A method of graphical analysis for unsymmetrical three-phase circuits
by Armin John Hill

A THESIS Submitted to the Graduate Committee in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering
Montana State University

Abstract:
This thesis presents a method of graphical analysis for electric circuits which is based upon some of the simple aspects of tensor analysis as recently applied in electrical problems* A development of the method is made in such a manner as to be understandable by undergraduate engineering students. The relationship 'between the homogenous linear transformation used in teaser analysis, and the ordinary system of three simultaneous linear equations found • in -elementary analytic geometry is shorn? and the graphical representation of this transformation is used as a basis of the later developments# It becomes possible through such a presentation to solve a system of three simultaneous linear equations graphically* by applying1 some principles Of descriptive geometry® A brief development of the tensor notation is included in order that this notation can be used in the more complex developments of the later parts® The possibility of the study of electric circuits through tx graphical analysis based upon such a presentation is discussed briefly* and a few of the fundamental methods of procedure for such an analysis are presented® An extension, of the principles to include equations with complex quantifies, is made* and these are applied to the study of alternating current circuits® General circuit problems in this form are found to be very complicated* but most of the actual problems are simplified enough that they can be handled on a practicable basis® The method is found to be particularly useful in handling unbalanced three-phase systems* either directly or by means of symmetrical components® With the latter* the transformation to such components can be made by aeons of a chart which has the acme form for all problems of this nature.

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Bozeman, Montana
June, 1938

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A METHOD OF GRAPHICAL ANALYSIS FOR UNBALANCED THREE-PHASE CIRCUITS

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Abstract

This thesis presents a method of graphical analysis for electric circuits which is based upon some of the simple aspects of tensor analysis as recently applied in electrical problems. A development of the method is made in such a manner as to be understandable by undergraduate engineering students. The relationship between the homogenous linear transformation used in tensor analysis, and the ordinary system of three simultaneous linear equations found in elementary analytic geometry is shown, and the graphical representation of this transformation is used as a basis of the later developments. It becomes possible through such a presentation to solve a system of three simultaneous linear equations graphically, by applying some principles of descriptive geometry. A brief development of the tensor notation is included in order that this notation can be used in the more complex developments of the later parts. The possibility of the study of electric circuits through a graphical analysis based upon such a presentation is discussed briefly, and a few of the fundamental methods of procedure for such an analysis are presented.

An extension of the principles to include equations with complex quantities is made, and these are applied to the study of alternating current circuits. General circuit problems in this form are found to be very complicated, but most of the actual problems are simplified enough that they can be handled on a practicable basis. The method is found to be particularly useful in handling unbalanced three-phase systems, either directly or by means of symmetrical components. With the latter, the transformation to such components can be made by means of a chart which has the same form for all problems of this nature. In the commonly occurring cases where the zero-phase components are absent, this chart takes the form of a convenient slide rule, on which the two remaining sets of components can be obtained from two settings of the slide.

As with the tensor analysis which it parallels, this method makes possible a simultaneous analysis of an entire network. It also offers a graphical introduction to tensor methods for the student who has had a limited mathematical background. Extensions of the graphical method to other fields are indicated, and it is hoped that such indications will lead to an increased application and interest in the valuable and important concepts given the engineer by tensor analysis.

A METHOD OF GRAPHICAL ANALYSIS FOR UNSYMMETRICAL THREE-PHASE CIRCUITS

"The key to the simplest analysis of alternating currents lies in the use of graphical methods."[1] This statement made by Dr. Kennelly in 1898 has been well verified since that time for graphical methods have been widely developed and applied in practically every branch of electrical engineering. In fact, in most cases the graphical development has paralleled closely the application of the analytical mathematics, and often has proved so valuable that it has become the accepted method of proceedure.

For instance, we find the concept of the time vector used by Steinmetz[2] as a pictorial representation of the complex number which had been so successfully introduced by Kennelly[3] and Steinmetz[4] to represent alternating current quantities. A study of the effect on such vectors of the changes which take place under certain operating conditions led to the discovery of the invaluable circle diagram[5] which since has come into general use in the analysis of the operating characteristics of alternating current machinery. Likewise charts and graphs proved themselves indispensable in the study of magnetic circuits when the corresponding mathematical equations were found to be too complicated for practical use. A survey of the field of power transmission reveals numerous applications of graphical proceedures, some of which, such as the Mershon diagram[6], and the Dwight[7], and the Thomas[8] charts, have become standard equipment in the handling of many of the problems in this field. Similar applications are to be found in practically every branch of the electrical engineering field.

The development presented in this thesis is an attempt to give a graphical interpretation of a few of the simpler applications which have recently been made of tensor analysis in the study of electric circuits. "For many years the concepts introduced by vector analysis were sufficient for handling the types of electromagnetic phenomena encountered in electrical circuits and apparatus; but with the later increasing complexities involved in machine design, Steinmetz's complex numbers became inadequate for universal application."[9] Consequently, Gabriel Kron of the General Electric Company issued in 1932 a series of mimeographed articles dealing with the applications of tensor analysis to electrical machinery[10] and gave an informal paper on the subject before the winter convention of the AIEE in January, 1933.[11] This is apparently the first attempt at such an application in the field of electrical engineering, but almost immediately a widespread interest in this new tool was apparent, and many articles began to appear concerning various applications of tensor and matrix methods to electrical problems.[12] Kron revised and enlarged his original work, publishing it as a series of articles in the General Electric Review[13], and in this form it is the most comprehensive treatment of the applications of tensor analysis to electric circuit problems found in available literature. It is primarily on some of the elementary parts of this work that the material of this thesis is based.

Tensor analysis, from the very beginning stages of its development demonstrated itself to be an extremely powerful and useful mathematical concept. It quickly proved its worth in the field of geometry, and the demands of relativistic and quantum physics showed that its methods were capable of a wide variety of applications. Now these recent attempts to apply it to the type of problems with which engineers are primarily concerned have clearly shown that here at last is a tool, powerful and versatile enough to cope with the increasingly complex problems of the engineering field.

The rapidly increasing interest brought about by the success of these applications has made it imperative that engineers who wish to keep pace with present literature acquire a working knowledge of tensor principles. In fact, it is safe to predict that before long a thorough knowledge of tensor analysis will be an indispensable requisite of the well trained engineer. Up to the present time, however, such an understanding is the special privilege of the few who have had an opportunity to study a considerable amount of advanced mathematics. Now it has been found that many of the applications presented in this thesis can be based directly upon the mathematical forms encountered in elementary algebra and analytic geometry, and therefore should be within the grasp of the undergraduate engineering student, or of the engineer whose mathematical background is limited. For this reason care has been taken to present the material in such a way that it can be understood by one who has had no more than the equivalent of one year of college mathematics. It is hoped that in this way, such a presentation may help to bring about a more general understanding of this powerful mathematical concept.

Therefore, while the primary purpose of this thesis is to present a graphical parallel of the application of tensor analysis to electric circuits, insofar as physical limitations will permit, it is also hoped that the form of presentation will accomplish two other results. First, the graphical presentation offers an excellent introduction to tensor methods. One of the chief obstacles in the path of a general understanding of tensors is the difficulty of obtaining a clear mental picture of the concepts involved.[14] Since some of the more elementary of these are here developed graphically, and are thus given a physical interpretation, it is hoped this will provide a ground work of such a nature that the mental hazards of the more advanced concepts are materially reduced. In the second place, when such an approach is made, the close connection between the simpler aspects of tensor analysis and the forms encountered in algebra and in analytic geometry is stressed.

