connected. If w is not a left special factor, then $G_1^{;w}$ has exactly one vertex, and is, thus, connected. If w is a left special factor, then $w = w_n$ for some $n \in \mathbb{N}$, and $G_1^{;w}$ contains two vertices associated to the words $w_n 1$ and $w_n 2$. These vertices have an edge connecting them given by the proximal pair $\ldots w_n \binom{1u_1}{2u_2} \ldots$.

Thus, all the Rauzy pieces are connected. □

Conjecture 6.19 *Suppose that φ is an IUP substitution on two letters with $\mathcal{R}_\varphi$ connected. If ψ is a rewriting of φ, then $\mathcal{R}_\psi$ is connected.*

Conjecture 6.20 *Suppose that φ is an IUPC substitution on two letters such that $\mathcal{R}_\varphi$ is connected. Let ψ be a primitive substitution such that $\mathcal{T}_\varphi \cong \mathcal{T}_\psi$. Then ψ is a RUPC substitution, and $\mathcal{R}_\psi$ is connected.*

The following example suggests that the same result is not true for three or more letters in the reducible case. In Figure 6.1, we have a picture of the Rauzy Fractal of the substitution φ given by $1 \mapsto 32$, $2 \mapsto 1$, $3 \mapsto 2$. The Rauzy Fractal R_φ is shown to be connected in [ST+09] (φ is the substitution σ_2 in [ST+09]). In Figure 6.2 we have the Rauzy Fractal of a rewriting φ_1 of φ. Figure 6.2 suggests that R_{φ_1} is not connected.

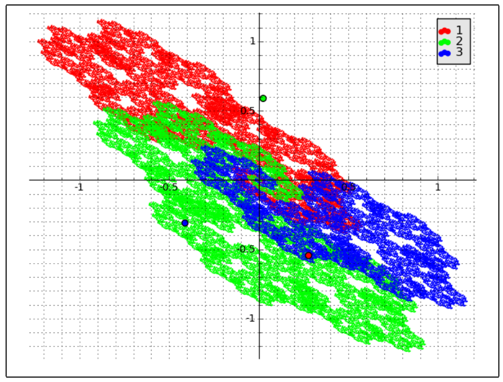


Figure 6.1: The Rauzy Fractal of the IUP substitution $1 \mapsto 32$, $2 \mapsto 1$ and $3 \mapsto 2$

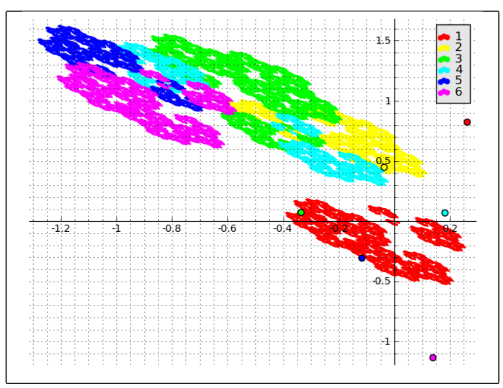


Figure 6.2: The Rauzy fractal of the RUP substitution $1 \mapsto 1234356$, $2 \mapsto 12356$, $3 \mapsto 136$, $4 \mapsto 14234356$, $5 \mapsto 142356$, $6 \mapsto 14236$ which is a rewrite of the substitution $1 \mapsto 32$, $2 \mapsto 3$, $3 \mapsto 1$. The rewriting was computed in Chapter 3.

6.2 Arnoux-Rauzy Substitutions

Recall that a A k-right-extension of a word w is a word of the form ws for some word s with $|s| = k$, and that a right special factor $w \in \mathcal{L}_\varphi$ is a word that admits more than one 1-right-extension.

Definition 6.21 *An* Arnoux-Rauzy *substitution is a substitution with complexity* $p(n) = 2n + 1$ *with the property that, for every* $k \in \mathbb{Z}^+$*, there is exactly one left special factor, and exactly one right special factor.*

By definition, all Arnoux-Rauzy substitutions are substitutions on three letters, since $p(1) = 3$.

As mentioned in the introduction, the Tribonacci substitution $1 \mapsto 12$, $1 \mapsto 13$, $3 \mapsto 1$ is an example of an Arnoux-Rauzy substitution. The condition of the existence of a unique left special factor, and a unique right-special factor means that for each n there exists a unique allowed word w of length n such that $w1$, $w2$ and $w3$ are allowed, and for each k there exists a unique allowed word w' of length k such that $1w$, $2w$ and $3w$ are allowed (see [AR91] and [Fog02],pages 6, 232 and 368, for example).

Throughout this section, we abuse notation and denote an proximal pair by its bi-infinite symbolic representation. In [AI01] it is shown that any Arnoux-Rauzy Substitution is an IUP substitution.

Recently, it was shown in [BJS11] that all Arnoux-Rauzy substitutions are connected through a different method than ours. This result had been announced by [Can03], but a proof with the method used in [Can03] has not been published.

The following proposition is well known.

Proposition 6.22 *Let* φ *be a primitive substitution. A finite allowed word* w *can be completed into an allowed left infinite word,* pw*, and into an allowed right infinite* *word ws. Also, any left-infinite or right-infinite allowed word can be completed to a bi-infinite allowed word.*

Proof First we show that a finite word w can be extended into a bi-infinite word.

Since φ is primitive, we can find a sequence $\{w_i\}_{i\in\mathbb{N}}$ such that $w_0 = w$, and $\varphi(w_{(i+1)}) = p_i w_i s_i$, where p_i, s_i are non-empty. Consequently, the word $\varphi^i(w_i) = \varphi^i(s_i)\ldots\varphi^2(p_2)\,\varphi(p_1)\,p_0 w_0 s_0 \varphi(s_1)\,\varphi^2(s_2)\ldots\varphi^i(s_1)$ provides instructions to extend w arbitrarily on both sides as i becomes larger and larger. Hence, we obtain a bi-infinite word having w as a factor.

Once we have a bi-infinite word, we can extract from it a left-extension or right extension as desired.

Consider now a left-infinite word w. By the previous case, each suffix of w can be completed to the right by one letter to the right. Since the alphabet is finite, there is a letter that is common to all extensions of suffixes of w. We can continue this procedure to build an allowed bi-infinite word.

The procedure for a right-infinite word w is similar, and is omitted. $\square$

If we are given two bi-infinite words w_1 and w_2 we say that w_1 and w_2 are *left asymptotic* provided that there exists a left infinite word p and two right infinite words s_1 and s_2 such that $w_1 = ps_1$ and $w_2 = ps_2$. We define *right asymptotic* similarly.

Definition 6.23 *Let φ be a RUP substitution. Let v_R be such that $E^u =< v_R >$. We say that two strands T_1, and T_2 are* left asymptotic *if* $\lim_{t\to-\infty} d(T_1 - tv_R, T_2 - tv_R) = 0$. *We say that two strands T_1, and T_2 are* right asymptotic *if* $\lim_{t\to\infty} d(T_1 - tv_R, T_2 - tv_R) = 0$.