With these points in mind, care has been taken to develop the material from the standpoint of one not acquainted with tensor methods. Ordinary algebraic notation is used for the first developments, with the tensor notation introduced for handling the more complex applications. Also an attempt is made at all times to keep the close connection between the graphical, the algebraic, and the tensor concepts in mind. For instance a system of simultaneous linear equations is shown graphically as a transformation from one set of coordinates to another, and this in turn is shown to be the equivalent of a fundamental transformation used in tensor analysis.

As might be expected, since it is based upon a different form of mathematics, the graphical analysis presented here has little in common with methods now in accepted use. Many more or less successful methods of graphical analysis for electric circuits have been developed, among which may be mentioned the one by Eddy[15], which is particularly applicable to variable frequency circuits, those presented by Lee[16], which give the effect of the variation of any circuit constant upon the other circuit values, and a recently developed method of handling graphically impedances in parallel, given by Boening.[17] However, all of these are based upon the equations for a single electric circuit, or portion of a circuit. As with the corresponding mathematics, the circuit constants are inextricably mixed with the values impressed upon the circuit, with the result that for each new condition a new equation must be set up, and likewise a new graphical plot must be made.

The strength of the tensor method lies in the fact that all the conditions within a complex machine or network can be represented by a single equation[18], and this equation not only is unchanged in form when a change takes place within the circuit, but it is similar in form for similar problems involving different networks or machines. Likewise the graphical application sets up a space structure which is useful in the analysis of all problems of a certain type. Variation of individual quantities within a given problem can be studied directly as a shift in the position of certain lines or points, in most cases not affecting many of the other values, and in no case affecting the general form of the problem. Further, within certain physical limitations, this method allows an analysis of an entire network at one time, a feature which could not be expected of a method based upon the mathematics of a single circuit only.

Applications in this thesis will be confined to the analysis of electric circuits only, though the same principles may easily be applied to other types of electrical problems as well as to problems in other fields of engineering where the corresponding tensor transformations are applicable. Steady state conditions only have been considered, though again there seems to be a possibility of an extension to cover some types of transient conditions. Extensions have been made to include complex quantities, however, making possible a study of alternating currents.

The most complete analyses are possible when not more than three independent equations are involved. Therefore the method is particularly adapted to a study of three-phase systems. When the principles are applied in obtaining the symmetrical components of an unbalanced three-phase system, immediate success is apparent for it becomes possible to construct a chart from which these components can be read easily, and in the commonly occurring case where the zero-phase components are absent, this chart takes the form of a very convenient slide-rule.

No attempt is made here to cover the field thoroughly or to exhaust the possibilities of any particular branch, as the subject appears too broad to permit more than a preliminary survey. Some of the possibilities are pointed out, however, and it is hoped that enough material is presented to give an incentive for a more complete study of this apparently useful method of presentation.

PART I: SIMULTANEOUS LINEAR EQUATIONS WITH REAL VALUES ONLY

Explicit and Implicit Forms:

Let us begin by considering a system of three simultaneous linear equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \qquad (1) \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

where the a's, b's, and c's are constant real numbers. The values of x, y, and z which will satisfy these equations may be found by using determinants as follows:

Let D represent the determinant of the a, b, and c values, and define r_1 as the minor of a_1 in D, r_2 as the minor of a_2, etc., with s and t to represent the minors of the b and c numbers respectively, with the added assumption that the symbol include not only the minor, but also the proper plus or minus sign according to the position of the respective a, b, or c value in the determinant D. Such minors with proper sign included are known as the "cofactors" of their respective a, b, or c number. Equations (1) may then be "solved" for x, y, and z in this form:

$$\begin{aligned} x &= \frac{k_1r_1 + k_2r_2 + k_3r_3}{D} \\ y &= \frac{k_1s_1 + k_2s_2 + k_3s_3}{D} \qquad (2) \\ z &= \frac{k_1t_1 + k_2t_2 + k_3t_3}{D} \end{aligned}$$

These equations may now be put in a form similar to that of equations (1) by setting up a determinant of the coefficients which would be the "inverse transpose" of D. This is done by determining nine sets of values (which we can represent by the letters d, e, and f) such that:

$$d_1 = \frac{r1}{D} \qquad e_1 = \frac{r2}{D} \qquad f_1 = \frac{r3}{D}$$
$$d_2 = \frac{s1}{D} \qquad e_2 = \frac{s2}{D} \qquad f_2 = \frac{s3}{D} \qquad (3)$$
$$d_3 = \frac{t1}{D} \qquad e_3 = \frac{t2}{D} \qquad f_3 = \frac{t3}{D}$$

giving us equations (2) in the form:

$$x = d_1k_1 + e_1k_2 + f_1k_3$$
$$y = d_2k_1 + e_2k_2 + f_2k_3 \qquad (4)$$
$$z = d_3k_1 + e_3k_2 + f_3k_3$$

Equations (1) and (4) are now in the same form, but we can see that the positions of the x, y, and z values and of the k_1, k_2, and k_3 values have been interchanged.

In order that no confusion may result in what follows, we will speak of the x, y, and z values as being in the "implicit" form when involved in the equations as they are in equations (1); and as being in the "explicit" form when in the positions they occupy in equations (4).

Interpretation of the Equations as a Transformation of Coordinates:

Select a system of coordinate linear axes in space and let the position of a point P with respect to them be defined by its coordinate distances, x, y, and z. Now determine another set of values, which may be called x', y', and z', in such a way that:

$$x' = a_1x + b_1y + c_1z$$
$$y' = a_2x + b_2y + c_2z \qquad (5)$$
$$z' = a_3x + b_3y + c_3z$$

where the a's, b's, and c's are once again constant values, i.e. they are not in any way affected by the position of the point P.

Now think of the x', y', and z' as representing values of the coordinates of P with respect to another set of axes, X', Y', and Z' respectively. From equations (5) it will be seen that when the values of x, y, and z are zero, the values of x', y', and z' are also zero. Thus X', Y', and Z' are coordinate axes having a common origin with the original set.

Equations (5) may therefore be said to represent a transformation of the coordinates of P from the original x, y, and z values to the new x', y', and z' values. In most of our applications it will be found more convenient to have x, y, and z expressed explicitly, however, and this can be done by the method shown in the preceeding section, thus:

$$\begin{aligned} x &= d_1x' + e_1y' + f_1z' \\ y &= d_2x' + e_2y' + f_2z' \\ z &= d_3x' + e_3y' + f_3z' \end{aligned} \qquad (6)$$

where the determinants of the d's, e's, and f's is the inverse transpose of that of the a's, b's, and c's as in equations (4).

From the similarity between equations (1) and (5), and between (4) and (6), it may be seen that any system of three simultaneous linear equations can be interpreted as a transformation, more specifically as a homogeneous transformation, from one system of coordinates to another. It will therefore be in order for us to examine this transformation in greater detail.

<u>The Unit Points:</u>

One of the first problems will be to determine the form of the second set of axes. This will involve first of all a location of the "unit points" on these axes, i.e. those points which will have as their x', y', and z' coordinates (1,0,0), (0,1,0), and (0,0,1) respectively.

Let us consider the first of these, the unit point on the X' axis with coordinates (1,0,0). In order to locate this point with reference to the original set of axes, i.e. to find its x, y, and z coordinates, it is merely necessary to substitute the given values of x', y', and z' in equations (6). The three coordinates of the X' unit point are immediately seen to be (d_1,d_2,d_3). Likewise the Y' unit point will be found to have the coordinates (e_1,e_2,e_3), and the Z' unit point the coordinates (f_1,f_2,f_3).

Let us now investigate the point with x', y', and z' coordinates of (2,0,0). By substituting as before, it will be seen that its coordinates in the original system are $(2d_1,2d_2,2d_3)$, or in other words, it is a point on a straight line through the origin and the X' unit point. More generally speaking, the point whose transformed coordinates are (x',0,0) will have coordinates in the original system of $(x'd_1,x'd_2,x'd_3)$; and if the values of the d's are constant, it can be seen that these points all fall on the same straight line. But these points are also on the X' axis, and if x' is allowed to take on all real values, all points on this axis will satisfy these conditions. In other words, the X' axis is a straight line.

Likewise it can be shown that the Y' and Z' axes are straight lines and we can make the general statement that if the coefficients in the equations (5) or (6) are constants, both sets of axes are linear, and conversely, if all the axes are linear, the coefficients of the equations must have constant values. Therefore these equations may be said to represent a homogeneous linear transformation of coordinates.

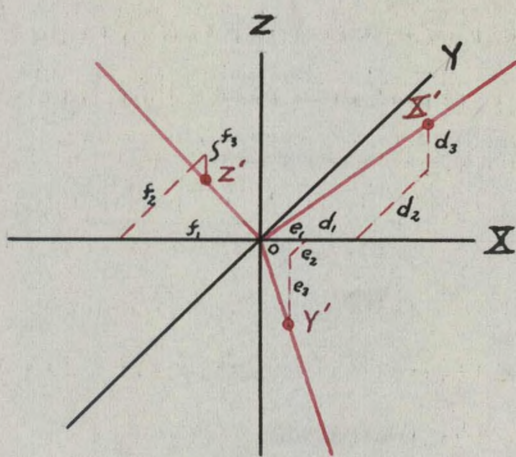


Fig. 1 Unit Points and Axis System for Linear Transformation of Coordinates

A space representation of such a transformation is shown in fig. 1. A right hand system of axes in a rectangular Cartesian form is chosen for the original set for simplicity and convenience in plotting. It may be remarked here that the discussion given above is perfectly general and that neither set of axes need be either rectangular or Cartesian in form. However, we usually have a choice in the form of one or the other of the sets, and throughout this thesis the original set will be given the rectangular Cartesian form. This choice should be made wherever possible for the added simplicity it gives.

The coordinate distances of the unit points representing the various coefficient values of the equations (6) are also indicated on fig. 1. It will be noticed that these are measured in the units of the original system in each case. Also, it may be noticed that the transformed axes are labelled at the unit points instead of at the positive end as is customary. For convenience the unit point and the axis will be given the same symbol here. As it is always apparent to which reference is made no confusion should result.

Danger of crowding and the relative unimportance of the unit points on the original Cartesian axes makes it impractical to apply this convention to them, and they will be labelled at their positive ends in the usual manner. Also to prevent too crowded a picture, the positive rays only of the transformed axes will be shown, and these axes will always be drawn in color. The X', Y', and Z' symbols will always refer to the transformed system, and these axes may take any position as long as they are linear and the origin is common for both systems. Likewise X, Y, and Z will always refer to the original rectangular Cartesian system. This notation will be used until the more convenient tensor notation is introduced.

Coefficients and Coordinates:

It will be seen from this analysis that the position of the unit points depends only upon the values of the coefficients in the equations of transformation. Thus, since all axes are linear, the relative positions of the two sets are definitely fixed by the values of these coefficients. These values may therefore be said to determine the transformation, and when arranged as a matrix or determinant will be known as the matrix or determinant, respectively, of the transformation. It is thus a very simple process to represent the coefficients in any system of two or three simultaneous linear equations as two sets of coordinate axes.

On the other hand it can be seen that the values of the coordinates, x, y, z, and x', y', z', depend only upon the position of the point P and the form of the particular axis system referred to, not being involved in any way in the transformation from one system to the other.

The Space-Vector:

It has already been noticed that the two systems of axes have a common origin. It may also be observed that the position of the point P with respect to this origin is not affected by the position of these axes. Let us now draw a space-vector OP, and it will be seen at once that the magnitude and direction of this vector is in no way dependent upon the system of axes selected. It is thus said to be "invariant" under any transformation of the type we have been discussing. In tensor analysis such invariant forms are of the utmost importance, for about them the whole scheme of analysis is built. This vector will therefore be of primary importance in this discussion for we will find that while it will always be referred to some system of axes, its properties are more fundemental than those of any such system, for it remains unchanged in any system that may be selected.

The coordinate values now take on a new meaning for they become the components of the space-vector parallel to their respective axes. Such a vector thus gives us a graphical representation of the relations between the two sets of coordinates, and since it is invariant it makes it immediately possible to find one of the sets, having given the other set and the equations of transformation.

Graphical Interpretation:

We can thus represent three simultaneous linear equations as a homogeneous linear transformation of the component values of a space-vector, and can represent this transformation graphically as is done in fig. 8. It now becomes possible to give a graphical interpretation to each of the component parts of equations (6). By selecting the two sets of axes as was done for fig. 1, the nine constant coefficient values become the three sets of coordinates of the three unit points, X', Y', and Z'. The two sets of variable or coordinate values, x, y, z, and x', y', z', become the two sets of three components each of the vector OP along their respective axes. Thus the fifteen component parts of equations (6) are represented by fifteen separate values in the graphical picture.

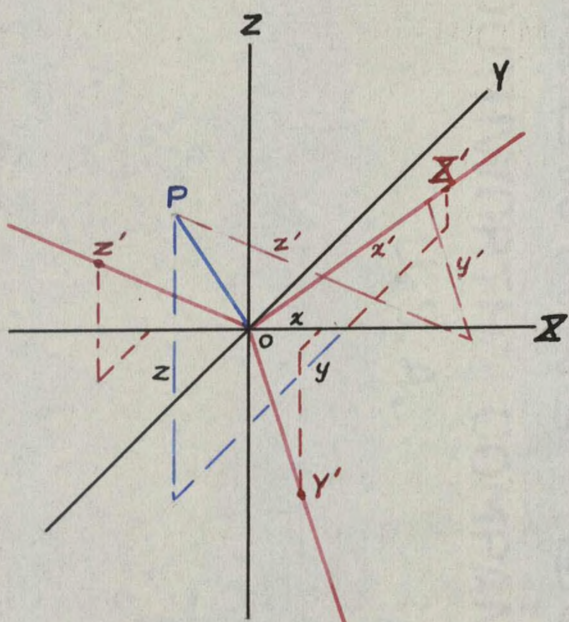


Fig. 2 Simultaneous Linear Equations Represented Graphically as a Transformation of Components of a Vector

Not only are the components of the equations kept separate, but of more importance is the fact that the coefficient and the variable values are represented by two distinct types of configuration, for while the coefficient values determine the form of space structure which is to be used, the variable values represent the relationship of an invariant space-element to these reference forms.

Evaluation of the Explicit Variables:

Assuming that the transformation determinant is known and that the x', y', and z' coordinates of the point P are given, it is a very simple matter to determine the values of x, y, and z graphically when these are expressed explicitly as in equations (6). If two equations only are involved, the representation may be made on one plane. If three equations are given, it becomes necessary to have a space figure, and this is best shown for these purposes on two projections as shown in fig. 3. A third quadrant projection of the right hand system of axes is used, which places the horizontal XY plane above the vertical XZ plane, and keeps the positive ends of the Y and Z axes upward, and of the X axis to the right.