The definition of left asymptotic for strands implies that if two strands T_1 and T_2 are left asymptotic, then $[T_1]$ and $[T_2]$ share a common left-infinite prefix. That is, there is a left infinite word p such that $[T_1] = ps_1$ and $[T_2] = ps_2$

Note that the definition of an Arnoux-Rauzy substitution implies that any left special factor, and any right special factor has three possible extensions. Let us denote the *sequence of left special factors* to be the sequence $\{w_i : w_i \text{is a right special factor, and} |w_i| = i\}$. For Arnoux-Rauzy substitutions, the right special factors are nested, as shown below. This fact is well known.

Proposition 6.24 *If φ is a Arnoux-Rauzy substitution, and $\{w_i\}_{i\in\mathbb{N}}$ is the sequence of right special factors, then w_i is a suffix of w_{i+1}.*

Proof Consider the right special factors $w_1, w_2, \ldots, w_n$ such that $|w_i| = i$. Since there is a unique right special factor of a given length, and the last i letters of w_{i+1} are also a right-special factor of length i, it follows that w_i is a suffix of w_{i+1}. □

For the following, note that a left or right asymptotic pair form a proximal pair of composants.

Proposition 6.25 *If φ is an Arnoux-Rauzy substitution, then there is a left-infinite word p, and three left-asymptotic composants $p1s_1$, $p2s_2$, and $p3s_3$ such that the right special factors of φ are suffixes of p. Furthermore, any left-asymptotic pair has p as the common part.*

Proof Let $\mathcal{A} = \{1, 2, 3\}$ be the alphabet of φ, and consider the sequences $\{w_i1\}_{i\in\mathbb{N}}, \{w_i2\}_{i\in\mathbb{N}}, \{w_i3\}_{i\in\mathbb{N}}$, where $\{w_i\}_{i\in\mathbb{N}}$ is the sequence of right special factors. Since the w_i are nested, we have that these sequences converge to three left-infinite words $p1, p2, p3$. The completions $p1s_1, p2s_2, p3s_3$ of $p1, p2, p3$ into allowed bi-infinite words form the desired pairs of left asymptotic composants.

To show that p is unique, let qr_1 and qr_2 be left asymptotic composants with common part q, and $r_1 \neq r_2$. Then the suffixes of q are special factors, and, by uniqueness, they are the same as the suffixes of p. Hence $p = q$. □

Corollary 6.26 *Arnoux-Rauzy substitutions have exactly three left-asymptotic composants, each of which is left-asymptotic to the other two.*

Proof By the proposition 6.25, there are three composants T_1, T_2, and T_3 with symbolic representations $p1\ldots$, $p2\ldots$, $p3\ldots$, respectively. Suppose, on the contrary, that there is another asymptotic composant T_4, and let $pk_0\ldots$ be the symbolic for for T_4, where $k_0 \in \{1,2,3\}$. Since both share p as a common left infinite prefix, we must have that T_4 and T_{k_0} are left asymptotic. Since they are symbolically different, we must have that there is a word w and two letters i_1 and i_2 such that T_4 has symbolic representation $pk_0wi_1\ldots$and T_3 has symbolic representation $pk_0wi_2\ldots$. Thus, we must have that $p = pk_0w$, with $|k_0| \geqslant 1$; thus p is a proper prefix of itself, which implies that the substitution is p is a periodic word, which is impossible.

Thus, Arnoux Rauzy substitutions have exactly three left-asymptotic pairs. □

Corollary 6.27 *Let φ be an Arnoux Rauzy substitution, and let p the common prefix of the composants of φ, then each of the words $p1$, $p2$ and $p3$ has a unique k-extension to the right for any $k \in \mathbb{Z}^+$*

Proposition 6.28 *Arnoux-Rauzy substitutions have connected Rauzy Fractals.*

Proof We proceed by induction. By proposition 6.25, there are asymptotic, hence proximal, composants $p1s_1$, $p2s_2$ and $p3s_3$. This means that the first connectivity graph G_1 is connected. Now, given $n \in \mathbb{N}$, assume that the connectivity graph G_n is connected.

Since vertices of G_n correspond to the allowed factors of length n, and the vertices of G_{n+1} correspond to the allowed factors of length $n+1$, each vertex of G_{n+1} is a right extension of a vertex of G_n. Notice that every edge in G_n connecting two vertices induces an edge connecting at least one of their right extensions by Proposition 5.16.

If the vertices of G_n are $\{v_1, v_2, \ldots, v_k, p_n\}$, where only p_n, a suffix of p, is a right special factor, then the vertices of G_{n+1} are $\{v_1 r_1, \ldots, v_k r_k, p_n 1, p_n 2, p_n 3\}$, where the r_i are letters uniquely determined by v_i. Since $p_n 1$, $p_n 2$ and $p_n 3$ can be completed to $p1t_1$, $p2t_2$ and $p3t_3$, respectively, we obtain a triangle formed by the vertices $p_n 1$, $p_n 2$ and $p_n 3$ in G_{n+1}. The graph G_n is the graph G_{n+1} after collapsing the triangle $p_n 1$, $p_n 2$ and $p_n 3$ into a point. Since the quotient G_n is connected, and the graph induced by $p_n 1, p_n 2$ and $p_n 3$ is also connected, we obtain that G_{n+1} is also connected.

Thus, all the connectivity graphs are connected, and the Rauzy Fractal is connected. □

Note that, in Proposition 6.28, all the analysis was done using only the right special factors.

Conjecture 6.29 *If φ is a Pisot substitution of complexity $2n+1$, and such that every pair of different 1-letter extensions wa and wb of a right special factor w can be extended to to an asymptotic pair with w as a suffix of the common part, then the Rauzy Fractal of φ is connected.*

Conjecture 6.30 *If φ is a Pisot substitution whose first connectivity graph is connected, of complexity $2n+1$, and such that every right-special factor is a suffix of the common part of a left-asymptotic pair, then the Rauzy Fractal of φ is connected.*

Note that Proposition 5.27 allow us to generalize slightly a result regarding the connectedness of the Rauzy Fractal.

Proposition 6.31 *If φ is an Arnoux-Rauzy substitution, then all the Rauzy pieces for every word are connected.*

Proof By the Proposition 5.27, it suffices to show that for Arnoux-Rauzy substitutions, all the graphs $G_1^{;w}$ are connected, for any word w.

Suppose first that w is not a special factor, then $G_1^{;w}$ consists of only one point, in which case it is connected.

If w is a special factor, then, by Proposition 6.25, there are three proximal composants $\ldots w1\ldots$, $\ldots w2\ldots$, and $\ldots w3\ldots$. This shows that $G_1^{;w}$ is a triangle, and hence, $G_1^{;w}$ is connected. Thus all the graphs $G_1^{;w}$ are connected, and, by Proposition 5.27, we have that all the Rauzy pieces are connected. $\square$

CHAPTER 7

REARRANGING THE RAUZY PIECES

One of the natural questions we are trying to answer is the following:

Suppose that two substitution tiling spaces are homeomorphic, what can we say about their corresponding Rauzy Fractals?

From the work we did in Chapter 3, we know that each method of producing new substitutions in that chapter is either a shift equivalence of substitutions or is obtained through rewriting with starting and stopping rules.

If α and β are substitutions whose tiling spaces are homeomorphic, we know, by the Rigidity Theorem 3.15, that there is a sequence of substitutions $\alpha = \varphi_1, \varphi_2, \ldots, \varphi_k = \beta$ such that, for each $i, i+1 \in \{1, 2, \ldots, k\}$, either (1) φ_{i+1} is obtained from φ_i via shift equivalence or via rewriting with stopping and starting rules, or (2) φ_i is obtained from φ_{i+1} via shift equivalence or via rewriting with stopping and starting rules.

7.1 Effects of Rewriting on the Rauzy Fractal

In this section we analyze the effects that shift equivalences and rewritings have on the Rauzy Fractal. Shift equivalence and rewriting produces a new strand space that often comes from a reducible Pisot substitution. We assume throughout that the geometric coincidence condition is satisfied.

First we note that by Corollaries 3.10 and 3.11, all rewritings considered in Chapter 3 transform RUP substitutions into RUP substitutions.

Definition 7.1 *We say that a non-empty word $v = v_1 \ldots v_{|v|} \in \mathcal{L}_\beta$* minimally covers *a word $w \in \mathcal{A}^*$ under $\pi : \mathcal{B} \to \mathcal{A}^+$ if there is a proper prefix $p_1 \in \mathcal{A}^*$*

of $\pi(v_1)$ *and a proper suffix* $s_{|v|} \in \mathcal{A}^*$ *of* $\pi\left(v_{|v|}\right)$ *such that* $\pi(v) = p_1 w s_{|v|}$. *Let* $\mathcal{V}_w = \{v : \ v \text{ covers } w \text{ minimally}\}$

The following property justifies the term minimal applied to the set $\mathcal{V}_w$.

Lemma 7.2 *Let* $x = x_1 \dots x_{|x|} \in \mathcal{B}^+$, $w \in \mathcal{A}^*$ *and* $\pi : \mathcal{B} \to \mathcal{A}^+$. *If there is a proper prefix* p *of* $\pi(x_1)$ *such that* $\pi(x) = pw\dots$, *then there is a word* $v \in \mathcal{V}_w$ *such that* v *is a prefix of* x.

Proof Consider the words $y_i = x_1 \dots x_i$ and find the first i, with $1 \leqslant i \leqslant |x|$, such that $\pi(y_i) = pw\dots$. The word y_i is a prefix of x and $y_i \in \mathcal{V}_w$. □

Note that if w is the empty word, then $\mathcal{V}_w = \mathcal{A}$, and if w is a singleton, then $\mathcal{V}_w = \{v \in \mathcal{A} : \ w \text{ is a factor of } \pi(v)\}$.

In the following proposition we use the following notation. For a substitution $\beta : \mathcal{B} = \{1, \dots, d\} \to \mathcal{B}^+$, we denote by $[\mathcal{B}]$, $[\mathcal{B}]_P$, $[\mathcal{B}]_s$ and $[\mathcal{B}]_u$ the spaces $\mathbb{R}^d$, V_β^P, V_β^s and V_β^u, respectively. We denote by $\mathcal{F}_{\varphi}^{\underline{w_1} w_2 \dots w_{|w|}}$ the set of stands $\{S_i\}_{i\in\mathbb{Z}}$ such that $(S_1 \setminus \{\max(S_1)\}) \cap E^s \neq \varnothing$ and such that the edges $S_1, S_2, S_3 \dots S_{|w|}$ have labels $w_1, w_2, \dots, w_{|w|}$, respectively. We denote by $\mathcal{F}_\beta^{.w_1 w_2 \dots w_{|w|}}$ the set of strands $\{S_i\}_{i\in\mathbb{Z}}$ such that $\min(S_1) \in \mathcal{R}_\beta^{.w_1 w_2 \dots w_{|w|}}$ and such that the edges $S_1, S_2, S_3, \dots, S_{|w|}$ have labels $w_1, w_2, \dots, w_{|w|}$, respectively. Finally, the Rauzy piece corresponding to the word w is denoted $\mathcal{R}_\beta^{.w}$

For the morphism $\pi : \mathcal{B} \to \mathcal{A}^+$, denote by $[\pi]_P$ the restriction of $[\pi]$ to V_β^P. Note that in the following proposition the hypothesis that $[\pi]_P : [\mathcal{B}]_P \to [\mathcal{A}]_P$ be bijective is satisfied, for example, when $\mathcal{B}$ is a rewriting of $\mathcal{A}$ or if α is shift equivalent to β.

Theorem 7.3 *Suppose that* $\beta : \mathcal{B} \to \mathcal{B}^+$ *and* $\alpha : \mathcal{A} \to \mathcal{A}^+$ *are RUPC substitutions, and that the morphism* $\pi : \mathcal{B} \to \mathcal{A}^+$ *is such that the diagram*

$$\begin{array}{ccc} \mathcal{B}^+ & \xrightarrow{\beta} & \mathcal{B}^+ \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{A}^+ & \xrightarrow{\alpha} & \mathcal{A}^+ \end{array}$$

commutes. Suppose that $[\pi]_P : [\mathcal{B}]_P \to [\mathcal{A}]_P$ *is bijective and preserves the stable space and the unstable space. Let* $w \in \mathcal{A}^*$*, and let* $\mathcal{V}_w \subset \mathcal{B}^+$ *the set of words that minimally cover* w*. Then*

$$\mathcal{R}_\alpha^{.w} = \bigcup \left\{ [\pi]_P \left(\mathcal{R}_\beta^{.v} + \mathrm{pr}_\beta^s \left([\pi]_P^{-1} [p] \right) \right) \right\},$$

where $v = v_1 v_2 \ldots v_{|v|} \in \mathcal{V}_w$*,* $\pi(v_1) = p \ldots$ *and* $\pi(v) = pw \ldots$

Proof Let $w = w_1 w_2 \ldots w_{|w|} \in \mathcal{L}_\alpha$ and let $\mathcal{V}_w$ be a set that minimally covers w. First, we show that $\mathcal{R}_\alpha^{.w}$ is contained in the union.

Let $q \in \mathcal{R}_\alpha^{.w}$, and let $S = \{S_i\}_{i \in \mathbb{Z}} \in \mathcal{F}_\alpha^{.w}$ such that $\min(S_1) \in [\mathcal{A}]_s$. Then there is a word $x = x_1 x_2 \ldots x_{|x|}$ such that w is a factor of $\pi(x)$ and a strand $T \in \mathcal{F}_\beta^{\underline{x_1} x_2 x_3 \ldots x_{|x|}}$ such that $\pi(T) = S$.