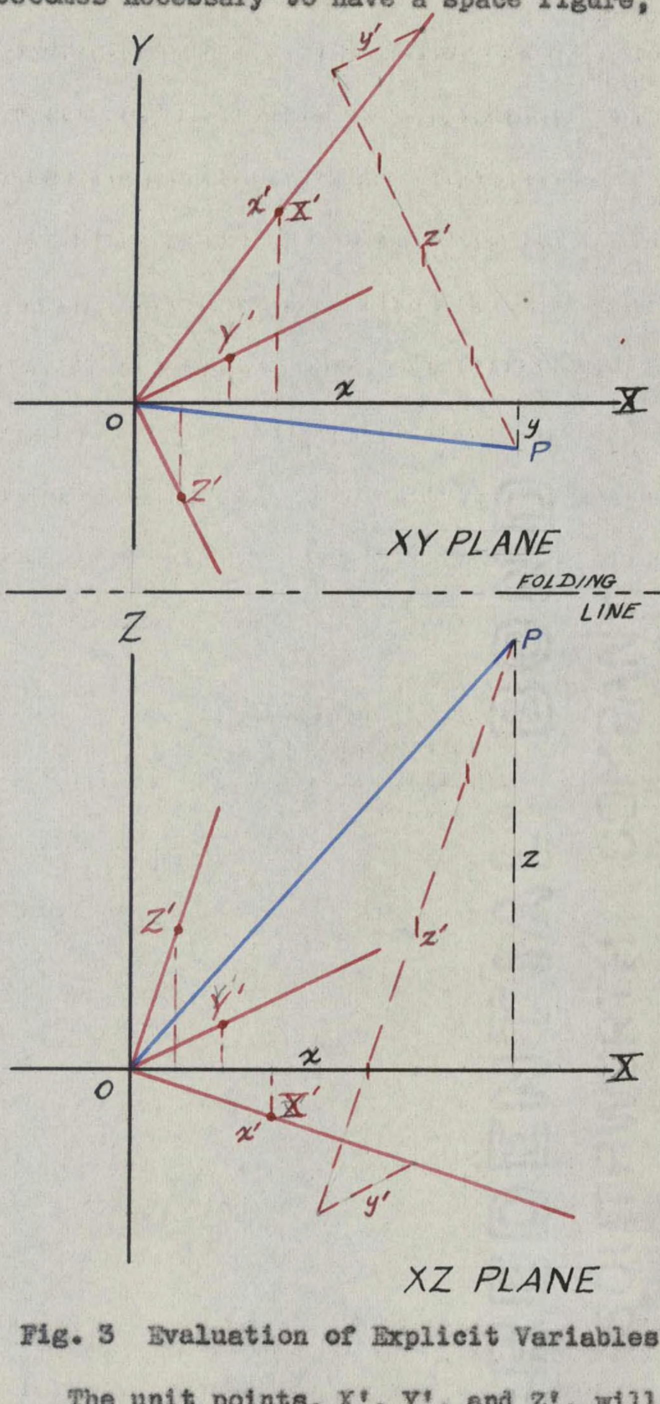


Fig. 3 Evaluation of Explicit Variables

Fig. 3 is a representation of a definite problem: Given the equations:

$$x = 3x' + 2y' + z'$$
$$y = 4x' + y' - 2z' \qquad (7)$$
$$z = -x' + y' + 3z'$$

Show the relationship between the two sets of axes, and evaluate x, y, and z for given values of x', y', and z'.

The unit points, X', Y', and Z', will be respectively (3,4,-1), (2,1,1), and (1,-2,3). Plot these and through each in turn and the origin construct the corresponding axis. Selecting some point P, located by its coordinates, say (2,-1,4), on the transformed axes, i.e. having given $x'=2$, $y'=-1$, $z'=4$, it will be found that the coordinates with respect to the original system can be obtained by taking twice each of the Cartesian components of X', minus one of each component of Y', plus four times each component of Z', then adding algebraically all of the components parallel to each of the original axes. This would give the value of x as $2 \cdot 3 + (-1) \cdot 2 + 4 \cdot 1$, or 8, which is identical with the value which would be obtained by substituting the given values of x', y', and z' in equations (7).

Graphically this process consists merely of reading off the rectangular Cartesian coordinates of the point P, once it has been located with reference to the transformed system. For instance in the problem given, the coordinates of P can be read from fig. 3 as (8,-1,9) or this would mean $x=8$, $y=-1$, $z=9$, which check the values obtained by substitution. Thus if the transformation constants remain unchanged, the values of x, y, and z for any set of values of x', y', and z' can be read directly from a diagram such as fig. 3. It will be shown later that this apparently very simple proceedure can be given some very useful applications.

Extension to Four and More Equations:

The method of using projection planes can easily be extended to cover the conditions which arise when four or more simultaneous equations are involved. Four equations, for example, will require four original and four transformed axes. No space figure of such an arrangement can be drawn in one view, of course, but by selecting projection planes properly, as shown in fig. 4, it is possible to evaluate the explicit variables by the process just described.

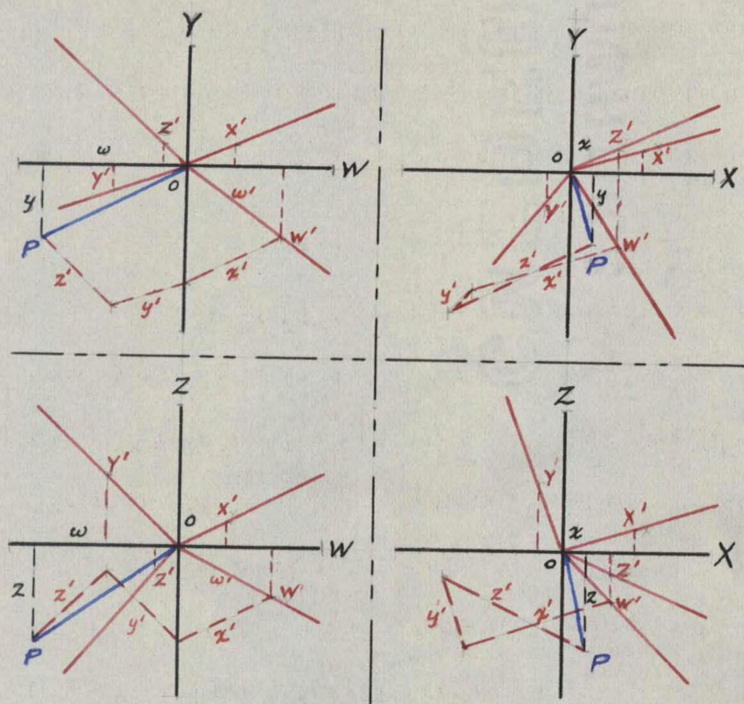


Fig. 4 Graphical Representation of Four Simultaneous Linear Equations

Let the four rectangular Cartesian axes be represented by W, X, Y, and Z, and the transformed axes by W', X', Y', and Z'. The point P will now be located by means of its four coordinate values w', x', y', and z', and the explicit values can be read off as the w, x, y, and z coordinates of this point.

It will be noticed that the fourth plane is used only for a check, as any three planes will give the desired information. Extensions can be made to any number of equations by adding another plane for each added equation.