By Lemma 7.2, there exists a word $v \in \mathcal{V}_w$ such that $x = v \ldots$, thus $T \in \mathcal{F}^{\underline{v_1} v_2 \ldots v_{|v|}}$.

Since $T \in \mathcal{F}^{\underline{v_1} v_2 \ldots v_{|v|}}$ and $\pi(T) = S$, there are words $p, s \in \mathcal{L}_\alpha$ such that $\pi(v_1) = ps$, $\pi(v) = ps\pi\left(v_2 \ldots v_{|v|}\right) = pw \ldots$, and such that $S \in \mathcal{R}^{p.sp\pi\left(v_2 \ldots v_{|v|}\right)}$.

Consider the state T_1 which has type v_1, and consider $\pi(T_1) = P_1 P_2 \ldots P_{|p|} S_1 \ldots$, where $P_1, \ldots, P_{|p|}$ are edges of S, and $\left[P_1 \ldots P_{|p|}\right] = p$. Then $\min(P_1) = q - [p]$. Thus $\min(T_1) = [\pi]_P^{-1}(q - [p])$, and $\mathrm{pr}^s(\min(T_1)) = \mathrm{pr}_\beta^s\left([\pi]_P^{-1}(q - [p])\right) \in \mathcal{R}^{.v_1 v_2 \ldots v_{|v|}}$; but $\left(\mathrm{pr}_\beta^s\right)[\pi]_P^{-1} q = [\pi]_P^{-1} q$ since $q \in [\mathcal{A}]_s$ and thus $[\pi]_P^{-1} q \in [\mathcal{B}]_s$. Hence $\left(\mathrm{pr}_\beta^s\right)[\pi]_P^{-1} q + \mathrm{pr}_\beta^s([p]) = [\pi]_P^{-1} q - \left(\mathrm{pr}_\beta^s\right)[p] \in \mathcal{R}_\alpha^{.v}$. Hence $q \in [\pi]\left(\mathcal{R}_\alpha^{.v} + \mathrm{pr}_\beta^s(p)\right)$.

Thus $R_\alpha^{.w} \subset \bigcup \left\{ [\pi] \left(\mathcal{R}_\beta^{.v} + \mathrm{pr}_\beta^s \left([\pi]_P^{-1} [p] \right) \right) \right\}$, where $v = v_1 v_2 \ldots v_{|v|} \in \mathcal{V}_w$, $\pi(v_1) = p \ldots$ and $\pi(v) = pw \ldots$

Conversely, take $v = v_1 v_2 \ldots v_{|v|} \in \mathcal{V}_w$ and let p be such that $\pi(v_1) = p\ldots$ and $\pi(v) = pw\ldots$, and let $r \in \mathcal{R}_\beta^{\cdot v}$ and $T \in \mathcal{F}_\beta^{\cdot v}$ such that $r = \min(T_1)$. Thus $\left[T_1 \ldots T_{|v|}\right] = v_1 \ldots v_{|v|}$.

We want to show that $[\pi]\left(\min(T_1) + \mathrm{pr}_\beta^s [\pi]_P^{-1}[p]\right) \in \mathcal{R}^{\cdot w}$. Since $[\pi]$ commutes with the projections, it suffices to show that $[\pi]\min(T_1) + \mathrm{pr}_\alpha^s [p] \in \mathcal{R}^{\cdot w}$, or, equivalently, that $[p] + [\pi]\min(T_1) - \mathrm{pr}_\alpha^u([p]) \in \mathcal{R}^{\cdot w}$

Let $q = [\pi]\left(r + \mathrm{pr}_\beta^s\left([\pi]_P^{-1}[p]\right)\right)$. Since $[\pi]\,\mathrm{pr}_\alpha^s = \mathrm{pr}_\beta^s [\pi]$, we obtain that $q = [\pi]\,r + \mathrm{pr}_\alpha^s([p])$. Since $r \in [\mathcal{B}]_s$, and $[\pi]$ preserves stable spaces, we obtain that $q \in [\mathcal{A}]_s$.

Now, $q - [p] = [\pi]\,r + \mathrm{pr}_\alpha^s([p]) - [p] = [\pi]\,r - \mathrm{pr}_\alpha^u([p]) = [\pi]\min(T_1) - \mathrm{pr}_\alpha^u([p])$. Thus, $q = [p] + [\pi]\min(T_1) - \mathrm{pr}_\alpha^u([p])$. It suffices to show that $q \in \mathcal{R}_\alpha^{\cdot w}$.

Since $[T_1] = v_1$ and $\pi(v_1) = p$ and $\pi(v) = pw$. We obtain that $\pi(T) - \mathrm{pr}_\alpha^u([p])$ is a strand in $\mathcal{R}^{\cdot w}$. Thus

$\mathcal{R}_\alpha^{\cdot w} \supset \bigcup\left\{[\pi]\left(\mathcal{R}_\beta^{\cdot v} + \mathrm{pr}_\beta^s\left([\pi]_P^{-1}[p]\right)\right)\right\}$, where $v = v_1 v_2 \ldots v_{|v|} \in \mathcal{V}_w$, $\pi(v_1) = p\ldots$ and $\pi(v) = pw\ldots$ $\square$

Suppose that α, β are substitutions that are shift-equivalent, and suppose that π and η are morphisms such that the diagram

$$\begin{array}{ccc} \mathcal{B}^+ & \xrightarrow{\beta} & \mathcal{B}^+ \\ \pi\downarrow & {}^{\eta}\nearrow & \downarrow\pi \\ \mathcal{A}^+ & \xrightarrow{\alpha} & \mathcal{A}^+ \end{array}$$

commutes. Then the result above also serves to represent the cylinders $\mathcal{R}_\beta^{\cdot v}$ in terms of $\mathcal{R}_\alpha^{\cdot w}$, by simply considering the commuting diagram

$$\begin{array}{ccccc} & & \mathcal{B}^+ & \xrightarrow{\beta} & \mathcal{B}^+ \\ & {}^{\eta}\nearrow & \downarrow\pi & {}^{\eta}\nearrow & \downarrow\pi \\ \mathcal{A}^+ & \xrightarrow{\alpha} & \mathcal{A}^+ & & \end{array}$$

We next address what happens with the Rauzy Pieces if we apply a rewriting with starting and stopping rules.

Suppose that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^+ & \overset{\beta}{\to} & \mathcal{B}^+ \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{A}^+ & \overset{\alpha}{\to} & \mathcal{A}^+ \end{array},$$

formed by rewriting with starting and stopping rules. For this particular case we also show above that $[\pi]$ commutes with the projections into the Pisot space, the stable space, and the unstable space. The previous theorem gives us a way to represent the Rauzy Pieces of the form $\mathcal{R}_\alpha^{.w}$ in terms of the Rauzy pieces of the form $R_\beta^{.v}$. Unlike the shift equivalent case, we do not have a commutative diagram, but one non-commutative diagram of the form

$$\begin{array}{ccc} \mathcal{B}^+ & \overset{\beta}{\to} & \mathcal{B}^+ \\ \pi\downarrow & {}^{\eta}\nearrow & \downarrow\pi \\ \mathcal{A}^+ & \overset{\alpha}{\to} & \mathcal{A}^+ \end{array}.$$

We address this case in the following proposition.