__Evaluation of the Implicit Variables - Solving the Equations:__

Since the space vector fixes the relationship between the two sets of coordinate values, it should be possible, having one set and the transformation given, to determine the other set, whether this set is in the explicit or in the implicit form. It will now be shown that it is possible to evaluate the implicit variables, in short to __solve__ two or three simultaneous linear equations, by a simple modification of these same graphical methods.

We will now have given the values of the x, y, and z in equations (6), or in other words the rectangular coordinates of the point P will be known. The problem now becomes one of determining the components of the vector OP parallel to each of the transformed axes when these axes may make any angle with each other, or with the original set. This can be done quite easily in a plane when only two equations are involved, but in order to find these components in three dimensions it is necessary to project the figure onto a third plane which will be perpendicular to the plane of two of the transformed axes. This may be done by applying the principles of orthogonal projection as found in any elementary text on descriptive geometry. For convenience the construction is shown in fig. 5, and is described here.

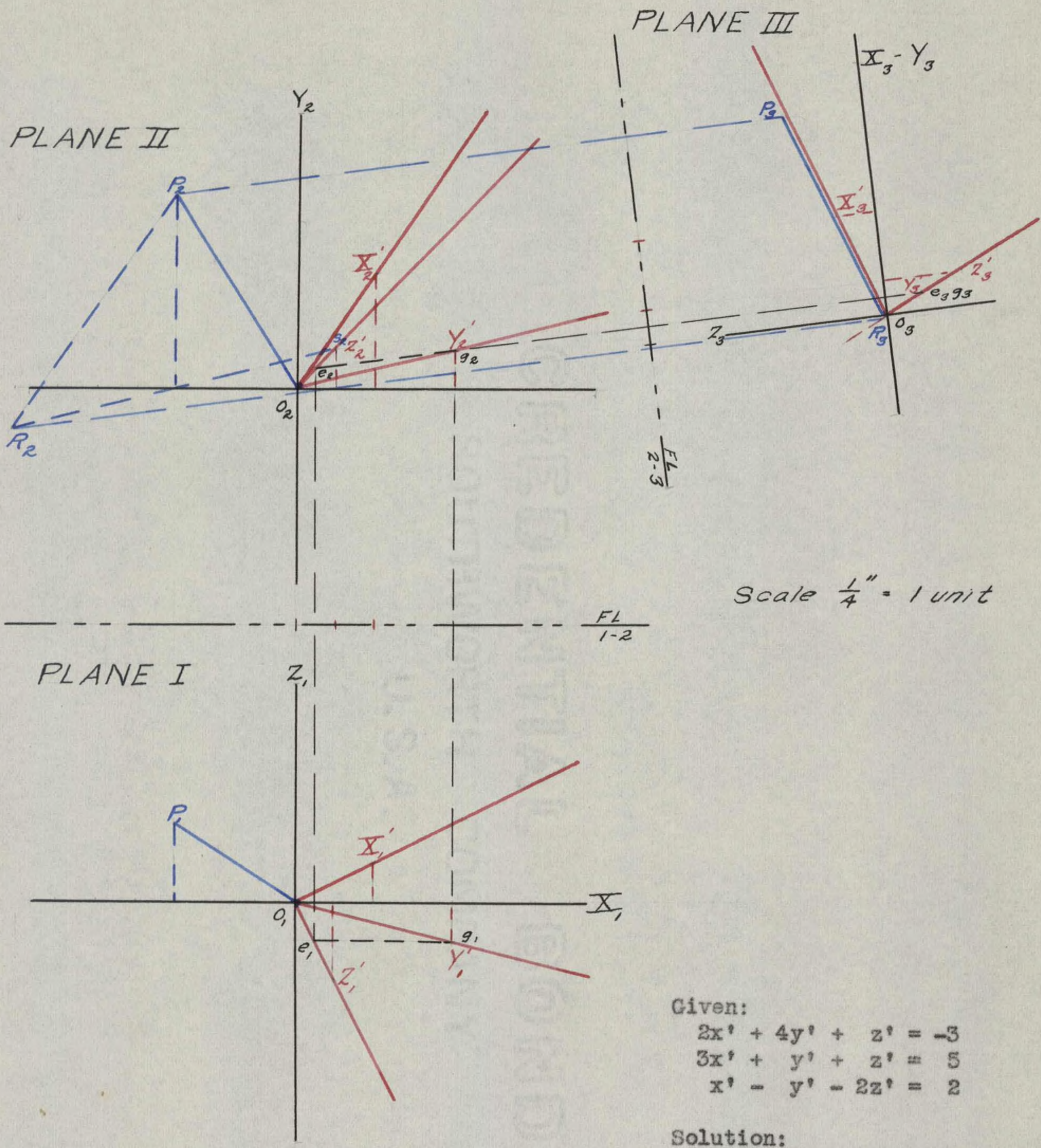
Construction for Graphical Solution of Three Equations:[19]

From the transformation equations construct the two sets of axes as was done for fig. 1, again keeping X, Y, and Z orthogonal and Cartesian. Use two planes as in fig. 3, and use third quadrant projection.

In the vertical Plane I, draw a construction line e-g, horizontally, in such a way that it intersect two of the axes, say Y' and Z', in the points e and g respectively. Project from the points e_1 and g_1 to Plane II, locating e_2 and g_2 as the intersections of the corresponding projection lines and axes. Draw the line e_2-g_2.

Select a third Plane III, adjacent to Plane II, by drawing the folding line Fl_{2-3} perpendicular to e_2-g_2. Now since the line e-g is horizontal, its projection on the horizontal plane will be parallel to it. Plane III will be perpendicular to it, and it will project on Plane III as a point e_3g_3. Further, Plane III will also be perpendicular to the plane Y'OZ', and this plane will project on Plane III as the line $O_3Y'_3Z'_3$. Now project OX' and the vector OP to Plane III.

The component of OP parallel to the OX' axis can now be found on Plane III by drawing the line P_3-R_3 parallel to $O_3X'_3$ until it intersects the plane Y'OZ', i.e. the line $O_3Y'_3Z'_3$. This will give one projection of the point R. The other may be found on Plane II as the intersection of a projection line from R_3 and a line drawn through P_2



Given:

$2x' + 4y' + z' = -3$

$3x' + y' + z' = 5$

$x' - y' - 2z' = 2$

Solution:

$x' = 2$) From

$y' = -2$) Plane

$z' = 1$) II.

Fig. 5 Graphical Solution of Simultaneous Linear Equations parallel to $O_2X'_2$. From the point R_2 thus located, a line parallel to $O_2Y'_2$ (or $O_2Z'_2$ if preferred) can be drawn until it intersects the other axis in S_2. Now OS, SR, and RP will be the three component lines required, and their respective lengths in terms of the unit lengths of each of the corresponding axes will be the required values of the respective coordinates.

Since each of these lines is parallel to its respective axis, it is not necessary that we have projections of each in its true length to ascertain these values, however, for each one will be foreshortened on any projection plane in exact proportion to the foreshortening of the unit distance of the corresponding axis on the same plane. In other words the required values of x', y', and z' can be found from Plane II as the lengths of the projections of the corresponding component lines on that plane in terms of the corresponding axial unit distances as shown on the same plane, or:

$$x' = \frac{R_2P_2}{O_2X'_2} \qquad y' = \frac{S_2R_2}{O_2Y'_2} \qquad z' = \frac{O_2S_2}{O_2Z'_2} \qquad (9)$$

These ratios can be easily evaluated by any one of several methods, the easiest perhaps being to measure the length of the projection of RP, for instance, with a divider set for the projected length of the OX' unit. Thus the equations can be solved by a purely graphical proceedure. Further, the method is perfectly general for all two and three equation systems with constant coefficients, having only physical limitations as we find common in all graphical proceedures.