Theorem 7.4 *Let $\alpha : \mathcal{A} \to \mathcal{A}^+$ and $\beta : \mathcal{B} \to \mathcal{B}^+$ be RUPC substitutions such that β is obtained from a rewriting of α following starting rules $B \subset \mathcal{L}_\alpha$ and stopping rules $E \subset \mathcal{L}_\alpha$ and such that the diagram*

$$\begin{array}{ccc} \mathcal{B}^+ & \overset{\beta}{\to} & \mathcal{B}^+ \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{A}^+ & \overset{\alpha}{\to} & \mathcal{A}^+ \end{array}$$

commutes. Let $c \in \mathcal{B}^+$. Then

$$[\pi]_s \mathcal{R}_\beta^{.c} = \bigcup \left\{ \mathcal{R}_\alpha^{e.\pi(c)} : e \in E \text{ and } e\pi(c) \in \mathcal{L}_\alpha \right\}$$

and, hence,

$$\mathcal{R}_{\beta}^{.c} = \bigcup [\pi]_P^{-1} \left\{ \mathcal{R}_{\alpha}^{.e\pi(c)} - \mathrm{pr}_{\alpha}^{s}([e]) : e \in E \text{ and } e\pi(c) \in \mathcal{L}_{\alpha} \right\}$$

Proof Clearly, it suffices to prove only the first equality of sets.

Let $p = [\pi] q$, where $q \in \mathcal{R}_{\beta}^{.c}$, and let $S \in \mathcal{F}^{.c_1 \ldots c_{|c|}}$, where $\min(S_1) = p$. Since π commutes with the projections, we have that $\pi(S) \in \mathcal{F}_{\alpha}^{.\pi(c)}$. Let $T = \pi(S) = \ldots T_{-1}.T_0 T_1 \ldots$, and let $b = [T_{-1}]$. Then $b\pi(c) \in \mathcal{L}_{\alpha}$ and $T \in \mathcal{F}^{b.c}$. Therefore, $\min(\pi(S_1)) = \pi(p) \in \mathcal{R}_{\alpha}^{b.\pi(c)}$.

Conversely, suppose that the word $e_0 \in E$ is a stopping rule and take a point $q \in \mathcal{R}_{\alpha}^{.e\pi(c)}$. Let $S = \ldots S_{-2} S_{-1}.S_0 S_1 \ldots \in \mathcal{F}^{e.\pi(c)}$. Thus $[S_{-1}] = e_0$, $q = \min(S_0)$ and $\left[S_{-1}.S_0 \ldots S_{|\pi(c)|}\right] = e_0.\pi(c)$.

Since the alphabet of β is obtained by the appearances in a periodic word W_a according to the occurrences of a word $e'b'$ with the word $e' \in E$, and the word $b' \in B$, we obtain that $\pi(c)$ must start with a starting rule. Since $[S]$ is uniquely factored in terms of words of the form $\pi(c_i)$, we must have that one of such breakings occurs between S_{-1} and S_0, Since $\left[S_{-1}.S_0 \ldots S_{|\pi(c)|}\right] = e_0.\pi(c)$ and the decomposition is unique, we must have that $q \in \mathcal{R}_{\beta}^{e.\pi(c)}$. □

From Theorems 7.3 and 7.4, we obtain immediately the following theorem.

Theorem 7.5 *If α and β are RUPC substitutions such that $\mathcal{T}_{\alpha} \cong \mathcal{T}_{\beta}$, then for any $w \in \mathcal{L}_{\alpha}$, there exists $n \in \mathbb{Z}^{+}$, words $x_1, x_2, \ldots, x_n \in \mathcal{L}_{\beta}$, linear invertible transformations $A_1, A_2, \ldots, A_n$ from $[\mathcal{B}]_P$ to $[\mathcal{A}]_P$, and points $p_1, p_2, \ldots, p_n$ in $[\mathcal{A}]_s$ such that*

$$\mathcal{R}_{\beta}^{.w} = \bigcup \left\{ A_i \left(\mathcal{R}_{\alpha}^{.x_i} - p_i \right) \right\}$$

Proof By the Rigidity Theorem 3.15, there is a sequence of substitutions $\alpha = \varphi_1, \varphi_2, \ldots, \varphi_k = \beta$ such that ether (1) φ_{i+1} is obtained from φ_i via shift equivalence or stopping and starting rules, or (2) φ_i is obtained from φ_{i+1} via shift equivalence or stopping and starting rules.

We proceed by induction on k. For $k = 1$, $\mathcal{R}_\beta^{\cdot w} = \mathcal{R}_\beta^{\cdot w}$,

Suppose that the result is true for k, and consider a $k+1$ sequence

$$\beta = \varphi_k, \varphi_{k-1}, \ldots, \varphi_2, \varphi_1.$$

Thus $\mathcal{R}_\beta^{\cdot w} = \bigcup \left\{ A_i \left(\mathcal{R}_{\varphi_2}^{\cdot x_i} - p_i \right) \right\}$. Since $\mathcal{R}_{\varphi_2}^{\cdot x_i} = \bigcup \left\{ B_j^i \left(\mathcal{R}_{\varphi_1}^{\cdot y_j^i} - p_j^i \right) \right\}$, we obtain that

$$\begin{aligned} \mathcal{R}_\beta &= \bigcup \left\{ A_i \left(\mathcal{R}_{\varphi_2}^{\cdot x_i} - p_i \right) \right\} \\ &= \bigcup \left\{ A_i \left(\mathcal{R}_{\varphi_2}^{\cdot x_i} - p_i \right) \right\} \end{aligned}$$

By Theorems 7.3 and 7.4, we can represent $\mathcal{R}_\beta^{\cdot w} = \mathcal{R}_{\varphi_1}^{\cdot w}$ as

$$\begin{aligned} \mathcal{R}_{\varphi_2}^{\cdot w} &= \bigcup \left\{ A_i \left(\bigcup \left\{ B_j^i \left(\mathcal{R}_{\varphi_1}^{\cdot y_j^i} - p_j^i \right) \right\} - p_i \right) \right\} \\ &= \bigcup \left\{ A_i B_j^i \left(R_\varphi^{\cdot y_j^i} - B_j^i p_j^i \right) - A_i B_j^i \left(B_j^i \right)^{-1} p_i \right\} \\ &= \bigcup \left\{ A_i B_j^i \left(R_{\varphi_1} - \left(B_j^i p_j^i + \left(B_j^i \right)^{-1} p_i \right) \right) \right\}, \end{aligned}$$

which is of the desired form. This concludes the proof. □

Analyzing the statement of the Rigidity Theorem, we note the following transformations are induced in the Rauzy Fractal:

$$\begin{aligned} \mathcal{R}_\alpha &\xrightarrow{\text{split and rewrite}} \mathcal{R}_{\alpha^*} = \mathcal{R}_{(\alpha^*)^n} \\ \mathcal{R}_{\alpha^*} = \mathcal{R}_{(\alpha^*)^n} &\xrightarrow{\text{return words}} \mathcal{R}_{(\alpha^*)_u^{n_1}} \\ \mathcal{R}_{(\alpha^*)_u^{n_1}} &\xrightarrow{\text{unreturn}} \mathcal{R}_{\gamma^{n_2}} = \mathcal{R}_\gamma \\ \mathcal{R}_{\gamma^{n_2}} = \mathcal{R}_\gamma &\xrightarrow{\text{shift equivalence}} \mathcal{R}_{(\beta^*)^m} = \mathcal{R}_{\beta^*} \end{aligned}$$

7.2 Consequences

Corollary 7.6 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a RUPC substitution, and suppose that $P = \{p_1, \ldots, p_2\}$ and valid stopping rules. Suppose that $\bigcup \left\{ \mathcal{R}_\varphi^{p\cdot} \mid p \in P \right\}$ is disconnected.*

Then the Rauzy fractal $\mathcal{R}_{\varphi_2}$ is disconnected for any substitution φ_2 produced from φ using stopping rules P and starting rules $\mathcal{A}$.

Proof It is immediate from Theorem 7.4. □

Since any Rauzy Piece for an invertible substitution is connected, and every Rauzy piece for an Arnoux-Rauzy substitution is also connected, we obtain the following proposition.

Corollary 7.7 *If $\varphi : \mathcal{A} \to \mathcal{A}^+$ is an Arnoux-Rauzy substitution or φ is an invertible substitution on two letters, then any rewriting of φ using starting rules $\mathcal{A}$ and any stopping rules has connected Rauzy pieces.*

Proof It is immediate from Theorem 7.4. □

Conjecture 7.8 *Suppose that φ and ψ are IUPC substitutions and that $\mathcal{T}_\varphi \cong \mathcal{T}_\psi$, then the Hausdorff dimension of the boundary of $\mathcal{R}_\eta$ is the same as the Hausdorff dimension of the boundary of $\mathcal{R}_\psi$.*

If Conjecture 7.8 were true, then we would have that the Hausdorff dimension of the boundary of the Rauzy fractal is a topological invariant for RUPC substitutions. The only missing information is to know if the Housdorff dimension along the boundary of the Rauzy Fractal is constant.

CHAPTER 8

FINDING ASYMPTOTIC COMPOSANTS

In this chapter we provide a method to find left-asymptotic composants. We recall that from Chapter 2 that the composant of a strand S is uniquely determined by the bi-infinite word $[S]$, where we disregard the location of the center.

Unlike the method presented in [BD01], our method does not require the substitution to be proper or needs to solve for a prefix or suffix problem. The method can easily be modified to find right-asymptotic composants. **We assume that substitutions are primitive and non-shift periodic**.

First we will state some definitions and the recognizability theorem by Mosse in [Mos92] and [Mos96], we also state well known corollaries of it. We state them as they are in [Fog02].

Definition 8.1 *Let φ be a primitive non shift-periodic substitution, and let W be a bi-infinite fixed point for φ. Let*

$$E_1 = \{0\} \cup \left\{ \left|\varphi\left(W_{[0,p-1]}\right)\right|, -\left|\varphi\left(W_{[-p,-1]}\right)\right| : p > 0 \right\}.$$

Let $B \in \mathcal{L}_W$ be a non-empty word, and $i, j \in \mathbb{Z}$ such that $B = W_{[i,i+|B|-1]} = W_{[j,j+|B|-1]}$. We say that B admits the same 1-decomposition *at ranks i and j, provided that $E_1 \cap \{i, \ldots, i+|B|-1\}$ and $E_1 \cap \{j, \ldots, j+|B|-1\}$ are images of one another under the translation by $(j-i)$.*

Definition 8.2 *Let φ and W as in Definition 8.1. Let $f_{E_1} : \mathbb{Z} \to \mathbb{Z}$ be given by*

$$f_{E_1}(i) = \begin{cases} -\left|\varphi\left(W_{[-p,-1]}\right)\right| & \text{if } i¡0 \\ 0 & \text{if } i = 0 \\ \left|\varphi\left(W_{[-p,-1]}\right)\right| & \text{if } i¿0 \end{cases}$$

Note that $f_{E_1}(\mathbb{Z}) = E_1$. The function f_{E_1} simply keeps track of the order of the elements of E_1.

Definition 8.3 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a primitive non shift-periodic substitution, and let W be a bi-infinite fixed point for φ. Let $B = W_{[i,i+|B|-1]}$, and let $E = E_1 \cap \{i, i+1, \ldots, i+|B|-1\}$, and assume that $E \neq \varnothing$. Let $m = \min E$, and $M = \max E$. We say that the word $B = W_{[i,i+|B|-1]}$* comes from the word *$x_0 x_1 \ldots x_{k+1} = W_{[j,j+k+1]}$* at rank *$i$ if*

1. *$f_{E_1}(j) < m \leqslant M < f_{E_1}(j+k)$, and*

2. *$B = S\varphi(x_1)\varphi(x_2)\ldots\varphi(x_k)P$, for some proper prefix P of x_0, and some proper suffix S of x_{k+1} such that $i = f_{E_1}(j) - |S|$ and $i + |B| - 1 = f_{E_1}(j+k) + |P|$.*

We also say that the word $x_0 x_1 \ldots x_{k+1}$ is an ancestor *of B.*

Theorem 8.4 *Let φ be a primitive non shift-periodic substitution, and let W be a right-infinite fixed point for φ. Then there exists an integer $L > 0$ such that if $W_{[i-L,j+L]} = W_{[j-L,j+L]}$, then $W_{[i,j]} = W_{[i',j']}$ have the same 1-decomposition, and have the same ancestors at ranks i and i'.*

Mosse's Theorem 8.4 has an important consequence for the tiling space $\mathcal{T}_\varphi$.

Corollary 8.5 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be primitive and non shift-periodic. The map $\varphi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ is injective.*

The following Corollary states that there exists an integer number $L > 0$ such that an ancestor u in U of any word v in V does not depend on the location of v, but only in the surrounding words of length L to the left and to the right of v.

Corollary 8.6 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a primitive non shift-periodic substitution. Let L as in theorem 8.4, and let $U, V \in X_\varphi$ be two bi-infinite words such that $V = \varphi(U)$. Suppose that $V_{[i-L,j+L]} = V_{[i'-L,j'-L]}$ for some integers $i < j$ and $i' < j'$. Then there are unique words $x_0 \dots x_{k+1}$, S and P, and unique ranks $l < r, l' < r'$ such that the word $x_0 \dots x_{k+1}$ at ranks l and l', respectively, is an ancestor of the word $V_{[i,j]} = V_{[i',j']}$ at ranks i and i', respectively with suffix S and prefix P.*

In the following corollary, stated as in [Fog02], σ denotes the shift map.