The Tensor Notation.[20]

We now reach a point where we can profitably introduce some of the valuable concepts of tensor notation, since this notation is particularly adapted to problems of the type we have been discussing, and will save us much time and trouble in the applications which are to follow.

Instead of indicating a series of variables as we have done heretofore by means of an "x", a "y", and a "z", only one symbol will be used, with different indices to distinguish between different variable quantities it may represent. Thus, instead of the above, we would use x_1, x_2, and x_3. We now make two additional modifications. First, the letter "y" is commonly used to indicate a system of variables in rectangular Cartesian coordinates, while "x" is used to represent variables in any system. Also, the type of vector we have been considering would be known as "contravariant", and superscripts rather than subscripts are commonly used in its representation. Our values would now be represented by y^1, y^2, y^3, for the x, y, and z, respectively, and x^1, x^2, x^3, for x', y', and z'.

Another convention is the use of a literal index to represent all the numerical index values from 1 to 3 in turn. Thus y^r would represent y^1, y^2, y^3, and our equations (5) now become:

$$\begin{aligned} x^1 &= C^1_1y^1 + C^1_2y^2 + C^1_3y^3 \\ x^2 &= C^2_1y^1 + C^2_2y^2 + C^2_3y^3 \\ x^3 &= C^3_1y^1 + C^3_2y^2 + C^3_3y^3 \end{aligned} \qquad (10)$$

or to shorten the notation by use of literal indices:

$$\begin{aligned} x^1 &= \sum_{r=1}^{3} C^1_ry^r \\ x^2 &= \sum_{r=1}^{3} C^2_ry^r \quad \text{or} \quad x^s = \sum_{r=1}^{3} C^s y^r \\ x^3 &= \sum_{r=1}^{3} C^3_ry^r \end{aligned} \qquad (11)$$

It has been further agreed that when one index appears twice in the same term -- as it does on the right side of equations (11) -- a summation of the three terms represented will be assumed, and the summation sign may therefore be omitted. These equations then become:

$$x^s = C^s_ry^r \qquad (12)$$

which is the usual form of tensor notation for equations such as those we have been considering. In tensor analysis these equations are said to represent a transformation, and in particular if all the values of the matrix C_r^s are constant, the transformation is linear. Also, each member of the above "tensor equation" is an invariant, since it is not changed by the transformation. These concepts are exactly what we observed in the study of equations (5) and (6) graphically, for the C_r^s is the set of constant coefficient values, while the members represent the space vector OP. It may be pointed out that this vector is represented by its components, as it always will be, and that it is these values, not the vector, which are changed during the transformation.

A vector such as OP, represented by three component values, is known as a tensor of the first order. The matrix C_r^s represents nine different terms, and is known as a tensor of the second order. Here it is the tensor of transformation, and will be known as the transformation tensor or matrix. The determinant of the nine values represented will be indicated by $|C|$ and will be known as the determinant of the transformation, being identical with the "D" of equations (2) and (3).

So far we have been discussing the tensor equivalent of equations (5) though it was found that the form of equations (6) was much more useful for our purposes. In order to reduce the equations represented by the tensor equation (12) to a form comparable with that of equations (6), it will be necessary to obtain a matrix or tensor which is the "inverse" of C_r^s, and which will be represented by the symbol γ_r^s.

This "inverse" form is defined such that each element of the determinant $|\gamma|$ is the value of the cofactor (see page 12) of the corresponding element in $|C|$ divided by the value of $|C|$. Using these symbols, equations (12) now become:

$$y_r = \gamma_r^s x_s \qquad (13)$$

However, here the indices are subscripts, indicating "covariant" tensors -- which are defined as those tensors which transpose inversely to the contravariant tensors as shown here. This covariant vector can be drawn in much the same way that the contravariant one was, but the component values are no longer the coordinates of the terminal point with respect to the various axes. Thus from a graphical standpoint, the covariant form is not useful to us, and it will be necessary to change this vector into a contravariant one. This can be done simply by "transposing" the transformation tensor, i.e. by interchanging the rows and columns in its determinant, and then shifting the indices of the component vector values. The equations now are in the desired form, equivalent to equations (6).

$$y^r = \gamma_s^r x^s \qquad (14)$$

Many problems in connection with electric circuits involve systems of simultaneous equations, and the tensor notation as well as the graphical methods developed above are very applicable to these. It will be noticed in equations (14) that the Cartesian coordinates are in the left hand member, and this member has only the one set of quantities involved. This is an indication that we are going to use the Cartesian coordinates as the original reference system (as we have done in the graphical development).

Thus it will be the most convenient to indicate the voltages in electric circuit problems by means of these "y" values, for in most cases the voltages are impressed upon, or fixed by conditions outside the network, while the currents depend upon the voltages and the network constants.

Since the coefficient values discussed in this part are real and constant, these will represent resistance values only, and we can represent the transformation matrix by R^m_n. Thus a system of three simultaneous equations in a direct current network would have the form:

$$E^m = R^m_n I^n \qquad (15)$$

which is similar to that for a single element in ordinary notation.

It can be shown that equations for complicated networks, or for the analysis of complicated machinery, can be reduced to some such simple tensor expression. This is perhaps one of the most valuable aspects of this application, for with this notation there is no longer a need to set up a new equation each time a new problem is attacked, or the conditions of a given problem are changed. The close relationship which exists between various types of problems is kept in evidence, and mental and mathematical processes are therefore kept at a minimum.

The transformation shown by equations (12), (14), and (15) is one of the fundamental ones in tensor analysis. We have therefore not only demonstrated a method of solving three simultaneous linear equations, but have also given a graphical interpretation of this important transformation as well as of the fundamental tensor concepts of a contravariant vector and an invariant.

Such a transformation, of course, has a wide-spread application in all types of problems, and in any of these the graphical method presented here may be used to supplement other types of analysis. In electric circuit problems, Kron[21] has shown that such a transformation may not only be used to obtain currents in terms of voltages and resistances, or vice versa, but it may also be used to calculate currents in a system after a rearrangement of system elements in terms of the currents before, voltages after in terms of voltages before, and other types of relationships. There will not be room here to mention all types of such applications. A few are presented, more to illustrate the proceedure than to cover any one branch of the field, and more extended applications will be made in later parts of this thesis.

Applications to Problems:

The problems presented here deal with applications to electric circuits only, though it should be kept in mind that the method used can be applied in other fields as well as to other types of electrical problems just as readily. Since, in this section, we are dealing with real numbers only, the problems will involve only resistances and direct current networks. Actual solutions are not carried out, as the methods are those presented above. In each case, however, the equations and transformation matrix are indicated.

It should be kept in mind that for purposes of graphical analysis and for problems where it is not necessary to evaluate the "implicit" variables, the method can be applied to any problem which involves a system of simultaneous linear equations. The problems presented here are confined to those which involve not more than three independent equations, and which therefore can be "solved" by the graphical method.

Problem 1. Given a network as shown in fig. 6, set up equations for voltages in terms of three of the currents, and arrange the transformation matrix.