Corollary 8.7 *Let φ be a primitive, non shift-periodic substitution. Let X_φ be the substitutive system. Then we have that for every word $w \in X_\varphi$, there exists a unique bi-infinite word $v \in X_\varphi$ such that $w = \sigma^k(\varphi(v))$, and $0 \leqslant k < \left|\varphi\left(v_{[0]}\right)\right|$*

Corollary 8.7 states that, up to a shift, we can uniquely "desubstitute" a word w as an image of a word v.

$$w = \dots | \underbrace{\dots}_{\varphi(v_{-1})} | \underbrace{w_{-k} \dots w_{-1}.w_0 \dots w_l}_{\varphi(v_0)} | \underbrace{\dots}_{\varphi(v_1)} | \underbrace{\dots}_{\varphi(v_2)} | \dots$$

For our algorithm we need the following lemmas and definitions.

Definition 8.8 *Let φ be a primitive non shift-periodic substitution. We say that L* makes φ have a disagreement, *if whenever $A = a\binom{u}{v}$ is a pair of finite allowed words such that $|a| = |u| = |v| = L$, and $u_{[0]} \neq v_{[0]}$. Then there are words $a', u', v' \in \mathcal{A}^+$ such that $u'_{[0]} \neq v'_{[0]}$, and $|a'| = |u'| = |v'| = L$ such that $\varphi\left(a\binom{u}{v}\right) = c'a'\binom{u'}{v'} \dots$ for some $c' \in A^*$.*

Lemma 8.9 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a primitive non shift-periodic substitution such that $|\varphi(l)| \geqslant 2$ for all $l \in \mathcal{A}$. If L is given by Theorem 8.4, then L makes φ have a disagreement.*

Proof Let $A = a\binom{u}{v}$ be a pair of finite allowed words such that $|a| = |u| = |v| = L$, and $u_{[0]} \neq v_{[0]}$. Since $|\varphi(a)| \geqslant 2$ for all $a \in \mathcal{A}$, we must have that $\varphi(a)$, $\varphi(u)$ and $\varphi(v)$ have length at least $2L$

$$\varphi(A) = \varphi(a)\begin{pmatrix}\varphi(u)\\ \varphi(v)\end{pmatrix} = \varphi(a)\begin{pmatrix}x_0x_1\dots x_Lx_{L+1}\dots x_{2L}\\ y_0y_1\dots y_Ly_{L+1}\dots y_{2L}\end{pmatrix}\dots.$$

If $x_0\dots x_L = y_0\dots y_L$, then, by Theorem 8.4, we must have that the ancestor of x_0 is the same as the ancestor of y_0, but that is impossible, since $u_{[0]} \neq v_{[0]}$. Therefore, there must be $k \leqslant L$ such that

$$\varphi(A) = \varphi(a)\begin{pmatrix}\varphi(u)\\ \varphi(v)\end{pmatrix} = \varphi(a)\,x_0\dots x_k\begin{pmatrix}x_{k+1}\dots x_{k+L}\\ y_{k+1}\dots y_{k+L}\end{pmatrix}\begin{pmatrix}x_{k+L+1}\dots x_{2L}\\ y_{k+L+1}\dots y_{2L}\end{pmatrix}\dots$$

Let c, a' be such that $ca' = \varphi(a)\,x_0\dots x_k$, and $|a'| = L$. Let $\binom{u'}{v'} = \binom{x_{k+1}\dots x_{k+L}}{y_{k+1}\dots y_{k+L}}$. The words c, a', u', v' satisfy the properties required. $\square$

Lemma 8.10 *Let $\varphi : \mathcal{A} \to \mathcal{A}^+$ be a primitive non shift-periodic substitution such that $|\varphi(l)| \geqslant 2$ for all $l \in \mathcal{A}$. Let L be such that L makes φ have a disagreement. Let $A = pa\binom{u}{v}\dots$ and $A'' = p''a''\binom{u''}{v''}\dots$ be asymptotic pairs such that $\varphi(A) = A''$. Suppose that $p \in \mathcal{A}^{\mathbb{Z}^-}$, and $a, u, v, a'', u'', v'' \in \mathcal{A}^+$ are such that $u_{[0]} \neq v_{[0]}$, $u''_{[0]} \neq v''_{[0]}$, and $|a| = |a''| = |u| = |v| = |u''| = |v''| = L$. Then $\varphi\left(a\binom{u}{v}\right) = c''a''\binom{u''}{v''}\dots$ for some $c'' \in \mathcal{A}^*$.*

Proof Note that $\varphi(pa)$ is a prefix of $p'a'$. By definition of L, we must have that there are words $c' \in \mathcal{A}^*$, a', u' and v' such that $|a'| = |u'| = |v'| = L$ such that

$$\varphi\left(pa\begin{pmatrix}u\\ v\end{pmatrix}\right) = \varphi(p)\,\varphi(a)\begin{pmatrix}\varphi(u)\\ \varphi(v)\end{pmatrix} = \varphi(p)\,\varphi(a)\,c'a'\begin{pmatrix}u'\\ v'\end{pmatrix}\dots$$

Since the pair $\binom{u''}{v''}$, has the first disagreement in the first letters of u'' and v'', we must have that $\binom{u'}{v'} = \binom{u''}{v''}$ and $a' = a''$, as desired. $\square$

Definition 8.11 *Let φ be a primitive non shift-periodic substitution, and let L make φ have a disagreement. We define P_{φ}^{L} to be the set of all pairs $a\binom{u}{v}$, where $au, av \in \mathcal{L}_{\varphi}$, $|a| = |u| = |v| = L$, and $u_{[0]} \neq v_{[0]}$.*

Definition 8.12 *Let $\varphi : \mathcal{A} \to \mathcal{A}^*$ be a primitive, non shift-periodic substitution, and let L make φ have a disagreement. Let $A = a\binom{u}{v}$ be a pair of finite allowed words such that $|a| = |u| = |v| = L$, and $u_{[0]} \neq v_{[0]}$. We define $\operatorname{dis}_{\varphi}^{L} : P_{\varphi}^{L} \to P_{\varphi}^{L}$ to be the function given by $\operatorname{dis}_{\varphi}^{L}\left(a\binom{u}{v}\right) = a'\binom{u'}{v'}$, where $\varphi\left(a\binom{u}{v}\right) = ca'\binom{u'}{v'}\ldots$, where $c, a', u', v' \in \mathcal{A}^*$, $|a'| = |u'| = |v'| = L$, and $u'_{[0]} \neq v'_{[0]}$.*

Note that P_{φ}^{L} is a finite set.

The following follows closely the argument given in [BD01], adapted to our setting.