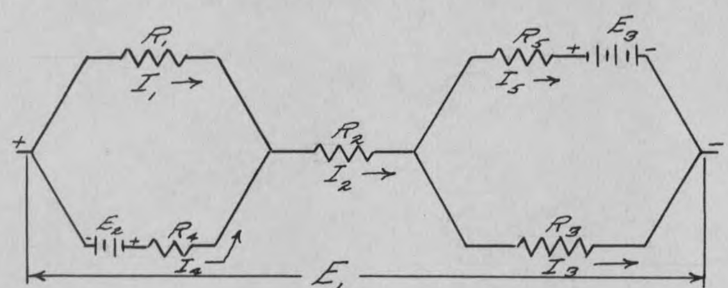


Fig. 6 Network for Problem 1.

Equations:

$$E_1 = I_1R_1 + I_2R_2 + I_3R_3$$
$$E_2 = -I_1R_1 - I_1R_4 + I_2R_4$$
$$E_3 = -I_2R_5 + I_3R_3 + I_3R_5$$

Matrix:

$R^m_n =$

R_1	R_2	R_3
$-(R_1+R_4)$	R_4	0
0	$-R_5$	R_3+R_5

Problem 2. Given the network in fig. 7. Proceed as for Problem 1.

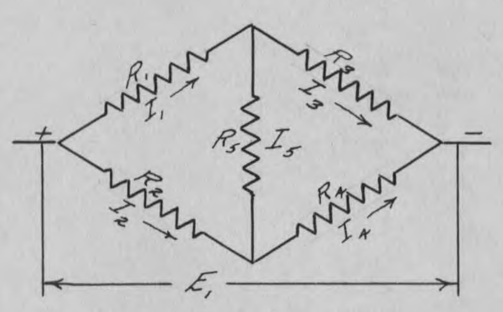


Fig. 7 Network for Prob. 2.

Equations:

$$E_1 = I_1R_1 + I_3R_3$$
$$0 = I_1R_1 + I_1R_2 + I_1R_5 - I_3R_5 - I_2R_2$$
$$0 = I_1R_5 - I_3R_3 - I_3R_4 - I_3R_5 + I_2R_4$$

Matrix:

$R^m_n =$

R_1	0	R_3
$R_1+R_2+R_3$	$-R_2$	$-R_5$
R_5	R_4	$-(R_3+R_4+R_5)$

Problem 3. Given the network in fig. 8, Proceed as for Problem 1.

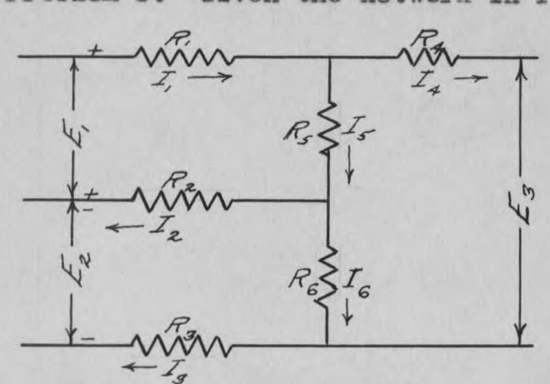


Fig. 8 Network for Prob. 3.

Equations:

$$E_1 = I_1R_1 + I_5R_2 + I_5R_5 - I_6R_2$$
$$E_2 = I_1R_3 - I_5R_2 - I_5R_3 + I_6R_2 + I_6R_3 + I_6R_6$$
$$E_3 = -I_1R_4 + I_5R_4 + I_5R_5 + I_6R_6$$

Matrix:

$R^m_n =$

R_1	R_2+R_5	$-R_2$
R_3	$-(R_2+R_3)$	$R_2+R_3+R_6$
$-R_4$	R_4+R_5	R_6

The equations in each of the above problems give us three current values only. Of course, it can be seen that by proper substitution any three of the various values can be found. However, it is also possible to obtain the other current values in terms of the three considered by means of a second transformation as shown in the example given here.

Problem 4. Given the network of Prob. 3, fig. 8. By means of a graphical transformation, obtain the values of the currents, I_2, I_3, and I_4, in terms of those values used in the equations, i.e. I_1, I_5, and I_6.

The equations relating these currents are:

$$\begin{aligned} I_2 &= \phantom{I_1 -{}} I_5 - I_6 \\ I_3 &= I_1 - I_5 + I_6 \\ I_4 &= I_1 - I_5 \end{aligned} \qquad \text{or} \quad {}_2I^s = \gamma^s_t I^t \qquad (16)$$

and the transformation matrix is:

$$\gamma^s_t = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} \qquad (17)$$

Such a transformation may be plotted separately, assuming I_2, I_3, and I_4 as values in the rectangular Cartesian system. However, we can easily find the inverse transpose of matrix (17) as:

$$C^t_s = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} \qquad (18)$$

and using this form we can work directly from the current axes of the graphical transformation for Prob. 3, finding a new set of axes whose coordinates will be the values of the three required currents. The graphical construction for this double transformation is shown in fig. 9, where values have been assumed for the resistances as follows:

$R_1 = 2$; $R_2 = 1$; $R_3 = 4$; $R_4 = 3$; $R_5 = 3$; $R_6 = 1$.

giving the transformation matrix:

$$R^m_n = \begin{vmatrix} 2 & 4 & -1 \\ 4 & -5 & 6 \\ -3 & 6 & 1 \end{vmatrix} \qquad (19)$$

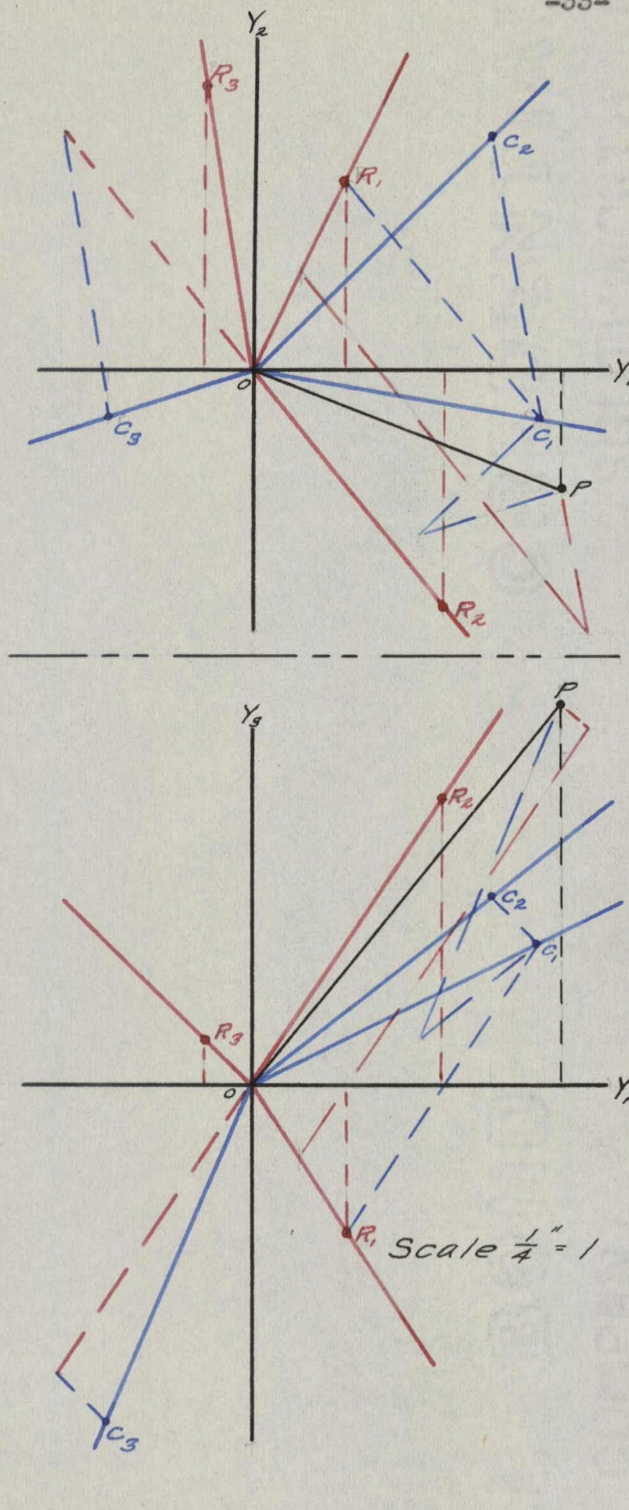


Fig. 9 Double Transformation

The space vector OP is shown in black, and the six components of OP parallel to each of the six transformed axes will be, respectively, the six required values of current. On the diagram the axes in red (labelled R_s) are parallel, respectively, to the currents I_1, I_5, and I_6; the blue axes (C_t) giving the values of I_2, I_3, and I_4 in order.

Since both sets of axes are at various angles, it will be necessary to draw two additional projection planes, one perpendicular to a pair of the Rs axes, and the other perpendicular to a pair of the C_t axes, if we are to make a complete study of the problem. Such a complicated figure results, that unless a particularly open form is possible, it will hardly be practicable for actual study; and two separate figures will usually be much more convenient.

This application does, however, show a case where both of the transformation axes are in a general rectilinear form, and it also demonstrates the possibility of transforming from one set of current values to another. In addition, it shows the possibility of applying a "double transformation", i.e. a second transformation to a set of transformed axes, and gives a graphical interpretation of the tensor form:[22]

$$E^m = C^m_n R^n_s I^s \qquad (20)$$

where C^m_n is the tensor represented by the matrix (18), R^n_s is the matrix (19), I^s represents the currents I_2, I_3, and I_4, and E^m are the three given voltage values.

Graphical Analysis:

One of the primary advantages of a graphical representation is that by means of it the effect of the change of one quantity on the other quantities may be observed directly without the necessity of mathematical calculation. We are particularly fortunate with this method in that each of the quantities involved is represented separately from the rest. It thus becomes an easy matter to make a graphical analysis of almost any nature desired. No attempt will be made here to discuss this phase of the development in detail, as this is an extensive study within itself. However, some of the types of analyses which can be made are indicated, with the idea in mind that they may at some time be studied more thoroughly, that the full value of these concepts may be realized.

Effect of Variation in One of the Implicit Coordinates:

A change in the value of any one of the six coordinates will affect only the position of P, and through this the value of some of the other coordinates. The positions of the axes, however, will not be affected. Interpreted electrically this merely means that changing the voltages or currents in a net-work will not affect the resistance values.

Such a change of implicit values can best be studied from a figure in two projection planes as is fig. 10. Care should be taken that the line representing the varying coordinate be drawn adjacent to P, and thus P will move along this coordinate line, or along the line PT in fig. 10, and the values of the other coordinates for any position such as P' can be observed easily.

It will be noticed that a change in one of the current values, such as I^3, will not affect the other current values, but will probably affect

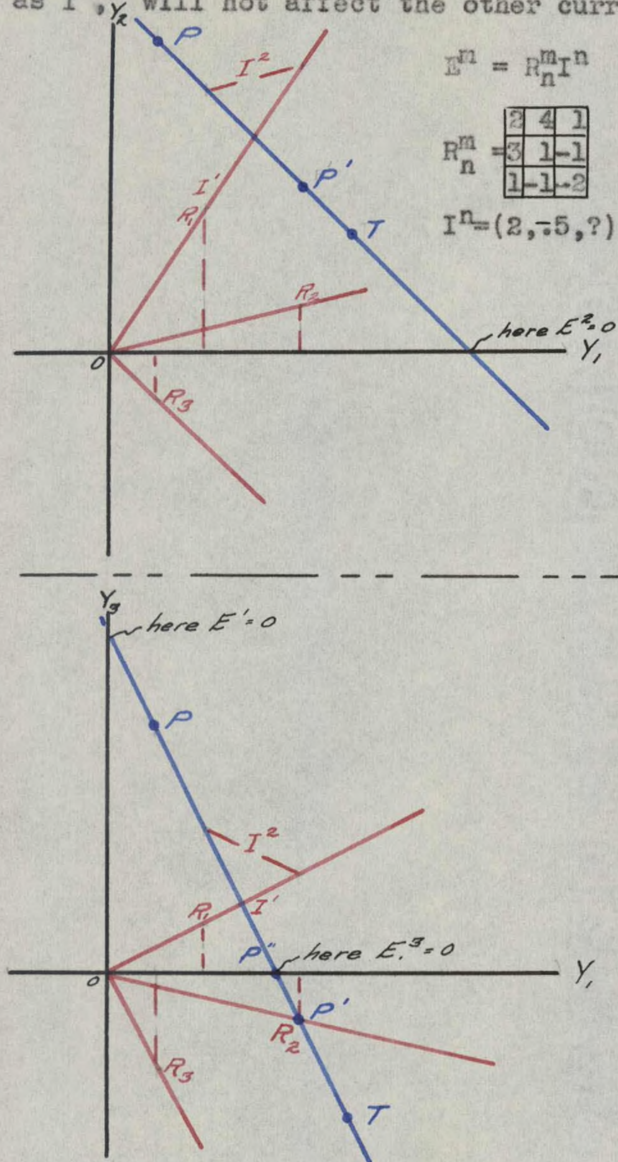


Fig. 10 Effects on Position of P as Coordinate I^3 is Varied

all three of the voltage components E^m. Zero value for one of the voltages will be found at some point such as P" where the line PT crosses the proper axis. In general the maximum values in such cases as shown here will be at infinity, though the relative rates of change of the various quantities can be obtained from an analytic study of the diagrams.

A point should be emphasized here, and that is that at this stage we are stressing the mathematical rather than the electrical aspects of the problems involved. With any change in an electric circuit, there is associated certain transient effects due to the inductance and capacity of the circuit. These effects are not considered in this thesis, but rather we will always consider a "change" as referring to a change in the steady state conditions of the circuit only.

Effect of Variation in Explicit Coordinates:

When the voltages are varied, the effect on the current values -- in the form we have been using -- is not so apparent, since now the problem becomes one of solving for the implicit variables for each new position of P. However, since the coefficient values are not changed, the position of Plane III (as in fig. 5) is not affected, and the necessity of working through this plane does not encumber the analysis unduly. Once again we notice that a change in one coordinate does not affect the other coordinates of the same set. In other words each of the voltages in a problem such as Prob. 1, page 31, may be varied independently of the others, but a variation in one of the voltages will usually affect all of the current values in one way or another.

Effect of Variation of the Coefficient Values:

When one of the coefficient values is altered, a more complex problem results, for while the position of the point P with respect to the origin is not necessarily changed, one or more of the transformed axes will be moved, and the coordinates of P in the transformed system will thus take on new values.

The best method of attack here is through a study of the change in position of the unit points. If but a single coefficient value is altered, all values will remain unchanged except one of the coordinates of one of the unit points. Such a variation must be linear as is shown in fig. 11 where the value of d_1 (the resistance R_1^1) is allowed to change.