Theorem 8.13 *Let φ be a primitive non shift-periodic substitution such that $|\varphi(l)| \geqslant 2$ for $l \in \mathcal{A}$, and let L make φ have a disagreement. Let A and A' be an asymptotic pair, then there there is a pair $P = a\binom{u}{v} \in P_{\varphi}^{L}$ and a number k such that $\left(\operatorname{dis}_{\varphi}^{L}\right)^{k}(P) = P$, and such that A and A' are obtained by iterating φ^{k} on the pair P. Conversely, if $P = a\binom{u}{v} \in P_{\varphi}^{L}$ is a pair such that $\left(\operatorname{dis}_{\varphi}^{L}\right)^{k}(P) = P$ for some k, then iterating φ^{k} on the pair P, we obtain an asymptotic pair.*

Proof Let $(A_0, A'_0) = p_0 a_0 \binom{u_0}{v_0}\ldots$ be an asymptotic pair such that $a_0\binom{u_0}{v_0} \in P_{\varphi}^{L}$, and let $P_0 = a_0\binom{u_0}{v_0}$. Since $\varphi : \mathcal{T}_{\varphi} \to \mathcal{T}_{\varphi}$ is bijective, we can apply φ^{-i} to (A_0, A'_0) to obtain a pair $P_i = a_i\binom{u_i}{v_i} \in P_{\varphi}^{L}$ such that $\varphi^{-1}(A_0, A'_0) = p_i a_i P_i \ldots$. Since P_{φ}^{L} is finite, we must have that there is a patch $P \in P_{\varphi}^{L}$, and infinitely many i such that $P_i = P$. Thus, the asymptotic composant is periodic under φ with some period k, which implies that P is in the eventual range of $\operatorname{dis}_{\varphi}^{L}$, and A_0 is obtained from iterating φ^{k} on the patch represented by P_i.

The converse is immediate. □

Lemma 8.9 and Theorem 8.13 are the basis for the following algorithm to find asymptotic composants for primitive non shift-periodic substitutions $\varphi : \mathcal{A} \to \mathcal{A}^+$ such that $|\varphi(l)| \geqslant 2$ for all $l \in \mathcal{A}$:

1. For each $L = 1, 2, \ldots$, verify if L makes φ have a disagreement. This is guaranteed to succeed at some point by Lemma 8.9.

2. Construct the set P_{φ}^{L} by letting a be a L-right-special factor, and trying all the L extensions in pairs.

3. Find the eventual range $R$of P_{φ}^{L} under dis_{φ}^{L}.

4. Each of the pairs in R lead to an asymptotic composant, by Theorem 8.13

We note that the algorithm might not be efficient, since the L guaranteed by Lemma 8.9 is large.

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APPENDICES

APPENDIX A

FUTURE WORK

In this chapter we are going to summarize some of the projects for future work and questions that occurred during the development of the dissertation.

1. Finish implementing a program that outputs the Connectivity Graphs via computation of the essential balanced pairs and the substitution on Balanced pairs
2. Find similar theorems to the ones we have for Arnoux Rauzy substitutions but applied to β-shifts
3. Find sets analogous to the Rauzy Fractal for self similar substitution tilings of higher dimensions.
4. Via return words, find an induction argument to determine, with only finitely many graphs, the connectedness of a broader class of substitutions.
5. If φ is a RUPC substitution, is there a IUPC substitution ψ such that $\mathcal{F}_\varphi \cong \mathcal{F}_\psi$?
6. Analyze in detail the relation of the connectivity graphs between a RUPC φ and $(\varphi^*)_u$
7. Is there a result analogous to Vuillon's for Rauzy Fractals in more than two letters?
8. Given a RUPC substitution φ is there always a substitution ψ such that $\mathcal{F}_\varphi \cong \mathcal{F}_\psi$ and $\mathcal{R}_\psi$ is disconnected.
9. Given a RUPC substitution φ, is there a RUPC substitution ψ such that $[\varphi] = [\psi]$ and R_ψ is connected?
10. By a result of Kurosh, any compact space is the limit superior of the projection spectra of a sequence of nerves. Is there a nice representation of the Rauzy Fractal in this setting?

11. Analyze how the essential balanced pairs change upon rewriting and shift equivalence.

12. Find results of connectedness using an approximation to the Rauzy Fractal by a finite number of projection of vertices from strand associated to a periodic word for φ.

13. If r is the minimal radius such that $R_\varphi \subset \overline{B_r(0)}$, what can we say about the r for a rewriting of φ

14. Can connectedness of the Rauzy Fractal be inferred only using Essential balanced pairs coming from asymptotic pairs?

15. Is there a connection between Siegel's method using Algebraic Number Theory and ours?

APPENDIX B

ALGORITHMS AND IMPLEMENTATIONS IN SAGE

To request CD copies contact your local public or university library to place an interlibrary loan request to Montana State University. For questions call (406) $994-3161$.

We attach to this dissertation some useful programs that we developed for dealing with substitutions and to draw the Rauzy Pieces of reducible Pisot substitutions. The programs are written in the computer algebra system Sage (see [S+11]).

The next proposition is probably well known. It is useful for a future implementation of an algorithm to compute the connectivity graphs of Chapter 5. In the following proof, $[a]_{\{v\}}$ denotes the coordinates of the vector $a \in E^u$ with respect to the basis $\{v\}$.

Proposition B.1 *Let φ be an IUP substitution, with unstable space $E^u = < v >$ for some vector $v \in \mathbb{R}^d$. Let a, b be the initial and the final vertex of a edge S, and let a', b' be the initial and the final vertex of an edge S'. Let $\alpha = [\mathrm{pr}^u(a)]_{\{v\}}$, $\beta = [\mathrm{pr}^u(b)]_{\{v\}}$, $\alpha' = [\mathrm{pr}^u(a')]_{\{v\}}$, $\beta' = [\mathrm{pr}^u(b')]_{\{v\}}$A necessary and sufficient condition that the edges S and S' are stable related is that $(\alpha - \beta')(\alpha' - \beta) \geqslant 0$.*

Proof Suppose that there exists a point q such that $\alpha \leqslant q \leqslant \beta$, and $\alpha' \leqslant q \leqslant \beta'$. Then $\alpha - \beta' \leqslant 0 \leqslant \beta - \alpha'$. Thus $(\alpha - \beta')(\beta - \alpha') \leqslant 0$, and hence $(\alpha - \beta')(\alpha' - \beta) \geqslant 0$.

Conversely, suppose that $(\alpha - \beta')(\alpha' - \beta) \geqslant 0$. If the intervals $[\alpha, \beta]$ and $[\alpha', \beta']$ would not intersect, then either $\alpha < \beta < \alpha' < \beta'$ or $\alpha' < \beta' < \alpha < \beta$. In either case $(\alpha - \beta')(\alpha' - \beta) < 0$, which is contrary to our supposition. Hence the intervals $[\alpha, \beta]$ and $[\alpha', \beta']$ intersect. $\square$